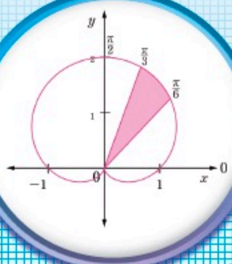
Advanced Mathematics

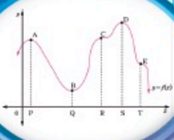
for Secondary Schools
Student's Book

Form Five



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$



Tanzania Institute of Education

Advanced Mathematics

for Secondary Schools Student's Book

Form Five

THE UNITED REPUBLIC OF TANZANIA
MINISTRY OF EDUCATION,
SCIENCE AND TECHNOLOGY

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Dr Aneth A. Komba
Director General
Tanzania Institute of Education

Preface

This textbook, *Advanced Mathematics for Secondary Schools* is written specifically for Form Five Students in the United Republic of Tanzania. It is written in accordance with the 2010 Advanced Mathematics Syllabus for Advanced Level Secondary Schools Education, Form V - VI issued by then, the Ministry of Education and Vocational Training.

The book consists of ten chapters, namely; Calculating devices, Sets, Logic, Coordinate geometry 1, Functions, Algebra, Trigonometry, Linear programming, Differentiation, and Integration. Each chapter contains illustrations, activities, and exercises. The answer for questions for proofs, show, verifications, and some illustrations are not provided. You are encouraged to do all activities and exercises together with other assignments provided. Doing so, will enable you to develop the intended competencies.

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Chapter One

Calculating devices

Introduction

There are several calculating devices such as an abacus, slide rule, computer, electrical calculator, and electronic calculators. The useful devices for calculations are electronic calculators. Electronic calculators are made to serve different purposes like in businesses, banking, mathematics, science, and engineering. In this chapter, you will learn about scientific calculators and computer packages. Calculating devices are widely used in commercial and numerical computations. The competencies developed will help you to compute different calculations in mathematics, businesses, and other related fields.

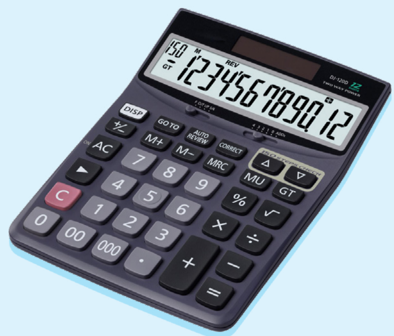
Scientific calculators

Sientific calculators are designed to compute numerical values in science, engineering, business, and mathematics. They can handle various functions like polynomial, trigonometric, exponential, logarithmic, and many other functions.

There are two types of scientific calculators. These are programmable and non-programmable scientific calculators. Their differences are based on their functionalities. Programmable scientific calculators can write, store, and graph the input data, whereas non-programmable scientific calculators are not capable of graphing the input data. In this case, non-programmable scientific calculators will be emphasized as a part of calculating devices.

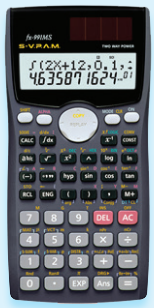
Activity 1.1: Identifying different types of calculators

Individually or in a group, study carefully each of the following electronic calculators and perform the tasks that follow:



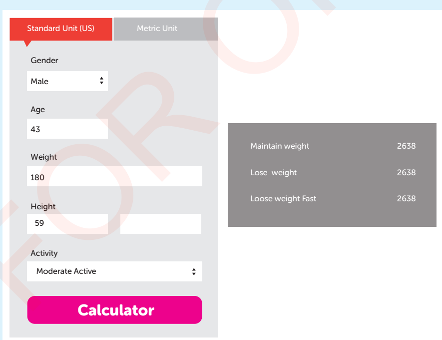
(a) Simple calculator

(b) Graphing calculator



(c) Scientific calculator

(d) Printing calculator



(e) Online calculator

1. List non-programmable and programmable scientific calculators.
2. Give reasons for your choices in task 1.
3. What challenges did you face in performing tasks 1 and 2?
4. Share your results with your fellow students for more inputs.

Basic features of non-programmable scientific calculators

Scientific calculators have many features that may differ depending on the manufacturers as well as on the models. However, their basic operations remain the same. Figure 1.1 shows the basic features of a non-programmable scientific calculator.

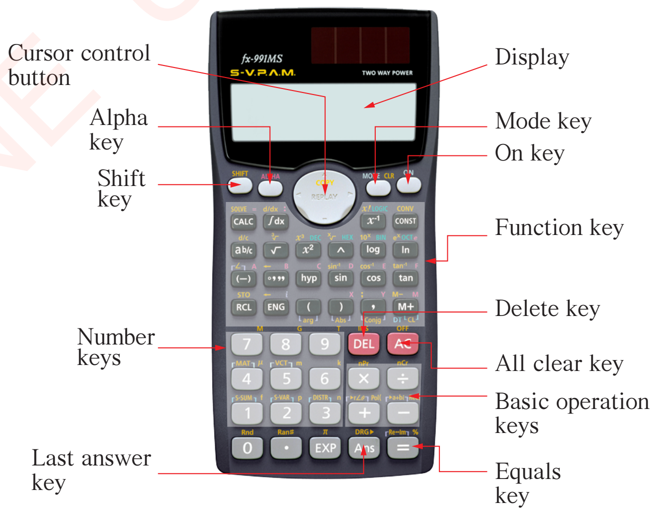
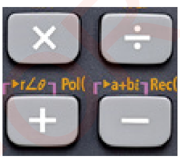


Figure 1.1: *Basic features for non-programmable scientific calculator*

Keys and functions of non-programmable scientific calculators

In most scientific calculators, buttons have more than one function. The symbols for the functions are differentiated by colours. Table 1.1 shows some basic keys / buttons of a non-programmable scientific calculator and their functions.

Table 1.1: *Keys and functions of a non-programmable scientific calculator*

Keys/Buttons	Name	Second function /shift	Example
	Switching on	Making a calculator ready for use	
	Clear all	Switching off	
	Deletion	Insertion	
	Replay	Copy	
	Mode	Clear	
	Alpha		A, B, C, D, E, F, X, Y, and M
	Inverse	Factorial notation	3^{-1}, 6!
	Basic arithmetic operations symbols		$4+6-2\times5\div2$
	Brackets insertion	Argument and modulus of complex numbers	$7\times(2\times4+116)\div3$ Arg $(2+i)$, Abs $(2+i)$

	Number in mixed fractions	Number in simple fractions	$5\frac{1}{2}, \frac{1}{4}$
	Exponents	Used for calculating the square, cube, n^{th} root, and numbers raised to other exponents	$4^2, 3^3, 5^4$
	Equal sign	symbol for executing the answer of certain inputs	$2+5=7$
	Square root	Cube root	$\sqrt{1258}, \sqrt[3]{217}$
	Logarithms and natural logarithm	Anti-logarithms exponential functions	$\log 1000, \ln 1000$ 105, e5
	Trigonometric functions	Inverse of trigonometric functions	$\sin 40^\circ, \sin^{-1}(0.5)$
	Standard form/notation	Value of π	$4\text{E}-4, 4\pi$
	Hyperbolic functions		sinh(2)
	Comma	Data separation in statistics and conjugate of a complex number	
	Degree/ minutes/ seconds	Degree, minutes, and second convention to degrees	$15°40'59''$, 15.6831
	Integration functions	Derivative functions	$\int_1^4 \pi y dy, d/dx(x^2, 1)$
	Calculate	Solve	$\begin{cases} 5x+2y=14 \\ 3x+4y=14 \end{cases}$

Steps of operating some basic keys of non-programmable scientific calculators

An electronic calculator with brand name CASIO *fx-991MS* is a non-programmable scientific calculator which is used to perform different calculations. This calculator is common due to its affordability and availability in the market. Most of these calculators have multi-line displays, which enables the user to see both the expression under operations and the result.

(a) Replay (cursor control) button

A usefully non-programmable scientific calculators have a replay button which is used to review previous calculations. The replay button is marked with four arrows as shown in Table 1.1. It is used for moving a cursor to the left, right, top, and bottom following the directions of the arrows. The "REPLAY" button for some non-programmable scientific calculators has a functionality of "COPY" as a second function.

(b) Preceding keys

A non-programmable scientific calculator has preceding keys which are used to precede the other keys depending on their colours. There are two preceding keys in scientific calculators, namely "SHIFT" key and "ALPHA" key. The "SHIFT" key precedes all keys with yellow colour while the "ALPHA" key precedes all keys with pink colour to access their functions. The steps for using these keys are as follows:

Step 1: Press "SHIFT" or "ALPHA" key.

Step 2: Press the key corresponding to the colour which is preceded in Step 1.

(c) Exponent and roots

A non-programmable scientific calculator can calculate the exponent of a number or the x^{th} root of a number. The buttons for performing these two operations are labelled "$\wedge$" and "$\sqrt[x]{\ }$", respectively.

The following are steps for finding exponent of a number:

Step 1: Enter the number.

Step 2: Press the "$\wedge$" key followed by exponent. If the exponent is in fraction, enclose it with parenthesis.

Step 3: Press the "=" key to obtain the value of exponential.

The steps for finding the x^{th} root of a number using a non-programmable scientific calculator are as follows:

Step 1: Enter the root index number (2 for square root, 3 for cube root, 4 for fourth root).

Step 2: Press the "SHIFT" key.

Step 3: Press the "$\wedge$" key and the symbol "$\sqrt[x]{\ }$" will be displayed.

Step 4: Enter the number whose root is required.

Step 5: Press the "=" key.

Example 1.1

Use a non-programmable scientific calculator to evaluate each of the following:

(a) $\sqrt[12]{9788}$ (c) $(8.9)^3$

(b) $\sqrt[5]{243}$ (d) $\sqrt[4]{135}$

Solution

Following the stated steps, the displayed answers are:

(a) $\sqrt[12]{9788} = 2.15059103$

(b) $\sqrt[5]{243} = 3$

(c) $(8.9)^3 = 704.969$

(d) $\sqrt[4]{135} = 3.408658099$

(d) Memory keys

A non-programmable scientific calculator can store answers for a while and retrieve using "RCL" when needed for use in calculations.

The following are the steps used to store results for a while in a non-programmable scientific calculator:

Step 1: Type a number to be stored.

Step 2: Press "SHIFT" key.

Step 3: Press "STO" key and then "M+" to store the number displayed.

Step 4: Press "RCL" followed by "M+" key to recall the stored number.

Step 5: Press "=" key to obtain the result.

Example 1.2

Use appropriate memory keys of a non-programmable scientific calculator to evaluate $\dfrac{200}{-31.5+2.5}$.

Solution

Step 1: Type -31.5 and press "+" key.

Step 2: Type 2.5 and press "=" key.

Step 3: Press "SHIFT" key.

Step 4: Press "STO" and then press "M+" key.

Step 5: Type 200 and press "$\div$".

Step 6: Press "RCL" key and then "M+" key.

Step 7: Press "=" key to obtain the answer.

After completing the steps, the result is -6.896551724.

(e) Scientific notation

Scientific notation, also referred to as standard notation (form), is a way of expressing numbers in the form of $m \times 10^n$, where n, the exponent, is an integer and m satisfies the condition $1 \leq m < 10$.

The steps for expressing a number in scientific notation are as follows:

Step 1: Type the number using number key.

Step 2: Press "=" key.

Step 3: Press "ENG" key.

Activity 1.2: Writing numbers in scientific notation with specified decimal places

Individually or in a group, perform the following tasks:

1. Set your non-programmable scientific calculator to fix the answers in any number of decimal places.
2. Use a non-programmable scientific calculator to evaluate each of the following:

   (a) $\dfrac{\sqrt{2}}{(46 \times 10^{-2})}$ in 2 decimal places

   (b) $\dfrac{\left(3+\sqrt{2}\right)}{4}$ in 4 decimal places

   (c) $\dfrac{0.95 \tan 23}{9.1245}$ in 3 decimal places
3. By using a non-programmable scientific calculator, write all the steps of fixing numbers with specified number of decimal places.

Example 1.3

Express 1245384791 in scientific notation by using a non-programmable scientific calculator.

Solution

Step 1: Type 1245384791.

Step 2: Press "=" key.

Step 3: Press "ENG" key.

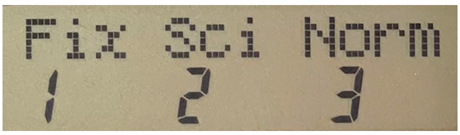
After following the steps, the result is $1.245384791 \times 10^{09}$.

Also, a number can be expressed in scientific notation with a specified number of significant figures or decimal places.

The following are steps for expressing numbers in scientific notation with specified number of significant figures or decimal places.

Step 1: Press the “MODE” key five times.

Step 2: Select “Fix“ for writing figure in a certain number of decimal places, “Sci” for writing figure in a certain number of significant figures, and “Norm” for writing number in a normal form. The screen of the calculator looks as follows:



Step 3: Type the number of decimal places, or significant figures from the given limit $0 \sim 9$.

Step 4: Use the number button to type the number of decimal places, or significant figures specified.

Step 5: Press “=” key to get the result in decimal places, or significant figures specified.

Example 1.4

Use a non-programmable scientific calculator to express 0.0024587 in scientific notation with 3 significant figures.

Solution

Step 1: Type the number 0.0024587 and press “=” key.

Step 2: Press the “MODE” key five times.

Step 3: Select “Sci” key by pressing the key numbered 2 as instructed on the calculator.

Step 4: Type number 3 from the limit $0 \sim 9$.

Step 5: Press “=” key to get the result in 3 significant figures.

After following the steps, the result is 2.46×10^{-03}.

(f) Logarithmic functions to base 10 and base *e*

Scientific calculators can compute common logarithms (base ten) and natural logarithms (base e) of numbers and their inverses.

(i) ***Base 10 logarithms***

The following steps are used to find the logarithm of a number to base ten:

Step 1: Press the “log” key.

Step 2: Enter the number whose logarithm is to be determined.

Step 3: Press the “=” key, then the logarithm of a number will be displayed.

Example 1.5

Evaluate $\log 45000$ using a non-programmable scientific calculator.

Solution

Step 1: Press the "log" key.

Step 2: Type 45000.

Step 3: Press the "=" key.

After following the steps, the result for $\log 45000 = 4.653212514$.

Note that, if the base is not 10 then apply base change, by using,

$$\log_b a = \frac{\log a}{\log b}.$$

Example 1.6

Using a non-programmable scientific calculator, evaluate $\log_5 6$.

Solution

$\log_5 6$, this can be written as,

$$\log_5 6 = \frac{\log 6}{\log 5}.$$

Then, following the steps as in Example 1.5, the result is 1.113282753.

Therefore, $\log_5 6 = 1.113282753$.

(ii) *Anti-logarithms to base 10 logarithms*

The following steps are used to find the anti-logarithm of a number to base 10:

Step 1: Press the "SHIFT" key.

Step 2: Press the "log" key.

Step 3: Enter the number whose anti-logarithm is required.

Step 4: Press the "=" key, then the logarithm of a number will be displayed.

Example 1.7

Find the anti-logarithm of 0.1284 using a non-programmable scientific calculator.

Solution

Step 1: Press the "SHIFT" key.

Step 2: Press the "log" key.

Step 3: Enter 0.1284.

Step 4: Press the "=" key.

After following the steps, the result for anti-logarithm of $0.1284 = 1.344002263$.

(iii) *Natural logarithms (log to base e)*

The natural logarithm of any number is its logarithm to the base of the mathematical constant "e" which is an irrational number approximately equal to 2.718281828459. The natural logarithm of x is generally written as $\ln x$ or $\log_e x$.

The following steps are used to find the natural logarithm of a number:

Step 1: Press the "ln" key.

Step 2: Enter the number whose natural logarithm is required.

Step 3: Press "=" key to obtain the result.

Example 1.8

Find $\ln 628$ by using a non-programmable scientific calculator.

Solution

Step 1: Press the "ln" key.

Step 2: Enter 628.

Step 3: Press the "=" key to obtain the result.

After following the steps, the result for $\ln 628 = 6.442540166$.

Note that, $\ln e^{x} = e^{\ln x} = x$. For instance, $\ln e^{2} = e^{\ln 2} = 2$.

Activity 1.3: Computing various expressions with an exponential numbers exponential numbers to base *e*

Individually or in a group, perform the following tasks:

1. Study careful the following rational expression where e is an exponetial number from a non- programmable scientific calculator and the variables a, b, c, d, and f, where e is an exponential number and $a, b, c, d, f \in \mathbb{Z}^{+}$

   (a) $\dfrac{\left(ae^{b}+ce^{d}\right)^{\frac{1}{b}}}{\sqrt{be^{fe}}}$

   (b) $\dfrac{e^{a}+be^{e^{a}}}{\sqrt{be^{c}}}$

   (c) $\dfrac{e^{\sqrt{a}+b^{\sqrt[a]{b}}}+\log e}{c\ln e^{c}-\ln\left(a\ln e^{\log_b e^{b}}\right)}$

2. Use a non-programmable scientific calculator to compute each rational expressions in task 1 for $a=2$, $b=3$, $c=4$, $d=2$, and $f=1$.

3. Write the steps you have used to compute the expressions in task 1 and 2.

4. Share your results with your fellow students for further discussion.

Exercise 1.1

1. Use a non-programmable scientific calculator to evaluate each of the following expressions. Write the results in standard notation (in four significant figures):

   (a) $\sqrt[7]{0.064}$

   (b) $\log_{3} 13$

   (c) $\dfrac{e^{4}\times\sqrt[3]{\ln 19\times\sqrt{279}\times\log 325}}{\sqrt{21.7}}$

   (d) $\ln\left(\dfrac{8\times10^{2}}{2\log\sqrt{9}}\right)$

   (e) $\dfrac{5.23\times\ln\sqrt{7}\div\ln\sqrt{3}}{186\log 2021}$

   (f) $\left(\sqrt{\dfrac{\log_{7} 631}{\ln 121+4\ln 2}}\right)^{3}$

   (g) $\left(\dfrac{303\times4+16\times1.26543}{1.03\times10^{-8}\times\ln 3}\right)^{3}$

(h) $\sqrt[5]{\dfrac{e^{\log 3}+\sqrt{\log_5 \sqrt{5}}}{e^{\ln 3}}}$

2. Evaluate each of the following expressions by using a non-programmable scientific calculator. Correct the answer to the stated decimal places or significant figures

(a) $\dfrac{\log 178 \times \sqrt{\ln 190}}{\sqrt[3]{3.87935 \times 27^3}}$ in 4 decimal places.

(b) $\dfrac{23.37+\log_5 312.34}{517 \times e^{\ln 316}}$ in 3 significant figures.

(c) $\dfrac{\sqrt[3]{0.3854} \times (12.3456)^3}{(0.056749)^4 \times \sqrt[5]{987}}$ in 6 significant figures.

(d) $6\left(\dfrac{e^{0.3}-e^{-0.3}}{e^{0.3}+e^{-0.3}}\right) \times \left(\dfrac{3.946 \times e^{\ln 2.67}}{\log 10^{1.5}}\right)^{\frac{1}{3}}$ in 4 decimal places.

(e) $\dfrac{\sqrt{0.06709}-0.2347^2}{6.87^4+\sqrt{734.8}}$ in 5 significant figures.

(f) $\sqrt{\dfrac{\log 3.14 \times \ln 2.3567}{2.5 \times 10^2}}$ in 6 decimal places.

(g) $\dfrac{0.316^{\frac{3}{5}} \times 1.6754^{\frac{4}{3}}}{4.567^{\frac{1}{4}}}$ in 4 significant figures.

(h) $\sqrt[3]{\left(\dfrac{\sqrt{2.17}+\sqrt{5.176}}{\sqrt{3.16} \times \sqrt{0.984}}\right)^4}$ in 2 significant figures.

(i) $\left(\log\left(\dfrac{1.356^3+0.924^2}{46.89-22.78}\right)\right)^3$ in 6 significant figures.

(j) $\ln\left(\dfrac{4 \times 10^{-3}}{3 \times 10^{-3}}\right)^2$ in 5 decimal places.

(g) Degree button

The degree button " °''' " is used to insert a number of degree format into the calculator. The number in degree format has three parts which are Degree " ° ", Minutes " ' ", and Seconds " '' ".

The following are the steps of inserting a number in degree format in a non-programmable scientific calculator.

Step 1: Enter the degree part number followed by " °''' " key.

Step 2: Enter the minutes part of the number followed by " °''' " key.

Step 3: Enter the seconds part of the number followed by " °''' " key.

Step 4: Press "=" key to obtain the result in degree format.

Example 1.9

Use a non-programmable scientific calculator to write the following numbers in the standard degree form:

(a) $20°45'18''$ (d) $15°46''$
(b) $25'55''$ (e) $45°80'$
(c) $79''$

Solution

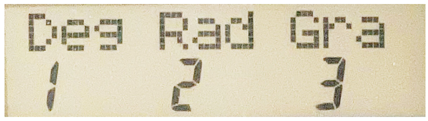
After following the required steps for inserting numbers into degree format gives;

(a) $20°45'18'' = 20°45'18''$
(b) $25'55'' = 0°25'55''$
(c) $79'' = 0°1'19''$
(d) $15°46'' = 15°0'46''$
(e) $45°80' = 46°20'0''$

A number in degree format can be converted to radian format and vice versa by using a non-programmable scientific calculator. The conversion involves the relationship between radian and degree, that is; 2π radian $= 360°$.

(i) ***Steps of converting a number from degree format to radian***

Step 1: Change the calculator into radian by pressing the "MODE" button four times and then press " 2 ". The screen of the calculator looks as follows.



Step 2: Enter the required degree number to be converted to radians.

Step 3: Press the "SHIFT" key followed by "ANS" key.

Step 4: Press the "1" key to access degree symbol in the display.

Step 5: Press the "=" key to obtain the result in radian format.

Note that, convert minutes and seconds into degrees if the given angle contains them.

The following are steps for conversion of minutes and seconds into degrees:

Step 1: Enter the required angle to be converted.

Step 2: Press the "=" key.

Step 3: Press the "SHIFT" key followed by " °’’’ " key.

Step 4: Press the "SHIFT" key followed by "ANS" key.

Step 5: Press the "1" key to access degree symbol in the display.

Step 6: Press the "=" key to obtain the result in radian format.

Activity 1.4: Comparing trigonometric ratios in degrees and in radians mode

Individually or in a group, perform the following tasks:

1. Consider the following special angles of trigonometric ratios

Table 1.2: *Values of special trigonometric ratios*

Trigonometric ratios	Value
$\sin 45°$	$\frac{\sqrt{2}}{2} = 0.707106781$
$\cos 45°$	$\frac{\sqrt{2}}{2} = 0.707106781$
$\tan 45°$	1

2. Use a non-programmable scientific calculator in "Rad" mode to compare the value of trigonometric ratios in Table 1.2.
3. Are the values in task 2 the same as in Table 1.2? if not why?
4. From your calculator, what can you do to make the values to be the same as in Table 1.2?
5. Share your results with your fellow students for further discussion.

(ii) ***Steps in converting a number from radian to degree measure***

Step 1: Make sure your calculator is in degree by pressing the "MODE" button four times and then press "1".

Step 2: Enter the radian number.

Step 3: Press the "SHIFT" key followed by "ANS" key.

Step 4: From the menu, press "2" followed by "=".

Step 5: Press the "SHIFT" key followed by "°,,," key to display the answer in degrees.

Example 1.10

Use a non-programmable scientific calculator to convert each of the following numbers to their respective form:

(a) $40°35'18''$ to radian form (c) 0.27854 to degree form

(b) 0.345 to degree form (d) $46°20'$ to radian form

Solution

(a) $40°35'18'' = 0.708400054$ to radiation

(b) $0.345 = 19°46'1.36''$ to radian form

(c) $0.27854 = 15°57'33''$ to degree form

(d) $46°20' = 0.80866922$ to radian form

Exercise 1.2

1. Use a non-programmable scientific calculator to convert each of the following degree to radian form:

   (a) $45°$ (e) $540°7'45''$
   (b) $30°$ (f) $78°6'23''$
   (c) $360°$ (g) $7'54''$
   (d) $19°$ (h) $45°0'18''$

2. Use a non-programmable scientific calculator to convert each of the following radians to degree form:

   (a) $\frac{2}{3}\pi$ (e) $0.3\,\text{rad}$
   (b) $\frac{5}{3}\pi$ (f) $0.1235\,\text{rad}$
   (c) 3π (g) $0.78\,\text{rad}$
   (d) $\frac{\pi}{4}$ (h) $0.786\,\text{rad}$

3. Evaluate each of the following using a non-programmable scientific calculator. Providing the answers in the stated form:

   (a) $24°6'34'' + 0.9\,\text{rad}$ in radian form.
   (b) $34°56'4'' + 0.983\,\text{rad}$ in radian form.
   (c) $60°46'44'' - 0.067\,\text{rad}$ in degree form.
   (d) $76°29'34'' + 0.215\,\text{rad}$ in degree form.
   (e) $96°16'54'' + 0.135\,\text{rad}$ in radian form.
   (f) $208°26'33'' - 1.767\,\text{rad}$ in radian form
   (g) $247°2'11'' + 0.785\,\text{rad}$ in degree form.
   (h) $324°6'34'' + 1.9\,\text{rad}$ in radian form.

(h) Trigonometric functions

The values of trigonometric functions such as sine, cosine, tangent, and their inverses can be determined using scientific calculators. The buttons for these functions are written as “sin”, “cos”, and “tan” on the calculator. For a non-programmable scientific calculator, the steps for computing values of trigonometric ratios are as follows:

Step 1: With the scientific calculator in degree mode “D”, press the key of the trigonometric ratio required. For example, sin, cos, and tan.

Step 2: Enter the angle whose trigonometric ratio is required.

Step 3: Press the "=" key.

If the trigonometric function has an angle in radian format, for instance 3 radians, or $\frac{\pi}{3}$ radians, then the mode of the calculator should be in radians.

Activity 1.5: Identifying the useful keys/buttons on a scientific calculator used for determining trigonometric ratios and their inverses

Individually or in a group, perform the following tasks:

1. Set your calculator in radian mode.
2. Identify all keys/buttons with trigonometric ratios and their inverses.
3. Use the identified buttons in task 2 to determine $\sin 40°$, $\cos 40°$, $\tan 40°$ $\sin^{-1}(0.5)$, $\cos^{-1}(0.5)$, and $\tan^{-1}(0.5)$.
4. Write all steps of finding the trigonometric ratios and their inverses.
5. Share the identified steps in task 4 with other students.

Example 1.11

Use a non-programmable scientific calculator to evaluate each of the following trigonometric ratios:

(a) $\cos\frac{2\pi}{5}$ (d) $\sin 16$

(b) $\cos^{-1} 0.257$ (e) $\tan^{-1} 1.4$

(c) $\tan 54°5'34''$ (f) $\tan 330°35'57''$

Solution

(a) $\cos\frac{2\pi}{5} = 0.309016994$

(b) $\cos^{-1} 0.257 = 1.310879679$

(c) $\tan 54°5'34'' = 1.381079275$

(d) $\sin 16 = -0.287903316$

(e) $\tan^{-1} 1.4 = 0.95054684$

(f) $\tan 330°35'57'' = -0.563490204$

Activity 1.6: Identifying the reciprocal of trigonometric ratios

Individually or in a group, perform the following tasks:

1. Use a non-programmable scientific calculator to evaluate the following; $\sin(45°)$, $\cos(45°)$, and $\tan(45°)$.
2. Find the reciprocal of all trigonometric functions evaluated in task 1.
3. Evaluate the following trigonometric functions using the knowledge obtained in tasks 1 and 2.
   (a) $\sec(35°)$ (c) $\cot(-80°)$
   (b) $\operatorname{cosec}(320°)$
4. Identify the steps you have used to evaluate the trigonometric functions in task 3.
5. Share the identified steps with your fellow students for more inputs.

If $|x| \geq 1$, then the inverses of the reciprocals of the trigonometric ratios are given by,

$$\sec^{-1} x = \cos^{-1}\left(\frac{1}{x}\right)$$

$$\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1} x = \tan^{-1}\left(\frac{1}{x}\right)$$

Exercise 1.3

1. Use a non-programmable scientific calculator to evaluate each of the following:

   (a) $\sin 30°4'54''$ (f) $\operatorname{cosec} \frac{3}{5}\pi$

   (b) $\cos \frac{\pi}{4}$ (g) $\tan \tan^{-1} \frac{2}{3}$

   (c) $\cos^{-1} \frac{1}{2}$ (h) $\sec^{-1} \cos \pi$

   (d) $\sec^{-1} 2$ (i) $\cot^{-1} \tan 45°6'27''$

   (e) $\cot^{-1} \frac{3}{4}$

2. Use a non-programmable scientific calculator to evaluate each of the following expressions. (Give the answer correct to the stated number of decimal places or significant figures):

   (a) $\dfrac{\sin^{-1} 0.5 + \log 5}{\sqrt[3]{\ln 5\left(\frac{22}{7} - \sin 45°0'30''\right)}}$

   (to 4 significant figures).

   (b) $\left(\dfrac{\tan^{-1} 0.345 + \ln \sqrt[3]{19.78}}{\cos 78°39'40'' + \log \pi}\right)^2$

   (to 3 decimal places).

   (c) $\dfrac{\tan \frac{\pi}{4} + \cos \cos^{-1} \ln 2}{\cos^{-1} 0.3 + \sin^{-1} 0.467}$

   (to 5 decimal places).

   (d) $\dfrac{\sqrt[3]{0.5677}}{\pi \sec^{-1}(2.34) + \sqrt{5.7 \times 10^{-2}}}$

   (to 3 significant figures).

   (e) $\dfrac{5.67^2 \cos 40°27'29''}{\sqrt{90.34 + \tan 67°1'12''}}$

   (to 6 significant figures).

   (f) $\dfrac{\cos ec^{-1} \sin \frac{\pi}{2} + \log_5 \sqrt{3 \times 10^{-3}}}{\ln 2 + \tan^{-1} \tan 30°21'41''}$

   (to 6 decimal places).

   (g) $\dfrac{\log_7 10.5 + \cot^{-1} 3.14}{\sqrt{23.90} + \cos \pi}$

   (h) $\dfrac{\sqrt[3]{\tan 45°} + \ln \operatorname{cosec}^{-1} \frac{3}{2}}{\cos^{-1} 0.345 + 0.367 \times 10^{-3}}$

3. If $\tan \theta = \dfrac{14.32 \times \tan 20°36'45''}{78.90 \times \cos 17°5'37''}$ using a non-programmable scientific calculator determine the value of θ.

   (a) In degrees, correct to 4 decimal places

   (b) In radian, correct to 4 decimal places.

(i) Operations on fractions

A non-programmable scientific calculator can perform operations on fractions using the fraction key " $a\frac{b}{c}$ ".

The following steps can be used to enter the fraction in the scientific calculator:

Step 1: Type the numerator.

Step 2: Press the fraction button key, that is " $a\frac{b}{c}$ ".

Step 3: Type the denominator.

Step 4: Press the "=" key to obtain the result in fraction format.

The following steps can be used to enter a mixed fraction in the scientific calculator:

Step 1: Type the whole number part.

Step 2: Press the fraction button key, that is " $a\frac{b}{c}$ ".

Step 3: Type the numerator.

Step 4: Press the fraction key, that is " $a\frac{b}{c}$ ".

Step 5: Type the denominator.

Step 6: Press the "=" button key to obtain the result in fraction format.

Example 1.12

Use a non-programmable scientific calculator to obtain the value of each of the following expressions:

(a) $\frac{18}{5} \div \frac{3}{12}$ (b) $9\frac{1}{5} + 2\frac{3}{8} - 2\frac{4}{5}$

Solution

Following the stated steps,

(a) $\frac{18}{5} \div \frac{3}{12} = 14\frac{2}{5}$

(b) $9\frac{1}{5} + 2\frac{3}{8} - 2\frac{4}{5} = 8\frac{31}{40}$

A non-programmable scientific calculator can convert numbers from fractions to decimals and vice versa.

The following steps are used to convert fractions to decimal numbers:

Step 1: Type the numerator.

Step 2: Press the fraction button key, that is " $a\frac{b}{c}$ ".

Step 3: Type the denominator.

Step 4: Press the "=" button key to obtain the result in fraction format.

Step 5: Press the fraction button again, that is " $a\frac{b}{c}$ " key to obtain the result in decimal format.

When converting numbers from decimal to fraction form, follow the following steps:

Step 1: Type the decimal number.

Step 2: Press the "=" key to obtain the result in decimal form.

Step 3: Press the fraction button key, that is " $a\frac{b}{c}$ " to obtain the result in fraction form.

Example 1.13

Use a non-programmable scientific calculator to convert each of the following numbers into the indicated forms:

(a) $7\frac{8}{21}$ into decimal form.

(b) 2.6 into fraction form.

Solution

Following the stated steps, the displayed answers are:

(a) $7\frac{8}{21} = 7.380952381$

(b) $2.6 = 2\frac{3}{5}$

Exercise 1.4

1. By using a non-programmable scientific calculator, evaluate each of the following expressions:

(a) $\left(19\frac{1}{2}\times 7\frac{1}{4}\right)-\left(4\frac{1}{8}\div 2\frac{1}{2}\right)$

(b) $2\frac{3}{4}+\frac{7}{9}-1\frac{3}{7}\div 3\frac{3}{5}$

(c) $\sqrt[5]{\dfrac{4\frac{2}{3}+7\frac{5}{9}-6\frac{1}{5}}{\frac{2}{7}\times 1\frac{7}{9}}}$

(d) $\sqrt[7]{\log_7\left(8\times 10^2\right)+\frac{\pi}{4}}$

(e) $\dfrac{\sin 30°7'}{\cos^{-1}0.5}+\dfrac{\log_2 5.3}{e^{-1}}$

(f) $2\frac{5}{7}+\sqrt[5]{e^2+\sin^{-1}\sqrt{0.3}}$.

2. By using a non-programmable scientific calculator, compute each of the following expressions:

(a) $\dfrac{\sqrt{240}\times e^{\ln\frac{1}{3}}\sin 22°}{\sqrt{\sec^{-1}(17)}\times 3^{4\ln 11}}$

(to 3 significant figures).

(b) $\log\left(\dfrac{\sqrt{98.2}\times(0.0076)^{-1}\times 10^7}{\tan\frac{\pi}{3}\times\cos^3\frac{\pi}{4}}\right)$

(to 6 significant figures).

(c) $\dfrac{\sin^{-1}(\ln 2)+\tan^2(\cos 45°23'12'')}{(\log_e 24)^{\frac{6}{5}}}$

(to 4 significant figures).

(d) $\sqrt{\dfrac{0.485^6+\tan^{-1}(1.54)\times e^3}{62.54^4\times\operatorname{cosec}^{-1}3.5}}$

(to 4 decimal places).

(e) $\left(\dfrac{68.48\times\sin 35.56}{\tan 46.65\times\sqrt[3]{751}}\right)^{\frac{3}{5}}$

(to 3 decimal places).

(f) $\sqrt[6]{e^3+(\log 23.4)^5\sqrt[3]{375.56}}$

(to 3 significant figures).

(g) $\dfrac{\sin 25°20'-\sqrt[3]{0.05e^{-3}}}{\log_2 3.2+0.006e^{0.3}}$

(to 6 significant figures).

(h) $\left(\sqrt[5]{\dfrac{(9.621)(7.0678)}{35.34+0.34605}}\right)^{\frac{5}{7}}$

(to 3 decimal places).

(i) $\log_e(e^4+2\ln 7.36)+\log_2 7$

(to 3 decimal places).

(j) $\left(\dfrac{200.31\times\sqrt{2000}}{1721\times\log_7 3}\right)^{\frac{2}{3}}$

(to 2 significant figures).

(k) $\sqrt[3]{\dfrac{2\frac{2}{5}+\log_3 122\times\ln 315}{e^{0.9}}}$

(to 6 significant figures).

(l) $\left(\dfrac{\left(5\frac{2}{7}\right)^2\times\operatorname{cosec}^{-1}2.3}{\log_e\sqrt{168.9}}\right)^{\frac{7}{9}}$

(to 5 significant figures).

(j) Statistical calculations

In a non-programmable scientific calculator, statistical data in numerals can be processed to determine various statistical parameters such as sum, mean, and standard deviation for ungrouped and grouped data.

(i) *Ungrouped statistical data*

Use the following steps in computations involving ungrouped statistical data:

Step 1: Press the "MODE" key two times.

Step 2: Select "SD" mode by pressing the button numbered 1 as instructed on the screen of the calculator. The screen of the calculator looks as follows.

Step 3: Type the number (value) then press the "M+" key.

Step 4: Repeat Step 3 for each of the values.

After entering the data into the calculator on the "SD" mode, the steps to determine sum of data values $\left(\sum x\right)$, sum of squared values $\left(\sum x^2\right)$, mean $(\bar{x})$, and standard deviation (δ_x) are as follows:

Step 1: Press the "SHIFT" button.

Step 2: Press the button key numbered 1 or 2 to obtain different options of statistical values.

Step 3: Select the statistical value needed by pressing the relevant button as instructed on the screen of the calculator.

Step 4: Press the "=" key to obtain the result.

Activity 1.7: Computing the central tendency and dispersion of data

Individually or in a group, perform the following tasks:

1. Use a tape measure to measure the heights of your fellow students.
2. Enter the heights into a non-programmable scientific calculator. Then,
   compute the following values:
   (a) Mean, $\bar{x}$
   (c) Sum of data values, $\sum x$
   (b) Standard deviation, δ_x
   (d) Sum of squared data values, $\sum x^2$
3. Did you face any challenges in doing the computations? If Yes, discuss with your fellow students on how to solve the challenges.

Example 1.14

Given the following data 3, 4, 6, 2, 8, 7, 5, 9, 1. Use a non-programmable scientific calculator to find each of the following:

(a) $\bar{x}$ (b) δ_x (c) $\sum x$

Solution

Following the required steps, the results are:

(a) $\bar{x} = 5$ (b) $\delta_x = 2.581988897$ (c) $\sum x = 45$

(ii) ***Grouped statistical data***

For the case of grouped data, apply the following steps:

Step 1: Press the "MODE" key two times.

Step 2: Select "SD" by pressing the key number 1.

Step 3: Type the class mark data.

Step 4: Press the "SHIFT" key followed by comma to obtain "**;**".

Step 5: Type the corresponding frequency of the class mark.

Step 6: Press "M+" key.

Step 7: Repeat steps 3 to 6 for each class mark and its corresponding frequency.

Step 8: Press the "SHIFT" key, then press key number 1 or 2 to obtain different options of statistical value.

Step 9: Select the statistical value needed by pressing the relevant key as instructed on the display of the calculator.

Step 10: Press the "=" key to obtain the result.

Example 1.15

Given the following frequency distribution table, calculate the mean and standard deviation using a non-programmable scientific calculator.

Class interval	Class mark (x)	Frequency (f)
11 – 20	15.5	15
21 – 30	25.5	9
31 – 40	35.5	7
41 – 50	45.5	11
51 – 60	55.5	6
61 – 70	65.5	8

Solution

Following the required steps, the results are:

(a) $\overline{x} = 36.92857143$.

(b) $\delta_x = 17.67045268$.

Note that, once the statistical data are stored in a non-programmable scientific calculator, they cannot be cleared by turning off the calculator or by pressing the "ON" or "AC" keys. This can only be erased by either changing the "MODE" or by running the steps for clearing statistical data.

The following are steps for clearing statistical data in a non-programmable scientific calculator:

Step 1: Press the "SHIFT" key.

Step 2: Press the "MODE" key.

Step 3: Press the "3" key.

Step 4: Press the "=" key two times.

Exercise 1.5

1. Use non-programmable scientific calculator to find the mean of the following data:

2	5	2	4	1	1	1	2	1	3
3	2	1	2	1	1	2	4	3	2
1	2	3	1	4	2	3	1	1	2

2. Use a non-programmable scientific calculator to find $\overline{x}$, $\sum x$, $\sum x^2$, δ_x, and δ_x^2 for each of the following data:

(a) $44\frac{1}{2}$, $47\frac{1}{2}$, $50\frac{1}{2}$, $53\frac{1}{2}$, $56\frac{1}{2}$, $59\frac{1}{2}$, and $62\frac{1}{2}$

(b) 0.85, 0.88, 0.89, 0.93, 0.94, and 0.96

(c)

x	121	122	123	124	125
f	14	25	32	23	6

(d)

x	27	28	29	30	31	32
f	30	43	51	49	42	35

3. The speeds to the nearest kilometre per hour of 120 vehicles passing at a check point were recorded and grouped as shown in the following table:

Speed km/h	21 – 25	26 – 30	31 – 35	36 – 40	41 – 45
Number of vehicles	22	48	25	16	9

Use a non-programmable scientific calculator to find the mean and standard deviation of the speed of the vehicles.

4. Use a non-programmable scientific calculator to compute the mean and standard deviation for the following data:

x	0	1	2	3	4	5	6	7	8	9	10
f	4	10	16	28	34	44	32	16	10	6	0

5. Use a non-programmable scientific calculator to find the mean and the standard deviation of the following test scores of Form Five students.

Scores	1 – 10	11 – 20	21 – 30	31 – 40	41 – 50
Frequency (f)	13	4	20	21	10

6. Outline the steps for:
   (a) Editing statistical data entered in a non-programmable scientific calculator.
   (b) Clearing statistical data in a non-programmable scientific calculator.

7. Use a non-programmable scientific calculator to evaluate $\sum x$, $\sum x^2$, and $\bar{x}$ of the following data set:

x	61	64	67	70	73
Frequency	5	18	42	27	8

(k) Probability calculations

A non-programmable scientific calculator can also be used to perform various probability calculations.

(i) *Combination of items*

To find the number of combinations of n items in groups of r items by using a non-programmable scientific calculator, observe the following steps:

Step 1: Type the n value.

Step 2: Press the "SHIFT" key.

Step 3: Press the " ÷" key to obtain ${}^nC_r$ function.

Step 4: Type the r value to complete the expression for combination.

Step 5: Press the "=" key to obtain the result.

(ii) *Permutations of items*

To find the number of permutations of n items in groups of r items by using a non-programmable scientific calculator, observe the following steps:

Step 1: Type the n value.

Step 2: Press the "SHIFT" key.

Step 3: Press the "$\times$" key to obtain ${}^{n}P_{r}$ function.

Step 4: Type the r value to complete the expression for permutations.

Step 5: Press the "=" key to obtain the result.

(iii) *Factorial of numbers*

To find the factorial of a number x by using a non-programmable scientific calculator, observe the following steps:

Step 1: Enter the x value.

Step 2: Press the "SHIFT" key.

Step 3: Press the "x^{-1}" key to obtain $x!$.

Using a non-programmable scientific calculator, calculate each of the following:

(a) ${}^{6}C_{3}$ (c) ${}^{15}P_{5}$ (e) $8! + 2(2!)$

(b) ${}^{7}C_{2}$ (d) ${}^{8}P_{2}$

Solution

(a) ${}^{6}C_{3} = 20$ (c) ${}^{15}P_{5} = 360360$ (e) $8! + 2(2!) = 40324$

(b) ${}^{7}C_{2} = 21$ (d) ${}^{8}P_{2} = 56$

(iv) *Normal distribution*

To find the probability from normal distribution by using a non-programmable scientific calculator, observe the following steps:

Step 1: Press the "MODE" key two times.

Step 2: Select "SD" by pressing number 1.

Step 3: Press "SHIFT" key followed by number 3 to obtain "DISTR". The screen of the calculator looks as follows:

Step 4: Select "P" by pressing number 1 if $P(z \le a)$, "Q" by pressing number 2 if $P(a \le z \le b)$, and "R" by pressing number 3 if $P(z \ge a)$.

Step 5: Enter the value of z as required.

Step 6: Press the "=" key to obtain the result.

Note that,

(i) P gives the probability from the variable to the left, that is $P(z \leq a)$. For instance, $P(z \leq -1.83) = 0.03362$.

(ii) R gives the probability from the variable to the right, that is $P(z \geq a)$. For instance, $P(z \geq 1.2) = 0.11507$.

(iii) Q gives the probability from zero to the variable, that is $P(0 \leq z \leq a)$. For instance, $P(z \leq 1.5) = 0.5 + Q(1.50) = 0.93319$

Example 1.17

Using a non-programmable scientific calculator, calculate each of the following:

(a) $P(z \leq -1.53)$

(b) $P(-2 \leq z \leq 0.27)$

(c) $P(z \geq 0.8)$

Solution

(a) $P(z \leq -1.53) = 0.06301$

(b) $P(-2 \leq z \leq 0.27) = Q(-2) + Q(0.27)$
$= 0.58367$

(c) $P(z \geq 0.8) = 0.21186$

Exercise 1.6

1. Use a non-programmable scientific calculator to evaluate each of the following:
   (a) $5!$
   (b) $7! - 2!$
   (c) $\frac{3!}{4!2!}$

2. Using a non-programmable scientific calculator evaluate the following:
   (a) ${}^{10}C_2 \times {}^{7}C_2$
   (b) ${}^{11}C_4 + {}^{6}C_3$

3. Use a non-programmable scientific calculator to evaluate each of the following:
   (a) $P(z \leq 0.85)$
   (b) $P(z \leq -1.377)$
   (c) $P(z > -1.377)$
   (d) $P(0.345 \leq z \leq 1.751)$
   (e) $P(z \leq 1.5)$

(l) Solutions of simultaneous, quadratic, and cubic equations

A non-programmable scientific calculator can be used to calculate the unknowns in the system of simultaneous equations with two or three unknowns, quadratic, and cubic equations.

Activity 1.8: Solving simultaneous equations with two and three unknowns

Individually or in a group, perform the following tasks:

1. Set a non-programmable scientific calculator in the mode that allows to solve simultaneous equations with two or three unknowns.
2. Use a non-programmable scientific calculator to solve each of the following simultaneous equations:

   (a) $\begin{cases} 5x - y = 2 \\ x + 3y = 10 \end{cases}$

   (b) $\begin{cases} x - 3y + 2z = 8 \\ 2x + y + 2z = 19 \\ x - y - 2z = 2 \end{cases}$
3. Write all the steps you have used to solve simultaneous equations with two or three unknowns in task 2.
4. Share the identified steps in task 3 with your fellow students for more inputs.

Steps for solving simultaneous equations

Step 1: Press the "MODE" key three times until the option for choosing "EQN" (equation) appears on the display.

Step 2: Choose "EQN" by pressing the key of its option number.

Step 3: Choose the number of unknowns, 2 for simultaneous equations with two unknowns or 3 for simultaneous equations with three unknowns will appear on the display. Press the number key corresponding to the number of unknowns in the system of simultaneous equations. That is either the key labelled 2 or 3. Coefficients of the unknown x, y, and z in the equations are named as follows:

A system of linear simultaneous equations with two unknowns has the form;

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

A system of linear simultaneous equations with three unknowns has the form;

$$\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

Step 4: Once the number of unknown is set, the calculator's display will indicate the value of a_1.

Step 5: Insert the number representing the coefficient a_1, and then press the "=" key.

Step 6: Continue with step 5 for the rest of the coefficients until you obtain their corresponding last constants (c_2 for 2-unkowns and d_3 for 3-unkowns).

Step 7: Press the "=" key to display the values of the unknowns.

Activity 1.9: Solving quadratic and cubic equations

Individually or in a group, perform the following tasks:

1. Set your non-programmable scientific calculator in the mode that allows to solve quadratic or cubic equations.
2. Use the non-programmable scientific calculator to calculate the values of each of the following:
   (a) $x^2 + 3x - 4 = 0$
   (b) $x^3 + 6x + 6 = 0$
3. Write the steps you have used to solve the equations in task 2.
4. Share the identified steps in task 3 with your fellow students for more inputs.

Steps for solving quadratic and cubic equations

Step 1: Press the "MODE" button three times until the option for choosing "EQN" appears on the display.

Step 2: Choose "EQN" and press the right arrow of the "REPLAY" key to choose the degree of the equation.

Step 3: Choose the degree of the equation to be solved (that is, 2 for quadratic equation and 3 for cubic equation).

Step 4: Arrange the equation in an acceptable format for inserting coefficients in the calculator. For a quadratic equation it should be $ax^2 + bx + c = 0$ and for a cubic equation it should be $ax^3 + bx^2 + cx + d = 0$.

Step 5: Insert the coefficient a, and then press the "=" key.

Step 6: Continue with step 5 for coefficients b up to c or d depending on the type of the equation.

Step 7: Once the last value, c or d is inserted in the calculator, the first solution x_1 will be displayed. Press the "=" button to get other solutions.

Exercise 1.7

Use a non-programmable scientific calculator to evaluate each of the following equations:

1. $x^3 - 23x^2 + 120x = 0$
2. $3x^2 - 46x + 120 = 0$
3. $x^3 - 2x^2 - x + 2 = 0$
4. $x^2 - 2\sqrt{2}x + 2 = 0$
5. $x^2 + x + \frac{3}{4} = 0$
6. $\begin{cases} x + 2y = 3 \\ 2x + 3y = 4 \end{cases}$
7. $\begin{cases} x - y + z = 2 \\ x + y - z = 0 \\ -x + y + z = 4 \end{cases}$

8. $\begin{cases} 7x+2y=11 \\ 4x+y=7 \end{cases}$

9. $\begin{cases} 6x+y=9 \\ -y+4x=7 \end{cases}$

10. $\begin{cases} 3x-y+2z=13 \\ 2z+3y+x=15 \\ y+4x-z=4 \end{cases}$

11. $\begin{cases} 3x+4y+z=5 \\ 2x-y-z=4 \\ x+3y+z=1 \end{cases}$

12. $\begin{cases} x+y+z=6 \\ 3x+y+z=8 \\ x+2z-y=5 \end{cases}$

(m) Calculations involving matrices

A non-programmable scientific calculator stores matrices and performs different operations such as addition, subtraction, and multiplication. In order to store the matrix in the calculator, some steps are required.

Steps for storing matrices in a non-programmable scientific calculator

Step 1: Press the "MODE" key three times until the matrix mode "MAT" appears on the display and then choose it by pressing its option number.

Step 2: While the calculator is in matrix mode press the "SHIFT" key, then press the 4 key "MAT".

Step 3: Press key number 1 and choose the name of the matrix, by pressing " 1" for naming it as matrix A, " 2" for B, and "3" for C.

Step 4: Set the dimension $m \times n$ of the matrix by typing the value of m (number of rows). Then press "=" key and type again the dimension n (number of columns) and then press "=" key. If the matrix is 2 by 2, then m is 2 and n is 2 or if the matrix is 3 by 2, then m is 3 and n is 2.

Steps for inserting entries of a matrix

After the name and dimension of the matrix have been set, entries of the matrix are required. The following are steps of recalling the stored matrices:

Step 1: Press the first entry " MatA_{11}" if the matrix is named as A, then, press "=" key.

Step 2: Press the second entry " MatA_{12}" then press "=" key. Following the same steps until the last entry of the entire matrix is entered.

The entries of matrices are named as illustrated in the following cases:

For a 2 by 2 matrix A, the entries are named as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

For a 3 by 3 matrix A, the entries are named as:

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

In other words, A_{11} means an entry of matrix A at the position of first row and first column. Similarly, A_{12} refers to the entry of matrix A at first row and second column.

Steps for editing entries of a matrix

Step 1: Press the "SHIFT" key, then press 4, "MAT" key.

Step 2: Select "Edit".

Step 3: Select the name of the matrix whose dimension has been set.

Step 4: Enter the correct value of the entry and then press the "=" key. Continue with these steps until all the entries are edited.

Determinant, transpose, and inverse of a matrix

After inserting the entries of a matrix, the steps to obtain the determinant of the matrix are as follows:

Step 1: Press "SHIFT" key.

Step 2: Press 4, "MAT" key.

Step 3: Scroll the right arrow of the replay button and select "Det" by pressing number 1 key.

Step 4: Again press "SHIFT" key.

Step 5: Press 4, "MAT" key, then select "MAT" by pressing number 3 key.

Step 6: Select the matrix whose determinant is required.

Step 7: Press the "=" key to obtain the result.

Note that, the same steps mentioned are used for finding transpose of the matrix by choosing transpose "Trn" instead of "Det".

Inverse of a matrix

After inserting the entries of a matrix, use the following steps to determine its inverse:

Step 1: Press the "SHIFT" key.

Step 2: Press 4, "MAT" key.

Step 3: Select "MAT" then press number 3 for the matrix.

Step 4: Select the matrix whose inverse is required.

Step 5: " MatA " will appear on the display, then press x^{-1} to obtain " MatA^{-1}" which is the inverse of matrix then press "=" key. The first entry for MatAns_{11} will appear, continue scrolling the right key until the last value is displayed.

Exercise 1.8

1. Given the following matrices, use a non-programmable scientific calculator to answer each of the questions which follows:

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{pmatrix}$$

(a) Compute the determinant and transpose of matrix A.

(b) Compute the inverse of each of the matrix A.

(c) Compute $A + A^{-1}$

(d) Compute A^2

2. If $A = \begin{pmatrix} 2 & 3 & -1 \\ 2 & 0 & 8 \\ 2 & 4 & 5 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 6 & -1 \\ 0 & 0 & 8 \\ 2 & 4 & 3 \end{pmatrix}$, and $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, use a non-programmable calculator to:

(a) compute the value of $2\det(A) + \det(B)$

(b) compute the value of $2A + 3B - 4I$

(c) Verify that,

(i) $AA^{-1} = A^{-1}A = I$

(ii) $\det(B) = \det(B^T)$

(iii) $AB \neq BA$

(d) Compute $4AB + 5BA^T$

(n) Evaluating derivatives and definite integrals of functions

A non-programmable scientific calculator can be used to calculate the definite derivatives and integral of functions. The key for integral of functions is labelled as $\int dx$ with its alternate functional key labelled as $\frac{d}{dx}$. The functional key $\int dx$ is used for performing calculations which require integration, while $\frac{d}{dx}$ is for differentiation.

The following are steps for calculating definite integral by using a non-programmable scientific calculator:

Step 1: Press "$\int dx$" key.

Step 2: Press the "ALPHA" key in order to insert a function in terms of x. Every variable of the function to be inserted must be preceded by pressing "ALPHA" key.

Step 3: Press comma, then insert lower limit of integration.

Step 4: Press again comma to insert upper limit of integration and close bracket.

Step 5: Press the "=" key to obtain the result.

Also, the following are the steps for calculating the derivative of a function at a point by a non-programmable scientific calculator.

Step 1: Press the "SHIFT" button then press "$\int dx$" key.

Step 2: Press the "ALPHA" key to insert a function in terms of x.

Step 3: Press comma, then insert a point given and close bracket.

Step 4: Press the "=" key to obtain the result.

Exercise 1.9

Using a non-programmable scientific calculator, evaluate each of the following:

1. $\int_0^5 \frac{dx}{1+x^2}$
2. $\frac{d}{dx}\left(x^3-4e^x+2x-\frac{3}{2}\right)$, when $x=1$
3. $\int_{-1}^{3}(6x^2-3x+7)dx$
4. $\frac{d}{dx}(\tan x-\sin x)$, when $x=\frac{\pi}{4}$
5. $\int_0^{\frac{\pi}{2}}\cos^2 x\,dx$
6. $\frac{d}{dx}\left(3\log x^2-\frac{\sqrt{x^2-1}}{x^3}\right)$, when $x=2$

(o) Conversion of units

Unit conversion is one of the functions of a scientific calculator. A non-programmable scientific calculators converts units under metric systems. Types of unit conversion to be carried out by a particular calculator are listed on the user's manual and on the inner part of the plastic cover of the calculator. Thus, before converting the units, identify the number which represent the type of conversion needed.

The following are steps for unit conversions.

Step 1: Press a number to be converted.

Step 2: Press the "SHIFT" key.

Step 3: Press the "CONST" key.

Step 4: Press a number to recall the unit.

Step 5: Press "=" key to obtain the result.

Activity 1.10: Conversion of units

Individually or in a group, perform the following tasks by using a non-programmable scientific calculator:

1. Identify all types of units' conversion that can be performed by a non-programmable scientific calculator.
2. Convert each of the following:
   (i) 10 ft $\rightarrow$ inch
   (ii) $4\,m^2 \rightarrow$ acres
   (iii) 80 km/h $\rightarrow$ m/s
   (iv) 60 cal $\rightarrow$ J
3. Identify the steps you have used to convert units in task 2.
4. Share the identified steps in task 3 with your fellow students for more inputs.

(p) Physical constants and quick access to constants such as Pi and e

Physical constants are useful in calculations. Sometimes, constants are not given hoping that the learners can obviously get them from their scientific calculators. Constants which can be recalled by a particular scientific calculator are listed on the inner part of the plastic cover of the calculator as well as in the user's manual. As in the case of units' conversion, each constant is assigned a number in order to recall it.

For example, the constant for acceleration due to gravity (g) is assigned number 35. Therefore, to recall it press the button "CONST", then type 35 and press (=) key to view its value.

Activity 1.11: Operation with constants

Individually or in a group, perform the following tasks by using a non-programmable scientific calculator:

1. Identify any constants of your choice and their assigned numbers.
2. Perform each of the following:
   (a) Add any two constants.
   (b) Subtract any two constants.
   (c) Multiply any two constants.
   (d) Divide any two constants.
3. Identify all the steps you have used to obtain the values of the constants in task 2.
4. Share the identified steps in task 3 with your fellow students for more inputs.

(q) Calculations involving vectors

A non-programmable scientific calculator stores vectors and performs different operations such as addition, subtraction and multiplication. In order to store a given vector in the calculator, some steps are required.

The following are steps for storing vectors in a non-programmable scientific calculator

Step 1: Press the "MODE" key three times until vector mode "VCT" appears on the display and then choose it by pressing a key number 3.

Step 2: Press the "SHIFT" button, and then press a key numbered 5.

Step 3: Press the key numbered 1 to set dimension.

Step 4: Press the key numbered 1 to choose the name of the vector, by pressing 1 for naming it as vector A, 2 for B, and 3 for C.

Step 5: Press the dimension m of the vector by typing the value of m. If the given vector is 2-dimensional then the value of m is 2 and m is 3 for a 3-dimensional vector.

Steps for inserting components of a vector

Once the name and dimension of a vector have been set, the next step is to enter the entries of the given vector, which is done using the following steps:

Step 1: Press the first entry "VctA_1" if the vector is named A. Then, press "=" key.

Step 2: Press the second entry "VctA_2", then press "=" key. If the vector is of a 3-dimension follow the same steps until the final entry is entered.

The components of vectors are named as illustrated in the following cases.

For 2-dimension vector A, the entries are named as $\text{VctA}_1\underline{i} + \text{VctA}_2\underline{j}$.

For 3-dimension vector A, the entries are named as $\text{VctA}_1\underline{i} + \text{VctA}_2\underline{j} + \text{VctA}_3\underline{k}$.

The following are steps for editing components of a vector

Step 1: Press "SHIFT" key, then press the key number 5.

Step 2: Select "Edit" by pressing its option number.

Step 3: Select the name of the vector whose dimension has been set.

Step 4: Enter the correct value of the entry, then press "=" key. Continue with these steps until all the components are edited.

(i) *Addition and subtraction of vectors*

After inserting the components of a vector, the steps to add or subtract the given vector are as follows:

Step 1: Press "SHIFT" button.

Step 2: Press a key number 5, select the first vector by pressing a key numbered 3. Select the name of the vector to be added or subtracted by pressing its option numbers.

Step 3: Press "+" key for addition or "−" key for subtraction.

Step 4: Press a key numbered 5, then press a key numbered 3, select the name of vector by pressing its option number.

Step 5: Press "=" key to obtain the result.

(ii) ***Modulus of a vector***

After entering the components of vector, use the following steps to determine its modulus:

Steps 1: Press the "SHIFT" key, then press ") " key to obtain "Abs".

Step 2: Press the"SHIFT" key, then press a key numbered 5.

Step 3: Select "Vct" by pressing a key numbered 5.

Step 4: Select the name of the vector whose modulus is required by pressing its option number.

Step 5: Press "=" key to obtain the result.

(iii) ***Dot product of vectors***

After entering the components of vectors, use the following steps to determine the dot product of vectors:

Step 1: Press the "SHIFT" key, then press 5 "VCT" key.

Step 2: Select "Vct" by pressing the key numbered 3.

Step 3: Select the first vector to be dotted.

Step 4: Scroll the right arrow of replay button and select "Dot" by pressing the key numbered 1.

Step 5: Again press the "SHIFT" key.

Step 6: Press the key numbered 5, then select "Vct" by pressing the key numbered 3.

Step 7: Select vectors to be dotted.

Step 8: Press "=" key to obtain the result.

(iv) ***Cross product of vectors***

After entering the components of vectors, use the following steps to determine the cross product of vectors:

Step 1: Press the "SHIFT" key, then press a key numbered 5.

Step 2: Select "Vct" by pressing a key numbered 3.

Step 3: Select the first vector to be crossed.

Step 4: Press "×" key, then repeat Step 1 to Step 2 and select the second vector to be crossed. Follow the same steps until the last vector to be crossed is inserted.

Step 5: Press "=" key to obtain the result.

Exercise 1.10

1. If $\underset{\sim}{a}=2\underset{\sim}{i}+3\underset{\sim}{j}+4\underset{\sim}{k}$ and $\underset{\sim}{b}=\underset{\sim}{i}-\underset{\sim}{j}+\underset{\sim}{k}$, use a non-programmable scientific calculator to find:

   (a) $\underset{\sim}{a}+\underset{\sim}{b}$ (b) $\underset{\sim}{b}-\underset{\sim}{a}$

2. Given $\overrightarrow{OA}=4\underset{\sim}{i}-3\underset{\sim}{k}$ and $\overrightarrow{OB}=-2\underset{\sim}{i}+4\underset{\sim}{j}+\underset{\sim}{k}$, using a non-programmable scientific calculator, evaluate each of the following:

   (a) $\overrightarrow{OA}\cdot\overrightarrow{OB}$ (c) $\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$

   (b) $\overrightarrow{OA}\times\overrightarrow{OB}$

3. Use a non-programmable scientific calculator to determine vector $2\underset{\sim}{a}+\underset{\sim}{b}-3\underset{\sim}{c}$ such that

$\underset{\sim}{a} = \underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{b} = 4\underset{\sim}{j} + \underset{\sim}{k}$ and

$\underset{\sim}{c} = 4\underset{\sim}{i} - 3\underset{\sim}{j} + 6\underset{\sim}{k}$

4. If $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$, $\underset{\sim}{v} = \underset{\sim}{i} + 2\underset{\sim}{j} + \underset{\sim}{k}$ and $\underset{\sim}{r} = 5\underset{\sim}{i} + 3\underset{\sim}{j} + \underset{\sim}{k}$, using a non-programmable scientific calculator, verify that:

   (a) $\underset{\sim}{u} \times \underset{\sim}{v} \neq \underset{\sim}{v} \times \underset{\sim}{u}$ (b) $\underset{\sim}{u} \cdot \underset{\sim}{r} = \underset{\sim}{r} \cdot \underset{\sim}{u}$

(r) Calculations involving complex numbers

A complex number is a number in a form $a + bi$ where a is the real part and b is an imaginary part of a complex number. A non-programmable scientific calculator can perform different operations on complex numbers such as addition, subtraction, multiplication, and division.

(i) *Addition and subtraction of complex numbers*

To add or subtract complex numbers using a non-programmable scientific calculator use the following steps:

Step 1: Press the "MODE" key once, then select complex mode "CMPLX" by pressing number 2 key.

Step 2: Insert open bracket by pressing "(" key, insert real part of the complex number followed by "+" or "−" sign, then by imaginary part, press the "ENG" key to insert i then insert close bracket by pressing ")" key.

Step 3: Press "+" key for addition or "−" for subtraction.

Step 4: Repeat step 2 to insert the second complex to be added or subtracted.

Step 5: Press "=" key to obtain the real part of the result.

Step 6: Press the "SHIFT" key, then "=" key to obtain the imaginary part of the result.

(ii) *Multiplication and division of complex numbers*

Step 1: Press the "MODE" key once, then select complex mode "CMPLX" by pressing a key numbered 2.

Step 2: Insert open bracket by pressing "(" key, insert a real part of the complex number followed by "+" or "−" sign, then by imaginary part, press "ENG" key to insert i then insert close bracket by pressing ")" key.

Step 3: Press "×" key for multiplication or "÷" key for division.

Step 4: Repeat step 2 to insert the second complex number multiplied or divided.

Step 5: Press "=" key to obtain the real part of the result.

Step 6: Press "SHIFT" key, then "=" key to obtain the imaginary part of the results.

(iii) *Conjugate of a complex number*

To obtain the conjugate of a complex number use the following steps:

Step 1: Press the "MODE" key once select complex mode "CMPLX" by pressing a key numbered 2.

Step 2: Press the "SHIFT" key then "," key to obtain the "Conjg" key.

Step 3: Insert the complex number whose conjugate is required, enclose it in brackets.

Step 4: Press "=" key to obtain the real part of the result.

Step 5: Press the "SHIFT" key, then "=" key to obtain the imaginary part of the results.

(iv) *Modulus of a complex number*

To obtain the modulus of a complex number, use the following steps:

Step 1: Press the "MODE" key once, then select complex mode "CMPLX" by pressing number 2 key.

Step 2: Press the "SHIFT" key then ")" to obtain "Abs".

Step 3: Insert the complex number whose modulus is required, enclose it in brackets.

Step 4: Press "=" key to obtain the results.

(v) *Argument of a complex number*

To obtain the argument of a complex number use the following steps.

Step 1: Press the "MODE" key once, then select complex mode "CMPLX" by pressing a key numbered 2.

Step 2: Press the "SHIFT" key then "(" key to obtain "arg" key.

Step 3: Insert the complex number whose argument is required, enclose it in brackets. Press "=" key to obtain the result.

Exercise 1.11

1. Given $z_1 = 2+3i$, $z_2 = 4-i$, and $z_3 = 1-i$, and use a non-programmable scientific calculator, evaluate each of the following:

   (a) $z_1 + z_2 - z_3$ (c) $\dfrac{z_1}{z_3}$

   (b) $z_1 z_2$ (d) $6z_1 - 4z_1 z_2 + \dfrac{z_3}{z_1}$

2. If $z_1 = i$, $z_2 = 3-i$, and $z_3 = 2+4i$, using a non-programmable scientific calculator, compute each of the following:

   (a) $|z_1 z_2|$ (c) $\text{Conjg}\left(\dfrac{z_1 z_2}{z_3}\right)$

   (b) $\arg\left(\dfrac{z_2}{z_1}\right)$

3. If $z_1 = 3 + 2i$, $z_2 = 1 + i$, and $z_3 = 4 - 3i$, using a non-programmable scientific calculator, verify each of the following:
   (a) $|z_1z_2| = |z_1||z_2|$
   (b) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$
   (c) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$
   (d) $\arg(z_1z_3) = \arg z_1 + \arg z_3$

(s) Errors in a non-programmable scientific calculator

A non-programmable scientific calculator displays an error message whenever an error is made. The replay button can be used to move the cursor within the calculation on the calculator screen, and then the character can be inserted at the location simply by pressing the appropriate buttons. In some cases it may be easier to abandon what has been typed and start again by pressing the all clear "AC" key. The following are messages that a calculator displays to alert an error that has been occurred.

(i) *Math error*

This occurs when the calculation inserted make sense but the result cannot be calculated, such as attempting to divide by zero or when the calculation is too large for the calculator to handle. For instance, $5 \div 0$, calculator will display "Math ERROR", because a number cannot be divided by zero.

(ii) *Syntax error*

This occurs when the format of the calculation inserted does not make sense. For instance,

$45°$ tan, calculator will display "syntax ERROR" since there is problem with the format of calculation inserted.

(iii) *Dimension error*

This will occur especially when preforming calculation with matrices or vectors whose dimensions do not allow that type of calculations. For instance,

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 2 & 0 & 6 \\ 6 & 0 & 5 \\ 2 & 1 & 3 \end{pmatrix},$$

the calculator will display "Dim ERROR" since it is not possible to multiply a 3×1 matrix by 3×3 matrix.

(iv) *Stack error*

This occurs when the capacity of numeric stack or command stack exceeded during calculation, for instance, find $14\text{BC} + 6\text{CB}^{\text{T}}$ given matrices

$$\text{B} = \begin{pmatrix} 3 & 2 & -1 \\ 3 & 0 & 8 \\ 3 & 4 & 5 \end{pmatrix}, \text{C} = \begin{pmatrix} 7 & 6 & -1 \\ 0 & 0 & 8 \\ 2 & 4 & 3 \end{pmatrix}.$$

A non-programmable calculator will display "Stack ERROR".

Computer packages

Computer packages are used in various disciplines depending on the application relevant to the field. There are various computer packages which are useful in computations, simulation, and analysis of the mathematical models. The useful packages include Maple, MATLAB, Mathematica, and Spreadsheet which are more advanced in executing mathematical computations and graphing. In this section, only installed computer calculator and spreadsheet are discussed.

The computer's calculator

The computer's calculator offers precise calculations and has a powerful interface. It has integrated the basic standard calculations with programming, scientific calculations, and statistics. Furthermore, there are also other features which are very useful like mortgage calculation and multifunctional converter.

Opening the computer's calculator

The following are useful steps to access the computer's calculator in Window 10 (See Figure 1.2):

Step 1: Click the Start Menu.

Step 2: Choose the calculator in the given options.

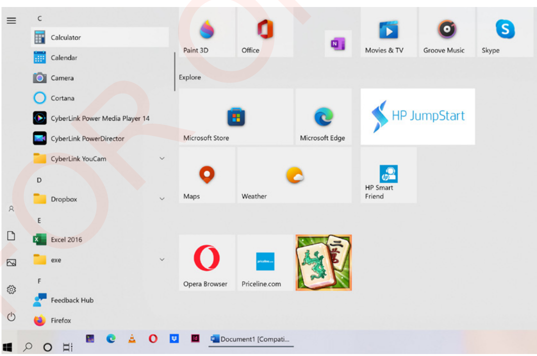


Figure 1.2: *Start menu of the computer*

Computer's calculation modes

Even though Windows 10 has a different version as compared to Windows 7, the computer calculators are the same in both operating systems. The interfaces are the same and their features are identical.

The computer calculator in Windows 10 has four (4) main modes for which calculations can be done:

(a) The standard mode
(b) The programmer mode
(c) The scientific mode
(d) The statistics mode

Each mode enables the user to perform basic operations, unit conversion, time conversion, or worksheet calculations. Thus, the mode and type of operations have to be set first before calculations so that the particular kind of functional keys can be seen and used.

(a) The standard mode

When the computer's calculator is opening, the standard mode will be selected by default as shown in Figure 1.3. The keyboard number values, the keypad (with the number key activated) or the mouse can be used to do calculations.

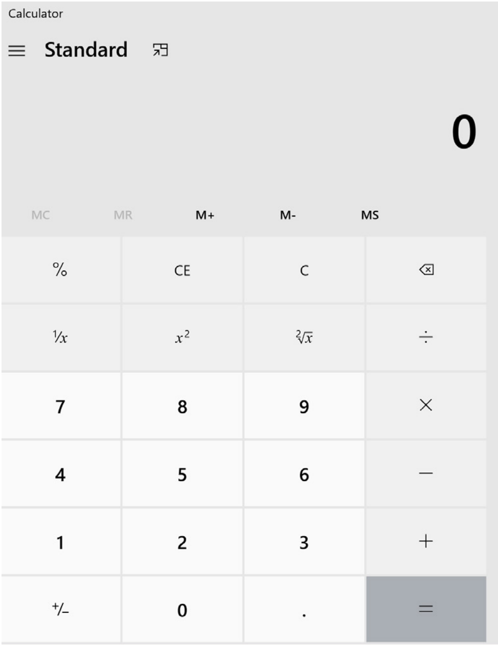


Figure 1.3: *Standard mode of the computer's calculator*

(b) The programmer mode

This mode gives the function keys of a programmable scientific calculator. It offers the possibility to do operations with bases such as binary, octal, hexadecimal, and decimal. One can convert values from one base to another. For example, conversion from a base two number system (binary 0, 1) to a base ten number system (decimal 0 – 9). This mode also offers the logical bit operations such as XOR, OR, and AND.

To access this mode, click on the option menu in the computer's calculator and select the programmer option. (See Figure.1.4).

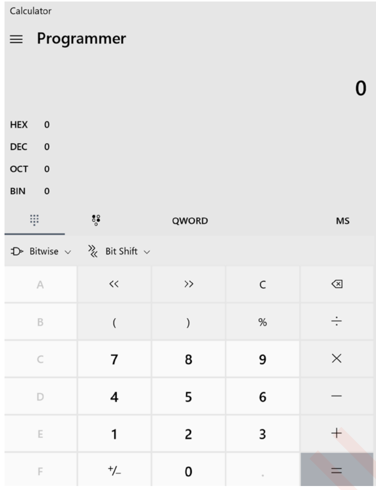


Figure 1.4: *The programmer mode of the computer's calculator*

(c) The scientific mode

This mode resembles a non-programmable scientific calculator. In this mode, any calculation which can be performed using a scientific calculator may be executed. To access this mode, click on the option menu in the computer's calculator and select the scientific option. (See Figure. 1.5).

Figure 1.5: *The scientific mode of the computer's calculator*

(d) Statistics mode

Like in the standard mode of the computer's calculator, operations in statistics mode are done by clicking on a number, followed by an operation then insert the second number, and finish up by clicking the equal sign to obtain the final answer. (See Figure 1.6)

Figure. 1.6: *Statistics mode of the computer's calculator*

How to create a dataset in statistics mode

A dataset is a collection of data in numbers that can be used in performing different operations. To create a dataset, click the number (s) which form the set of a particular data and then click the **add** button. Repeat these steps until all the data are entered. Each time you press the add button, the number of count changes. "Count" represents the total number of data entered. The data will be displayed on the dataset list. When all the data are entered in the dataset list, click on any operation such as sum, mean, or standard deviation and the corresponding result will be displayed at the bottom of the dataset list.

Note that, clicking the **C** button in the statistics mode deletes the current data entered before clicking the **add** button and the CAD button clears all the data from the dataset.

Other options in the computer's calculator

A computer's calculator includes other tools which are used in measurements such as mass, time, volume, weight, light intensity, and distance. Thus, a computer's calculator has more functions to do than a pocket or hand- scientific calculator.

Exercise 1.12

1. By using a computer's calculator, evaluate each of the following expressions:

(a) $\left(\left(\frac{13}{7}\right)^{7}\right)^{4} \div \left(\frac{1}{4}-13\right)^{6}$

(b) $\dfrac{(-1.76)^{5}+(3.0006)^{7}}{\left(\left((2.12)^{-3}\times 44\right)-\left(0.0009+\left(\frac{4}{7}\right)^{8}\right)\right)}$

2. By using a computer's calculator, determine the value for each of the following expressions:

   (a) $\sqrt{3\tan 60^\circ} + \frac{\sqrt[4]{15}}{7}(\cos 45^\circ - \sin 120^\circ)$

   (b) $\tan^{-1}\left(\sin\left(\cos\left(\frac{\log 315}{\ln 2}\right)\right)\right)$, compute in radian.

3. Compute the mean, sum of squares, and the standard deviation of the following data using a computer's calculator:
   12, 74, 29, 10, 70, 12, 36, 90, 10, 52, 39, 70, and 12

4. Convert each of the following using a computer's calculator:

   (a) 60° into radians
   (b) 100 acres into square metres
   (c) 1000 cm^3 into m^3
   (d) 1000 cm^3 into litres

5. Use a computer's calculator to find the difference in years, weeks, and days between the following periods:

   (a) 9/12/1961 – 12/4/1962
   (b) 07/7/1964 – 30/10/1995
   (c) 14/9/1999 – 8/9/2017

Spreadsheets

A spreadsheet is an interactive computer application program for organizing, analyzing, and storing data. The program operates on data represented as cells of an array organized in rows and columns. Each cell of the array may contain either numeric or text data.

The results from the formulae can automatically be calculated and displayed in the other cell within the worksheet.

Microsoft Excel is one of the spreadsheet programs offered in the Microsoft Office software package. This program allows the user to perform calculations such as average, addition, subtraction, and finding the maximum and minimum values of numbers. In addition, Microsoft Excel can be used to create histograms, pie charts, and plot graphs of any function.

Starting Microsoft Excel

The steps for opening Microsoft Excel depend on the type of a computer's operating system (Microsoft Windows). In this section, Microsoft Excel as one of the applications (Microsoft Offices 2016) will be used as an example. The following steps shows how to open Excel in Microsoft Windows 10:

1. Click the start menu.
2. Select the Microsoft Office.
3. Select the Microsoft Excel.

Then, the program opens in a display similar to that shown in Figure1.7.

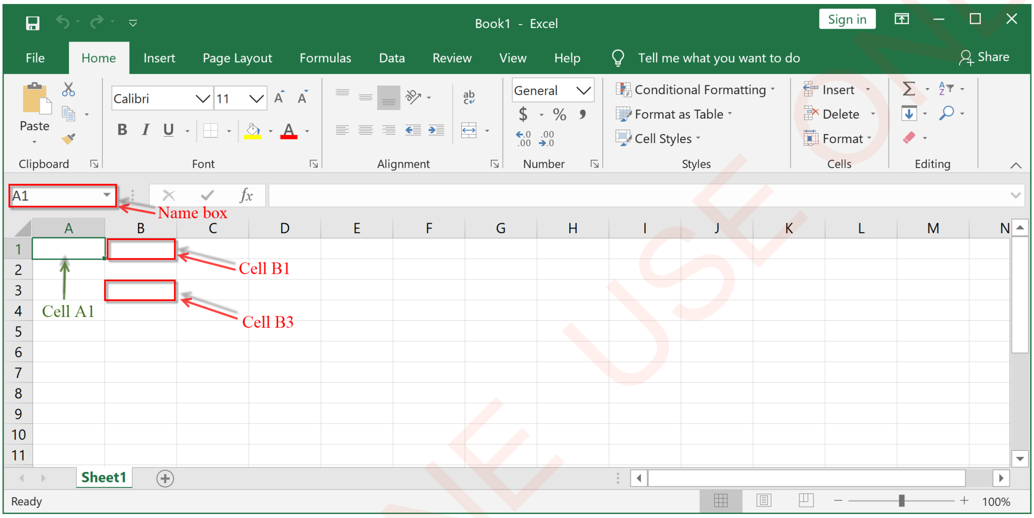


Figure 1.7: *Microsoft Excel spreadsheet display*

The Excel sheet is divided into grids called cells where the data can be entered. The cell is named by notation representing column and row. The columns are assigned capital letters of the alphabet while the rows are assigned numbers. Thus, the first cell is named as A1 which means the cell is in column A and in row 1 as shown in Figure.1.7, while B1 means the cell is in another column B in the same row (row 1), and the fourth cell vertically below the B1 cell is B4. Before performing any calculations, data (numbers) have to be entered to the cells. Normally, the cell name appears in the "Name box' as shown in Figure 1.7.

To enter the data, click on the cell and type the data. The next data can be entered along the same column or row. The vertical and horizontal navigation arrows are used for moving the cursor to another cell. You may also use the "mouse pointer" by clicking the required cell.

Activity 1.12: Entering and analyzing data in Excel

Individually or in a group, perform the following tasks:

1. Use Microsoft Excel to enter the following scores in a worksheet; 30, 50, 60, 70, 25, 80, 20, 50, 40, 90, 30, and 70.
2. Sort the score in descending order.
3. Find the sum, average, maximum, and minimum score.
4. Identify the steps you have used to insert data, sort and analyse the data in tasks 1, 2, and 3, respectively.
5. Share the identified steps with your fellows students.

Note that, in Excel sheet, for easy operations, the data have to be entered in one column or one row. For example, the column of the data in Activity 1.12 should appear as shown in Figure 1.8.

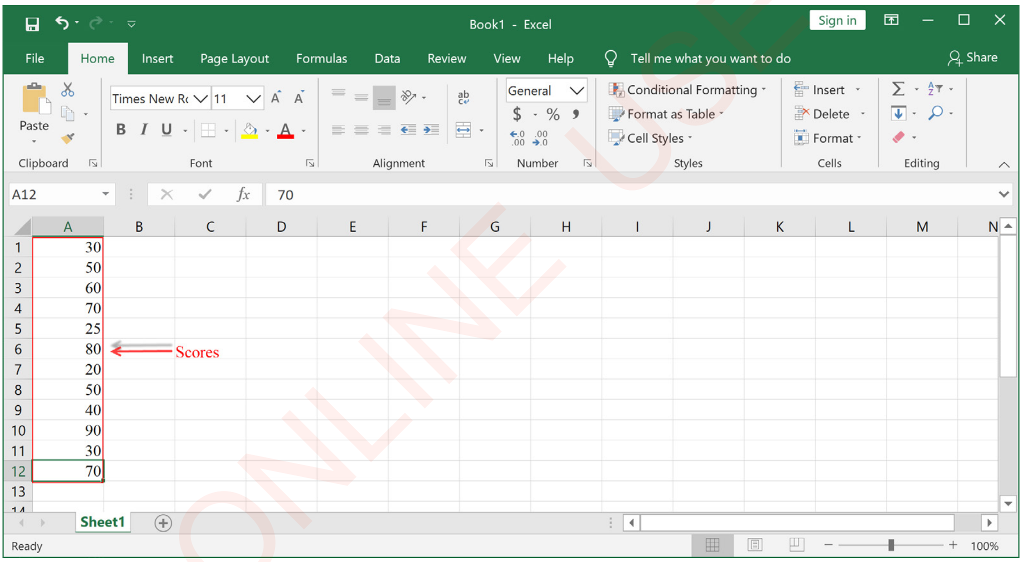


Figure 1.8: *Scores of students in a certain test*

Rearrangement of data

To rearrange the data in Excel, first select the data, then on home tab, click the sort button and select the type of arrangement preferred. Observe the illustration in Figure.1.9.

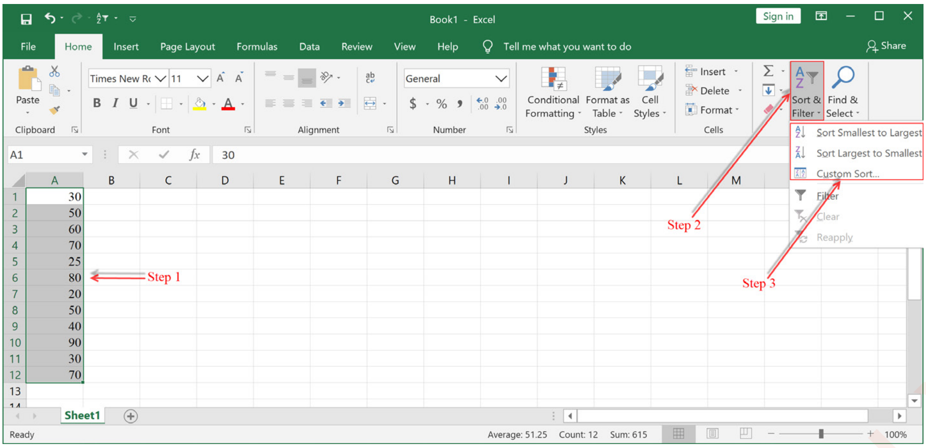


Figure 1.9: *Scores rearrangement*

Calculation of the sum and average of data

To find the sum or average of data, select the data and then click the sum/average button. On the menu that appears after clicking, select the option either sum or average. The result will be displayed at the next cell from the last. Figure.1.10 illustrates the steps for calculating the sum and average of data.

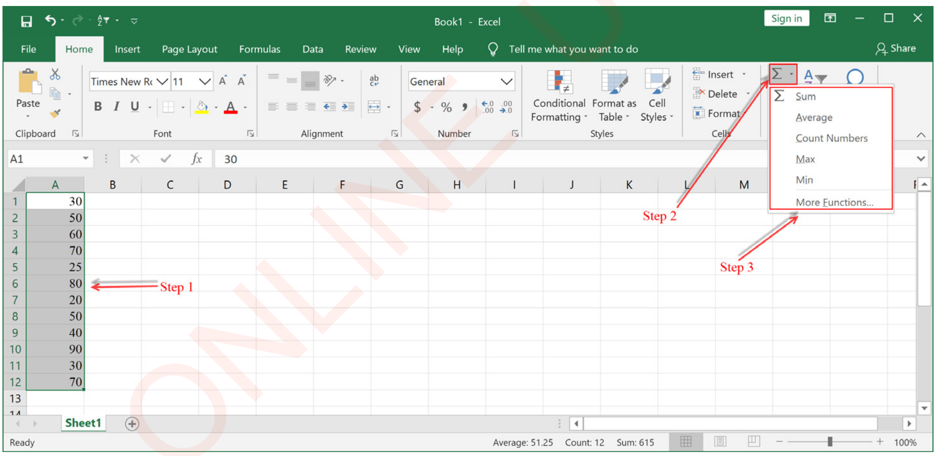


Figure 1.10: *Determination of the sum or average of data*

Setting a formula for performing calculations in Excel

In Excel, a formula can be set and applied to execute operations. In other words, Excel provides room for processing data using a formula of user's choice. Thus, the user may create a formula that involves some steps which depend on the type and length of the formula.

Before creating a formula, the user should be aware of the basic symbols. Table 1.2 shows basic symbols used in defining a formula.

Table 1.3: *Basic calculation symbols in Excel*

Symbol	Meaning
+	Addition
−	Subtraction
* or ×	Multiplication
÷ or /	Division
^	Exponent

Excel formula for addition

The process of adding numbers in Excel starts with an equal sign (=), followed by the first term/number, then plus sign (+), and then the second term and so on. For example, to add a number in A1 and the other in B1, use the following steps:

Step 1: Click the cell where you want the result to appear.

Step 2: Put the equal sign (=).

Step 3: Click the cell of the number before the addition sign.

Step 4: Type the addition sign (+).

Step 5: Click the cell of the number to be added.

Step 6: Press enter key to obtain the answer.

While following the addition steps, the formula is created and displayed along the fx area (formula bar) as illustrated in Figure 1.11.

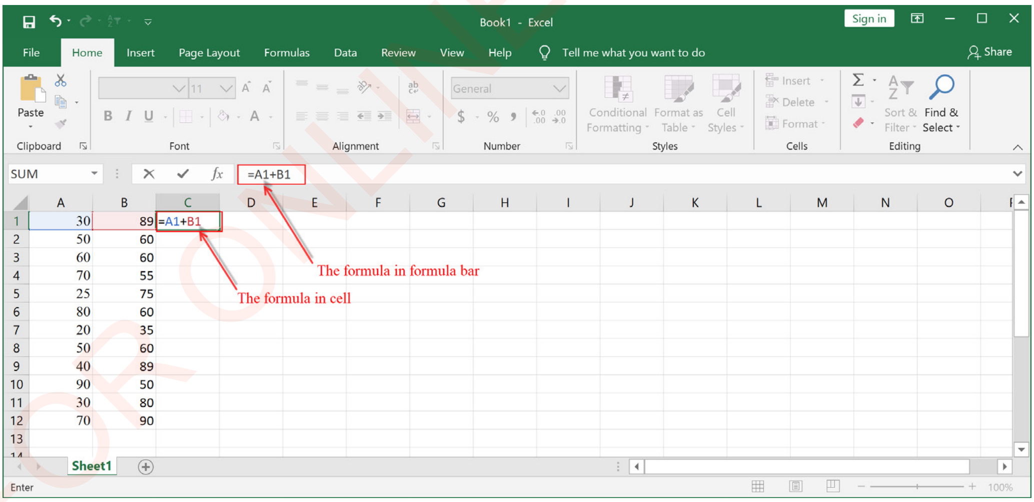


Figure 1.11: *Cell selection and formula display*

The cell at which the formula is typed can be used as a reference. If the same formula applies to other data, place the cursor at the bottom of the right corner of the reference cell (the symbol + will appear), click, and hold the cursor and drag down across all the rows. After releasing the mouse, the formula will be applied to all the cells through which the dragging process took place. (See Figure 1.12).

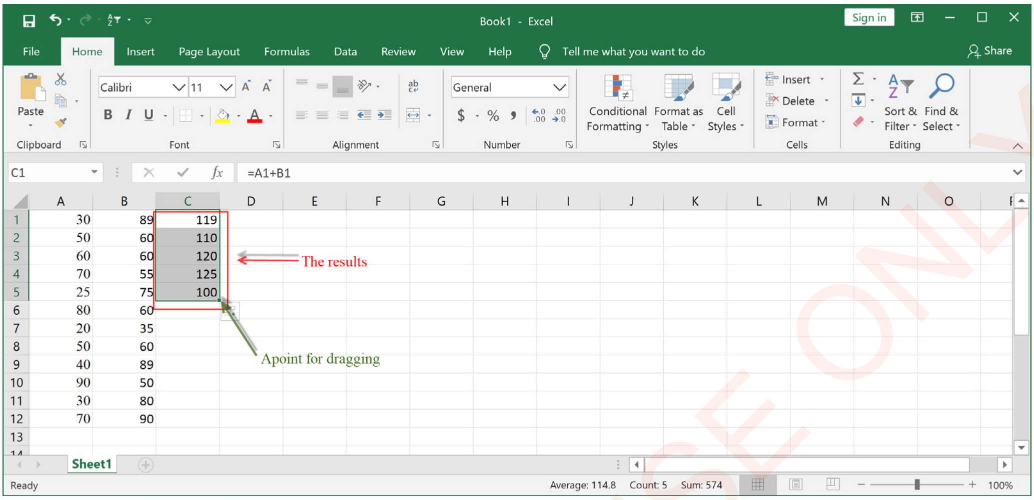


Figure 1.12: *Sum of A1 and B1*

Example 1.18

1. Use Excel to evaluate each of the following:

 (a) $(12+6)^4$

 (b) $(40\times 35)^3 \div (4^5 - 250)$

Solution

(a) To evaluate $(12+6)^4$ use the following steps:

Step 1: Enter 12, 6, and 4 in three consecutive Excel cells (A1, B1, and C1)

Step 2: Write an Excel formula using symbols and cell names $(A_1 + B_1)^{C1}$ as shown in Figure 1.13(a).

Step 3: Press enter to obtain the answer as shown in Figure 1.13(b).

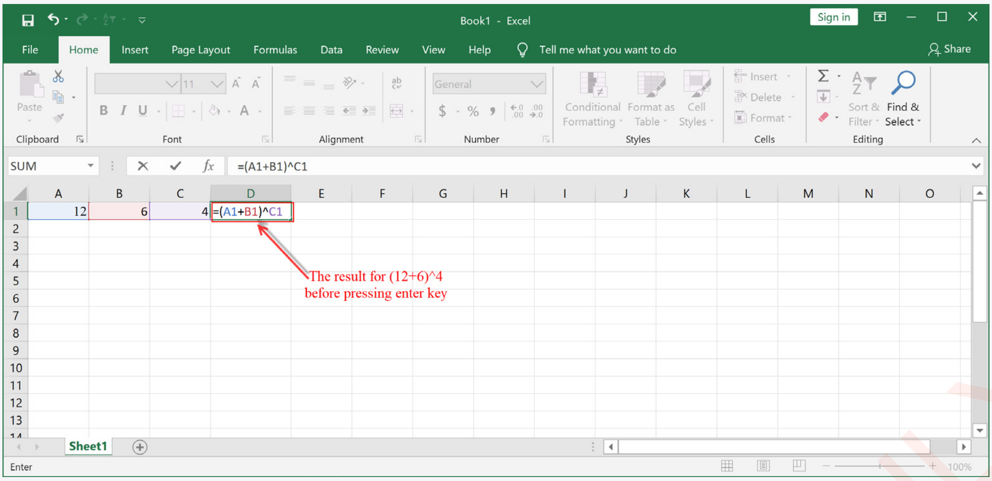


Figure 1.13(a): *Illustrating the formula in Excel*

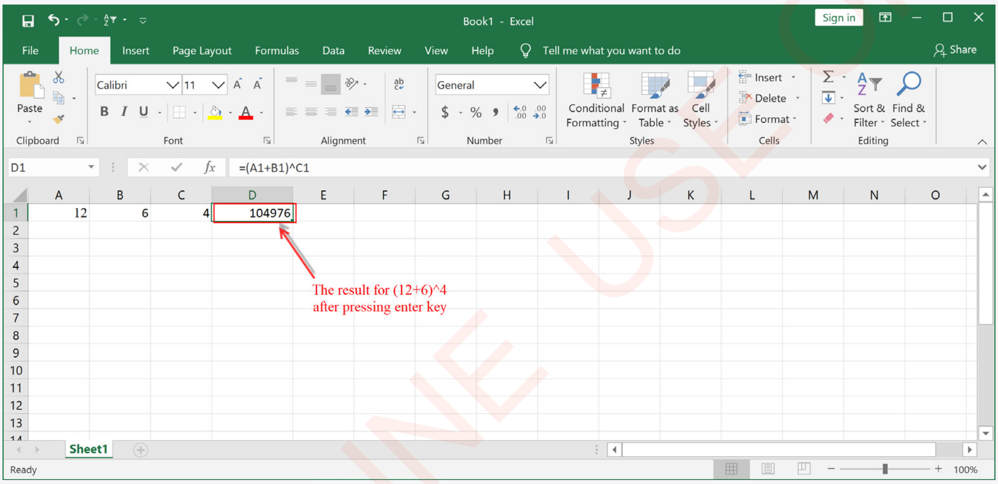


Figure 1.13(b): *Illustrating the answer obtained in Excel*

Therefore, following the required steps, the result is 104,976.

(b) To evaluate $(40\times35)^3 \div (4^5 - 250)$ use the following steps:

Step 1: Enter the numbers $40, 35, 3, 4, 5,$ and 250 in Excel.

Step 2: Make an Excel formula of your choice for representing the expression $(40\times35)^3 \div (4^5 - 250)$ as shown in Figure 1.14(a).

Step 3: Press enter to obtain the answer as shown in Figure 1.14(b).

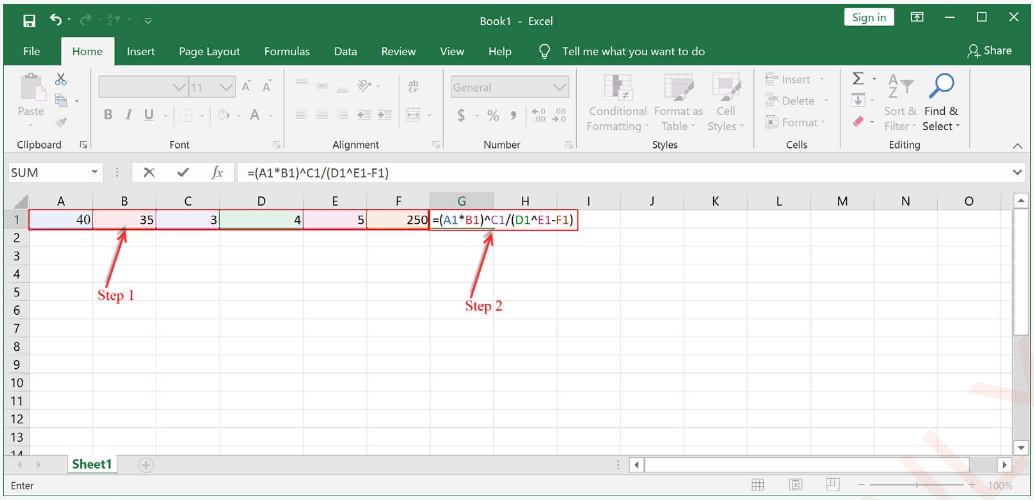


Figure 1.14(a): *Illustrating the formula in Excel*

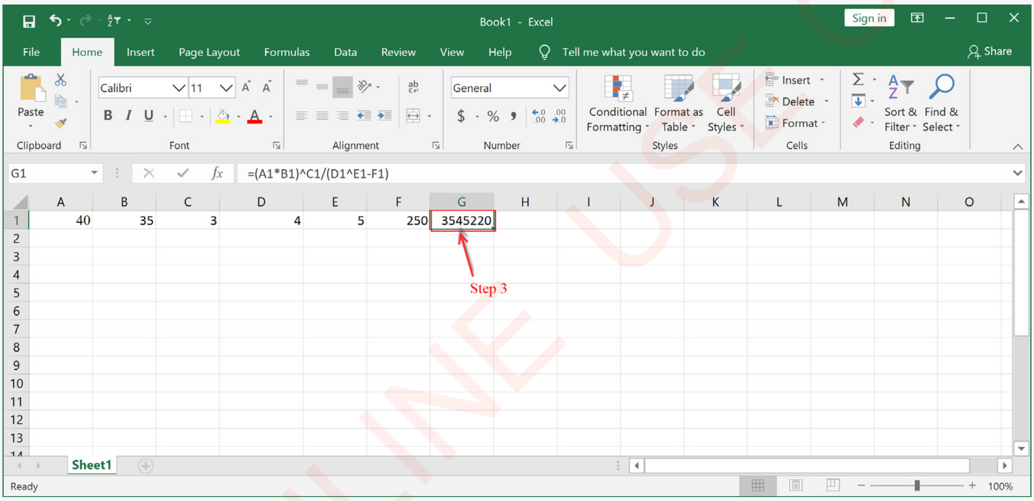
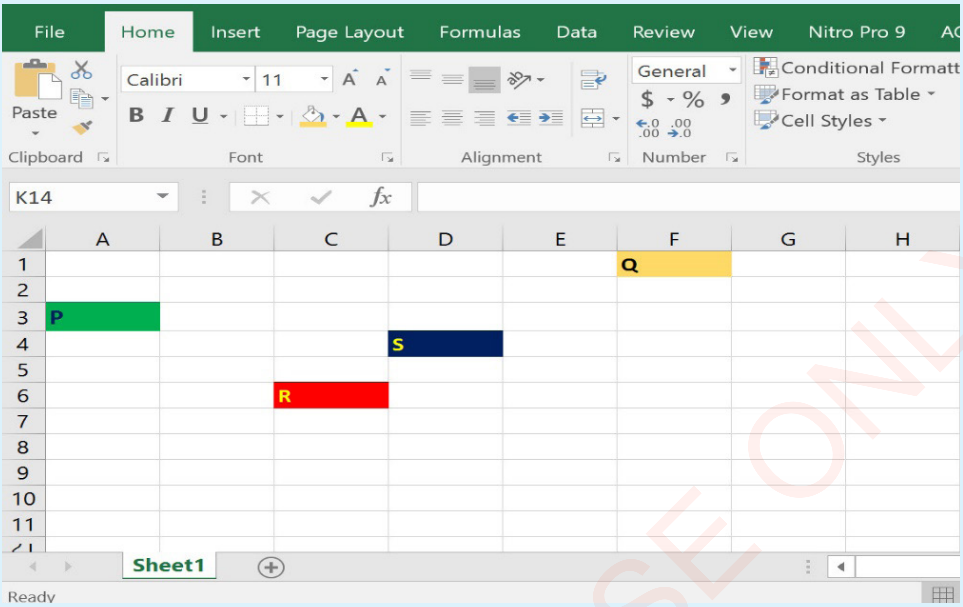


Figure 1.14(b): *Illustrating the answer obtained in Excel*

Therefore, following required steps, the result is 3,545,220.

Exercise 1.13

1. Study the following figure and then answer the questions that follow.



(a) What does the figure represent?
(b) Provide the grid name for the cells with letters P, Q, R, and S.

2. Write the steps for setting an Excel formula to calculate, $(12\times13)+(8\div2)-27$.

3. Use Excel to compute each of the following:

(a) $\dfrac{2(23+11)}{(7-10)\div0.9}$ (c) $\sqrt{21.8}+\sqrt[4]{781}$

(b) $(12\times13)-(4\div8)+270$ (d) $\left(1+(0.00031)^6\right)^2$

4. The following table presents the examination scores of ten best candidates in five subjects.

(a) Write the steps for finding the total and mean of the scores using Excel.
(b) Determine the total and the mean score of each candidates.
(c) Rearrange the scores in descending order of total score and then state the best three candidates.

S/N	Candidates	BAM	PHYSICS	CHEMISTRY	BIOLOGY	G.S
1.	Candidate 1	80	77	63	90	78
2.	Candidate 2	40	94	44	78	90
3.	Candidate 3	55	79	98	57	69
4.	Candidate 4	77	96	55	88	65
5.	Candidate 5	49	88	81	50	33
6.	Candidate 6	73	98	59	85	55
7.	Candidate 7	45	71	94	70	88
8.	Candidate 8	92	43	77	97	72
9.	Candidate 9	68	39	68	99	56
10.	Candidate 10	88	29	69	64	91

5. Complete the following table by using an Excel Spreadsheet:

x	y	x^2	y^2	$(xy)^2$	x^2y^2
11	101				
21	91				
31	81				
41	71				
51	61				
$\sum x$	$\sum y =$	$\sum x^2 =$	$\sum y^2 =$	$\sum (xy)^2 =$	$\sum x^2y^2 =$

6. Use a computer spreadsheet to find the mean, median, mode, variance, and standard deviation of the following data: 3, 13, 63, 43, 23, 83, 53, 33, 93, 73, 13, 53, 73, 15, 78, 45, 73, and 63.

Chapter summary

1. Scientific calculators are calculators designed to compute numerical values in science, engineering, business, and mathematics.
2. Scientific calculators have many features that may differ depending on the manufacturers as well as the model.
3. Most of the buttons of a scientific calculator have more than one function.
4. A proposed electronic calculator named CASIO *fx-991MS* is a non-programmable scientific calculator which performs different calculations in mathematics.

5. A non-programmable scientific calculator displays an error message whenever an error is made.

6. Computer packages are used in various disciplines depending in their applications in the intended field.

7. A computer's calculator gives correct calculations and has a powerful interface.

8. A spreadsheet is an interactive computer application program for organizing, analyzing, and storing data.

Revision exercise 1

1. Use non-programmable scientific calculator to compute each of the following expressions:

(a) $\dfrac{(0.6284)^2(62.45)^3(142.72)^4}{\sqrt{(12.68)^3(150.76)^5}}$

(b) $\dfrac{\sin 62^\circ \times \cos 43.5^\circ}{10^{-2} \times \log 140.5 \times \tan 75.2^\circ}$

(c) $\dfrac{{}^{12}C_8 \times {}^{10}P_8 \times \sin^{-1}(0.8695)}{\sqrt[3]{812.5} \times \tan^{-1}(1.5) \times \ln(18.62)}$

2. Use non-programmable scientific calculator to evaluate each of the following:

(a) $\dfrac{d}{dx}(x^4 + 4x^3 + 2x^2 - 10x + 6)$, when $x = 0.5654$

(b) $\dfrac{d}{dx}\sqrt{1+4x+x^2}$, when $x = 6$

3. Use non-programmable scientific calculator to evaluate each of the following definite integrals:

(a) $\int_0^1 (2x+1)e^{x^2+x+1}dx$ (b) $\int_{-2}^3 (2x^3 + 8x^2 - 6x - 16)dx$

4. By using a non-programmable scientific calculator, solve each of the following systems of simultaneous equations:

(a) $\begin{cases} 5x + 6y = 450 \\ 2x + 2y = 160 \end{cases}$ (b) $\begin{cases} 6x + 6y = 900 \\ 12x + 8y = 1440 \end{cases}$

5. By using a non-programmable scientific calculator, solve each of the following systems of simultaneous equations:

(a) $\begin{cases} 3x+3y+3z=21 \\ 5x+10y+15z=80 \\ 7x+21y+28z=154 \end{cases}$ (b) $\begin{cases} 5x+10y-15z=-20 \\ 8x+12y+8z=8 \\ 24x-24y-32z=88 \end{cases}$

6. Use non-programmable scientific calculator to find the solution of each of the following equations:

(a) $24x^2-38x+15=0$ (c) $x^3-5x^2-102x+216=0$

(b) $x^2+9x-400=0$ (d) $30x^3-43x^2-7x+6=0$

7. Enter the following statistical data in a non-programmable scientific calculator and answer the questions that follow:

x	15.5	25.5	35.5	45.5	55.5	65.5	75.5	85.5	95.5
f	25	10	8	12	48	4	26	34	33

Find:

(a) $\overline{x}$ (b) δx (c) $\sum x$ (d) $\sum x^2$

8. Complete the following table by using Excel Spreadsheet:

x	8	12	24	16	10	30	25	5	$\sum x =$
y	16	18	20	14	12	22	32	48	$\sum y =$
x^2									$\sum x^2 =$
y^2									$\sum y^2 =$
$(xy)^2$									$\sum (xy)^2 =$
x^2y^2									$\sum (x^2y^2) =$

9. Using Excel Spreadsheet, complete the following table of scores for ten best candidates in HGE Combination at a certain school.

S/N	History	Geography	Economics	GS	BAM	Total	Average	Rank
1	70	72	68	81	46			
2	65	52	67	35	30			
3	54	48	72	69	15			
4	82	88	80	64	75			
5	76	75	62	51	40			
6	50	86	47	38	42			
7	38	54	60	73	21			
8	45	61	35	64	25			
9	86	92	70	65	83			
10	72	60	57	85	31			

10. The length (in cm) of 15 rods in a shop are given as follows; 38, 52, 80.5, 75.8, 47, 68.4, 16.7, 29.8, 37.5, 80.6, 78.2, 65.9, 18.7, 29.7, 83.2. Compute the following using Microsoft Excel:

(a) $\bar{x}$ (b) δ_x (c) $\sum x^2$ (d) $\sum x$

11. Use Microsoft Excel to compute each of the following expressions:

(a) $\dfrac{\sqrt{64.3}+\sqrt[5]{162.8}-(2.68)^3}{(71.6-19.9)\div 0.58}$

(c) $\left(16.59+(0.00045)^7\right)^4$

(b) $(46\times 95)-(16\div 0.05)+(292-12)$

(d) $\dfrac{(72.5\times 12.8)^6\div(8^4+267)}{(6^4\times 4^3)-(19^2+2^3)}$

12. Using a computer calculator, determine the value of each of the following:

(a) $\sqrt[3]{6\sin 32.5^\circ}+\sqrt[4]{\cos 82.5^\circ}-\dfrac{1}{3}\sqrt{12.8}$

(b) $\tan^{-1}[(0.6875)^2+(1.028)^3]$, give answer in radians.

(c) The mean and standard deviation of the following data:

126	158	162	145	136	192	181	175	168	170
140	160	154	164	170	161	191	194	141	132

13. Given the matrices $A=\begin{pmatrix} 8 & 12 & 14 \\ 5 & 2 & 3 \\ 6 & 7 & -1 \end{pmatrix}$ and $B=\begin{pmatrix} -2 & -4 & 10 \\ 9 & -5 & -7 \\ 1 & 13 & -3 \end{pmatrix}$,

use a non-programmable scientific calculator to find the following:

(a) AB (b) $A^{-1}+B^{-1}$ (c) $(AB)^T$ (d) $|AB|$

14. The following is a frequency distribution table of scores in a certain mathematics test.

Class Interval	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 –70	70–80	80–90
Frequency	25	19	18	26	28	20	43	21

Use a non-programmable scientific calculator to find each of the following:

(a) Mean of the scores (b) Standard deviation (c) $\sum x^2$

15. Using a non-programmable scientific calculator, convert each of the following:

(a) 315°F into degree of Celsius

(b) 400 miles to kilometres

(c) 160 m/s into km/h

16. If $\underline{a}=-7\underline{i}+4\underline{j}+\frac{1}{2}\underline{k}$ and $\underline{b}=6\underline{i}-5\underline{j}-\underline{k}$, use a non-programmable scientific calculator to evaluate each of the following;

(a) $\underline{a}\cdot\underline{b}$ (b) $\underline{a}\times\underline{b}$ (c) $|\underline{a}\times\underline{b}|$ (d) $-6\underline{a}+7\underline{b}$

17. If $z_1=4-3i$ and $z_2=5-12i$, use a non-programmable scientific calculator to evaluate each of the following:

(a) z_1z_2 (c) $|4z_1-5z_2|$ (e) $\text{conjg}(z_1z_2)$

(b) $\frac{z_2}{z_1}$ (d) $\arg\left(\frac{z_1}{z_2}\right)$

18. Use a non-programmable scientific calculator to evaluate each of the following:

(a) $\int_0^{\frac{\pi}{2}}\cos 4x\cos 3x\,dx$ (b) $\int_0^1\frac{3y}{(4y^2-1)^5}dy$ (c) $\int_0^1\frac{4u^2+9u+8}{(u+2)(u+1)^2}du$

Chapter Two

Sets

Introduction

A set is a collection of objects which have common properties. The set is usually denoted by a capital letter, and its objects are listed between curly brackets { } separated by commas. Each object inside the set is called an element of that set. Some basic mathematical concepts, such as relations and vectors, are precisely specified using theoretical concepts of sets. In this chapter, you will learn about the number of elements in the set, basic set operations, Venn diagrams, and simplification of set expressions. The competencies developed can be applied to perform various tasks such as organizing, creating, and categorizing objects.

Methods of representing sets

The useful methods which are used to represent sets are descriptive or statement form method, roster or listing form method, and rule or set builder notation.

Descriptive method

In this method, the elements of a set are described by words and the description is enclosed in curly brackets. For instance, if A is the set of positive even numbers less than or equal to twenty, it can be written as A = {positive even numbers less than or equal to twenty}.

Roster method

In a roster or listing method, the elements of a set are listed inside the curly brackets. The elements are separated by using commas. For example, if B is a set of all factors of 42, then it can be represented as B = {1, 2, 3, 6, 7, 14, 21, 42}.

Set builder notation

This is a method that uses formulas to describe sets. The elements of the set are described by using symbols, usually x followed by a colon (:) which is read as "such that", and followed by a description of the elements. The whole description is enclosed in curly brackets. For instance, if C is a set of odd numbers, it can be described as $C = \{x : x = 2n - 1, \text{ where } n \in \mathbb{N}\}$; which is read as, "C is the set of all elements x such that x is an odd positive number".

Exercise 2.1

1. By using the descriptive method, describe each of the following sets:
   (a) $A = \{1, 8, 27, \ldots\}$
   (b) $B = \{3, 6, 9, 12, \ldots\}$
   (c) $C = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$
   (d) $D = \{\text{dog, cat, sheep, goat, cow}\}$

2. Using the set builder notation, describe each of the following sets:
   (a) $A = \{1, 4, 9, 16, \ldots\}$
   (b) $B = \{1, 8, 27, 64, \ldots\}$
   (c) $C = \{-3, 3\}$
   (d) $D = \{1, 2, 3, 4, 5, \ldots\}$
   (e) $E = \{2, 4, 6, 8, \ldots\}$
   (f) $F = \{0, 1, \ldots\}$

3. If x is an integer, use the roster method to describe each of the following sets:
   (a) $A = \{x : x^2 - 1 = 0\}$
   (b) $B = \{x : x \geq 3\}$
   (c) $C = \{x : x^2 - 4x + 3 = 0\}$

Types of sets

There are various types of sets. These include the empty or null set, equal sets, equivalent sets, subsets, finite and infinite sets, proper and improper subsets, supersets, power sets, singleton set, and universal set.

Empty or null set

An empty set is a set which does not have any element. The empty set is denoted by $\varnothing$ and read as empty set. In roster form an empty set is denoted by $\{\ \}$. For example, A = {three sided rectangles} is an empty set because there is no rectangle with three sides. Also, set $B = \{x : 7 < x < 8;\ x \in \mathbb{N}\}$ is the empty set because there is no natural number between 7 and 8.

Example 2.1

Given that $K = \{x : -10 \leq x \leq -1\}$. Find the natural numbers $x \in \mathbb{N}$ from this set.

Solution

Natural numbers are counting numbers 1, 2, 3, 4, …

There are no counting numbers between $\{x : -10 \leq x \leq -1\}$.

Therefore, $K = \varnothing$ or $K = \{\ \}$.

Example 2.2

Given that $P = \{1, 3, 5, 7, 9, 11\}$. Identify a set of even numbers E in set P.

Solution

A number which is divisible by 2 without a remainder is an even number.

There are no even numbers in set $P = \{1, 3, 5, 7, 9, 11\}$.

Therefore, $E = \{\ \}$ or $E = \varnothing$.

Example 2.3

If A = {Hemedi, Shedrack, Peter, Jonathan, Patrick}, find the set of names of girls (G) from set A.

Solution

There are no girls' names in set A.

Therefore, $G = \{\ \}$ or $G = \varnothing$.

Equal sets

Two or more sets are said to be equal, if they have the same elements and the same number of elements and elements must be alike. For instance, if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then $A = B$. The two sets are equal meaning that the same elements in set A are found in set B, and the two sets have the same number of elements. The comparison written as A=B and it is read as set A is equal to set B.

Unequal sets are sets which have at least one uncommon element. Unequal sets C and D are denoted by $C \neq D$.

For example, if $C = \{a, b, c, d, e, g\}$ and $D = \{a, b, c, d, e, f\}$, then sets C and D are not equal.

Note that, not all elements in set C are in set D and vice versa, although the two sets have the same number of elements. Therefore, $C \neq D$.

Example 2.4

Given that A = {Halima, Angelina, Christian, Sarapia} and B = {Halima, Angelina, Christian, Sarapia}. Are the two sets equal?

Solution

Since each element of set A is in set B and vice-versa, then $A = B$.

Therefore, the two sets are equal.

Example 2.5

Given that $C = \{2, 4, 6, 8\}$ and $D = \{2, 4, 6, 8, 10, 12\}$. Are the two sets equal?

Solution

Since set D has the elements 10 and 12 which are not in set C, then $C \neq D$.

Therefore, the two sets are not equal.

Equivalent sets

Two sets A and B are said to be equivalent if they have the same number of elements. The elements do not need to be the same. What matters is the sets to have one-to-one correspondence of elements. Let A = {a, e, i, o, u} and B = {1, 2, 3, 4, 5}, If the number of elements in sets, A and B are denoted by $n(A)$ and $n(B)$, respectively, then sets A and B are equivalent if $n(A) = n(B)$.

Set equivalence is denoted by set $A \equiv B$.

Example 2.6

Given that $S = \{x : x$ is a counting number less than 7$\}$ and $G = \{x : 1 \le x \le 6, x$ is an integer$\}$. Determine whether or not the two sets S and G are equivalent.

Solution

Given S = {1, 2, 3, 4, 5, 6} and G = {1, 2, 3, 4, 5, 6}.

Since n(S) = 6 and n(G) = 6, then n(S) = n(G).

Therefore, sets S and G are equivalent sets.

Example 2.7

Given that $D = \{1, 3, 5, 7, 9, 11, 13\}$ and $G = \{x : 3 \le x \le 9,\ x \in \mathbb{N}\}$. Are the two sets equivalent?

Solution

Since $G = \{3, 4, 5, 6, 7, 8, 9\}$

Thus, $n(D) = 7$ and $n(G) = 7$, then $n(D) = n(G)$.

Therefore, the two sets are equivalent.

Subsets

Set A is said to be a subset of set B if all the elements of set A are also elements of set B. A symbol for subset is $\subset$. For instance, if set F = {all English alphabets} and set G = {all vowels}, then set G is a subset of set F. This is written as $G \subset F$.

Note that, the number of subsets in a set is given by 2^n, where n is the number of elements in the given set. The terms "proper" and "improper" subsets are used in sets. A proper subset, denoted by the symbol $\subset$, is a subset containing some of elements (not all elements) of the original set, whereas an improper subset denoted by the symbol $\subseteq$ is a set containing every element in the original set. Any set has one improper subset which is the set itself. Therefore, given set A, then $A \subseteq A$ read as "set A is an improper subset of set A". Other subsets of A are proper subsets.

Power set

The collection of all subsets of a set is called the power set. Given set A, the power set of A is denoted by P(A), and is read as "power set of A". Generally, if $n(A) = n$ then the number of subsets of set A is given by 2^n. Thus, $P(A) = 2^n$.

Example 2.8

Given set $B = \{1, 2, 3\}$,

(a) List all the subsets of set B.

(b) List all the improper and proper subsets of set B.

(c) Find P(B)

(d) Find n(P(B))

Solution

(a) The subsets of set B are $\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2, 3\}$, and $\{1, 2, 3\}$.

(b) Improper subsets is $\{1, 2, 3\}$

Proper subsets are $\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}$, and $\{2, 3\}$.

(c) $P(B) = \{\{\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

(d) $n(P(B)) = 2^3 = 8$.

Example 2.9

List all the subsets of set K given that $K = \{\text{Anna, Ally, Halima, John}\}$.

Solution

The subsets of set K are {}, {Anna}, {Ally}, {Halima}, {John}, {Anna, Ally}, {Anna, Halima}, {Anna, John}, {Ally, Halima}, {Ally, John}, {Halima, John}, {Anna, Ally, Halima}, {Anna, Ally, John}, {Ally, Halima, John}, {Anna, Halima, John}, and {Anna, Ally, Halima, John}.

Finite and infinite sets

A finite set is a set with a fixed number of elements which are countable and all elements can be listed. An infinite set is a set with uncountable or endless number elements. In other words, the infinite set is a set that is not finite.

For instance, if V = {a, e, i, o, u} and P = {prime numbers}, set V is called a finite set because it contains countable elements which are five vowels of the English alphabets. Set P is an infinite set because it contains uncountable prime numbers.

Example 2.10

Identify finite and infinite sets, given that $A=\{2y^2 : y \in \mathbb{N}\}$, $B=\{1, 2, 3, 4, 5, 6, \ldots\}$, and $C=\{x : 1 \le x \le 20,\ x \text{ is an integer}\}$

Solution

Sets A and B are infinite, since they have uncountable number of elements, while set C is finite, since it has countable number of elements.

Singleton sets

A set consisting of a single element is called a singleton set. For example,

(i) $A=\{0\}$ is a singleton set

(ii) $B=\{k\}$ is a singleton set

(iii) $C=\{x : x \in \mathbb{N} \text{ and } x^3 = 27\}$ is a singleton set with a single element 3.

Universal sets

A universal set is a set containing all elements or members of all related sets without any repetition of elements. The symbol U or ξ is used to denote a universal set. However, U is the mostly used symbol. For instance, if $A=\{1, 2, a, 4, 5\}$ and $B=\{1, 2, 3, a, b, c, d\}$, then the universal set associated with A and B is given by $U=\{1, 2, 3, 4, 5, a, b, c, d\}$.

Note that, $A \subset U$ and $B \subset U$.

Example 2.11

If $U=\{7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$, from set U list the elements of each of the following related sets:

(a) $A=\{x : x \text{ is a factor of } 60\}$

(b) $B=\{x : x \text{ is an even number}\}$

(c) $C=\{x : x \text{ is an odd number}\}$

Solution

The elements of sets A, B, and C from the universal set "U" are;

(a) $A=\{10, 12, 15\}$

(b) $B=\{8, 10, 12, 14, 16\}$

(c) $C=\{7, 9, 11, 13, 15\}$

Exercise 2.2

1. Which of the following sets are finite or infinite? Give reasons to support your answers.

   (a) $E=\{x : x$ is a plant on the earth's surface$\}$

   (b) $F=\{x : 10 \le x \le 30,\ x \in \mathbb{R}\}$.

   (c) $G=\{x : 10 < x < 20,$ x is not an odd number$\}$

2. If $H=\{6, 7, 8, 9, 10\}$, T = {Tony, James, Herry, Juma}, and R = {Juma, Herry, Tony, James}, determine which of the three sets are equivalent, equal or unequal?

3. Write true or false in each of the following:
   (a) $\{4\}$ is an element of $\{\{4\}, \{4\}\}$
   (b) $\{1\} \subseteq \{0,1\}$
   (c) $\{1\} \subseteq \{1\}$
   (d) $\varnothing$ is an element of $\{\}$
   (e) 0 is an element of $\{\}$
4. Given set K = {a, b, c, d}.
   (a) List all the subsets of K.
   (b) Find n(K)
5. Given set J = {dog, cat, lion, zebra}. Find:
   (a) The power set, P(J)
   (b) $n[P(J)]$
6. Find the universal set associated with V and S, if V = {all prime numbers less than 20} and $S = \{2, 3, 4, 6, 7, 8, 10, 11, 14\}$.

Basic operations of sets

Set operations are performed on two or more sets in order to build a relationship. There are four main set operations which are union of sets, intersection of sets, complement of a set, and difference of sets.

Union of sets

The symbol "$\cup$" is used to denote the union of sets. The union of two sets A and B is the set of all elements that are either in set A or in set B, or in both sets A and B without repetition of elements. In set notation it is written as

$A \cup B = \{x : x \in A \text{ or } x \in B\}$.

For example, if $A = \{2, 4, 6, 7\}$ and $B = \{1, 2, 3, 5\}$, then

$A \cup B = \{2, 4, 6, 7\} \cup \{1, 2, 3, 5\} = \{1, 2, 3, 4, 5, 6, 7\}$.

Example 2.12

Find $A \cup B$ if $A = \{a, b, c, d, e, f, g, h\}$ and $B = \{a, e, i, o, u\}$.

Solution

$A \cup B$ denotes the set of all elements that are either in set A or in set B or in both. That is, $A \cup B = \{a, b, c, d, e, f, g, h, i, o, u\}$.

Therefore, $A \cup B = \{a, b, c, d, e, f, g, h, i, o, u\}$.

Example 2.13

Find $A \cup B$ if $A = \{3, 5, 7, 8, 10, 11, 14, 16\}$ and $B = \{1, 2, 6, 9, 12\}$.

Solution

From the given sets, $A \cup B = \{3, 5, 7, 8, 10, 11, 14, 16\} \cup \{1, 2, 6, 9, 12\}$

Therefore, $A \cup B = \{1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 12, 14, 16\}$.

Intersection of sets

The symbol "$\cap$" is used to denote the intersection of sets. The intersection of two sets A and B is the set of elements that belong to both A and B. In set notation it is written as, $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

For instance, if $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and $B = \{1, 3, 5, 7, 9\}$, then $A \cap B = \{1, 3, 5, 7\}$.

If the intersection of sets is an empty set (no common elements), then the sets are said to be disjoint. Otherwise, they are joint sets.

Example 2.14

Find $A \cap B$, if $A = \{a, e, i, o, u\}$ and $B = \{a, b, c, d, e, f, g, h, i\}$.

Solution

From the given sets,

$$\begin{aligned} A \cap B &= \{a, e, i, o, u\} \cap \{a, b, c, d, e, f, g, h, i\} \\ &= \{a, e, i\}. \end{aligned}$$

Therefore, $A \cap B = \{a, e, i\}$.

Note that, A and B have three elements in common, thus they are joint sets.

Example 2.15

If $A=\{a, e, i, o\}$ and $B=\{b, c, f, g\}$, find $A \cap B$ and state whether A and B are joint or disjoint sets.

Solution

From the given sets,

$A \cap B=\{a, e, i, o\} \cap \{b, c, f, g\}$
$= \varnothing$.

Therefore, A and B are disjoint sets because they have no common element(s).

Complement of a set

The complement of a set A, denoted as A' or A^c is the set that contains all the elements of the universal that are not in set A, or it is the difference between the universal set and set A. This means that, if A is a subset of a universal set, then the elements of the universal set which are not in A form the complement of A. For instance, if $U=\{a, b, c, d, ..., z\}$ and $A=\{a, b, c, d, e\}$ then $A'=\{f, g, h, ..., z\}$.

Example 2.16

Given $U=\{125, 245, 365, 475, 585\}$ and $A=\{365\}$, find A.

Solution

Given $U=\{125, 245, 365, 475, 585\}$ and $A=\{365\}$.

Thus, $A=\{125, 245, 475, 585\}$

Therefore, $A'=\{125, 245, 475, 585\}$.

Example 2.17

Given $U=\{a, e, i, o, u\}$, $B=\{e, i, u\}$, find B'.

Solution

Given $B=\{e, i, u\}$, then $B' = \{a, o\}$.

Therefore, $B'=\{a, o\}$.

Example 2.18

Given $U=\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$, $A=\{5, 7, 11, 13, 15, 17\}$, and $B=\{4, 6, 8, 10, 12, 14\}$ find:

(a) $A' \cup B'$ (b) $A \cap B'$

Solution

The compliment of sets A and B are:

$A'=\{4, 6, 8, 9, 10, 12, 14, 16, 18\}$

and

$B' = \{5, 7, 9, 11, 13, 15, 16, 17, 18\}$

Therefore,

(a) $A' \cup B'=\{4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18\}$

(b) $A \cap B'=\{5, 7, 11, 13, 15, 17\}$.

Difference of sets

The difference of two sets A and B, written as $A-B$, is the set of all elements of set A that are not in set B. Consider two sets A and B whereas, $A=\{1, 2, 3, 4, 5\}$ and $B=\{3, 4, 5, 6, 7, 8\}$. The difference of these two sets is found by writing all the elements of set A, and then remove all element of A which are also elements of B. Since set A

shares the elements 3, 4, 5 with set B, then $A-B=\{1,\ 2\}$. The same results can be obtained by finding $A\cap B'$.
In general, $A-B=A\cap B'$.

Example 2.19

Given $A=\{4,\ 7,\ 8,\ 12, 15, 17\}$ and $B=\{4,\ 6,\ 8,\ 10,\ 12,\ 14\}$, find

(a) $A-B$ (b) $B-A$

Solution

(a) $A-B$ contains elements that are only in set A but not in set B, that is,

$A-B=\{7, 15, 17\}$.

(b) $B-A$ contains elements that are only in set B but not in set A, that is,

$B-A=\{6, 10, 14\}$.

Symmetric difference of sets

A symmetric difference of two sets A and B is the set $(A-B)\cup(B-A)$, and is denoted by $A\Delta B$ It is the set of all elements which belong to either set A or set B but not to both. This means that, if $x\in(A\Delta B)$, then $A\Delta B=\{x : x\in A \text{ or } x\in B \text{ and } x\notin A\cap B\}$.

Example 2.20

Determine $A\Delta B$ for $A=\{1, 2, 3,\ 9,\ 12,\ 15\}$ and $B=\{1, 3, 9, 14, 13,\ 20,\ 12, 25\}$.

Solution

The set $A\Delta B$ is formed by the elements which belong to either set A or B but not to both sets. Thus, $A-B=\{2, 15\}$ and $B-A=\{13, 14, 20, 25\}$.

$$\Rightarrow A\Delta B=(A-B)\cup(B-A)$$
$$=\{2, 13, 14, 15, 20, 25\}.$$

Therefore, $A\Delta B=\{2, 13, 14, 15, 20, 25\}$.

Example 2.21

Determine $A\Delta B$ for A = {mango, pineaple, orange, banana, apple} and B = {orange, mango, watermelon, avocado, apple}.

Solution

The set $A \Delta B$ contains the elements which belong to either set A or set B but not to both sets. That is, $A \Delta B = \{\text{pineapple, banana, watermelon, avocado}\}$.

Therefore, $A \Delta B = \{\text{pineapple, banana, watermelon, avocado}\}$.

Exercise 2.3

In questions 1 – 10, find the union and intersection of the given sets.

1. $A = \{\text{first five letters of the English alphabets}\}$, $B = \{a, b, c, d, e, f, g, h\}$
2. $A = \{\text{Even numbers}\}$, $B = \{\text{Counting numbers}\}$
3. $G = \{25, 30, 45\}$, $H = \{20, 25\}$
4. $J = \{0, \Delta, 3\}$, $K = \{\Delta\}$
5. $A = \{\text{All positive multiples of 6 less than 60}\}$,
   $B = \{\text{All positive multiples of 4 less than 48}\}$
6. $W = \{14, 16, 18, 20\}$, $Z = \{\ \}$
7. $A = \{94, 110, 120, 131, 140\}$, $B = \{94, 110, 265\}$
8. $A = \{\text{Prime factors of 72}\}$, $B = \{\text{Prime factors of 15}\}$
9. $A = \{\text{all even number less than 28}\}$, $B = \{\text{all multiples of 3 less than 27}\}$
10. $A = \{a, b, c, d\}$, $B = \{a, d, e\}$, and $C = \{\ \}$
11. Find the intersection of $M = \{a, b, c, d, e, g\}$ and $N = \{1, 2, 3, 4, 5, 6, 7\}$.
12. What is the name of sets which have no common elements?
13. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$, $M = \{1, 2, 3, 4\}$ and $N = \{1, 3, 7, 9, 11\}$, find M' and N'.
14. If $U = \{\text{mango, orange, tomato, cabbage, pineapple, watermelon}\}$,
    $A' = \{\text{mango, watermelon, tomato}\}$, list the elements of set A.
15. If $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$, list the elements of $A \cup B$.
16. Given that; $U = \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$, $A = \{3, 4, 5, 6\}$, and $B = \{5, 6, 7, 8\}$, find each of the following:

    (a) $(A \cap B')$ (c) $B' \cup A'$

    (b) $B \cup A'$ (d) $A' \cap B$

17. Given that $U=\{0, 11, 12, 13, 14, 15, 16, 17\}$, $P=\{0, 11, 13\}$, and $Q=\{15, 16, 17\}$ write T for a true statement and F for a false statement in each of the following:

(a) $P'=Q$ (c) $n(P\cap Q)=1$

(b) $P'\cap Q'=(P\cup Q)'$ (d) $n(P'\cup Q')=9$

18. If $U=\{\text{magenta, red, pupple, green, pink, blue, black, white}\}$, find A' if $A=\{\text{green, pink, white, pupple, black}\}$.

19. Given that $U=\{a, b, c, d, e, f, g, h, i, j\}$, $A=\{a, b, c, d, e, f\}$, and $B=\{a, d, e\}$, find each of the following:

(a) $A\cap B$ (c) $A'\cap B$ (e) $A'\cap B'$

(b) $A\cup B$ (d) $A\cap B'$ (f) $A'\cup B'$

20. Find the union and intersection in each of the following sets:

(a) $A=\{22, 24, 26\}$, $F=\{24, 26, 28, 30\}$

(b) $Y=\{x: 2<x\leq 18\}$, $W=\{x: 7\leq x<23\}$

21. Given that $A=\{13, 14, 15\}$ and $B=\{14, 15, 16, 17, 18\}$, determine
(a) $A-B$ (b) $B-A$

22. If $M=\{32, 34, 36, 38, 40, 42\}$ and $K=\{33, 34, 35, 36, 37, 38, 40\}$, find:
(a) $M-K$
(b) $K-M$
(c) $(M-K)\cup(K-M)$

23. Given three sets P, Q, and R, with $P=\{8, 9, 10, 11, 12\}$, $R=\{9, 11, 14, 16, 18\}$, and $Q=\{8, 9, 14, 16, 18, 20, 22\}$, find each of the following:
(a) $P-R$
(b) $R-P$
(c) $Q-P$

24. Given that $U=\{\text{all odd numbers less than } 20\}$, $A=\{7, 13, 15\}$, and $B=\{3, 5, 9\}$, find each of the following:
(a) $B-A$ (b) $(A-B)'$

25. If $X=\{2, 4, 6, 8, 10\}$, and $Y=\{15, 17, 20\}$, find $X-Y$.

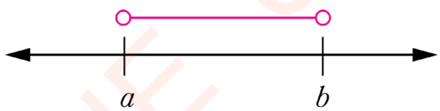
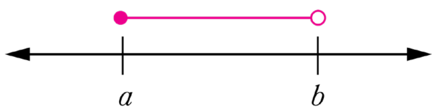
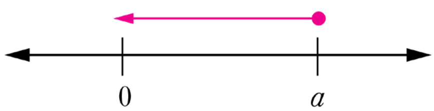
26. Given $\{6, 9, 12, 15\}-\{15\}=\{6, x, y\}$, find the values of x and y.

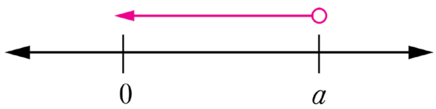
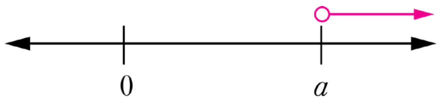
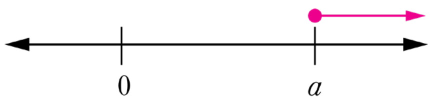
Representation of sets on a number line

A number line can be used to represent a set of values in a given interval. The number of elements within the interval cannot be counted or listed because they are infinite. An interval on the number line can be open, closed, half open, or half closed.

An open interval is an interval in which the end points are excluded. For example, if $a < b$, then the open interval from a to b, denoted by $\{a < x < b\}$ or (a, b) is a number line extending from point a to point b, excluding the end points a and b. A closed interval from point a to point b, denoted by $\{a \leq x \leq b\}$ or $[a, b]$ is a number line extending from point a to point b, including the end points a and b. An interval can include only one end point. This kind of interval is called half-open or half-closed. A half-open interval can be left closed, right open denoted as $\{a \leq x < b\}$ or $[a, b)$; or it can be left opened, right closed denoted as $\{a < x \leq b\}$ or $(a, b]$. Table 2.1 shows the forms of intervals on the number line.

Table 2.1: Forms of intervals on a number line

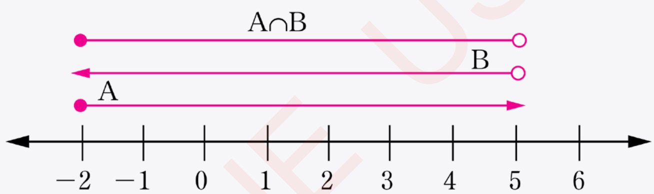
Interval notation	Number line representation	Meaning
$\{x : a < x < b\}$ or (a, b)		Left open, right open
$\{x : a \leq x \leq b\}$ or $[a, b]$		Left closed, right closed
$\{x : a < x \leq b\}$ or $(a, b]$		Left open, right closed
$\{x : a \leq x < b\}$ or $[a, b)$		Left closed, right open
$\{x : x \leq a\}$ or $(-\infty, a]$		Right closed

Interval notation	Number line representation	Meaning
$\{x : x < a\}$ or $(-\infty, a]$		Right open
$\{x : x > a\}$ or (a, ∞)		Left open
$\{x : x \geq a\}$ or $[a, \infty)$		Left closed

Example 2.22

Using the number line, find $A \cap B$, given that $A = \{x : x \geq -2\}$ and $B = \{x : x < 5\}$, where $x \in \mathbb{R}$.

Solution

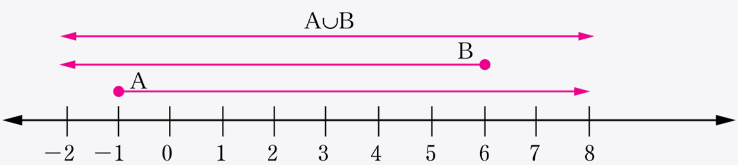


Therefore, $A \cap B = \{x : -2 \leq x < 5\}$ as represented on the number line.

Example 2.23

Use a number line to represent $A \cup B$, if $A = \{x : x \geq -1\}$ and $B = \{x : x \leq 6\}$, where $x \in \mathbb{R}$.

Solution

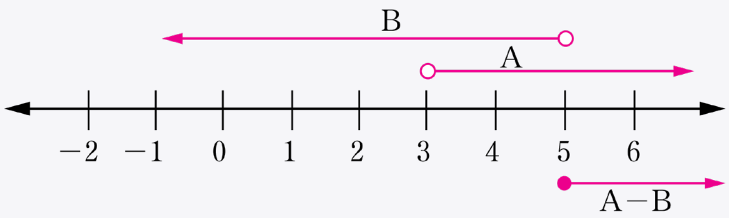


Therefore, $A \cup B = \{x : x \in \mathbb{R}\}$ as represented on the number line.

Example 2.24

Find $A - B$ by using a number line, where $A = \{x : x > 3\}$ and $B = \{x : x < 5\}$, $x \in \mathbb{R}$.

Solution



Therefore, $A - B = \{x : x \geq 5\}$ as represented on the number line.

Exercise 2.4

1. Given $A = \{x : 6 < x < 10\}$ and $B = \{x : x < 12\}$, where $x \in \mathbb{R}$, use a number line to represent each of the following:

   (a) A' (b) B' (c) $(A \cup B)'$ (d) $(A \cap B)'$

   (e) $A - B$ (f) $B - A$ (g) $A - B'$ (h) $(A - B')'$

   (i) $(A - B)'$ (j) $(B - A)'$ (k) $A' \cap B$ (l) $A' \cap B'$

   (m) $A' - B'$ (n) $(A - B') \cap (A' \cap B')$

2. If $A = \{x : 8 < x < 20\}$ and $B = \{x : x < 15\}$, where $x \in \mathbb{R}$, use a number line to represent each of the following:

   (a) $A' \cap B'$ (b) $(A \cup B)'$ (c) $(A \cap B)'$ (d) A'

   (e) $B - A$ (f) $(A - B)'$ (g) $(A \cup B)' \Delta (B - A)$ (h) $[(A \Delta B)' \cup (A \cap B)]'$

   (i) $A - B'$ (j) $(A' \Delta B)'$ (k) $A' \Delta B$ (l) $A \Delta B$

3. If $A = \{x : 8 \leq x \leq 20\}$ and $B = \{x : x \leq 18\}$ where $x \in \mathbb{R}$, use a number line to determine each of the following:

   (a) $(A - B)'$ (b) $(B - A)'$ (c) $(A - B')'$

Fundamental laws of set algebra

The laws of algebra of sets are the properties of set operations and set relations. These properties provide insight into the fundamental nature of sets. They are used in writing set expressions in the most simplified form without changing the meaning of the original value of the expression.

Proof of the fundamental laws of algebra of sets

The laws of algebra of sets can be proved using basic operations. Table 2.2 shows some laws of algebra of sets, where A, B, and C are non-empty sets.

Table 2.2: *Laws of algebra of sets*

S/N	Set notations	Laws
1.	$A \cup \varnothing = A$ $A \cap U = A$ $A \cup U = U$ $A \cap \varnothing = \varnothing$	Identity/Domination
2.	$A \cup A = A$ $A \cap A = A$	Idempotent
3.	$(A')' = A$	Double complement
4.	$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative
5.	$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative
6.	$(A \cup B)' = A' \cap B'$ $(A \cap B)' = A' \cup B'$	De Morgan's
7.	$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption
8.	$A \cup A' = U$; $\varnothing' = U$ $A \cap A' = \varnothing$; $U' = \varnothing$	Complement

S/N	Set notations	Laws
9.	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive

Note that, two or more sets are equal if and only if the sets are improper subsets of each other, that is $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

(a) Idempotent law

Given set A, then $A \cup A = A$ and $A \cap A = A$.

Proof

Given the set A, then

$A \cap A = \{x : x \in A \text{ and } x \in A\}$, which implies $\{x : x \in A\} = A$ and
$A \cup A = \{x : x \in A \text{ or } x \in A\}$, which implies $\{x : x \in A\} = A$.

Therefore, $A \cup A = A$ and $A \cap A = A$.

(b) Commutative law

Given two sets A and B, then $A \cup B = B \cup A$ and $A \cap B = B \cap A$.

Proof

If $x \in A \cup B$, then either $x \in A$ or $x \in B$, which implies $x \in A$ or $x \in B$ and hence $x \in B \cup A$. Similarly, it can be shown that $B \cup A \subseteq A \cup B$.
Therefore, $A \cup B = B \cup A$.
Also, if $x \in A \cap B$, then $x \in A$ and $x \in B$, which implies $x \in B$ and $x \in A$ and hence $x \in B \cap A$. Similarly, it can be shown that $B \cap A \subseteq A \cap B$.
Therefore, $A \cap B = B \cap A$.

(c) Associative law

Suppose A, B, and C are three sets, then $A \cup (B \cup C) = (A \cup B) \cup C$ and $A \cap (B \cap C) = (A \cap B) \cap C$.

Proof

If $x \in (A \cup B) \cup C$, then $x \in A \cup B$ or $x \in C$ which implies $(x \in A \text{ or } x \in B)$ or $x \in C$. Thus, $(x \in A)$ or $(x \in B \cup C)$. Hence, $x \in A \cup (B \cup C)$. Therefore, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$.

Similarly, it can be proved that $(A \cup B) \cup C \subseteq A \cup (B \cup C)$.
Therefore, $A \cup (B \cup C) = (A \cup B) \cup C$.

Activity: Identifying associative laws of sets

Individually or in a group, perform the following tasks:

1. Select three sets A, B, and C where A and B are joint sets, as well as B and C. Also, make sure that, A and $B \cap C$ have at least one common member so as C and $A \cap B$.
2. List the elements of $A \cap B$, and elements of $B \cap C$.
3. List the elements of $A \cap (B \cap C)$.
4. List the elements of $(A \cap B) \cap C$.
5. Compare the lists in tasks 3 and 4.
6. What did you observe in task 5? Give reasons.

(d) Distributive law

Given three sets A, B, and C, then $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Proof

If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B \cup C$. Thus, $x \in A$ and ($x \in B$ or $x \in C$), which implies ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$). Hence, $x \in (A \cup B)$ and $x \in (A \cup C)$.
$\Rightarrow x \in [(A \cup B) \cap (A \cup C)]$.

Thus,
$A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly, it can be shown that
$(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Therefore,
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

(e) De Morgan's law

Given two sets A and B, then
$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$.

Proof

Suppose $K = (A \cup B)'$ and $N = A' \cap B'$.
Let x be an arbitrary element of K, then
$x \in K \Rightarrow x \in (A \cup B)'$.
This implies that, $x \notin (A \cup B), x \notin A$, and $x \notin B$, $x \in A'$, and $x \in B'$.
$\Rightarrow x \in A' \cap B'$, and $x \in N$.
Thus, $K \subseteq N$ .... (i)

Also, let y be an arbitrary element of N, then $y \in N \Rightarrow y \in A' \cap B'$.

This implies that,
$y \in A'$ and $y \in B'$, $y \notin A$ and $y \notin B$
$y \notin (A \cup B), y \in (A \cup B)'$, and $y \in K$.
Thus, $N \subseteq K$ .... (ii)

Therefore, a combination of equations (i) and (ii) proves that, $(A \cup B)' = A' \cap B'$.

Note that, $(A \cap B)' = A' \cup B'$ can be proved in the same way.

(f) Complement law

If U is a universal set and A is a finite set, then the following hold true:

(i) $A \cup A' = U$ (iii) $\varnothing' = U$

(ii) $A \cap A' = \varnothing$ (iv) $U' = \varnothing$

Proof

(i) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$
(ii) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \varnothing$
(iii) $\varnothing' = \{x \in U : x \notin \varnothing\} = U$
(iv) $U' = \{x \in \varnothing : x \notin U\} = \varnothing$

(g) Identity/Domination law

If U is a universal set and A is a finite set, then the following holds:

(i) $A \cup \varnothing = A$ (ii) $A \cap U = A$ (iii) $A \cup U = U$ (iv) $A \cap \varnothing = \varnothing$

(h) Double complement law

If U is a universal set and A is a finite set, then $(A')' \subseteq A$.

Proof

If $x \in (A')'$, then x does not belong to A', that is $x \notin A'$, then $x \in A \Rightarrow (A')' \subseteq A$.

Similarly, it can be shown that $A \subseteq (A')'$.

Therefore, $(A')' = A$.

(i) Absorption law

If A and B are two sets, then $A \cap (A \cup B) = A$ and $A \cup (A \cap B) = A$

Proof

Given $A \cap (A \cup B)$

Then $A \subseteq A$ and $A \subset A \cup B$.

Therefore, $A \cap (A \cup B) = A$. Also, given $A \cup (A \cap B)$

Then $A \subseteq A$ and $A \subset A \cap B$.

Therefore, $A \cup (A \cap B) = A$.

Example 2.25

Use the laws of algebra of sets to simplify $(A \cup B)' \cap A$.

Solution

Given $(A \cup B)' \cap A$, then

$(A \cup B)' \cap A = (A' \cap B') \cap A$	De Morgan's law
$= A \cap (A' \cap B')$	Commutative law
$= (A \cap A') \cap B'$	Associative law
$= \varnothing \cap B'$	Complement law
$= \varnothing$	Identity law

Therefore, $(A \cup B)' \cap A = \varnothing$.

Example 2.26

By using the laws of algebra of sets, simplify $[A-(A\cap B')]\cup(A\cap B)$.

Solution

Given $[A-(A\cap B')]\cup(A\cap B)$, then

$[A-(A\cap B')]\cup(A\cap B)=[A\cap(A\cap B')']\cup(A\cap B)$	By definition $A-B = A\cap B'$
$=[A\cap(A'\cup B)]\cup(A\cap B)$	De Morgan's law
$=[(A\cap A')\cup(A\cap B)]\cup(A\cap B)$	Distributive law
$=[\varnothing\cup(A\cap B)]\cup(A\cap B)$	Complement law
$=(A\cap B)\cup(A\cap B)$	Identity law
$=A\cap B$	Idempotent law

Therefore, $[A-(A\cap B')]\cup(A\cap B)=A\cap B$.

Example 2.27

Use the laws of algebra of sets to simplify $A\cap(A\cup B)$.

Solution

Given $A\cap(A\cup B)$, then

$A\cap(A\cup B)=(A\cup\varnothing)\cap(A\cup B)$	Identity law
$=A\cup(\varnothing\cap B)$	Distributive law
$=A\cup\varnothing$	Domination law
$=A$	Identity law

Therefore, $A\cap(A\cup B)=A$.

Exercise 2.5

1. By using the laws of algebra of sets prove that:
   (a) $(B\cap A)\cup(U\cap A)=A$ (b) $(\varnothing\cup B)\cap(B\cup A)=B$

2. If A, B, and C are any three sets, show that:
   (a) $(C-A)\cup(B\cap A')=(C\cup B)-A$
   (b) $(A\cap B)\subseteq B$
   (c) $B\subseteq(A\cup B)$

3. Use the laws of algebra of sets to simplify each of the following expressions:
   (a) $(A\cup B)\cap(A\cup C)$
   (b) $(A\cup B)'\cap(A\cap B)'$
   (c) $(\varnothing\cup A)\cap(B\cup A)$
   (d) $(A\cap B')\cap(B\cap A')$
   (e) $[A\cap(A\cap B)']'$
   (f) $(A-B)\cap B$
   (g) $(A\cup B')\cup[(A-C)\cup(B'-C)]$
   (h) $(P - M) - H$
   (i) $(A\cup B)\cap(A'\cup(A\cap B))$
   (j) $(A\cup B)'\cap(A\cup B)\cap(A\cup B)'$

4. Use the laws of algebra of sets to prove each of the following identities.
   (a) $A-B=A-(A\cap B)$
   (b) $(A\cap C)-(B\cap C)=(A-B)\cap C$
   (c) $(A\cup B)=A\cup(B-A)$

5. Show each of the following identities by using the laws of algebra of sets:
   (a) $A\cap B=A\cap(A'\cup B)$
   (b) $(A-B)\cap(A-C)=A-(B\cup C)$
   (c) $A=(A\cap B)\cup(A\cap B')$

Venn diagrams

Sets can be geometrically shown by drawings called Venn diagrams. Venn diagrams were introduced by the English Mathematician John Venn. Venn diagrams consist of overlapping circles or ovals inscribed in a rectangle which represents a universal set. Each circle or oval represents a set. The points inside a circle represent elements of the set, while the points outside the boundary of the circle represent elements that are not in the set. Venn diagrams provide a quick way of showing relationships of the sets and thus, are useful in presentations and reports.

For example, $U=\{a, b, c, d, e\}$ can be presented in a Venn diagram as shown in Figure 2.1.

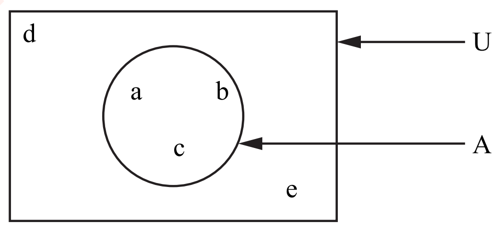


Figure 2.1: *Venn diagram*

In Figure 2.1, U is the universal set containing the elements a, b, c, d, and e. A is the set of alphabetical letters a, b, and c. It is clearly seen that, A is a subset of U and A consists of the elements d and e.

If the sets have any element in common, then the ovals overlap. Such sets are called joint sets. For example, if $A = \{a, b, c\}$ and $B = \{a, b, d\}$, then the relation betwen A and B is represented as in Figure 2.2.

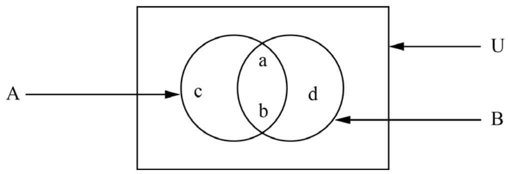


Figure 2.2: *Joint sets in a Venn diagram*

If the sets do not overlap in a Venn diagram, they are called disjoint sets. For example, if $A = \{a, b\}$ and $B = \{1, 2\}$ the relation between A and B is shown in Figure 2.3.

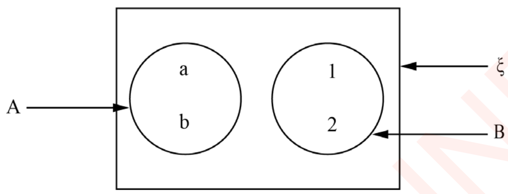


Figure 2.3: *Disjoint sets in a Venn diagram*

If all elements of set A are also elements of set B, then set A is a subset of set B. For example, if $A = \{7, 8, 11\}$ and $B = \{6, 7, 8, 9, 10, 11, 12\}$, then the Venn diagram which shows set A is contained inside set B or A is a proper subset of B as shown in Figure 2.4.

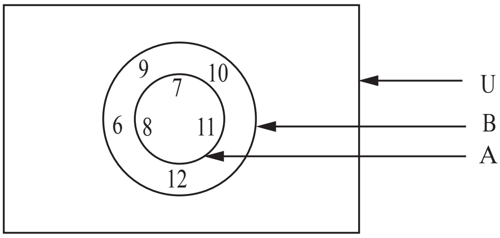


Figure 2.4: *Subsets in a Venn diagram*

Operations of sets using Venn diagrams

A Venn diagram is designed in such a way that it depicts the sets and how its subsets are related to each other. The number of circles and the way they intersect should focus on the entire problem. The regions are then labelled according to the problem specifications. Usually, the region corresponding to a given set is shown by a shaded region with the elements belonging to the set.

Intersection of sets

The intersection of two or more sets is shown in a Venn diagram by shading the overlapping region of the given sets. The shaded region in Figure 2.5 shows the intersection of sets A and B.

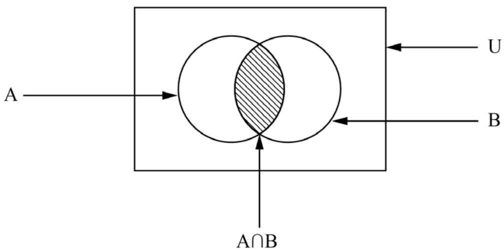


Figure 2.5: *Shaded region showing the intersection of sets A and B*

Union of sets

The union of sets in a Venn diagram is represented by the combined regions of all sets under consideration. For example, in Figure 2.6; the union of sets, A and B, is the shaded region.

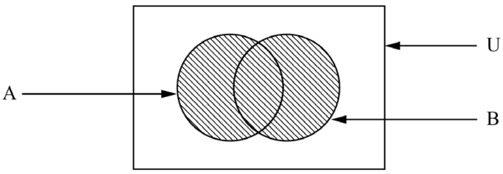


Figure 2.6: *Shaded region showing the union of sets A and B*

Symmetric difference of sets

The symmetric difference of two sets A and B is also known as the disjunctive union set of elements which belong either to set A or B but not in the intersection. In set notation, the symmetric difference of two sets, A and B is given by $A\Delta B = (A - B) \cup (B - A)$. This combination of sets is represented by the region that does not include the intersection in the Venn diagram. For example, the shaded region in Figure 2.7 represents the symmetric difference of two sets, A and B.

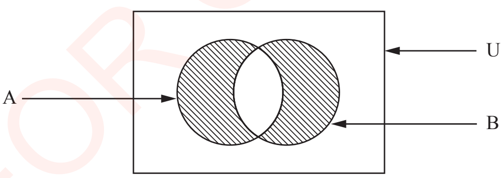


Figure 2.7: *Shaded region showing the symmetric difference of sets A and B*

Relative difference

The relative difference of set A with respect to set B, that is $A - B$ is the region in A that does not include B. The relative difference of set A with respect to set B is the shaded region in Figure 2.8.

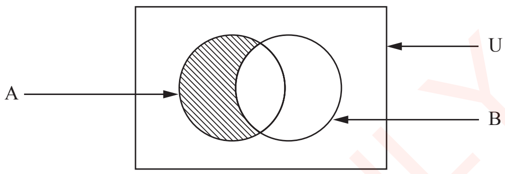


Figure 2.8: *Shaded region showing the relative difference of set A with respect to set B*

Complement of a set

As it was described earlier, the complement of set A, denoted by A' or A^c is the set containing the elements that are in the universal set but not in set A. In the Venn diagram, A' is shown by the region outside A within the universal set as shown in Figure 2.9.

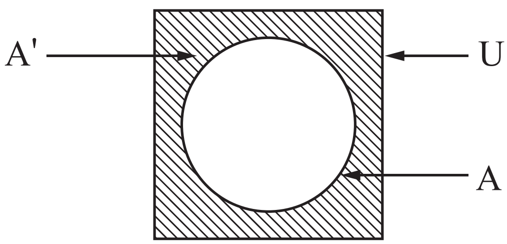


Figure 2.9: *Shaded region showing A'.*

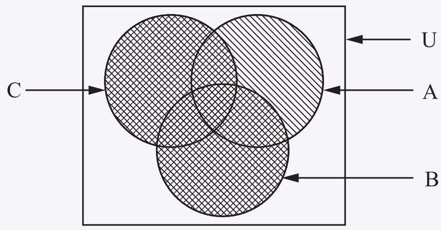
Example 2.28

Represent each of the following set expressions using a Venn diagram:
(a) $A \cap (B \cup C)$ (b) $(A \cup B) \cap (A \cup B')'$

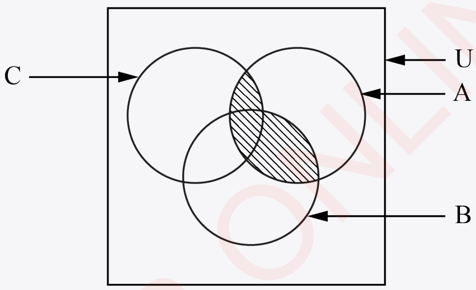
Solution

(a) Given $A \cap (B \cup C)$.

First shade the region A and then the region $B \cup C$ as shown in the following figure.

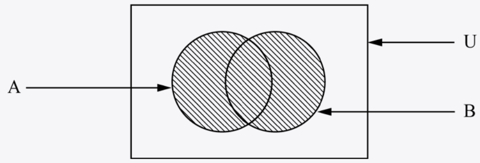


The required region is the intersection of the two shadings; that of A and that of $B \cup C$ as shown in the following figure.

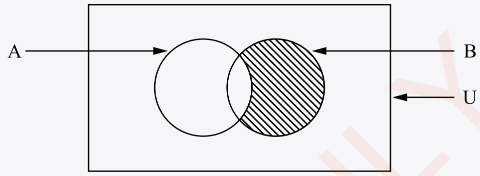


The shaded part of the Venn diagram represents $A \cap (B \cup C)$.

(b) Given $(A \cup B) \cap (A \cup B')'$, then Shade the region $A \cup B$ and the region $(A \cup B')'$ as shown in the following figures.

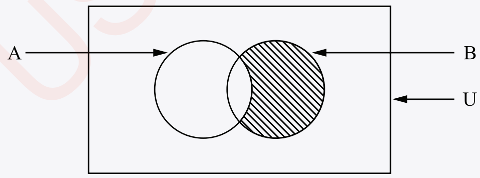


The shaded part of the Venn diagaram represents $A \cup B$.



The shaded region represents $(A \cup B')'$.

The required region is the intersection of the two shadings; that of $A \cup B$ and that of $(A \cup B')'$ as shown in the following figure.



The shaded region represents
$(A \cup B) \cap (A \cup B')'$

Exercise 2.6

Use the Venn diagrams to represent each of the following set expressions:

1. $(A \cup B)' \cap (A \cup C)$
2. $(A \cup B)'$
3. $A' \cap B'$
4. $A \cap B \cap C$
5. $A - B$
6. $(A - B) \cup (A - C) \cup (B - C)$

Number of elements/cardinality of sets

The cardinality of sets is a measure of sets size, that is the number of elements in the set. Thus if A is a finite set with *n* elements, then the cardinality of set A is *n*. Therefore, the cardinality of an empty set is zero. The cardinality of a combined set from two or more sets can be determined by using a general formula.

To derive the general formula, consider two arbitrary finite sets A and B enclosed in a universal set as represented in Figure 2.10.

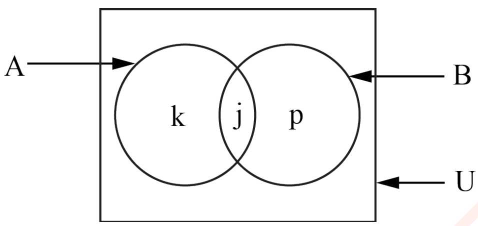


Figure 2.10: *Two enclosed sets in the universal set*

From Figure 2.10;

$$n(A) = k + j$$

$$n(B) = j + p$$

$$n(A \cap B) = j$$

$$n(A \cup B) = k + j + p$$

Adding the cardinality of set A and set B gives:

$$\begin{aligned} n(A) + n(B) &= (k + j) + (j + p) \\ &= (k + j + p) + j \\ &= n(A \cup B) + n(A \cap B) \end{aligned}$$

By rearranging,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

Thus, if U is the universal set with subsets A and B, then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B).$$

This is a general formula commonly used in solving problems involving two sets.

Example 2.29

If n (A) = 28, n(B) = 10, and $n(A \cup B) = 33$, find $n(A \cap B)$.

Solution

Given n(A) = 28, n(B) = 10 and $n(A \cup B) = 33$, from $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we have,

$$\begin{aligned} n(A \cap B) &= n(A) + n(B) - n(A \cup B) \\ &= 28 + 10 - 33 \\ &= 5 \end{aligned}$$

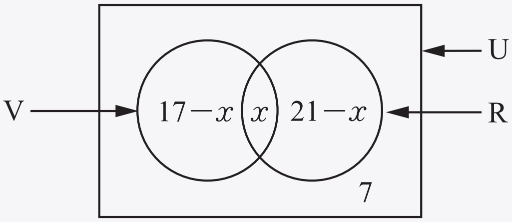
Therefore, $n(A \cap B) = 5$.

Example 2.30

There are 21 red flowers and 17 variegated flowers in a bouquet of 44 blooms. How many red and variegated flowers are there if 7 are neither red nor variegated?

Solution

Let R stands for red flowers and V stands for variegated flowers. The following figure shows the information given with *x* as the number of elements in the intersection of sets.



From $n(V\cup R)+n(V\cup R)'=n(U)$

But $n(V\cup R)=n(V)+n(R)-n(V\cap R)$

$\Rightarrow n(V)+n(R)-n(V\cap R)+n(V\cup R)'=n(U)$

$\Rightarrow (17-x+x)+(21-x+x)-x+7=44$

$A\cup B\cup C$

$\Rightarrow x=1$

Therefore, 1 flower is both red and variegated.

Cardinality of union of three sets

The formula for the cardinality of union of two sets can be extended to suit the problems involving three sets by using the laws of algebra of sets. If A, B, and C are finite sets, then $A\cup B\cup C$ and $A\cap B\cap C$ are also finite. The cardinality of $A\cup B\cup C$ is given by

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$$

Proof

$$n(A\cup B\cup C)=n[(A\cup B)\cup C];\text{ associative law}$$

$$=n(A\cup B)+n(C)-n[(A\cup B)\cap C]$$

$$=n(A)+n(B)-n(A\cap B)+n(C)-n[(A\cap C)\cup(B\cap C)]$$

$$=n(A)+n(B)-n(A\cap B)+n(C)-[n(A\cap C)+n(B\cap C)]-n[(A\cap C)\cap(B\cap C)]$$

$$=n(A)+n(B)-n(A\cap B)+n(C)-n(A\cap C)-n(B\cap C)+n(A\cap B\cap C).$$

Therefore,

$$n(A\cup B\cup C)=n(A)+n(B)+n(C)-n(A\cap B)-n(B\cap C)-n(A\cap C)+n(A\cap B\cap C).$$

Example 2.31

Given that

$n(A)=10$, $n(B)=7$, $n(C)=9$, $n(A\cap B)=4$, $n(B\cap C)=3$, $n(A\cap C)=3$,

and $n(A\cap B\cap C)=1$, find $n(A\cup B\cup C)$.

Solution

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C),$

Substituting the values in the formula gives;

$$
\begin{aligned}
n(A \cup B \cup C) &= 10 + 7 + 9 - 4 - 3 - 3 + 1 \\
&= 27 - 10 \\
&= 17
\end{aligned}
$$

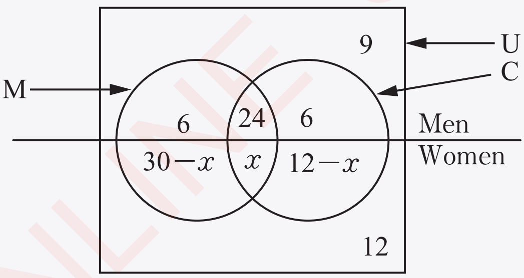
Therefore, $n(A \cup B \cup C) = 17$.

Example 2.32

A factory has 45 men and 45 women. A survey showed that 60 workers in the factory were machine operators, 42 workers were cleaners, 30 of the men were operators, 30 of the men were cleaners, 24 of the men were both operators and cleaners and 12 of the women were neither operators nor cleaners. How many workers in the factory were both operators and cleaners?

Solution

Let M and C represent the sets of men and women who are machine operators and cleaners, respectively, and x represents the number of women who are both machine operators and cleaners, then the information is summarized in the following figure:



It is clear that,

$(30 - x) + x + (12 - x) + 12 = 45$

$\Rightarrow 30 + 12 + 12 - x = 45$

$\Rightarrow 54 - x = 45$

$\Rightarrow x = 54 - 45$

Thus, $x = 9$.

Hence, n(Mnc) = 24 + 9 = 33

Therefore, the number of workers who were both cleaners and operators is 33.

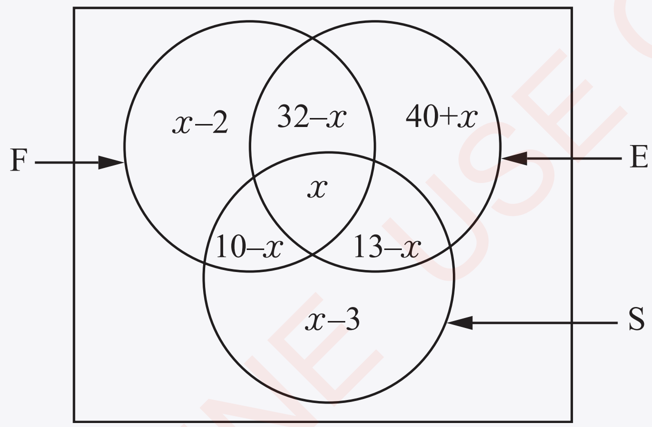
Example 2.33

In a certain city, 85% of the people speak English, 40% speak French, and 20% speak Spanish. Also, 32% speak English and French, 13% speak English and Spanish, and 10% speak French and Spanish. Find the percentage of people who speak all the three languages.

Solution

Let E, F, and S represent the sets of people who speak English, French, and Spanish, respectively.

Let x be the percentage of people who speak all three languages. The Venn diagram which summarizes the given information is shown as follows:



The total percentage is 100, that is $n(E \cup F \cup S) = 100$.

From the Venn diagram,

$100 = x - 2 + x - 3 + 40 + x + 10 - x + 13 - x + 32 - x + x$

$\Rightarrow 100 = 90 + x$

$\Rightarrow x = 10$

Therefore, the percentage of people who speak all the three languages is 10.

Example 2.34

A survey of 300 summer movie customers found that most movie customers watched one of three types of movies, namely comedy, story, and action.

Let C represents the comedy movie customers,

S represents the story movies customers, and

A represents the action movies customers.

If $n(C) = 156$, $n(S) = 106$, $n(A) = 133$, $n(C \cap S \cap A) = 8$,

$n(C \cap S) = 53$, $n(S \cap A) = 41$,

$n(C \cap A) = 87$,

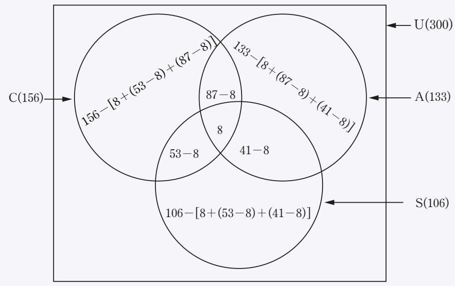
(a) how many summer movie customers did not watch comedy or stories, or action movie?

(b) how many watched comedy or action movie?

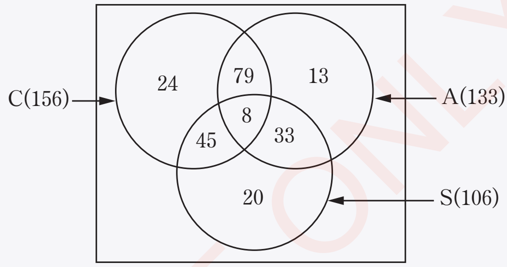
(c) how many watched story movie only?

Solution

The given information can be represented in the Venn diagram as follows:



Simplification yields



From the previous figures,

(a) If U is the universal set, then customers who did not watch a comedy, stories, or action movies

$= n(\cup) - n(A \cup S \cup C)$

$= 300 - (20 + 45 + 8 + 33 + 13 + 79 + 24)$

$= 300 - 222$

$= 78$

Therefore, 78 customers did not watch a comedy or stories, or action movie.

(b) Those who watched a comedy or action movies

$= n(A \cup C)$

$= 45 + 33 + 8 + 13 + 79 + 24$

$= 202$

Therefore, 202 customers watched the comedy or the action movie.

(c) Those who watched story movies only

$= n(S) - n(S \cap A) - n(S \cap C) - n(S \cap A \cap C)$

$= 106 - 33 - 45 - 8 = 20$

Therefore, 20 customers watched the story movies only.

(b) Number of girls in the class

$= 9 + 3 + 2 + 1 + 3$

$= 18$

Therefore, the number of girls in the class is 18.

(c) 3 girls are white but not inteligent.

Example 2.35

In a certain class there are 15 girls who like chocolate, 5 girls who are white, and 6 girls who are intelligent. Every white girl likes chocolate, 3 intelligent girls do not like chocolate. If 2 girls are both white and intelligent,

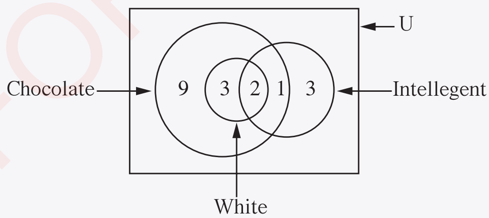
(a) present the above information in a Venn diagram,

(b) find the number of girls in the class,

(c) find the number of girls who are white but not intelligent.

Solution

(a) The given information can be represented in the Venn diagram as follows:



Example 2.36

In a certain school there are 20 students who are girls and 20 who are boys. 15 students study Mathematics and 16 study Chemistry. 10 girls study Mathematics and 11 boys study Chemistry. If 8 girls and 7 boys study neither Mathematics nor chemistry, find the number of students who study:

(a) both subjects,

(b) Mathematics but not Chemistry,

(c) only one subject,

(d) at least one subject.

Solution

Let C stands for Chemistry

M stands for Mathematics

Given;

Number of girls = 20

Number of boys = 20

$n(M) = 15$

$n(C) = 16$

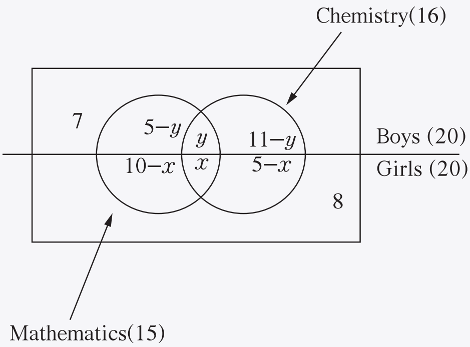
Number of girls who study Mathematics =10

Number of boys who study Chemistry = 11

8 girls study neither subject

7 boys study neither subject.

The given information can be represented in the Venn diagram as follows:



If y represents the number of boys who study both Mathematics and Chemistry, then the value of y is obtained by equating the sum of all members to the total number of boys; that is,

$$5-y+y+11-y+7=20$$

$$\Rightarrow 23-y=20$$

$$\Rightarrow y=23-20$$

Thus, $y=3$

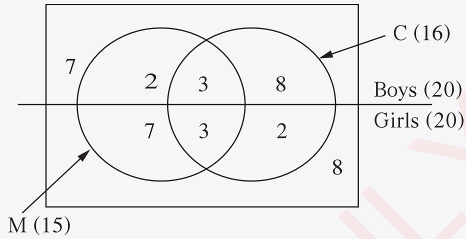
If x represents the number of girls who study Mathematics and Chemistry, then the value of x is obtained by equating the sum of all members to the total number of girls; that is,

$$10-x+x+5-x+8=20$$

$$\Rightarrow 23-x=20$$

$$\Rightarrow x=23-20$$

$$\Rightarrow x=3$$



(a) $n(M\cap C)=3+3=6$ students

Therefore, 6 students study both subjects.

(b) $n(M\cap C')=2+7=9$ students

Therefore, 9 students study Mathematics but not Chemistry.

(c) Only one subject

$=2+7+8+2=19$ students

Therefore, 19 students study only one subject.

(d) At least one subject

$=2+7+3+3+8+2$

$=25$ students

Therefore, 25 students study at least one subject.

Exercise 2.7

In questions 1 to 5, find

(a) $n(A\cap B)$ (c) $n(A\cup B)$

(b) $n(A\Delta B)$ (d) $n(A-B)$

1. $A=\{6, 7, 4, 3\}$ and $B=\{3, 7, 4, 10\}$.

2. $A = \{x : 3 \le x \le 10\}$ and
   $B = \{x : x \le 6\}$; where $x \in \mathbb{N}$.
3. $A = \{0\}$ and $B = \{5, 10\}$.
4. $A = \{6\}$ and $B = \{4, 14, 6\}$.
5. $B = \{6\}$ and $A = \{14, 6\}$.
6. Given that $n(A \cup B)' = 63$, $n(A') = 30$, and $n(B') = 40$, find $n(A \cap B)'$.
7. Given that, $n(A \cap B) = 50$, $n(A \cap B') = 100$, and $n(A' \cap B) = 50$. If A and B are the only subsets of the universal set, find
   (a) n(A) (b) n(B).
8. In a certain hospital, 150 patients were diagnosed to be suffering from various diseases as follows: 93 had Covid-19, 90 had Malaria, 64 had Tuberculosis, 32 had Tuberculosis and Covid-19, 60 had Covid-19 and Malaria, 25 had Malaria and Tuberculosis. Find the number of patients who had all the three diseases.
9. A survey on the type of crops grown in a certain village revealed that out of 210 families, 106 grow rice, 65 grow maize, 48 grow rice and maize, 22 grow millet and maize, 14 grow rice and millet only. The number of families who grow rice only is twice the number of those who grow millet only and 7 families grow none of the crops. Determine the number of families growing;
   (a) all three crops
   (b) exactly one crop
   (c) at most one crop
   (d) rice or maize but not millet.
10. Given that $A = \{x \in \mathbb{R} : x \le -1\}$, $B = \{x \in \mathbb{R} : -2 \le x \le 3\}$, and $C = \{x \in \mathbb{R} : x \ge -2\}$. Represent on a number line the solution which defines each of the following sets:
   (a) $A \cap B$ (b) $A \cap B \cap C$ (c) $(A \cap B \cap C)'$ (d) $(A \cap B') \cup C'$

Chapter summary

1. A set is a collection of objects which have common properties.
2. Sets can be presented in various forms. The common forms include descriptive or statement, roster or listing, and set builder notation.
3. There are different types of sets, namely; the empty set or null set, equal sets, equivalent sets, finite and infinite sets, singleton sets, power sets, proper and improper subsets, and universal sets.

4. The basic operations of sets include union of sets, intersection of set, complement of sets, and difference of sets.
5. The common laws of algebra of sets are; Identity law, idempotent law, double complement law, commutative law, associative law, De Morgan's law, absorption law, distributive law, and complement law.
6. The principal set operations are intersection, union, difference, symmetric difference, and the complement of sets.
7. If A and B are non empty sets, then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.
8. If A, B, and C are non empty sets, then $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$.
9. A symmetric difference of two sets, A and B is defined by $A \Delta B = (A - B) \cup (B - A)$.

Revision exercise 2

1. Let $A = \{0, 4, 8, 12, 16\}$, $B = \{0, 2, 4, 6, 8\}$, and $C = \{0, 6, 12, 18\}$: Find each of the following:
   (a) $A \cup B \cup C$
   (b) $A \cap B \cap C$
   (c) (i) $n(A \cup B \cup C)$ (ii) $n(A \cap B \cap C)$
2. If $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{8, 2, 4, 6, 9, 10\}$, and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, in each of the following show that:
   (a) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
   (b) $n(A \cap B)' = n(A \cap B') + n(B \cap A') - n(A' \cup B')'$
3. How many subsets does each of the following set have?
   (a) $\{1, 2, 3\}$ (b) $\{6, 7, 8, 9\}$ (c) $\{10, 12, 14, 16\}$
4. (a) Show the region representing by each of the following sets in a Venn diagram:
   (i) $B - A'$
   (ii) $(A \cap B) \cup (B' \cap C)$
   (iii) $A \cap B' \cap C$
   (iv) $A \cap B \cap C'$
   (v) $(A \cup B) \cap C'$
   (vi) $A' \cap B' \cap C'$
   (vii) $A' \cap (B \cup C)$
   (viii) $A' \cap B' \cap C$
   (ix) $A \cap (B' \cup C)$

   (b) Verify that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ by using Venn diagram.

5. If $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{d, e, f, g\}$, find each of the following:

(a) $B - C$ (c) $A - B$ (e) $B - A$

(b) $A - C$ (d) $C - B$ (f) $C - A$

6. Given that $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{1, 2, 4, 7, 6\}$,

and $C = \{2, 5, 6\}$, show that

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

7. Determine whether each of the following is true or false.

(a) $A \Delta B = B \Delta C$

(b) $(A \Delta B) \Delta B = A$

(c) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$

(d) $A \Delta C = (A \cup C) - (A \cap C)$

(e) $A \Delta C = (A - C) \cup (B - A)$

(f) $(A - B) - C = A - (B - C)$

(g) $A - C = C - B$

(h) $(A - C) \Delta (B - C) = B$

8. (a) Let E be the set of all positive integers less than 50, and A and B be subsets of E such that:

A = {subset of E whose elements are multiples of 5},
B = {subset of E whose elements are multiples of 4}.

(i) List all the elements of A, B, $A \cap B$ and $A \cup B$.

(ii) Describe in words the members of $A \cap B$.

(iii) Find n(A), n(B), and $n(A \cap B)$.

(b) Find the elements of sets A, B, and C given that

$A \cup B = \{p, q, r, s\}$, $A \cup C = \{q, r, s, t\}$, $A \cap B = \{q, r\}$, and $A \cap C = \{q, s\}$

9. Given $A = \{x : x \in \mathbb{R}, x \geq 2\}$ and $B = \{x : x \in \mathbb{R}, -2 < x \leq 5\}$, using the number line find;

(a) $A' \cup B$
(b) $A' \cap B$
(c) $(A \cap B)'$
(d) $A - B$

10. By using the laws of algebra of sets simplify each of the following expressions:

(a) $A \cup (A \cap B')'$
(b) $(A - B) - (A \cap B')$
(c) $A \cap (A' \cap B)'$
(d) $B \cap (B \cup A)$
(e) $(A \cup B) \Delta B'$
(f) $A \cap (A' \cup B')$
(g) $\left(((A \cup B) \cap C)' \cup B'\right)' \cup C$
(h) $(X \cap Y') \cup (X \cup Y')$
(i) $[A - (B - A)] - [A \cup (B \cap A)]$
(j) $(A' \cap B) \Delta A$

11. (a) Use appropriate laws of algebra of sets to simplify $\left[A \cap (A \cup B)'\right]$,

(b) Two sets A and B are said to be equivalent if $n(A) = n(B)$. If A, B, and C are equivalent sets of which A and C are disjoint sets, prove that $n(A \cup B \cup C) = 3n(A) - n(A \cap B) - n(B \cap C)$.

12. If $U = \{1, 2, 3, \ldots, 10\}$, and $B = \{2, 4, 6\}$, in each of the following show that:

(a) $B \Delta B = \varnothing$
(b) $B \Delta U = B'$
(c) $B \Delta \varnothing = B$
(d) $B \Delta B' = U$

13. If A, B, and C are any three sets, prove that:

(a) $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
(b) $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$
(c) $U \cap A \subseteq (A \cup B \cup C)$
(d) $[(A \cap B') - C] \subseteq A - C$
(e) $(A \cap C') \cap (C \cap B') = \varnothing$
(f) $(B \cap A') \cup (C - A) = (B \cup C) \cap A'$
(g) $A \cap B' \subseteq A$
(h) $(B \cap C') \cup (B' \cup C)' = B \cap C'$
(i) $(A \cap B \cap C)' = A' \cup B' \cup C'$

14. Out of 880 boys in a school, 224 played cricket, 240 played hockey, and 336 played basketball. Of the total, 64 played basketball and hockey, 80 played cricket and basketball, and 40 played cricket and hockey, 24 boys played all the three games. Determine the number of boys who;
   (a) did not play any game,
   (b) played only one game,
   (c) played cricket or hockey but not basketball,
   (d) played basketball and hockey but not cricket.

15. A class contains 15 boys and 15 girls. If 20 students take Science, 14 students take Mathematics, of the boys, 10 take Science and, 10 take Mathematics, 8 of the boys take both Science and Mathematics, 4 of the girls take neither Science nor Mathematics. Find the number of students in the class who take;
   (a) both Science and Mathematics,
   (b) Mathematics but not Science,
   (c) exactly one subject,
   (d) neither of the subjects.

16. In a certain high school, 100 students were asked to mention the foreign languages they learn. Their responses showed that 45 students learn Spanish, 28 learn French, 22 learn Chinese, 12 students learn Spanish and French, 8 learn Spanish and Chinese, 10 learn French and Chinese, and 30 students do not learn any language. Find the number of students who learn;
   (a) all the three languages,
   (b) exactly one of the three languages,
   (c) at most two of the three languages.

17. In an examination, 40% of the students passed in Mathematics, 45% in Chemistry, and 55% in Physics. If 10% passed in Mathematics and Chemistry, 20% in Chemistry and Physics, and 15% in Physics and Mathematics, determine the pass percentages in;
   (a) all the three subjects
   (b) at least two subjects,
   (c) Chemistry or Physics but not Mathematics.

18. Spare parts manufactured by a certain factory were subjected to three types of defects A, B, and C. A sample of 4,000 items were inspected and it was found that 6.2% had defect A, 7.4% had defect B, 8.2% had defect C, 2.2% had defect A and B, 2.6% had defect B and C, 2.0% had defect A and C, 1.2% had all the three defects. Find the percentage of items which had,
(a) none of the defects
(b) at least one of the defects
(c) not more than one defect.

19. Out of 35 students in a certain school, 2 students study Physics, Chemistry and Mathematics. It is given that 6 of them study only Physics and Chemistry, 5 students study only Physics and Mathematics, and 4 students study only Chemistry and Mathematics. The number of students who study only one subject is the same for all the three subjects. Find the number of students who study:
(a) Mathematics only.
(b) Physics or Chemistry but not Mathematics.
(c) At most two subjects.

20. Given two sets A and B, such that n(A) = 10 and n(B) = 15, find the
(a) maximum and minimum values of $n(A \cup B)$ can have,
(b) maximum and minimum values of $n(A \cap B)$ can have.

21. In a group of 72 students who were surveyed, 29 students liked mangoes, 41 liked bananas, 48 liked oranges, 22 liked oranges and bananas, 16 liked bananas and mangoes, and 20 liked oranges and mangoes. Find how many students liked,
(a) mangoes but not bananas or oranges,
(b) all the three fruits,
(c) oranges or bananas but not mangoes.
(d) exactly two fruits.

22. Simplify each of the following expressions by using the laws of algebra of sets:
(a) $(A \cup B)-(A-B)$
(b) $(A' \cap B') \cap (A \cap B)'$
(c) $A \cap (A \cup B)-(A-B)$
(d) $(A \cap (A' \cup B)) \cup (B \cap (A' \cup B'))$

23. (a) Prove that if A is a subset of B and B is a subset of C, then A is a subset of C.

(b) Given that $A = \{x \in \mathbb{R} : 1 \le x \le 3\}$, $B = \{x : 2 \le x \le 4, x \in \mathbb{R}\}$, and $C = \{x \in \mathbb{R} : x > 3\}$, use set builder notation to find the expression for each of the following sets:

(i) $(A' \cup B') \cap C$

(ii) $A' \cap B \cap C$

24. (a) Use a Venn diagram to express each of the following sets:

(i) $(B - A)'$

(ii) $(A \cap B) \cup (B' \cap C)$

(iii) $(A \cap B' \cap C) \cup (A' \cap B \cap C')$

(b) By using the laws of algebra of sets simplify $(A - B)' \cap (A \cup B)$.

25. In a certain teachers' college, there are 110 trainees taking Economics (E), History (H), and Geography (G), where, $n(E \cup H) = 70$, $n(H \cup G) = 80$, $n(H \cap E' \cap G') = 10$, $n(G \cup E) = 85$, $n(H) = 40$, and $n(G) = 55$, $n(H \cap E \cap G) = 5$, $n(E) = 50$. Find the number of trainees who take;

(a) History and Geography

(b) History and Geography but not Economics

(c) History or Geography but not Economics

(d) exactly two subjects

Chapter Three

Logic

Introduction

The term "logic" originates from the Greek word "Logos" meaning idea, word, thought, argument, account, reason, or principle. Logic can be defined as a science or an art which deals with the study of truth, principles of correct reasoning, and making good decisions. In this chapter, you will learn about statements, logical connectives, laws of algebra of propositions, validity of the arguments, and electrical networks of statements. The competencies developed have a number of applications in real life situations such as to distinguish between valid and invalid arguments, correct reasoning and making proper decisions in daily life activities, constructions of circuit diagrams in the field of electronics, making judgements in fields of law, among many other applications.

Concept of logic

Logic in everyday life refers to a correct reasoning. In mathematics, it is the study of truth and how the truth of a statement or proposition can be obtained from mathematical deduction. In logic, the interest is not on the statements themselves, but how the true and false statements are related to each other. In most cases, the propositions are represented by letters like p, q, r, and s. These propositions have to be either true (T) or false (F).

Statements

In mathematics, a statement or proposition can be defined as a declarative sentence which can be either "true" or "false" but not both. For example, "The sum of interior angles of a rectangle is $180°$". The truth value of this statement is false.

The following sentences are statements:

(i) $6+2=9$

(ii) $3+2=5$

(iii) Tanga is the capital city of Tanzania.

(iv) Lusaka is the largest city in Tanzania.

The truth value of the statement in (ii) is true, whereas the truth value of statements in (i), (iii), and (iv) is false.

Note that, not all sentences are statements. For example;

(i) What is your name?

(ii) Good night daughter.

(iii) $x-7=12$

(iv) $2x+y=5z$

The sentences in (i) and (ii) are not statements because they are not declarative sentences, and the sentences in (iii) and (iv) are not statements because they are neither true nor false.

Activity 3.1: Identifying a statement

Individually or in a group, perform the following tasks:

1. Construct any six sentences of your choice.
2. From the sentences constructed in task 1, identify mathematical and non-mathematical statements.
3. Share the results you obtained in task 2 with other students.
4. Discuss with your fellow students how to differentiate between a mathematical and a non-mathematical statements.
5. Discuss the challenges you faced in task 2.

Sentences which involve exclamative words, questioning sentences, and instructional sentences are not mathematical sentences. If a sentence mentions a particular place or a particular person it becomes a statement. For instance, "Is Juma a lazy boy?" is not a statement, but "Juma is a lazy boy" is a statement. In writing mathematical statements, the letters p, q, r, s, …. are usually used to denote propositions in order to reduce the requirement of writing long sentences. The truth value of a true proposition is denoted by **T** and the truth value of a false proposition is denoted by **F**.

Note that, the truth value of a statement is either true or false.

Types of mathematical statements

There are two types of mathematical statements, namely; simple mathematical statements and compound mathematical statements. A simple mathematical statement is formed by one declarative sentence, that is either true or false. For instance, the statement "Asha is swimming" is a simple statement, while a compound statement is formed by more than one declarative sentence. The statement "If Elia studies hard, then he will pass the test" is a compound statement, since it is formed by two simple statements which are "Elia studies hard" and "he will pass the test".

Truth table

A truth table is a table that used to show the validity of a compound statement depending on the truth values of the simple statements. The number of cases that describe the given compound statement depends on the number n of propositions. For instance, if the compound statement has 1 proposition $(n=1)$, then the number of cases is 2. If it has 2 propositions $(n=2)$, then the number of cases is $2^2=4$, and if it has 3 propositions $(n=3)$, then the number of cases is $2^3=8$.

Generally, if there are n propositions, then the number of cases is 2^n.

Tables 3.1 show the truth tables of the statement formed by 1, 2, and 3 propositions.

Table 3.1(a): *Number of cases for one proposition p*

(a) One proposition p

	p
Case 1	T
Case 2	F

Table 3.1(b): *Number of cases for two propositions p and q*

	p	q
Case 1	T	T
Case 2	T	F
Case 3	F	T
Case 4	F	F

Table 3.1(c): *Number of cases for three propositions p, q, and r*

	p	q	r
Case 1	T	T	T
Case 2	T	T	F
Case 3	T	F	T
Case 4	T	F	F
Case 5	F	T	T
Case 6	F	T	F
Case 7	F	F	T
Case 8	F	F	F

Negation of a statement

Negation refers to a way of forming a mathematical statement that has the opposite truth value. For example, the statement "It is not p" is called the negation of p, mathematically it is written as "$\sim p$" and reads "negation of p". One way of writing a negation statement is to put the word "not" with the verb. Another way is to precede it with the phrase "it is not true that" or "it is false that". For instance, the negation of the proposition "Today is Friday" is "Today is not Friday" or "It is not true that Today is Friday" or "It is false that Today is Friday". Similarly, the negation of the proposition "Issa does not love Rita" is "Issa loves Rita".

Example 3.1

Draw a truth table of a mathematical statement $\sim p$.

Solution

Truth table for $\sim p$

p	$\sim p$
T	F
F	T

Exercise 3.1

1. Which of the following sentences are statements?
   (a) $26-5=21$.
   (b) $7(8\div 2)+8=36$.
   (c) There exist integers x and y such that $3x+7y=2$.
   (d) Will you come tommorow?
   (e) Given that PQR is a right-angled triangle with a right angle at vertex Q, and if M is the midpoint of the hypotenuse, then the line segment connecting vertex Q to M is half the length of the hypotenuse.
2. State which of the following sentences is a simple or compound statement:
   (a) Five is less than eight.
   (b) John eats rice or meat.
   (c) Dodoma is the capital city of Tanzania.
   (d) Juma is singing and Thomas is reading a novel.
   (e) Ester will pass the examination if and only if she studies hard.
3. Draw the truth table whose number of propositions is:
   (a) $n=3$
   (b) $n=4$
4. Write the negation of each of the following propositions.
   (a) Pili is not a woman.
   (b) It is raining now.
   (c) Maasai maintain their culture.
   (d) Summer comes after spring.
   (e) Industries are friendly to the environment.
   (f) Tomorrow is Saturday.

Logical connectives

Logical connectives are symbols used to connect two or more propositions of a compound statement. Logical connectives include conjunction, disjunction, conditional (implication), and biconditional (double implications).

Conjunction

In mathematics, a conjunction is described as a statement formed by adding two logical statements p and q with a connector "and", symbolically denoted by "$\wedge$" and reads as "and". When the values of two propositions p and q are combined together by the connector, it is expressed symbolically as $p \wedge q$ and is read "p and q". A conjunction has truth value true if and only if both of its statements are true, otherwise it is false. In other words, conjunction of two statements is true only in case when each sub-statement is true. The truth table for $p \wedge q$ is shown in Table 3.2.

Table 3.2: *Truth table for* $p \wedge q$

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Example 3.2

Write the conjunction of the propositions p and q, where p and q are the propositions "Halima likes cold coffee" and "Alex likes milkshake", respectively. Hence, comment on its truth value.

Solution

Given the following propositions:

p: Halima likes cold coffee.

q: Alex likes milkshake.

The conjunction is written as $p \wedge q$. The proposition $p \wedge q$ is a compound statement "Halima likes cold coffee and Alex likes milkshake". The formed compound statement $p \wedge q$ is true when only p and q are both true, otherwise it is false.

Example 3.3

Write the conjunction of the propositions p and q, where p is the proposition "Mwanaidi likes singing" and q is "Tumaini likes swimming". Hence, comment on its truth value.

Solution

Given the following propositions:

p: Mwanaidi likes singing.

q: Tumaini likes swimming.

The conjunction is given as $p \wedge q$. The conjunction $p \wedge q$ is a compound statement, "Mwanaidi likes singing and Tumaini likes swimming". The compound statement $p \wedge q$ is true if and only if p and q are both true, otherwise it is false.

Example 3.4

Determine the truth value in each of the following statements:

(a) $8+25=43$ and $72 \div 8 \leq 9$

(b) 2 is a prime number but 4 is not a prime number

(c) Dodoma is the capital city of Tanzania and it is false that Mwanza is a city in Tanzania.

Solution

(a) Let p: $8+25=43$

q: $72 \div 8 \leq 9$.

The truth value of $8+25=43$ is false(F) and the truth value of $72 \div 8 \leq 9$ is true (T), then the truth value of $p \wedge q$ is $\mathrm{F} \wedge \mathrm{T} = \mathrm{F}$.

Therefore, the truth value of $p \wedge q$ is False.

(b) Let p: 2 is a prime number.

q: 4 is not a prime number.

The truth value of "2 is a prime number" is true (T) and the truth value of "4 is not a prime number" is true (T), then the truth value of $p \wedge q$ is $\mathrm{T} \wedge \mathrm{T} = \mathrm{T}$.

Therefore, the truth value of $p \wedge q$ is true.

(c) Let p: Dodoma is a capital city of Tanzania.

q: Mwanza is a city in Tanzania.

The truth value of p is true (T) and the truth value of q is true (T), then the truth value of $\mathrm{T} \wedge \sim \mathrm{T} = \mathrm{T} \wedge \mathrm{F} = \mathrm{F}$.

Therefore, the truth value of $p \wedge q$ is False.

Example 3.5

Write the conjunction of the propositions p and q, when p is the proposition "Subira likes novels" and q is a proposition "Jack likes movies". Hence, comment on its truth value.

Solution

Given the following:

p : Subira likes novels.

q : Jack likes movies.

The conjunction is written as $p \wedge q$. The proposition $p \wedge q$ is the compound statement "Subira likes novels but Jack likes movies". This compound statement is true only when p and q are true, otherwise it is false.

Disjunction

Disjunction is a compound statement which is formulated by using a connector "or" and its symbol is "$\vee$". When the two statements p or q are joined together, the combination is symbolically expressed as "$p \vee q$". The truth value of a disjunction is false if and only if both statements are false, otherwise it is true. The truth table for $p \vee q$ is as shown in Table 3.3.

Table 3.3: *Truth table for $p \vee q$*

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Example 3.6

If p is the proposition "John eats beans" and q is the proposition "John eats meat", write the statement for each of the following:

(a) $p \vee q$ (b) $\sim(p \vee q)$

Solution

(a) "John eats beans or meat".

(b) "It is false that John eats beans or meat".

Example 3.7

If p is the proposition "A giraffe eats meat" and q is the proposition "A lion eats meat", write the truth value in each of the following statements.

(a) $p \vee q$ (c) $\sim(p \wedge q)$

(b) $p \wedge q$ (d) $\sim(p \vee q)$

Solution

Given p the proposition that "A giraffe eats meat", its truth value is false (F), and q the proposition that "A lion eats meat" and its truth value is true, then

(a) The truth value of $p \vee q$ is $\text{F} \vee \text{T} = \text{T}$.

Therefore, the truth value of $p \vee q$ is true.

(b) The truth value of $p \wedge q$ is $\text{F} \wedge \text{T} = \text{F}$.

Therefore, the truth value of $p \wedge q$ is false.

(c) The truth value of $\sim(p \wedge q)$ is $\sim(\text{F}) = \text{T}$.

Therefore, the truth value of $\sim(p \wedge q)$ is true.

(d) The truth value of $\sim(p \vee q)$ is $\sim(\text{T}) = \text{F}$.

Therefore, the truth value is false.

Conditional statements (implications)

A relationship between two statements in which the second statement is a logical consequence of the first statement is referred to as a conditional or implication statement. If p and q are propositions, then the implication is written as "$p \rightarrow q$", and its truth value is false when p is true and q is false, otherwise it is true. The proposition "$p \rightarrow q$" reads as p implies q, if p then q, or sometimes q if p. The truth table of $p \rightarrow q$ is as shown in Table 3.4.

Table 3.4: *Truth table for $p \rightarrow q$*

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The following phrases are used to represent the condition statement, "$p \rightarrow q$".

(i) "If p, then q"

(ii) "p is sufficient for q"

(iii) "If p, q"

(iv) "a necessary condition for p is q"

(v) "q if p"

(vi) "q when p"

(vii) "p implies q"

(viii) "p only if q"

(ix) "q is necessary for p"

(x) "q whenever p"

(xi) "q follows from p"

Example 3.8

Write each of the following statements in symbolic form:

(a) If the car is gone, then Leah has left.

(b) If you get a degree, then you can get a job.

Solution

(a) Let p : The car is gone

q : Leah has left

Therefore, $p \to q$.

(b) Let p : You get a degree

q : You can get a job

Therefore, $p \to q$.

Example 3.9

Let p: Angel goes abroad

q: Angel has a passport

Write down a verbal sentence in each of the following:

(a) $p \to \sim q$

(b) $\sim p \to \sim q$

(c) $\sim p \to q$

Solution

(a) If Angel goes abroad, then she does not have a passport.

(b) If Angel does not go abroad, then she does not have a passport.

(c) If Angel does not go abroad, then she has a passport.

Biconditional statements

A biconditional statement is a statement which is written in the form " p if and only if q ". It is denoted by a double-headed arrow "$\leftrightarrow$". If p and q are propositions, then the biconditional proposition is written as " $p \leftrightarrow q$ " which is read as "p if and only if q". A biconditional statement is said to be true if and only if both parts have the same truth value, otherwise it is false. The truth table of $p \leftrightarrow q$ is shown in Table 3.5.

Table 3.5: *Truth table for $p \leftrightarrow q$*

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Also, the truth value of $p \leftrightarrow q$ is true if the implication $(p \to q)$ is true and $(q \to p)$ is also true. The truth value of the statement $(p \to q) \wedge (q \to p)$ in Table 3.6 is the same as that of $p \leftrightarrow q$ shown in Table 3.5.

Table 3.6: *Truth table for* $(p \to q) \wedge (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note that; if p and q have the same truth value, then $p \leftrightarrow q$ is true, if p and q have opposite truth values, then $p \leftrightarrow q$ is false.

The following phrases are used to represent the biconditional statement, "$p \leftrightarrow q$":

(i) "q if and only if p"

(ii) "p if and only if q"

(iii) "p is a necessary and sufficient condition for q"

(iv) "If p, then q and conversely"

(v) "p double implication q"

(vi) "p implies q and q implies p"

Example 3.10

Write each of the following statements in symbolic form:

(a) You will pass the examination if and only if you work hard.

(b) I will give you my car if and only if you know how to drive.

Solution

(a) Let p : You will pass the examination.

q : You will work hard.

Therefore, $p \leftrightarrow q$.

(b) Let p : I will give you my car.

q : You know how to drive.

Therefore, $p \leftrightarrow q$.

Example 3.11

Determine the truth value in each of the following statements:

(a) $2+9=12$ if and only if 2 is a factor of 12.

(b) It is not true that $6+14=20$ if and only if $18\times5=90$.

Solution

(a) Let p : $2+9=12$

q : 2 is a factor of 12.

The truth value of $2+9=12$ is false (F) and the truth value of 2 is a factor of 12 is true (T).

Thus, the truth value of $p \leftrightarrow q$ is $\text{F} \leftrightarrow \text{T} = \text{F}$.

Therefore, the truth value of $p \leftrightarrow q$ is false.

(b) Let p : $6+14=20$

q : $18\times5=90$

The truth value of $6+14=20$ is true (T) and the truth value of $18\times5=90$ is true (T).

The truth value of $p \leftrightarrow q$ is $\text{T} \leftrightarrow \text{T} = \text{T}$.

Thus the truth value of $\sim(p \leftrightarrow q)$ is $\sim(\text{T}) = \text{F}$

Therefore, the truth value of $\sim(p \leftrightarrow q)$ is false.

Exercise 3.2

1. Determine the truth value in each of the following statements:
   (a) Paris is in France and Mombasa is in Uganda.
   (b) $6 < 9$ if and only if $-6 \div 2 > -2$
   (c) If $2 + 4 = 6$ and $6 - 5 < 1$, then $\sqrt{71} < 8.5$
2. Let p: She is tall and q: She is beautiful. Write each of the following statements in symbolic form:
   (a) She is tall and beautiful.
   (b) She is tall but not beautiful.
   (c) It is false that she is short or beautiful.
   (d) She is neither tall nor beautiful.
   (e) She is tall or she is short and beautiful.
3. Construct a truth table for the proposition
   $(\sim p \leftrightarrow \sim q) \leftrightarrow \sim (p \rightarrow q)$.
4. Let p: It is cold and q: It rains. Write each of the following statements in symbolic form:
   (a) A necessary condition for it to be cold is that it rains.
   (b) A sufficient condition for it to be cold is that it rains.
   (c) It is not true that it is not cold if and only if it does not rain.
5. Given that p: You like Physics, q: You like Chemistry, and r: You like Biology. Give a verbal sentence which describes each of the following:
   (a) $(p \vee q) \wedge \sim r$
   (b) $(p \wedge q) \vee (\sim p \wedge \sim r)$
   (c) $\sim (p \wedge \sim r)$
6. Construct a truth table for the proposition $(p \wedge q) \rightarrow (p \vee q)$.
7. Express each of the following statements into symbolic form:
   (a) It is false that "Grace speaks French but not Germany".
   (b) Grace does not speak French or she does not speak Germany.
8. Construct a truth table for the proposition $\sim p \leftrightarrow (q \rightarrow r)$.
9. Determine the truth value in each of the following statements:
   (a) Either $69 \times 12 = 13 \times 36$ or $12 - 18 \neq -6$.
   (b) If 3 is an odd number, then neither 3 is a prime number nor an even number.
   (c) $\sqrt[3]{64} = \sqrt[4]{256}$ if and only if $2 + 4 = 6$ and $16 - 10 - 56 - 50$.
10. Express each of the following compound statements in symbolic notation using letters P, Q, and R to stand for the statements:

(a) Either the manufactured drug is not faulty and accepted by the Tanzania Food and Drug Authority (TFDA) or the drug is faulty and is not accepted by TFDA.

(b) If Jonathan is a member of a social committee, then the committee is strong. The committee is strong if and only if Jonathan's argument is accepted by other members. Therefore, Jonathan's argument is not accepted and the committee is not strong.

11. If p stands for "Halima is poor" and q stands for "Halima works hard", write statements to represent each of the following propositions:

(a) $q \vee p$ (e) $\sim p \rightarrow \sim q$
(b) $\sim(p \wedge q)$ (f) $\sim p \vee \sim q$
(c) $p \rightarrow q$ (g) $q \leftrightarrow p$
(d) $p \wedge q$ (h) $\sim q \vee (p \wedge q)$

12. If p stands for "Swimming in the pool is not dangerous" and q stands for "Few people have been drowned in the swimming pool". Write a sentence for each of the following logical expressions:

(a) $\sim p$ (c) $\sim p \rightarrow q$
(b) $\sim q \leftrightarrow p$

13. Construct a truth table for each of the following compound statements:

(a) $(p \wedge q) \wedge \sim (p \wedge q)$
(b) $(q \rightarrow p) \leftrightarrow (p \rightarrow q)$
(c) $(p \vee q) \wedge r$
(d) $(p \vee q) \rightarrow \sim r$
(e) $[(\sim p \leftrightarrow q) \wedge r] \rightarrow t$

Converse, inverse, and contrapositive

A conditional statement can be written in its converse, inverse or contrapositive.

Converse of a conditional statement

The converse of a conditional statement "$p \rightarrow q$" is written by interchanging the roles of the statements. Therefore, the converse of $p \rightarrow q$ is the statement "$q \rightarrow p$", as shown in Table 3.7.

Table 3.7: *Truth table for* $q \rightarrow p$

p	q	$q \rightarrow p$
T	T	T
T	F	T
F	T	F
F	F	T

Inverse of a conditional statement

The inverse conditional statement "$p \rightarrow q$" is written by negating the original conditional statement. Thus, the inverse of $p \rightarrow q$ is the implication "$\sim p \rightarrow \sim q$", as shown in Table 3.8.

Table 3.8: *Truth table for* $\sim p \to \sim q$

p	q	$\sim p$	$\sim q$	$\sim p \to \sim q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Contrapositive of a conditional statement

The contrapositive of the conditional statement "$p \to q$" can be written by exchanging the roles of the inverses of the given conditional statement. Hence, the contrapositive of $p \to q$ is $\sim q \to \sim p$, and it has the same truth table as $p \to q$, as shown in Table 3.9.

Table 3.9: *Truth table for* $\sim q \to \sim p$

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Example 3.12

Write the converse, the contrapositive, and the inverse of the statement "A child cries whenever it is raining".

Solution

Let p : A child cries.

q : It is raining.

Symbolically the given statement is $p \to q$ and its converse is $q \to p$, that is "If it is raining, then the child cries". Its contrapositive is $\sim q \to \sim p$, that is "If it is not raining, then the child does not cry."

Its inverse is $\sim p \to \sim q$ that is "If the child does not cry, then it is not raining".

Example 3.13

Write the converse, the contrapositive, and the inverse of the statement "If Flora is happy, then she eats food and drinks milk".

Solution

Let p: Flora is happy.

q : Flora eats food and drinks milk.

Its converse is $q \to p$. That is, "If Flora eats food and drinks milk, then she is happy".

Its contrapositive is $\sim q \to \sim p$: "If Flora does not eat food and drink milk, then she is not happy ".

Its inverse is $\sim p \to \sim q$. That is, "If Flora is not happy, then she does not eat food and drink milk".

Example 3.14

Given the proposition

$$[(p \to q) \wedge (q \to r)] \to (p \to r),$$

write down the:

(a) converse

(b) inverse

(c) contrapositive

Solution

(a) Its converse is

$$(p \to r) \to [(p \to q) \wedge (q \to r)]$$

(b) Its inverse is

$$\sim [(p \to q) \wedge (q \to r)] \to \sim (p \to r)$$

(c) Its contrapositive is

$$\sim (p \to r) \to \sim [(p \to q) \wedge (q \to r)]$$

Exercise 3.3

1. Write the inverse, converse, and contrapositive in each of the following propositions:
   (a) If tomorrow is Saturday, then Paul will go to the beach.
   (b) If it is raining, then the shop is closed.
   (c) Industries are environmentally friendly if they are in harmony with the surroundings.
   (d) If ABC is an equilateral triangle, then it is a right-angled triangle.
   (e) If $f(x)$ is a rational function, then it has asymptotes.

2. Given the statement $p \to (q \to \sim p)$.
   (a) Construct the truth table for the converse.
   (b) Construct the truth table for the contrapositive.

3. Given the statement "If two vectors are orthogonal, then their dot product is zero". Write the verbal sentence for its:
   (a) inverse
   (b) converse
   (c) contrapositive.

4. For each of the following statements, write down the corresponding contrapositive statement:
   (a) If Halima has courage, then she will win.
   (b) It is necessary to be strong in order to be a sailor.
   (c) To be a square is a sufficient condition for a geometrical figure to be a rectangle.

5. Let p: Kiswahili is interesting,
   q: Physics has applications, and
   r: Physics depends on Mathematics.

   Write in symbolic form the converse, inverse, and contrapositive of the statement "If Kiswahili is interesting, then Physics has applications and depends on Mathematics".

6. For each of the following write:
   (a) The contrapositive of the inverse of $p \to q$.
   (b) The inverse of the converse of $\sim p \to (q \to \sim r)$
   (c) The converse of the contrapositive of $(p \leftrightarrow q) \to (\sim p \leftrightarrow \sim r)$

Logic symbols dominance

In algebra, whenever arithmetic operations $+, -, \times$, and $\div$ are used in the expression, then the expression is evaluated by applying the BODMAS rule. That is, Bracket of, Division, Multiplication, Addition, and Subtraction. In logical expressions, brackets are also used. However, if brackets in logical expressions are many, then the order shown in Table 3.10 should be followed. For instance, when evaluating $p \wedge q \vee r$, workout "$\wedge$" first, that is $(p \wedge q) \vee r$, and not $p \wedge (q \vee r)$.

Table 3.10: *Order of dominance in logic expression*

Connective	$\sim$	$\wedge$	$\vee$	$\to$	$\leftrightarrow$
Priority	1st	2nd	3rd	4th	5th

Activity 3.2

Individually or in a group, perform the following tasks:

1. Using the logical connectives $\sim$, $\wedge, \vee, \to$, and $\leftrightarrow$, construct five propositions of your choice.
2. From the propositions constructed in task 1, suggest the order of evaluation using the order of dominance as shown in Table 3.10.
3. Share the results you obtained in task 2 with other students for more inputs.
4. With your fellow students discuss the challenges you met in task 2, if any.

Logical equivalences

Two or more logical expressions are logically equivalent if and only if they have the same truth table. This means, if p and q are propositions that are equivalent, then $p \leftrightarrow q$ will have the truth value T in all cases. A logical expression or statement having only truth values true (T) in the last column of its truth table is called a tautology.

Example 3.15

Show that $p \to q$ and $\sim p \vee q$ are logically equivalent.

Solution

The expressions $p \to q$ and $\sim p \vee q$ are logically equivalent if $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology as shown in the following table.

p	q	$p \to q$	$\sim p$	$\sim p \vee q$	$(p \to q) \leftrightarrow (\sim p \vee q)$
T	T	T	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

Since the truth values in the column of $(p \to q) \leftrightarrow (\sim p \vee q)$ in the truth table contain only the truth values T, then it is a tautology.
Therefore, $p \to q$ and $\sim p \vee q$ are logically equivalent.

Note that, if a compound proposition or a logical expression has truth value "F" for all combinations of the truth values of the proposition variables which it contains, it is said to be a contradiction. In other words, the contradiction is the negation of a tautology. For example, in Example 3.15, it was shown that $(p \to q) \leftrightarrow (\sim p \vee q)$ is a tautology.
Therefore, $\sim[(p \to q) \leftrightarrow (\sim p \vee q)]$ is the contradiction.

Example 3.16

Use the truth table to verify that the statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Solution

Given $(p \vee q) \wedge (\sim p \wedge \sim q)$, then its truth table is as follows.

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \wedge \sim q$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

From the table, all truth values in the last column are false, the statement is a contradiction.

Exercise 3.4

Determine whether or not the logical statements in each of the following pairs are logically equivalent:

1. $(p \rightarrow q)$ and $(\sim q \rightarrow \sim p)$
2. $(\sim p \vee \sim q)$ and $p \vee q$
3. $(p \wedge q)$ and $\sim (p \rightarrow \sim q)$
4. $(p \rightarrow q) \wedge (p \rightarrow r)$ and $p \rightarrow (q \wedge r)$
5. $(q \rightarrow r) \vee (p \rightarrow r)$ and $(p \wedge q) \rightarrow r$
6. $(p \leftrightarrow q)$ and $(p \rightarrow q) \wedge (q \rightarrow p)$
7. $p \leftrightarrow q$ and $(\sim p \leftrightarrow \sim q)$
8. $p \leftrightarrow r$ and $(p \wedge r) \vee (\sim p \wedge \sim r)$
9. $\sim (p \leftrightarrow r)$ and $p \leftrightarrow \sim r$
10. $\sim p \vee (\sim p \wedge q)$ and $\sim p \wedge \sim q$

Laws of algebra of propositions

The laws of algebra of propositions are sets of pairs of equivalent logical propositions. The laws are used to simplify complicated logical statements. Table 3.11 presents the laws of algebra of propositions with their names.

Table 3.11: *Laws of algebra of propositions*

Equivalent statements	Laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent
$p \wedge T \equiv p$ $p \vee F \equiv p$ $p \vee T \equiv T$ $p \wedge F \equiv F$	Identity
$\sim (\sim p) \equiv p$	Double negation
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative
$\sim (p \vee q) \equiv \sim p \wedge \sim q$ $\sim (p \wedge q) \equiv \sim p \vee \sim q$	De Morgan's
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive
$p \vee \sim p \equiv T$ $p \wedge \sim p \equiv F$	Complement (Negation)
$p \rightarrow q \equiv \sim p \vee q$ $p \leftrightarrow p \equiv (p \rightarrow q) \wedge (q \rightarrow p)$ $\equiv (\sim p \vee q) \wedge (\sim q \vee p)$	Condition and Bi-conditional

Example 3.17

Use the laws of algebra of propositions to simplify $\sim(p \vee q) \vee (\sim p \wedge q)$.

Solution

$$\begin{aligned} \sim(p \vee q) \vee (\sim p \wedge q) &\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q) && \text{De Morgan's law} \\ &\equiv \sim p \wedge (\sim q \vee q) && \text{Distributive law} \\ &\equiv \sim p \wedge \mathrm{T} && \text{Complement law} \\ &\equiv \sim p && \text{Identity law} \end{aligned}$$

Therefore, $\sim(p \vee q) \vee (\sim p \wedge q) \equiv \sim p$.

Example 3.18

Use the laws of algebra of propositions to show that $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

Solution

The two expressions, $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent if and only if $\sim(p \vee (\sim p \wedge q))$ can be simplified to $\sim p \wedge \sim q$.

$$\begin{aligned} \sim\left(p \vee (\sim p \wedge q)\right) &\equiv \sim p \wedge \sim(\sim p \wedge q) && \text{De Morgan's law} \\ &\equiv \sim p \wedge [\sim(\sim p) \vee \sim q] && \text{De Morgan's law} \\ &\equiv \sim p \wedge (p \vee \sim q) && \text{Double negation law} \\ &\equiv (\sim p \wedge p) \vee (\sim p \wedge \sim q) && \text{Distributive law} \\ &\equiv \mathrm{F} \vee (\sim p \wedge \sim q) && \text{Complement law} \\ &\equiv \sim p \wedge \sim q && \text{Identity law} \end{aligned}$$

Since $\sim(p \vee (\sim p \wedge q))$ has been simplified to $\sim p \wedge \sim q$, then $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

Example 3.19

Using the laws of algebra of proposition, show that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.

Solution

If the statement is a tautology, it can be simplified to one equivalent statement 'T' using the laws of algebra of propositions. Otherwise, it is not a tautology.

$$
\begin{aligned}
(p\wedge q)\rightarrow(p\vee q) &\equiv \sim(p\wedge q)\vee(p\vee q) && \text{Conditional law}\\
&\equiv(\sim p\vee\sim q)\vee(p\vee q) && \text{De Morgan's law}\\
&\equiv\sim p\vee p\vee\sim q\vee q && \text{Commutative law}\\
&\equiv(\sim p\vee p)\vee(\sim q\vee q) && \text{Associative law}\\
&\equiv \mathrm{T}\vee \mathrm{T} && \text{Complement law}\\
&\equiv \mathrm{T} && \text{Idempotent law}
\end{aligned}
$$

Since, the last result is T, then $(p\wedge q)\rightarrow(p\vee q)$ is a tautology.

Exercise 3.5

1. Use the laws of algebra of propositions to establish whether or not each of the following propositions is a tautology:
   (a) $(p\wedge\sim q)\rightarrow\sim q$
   (b) $p\vee(p\wedge q)\leftrightarrow p$
   (c) $[(p\rightarrow\sim q)\wedge(\sim p\wedge q)]\rightarrow q$
   (d) $[(p\rightarrow\sim q)\wedge(r\rightarrow q)\wedge q]\rightarrow r$
   (e) $[(p\rightarrow q)\wedge(q\rightarrow r)]\rightarrow(p\rightarrow r)$

2. Using the laws of algebra of propositions simplify the proposition $\sim[(p\vee q)\vee(\sim p\wedge q)]$.

3. Use the laws of algebra of propositions to show that $(P\wedge Q)\vee[\sim R\wedge(Q\wedge P)]\equiv P\wedge Q$.

4. Simplify each of the following by using the laws of algebra of propositions:
   (a) $q\rightarrow(\sim p\rightarrow\sim q)$
   (b) $\sim(p\vee q)\vee(\sim p\wedge q)$
   (c) $p\vee(p\wedge q)$
   (d) $[(p\rightarrow q)\wedge\sim q]\rightarrow\sim p$

5. Use the laws of algebra of propositions to simplify the proposition $[(p\rightarrow(q\vee\sim r))\wedge(q\rightarrow(p\wedge r))]\rightarrow[((q\vee\sim r)\wedge(p\wedge r))\rightarrow r]$

Arguments

An argument is a series of connected propositions that form a definite statement. Arguments are normally made up of two parts, the initial statements, called "premises", followed by a last statement called "conclusion". For instance, the following is an example of the argument: "If I go to the cinema, I will not do my homework. I did my homework. Therefore, I did not go to the cinema". The first two statements *"If I go to the cinema, I will not do my homework"* and *"I did my homework"* are premises, and the last statement *"Therefore, I did not go to the cinema"* is the conclusion. Remember, an argument is valid if and only if whenever all the premises are true, the conclusion is also true.

Activity 3.3: Identifying premises and conclusion of the argument

Individually or in a group perform the following tasks:

1. Construct at least four mathematical sentences of your choice.
2. Using mathematical sentences constructed in task 1, identify statements that form some arguments and hence state the premises and conclusion for each of the argument constructed.
3. Formulate the mathematical argument and state the premises and conclusion drawn from the argument.
4. Share your results with other students for more inputs.

Testing the validity of an argument

If P_1, P_2, and P_3 are premises and q is the conclusion of an argument, then the full argument is written as $P_1; P_2; P_3 \therefore q$. In testing the validity of the argument, connect the premises P_1, P_2, and P_3 with a conjunction $\wedge$ and connect the premises with the conclusion by an implication sign $(\rightarrow)$. If the compound statement formed is a tautology, then the argument is valid, otherwise it is not valid, that is, $[P_1 \wedge P_2 \wedge P_3] \rightarrow q$ must be a tautology for the argument in question to be valid. The argument can be valid for false premises and/or conclusion.

Example 3.20

Use the laws of algebra of proposition to verify the validity of the following argument:
If I study, then I will not fail the examination. If I do not play football, then I will study. But I failed the examination. Therefore, I played football.

Solution

Let p: I study

q: I will not fail the examination

r: I play football

The premises and conclusion of the argument are written as:
$p \rightarrow q; \sim r \rightarrow p; \sim q \therefore r$

Connecting the premises and conclusion gives $p \to q; \sim r \to p; \sim q \therefore \sim r$,

$\equiv[(\sim p \vee q) \wedge (\sim r \vee p) \wedge \sim q] \to r$	Implication law
$\equiv[\sim q \wedge (\sim p \vee q) \wedge (r \vee p)] \to r$	Commutative law
$\equiv[((\sim q \wedge \sim p) \vee (\sim q \wedge q) \wedge (r \vee p)] \to r$	Distributive law
$\equiv[((\sim q \wedge \sim p) \vee F \wedge (r \vee p)] \to r$	Complement law
$\equiv[(\sim q \wedge \sim p) \wedge (r \vee p)] \to r$	Identity law
$\equiv[\sim q \wedge (\sim p \wedge r) \vee (\sim p \wedge p)] \to r$	Distributive law
$\equiv[\sim q \wedge (\sim p \wedge r) \vee F] \to r$	Complement law
$\equiv[\sim q \wedge \sim p \wedge r] \to r$	Identity law
$\equiv \sim (\sim q \wedge \sim p \wedge r) \vee r$	Implication law
$\equiv q \vee p \vee \sim r \vee r$	De Morgans law
$\equiv q \vee p \vee T$	Complement law
$\equiv T$	Identity law

Since the truth value is a tautology, then the argument is valid.

Example 3.21

Test the validity of the argument:
If I read my textbook, I will understand how to do my homework. I did not understand how to do my homework. Therefore, I did not read my textbook.

Solution

Let p : I read my textbook.
q : I will understand how to do my homework.

The premises and conclusion of the argument are written as: $p \to q;\ \sim q \therefore \sim p$. Connection of the premises and conclusion is $[(p \to q) \wedge \sim q] \to \sim p$. Its truth table is as shown in the following table.

p	q	$\sim p$	$\sim q$	$p \to q$	$(p \to q) \wedge \sim q$	$[(p \to q) \wedge \sim q] \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

Since the truth values in the last column are all true, then it is a tautology. Therefore, the argument is valid.

Example 3.22

Test whether or not the following argument is valid.
Either Halima reaches home early or the traffic jam is not there. The traffic jam is not there. Therefore, Halima does not reach home early.

Solution

Let p: Halima reaches home early.
q : The traffic jam is there.

The premises and conclusion of the argument can be written as $p \vee \sim q; \sim q \therefore \sim p$
Connecting the premises and conclusion gives $[(p \vee \sim q) \wedge \sim q] \to \sim p$. Its truth table is as shown in the following table.

p	q	$\sim p$	$\sim q$	$p \vee \sim q$	$(p \vee \sim q) \wedge \sim q$	$[(p \vee \sim q) \wedge \sim q] \to \sim p$
T	T	F	F	T	F	T
T	F	F	T	T	T	F
F	T	T	F	F	F	T
F	F	T	T	T	T	T

The truth values in the last column are not all true. Therefore, the argument is not valid.

Exercise 3.6

Test whether or not each of the following argument is valid:

1. If a girl is cute, then she is not humble. The girl is not humble. Therefore, she is cute.
2. If it rains or snows, then my rooftop leaks. My rooftop is leaking. Therefore, it is raining and snowing.
3. I wash the glasses or I do not drink juice. I drink juice. Therefore, I wash the glasses.
4. My car tyres will get a puncture if I do not change the tyres regularly. My car tyres got a puncture. Therefore, I did not change my car tyres regularly.
5. You will not be treated with respect if you are not humble. You are not treated with respect. Therefore, you are not humble.
6. If you are kind to a cat, then it will be your friend. You were not kind to a cat. Hence, it is not your friend.
7. If Nairobi is the capital city of Tanzania, then it is in Tanzania. Nairobi is not in Tanzania. Therefore, Nairobi is not the capital city of Tanzania.
8. If I am illiterate, then I can't read and write. I cannot read but I can write. Thus, I am not illiterate.
9. My finger nails will become dirty if I plant flowers. My finger nails did not become dirty. Therefore, I did not plant flowers.
10. People will not fail to understand me if I speak to them. They did not understand me. Therefore, I did not speak to them.

Electrical networks

An electrical network is an arrangement of a battery, lamp, and switches in series or in parallel connection using wires. Electrical switches are used in electrical networks to allow or stop the flow of electric current.

Electrical switches and statements

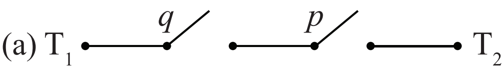
When a switch is "ON", the electric current can flow between terminals and the lamp will be switched "ON". Otherwise, the switch is "OFF" and current cannot flow. The terms "CLOSED" and "OPEN" are also used to mean "ON" and "OFF", respectively.

Series and parallel switches

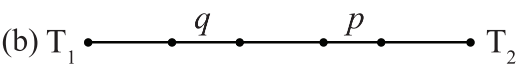
Connections of electrical switches can be done in series or in parallel.

Series switches

If p and q are two switches connected in series, then the logical expression to represent the switches is the conjunction "$p \wedge q$". This means that both p and q must be closed for electricity to flow in the electrical network. If both switches are open or one is open and other is closed, the electric current will not flow. Figure 3.1(a) and Figure 3.1 (b) shows the series connection of the open and closed switches. T_1 and T_2 are terminals bounding the switches.



Open switches

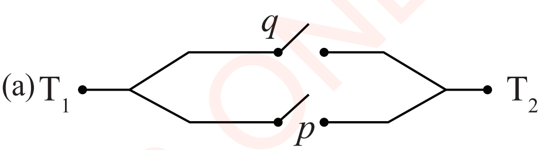


Closed switches

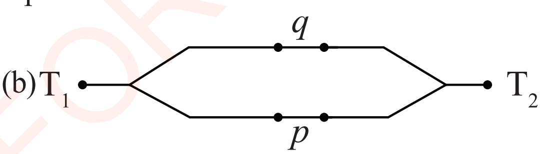
Figure 3.1 *Open and closed switches in series connection*

Parallel switches

If two switches, p and q are connected in parallel, the logical expression that represents the two switches is the disjunction, that is "$p \vee q$". This means that either p or q or both p and q must be closed for electricity to flow in the electrical network, otherwise the electric current will not flow. Figure 3.2 (a) and Figure 3.2 (b) shows the parallel connection of open switches and closed switches. T_1 and T_2 are terminals bounding the switches.



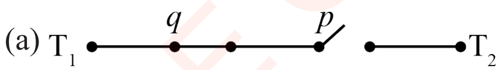
Open switches



Closed switches

Figure 3.2: *Open and closed switches in parallel connection*

In some applications, the switches can be connected in opposite positions. This means that, if one switch is open, the other must be closed, and vice versa. Two switches that have opposite positions are called complementary switches. If these switches are connected in series, electricity will not flow between two terminals. If they are in parallel, electricity will flow between the two terminals. Figure 3.3(a) and Figure 3.3(b) show the two cases.



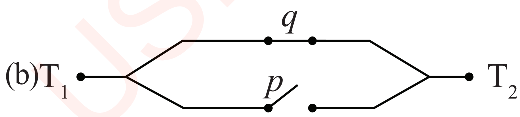


Figure 3.3: *Switches connected in opposite positions*

Logic can be used in constructing and simplifying networks. It can also be used in reducing the switches in the circuit, and hence perform the same function as a complex circuit. This may be done using the idea of equivalence of propositions. In simplifying the complex electrical network, two steps are important to follow:

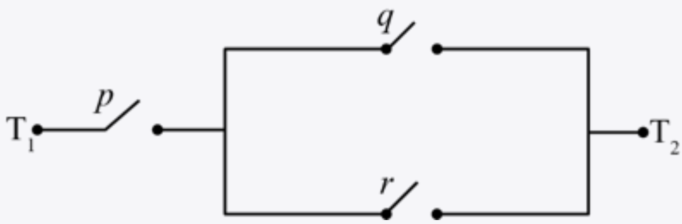
(1) Construct a compound proposition of the network, and (2) simplify the compound proposition using the laws of algebra of propositions.

Example 3.23

Draw an electrical network for $p \wedge (q \vee r)$.

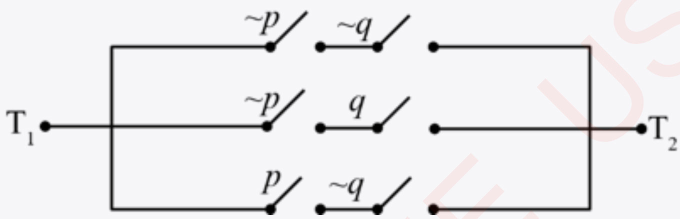
Solution

The following is the electric network for $p \wedge (q \vee r)$.



Example 3.24

Draw a simplified electric network using the following circuit:



Solution

The compound statement for the electrical network is

$(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge \sim q)$.

Thus, $(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge \sim q) \equiv \sim p \wedge (\sim q \vee q) \vee (p \wedge \sim q)$ Distributive law

$\equiv (\sim p \wedge T) \vee (p \wedge \sim q)$ Complement law

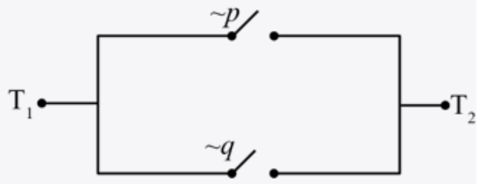
$\equiv \sim p \vee (p \wedge \sim q)$ Identity law

$\equiv (\sim p \vee p) \wedge (\sim p \vee \sim q)$ Distributive law

$\equiv T \wedge (\sim p \vee \sim q)$ Complement law

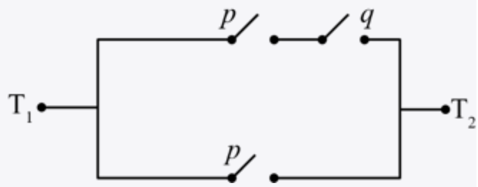
$\equiv \sim p \vee \sim q$ Identity law

Therefore, the following is a simplified electric network:



Example 3.25

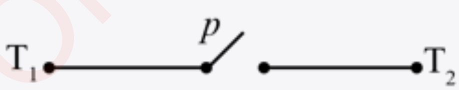
Draw an electrical network that is simpler than the following.



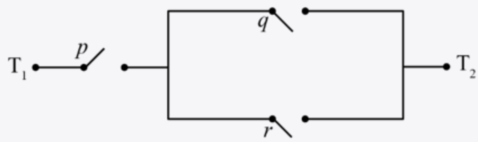
Solution

The compound statement for the electrical network is $(p \wedge q) \vee p$. This statement is equivalent to p (by absorption law).

This implies that, electric current can flow between T_1 and T_2 when p is closed regardless of whether q is closed or not. This is to say that q can be ignored and still achieve the desired goal. The following is the resulting electrical network:



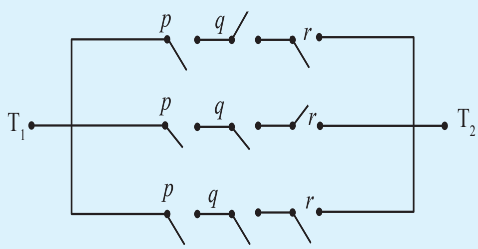
Note that, sometimes the switch is drawn opening downwards to indicate negation. Negation symbol is not placed in such drawings as shown in the following electric network for $p \wedge (\sim q \vee \sim r)$.



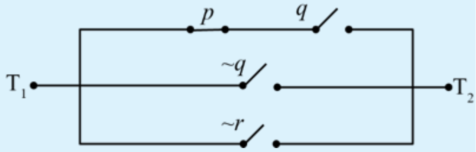
Exercise 3.7

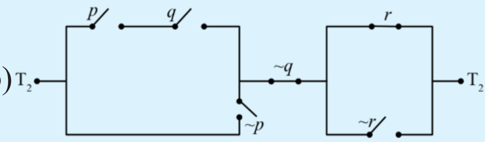
1. Draw an electrical network for each of the following compound statements:
   (a) $p \wedge q$
   (b) $p \wedge \sim q$
   (c) $p \wedge (q \vee r) \vee s$
   (d) $(p \wedge q) \vee (p \wedge r)$
   (e) $(p \vee q) \wedge (r \vee s) \vee y$
   (f) $(\sim p \vee r) \wedge (\sim p \wedge \sim r)$
   (g) $p \vee q \wedge \sim p$
   (h) $\sim (p \vee q) \vee (q \vee p)$
   (i) $[\sim (p \vee q) \vee (q \vee p)] \wedge r$

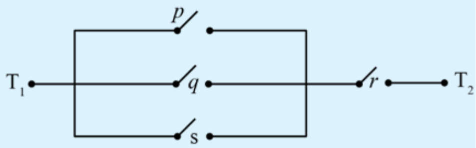
2. Draw a simplified diagram for the following electrical network.



3. Write down the compound statement in each of the following electrical network:

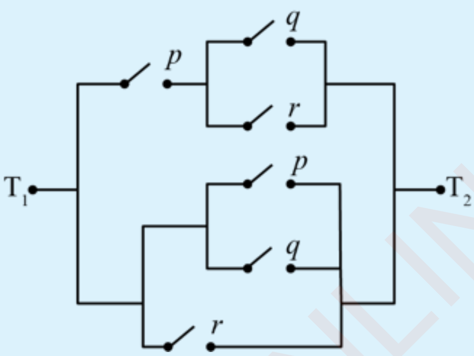
(a) 

(b) 

(c) 

4. Draw the simplified networks for the electrical networks in question 3.

5. Draw a simplified electrical network for the following:



Construction of a compound proposition from a truth table

It is possible to determine an equivalent compound proposition corresponding to the given truth table. The steps are as follows: First, identify the true values in the last column of the truth table. Next, construct basic conjunction for all true values in the last column. Finally, connect the conjunctions by disjunction(s).

Example 3.26

Construct a compound proposition equivalent to the following truth table:

p	q	r	B
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	F

Solution

Locate the true values in the last column (column B). The basic conjunctions are written for each true value T as follows:

p	q	r	B	Basic conjunction
T	T	T	F	
T	T	F	F	
T	F	T	F	
T	F	F	T	$p\wedge{\sim q}\wedge{\sim r}$
F	T	T	F	
F	T	F	T	${\sim p}\wedge q\wedge{\sim r}$
F	F	T	T	${\sim p}\wedge{\sim q}\wedge r$
F	F	F	F	

The required equivalent proposition is obtained by joining the three "basic conjunctions by the symbol "$\vee$", that is, $(p \wedge \sim q \wedge \sim r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r)$

Exercise 3.8

Write down the compound proposition corresponding to columns S, G, L, M, and D in the following truth tables:

1.

p	q	S
T	T	F
T	F	T
F	T	F
F	F	F

2.

p	q	G
T	T	T
T	F	F
F	T	F
F	F	T

3.

p	q	L
T	T	F
T	F	F
F	T	F
F	F	T

4.

p	q	M
T	T	T
T	F	T
F	T	F
F	F	T

5.

p	q	D
T	T	T
T	F	F
F	T	F
F	F	F

6. Construct the compound sentence for s_1 and s_2 having the following truth table and simplify s_1 using the laws of propositions.

p	q	r	s_1	s_2
T	T	T	F	T
T	T	F	F	F
T	F	T	T	F
T	F	F	F	F
F	T	T	F	T
F	T	F	F	F
F	F	T	T	F
F	F	F	F	F

Chapter summary

1. Logic is a science or an art which deals with the study of truth, principles of correct reasoning, and making good decisions.
2. A mathematical statement is a declarative sentence which can be either true or false, but not both.
3. The truth value of a true proposition is denoted by a letter "T" and the truth value of a false proposition is denoted by a letter "F".
4. There are two types of statements, namely; simple statements and compound statements.
5. The number of cases that describe a given compound statement depends on the number of propositions contained in the compound statement.
6. Negation of a statement is written by putting the word "NOT" with the verb, or to begin the sentence with the phrase "it is not true that" or "it is false that".
7. A conjunction is a type of compound statement that is comprised of two propositions joined by the "AND".
8. A disjunction is a compound statement which comprises of two simple statements formed by joining the statements with the "OR".
9. An implication statement is a type of compound statement that is formed by joining two simple statements with the logical implication connective.
10. A double implication statement denoted by a double-headed arrow "$\leftrightarrow$" is a type of compound statement formed by a combination of a conditional statement and its converse. This type of statement is also known as a biconditional statement. The truth value of the biconditional statement $p \leftrightarrow q$ is true when simple statements p and q are both true or both false. Otherwise, $p \leftrightarrow q$ is false.

Revision exercise 3

1. "Is it true that science subject are poorly performed by most of the students?" Is this a mathematical statement or not? Give a reason.
2. Write the converse, contrapositive, and inverse each in the following statements:
   (a) If a person is 20 years old, then is an adult.
   (b) If today is Friday, then I have a test.
   (c) If you buy our clothes, then you are attractive.
   (d) If today is Saturday, then it is a holiday.
   (e) If Nuru is intelligent, then she will pass the examination.
3. Construct the truth table in each of the following statements:
   (a) $(p\vee q)\rightarrow q$
   (b) $q\rightarrow(q\vee r)$
   (c) $[\sim(q\vee r)]\leftrightarrow[\sim q\wedge r]$
   (d) $\left[(p\rightarrow q)\wedge(q\vee r)\wedge p\right]\rightarrow r$
   (e) $\left[p\rightarrow\left[(q\vee\sim r)\right]\right]\rightarrow(p\wedge q)$
   (f) $p\wedge(q\vee r)$
   (g) $\left[\sim(p\wedge q)\vee(p\vee q)\right]$
   (h) $p\vee\sim q\rightarrow q$
   (i) $\sim r\rightarrow(p\rightarrow q)$
4. Without using a truth table, determine whether or not each of the following propositions is a tautology:
   (a) $\sim(r\rightarrow t)\rightarrow\sim r$
   (b) $p\rightarrow(p\vee q)$
   (c) $(r\wedge t)\rightarrow(r\rightarrow t)$
   (d) $\sim(r\rightarrow t)\rightarrow r$
5. Suppose that $S_1\leftrightarrow S_2$ is a tautology. What can you say about the sentences S_1 and S_2?
6. Use the truth tables to show whether or not each of the following statements is a tautology:
   (a) $\sim(\sim p\vee p)\rightarrow\sim(\sim q\vee q)$
   (b) $(p\rightarrow q)\vee(p\rightarrow r)\leftrightarrow(p\rightarrow(q\vee r))$
   (c) $(p\rightarrow q)\rightarrow\sim(p\vee\sim q)$
   (d) $\sim(\sim p\vee\sim q)\rightarrow\sim(p\rightarrow\sim q)$
   (e) $(\sim p\wedge(p\rightarrow q))\leftrightarrow\sim q$
   (f) $(p\leftrightarrow q)\wedge(p\rightarrow r)\leftrightarrow(p\rightarrow(q\wedge r))$
   (g) $p\vee(p\wedge q)\leftrightarrow p$
   (h) $[(p\wedge q)\rightarrow(\sim r\wedge p)\vee r]$
   (i) $(p\wedge q)\rightarrow(q\wedge r)\rightarrow(r\wedge s)$
   (j) $p\rightarrow q\rightarrow\sim q\rightarrow\sim p$

7. If p and q represent any two sentences, and s represents a tautology, which of the following pairs of sentences are equivalent to each other?
   (a) $p \wedge s;\ p$
   (b) $\sim s \vee p;\ p$
   (c) $s \rightarrow p;\ p$
   (d) $p \vee s;\ p$
   (e) $p \vee (q \vee \sim s);\ q \vee p$

8. If H_1 and H_2 are equivalent sentences, determine whether or not the sentences, $H_1 \rightarrow H_2$ and $H_2 \rightarrow H_1$ are equivalent to each other? Explain your answer.

9. Using truth tables, determine whether or not the following pairs of statements are logically equivalent:
   (a) $p \wedge (q \vee r)$ and $[p \rightarrow (q \vee \sim r)] \rightarrow (p \wedge q)$
   (b) $\sim (p \wedge \sim q) \wedge (\sim q)$ and $\sim (p \vee q)$
   (c) $(p \rightarrow q) \vee (p \rightarrow r)$ and $p \rightarrow (q \vee r)$
   (d) $(p \wedge q) \vee (\sim p \wedge \sim q)$ and $p \leftrightarrow q$

10. Which of the following pairs of sentences are equivalent?
   (a) $\sim p;\ p$
   (b) $p \wedge q;\ \sim (p \vee \sim q)$
   (c) $(p \wedge q) \wedge r;\ r$
   (d) $p \rightarrow q;\ \sim p \vee q$
   (e) $p \rightarrow (q \rightarrow r);\ p$
   (f) $\sim (p \rightarrow q);\ p \wedge \sim q$

11. Using the laws of algebra of propositions, simplify each of the following expressions:
   (a) $[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$
   (b) $(p \wedge q) \rightarrow (p \vee q)$
   (c) $[(p \rightarrow \sim q) \wedge (r \rightarrow q) \wedge q] \rightarrow r$
   (d) $[(p \rightarrow q) \wedge (\sim q)] \rightarrow \sim p$

12. State whether or not each of the following compound statements is a tautology:
   (a) $(p \rightarrow q) \wedge (q \rightarrow r) \leftrightarrow (p \rightarrow r)$
   (b) $\sim [(p \wedge \sim r \wedge \sim q)] \wedge (\sim p \vee q \vee \sim r)$

13. Let p be "Lightness is clever", q be "Lightness is polite", and r be "Lightness is humble":
   (a) Write the verbal sentence representation in each of the following:
      (i) $\sim (p \rightarrow \sim q) \vee r$
      (ii) $\sim p \vee (p \wedge q)$
   (b) Write each of the following compound statements in symbolic form:
      (i) Lightness is either clever or polite, but she is not polite if she is clever.
      (ii) Lightness is either polite or humble, but she is not clever.
      (iii) Lightness is neither polite nor humble as long as she is not clever.

14. Test the validity of each of the following arguments:

(a) If I am clever, then I know how to reason. I do not know how to reason. Therefore, I am not clever.

(b) Maize grows well whenever there is rain. If the maize grows well, there is no famine. But there is famine. Therefore, there is no rain.

(c) If Amos goes to school, then he must have a good life. But he will not have good life if he fails examinations. Therefore, Amos must go to school.

(d) The game will be off if it rains or if there are no players. The game was on. Therefore, it did not rain.

(e) If my smart phone crashes, I will loose all my contacts. I have not lost all my contacts. Therefore, my smart phone has not crashed.

(f) $q; r; \sim q \rightarrow p \therefore \sim r$

(g) $p \rightarrow \sim q, \sim r \vee q, r, \mapsto \sim p$

(h) $p \rightarrow q, q \vee r \therefore \sim r \rightarrow \sim p$

(i) $r \rightarrow q; r \rightarrow p, q \therefore \sim p$

(j) $p \rightarrow q, q \vee r \therefore r$

15. Construct an electrical network corresponding to each of the following propositions:

(a) $[(p \wedge q \wedge r) \vee s] \wedge t$

(b) $[p \vee (p \wedge r) \vee s] \wedge t$

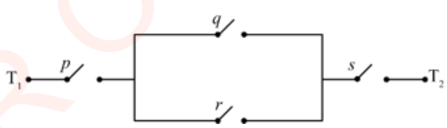
(c) $p \vee (\sim q \wedge r) \vee p$

(d) $(p \vee (q \wedge r)) \vee s$

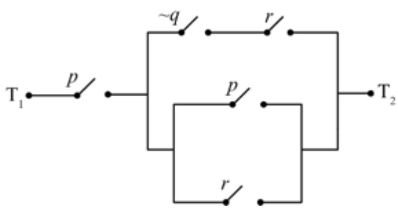
(e) $p \vee (q \wedge (r \wedge s))$

16. Write down the compound statement corresponding to each of the following electrical networks:

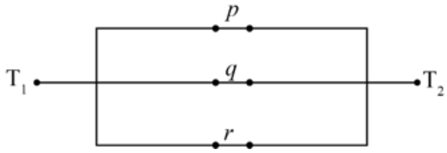
(a)



(b)



(c)



17. Find the compound statement having the truth values shown in columns "M and S" in the following truth tables:

(a)

p	q	M
T	T	F
T	F	T
F	T	T
F	F	F

(b)

p	q	r	S
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	F
F	F	T	F
F	F	F	T

Chapter Four

Coordinate geometry 1

Introduction

Coordinate geometry is a branch of geometry in which the positions of the points of geometric figures in two-dimensional plane are defined with the help of ordered pairs of numbers called coordinates. In this chapter, you will learn about the rectangular Cartesian coordinate system, ratio theorem, and circle. The competencies developed will help you to solve problems in the fields of trigonometry, calculus, and dimensional geometry. Also, in real life, coordinate geometry is used in space activities such as location of air transport, map projection, describing the position of objects, and in many other applications.

Rectangular Cartesian coordinate system

The rectangular Cartesian coordinate system in a plane specifies every point uniquely by a pair of two coordinates, x-coordinate which is also referred as *abscissa* and y-coordinate which is also referred to as *ordinate*. The x-coordinate is the perpendicular distance of a point from the y-axis which is parallel to the x-axis and y-coordinate is the perpendicular distance of the point from the x-axis, which is parallel to the y-axis. The axes meet at a point called the *origin*. At the *origin*, the values of x and y-coordinates are always (0,0) as shown in Figure 4.1.

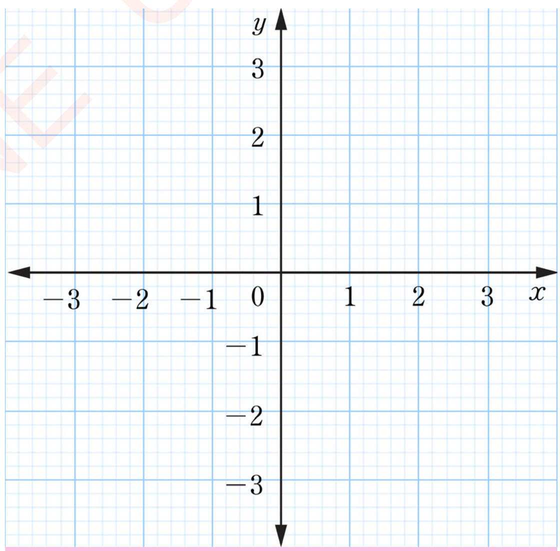


Figure 4.1: *The xy - coordinate system*

The coordinate system having the same units on both axes is referred to as rectangular Cartesian coordinate system. Usually, the coordinates x and y at any point P in the coordinate system are represented as P(x, y).

Plotting points on the Cartesian coordinate system

A point in a plane is represented by a pair of numbers (x, y) and can be plotted as follows:

(i) Draw a perpendicular line from the x-axis at the point corresponding to the x value.

(ii) Draw a perpendicular line to the y-axis at the point corresponding to the y value as shown in Figure 4.2.

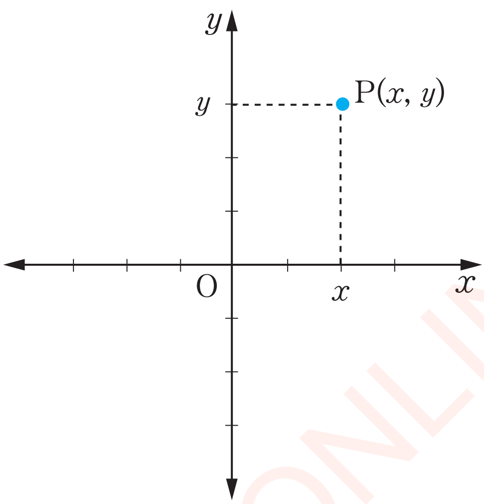


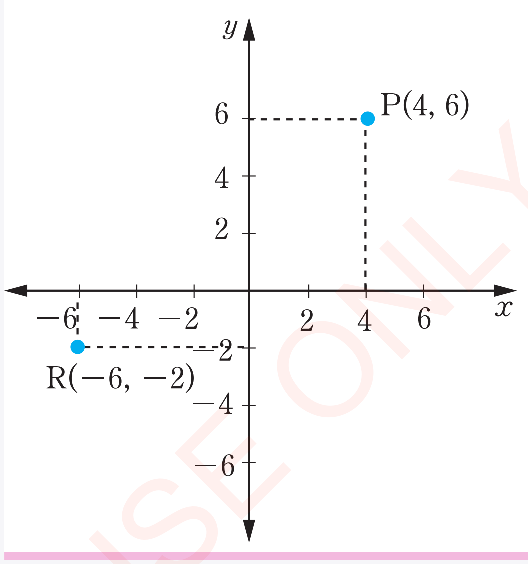
Figure 4.2: *Position of the point* P (x, y)

Example 4.1

Plot the points P(4, 6) and R(−6, −2) on the same rectangular Cartesian coordinate system.

Solution

The positions of points P(4, 6) and R(−6, −2) on the rectangular Cartesian coordinate system are shown in the following figure.



Exercise 4.1

Locate each of the following points on the Cartesian coordinate system and join them with a straight edge. What kind of a polygon is formed in each case?

1. A(9, −5), B(9, −1), and C(2, 3).
2. P(−6, 3), Q(3, −3), and R(9, 6).
3. K(−4, 6), L(−1, 3), and M(−5, 2).
4. D(−2, 2), E(−6, 2), F(−2, −8), and G(−6, −8).
5. W(0, 0), X(5, 0), Y(0, 5), and Z(5, 5)
6. L(4, 4), M(4, 8), N(10, 4), and P(10, 8).

Area of a rectangle by coordinates of vertices

Given four vertices of a rectangle in the Cartesian coordinate system, the vertices can be used to evaluate the area of the rectangle.

Let the four vertices of a rectangle be $P(x_1, y_1)$, $Q(x_2, y_1)$, $R(x_2, y_2)$, and $S(x_1, y_2)$ as shown in Figure 4.3.

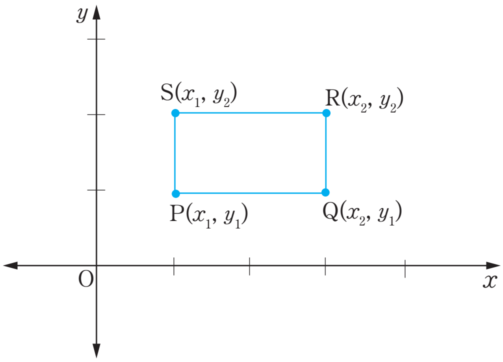


Figure 4.3: *Rectangle PQRS*

Use the vertices P and Q to compute the length of the rectangle, and Q and R to compute the width. Recall that the area of a rectangle is given by:

Area of rectangle PQRS = Length × Width

$$= \overline{PQ} \times \overline{QR}$$

Using the distance formula,

Area of rectangle PQRS $= \sqrt{(x_2 - x_1)^2 + (y_1 - y_1)^2} \times \sqrt{(x_2 - x_2)^2 + (y_2 - y_1)^2}$

$$= \sqrt{(x_2 - x_1)^2} \times \sqrt{(y_2 - y_1)^2}$$

$$= (x_2 - x_1)(y_2 - y_1)$$

$$= (x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1) \text{ square units.}$$

Therefore, the area of a rectangle PQRS $= (x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1)$ square units.

Similarly, the vertices S and R for the length and S and P for the width can be used fo find the area of rectangle PQRS as follows:

Area of rectangle PQRS $= \overline{SR} \times \overline{PS}$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_2)^2} \times \sqrt{(x_1 - x_1)^2 + (y_2 - y_1)^2}$$

$$= \sqrt{(x_2 - x_1)^2} \times \sqrt{(y_2 - y_1)^2}$$

$$= (x_2 - x_1)(y_2 - y_1)$$

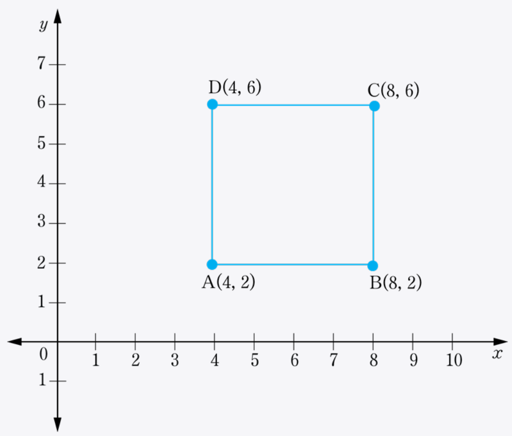
$$= (x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1) \text{ square units.}$$

Therefore, the area of rectangle PQRS $= (x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1)$ square units.

Example 4.2

Locate the points A(4, 2), B(8, 2), C(8, 6), and D(4, 6) on a rectangular Cartesian coordinate system and find the area of the resulting figure.

Solution



The area of figure ABCD can be computed by using the formula of area of rectangle as follows:

Let $(x_1, y_1) = (4, 2)$ and $(x_2, y_2) = (8, 6)$.

Area of the figure ABCD

$= x_2y_2 - x_1y_2 - x_2y_1 + x_1y_1$

$= (8)(6) - (4)(6) - (8)(2) + (4)(2)$

$= 48 - 24 - 16 + 8$

$= 16$ square units.

Therefore, the area of the figure ABCD is 16 square units.

Proof of parallelogram properties by using rectangular Cartesian coordinates

The points in a rectangular Cartesian coordinate system can be used to verify the properties of various shapes, such as the shape of a parallelogram. The first property of a parallelogram is that, the opposite sides are equal as shown in Figure 4.4.

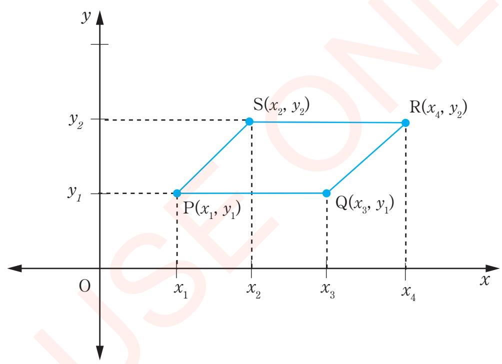


Figure 4.4: *Parallelogram* PQRS

Figure 4.4 shows the vertices of the parallelogram PQRS which are $P(x_1, y_1)$, $Q(x_3, y_1)$, $R(x_4, y_2)$, and $S(x_2, y_2)$.

$\overline{PS}$ is opposite to $\overline{QR}$ and $\overline{SR}$ is opposite to $\overline{PQ}$.

Using distance formula,

$$\overline{PS} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \qquad (4.1)$$

$$\overline{QR} = \sqrt{(x_4 - x_3)^2 + (y_2 - y_1)^2} \qquad (4.2)$$

But $|x_2 - x_1| = |x_4 - x_3|$. Substituting this in equation (4.2) gives;

$$\overline{QR} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Therefore,

$$\overline{QR} = \overline{PS} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \quad (4.3)$$

Similarly, $\overline{SR} = |x_4 - x_2|$ and $\overline{PQ} = |x_3 - x_1|$

But $|x_4 - x_2| = |x_3 - x_1|$.

Therefore, $\overline{SR} = \overline{PQ}$. (4.4)

Equations (4.1) to (4.4) can be used to prove that, the opposite sides of a parallelogram are equal.

Activity 4.1: Recognizing that the opposite sides of a parallelogram are parallel to each other

Individually or in a group, perform each of the following tasks:

1. Sketch a parallelogram EFGH with vertices of your choice.
2. Compute the gradients of the line segments $\overline{EF}$, $\overline{EH}$, $\overline{HG}$, and $\overline{FG}$.
3. Give a your opinion regarding the gradients of each of the following pairs:
   (a) $\overline{EF}$ and $\overline{HG}$
   (b) $\overline{EH}$ and $\overline{FG}$
4. Share your findings with your fellow students for further discussion and improvements.

From Activity 4.1 it can be observed that, the opposite sides of a parallelogram are parallel.

The second property of a parallelogram is that, the opposite sides are parallel.

Figure 4.4 shows that $\overline{PS}$ is parallel to $\overline{QR}$ and $\overline{PQ}$ is parallel to $\overline{SR}$. If the line segments are parallel to each other, it means that the segments have the same gradient. Using the gradient of a line segment,

The gradient of

$$\overline{PS} = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$$

$$= \frac{y_2 - y_1}{x_2 - x_1} \quad (4.5)$$

The gradient of

$$\overline{QR} = \frac{\text{change in } y\text{-coordinates}}{\text{change in } x\text{-coordinates}}$$

$$= \frac{y_2 - y_1}{x_4 - x_3} \quad (4.6)$$

But $|x_4 - x_3| = |x_2 - x_1|$, substituting in equation (4.6) gives

$$\text{The gradient of } \overline{QR} = \frac{y_2 - y_1}{x_2 - x_1} \quad (4.7)$$

Therefore, the gradient of

$\overline{PS}$ = the gradient of $\overline{QR}$.

Equations (4.5) and (4.7) can be used to prove that, the opposite sides of a parallelogram are parallel.

The third property of a parallelogram is that, the opposite interior angles are equal as shown in Figure 4.5

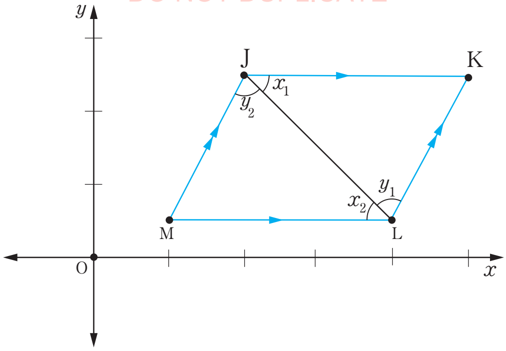


Figure 4.5. *Parallelogram JKLM*

Consider triangles JKL and JLM in Figure 4.5:

	Argument	Reason
1.	$\hat{x}_1 = \hat{x}_2$	Alternate interior angles, $\overline{JK} // \overline{ML}$.
2.	$\hat{y}_1 = \hat{y}_2$	Alternate interior angles, $\overline{JM} // \overline{KL}$.
3.	$\overline{JL} = \overline{JL}$	The common line to both tringles.
4.	$\Delta JKL = \Delta JLM$	By Angle, Angle, Side theorem (AAS)
5.	$\hat{J} = \hat{x}_1 + \hat{y}_2 = \hat{x}_2 + \hat{y}_2$	From step 1
6.	$\hat{L} = \hat{x}_2 + \hat{y}_1 = \hat{x}_2 + \hat{y}_2$	From step 2
7.	$\hat{J} = \hat{L}$	From steps 5 and 6

Therefore, the opposite interior angles of a parallelogram are equal. Similarly, the same steps can be used to prove that $\hat{M} = \hat{K}$. The fourth property of a parallelogram is that, the diagonals bisect each other as shown in Figure 4.6.

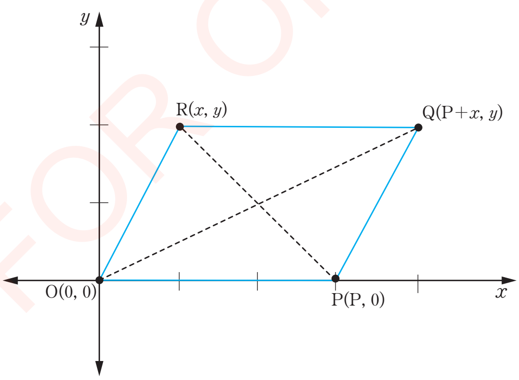


Figure 4.6: *Parallelogram OPQR*

Midpoint of the diagonal $\overline{OQ}$ is

$$\left(\frac{p+x+0}{2}, \frac{y+0}{2}\right) = \left(\frac{p+x}{2}, \frac{y}{2}\right)$$

Midpoint of the diagonal $\overline{PR}$ is

$$\left(\frac{p+x}{2}, \frac{y+0}{2}\right)$$

$$= \left(\frac{p+x}{2}, \frac{y}{2}\right)$$

Since the midpoint of the diagonals $\overline{OQ}$ and $\overline{PR}$ is a unique point, both line segments intersect at the midpoint.

Hence, $\overline{OQ}$ and $\overline{PR}$ bisect each other.

Activity 4.2 Recognizing that the diagonals of a parallelogram bisect each other

Individually or in a group, perform each of the following tasks:

1. Draw any parallelogram of your choice and label all its vertices.
2. Construct two diagonals and label their intersection point.
3. Verify that the first triangle is the rotation of the second triangle about the intersection of the diagonals through $180°$.
4. Measure all the lengths from each of the vertex to the intersection of the diagonals.
5. Identify the pair of lengths which are equal to each other in task 4.
6. What did you observe in task 5? Give comments
7. Share your findings with your fellow students for more inputs.

Example 4.3

Given the points P(2, 2), Q(6, 2), R(8, 3), and S(4, 3). Locate the points on a rectangular Cartesian coordinate system and find the following:

(a) The distance $\overline{PQ}$

(b) The distance $\overline{RS}$

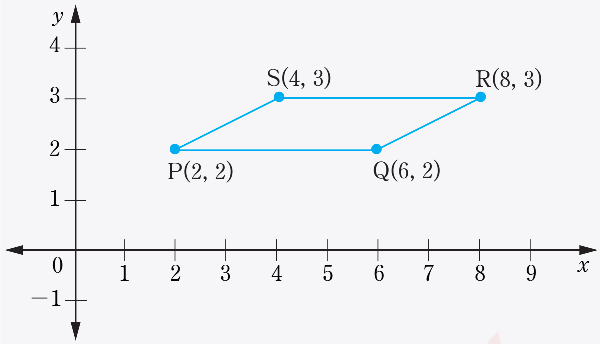
(c) The distance $\overline{PS}$

(d) The distance $\overline{QR}$

(e) What is the name of the resulting geometrical shape?

Solution

The points are plotted as shown in the following figure



(a) $\overline{PQ} = \sqrt{(6-2)^2} = \sqrt{16} = 4$ units

(b) $\overline{RS} = \sqrt{(4-8)^2} = \sqrt{16} = 4$ units

(c) $\overline{PS} = \sqrt{(4-2)^2 + (3-2)^2}$

$= \sqrt{4+1} = \sqrt{5}$ units

(d) $\overline{QR} = \sqrt{(8-6)^2 + (3-2)^2}$

$= \sqrt{4+1} = \sqrt{5}$ units

(e) $\overline{PQ}$ is opposite to $\overline{RS}$ and $\overline{PS}$ is opposite to $\overline{QR}$. Since, $\overline{PQ} = \overline{RS}$ and $\overline{PS} = \overline{QR}$.

Therefore, figure PQRS is a parallelogram.

Example 4.4

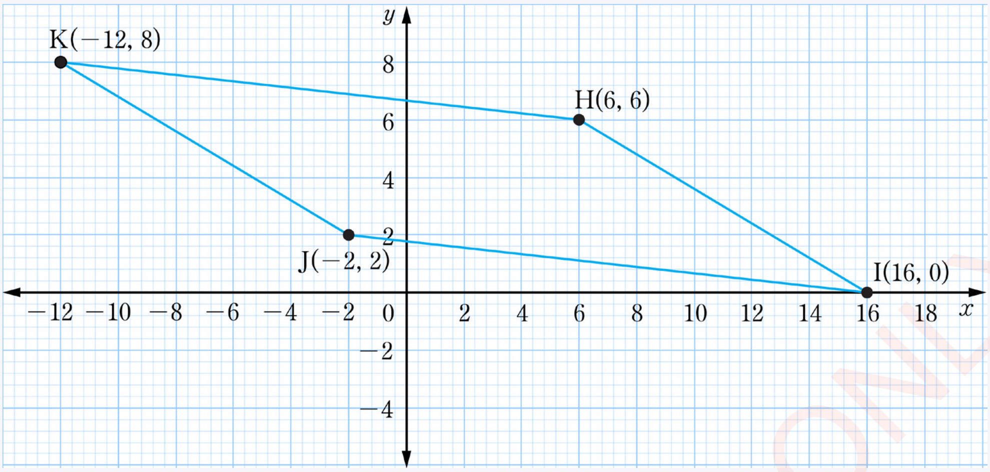
The points H, I, J, and K have coordinates $(6,6)$, $(16,0)$, $(-2, 2)$, $(-12, 8)$, respectively.

(a) Find the gradients of the lines $\overline{HI}$, $\overline{IJ}$, $\overline{JK}$, and $\overline{KH}$, then verify that HIJK is a parallelogram.

(b) Show that the diagonals $\overline{HJ}$ and $\overline{KI}$ bisect each other.

Solution

(a) The points are plotted as shown in the following figure:



The gradient of $\overline{HI} = \frac{6-0}{6-16} = -\frac{3}{5}$

The gradient of $\overline{IJ} = \frac{2-0}{-2-16}$

$= -\frac{1}{9}$

The gradient of $\overline{JK} = \frac{8-2}{-12+2}$

$= -\frac{3}{5}$

The gradient of $\overline{KH} = \frac{6-8}{6+12}$

$= -\frac{1}{9}$

Since, the gradient of $\overline{HI}$ = the gradient of $\overline{JK}$, then $\overline{HI} // \overline{JK}$

Also, the gradient of $\overline{IJ}$ = the gradient of $\overline{KH}$, then $\overline{IJ} // \overline{KH}$.

Since, the opposite sides are parallel to each other, then HIJK is a parallelogram.

(b) The midpoint of the diagonal

$\overline{HJ} = \left(\frac{6+(-2)}{2}, \frac{6+2}{2}\right)$

$= (2, 4)$

and the midpoint of the diagonal

$\overline{KI} = \left(\frac{-12+16}{2}, \frac{8+0}{2}\right)$

$= (2, 4)$

Since, the midpoint of the diagonals $\overline{HJ}$ and $\overline{KI}$ is a unique point, both line segments intersect at the midpoint.

Therefore, the diagonal $\overline{HJ}$ and $\overline{KI}$ bisect each other.

Exercise 4.2

1. Use the distance formula to show that each of the following points form the vertices of an isosceles triangle.
   (a) A(4, –4,), B(10, 0), and C(3, 4)
   (b) P(–6, 3), Q(3, –3), and R(9, 6)
   (c) T(–5, 5), U(–2, 2), and V(–6, 1)
   (d) W(–1, 0), X (0, –1), and Y(2, 2)
   (e) L(0, 0), M(6, 2), and N(5, –5)
   (f) E(2, –3), F(8, 6), and G(–7, –9)

2. Show that each of the following points are vertices of a parallelogram:
   (a) A(–1, 3), B(–3, 7), C(–5, 3), and D(– 3, –1)
   (b) E(– 6, 3), F(– 6, 1), G(– 2, 3), and H(– 2, 5)
   (c) J(– 5, 1), K(– 2, – 2), L(1, 7), and M(4, 4)
   (d) N(–1, –1), P(2, –4), Q(6, 0), and R(3, 3)
   (e) S(3, 3), T(3, 0), U(– 3, – 3), and V(– 3, 0)

3. The coordinates of a parallelogram are A(1, 1), B(*x*, *y*), C(4, – 1), and D(– 1, – 1). Find the values of *x* and *y*.

4. Show that the figure formed by the lines $y-2x=4$, $x+y=-1$, $y=2x-4$, and $x+y-1=0$ is a parallelogram. Hence, find the equations of its diagonals.

5. A quadrilateral EFGH has vertices E(–2, –3), F(1, –1), G(7, –10), and H(2, –9)
   (a) Prove that $\overline{EH}$ is parallel to $\overline{FG}$
   (b) Prove that $F\hat{E}H = F\hat{G}H$
   (c) Find the area of the quadrilateral EFGH

6. Prove that the quadrilateral with vertices H(4, 0), I(7, –3), J(–2, –2), and K(–5, 1) is a parallelogram. Hence, calculate the point at which the diagonals bisect each other.

Angle between two lines

Given two lines L_1 and L_2 whose slopes are m_1 and m_2, respectively. The angle of intersection between the two lines, L_1 and L_2 can be obtained in terms of their slopes. Consider two intersecting lines L_1 and L_2 as shown in Figure 4.7.

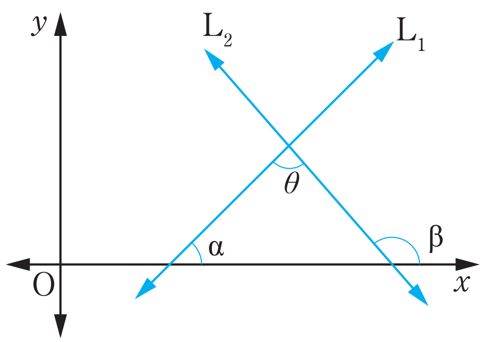


Figure 4.7: *Angle of intersection between two lines* L_1 *and* L_2

Let: α be the angle between L_1 and the *x*-axis.

β be the angle between L_2 and the x-axis. θ be the angle between the two lines, L_1 and L_2. In Figure 4.7,

since, $\alpha + \theta + 180° - \beta = 180°$, then $\theta = \beta - \alpha$.

Introducing tangent on both sides of the equation $\theta = \beta - \alpha$, to obtain,

$\tan\theta = \tan(\beta - \alpha)$

$$\Rightarrow \tan\theta = \frac{\tan\beta - \tan\alpha}{1 + \tan\beta\tan\alpha}$$

But $\tan\alpha$ = slope of L_1 and $\tan\beta$ = slope of L_2.

Thus, $\tan\theta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$.

Therefore, $\theta = \tan^{-1}\left(\dfrac{m_2 - m_1}{1 + m_1 m_2}\right)$.

Note that; for an acute angle between two lines, $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$.

Example 4.5

Find the acute angle between the lines $2x + 4y = 15$ and $3y - 2x + 6 = 0$.

Solution

Let L_1 be the line $2x + 4y = 15$, and L_2 be the line $3y - 2x + 6 = 0$, then $m_1 = -\dfrac{1}{2}$ and $m_2 = \dfrac{2}{3}$, respectively.

The acute angle between the lines L_1 and L_2 is obtained from the formula;

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

$$\Rightarrow \theta = \tan^{-1}\left|\frac{\frac{2}{3} - \left(-\frac{1}{2}\right)}{1 + \frac{2}{3}\left(-\frac{1}{2}\right)}\right|$$

$$\Rightarrow \theta = \tan^{-1}(1.75) = 60.26°.$$

Therefore, the angle between the lines L_1 and L_2 is 60.26°.

Example 4.6

Two straight lines have the equations $3x - 2y = 5$ and $4x + 5y = 1$, respectively. Find the tangent of the angle between the lines.

Solution

Let L_1 be the line $3x - 2y = 5$, and L_2 be the line $4x + 5y = 1$, then $m_1 = \dfrac{3}{2}$ and $m_2 = -\dfrac{4}{5}$, respectively.

The angle between the two lines given by:

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

$$\Rightarrow \tan\theta = \left(\frac{-\frac{4}{5} - \left(\frac{3}{2}\right)}{1 + \left(\frac{3}{2} \times -\frac{4}{5}\right)}\right)$$

$$=\begin{pmatrix} -\frac{23}{10} \\ \frac{2}{-\frac{2}{10}} \end{pmatrix}$$

Thus, $\tan\theta = 11.5$

Therefore, the tangent of the angle between the two lines is 11.5.

Example 4.7

Find the equation of the line passing through the point (4, 6) and inclined with a line $2x - y = 2$ at 45°.

Solution

Suppose L_2 represents the line $2x - y = 2$. Then, $m_2 = 2$.

From $\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$,

where $\theta = 45°$, then

$$\tan 45° = \frac{2 - m_1}{1 + 2m_1}$$

But $\tan 45° = 1$. Thus,

$$1 = \frac{2 - m_1}{1 + 2m_1}$$

$$\Rightarrow 1 + 2m_1 = 2 - m_1$$

$$\Rightarrow m_1 = \frac{1}{3}$$

Hence, the slope of the line is $\frac{1}{3}$.

The equation of the line is given by:

$$\frac{1}{3} = \frac{y-6}{x-4}$$

$$\Rightarrow x - 4 = 3y - 18$$

$$\Rightarrow y = \frac{1}{3}x + \frac{14}{3} \Rightarrow x - 3y = -14$$

Therefore, the equation of the line is $x - 3y = -14$.

Exercise 4.3

1. Find the obtuse angle between the lines L_1: $8x + 6y - 13 = 0$ and L_2: $-6x + 4y + 1 = 0$.
2. Find the acute angle between the line L_1 which passes through the points A(2, 2) and B(8, 6) and the line L_2 which passes through the points C(0, 3) and D(6, – 3).
3. Find the interior angles of each of the triangles having the following vertices:
   (a) A(–4, –5), B(–1, 1), C(0, –3)
   (b) P(– 2, – 1), Q(4, – 3), R(1, 2)
   (c) L(– 4, 4), M(– 3, 1), N(6, 2).
4. Find the equation of the line through the point (2, 3), which makes an angle 45° with the line $2x - y + 4 = 0$.
5. Let $nx + my = 8$ and $3nx - 2my = 6$ be equations of two lines which intersect at the point (3, 1). Find the values of m and n, hence determine the angle between the lines at the point of intersection.
6. Determine the angles formed between the lines with equation
   $$2x^2 + 5xy - 12y^2 = 0$$

Perpendicular distance of a point from a line

The shortest distance of a point $P(x_1, y_1)$ from a given line is called the perpendicular distance. In practical situation, it is common to have a line and any point from the line.

Consider a point $P(x_1, y_1)$ and a line $ax + by + c = 0$. The perpendicular distance (r) of the point $P(x_1, y_1)$ from the line can be derived as follows:

Suppose Q(x, y) is any point on the line, such that $\overline{PQ}$ is perpendicular to the line $ax + by + c = 0$ as shown in Figure 4.8

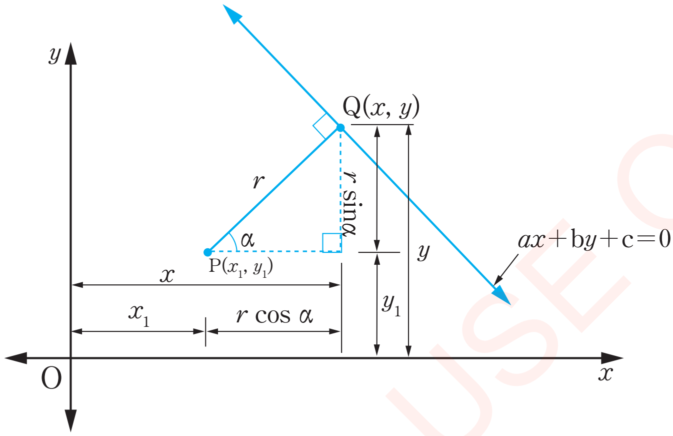


Figure 4.8: *Perpendicular distance of a point from a line*

The slope of $ax + by + c = 0$ is $-\dfrac{a}{b}$. For perpendicular lines, the slope of one line is the negative reciprocal of the slope of the other line, or the product of the slopes of two perpendicular lines equals –1. Therefore, the slope of the line $\overline{PQ}$ is $\dfrac{b}{a}$. Let α be the angle between line $\overline{PQ}$ and the horizontal axis. The tangent of the angle between the line $\overline{PQ}$ and the horizontal axis equals the slope of the line $\overline{PQ}$, that is, $\tan\alpha = \dfrac{b}{a}$. Consider the righ-angled triangle in Figure 4.9.

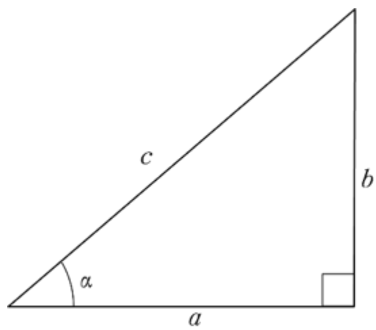


Figure 4.9: *Right-angled triangle*

From Figure 4.9, using trigonometric ratios, $\cos\alpha = \dfrac{a}{c}$ and $\sin\alpha = \dfrac{b}{c}$.

Using Pythagoras' theorem,

$c^2 = a^2 + b^2$

$\Rightarrow c = \sqrt{a^2 + b^2}$

$\Rightarrow \cos\alpha = \dfrac{a}{\sqrt{a^2 + b^2}}$ and $\sin\alpha = \dfrac{b}{\sqrt{a^2 + b^2}}$

But x and y are the coordinates of Q. Then $x = x_1 + r\cos\alpha$ and $y = y_1 + r\sin\alpha$.

Substituting the values of $\cos\alpha$ and $\sin\alpha$ in x and y results to, $x = x_1 + \dfrac{ar}{\sqrt{a^2 + b^2}}$ and $y = y_1 + \dfrac{br}{\sqrt{a^2 + b^2}}$.

The point Q(x, y) lies on the line $ax + by + c = 0$. This means that x and y satisfy the equation $ax + by + c = 0$.

That is,

$$a\left(x_1 + \frac{ar}{\sqrt{a^2 + b^2}}\right) + b\left(y_1 + \frac{br}{\sqrt{a^2 + b^2}}\right) + c = 0$$

$$\Rightarrow ax_1 + by_1 + c = -r\left(\frac{a^2 + b^2}{\sqrt{a^2 + b^2}}\right)$$

$$\Rightarrow ax_1 + by_1 + c = -r\sqrt{a^2 + b^2}$$

$$\Rightarrow -r = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow |-r| = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

The absolute value sign is introduced since distance always has positive magnitude.

Therefore, the formula for finding the perpendicular distance of a point $P(x_1, y_1)$ from a line $ax + by + c = 0$ is given by,

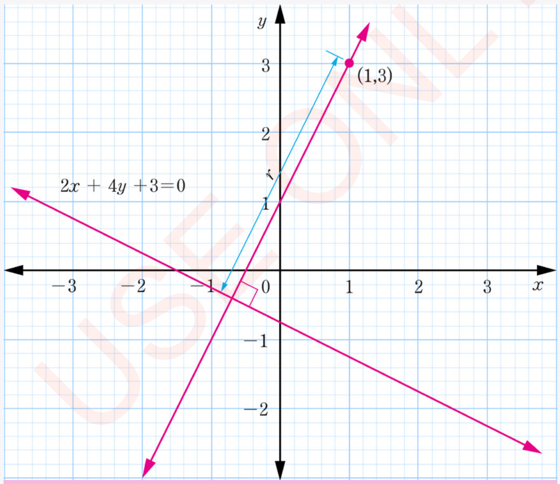
$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$

Example 4.8

Find the perpendicular distance from the point (1, 3) to the line $2x + 4y + 3 = 0$.

Solution

Given the equation $2x + 4y + 3 = 0$.
Comparing with $ax + by + c = 0$, the constants are $a = 2$, $b = 4$ and $c = 3$.
Consider the following graph:



Let $(x_1, y_1) = (1, 3) \Rightarrow x_1 = 1$ and $y_1 = 3$.

Using the perpendicular distance formula;

$$r = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$r = \frac{|2\times1 + 4\times3 + 3|}{\sqrt{2^2 + 4^2}} = 3.8 \text{ units}$$

Therefore, the perpendicular distance is 3.8 units.

Example 4.9

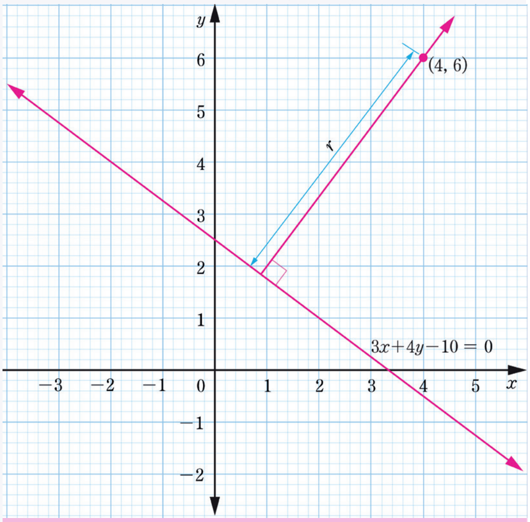
Find the length of the line segment perpendicular to $3x + 4y - 10 = 0$ drawn from the point (4, 6).

Solution

Given the equation, $3x+4y-10=0$.

Comparing with $ax+by+c=0$, the constants will be, $a=3$, $b=4$ and $c=-10$.

Consider the following diagram:



$(x_1, y_1)=(4, 6) \Rightarrow x_1=4$ and $y_1=6$

Using the perpendicular distance formula,

$$r = \frac{|ax_1+by_1+c|}{\sqrt{a^2+b^2}}$$

$$r = \frac{|3\times4+4\times6-10|}{\sqrt{3^2+4^2}} = 5.2 \text{ units}$$

Therefore, the length of the perpendicular line segment is 5.2 units.

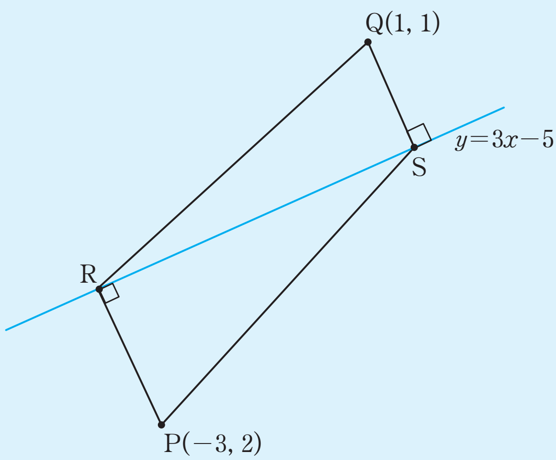
Exercise 4.4

1. Find the perpendicular distance:
   (a) From the point (3, 2) to the line $3x-4y=-4$
   (b) From the point $(0,-3)$ to the line $x=-5y-2$
   (c) From the point $(0, a)$ to the line $4x=3y$
   (d) From the point (h, k) to the line $3x+4y-h=0$
   (e) From the point (0, 0) to the line $3x=4y$, and hence comments on your results.

2. The perpendicular distance of the point $(p, 2)$ from the line $3x+4y+1=0$ is $\frac{\sqrt{2}}{5}$ units. Find the values of p.

3. If the line $Ax+By+C=0$ and $-Bx+Ay+Bs-At=0$ are perpendicular at point Q, where A, B, C, s and t are constants, show that the coordinates of the point Q is given by,
$$\left(\frac{B^2s-ABt-AC}{A^2+B^2}, \frac{-ABs+A^2t-BC}{A^2+B^2}\right).$$

4. Given the following figure, find the distances $\overline{PR}$, $\overline{QS}$, and $\overline{RS}$.



5. If m and n are lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$, and $x\sec\theta + y\operatorname{cosec}\theta = k$, respectively, prove that $m^2 + 4n^2 = k^2$.

Equations of angle bisectors

If L_1 and L_2 are two intersecting lines, and θ is one of the angles between the lines, then the other angle between the lines is $180° - \theta$. There are two equations of angle bisectors for these lines which are the bisector of the angle θ and the bisector of the angle $180° - \theta$ as shown in Figure 4.10. If the lines are not perpendicular, θ will be an acute angle, which implies that $180° - \theta$ is an obtuse angle. Then the bisector of angle θ is called the acute angle bisector while the bisector of angle $180° - \theta$ is called the obtuse angle bisector.

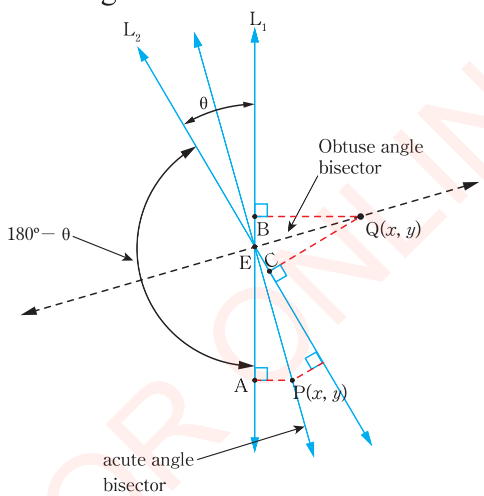


Figure 4.10: *Angle bisector with lines* L_1 *and* L_2

Suppose L_1 and L_2 are two lines with equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$, respectively.

If $P(x, y)$ is a point on the bisector of the angle AEC, then p will be equidistant from both lines and E is the point of intersection of the bisectors.

(a) The perpendicular distance of point $P(x, y)$ from the line $a_1x + b_1y + c_1 = 0$ is given by $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}$.

(b) The perpendicular distance of point $P(x, y)$ from the line $a_2x + b_2y + c_2 = 0$ is given by $-\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

The negative sign indicates that the point $P(x, y)$ lies on opposite sides of the two lines. Hence, the coordinates of P will satisfy $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

If $Q(x, y)$ is the point on the bisector of the angle AED, then Q will be equidistant from both lines and will lie on the same side of the lines as the origin.

(c) The perpendicular distance of point $Q(x, y)$ from the line $a_1x + b_1y + c_1 = 0$ is given by $\dfrac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}$.

(d) The perpendicular distance of point $Q(x, y)$ from the line $a_2x + b_2y + c_2 = 0$ is $\dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$.

The perpendicular distance of the point $Q(x, y)$ from the two lines are of the same sign, indicating that this point lies on the same side of the two lines. Hence, the coordinates (x, y) of Q will satisfy $\dfrac{a_1x+b_1y+c_1}{\sqrt{a_1^{\,2}+b_1^2}}=\dfrac{a_2x+b_2y+c_2}{\sqrt{a_2^{\,2}+b_2^2}}$.

Therefore, the equations to the bisectors of the angles between the two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are written as

$$\frac{a_1+b_1y+c_1}{\sqrt{a_1^2}}=\pm\left(\frac{a_2+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right).$$

Example 4.10

Find the equations of the angle bisectors between the lines $x+2y-5=0$ and $y=3x$.

Solution

The equations of the angle bisectors are obtained from the formula;

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^{\,2}+b_1^{\,2}}}=\pm\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^{\,2}+b_2^{\,2}}}\right).$$

$$\Rightarrow \frac{x+2y-5}{\sqrt{1^2+2^2}}=\pm\left(\frac{3x-y}{\sqrt{3^2+(-1)^2}}\right)$$

$$\Rightarrow \frac{x+2y-5}{\sqrt{5}}=\pm\left(\frac{3x-y}{\sqrt{10}}\right)$$

$$\Rightarrow \sqrt{2}(x+2y-5)=\pm(3x-y).$$

Therefore, the equations of the angle bisectors are,

$\left(-3+\sqrt{2}\right)x+\left(1+2\sqrt{2}\right)y-5\sqrt{2}=0$ and

$\left(-3+\sqrt{2}\right)x+\left(-1+2\sqrt{2}\right)y-5\sqrt{2}=0$.

Example 4.11

Find the equations of the bisectors of the angles between the lines $3x+4y-7=0$ and $y-2=0$.

Solution

The equations of the angle bisectors are given by

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^{\,2}+b_1^{\,2}}}=\pm\left(\frac{a_2x+b_2y+c_2}{\sqrt{a_2^{\,2}+b_2^{\,2}}}\right).$$

Thus,

$$\frac{3x+4y-7}{\sqrt{3^2+4^2}}=\pm\left(\frac{y-2}{\sqrt{0^2+1^2}}\right)$$

$$\Rightarrow \frac{3x+4y-7}{\sqrt{25}}=\pm\left(\frac{y-2}{\sqrt{1}}\right)$$

$$\Rightarrow \frac{3x+4y-7}{5}=\pm\left(\frac{y-2}{1}\right)$$

$$\Rightarrow \frac{3x+4y-7}{5}=\pm(y-2)$$

The equations of the angle bisectors are $\dfrac{3x+4y-7}{5}=(y-2)$ and $\dfrac{3x+4y-7}{5}=-(y-2)$

$$\Rightarrow 3x-y+3=0 \text{ and } 3x+9y-17=0$$

Therefore, the equations of the angle bisectors are $3x - y + 3 = 0$ and $3x + 9y - 17 = 0$.

Exercise 4.5

1. Find the equations of the angles bisectors for each of the following pairs intersecting lines:
   (a) $3x + 4y - 7 = 0$ and $y - 1 = 0$
   (b) $x = -y$ and $y = x$
   (c) $3x + 4y = 0$ and $2x - 1 = y$
2. Find the equation of an obtuse angle bisector of lines $4x - 3y + 10 = 0$ and $8y - 6x - 5 = 0$.
3. Show that $7x + 56y = 67$ is the equation of the angle bisectors of the lines with the equations $3x + 4y = 8$ and $5x - 12y + 6 = 0$.
4. Find the equations of the bisectors of the angles between the lines $x + 7y = 3$, $17x - 7y + 3 = 0$, and $x - y + 1 = 0$

Locus

When a point moves in such a way that it satisfies a given condition, the path it traces is called a locus. For example, if a point moves in the plane in such a way that it is always equidistant from the origin, then its locus is a circle. Also, if a set of points in three-dimensional are equidistant from a given point, its locus is a sphere. If a point (x, y) moves so that it satisfies any given condition, it will describe a locus or curve.

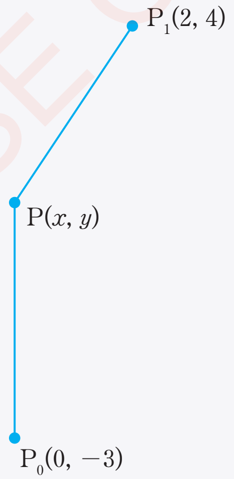
The relation between the coordinate (x, y) of any point will only be true if the point lies on the locus.

Example 4.12

Find the equation of the locus of a point which moves in such a way that it is equidistant from the point (2, 4) and the point (0, –3). What is the name of the equation of the locus formed?

Solution

Let P(x, y) be the point equidistant from the point (2, 4) and the point (0, –3) as shown in the following figure.



By the distance formula,

$$\sqrt{(x-2)^2 + (y-4)^2} = \sqrt{(x-0)^2 + (y-(-3))^2}$$

Squaring both sides gives,

$$(x-2)^2 + (y-4)^2 = (x-0)^2 + (y+3)^2$$

$$\Rightarrow x^2 - 4x + 4 + y^2 - 8y + 16 = x^2 + y^2 + 6y + 9$$

$$\Rightarrow -4x - 14y + 11 = 0$$

$$\Rightarrow 4x + 14y - 11 = 0$$

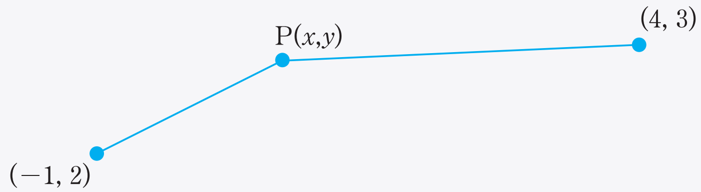
Therefore, the locus is a straight line whose equation is $4x + 14y - 11 = 0$.

Example 4.13

A point moves in such a way that its distance from point (–1, 2) is two times its distance from point (4, 3). Find the equation of the locus formed.

Solution

Consider the following figure:



Let $P(x, y)$ be the point from points (–1, 2) and (4, 3) that satisfies the relation.

The distance from point (–1, 2) = 2 × distance from the point (4, 3)

Using the distance formula, gives

$$\sqrt{(x-(-1))^2+(y-2)^2} = 2\sqrt{(x-4)^2+(y-3)^2}$$

Squaring both sides gives

$$(x+1)^2+(y-2)^2=4[(x-4)^2+(y-3)^2]$$

Expanding both sides gives,

$$x^2+2x+1+y^2-4y+4 = 4[x^2-8x+16+y^2-6y+9]$$

Simplification of the equation gives,

$$3x^2+3y^2-34x-20y+95=0$$

Therefore, the locus is $3x^2+3y^2-34x-20y+95=0$.

Exercise 4.6

1. In each of the following, find the locus of a point which moves in such a way that:
   (a) It is 4 units to the right of the y-axis.
   (b) It is 5 units above the x-axis.
   (c) Its distance from the origin is 5.
   (d) Its distance from the point (–2, –3) is 8.
   (e) The sum of the squares of its distances from the points (2, 0) and (6, 0) is 16 units.
   (f) Its distance from the point (0, 0) is twice its distance from the line $y = 4$.
   (g) Its distance from the point (0, 0) is half its distance from the line $x = 3$.
   (h) The difference between the squares of its distance from the points (–2, 0) and (2, 0) is 3 units.

2. Given the coordinate points A(4, 0) and B(0, 3). Find the equation of the locus of a point $P(x, y)$ which moves in such a way that $\overline{PA} = \overline{PB}$.
3. A point $P(x, y)$ moves in such a way that its distance from a line parallel to the y-axis through the point $(-a, 0)$ is equal to its distance from the point $(a, 0)$. Find the equation of the locus formed.
4. A point $P(x, y)$ is twice as far from the point A(3, 0) as it is from the line $x = 5$. Find the equation of the locus of P.
5. Find the locus of the point which moves such that its perpendicular distances from the lines $4x - 3y = 0$ and $5x + 12y = 0$ are equal.

Ratio theorem

A line segment can be divided internally or externally by a point in any given ratio. In each division, if the ratio is provided then the coordinates of the point dividing the line segment can be found.

Internal division

Consider a line segment $\overline{AB}$ with two fixed points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let $P(x, y)$ be a point dividing the line $\overline{AB}$ in the ratio $m : n$ on rectangular Cartesian coordinates as shown in Figure 4.11.

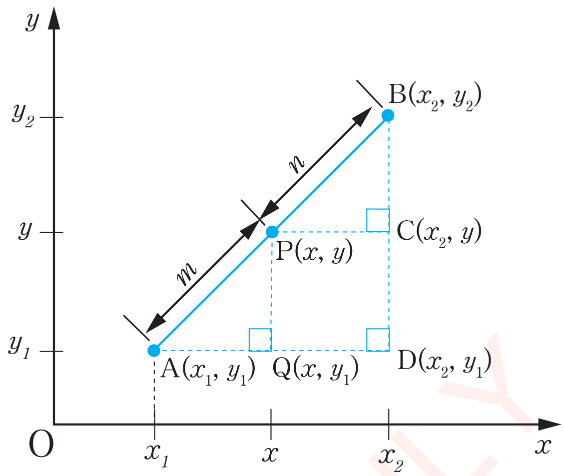


Figure 4.11: *Internally division of a line*

In Figure 4.11, it can be observed that ΔAPQ and ΔPBC are similar.

Hence, $\dfrac{\overline{AQ}}{\overline{PC}} = \dfrac{\overline{AP}}{\overline{PB}}$.

But $\overline{AQ} = x - x_1$ and $\overline{PC} = x_2 - x$.

Thus, $\dfrac{x - x_1}{x_2 - x} = \dfrac{m}{n}$.

Making x the subject of the formula gives

$$x = \frac{mx_2 + nx_1}{m + n} \qquad (4.8)$$

Similarly, $\dfrac{\overline{QP}}{\overline{CB}} = \dfrac{\overline{AP}}{\overline{PB}}$

But $\overline{QP} = y - y_1$ and $\overline{CB} = y_1 - y$

Thus, $\dfrac{y - y_1}{y_2 - y} = \dfrac{m}{n}$.

Making y the subject of the formula gives

$$y = \frac{my_2 + ny_1}{m + n} \qquad (4.9)$$

The expressions of x and y in equations (4.8) and (4.9) give the coordinates of the point $P(x, y)$ which divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ internally in the ratio $m : n$, respectively. That is,

$$P(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right).$$

Example 4.14

Find the coordinates of the point that divides the line joining the points A(– 4, 10) and B(8, 4) in the ratio of 1:3 internally.

Solution

Given the ratio 1:3 $\Rightarrow m = 1$ and $n = 3$.

Also,

$(x_1, y_1) = (-4, 10)$, and $(x_2, y_2) = (8, 4)$.

From

$$\text{P}(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right),$$

gives

$$\begin{aligned} \text{P}(x, y) &= \left(\frac{1(8) + 3(-4)}{1+3}, \frac{1(4) + 3(10)}{1+3} \right) \\ &= \left(\frac{8-12}{4}, \frac{4+30}{4} \right) \\ &= \left(-1, \frac{17}{2} \right) \end{aligned}$$

Therefore, the point of internal division is $P(x, y) = \left(-1, \frac{17}{2} \right)$.

Example 4.15

A point P divides internally the line segment joining the points A(4, 6) and B(– 5, 2) in the ratio $\overline{\text{AP}} : \overline{\text{PB}} = 2:3$. Find the coordinates of P.

Solution

Given the ratio 2:3 $\Rightarrow m = 2$ and $n = 3$.

Also, $(x_1, y_1) = (4, 6)$, and $(x_2, y_2) = (-5, 2)$.

From

$$\text{P}(x, y) = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n} \right),$$

$$\begin{aligned} \Rightarrow \text{P}(x, y) &= \left(\frac{(2\times -5) + (3\times 4)}{2+3}, \frac{(2\times 2) + (3\times 6)}{2+3} \right) \\ &= \left(\frac{-10+12}{5}, \frac{4+18}{5} \right) \\ &= \left(\frac{2}{5}, \frac{22}{5} \right) \end{aligned}$$

Therefore, the point of internal division is $\text{P}(x, y) = \left(\frac{2}{5}, \frac{22}{5} \right)$.

External division

Let $\text{P}(x, y)$ be a point which divides the line segment joining the points $\text{A}(x_1, y_1)$ and $\text{B}(x_2, y_2)$ externally in the ratio $m : n$. The formula for computing the coordinates of P can be derived by considering the sketch in Figure 4.12.

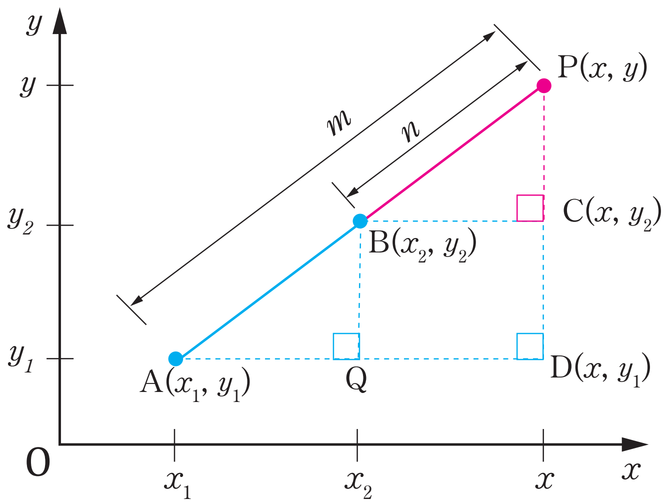


Figure 4.12: *External division of a line*

In Figure 4.12, ΔAPD is similar to ΔBPC.

Hence, $\dfrac{\overline{BC}}{\overline{AD}} = \dfrac{\overline{BP}}{\overline{AP}}$.

Thus, $\dfrac{x - x_2}{x - x_1} = \dfrac{n}{m}$.

Express x as subject of the formula to get $x = \dfrac{nx_1 - mx_2}{n - m}$. (4.10)

Similarly, $\dfrac{\overline{PC}}{\overline{PD}} = \dfrac{\overline{BP}}{\overline{AP}}$.

Thus, $\dfrac{y - y_2}{y - y_1} = \dfrac{n}{m}$.

Express y as subject of the formula to obtain $y = \dfrac{ny_1 - my_2}{n - m}$ (4.11)

If equations (4.8) and (4.10) are compared, it can be observed that they are similar except that m in equation (4.8) has been replaced by $-m$ in equation (4.10). Similarly, $-m$ in equation (4.11) has replaced m in equation (4.9). The expressions for x and y in equations (4.10) and (4.11), respectively, are the coordinates of the point $P(x, y)$ which divides the line joining the points $A(x_1, y_1)$ and $B(x_2, y_2)$ externally in the ratio $m : n$.

Thus, $P(x, y) = \left(\dfrac{nx_1 - mx_2}{n - m}, \dfrac{ny_1 - my_2}{n - m} \right)$

Example 4.16

Find the coordinates of the point that divides externally the line joining the points A(–2, 8) and B(7, 3) in the ratio of 2:3.

Solution

Let $P(x, y)$ be the point dividing the line in the ratio of $2 : 3 \Rightarrow m = 2$ and $n = 3$.

Also, let $(x_1, y_1) = (-2, 8)$ and $(x_2, y_2) = (7, 3)$.

From $P(x, y) = \left(\frac{nx_1 - mx_2}{n-m}, \frac{ny_1 - my_2}{n-m}\right)$,

$$P(x, y) = \left(\frac{3(-2)-2(7)}{3-2}, \frac{3(8)-2(3)}{3-2}\right)$$
$$= (-20, 18).$$

Therefore, the point of external division is $P(x, y) = (-20, 18)$.

Example 4.17

If the point $P(x, 6)$ divides externally the line segment joining the point A (–1, 3) to point B $(4, y)$ in the ratio of $-3:2$, find the values of x and y.

Solution

Given the ratio
$-3:2 \Rightarrow m = -3$ and $n = 2$.

Let $(x_1, y_1) = (-1, 3)$ and
$(x_2, y_2) = (4, y)$

From $P(x, y) = \left(\frac{nx_1 - mx_2}{n-m}, \frac{ny_1 - my_2}{n-m}\right)$

$$\Rightarrow (x, 6) = \left(\frac{2(-1)-(-3)(4)}{2-(-3)}, \frac{2(3)-(-3)y}{2-(-3)}\right)$$

The value of x is given by;

$$x = \frac{-2+12}{5} \Rightarrow x = 2$$

Also, the value of y will be given by;

$$6 = \frac{2(3)-(-3)y}{2-(-3)} \Rightarrow 6 = \frac{6+3y}{5}$$

$$\Rightarrow 6 + 3y = 30 \Rightarrow y = 8$$

Therefore, the values of $x = 2$ and $y = 8$.

Exercise 4.7

1. For each of the following pairs of points, find the points which divides the line segment both internally and externally:
   (a) $A(3, 6)$ and $B(-8, -2)$ in the ratio of 2:3
   (b) C(1, 1) and $D(-3, 5)$ in the ratio of 1:1
   (c) $Q(-3, 12)$ and $R(-8, -2)$ in the ratio of 2:3
   (d) S(3, 6) and T(7, 6) in the ratio of 5:1
2. Points A and B have coordinates (7, 6) and (– 6, –10), respectively. If the point C divides $\overline{AB}$ externally in the ratio of 1:2, find the coordinates of C.
3. A line joins the points $P(-4, -3)$ and $Q(3, 2)$. Given that point B divides $\overline{PQ}$ internally in the ratio of 1:2 and point R divides $\overline{PQ}$ internally in the ratio of 2:5, find the length of $\overline{BR}$.
4. A line $\overline{PQ}$ is divided internally in the ratio 2:1 by the point A(1, 1), and externally in the ratio $-5:2$ by the point B(4, 7). Find the coordinates of P and Q.
5. A line $\overline{PQ}$ is divided internally in the ratio $m:n$ by a point B. Given that P, Q, and B have the coordinates (–2, –4), (3, 11), and (1, 5) respectively, find the values of m and n.

6. The line $2x+y-4=0$ divides the line segment joining the points A(2,−2) and B(3,7). Find the ratio of the line segment.

Applications of the ratio theorem

The ratio theorem is used in other areas of mathematics such as showing similarity of two geometrical objects, and in vector calculations.

A circle

A circle is the locus of a point which moves such that its distance from a fixed point is constant. The fixed point is called the centre of the circle and the constant distance is called the radius of the circle.The diameter of a circle is any line through the centre of the circle and with endpoints at the circumference of the circle. Objects such as a coin, a wheel of a bicycle or vehicle, camera lenses, pizzas, Ferris wheels, rings, steering wheels, and bottle tops, are all examples of circular objects. If the middle point of a circular object is exactly located, then the distance from the middle point towards any point of the edge will always be the same.

Standard equation of a circle

Suppose $C(a,b)$ is the point at the centre of a circle, r is its radius, and $P(x, y)$ is any point on the edge of the circle as shown in Figure 4.13, then the standard equation of the circle can be obtained.

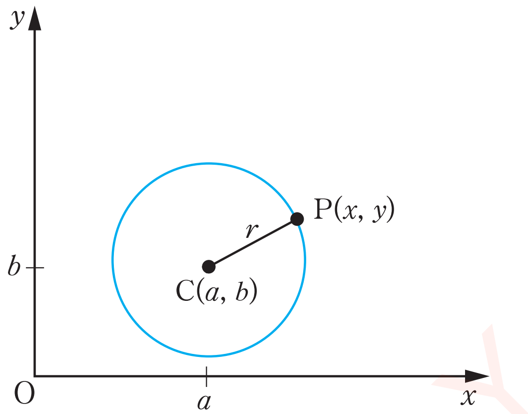


Figure 4.13: *A circle*

In Figure 4.13, the radius, $r = \overline{CP}$.
Using the distance formula gives;

$$\overline{CP} = \sqrt{(x-a)^2 + (y-b)^2}$$

But, $\overline{CP} = r$.

Thus, $\sqrt{(x-a)^2 + (y-b)^2} = r$.

Squaring both sides, gives

$$(x-a)^2 + (y-b)^2 = r^2. \qquad (4.12)$$

Equation (4.12) is the standard equation of a circle.

Example 4.18

Write the standard equation of the circle given the following:

(a) Centre (3, 2) and radius 4 units.
(b) Centre (0, 4) and diameter 6 units.

Solution

(a) Given the centre $(a, b) = (3, 2)$ and $r = 4$, using the standard equation of the circle, gives

$(x-a)^2 + (y-b)^2 = r^2$

$\Rightarrow (x-3)^2 + (y-2)^2 = 4^2$

Therefore, the standard equation is $(x-3)^2 + (y-2)^2 = 16$.

(b) Given the centre $(a,b) = (0,4)$ and $d = 2r = 6$ units $\Rightarrow r = 3$ units. The standard equation of a circle is given by

$(x-0)^2 + (y-4)^2 = 3^2$

$\Rightarrow (x-0)^2 + (y-4)^2 = 9$

Therefore, the standard equation is $x^2 + (y-4)^2 = 9$.

The general equation of a circle

The general equation of a circle is an expansion of the standard form of equation of a circle of radius r and centre (a, b).

Now, from equation (4.12), the general equation is obtained as follows:

$(x-a)^2 + (y-b)^2 = r^2$

$\Rightarrow x^2 - 2xa + a^2 + y^2 - 2yb + b^2 = r^2$

$\Rightarrow x^2 + y^2 - 2xa - 2yb + a^2 + b^2 = r^2$

$\Rightarrow x^2 + y^2 - 2xa - 2yb + a^2 + b^2 - r^2 = 0$

$$\Rightarrow x^2 + y^2 + 2(-a)x + 2(-b)y + a^2 + b^2 - r^2 = 0 \qquad (4.13)$$

Let $g = -a$, $f = -b$ and $c = a^2 + b^2 - r^2$. Then, equation (4.13) becomes

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad (4.14)$$

Equation (4.14) is called the general equation of the circle.

Note that; when comparing the coordinates of the centre and radius of the circle, the coefficients of x^2 and y^2 in any form of equation of a circle must be the same and positive. Otherwise, it will not be an equation of a circle.

Since $c = a^2 + b^2 - r^2$, then

$r^2 = a^2 + b^2 - c$.

$\Rightarrow r = \sqrt{a^2 + b^2 - c}$

Since $g = -a$ and $f = -b$, then $a = -g$ and $b = -f$.

Hence, the centre and radius of a circle are $(-g, -f)$ and $\sqrt{g^2 + f^2 - c}$, respectively.

Example 4.19

Find the centre and the radius of the circle whose equation is $x^2 + y^2 - 4x + 2y + 1 = 0$.

Solution

Given the equation

$x^2 + y^2 - 4x + 2y + 1 = 0$.

Compare the given equation with the general equation of a circle.

The values are $g = -2$, $f = 1$, and $c = 1$. The radius, r and the centre of the circle are given by

$r = \sqrt{g^2 + f^2 - c}$.

$= \sqrt{(-2)^2 + (1)^2 - 1} = \sqrt{4 + 1 - 1}$

$\Rightarrow r = 2$

Centre $= (-g, -f)$

$= (2, -1)$

Therefore, the centre of the circle is $(2, -1)$ and radius is 2 units.

Alternatively,
it is possible to find the centre and radius if the general equation of a circle is changed into the standard equation by completing the square. That is,

$x^2+y^2-4x+2y+1=0$

$\Rightarrow x^2-4x+4+y^2+2y+1=4$

$\Rightarrow (x-2)^2+(y+1)^2=4$

$\Rightarrow (x-2)^2+(y+1)^2=2^2$

Compare the resulting equation with the standard equation of a circle.

$(x-a)^2+(y-b)^2=r^2$

Thus, centre $(2,-1)$ and $r=2$.
Therefore, the centre of the circle is $(2,-1)$ and radius is 2units.

Example 4.20

Find the general equation of the circle whose centre and radius are $(1, 3)$ and 2 units, respectively.

Solution
From the general form of equation of a circle $x^2+y^2+2gx+2fy+c=0$
Centre = (1, 3), then $g=-1$ and $f=-3$.

The radius $r=\sqrt{g^2+f^2-c}$

$2=\sqrt{(-1)^2+(-3)^2-c}$

$\Rightarrow 2=\sqrt{10-c},\ c=6.$

Substituting all the values into the general equation $x^2+y^2+2gx+2fy+c=0$ gives

$x^2+y^2-2x-6y+6=0.$

Therefore, the general equation of the circle is $x^2+y^2-2x-6y+6=0$.

Exercise 4.8

1. Find the expanded form of the equation of a circle using each of the following information:
   (a) Centre $=(0,2)$ and radius = 4 units
   (b) Centre = (–2, 2) and radius = 5 units
   (c) Centre = (0, 0) and diameter = 4 units
   (d) Centre $=\left(-\frac{1}{2},5\right)$ and radius = 10 units
   (e) Centre = (2, –3) and diameter $=\frac{\sqrt{2}}{2}$ units
2. Find the centre and radius of each of the circles given by the following equations:
   (a) $(x-1)^2+(y+2)^2=45$
   (b) $x^2+y^2=25$
   (c) $x^2+y^2-4y=0$
   (d) $(x+1)^2+y^2=4$
   (e) $(2x-1)^2+(2y+3)^2=4$
3. Find the equation of a circle which touches both axes at the distance 4 units from the origin.
4. Find the equation of the circle of radius 12 units whose centre is at the point at which the lines $y=4x+3$ and $y=5x+44$ meet.
5. Find the equation of the circle whose centre is at (5, 2) and passes through the centre of the circle $x^2+y^2-3x-4y-1=0$

Equation of a circle given the end points of a diameter

If $\overline{AB}$ is a diameter of a circle with coordinates $A(x_1,y_1)$ and $B(x_2,y_2)$, and if $P(x,y)$ is any point on the circle, then $A\hat{P}B=90°$. Figure 4.14 illustrates the statement.

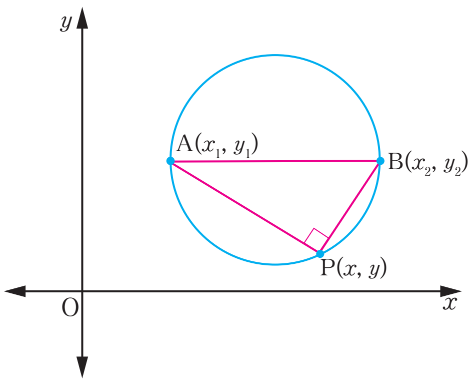


Figure 4.14: *Circle with end points of a diameter*

In Figure 4.14 the gradient of $\overline{AP} = \dfrac{y-y_1}{x-x_1}$

Thus, $m_1 = \dfrac{y-y_1}{x-x_1}$ (4.15)

Similarly, the gradient of $\overline{BP} = \dfrac{y-y_2}{x-x_2}$.

That is, $m_2 = \dfrac{y-y_2}{x-x_2}$. (4.16)

But $\overline{AP}$ is perpendicular to $\overline{BP}$, that is $m_1 m_2 = -1$. Substituting equations (4.15) and (4.16) gives;

Thus, $\left(\dfrac{y-y_1}{x-x_1}\right)\left(\dfrac{y-y_2}{x-x_2}\right) = -1.$

This implies that,

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0. \qquad (4.17)$$

Equation (4.17) can be used to find the equation of a circle when given the two end points of a diameter.

Example 4.21

Find the equation of the circle of diameter $\overline{AB}$ with end points $A(2,\ 1)$ and $B(-2,2)$.

Solution

Given

$A(x_1, y_1) = (2,1)$ and $B(x_2, y_2) = (-2,2)$

Use the equation of the circle of the form

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$

Substituting the given values in the equation gives

$(x-2)(x-(-2))+(y-1)(y-2)=0$

$\Rightarrow (x-2)(x+2)+(y-1)(y-2)=0$

$\Rightarrow x^2-2x+2x-4+y^2-y-2y+2=0$

$\Rightarrow x^2+y^2-3y-2=0$

Therefore, the equation of a circle is $x^2+y^2-3y-2=0$.

Equation of a circle passing through three given points

Suppose

$A_1(x_1, y_1)$, $A_2(x_2, y_2)$, and $A_3(x_3, y_3)$

are three points on a circle with the centre $(a,\ b)$ and radius r. Substitution of each of the given point in the standard equation of the circle, enables determination of the unknown values.

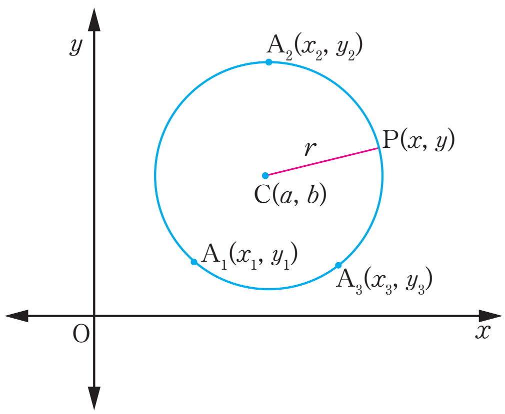


Figure 4.15: *Three points on a circle*

Using Figure 4.15, the standard equation of the circle is given by, $(x-a)^2+(y-b)^2=r^2$.

Substitute the three given points lying on a circle:

For a point $A_1(x_1, y_1)$, the equation will be:

$$(x_1-a)^2+(y_1-b)^2=r^2 \qquad (4.18)$$

For a point $A_2(x_2, y_2)$, the equation will be;

$$(x_2-a)^2+(y_2-b)^2=r^2 \qquad (4.19)$$

For a point $A_3(x_3, y_3)$, the equation will be

$$(x_3-a)^2+(y_3-b)^2=r^2 \qquad (4.20)$$

Combine the three equations (4.18), (4.19), and (4.20) to form a system of simultaneous equations

$$\begin{cases}(x_1-a)^2+(y_1-b)^2=r^2\\(x_2-a)^2+(y_2-b)^2=r^2\\(x_3-a)^2+(y_3-b)^2=r^2\end{cases}$$

Therefore, the system of equations that will be formed can be easily solved simultaneously to obtain the values of a, b, and r, which when plugged back into the standard equation of the circle. The same procedure can be used when the general equation of a circle is considered and gives the following system of simultaneous equations:

$$\begin{cases}x_1^2+y_1^2+2gx_1+2fy_1+c=0\\x_2^2+y_2^2+2gx_2+2fy_2+c=0\\x_3^2+y_3^2+2gx_3+2fy_3+c=0\end{cases}$$

The system of equations can be solved simultaneously to obtain the values of g, f, and c which when substituted back, the general equation of the circle will be formed.

Example 4.22

Find the equation of the circle which passes through the points A(3, 1), B(2, 6), and C(3, 2).

Solution

The given points $A(x_1, y_1) = (3, 1)$, $B(x_2, y_2) = (2, 6)$, and $C(x_3, y_3) = (3, 2)$ lie on a circle. Thus, they satisfy the system of general equations of a circle. The system of general equations of a circle formed by the given points is given by:

$$\begin{cases} x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \\ x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0 \\ x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0 \end{cases}$$

Substitute the given points to get the values of g, f, and c:

$$\begin{cases} 3^2 + 1^2 + 2(3)g + 2(1)f + c = 0 \\ 2^2 + 6^2 + 2(2)g + 2(6)f + c = 0 \\ 3^2 + 2^2 + 2(3)g + 2(2)f + c = 0 \end{cases} \Rightarrow \begin{cases} 6g + 2f + c = -10 \\ 4g + 12f + c = -40 \\ 6g + 4f + c = -13 \end{cases}$$

Solving the system of equations simultaneously, the values of g, f, and c are 7.5, -1.5, and -52, respectively.

Therefore, the equation of a circle is $x^2 + y^2 + 15x - 3y - 52 = 0$.

Alternatively, the three given points can be substituted in the standard equation of the circle.

$(x - a)^2 + (y - b)^2 = r^2$

For a point $A(x_1, y_1) = (3, 1)$, the equation will be

$(3 - a)^2 + (1 - b)^2 = r^2$ .... (i)

For a point $B(x_2, y_2) = (2, 6)$, the equation will be

$(2 - a)^2 + (6 - b)^2 = r^2$... (ii)

For a point $C(x_3, y_3) = (3, 2)$, the equation will be

$(3 - a)^2 + (2 - b)^2 = r^2$ ....(iii)

Combine equations (i), (ii), and (iii) gives the equations:

$$\begin{cases} (3 - a)^2 + (1 - b)^2 = r^2 \\ (2 - a)^2 + (6 - b)^2 = r^2 \\ (3 - a)^2 + (2 - b)^2 = r^2 \end{cases}$$

Equating equations (i) and (ii) gives

$(3 - a)^2 + (1 - b)^2 = (2 - a)^2 + (6 - b)^2$

$$\Rightarrow a^2-6a+9+b^2-2b+1=a^2-4a+4+b^2-12b+36$$

$$-2a+10b=30$$

$$a-5b=-15 \text{ ....(iv)}$$

Equating equations (ii) and (iii) gives

$$(2-a)^2+(6-b)^2=(3-a)^2+(2-b)^2$$

$$\Rightarrow a^2-4a+4+b^2-12b+36=a^2-6a+9+b^2-4b+4$$

$$2a-8b=-27 \text{ ....(v)}$$

Solve equations (iv) and (v) simultaneously to get, $a = -7.5$, $b = 1.5$.

Thus, the centre of the circle is (a, b) = (–7.5, 1.5). The radius of the circle is computed by substituting the values of a and b in one of the equations in the system of simultaneous equations. Using equation (i),

$(3-a)^2+(1-b)^2=r^2$, then,

$(3-(-7.5))^2+(1-1.5)^2=r^2$.

$$\Rightarrow\left(3+\frac{15}{2}\right)^2+\left(1-\frac{3}{2}\right)^2=r^2$$

$$\Rightarrow r^2=\frac{221}{2}$$

$$\Rightarrow r=\sqrt{\frac{221}{2}}$$

Thus, the circle has centre at $\left(-\frac{15}{2},\frac{3}{2}\right)$ and the radius, $r=\sqrt{\frac{221}{2}}$.

The equation of the circle is given by

$$\left(x+\frac{15}{2}\right)^2+\left(y-\frac{3}{2}\right)^2=\frac{221}{2}$$

$$\Rightarrow x^2+15x+\frac{225}{4}+y^2-3y+\frac{9}{4}=\frac{221}{2}$$

Thus, $x^2+y^2+15x-3y-52=0$.

Therefore, the equation of the circle is $x^2+y^2+15x-3y-52=0$.

Exercise 4.9

1. Find the equation of the circle with $\overline{AB}$ as a diameter when the coordinates of A and B are respectively:
   (a) $(-2,3)$ and $(-5,2)$
   (b) $(6,1)$ and $(-2,-5)$
   (c) $(-1,1)$ and $(0,-3)$
   (d) $(8,-3)$ and $(-7,-1)$
   (e) $(0,3)$ and $(-4,0)$
2. Find the equation of a circle which passes through the following points:
   (a) $(-1,-5)$, $(6,2)$, and $(-2,-2)$
   (b) $(1,2)$, $(0,-6)$, and $(4,0)$
   (c) $(-5,-10)$, $(-6,-5)$, and $(12,7)$
   (d) (1, 1), (1, 7), and (8, 8)
   (e) $(1,-5)$, $(1,2)$, and $(0,-2)$

3. A triangle has vertices (0, 6), (4, 0), and (6, 0). Find the equation of the circle passing through the three points.

Equation of a tangent to a circle

A tangent to a circle is a straight line which touches the circle at only one point. The point where it touches the circle is called the point of tangency. The line tangent to a circle is always perpendicular to the radius at the point of tangency as shown in Figure 4.16.

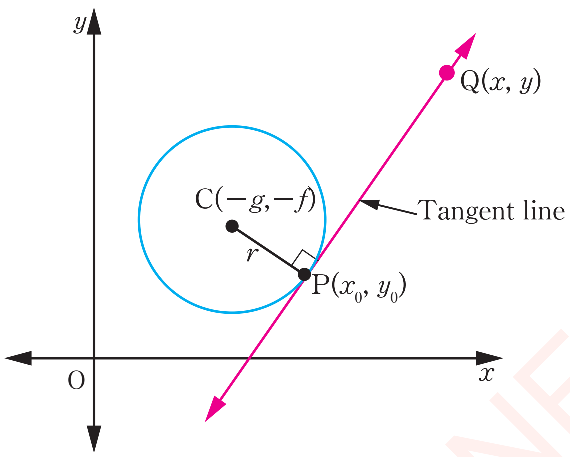


Figure 4.16: *Tangent line to a circle*

In Figure 4.16, C($-g, -f$) is the centre of the circle, $P(x_0, y_0)$ is the point of tangency and Q(x, y) is any other point on the tangent line.

The gradient from the centre to the tangency is given by $\overline{CP} = \dfrac{y_0+f}{x_0+g}$.

The tangent line is always perpendicular to the line drawn from the point of tangency to the centre of the circle (radius). The slope (gradient) of the tangent line equals the negative reciprocal of the slope of $\overline{CP}$. Therefore, the slope of the tangent line is $-\left(\dfrac{x_0+g}{y_0+f}\right)$.

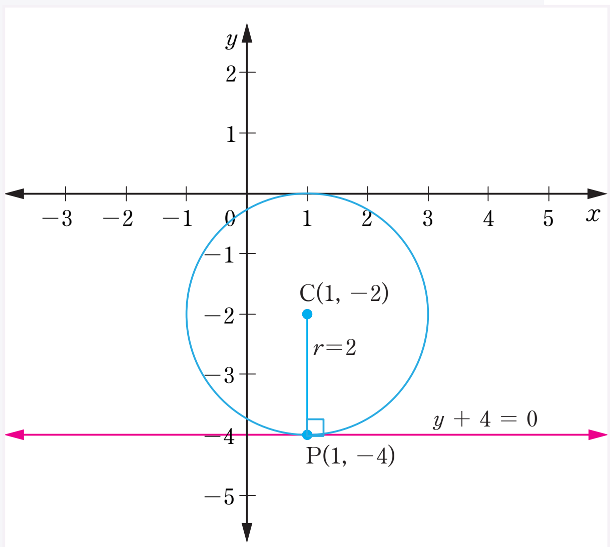
Thus, the equation of the tangent line is given by $\dfrac{y-y_0}{x-x_0} = -\left(\dfrac{x_0+g}{y_0+f}\right)$.

Example 4.23

Find the equation of a tangent to the circle $x^2 + y^2 - 2x + 4y + 1 = 0$ at the point P(1, – 4).

Solution

Comparing the given equation $x^2 + y^2 - 2x + 4y + 1 = 0$ with the general equation of the circle, the centre of the circle is $(-g, -f) = (1, -2)$ and the radius of the circle is $r = 2$. Consider the following figure:



From the previous Figure 4.16, the slope of $\overline{\text{CP}}$ is $m_1 = \dfrac{-4-(-2)}{1-1} = \infty$.

Undefined slope denotes that the line is parallel to the y-axis. Thus, the slope of the tangent is parallel to the x-axis, that is $m_2 = 0$

Thus, slope of the tangent, $m_2 = -\dfrac{1}{\infty} = 0.$

If Q(x, y) is any other point on the tangent, then

$$m_2 = \frac{y-(-4)}{x-1} \Rightarrow 0 = \frac{y+4}{x-1}.$$

Therefore, the equation of the tangent line is $y + 4 = 0$.

Example 4.24

Show that if $y = mx + c$ is a tangent to the circle $x^2 + y^2 = a^2$, then

$$a = \left|\frac{c}{\sqrt{1+m^2}}\right|.$$

Solution

Let $y = mx + c$ be the tangent to the circle $x^2 + y^2 = a^2$. But the tangent line satisfies the equation of the circle.
Thus, substituting $y = mx + c$ in the given circle, to obtain

$$x^2 + (mx+c)^2 = a^2.$$

$$\Rightarrow x^2 + m^2x^2 + 2mcx + c^2 = a^2$$

$\Rightarrow (1+m^2)x^2 + 2mcx + c^2 - a^2 = 0$, which is a quadratic equation.

Recall the general formula for solving a quadratic equation $\text{A}x^2 + \text{B}x + \text{C} = 0$. That is, $x = \dfrac{-\text{B} \pm \sqrt{\text{B}^2 - 4\text{AC}}}{2\text{A}}$.

$\Rightarrow \text{A} = 1 + m^2$, $\text{B} = 2mc$, and $\text{C} = c^2 - a^2$

Substituting the values of A, B, and C into the general quadratic formula gives,

$$x = \frac{-2mc \pm \sqrt{(2mc)^2 - 4(1+m^2)(c^2 - a^2)}}{2(1+m^2)}$$

$$=-\frac{2mc}{2(1+m^2)}\pm\frac{\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}}{2(1+m^2)}$$

Since the line is tangent to the circle, then it touches the circle, and hence the discriminant is zero.

Thus, $\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}=0$.

Squaring both sides, gives

$$\left(\sqrt{(2mc)^2-4(1+m^2)(c^2-a^2)}\right)^2=(0)^2$$

$$\Rightarrow 4m^2c^2-4(1+m^2)(c^2-a^2)=0$$

$$\Rightarrow m^2c^2-(1+m^2)(c^2-a^2)=0$$

$$\Rightarrow m^2c^2-(c^2+m^2c^2-a^2-a^2m^2)=0$$

$$\Rightarrow -c^2+a^2+a^2m^2=0$$

$$\Rightarrow c^2=a^2(1+m^2)$$

Thus, $a^2=\dfrac{c^2}{1+m^2}$

Take the square root both sides to get, $a=\pm\dfrac{c}{\sqrt{(1+m^2)}}$.

Therefore, $a=\left|\dfrac{c}{\sqrt{1+m^2}}\right|$.

Example 4.25

Find the equations of the tangents to the circle $x^2+y^2-6x+4y=12$ which are parallel to the straight line $4x+3y+5=0$.

Solution

Any straight line parallel to the given equation is in the form of $4x+3y+c=0$, where c is a constant.

The equation of the circle is given as

$$x^2+y^2-6x+4y=12$$

$$\Rightarrow (x-3)^2+(y+2)^2=5^2$$

Thus, the centre of the circle is $(3,-2)$ and the radius is 5.

For the straight line to be tangent to the circle, its distance from the point $(3,-2)$ must be equal to ± 5.

Hence, substituting $(x_1, y_1) = (3,-2)$, $(a,b) = (4,3)$, and $r = \pm 5$ in the perpendicular distance formula,

$$r = \frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}.$$

$$\Rightarrow \pm 5 = \frac{12 - 6 + c}{\sqrt{4^2 + 3^2}}$$

so that

$c = -6 \pm 25 \Rightarrow c = 19$ or $c = -31$.

Therefore, the equations of the tangents are $4x + 3y + 19 = 0$ and $4x + 3y - 31 = 0$.

Equation of a normal to a circle

The normal to a circle is a straight line drawn at 90° to the tangent at the point where the tangent touches the circle. On the other hand, the normal is referred as a perpendicular line to the tangent at the point of contact to the circle as shown in Figure 4.17.

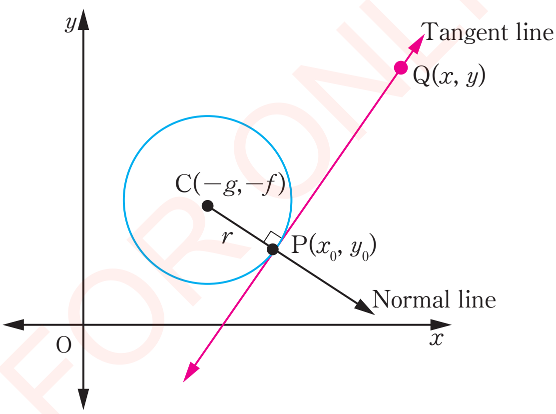


Figure 4.17: *Normal line to a circle*

From Figure 4.17, it can be observed that the slope of the normal is the same as the slope of $\overline{CP}$, that is, the slope of the line joining the centre of the circle and the given point of tangency. In Figure 4.17, $P(x_0, y_0)$ is the point of tangency on the circle and $Q(x, y)$ is any other point on the line normal to the circle.

Now, the slope of $\overline{CP} = \dfrac{y_0 + f}{x_0 + g}$.

Since the slope of the normal is equal to the slope of $\overline{CP}$, then the equation of the normal is given as $\dfrac{y - y_0}{x - x_0} = \dfrac{y_0 + f}{x_0 + g}$.

Example 4.26

Find the equation of the normal to the circle $x^2 + y^2 - 6x + 4y = 12$ at the point $(-1, -5)$.

Solution

Compare the equation of a circle $x^2 + y^2 - 6x + 4y = 12$ with the general equation of the circle. The centre of the circle is $(-g, -f) = (3, -2)$ and radius, $r = 5$.

Slope of the normal equation is given by, $m = \dfrac{-5+2}{-1-3} = \dfrac{3}{4}$.

Suppose (x, y) is any other point on the normal, then

$$m = \frac{y+5}{x+1}$$

Thus, $\frac{y+5}{x+1}=\frac{3}{4}$

$\Rightarrow -3x+4y+17=0$

Therefore, the equation of the normal is $-3x+4y+17=0$

Exercise 4.10

1. Find the equation of the tangent and the normal to each of the following circles at the point given:
   (a) $8x^2+8y^2=5, \left(\frac{3}{4},-\frac{1}{4}\right)$
   (b) $x^2+y^2=25, (-3,4)$
   (c) $(x-1)^2+(y+2)^2=5, (3,-3)$
   (d) $4x^2+4y^2-4x+8y-15=0, \left(1\frac{1}{2},1\right)$
   (e) $x^2+y^2-6x-2y-3=0, (5,4)$
   (f) $x^2+y^2+4x+6y=0, (0,0)$
2. A tangent to the circle $x^2+y^2=5$ at the point $(1, -2)$ also touches the circle $x^2+y^2-8x+6y+20=0$. Find the coordinates of the corresponding point of contact.
3. Find the values of k such that the line $3x+4y=k$ touches the circle $x^2+y^2=10x$.
4. Find the equation of the normal to the circle $2x^2+2y^2-2x-5y+3=0$ at the point (1, 1).
5. Show that the tangents to the circle $x^2+y^2+6x-2y+5=0$ at the points $P(-1, 2)$ and $Q(-2,-1)$, respectively meet at the origin O.
6. Show that the tangent to the circle $x^2+y^2-6x+2y+5=0$ at the point $(1, 0)$ touches the circle $5x^2+5y^2=4$.
7. Find the equation of the diameter of the circle $x^2+y^2-4x-6y+12=0$ that is parallel to the line $x+y=7$.
8. Find the equations of the tangents to the circle $x^2+y^2+4x-2y-24=0$ at the points where the circle crosses the line $y-x=0$.
9. Find the condition for line $y=mx+c$ to touch the circle $(x-a)^2+(y-b)^2=r^2$. Hence, find the point of contact.
10. Find the equations of the tangents to the circle $x^2+y^2=25$ which are parallel to the line $4x-3y-2=0$.
11. Find the equations of the tangents to the circle $x^2+y^2-6x+10y+29=0$ which are parallel to the line $2x+y+8=0$.
12. Show that the line $y=2x$ is tangent to the circle $x^2+y^2-8x-y+5=0$.
13. Find the equations of the tangents to the circle $x^2+y^2-6x+4y+8=0$ which are perpendicular to the line $2x-y-1=0$.

14. The tangent to the circle $x^2+y^2-2x-6y+5=0$ at the point $(3,\ 4)$ meets the x-axis at M. Find the distance of M from the centre of the circle.

15. Find the equation of the circle centred at $D(-3,-5)$ and that touches the line $12x+5y-4=0$.

Point of intersection of circles

There are three cases in which two circles can intersect. The first case is when two circles touch each other externally, the second case is when two circles of different sizes touch each other internally and the third case is when two circles intersect at two distinct points.

(i) External point of intersection of circles

Two circles touch each other externally if and only if the distance between their centres is equal to the sum of their radii. Consider two circles touching each other externally as shown in Figure 4.18.

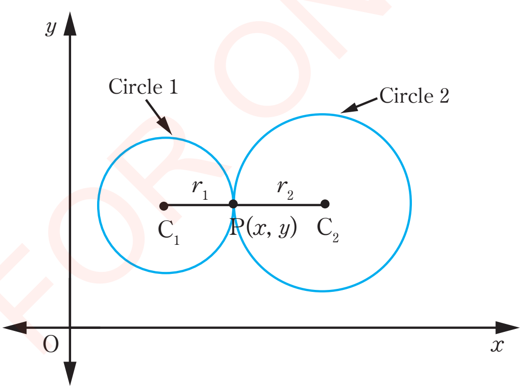


Figure 4.18: *Two circles intersecting externally*

Suppose $C_1(a_1,b_1)$ and $C_2(a_2,b_2)$ are the coordinates of the centres of the circles $C_1:(x-a_1)^2+(y-b_1)^2=r_1^2$ and $C_2:(x-a_2)^2+(y-b_2)^2=r_2^2$, where, r_1 and r_2 are radii of the circles.

From Figure 4.18, the distance between the centres of the circles C_1 and C_2 is given by

$$\overline{C_1C_2}=\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}.$$

The distance $\overline{C_1C_2}$ is equal to the sum of their radii which is given by $r=r_1+r_2$.

Thus, $\sqrt{(a_2-a_1)^2+(b_2-b_1)^2}=r_1+r_2$.
Therefore, the condition for two circles to intersect each other externally at one point is given by;

$$\overline{C_1C_2}=r_1+r_2.$$

Example 4.27

Show that the circles whose equations are

$x^2+y^2-4x-6y-3=0$ and

$x^2+y^2-12x-12y+71=0$,

intersect externally at one point.

Solution

If the two circles intersect externally, then they must satisfy the equation

$\overline{C_1C_2}=r_1+r_2$.

Given that,

$C_1:x^2+y^2-4x-6y-3=0$,

Its centre is C_1 (2, 3) and its radius is $r_1 = 4$ units

$C_2 : x^2 + y^2 - 12x - 12y + 71 = 0$,

Its centre is C_2 (6, 6) and its radius is $r_2 = 1$ unit

Thus, the distance between the centres,

$$\overline{C_1C_2} = \sqrt{(6-2)^2 + (6-3)^2}$$
$$= \sqrt{25}$$
$$= 5$$

Sum of the radii, $r_1 + r_2 = 4 + 1 = 5$

Thus, $\overline{C_1C_2} = r_1 + r_2 = 5$.

Therefore, the two circles intersect externally at one point because the distance between their centres is equal to the sum of their radii.

(ii) Internal point of intersection of the circles

Two circles touch each other internally if and only if the distance between their centres is equal to the difference between their radii. Consider two circles with different sizes intersecting each other internally as shown in Figure 4.19.

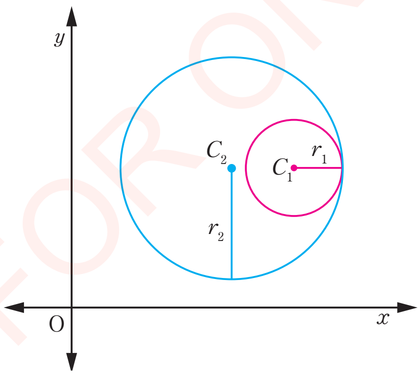


Figure 4.19: *Two circles intersecting internally*

In Figure 4.19, the distance between the centres of the circles $C_1(a_1, b_1)$ and $C_2(a_2, b_2)$ is given by $\overline{C_1C_2} = \sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2}$ (4.21)

Since $r_2 > r_1$, then the difference between their radii is given by $r = r_2 - r_1$.

Thus, $\sqrt{(a_2 - a_1)^2 + (b_2 - b_1)^2} = r_2 - r_1$ (4.22)

Equating equations (4.21) and (4.22) gives,

$$\overline{C_1C_2} = r_2 - r_1$$

Therefore, the condition for two circles to intersect each other internally at one point is given by; $\overline{C_1C_2} = r_2 - r_1$.

Example 4.28

Show that the two circles whose equations are $x^2 + y^2 + 2x - 8 = 0$ and $x^2 + y^2 - 6x + 6y - 46 = 0$ intersect internally.

Solution

The two circles intersect internally if they satisfy the equation $\overline{C_1C_2} = r_2 - r_1$

$C_1 : x^2 + y^2 + 2x - 8 = 0$,

centre is $C_1(-1, 0)$ and its radius is $r_1 = 3$ units.

$C_2 : x^2 + y^2 - 6x + 6y - 46 = 0$, its centre is $C_2(3, -3)$ and its radius is $r_2 = 8$ units.

The distance between the centres is

$$\overline{C_1C_2} = \sqrt{(3-(-1))^2 + ((-3)-0)^2}$$
$$= \sqrt{25}$$
$$= 5$$

The difference of the radii,

$r_2 - r_1 = 8 - 3 = 5$

Thus, $\overline{C_1C_2} = r_2 - r_1 = 5$.

Therefore, the circles intersect internally at one point because the distance between their centres is equal to the difference between their radii.

(iii) Intersection of two circles at two distinct points

The intersection of two circles at distinct points occurs when the circles overlap each other and meet at two distinct points A and B which satisfy both equations of circles as shown in Figure 4.20.

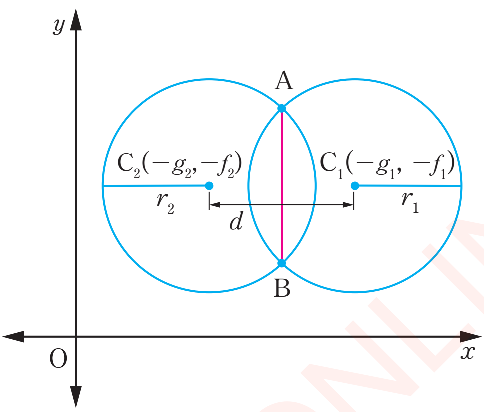


Figure 4.20: *Circles intersecting at two distinct points*

In Figure 4.20, the condition for the two circles to intersect at two distinct point is given by

$\overline{C_1C_2} < r_1 + r_2$

The line segment joining $\overline{AB}$ is called a common chord. The equation of a common chord is a straight line obtained by solving the two equations of the circles simultaneously. The common chord of the circles $C_1 : x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $C_2 : x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is obtained by solving the two equations simultaneously. That is,

$$-\begin{cases} x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \\ x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \end{cases}$$

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

Therefore, the equation of the common chord of the two circles is given by

$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$

Example 4.29

Find the equation of common chord of the two circles $x^2 + y^2 - 2y - 4 = 0$ and $x^2 + y^2 - x + y - 12 = 0$. Hence, find the two distinct points of intersection.

Solution

Given two circles $x^2 + y^2 - 2y - 4 = 0$ and $x^2 + y^2 - x + y - 12 = 0$.

Solve the two equations simultaneously as follows:

$$-\begin{cases} x^2 + y^2 + 0x - 2y - 4 = 0 \\ x^2 + y^2 - x + y - 12 = 0 \end{cases}$$

$$x - 3y + 8 = 0$$

Therefore, the equation of the common chord is $x - 3y + 8 = 0$.

The points of intersection are obtained by solving simultaneously the equation of the common chord with one of the equations of circles. In this case, take the equation

$x^2+y^2-2y-4=0$, then substitute the equation of a common chord into the selected equation of a circle as follows:

$(3y-8)^2+y^2-2y-4=0$

$\Rightarrow y^2-5y+6=0$

$\Rightarrow y=2$ or $y=3$.

Substitute back the values of y in the equation of the common chord to get the values of x. That is, when $y=2 \Rightarrow x=-2$ and when $y=3 \Rightarrow x=1$.

Therefore, the points of intersection of the circles are $(-2, 2)$ and $(1, 3)$.

Exercise 4.11

1. Show that the circles
   $x^2+y^2-2x-2y-7=0$ and
   $x^2+y^2-10x-8y+37=0$
   intersect each other externally.
2. Show that the circles
   $x^2+y^2-4x-2y=0$ and
   $x^2+y^2-8x-10y-4=0$ touch each other internally.
3. Find the equation of the common chord of the circles $x^2+y^2=4$ and $x^2+y^2-4x-2y-4=0$.
4. Find the coordinates of the point where the common chord of the circles $x^2+y^2-4x-8y-5=0$ and $x^2+y^2-2x-4y-5=0$ meets the line joining their centres.
5. Find the points of intersection of the circles given by the equations $x^2+y^2-4x-6y=0$ and $(x-1)^2+(y+1)^2=16$.
6. Find the equation of the circle which passes the points $(-2, 2)$, $(2, 4)$, and $(5, -5)$. Show that this circle touches the circle $2x^2+2y^2-17x+16y+65=0$ and find the coordinates of the point of contact.

Orthogonal circles

Two circles are said to be orthogonal if and only if they intersect in such a way that the tangents at the points of intersection are perpendicular, as shown in Figure 4.21.

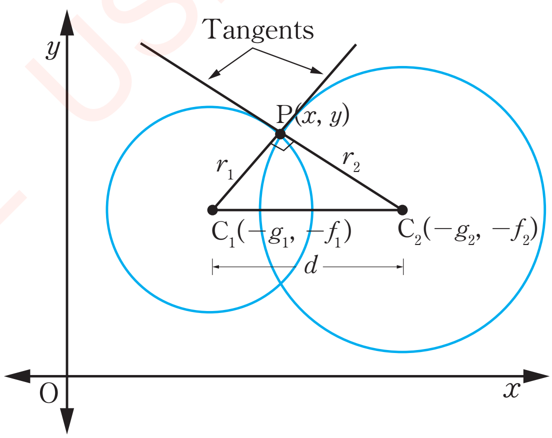


Figure 4.21: *Orthogonal circles*

In Figure 4.21, $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$ are the centres of the circles and $P(x, y)$ is a point of intersection. Let r_1 and r_2 be the radii of the circles with centres C_1 and C_2, respectively, and d be the distance between the centres of the circles.

For the circles to be orthogonal, triangle C_1PC_2 must be a right-angled triangle at P. Therefore, from Pythagoras' theorem, $d^2 = r_1^2 + r_2^2$.

That is, the square of the distance between the centres must be equal to the sum of the squares of their radii. Also, the tangent from C_2 to the other circle is equal to the radius, R. That is, if two circles are orthogonal, the length of the tangent drawn from the centre of first circle to the second circle is equal to the radius of the first circle.

Example 4.30

Prove that the two circles $x^2 + y^2 - 8x - 14y + 56 = 0$ and $x^2 + y^2 - 4x - 8y + 16 = 0$ are orthogonal.

Solution

The two circles are orthogonal if they satisfy the equation $d^2 = r_1^2 + r_2^2$.

$C_1 : x^2 + y^2 - 8x - 14y + 56 = 0$

Its centre is C_1 (4, 7) and its radius is $r_1 = 3$ units.

Also,

$C_2 : x^2 + y^2 - 4x - 8y + 16 = 0$

Its centre is C_2 (2, 4) and its radius is $r_2 = 2$ units

Distance between the centres,

$d^2 = (g_1 - g_2)^2 + (f_1 - f_2)^2$

$d^2 = (4-2)^2 + (7-4)^2$

$d^2 = 2^2 + 3^2$

$= 13$

Thus, $d^2 = 13$.

But $r_1^2 + r_2^2 = 3^2 + 2^2$

$= 9 + 4$

$= 13$

Hence, $d^2 = r_1^2 + r_2^2 = 13$.

Therefore, the circles are orthogonal.

Exercise 4.12

1. Determine whether or not the following circles are orthogonal:
   (a) $2x^2 + 2y^2 - 12x + 8y + 4 = 0$ and $2x^2 + 2y^2 + 16x + 4y - 44 = 0$
   (b) $x^2 + y^2 - 8x + 6y - 23 = 0$ and $x^2 + y^2 - 2x - 5y - 16 = 0$
   (c) $x^2 + y^2 - 8x + 6y + 21 = 0$ and $x^2 + y^2 - 2y - 15 = 0$
   (d) $x^2 + y^2 - 2x + 8y + 1 = 0$ and $x^2 + y^2 + 2y - 9 = 0$
   (e) $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 + 4y - 3 = 0$
2. Find the equation of the circle which passes through the point $(1, 2)$ and cuts orthogonally each of the circles $x^2 + y^2 - 2x + 8y - 7 = 0$ and $x^2 + y^2 = 9$.
3. For each of the following find the values of k if the circles:

(a) $x^2+y^2-2x+22y+5=0$ and $x^2+y^2+14x+6y+k=0$ intersect at right angles.

(b) $x^2+y^2+4x+6y+4=0$ and $x^2+y^2-2x+14y+k=0$ intersect orthogonally.

4. Find the equation of the circle which intersect orthogonally the circles $x^2+y^2-4x-6y+11=0$ and $x^2+y^2-10x-4y+21=0$ and has its centre on $2x+3y-7=0$.

Length of a tangent from a point to a circle

Consider a circle whose equation is $x^2+y^2+2gx+2fy+c=0$. The circle is centred at $(-g,-f)$ and has a radius of $\sqrt{g^2+f^2-c}$. Let $P(x_1, y_1)$ be a point outside the circle as shown in Figure 4.22.

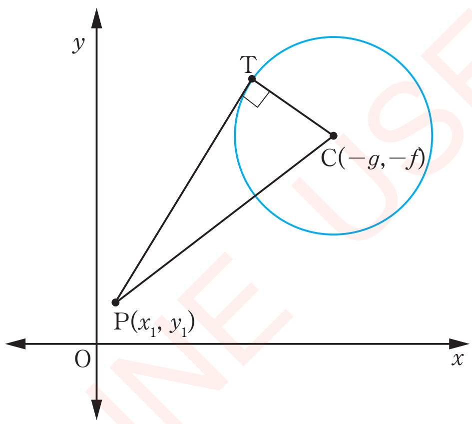


Figure 4.22: *Length of a tangent from a point to a circle*

The tangent line meets at right angle with the line passing through the centre at the point of contact.

Using the Pythagoras' theorem, it follows that $\overline{CP}^2=\overline{CT}^2+\overline{TP}^2$.

This implies that, $\overline{TP}^2=\overline{CP}^2-\overline{CT}^2$ (4.23)

But, $\overline{CT}$ is the radius of the circle, which is given by

$$\overline{CT}=\sqrt{g^2+f^2-c} \Rightarrow \overline{CT}^2=g^2+f^2-c \quad (4.24)$$

Applying the distance formula,

$$\overline{CP} = \sqrt{(x_1+g)^2+(y_1+f)^2} \Rightarrow \overline{CP}^2 = (x_1+g)^2+(y_1+f)^2 \quad (4.25)$$

Substituting equations (4.24) and (4.25) in equation (4.23) gives,

$$\overline{TP}^2 = (x_1+g)^2+(y_1+f)^2-(g^2+f^2-c)$$

$$= x_1^2+2gx_1+g^2+y_1^2+2fy_1+f^2-g^2-f^2+c$$

$$= x_1^2+y_1^2+2gx_1+2fy_1+c$$

Thus, $\overline{TP} = \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$.

Therefore, the length of a tangent from point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is given by,

$\sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$ units.

Example 4.31

Find the length of a tangent from the point $P(2,5)$ to the circle $x^2+y^2-2x-3y-1=0$.

Solution

Given the equation of a circle with the corresponding values $P(x_1, y_1)=(2,5)$, $g=-1, f=-\frac{3}{2}$, and $c=-1$.

Length of tangent $= \sqrt{x_1^2+y_1^2+2gx_1+2fy_1+c}$, substituting the values gives,

$$= \sqrt{2^2+5^2+2(-1)(2)+2\left(-\frac{3}{2}\right)(5)-1}$$

$$= 3 \text{ units}$$

Therefore, the length of a tangent is 3 units.

Exercise 4.13

1. Find the length of the tangent from the point (12, –9) to the circle $3x^2+3y^2-7x+22y+9=0$.
2. A tangent is drawn to a circle of radius 5 cm from a point 8 cm away from the end point of the radius on the circumference of the circle. Find the length of the tangent.

3. What is the length of a tangent from the point P(4, 2) to the circle with centre (0, 0) and radius of 4 units?
4. Find the length of a tangent to the circle $x^2+y^2-2x+4y-3=0$, from the centre of the circle $x^2+y^2+6x+8y-1=0$.
5. Compute the length of the tangent from the point $P(9, 8)$ to the circle which passes through the points $\mathrm{E}(5, 7)$, $\mathrm{F}(-2, 6)$, and $\mathrm{G}(6, 0)$.
6. Show that the length of the tangents from the point $(0, 5)$ to the circles $x^2+y^2+2x-4=0$ and $x^2+y^2-y+1=0$ are equal.
7. Find the equation of the circle such that the lengths of the tangents from the points $(-1, 0)$, $(0, 2)$, and $(-2, 1)$ are respectively 3, $\sqrt{10}$, and $3\sqrt{3}$ units.

Chapter summary

1. A parallelogram is a quadrilateral which obey the properties:
   (i) Opposite sides are parallel to each other
   (ii) The opposite line segments have equal length
   (iii) The opposite interior angles are equal
   (iv) The diagonals bisect each other.
2. The area of a rectangle whose vertices are (x_1, y_1), (x_2, y_1), (x_2, y_2), and (x_1, y_2) is defined by the formula, $\mathrm{A}=\left(x_2y_2-x_1y_2-x_2y_1+x_1y_1\right)$.
3. If L_1 and L_2 are two intersecting lines, and θ is one of the angle between the lines, then the other angle between the lines is $180°-\theta$, where $\theta=\tan^{-1}\left(\dfrac{m_2-m_1}{1+m_1m_2}\right)$.
4. The perpendicular distance of point $\mathrm{P}\left(x_1, y_1\right)$ from a line $ax+by+c=0$ is given by $\dfrac{|ax+by+c|}{\sqrt{a^2+b^2}}$.
5. The equations to the bisectors of the angles between the two lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are $\dfrac{a_1+b_1y+c_1}{\sqrt{a_1^2}}=\pm\left(\dfrac{a_2+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}\right)$.
6. The coordinates of the point $\mathrm{P}(x, y)$ which divides internally and externally the line joining the points $\mathrm{A}\left(x_1, y_1\right)$ and $\mathrm{B}\left(x_2, y_2\right)$ in the ratio $m:n$ are $\left(\dfrac{mx_2+nx_1}{m+n}, \dfrac{my_2+ny_1}{m+n}\right)$ and $\left(\dfrac{nx_1-mx_2}{n-m}, \dfrac{ny_1-my_2}{n-m}\right)$, respectively.

7. A circle is the locus of a point which moves so that its distance from a fixed point is always constant.
8. The equations of the tangent and normal to the circle $x^2+y^2+2gx+2fy+c=0$ at the point of the tangency $P(x_0, y_0)$ are $\frac{y-y_0}{x-x_0}=-\left(\frac{x_0+g}{y_0+f}\right)$ and $\frac{y-y_0}{x-x_0}=\frac{y_0+f}{x_0+g}$, respectively.
9. Two circles are said to be orthogonal if and only if they intersect in such a way that the tangents at the points of intersection are perpendicular.
10. A circle whose equation is $x^2+y^2+2gx+2fy+c=0$ is centred at $(-g, -f)$ and has a radius of $\sqrt{g^2+f^2-c}$.
11. If the circles $x^2+y^2+2g_1x+2f_1y+c_1=0$ and $x^2+y^2+2g_2x+2f_2y+c_2=0$ intersect orthogonally, then $d^2=r_1^2+r_2^2$ or $2g_1g_2+2f_1f_2=c_1+c_2$.
12. The equation of the common cord to the two circles c_1 and c_2 is defined by $2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$.
13. The common chord of two circles c_1 and c_2 is a line passing through the point of intersection of the two circles.

Revision exercise 4

1. Find the value of k if the circles $x^2+y^2+2x-4y+k=0$ and $x^2+y^2-6x+4y-19=0$ cut orthogonally.
2. A triangle has its vertices at points A(1, 7), B(5, 5), and C(8, –1). The point M is the midpoint of $\overline{AB}$, L is the point on $\overline{BC}$ such that $\overline{BL}:\overline{LC}=1:2$, and N is the point on $\overline{AC}$, such that $\overline{AN}:\overline{NC}=3:1$ Find the coordinates of L, M, and N.
3. Find the area of the triangle formed by the three lines $3x+4y=30$, $2y=x$, and $y=3x$
4. Show that the bisector of the acute angle between $y=x+1$ and the x-axis has the gradient of $-1+\sqrt{2}$.
5. Find the equation of the circle which intersects orthogonally the circles $x^2+y^2+2x-2y-2=0$ and $x^2+y^2+6x-4y+4=0$ and has the centre on the line $4x-3y+3=0$.
6. If the perpendicular distance of the point $(m, 3)$ from the line $x-2y+1=0$ is $\frac{3\sqrt{5}}{5}$ units, Find the value of m.
7. Find the coordinates of the line segment joining points $(-2, 3)$ and $(4, 6)$ in the ratio of 3:2 internally and externally division.

8. Find the coordinates of the point that divides the line joining the points (3, 4) and (4, 2):
   (a) Internally in the ratio 3:2.
   (b) Externally in the ratio 3:2.
9. Determine the equation to the bisector of the acute angle between the lines $3x+4y=1$ and $5x-12y+6=0$.
10. Show that the circles
    $x^2+y^2-16x+12y+75=0$ and
    $5x^2+5y^2-32x-24y+75=0$
    touch each other. Hence, find the equation of the common chord and the point of contact.
11. Find the points L and M which divide the line joining (1, 2) and (10, 8) into three equal parts.
12. Calculate the area of the triangle with vertices $(-1, 3)$, $(5, 2)$, and $(4, -1)$.
13. Find the area of a figure formed between the line $3y+2x=6$ and the axes.
14. The vertices of a parallelogram are $\mathrm{P}(-2, 6)$ $\mathrm{Q}(1, 2)$, $\mathrm{R}(10, 4)$, and $\mathrm{S}(7, 8)$
    (a) Show that PQRS is a parallelogram
    (b) Find the equations of the diagonal.
    (c) Find the area of the parallelogram
15. Find the distance of the point of intersection of the line $2x + 3y = 21$ and $3x - 4y + 11 = 0$ from the line $8x + 6y + 5 = 0$.
16. Find the angle between a pair of lines whose equation is $4x^2-24xy+11y^2=0$.
17. Find the equations of a line passing through the point (3, 2) and makes an angle of 45° with the line $y+3x-7=0$.
18. Determine the equations of two orthogonal lines through the point (3, 2), which make an angle of 45° with the line $-2x+y-1=0$.
19. Find an acute angle between the straight lines in $2x^2+5xy-12y^2=0$.
20. Four points (–2, –1), (4, 3), (a, b), and (0, – 4) are the vertices of a parallelogram. Find the values of a and b.
21. Find the equation of the circle which is circumscribed about the triangle whose vertices are $(-2, 3)$, $(5, 2)$, and $(0, -1)$.
22. Prove that A(2, 4), B(5, 3), C(2, 2), and D(–1, 3) are the vertices of a parallelogram.
23. Find the equation of the circle whose centre is $(2, -3)$ and passing through the point $(3, -5)$.
24. A line from a point $\mathrm{A}(1, 2)$ intersects the straight line $2y+3x-14=0$ at point B. If the perpendicular line is extended to point C in such a way that $\overline{\mathrm{AB}}=\frac{1}{2}\overline{\mathrm{BC}}$, determine the coordinates of point C.
25. If $\mathrm{A}=(9,13)$ and $\mathrm{C}=(4,-2)$, find the coordinates of point B given that $\overline{\mathrm{AB}}=\frac{3}{2}\overline{\mathrm{BC}}$.

26. Find the equation of the circle whose diameters are $x+y=2$ and $x-y=0$ and radius is 1 unit.

27. Find the equations of bisectors of the angles between the lines $3x+4y=7$ and $y-1=0$.

28. If a point is equidistant from (2, 1) and (–7, –5), find its locus.

29. Find the locus of P, such that it is equidistant from (3, 1) and the line $3x-4y+1=0$.

30. What is the locus of the point which moves in such a way that it is equidistant from point (2, 3) and the line $y=4$?

31. Determine the equation of the circle which passes through the points A (1, 1), B(1, 7), and C(8, 8).

32. Show the line $y=x+1$ touches the circle $x^2+y^2-4x+6y-13=0$.

33. Find the equation of the circle, which has its centre on the line $x+y=0$ and passes through the points of intersection of the two circles $(x-1)^2+(y+5)^2=50$ and $(x+1)^2+(y+1)^2=10$.

34. Find the equation of the circle joining A(0, 3) and B(4, 5) as a diameter.

35. Which values of a and b will make the equation $ax^2+2bxy+2y^2+8x+12y=0$ to be a circle?

36. Show that the line $y=2x$ touches the circle $x^2+y^2-8x-y+5=0$. Hence, find the coordinates of point of the contact.

37. Determine the equation of a circle of radius 5 units, which touches the x-axis and passes through the point (3, 1).

38. Find the centre and the radius of a circle given by $x^2+y^2+4x-8y+4=0$. Use the circle to find the length of the tangent from the point P(3, 8).

39. A circle centred at the point (2, 3) touches the line joining the points (0, 4) and (3, 1). Find the equation of the circle.

40. A line $2x+2y-3=0$ touches the circle $4x^2+4y^2+8x+4y-13=0$ at A. Find the equation of the line joining A to the origin.

41. Show that the tangents to the circle $x^2+y^2=169$ at (5, 12) and $(12,-5)$ are perpendicular.

42. Find the value of c so that the circles $x^2+y^2-6x+4y+12=0$ and $x^2+y^2+8x+2y+c=0$ are orthogonal.

43. Find the equations of the lines that pass through the point (3, 1) and tangent to the circle whose equation is $x^2+y^2-2=0$.

44. Find the equation of the tangent and normal to the circle $x^2+y^2-10x+4y-140=0$ at the point $(-7,-7)$.

45. Find the equations of the tangents from the origin to the circle whose equation is $x^2+y^2-8x-y+5=0$.

46. What is the length of the tangent from a point (8, 4) to the circle with centre (3, 0) and radius of 2 units?

47. Find the length of the tangents to the circle $2x^2+2y^2-4x+8y-6=0$, from the centre of the circle $3x^2+3y^2+18x+24y-3=0$.

48. Show that the line $y=x+1$ touches the circle $x^2+y^2-8x-2y+9=0$

49. Find the equation of the tangent to the circle $x^2+y^2-2x+y-5=0$ at the points $(3,-2)$. If this tangent crosses the axes at A and B, calculate the area of triangle OAB, where O is the origins.

50. Show that the line $y=px+q$ touches the circle $x^2+y^2=r^2$ if $q^2=r^2(1+p^2)$.

Chapter Five

Functions

Introduction

In mathematics, a function is a relation between a set of inputs and a set of outputs according to a certain rule. Functions have the property that each input is related to exactly one output. The set of inputs is called the domain, whereas the set of outputs is called the range. In this chapter, you will learn how to plot graphs of functions. The competencies developed will help you in designing machines, making predictions, studying growth relations, formulating mathematical models, developing computer programs, and in solving many other problems.

Graphs of functions

A function can be represented graphically. A function has an independent variable and a dependent variable. Normally, x is the independent variable and y is the dependent variable. To draw a graph of a function, a common method is to choose some values for the independent variable x, substitute them into the function to get a set of ordered pair (x, y). Tabulate the values, and locate the point on the xy-plane, then, connect the points to obtain the graph. Normally, using more points gives a better graph. In this section, you will learn how to plot graphs of polynomial functions up to 4^{th} degree, rational functions, composite functions, exponential functions, and logarithmic functions.

Activity 5.1: Identifying a function from the drawn graph

Individually or in a group, perform the following tasks:

1. Draw each of the following figures in the xy-plane:
   (a) A parabola opening upwards or downwards.
   (b) A parabola opening to the left or to the right.
   (c) A circle of reasonable radius.

2. In each of the figures drawn in task 1, draw a vertical line that intersects the figure at the middle.
3. What have you observed after performing task 2?
4. What conclusion can you draw from the figures?

Polynomial functions

A function $f:\mathbb{R}\to\mathbb{R}$ of the form $f(x)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_2x^2+a_1x+a_0$, where $n\in\mathbb{N}$, and a_n, a_{n-1}, $a_{n-2},\ldots$, a_0 are constants, n is a positive integer and $a_n\neq 0$ is called a polynomial function.

The degree of a polynomial function is the highest power of x appearing in the polynomial.

For example, $f_1(x)=x+2$, $f_2(x)=x^2$, $f_3(x)=x^3-2x+1$, $f_4(x)=3x^4-2x^3+5$ are polynomial functions of degree 1, 2, 3, and 4, respectively.

Polynomial functions are classified depending on their degrees, that is:

(i) If the degree is 0, then the polynomial is a constant function.
(ii) If the degree is 1, then the polynomial is a linear function.
(iii) If the degree is 2, then the polynomial is a quadratic function.
(iv) If the degree is 3, then the polynomial is a cubic function.
(v) If the degree is 4, then the polynomial is a quartic function.

Graphs of linear functions

A function defined by $f(x)=y=mx+c$, where m is the gradient, c is the y-intercept, and x is the independent variable, is called a linear function. The graph of a linear function is always a straight line, as shown in Figure 5.1. The x-intercept is denoted by "a" where $a=-\dfrac{c}{m}$.

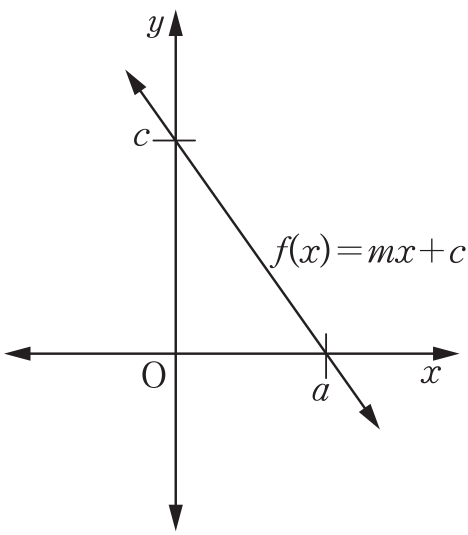


Figure 5.1: *Graph of $f(x)=mx+c$, for $m<0$*

Example 5.1

Draw the graph of $f(x) = 3x + 2$

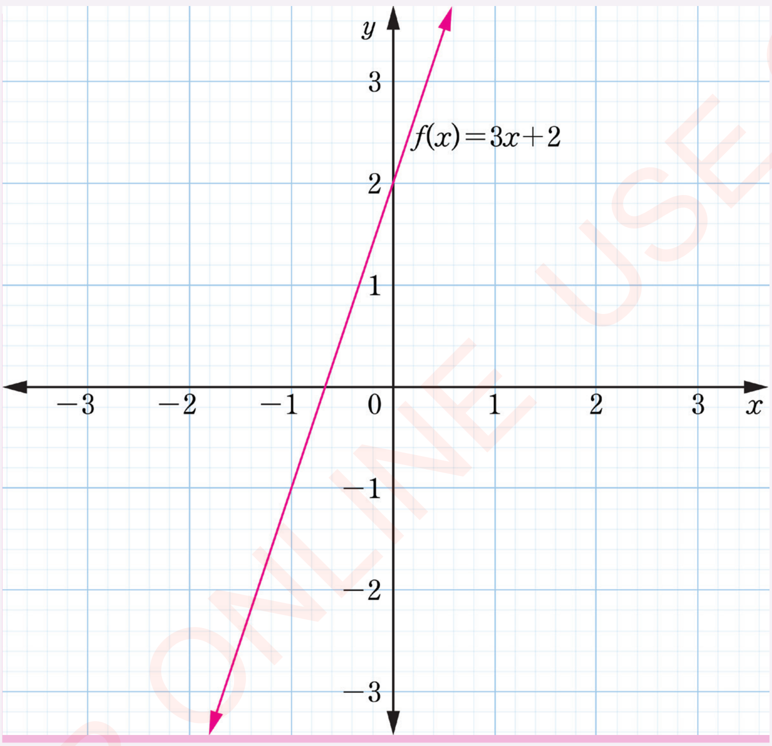
Solution

Given $f(x) = 3x + 2$.

The selected values for graphing the function are tabulated as follows:

x	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f(x)$	$-\frac{5}{2}$	-1	$\frac{1}{2}$	2	$\frac{7}{2}$

The graph of $f(x) = 3x + 2$ is shown in the following figure:



Alternatively:

The graph of $f(x) = 3x + 2$ can be sketched by using x and y-intercepts as follows:

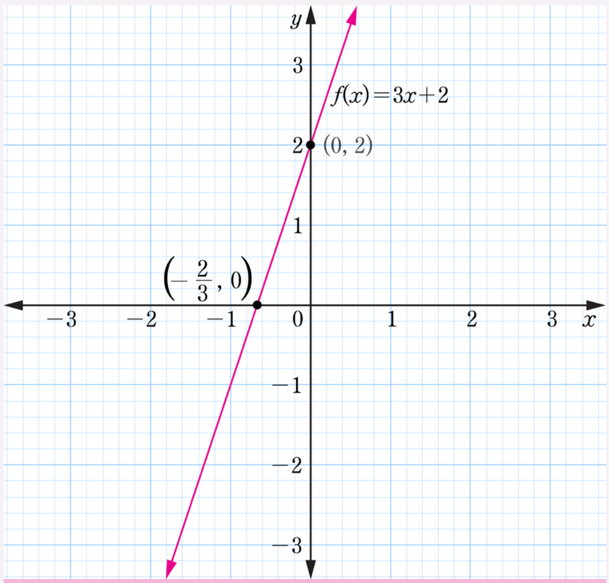
The x-intercept:

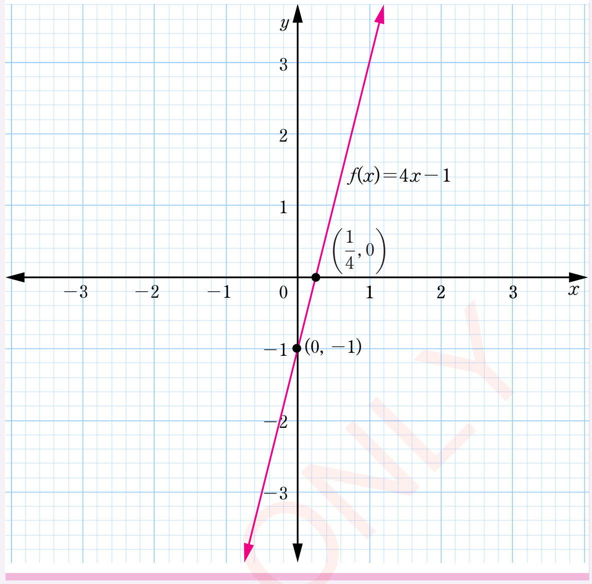
when $y = 0$, $x = -\frac{2}{3}$,

The y-intercept:

when $x = 0$, $y = 2$

The graph of $f(x) = 3x + 2$ is shown in the following figure:





Example 5.2

Draw the graph of $f(x)=4x-1$.

Solution

Given $f(x)=4x-1$.

The x and y-intercepts for graphing the function are tabulated as follows:

x	0	$\frac{1}{4}$
$f(x)$	-1	0

The graph of $f(x)=4x-1$ is shown in the following figure:

Exercise 5.1

Draw the graph of each of the following:

1. $f(x)=-2x+6$
2. $f(x)=\frac{1}{3}(4x+8)$
3. $f(x)=-3x+2$
4. $f(x)=4x+9$
5. $f(x)=-7x-3$
6. $f(x)=\frac{1}{2}x+8$
7. $f(x)=8x-\frac{1}{3}$
8. $f(x)=-9x$
9. $f(x)=\frac{2}{9}x+3$
10. $g(x)=\frac{4}{10}x$

Graphs of quadratic functions

A polynomial function of the form $f(x)=ax^2+bx+c$, where a, b, and c are constants, and $a \neq 0$ is called quadratic function. When $a>0$, the graph opens upwards and when a < 0, the graph opens downwards, as shown in Figure 5.2(a) and Figure 5.2 (b), respectively.

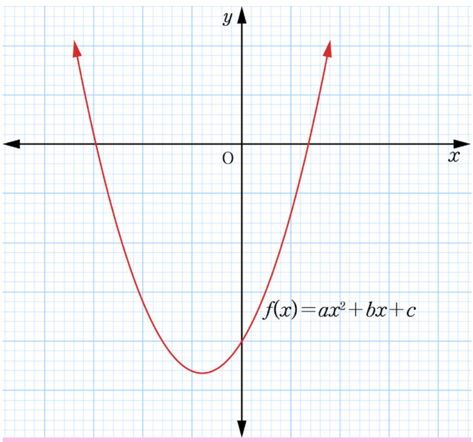


Figure 5.2(a) : *Graph of a quadratic function for* $a>0$

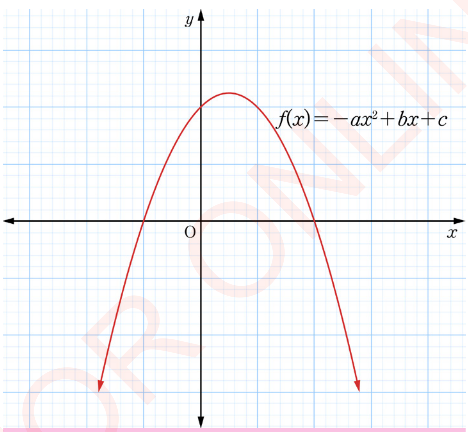


Figure 5.2(b) : *Graph of a quadratic function for* $a<0$

Example 5.3

Draw the graph of $f(x)=x^2+4x+2$.

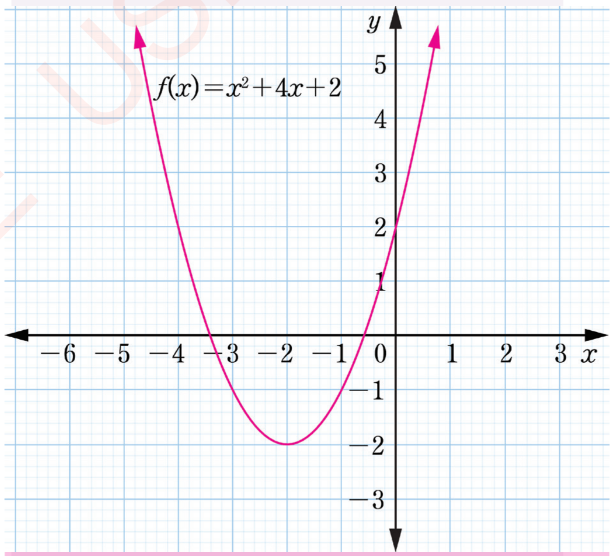
Solution

Given $f(x)=x^2+4x+2$.

The selected values for graphing the function are tabulated as follows:

x	-4	-3	-2	-1	0
$f(x)$	2	-1	-2	-1	2

The graph of $f(x)=x^2+4x+2$ is shown in the following figure:



Example 5.4

Draw the graph of $f(x)=-x^2+2x+1$.

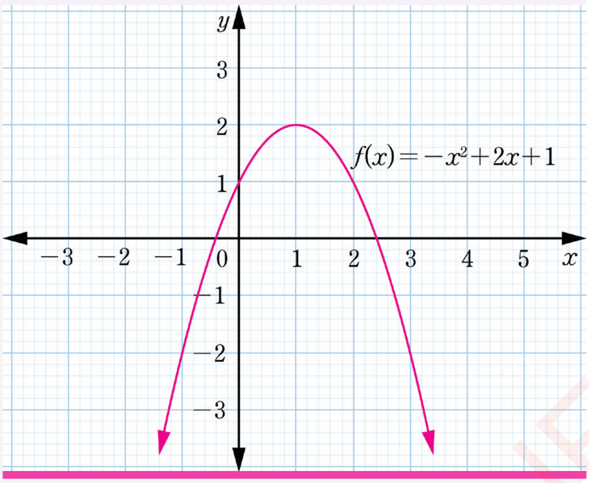
Solution

Given $f(x) = -x^2 + 2x + 1$.

The selected values for graphing the function are tabulated as follows:

x	-1	0	1	2	3
$f(x)$	-2	1	2	1	-2

The graph of $f(x) = -x^2 + 2x + 1$ is shown in the following figure:



Example 5.5

Draw the graph of $f(x) = 3x^2 + 5x + 4$.

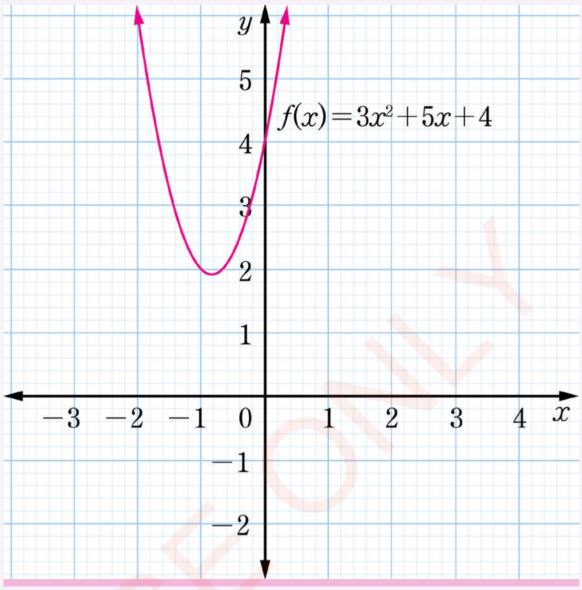
Solution

Given $f(x) = 3x^2 + 5x + 4$.

The selected values for graphing the function are tabulated as follows:

x	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0
$f(x)$	6	$\frac{13}{4}$	2	$\frac{9}{4}$	4

The graph of $f(x) = 3x^2 + 5x + 4$ is shown in the following figure:



Exercise 5.2

Draw the graph of each of the following:

1. $f(x) = (x+3)^2$
2. $f(x) = 3x^2 + 2x + 1$
3. $f(x) = x^2 + 12x + 3$
4. $f(x) = x^2 - 2x - 3$, and $g(x) = -2x^2 - 5x + 7$ on the same plane.
5. $f(x) = 3x^2 - \frac{3}{4}x + 1$
6. $f(x) = x^2$ and $g(x) = -2x + 7$ on the same plane.

7. $f(x) = -(x^2 - 3x - 2) - 5$
8. $f(x) = \frac{3}{5}\left(x^2 + 2x\right) + 2$
9. $f(x) = \frac{1}{2}\left(\sqrt{4}x^2 + 5x + 1\right)$
10. $f(x) = -4x^2 - x - 3$

Graphs of cubic functions

A polynomial function of the form $f(x) = ax^3 + bx^2 + cx + d$, where a, b, c, and d are constants and $a \neq 0$ is called a cubic function. The procedure for graphing a cubic function is the same as that used for graphing a quadratic function.

Example 5.6

Draw the graph of $f(x) = x^3 - 4x$ and state its domain and range.

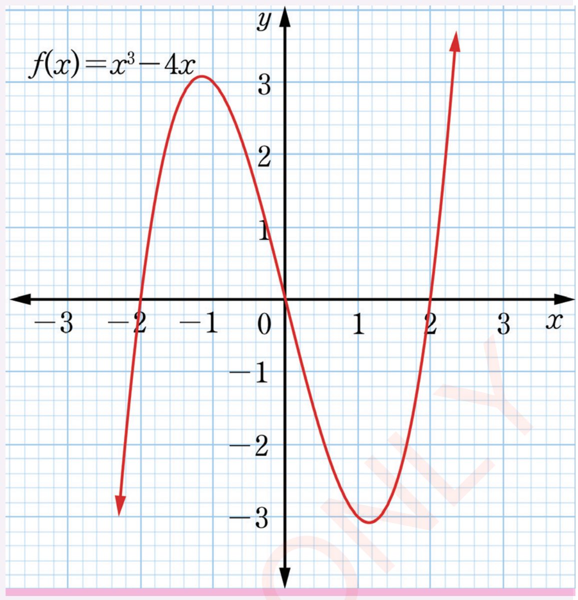
Solution

Given $f(x) = x^3 - 4x$.

The selected few values for graphing the function are tabulated as follows:

x	–2	–1	0	1	2
$f(x)$	0	3	0	–3	0

The graph of $f(x) = x^3 - 4x$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

Example 5.7

Draw the graph of $f(x) = x^3 + 3x^2 - 2x - 5$ and state how the graph behaves for large positive and negative values of x.

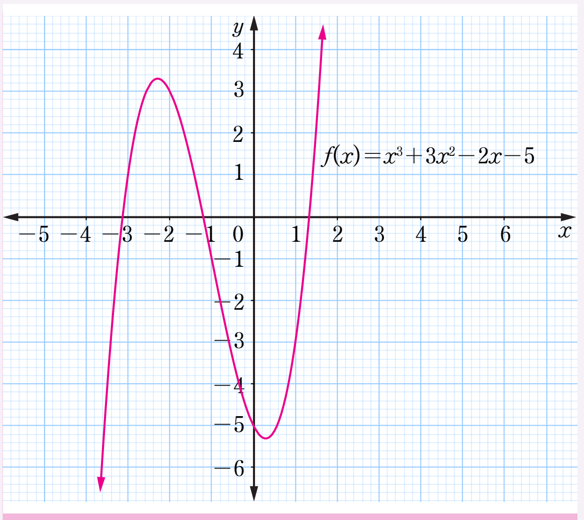
Solution

Given $f(x) = x^3 + 3x^2 - 2x - 5$

The selected values for graphing the function are tabulated as follows:

x	–3	–2	–1	0	1
$f(x)$	1	3	–1	–5	–3

The graph of $f(x) = x^3 + 3x^2 - 2x - 5$ is shown in the following figure:



As x increases indefinitely, $f(x)$ increases and the graph of $f(x)$ is opening upwards. As x decreases indefinitely $f(x)$ decreases and the graph of $f(x)$ is opening downwards.

Example 5.8

Draw the graph of $f(x)=-x^3-2x^2+x+1$ and state its domain and range.

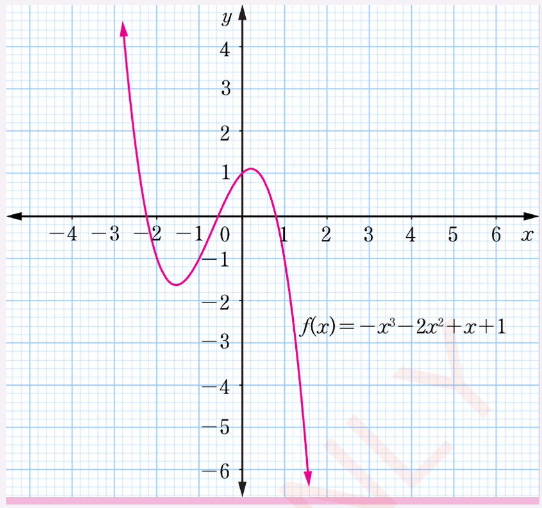
Solution

Given $f(x)=-x^3-2x^2+x+1$.

The selected values for graphing the function are tabulated as follows:

x	-2	-1	0	1
$f(x)$	-1	-1	1	-1

The graph of $f(x)=-x^3-2x^2+x+1$ is shown in the following figure:



Domain $=\{x: x\in\mathbb{R}\}$

Range $=\{y: y\in\mathbb{R}\}$

Exercise 5.3

Draw the graph of each of the following functions and hence state the domain and range:

1. $f(x)=x^3-3x$
2. $f(x)=2x^3+2x^2-4x+5$
3. $f(x)=2x^3+x^2-x-1$
4. $f(x)=x(x-2)(x+3)$
5. $f(x)=-(x^3-7x+3)$
6. $f(x)=x(x^2+3x)+4$
7. $f(x)=x^3-4x^2+x$ and $g(x)=-\frac{1}{3}x^3+\frac{3}{2}x^2+x$ on the same axes.
8. $f(x)=-x^3-5x^2+4$
9. $f(x)=(x-2)^3-4x$ and $h(x)=-x^2+4$ on the same axes.

Graphs of quartic functions

A polynomial function of the form $f(x) = ax^4 + bx^3 + cx^2 + dx + e$, where *a, b, c, d,* and *e* are constants and $a \neq 0$, is called a quartic function. The procedure for graphing a quartic function is the same as the procedure for graphing a cubic function.

Example 5.9

Draw the graph of $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$.

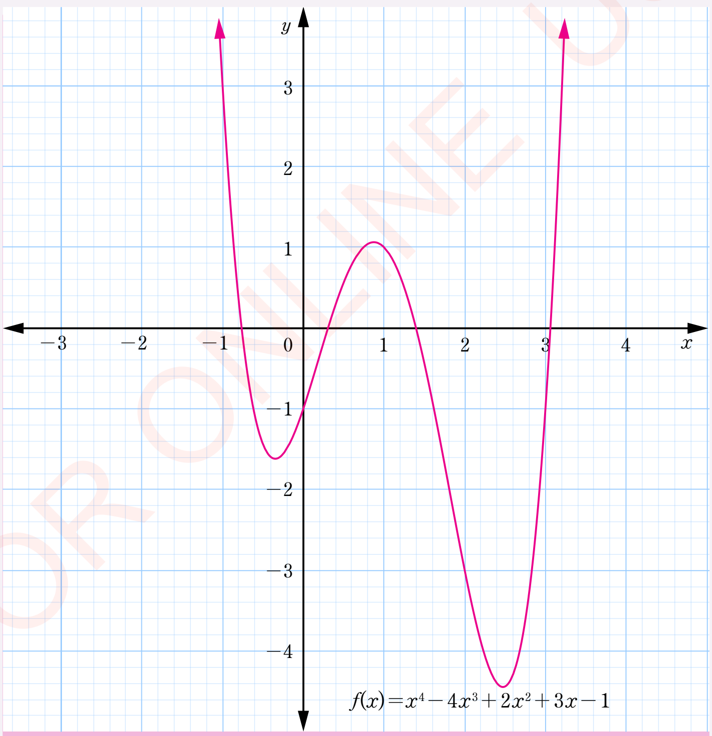
Solution

Given $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$.

The selected values for graphing the function are tabulated as follows:

x	-1	0	1	2	3
$f(x)$	3	-1	1	-3	-1

The graph of $f(x) = x^4 - 4x^3 + 2x^2 + 3x - 1$ is shown in the following figure:



Example 5.10

Draw the graph of $f(x)=(x^2-3)(x^2-8)$ and give its domain and range.

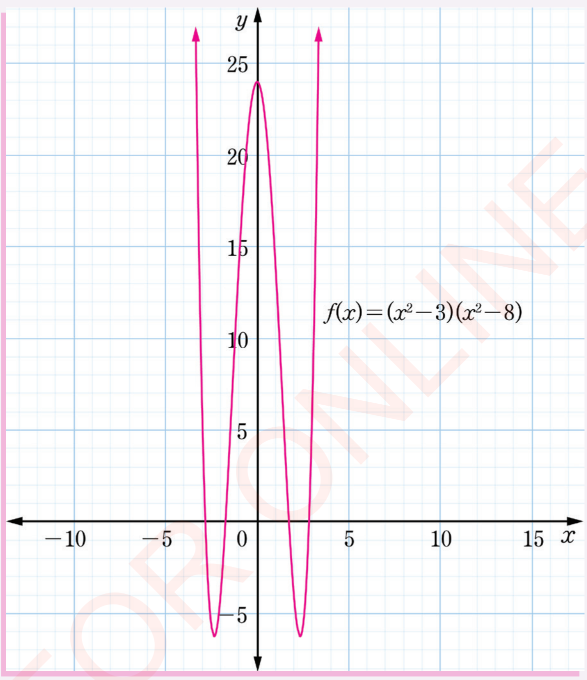
Solution

Given $f(x)=(x^2-3)(x^2-8)$.

The selected values for graphing the function are tabulated as follows:

x	-3	-2	-1	0	1	2	3
$f(x)$	6	-4	14	24	14	-4	6

The graph of $f(x)=(x^2-3)(x^2-8)$ is shown in the following figure:



Domain $=\{x:-\infty<x<\infty\}$

Range $=\left\{y: y\in\mathbb{R}, y\geq -\frac{25}{4}\right\}$

Exercise 5.4

1. Given that $f(x)=x^4-2x^3-x^2+2x$, find the zeros of $f(x)$.
2. Draw the graph of $f(x)=x^4-3x^3-4x^2+6x+4$ and then find its domain and range.
3. Draw the graph of each of the following:
   (a) $f(x)=x^4-5x^2+10$.
   (b) $f(x)=-x^4-4x^3-x^2+8x$.
   (c) $f(x)=-(x-2)(x+2)^2(x+1)$
4. Draw the graph of $f(x)=4x^4-3x^2$.
5. Draw the graph of $f(x)=x^2\left(x^2-1\right)$ over the interval $-2\leq x\leq 2$ and find its range.

Graphs of rational functions

A function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where the numerator $p(x)$ and denominator $q(x)$ are polynomial functions and $q(x)\neq 0$, is called a rational function. This kind of functions may not be defined at some values. The value at which the rational function is not defined is known as a hole. As the values of x get closer to or far from the hole, the behaviour of the function is determined.

Asymptote

An asymptote to a curve is defined as a curve towards which the curve approaches as the distance from the origin increases. There are three types of asymptotes that may exist in rational functions, namely; vertical asymptotes, horizontal asymptotes, and oblique asymptotes.

A vertical asymptote is a vertical line obtained by equating the denominator of a rational function to zero, that is $q(x)=0$. A horizontal asymptote is a horizontal line parallel to the axis of the independent variable. To find the horizontal asymptotes of the function $f(x)$, reduce the term with the highest degree to a constant. This is done by dividing each term in the rational function by the variable with highest degree, and then solve for y when the independent variable x approaches $\pm\infty$.

Generally, if $f(x)=\dfrac{p(x)}{q(x)}$ is a rational function, where p and q are polynomial functions with leading coefficients a and b, respectively, then the following hold:

(a) If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y=\dfrac{a}{b}$ is the horizontal asymptote of the graph $y=f(x)$.

(b) If the degree of $p(x)$ is less than the degree of $q(x)$, then $y=0$ is the horizontal asymptote of the graph of $y=f(x)$.

(c) If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y=f(x)$ has no horizontal asymptotes.

An oblique asymptote is a straight line that is inclined at a certain angle with the x-axis. The oblique asymptote is a function of the form $y=mx+b$ where $m\neq 0$. This kind of asymptote exists when the degree of the numerator is greater than that of the denominator. Normally, the oblique asymptote is the quotient obtained by dividing the numerator by the denominator using the long division method.

The graph of a rational function $f(x)=\dfrac{p(x)}{q(x)}$ has a hole at $x=a$ if:

(i) Both $p(x)$ and $q(x)$ have the common factor $(x-a)$.

(ii) The simplified denominator does not have the common factor $(x-a)$.

To find the y-coordinate of the hole in a rational function, substitute $x=a$ into a simplified expression of $f(x)$. In the graph, a hole (if any) is located using a small circle.

Steps for graphing a rational function

When graphing a rational function use the following steps:

1. Identify the holes, if any.
2. Find and locate the x and y-intercepts, if they exist.
3. Find the asymptotes (horizontal, vertical, or oblique) if any.
4. Draw the asymptotes on the graph, using dotted lines.
5. Sketch the graph by testing the behaviour of the graph.

Note that, the graph of a rational function may have many vertical asymptotes, but it will have at most one horizontal or oblique asymptote.

Example 5.11

Find the holes and vertical asymptotes of the function $f(x)=\dfrac{2x^2-4x}{x^2+x-6}$.

Solution

Given $f(x)=\dfrac{2x^2-4x}{x^2+x-6}$.

Factorization of the numerator and denominator gives;

$$f(x)=\frac{2x(x-2)}{(x+3)(x-2)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (i)$$

It can be observed that $(x-2)$ is a factor in both the numerator and denominator. Setting this factor equals to zero, that is, $x-2=0$, gives $x=2$.

Simplifying equation (i) and substituting $x=2$ gives

$f(2)=y=\dfrac{4}{5}$.

Therefore, a hole is at the point $\left(2,\ \dfrac{4}{5}\right)$.

Finding the vertical asymptotes, equate the denominator of the function to zero, that is, $x^2+x-6=0$.

Solving for x gives $x=2$ or $x=-3$. Therefore, the vertical asymptotes are the lines $x=2$ and $x=-3$.

Example 5.12

Find the vertical and horizontal asymptotes of the function

$f(x)=\dfrac{x}{x-3}$.

Solution

Given $f(x)=\dfrac{x}{x-3}$.

Vertical asymptotes:

Finding the vertical asymptotes, equate the denominator of the function to zero, that is,

$x-3=0 \Rightarrow x=3.$

Therefore, the vertical asymptote is the line $x=3$

Horizontal asymptote:

Since the degrees of the numerator and denominator are equal, then the horizontal asymptote can be found as follows:

$$y=\frac{\frac{x}{x}}{\frac{x}{x}-\frac{3}{x}} \quad =\frac{1}{1-\frac{3}{x}}$$

As $x\to\pm\infty, y\to 1$

Therefore, the horizontal asymptote is the line $y=1$.

Example 5.13

Find the oblique asymptote of the function $f(x)=\frac{4x^2+5}{x-2}$.

Solution

Given $f(x)=\frac{4x^2+5}{x-2}$.

Finding the oblique asymptote, divide the numerator by denominator using long division as follows:

$$\begin{array}{r} 4x+8 \\ x-2\overline{\big)\,4x^2+5} \\ \underline{-4x^2-8x} \\ 8x+5 \\ \underline{-8x-16} \\ 21 \end{array}$$

Thus, $y=\frac{4x^2+5}{x-2}=(4x+8)+\frac{21}{x-2}$.

Therefore, the oblique asymptote is the line $y=4x+8$.

Example 5.14

Sketch the graph of $f(x)=\frac{x}{x^2-9}$, and hence determine the domain and range.

Solution

Given $f(x)=\frac{x}{x^2-9}$.

Find the x and y-intercepts:

The x-intercept is obtained when $y=0$.

So, $0=\frac{x}{x^2-9} \Rightarrow x=0$

Thus, the x-intercept is at the point $(0,\ 0)$.

The y-intercept is obtained when $x=0$. That is, $y=\frac{0}{0-9}$

$\Rightarrow y=0$

Thus, the y-intercept is at the point $(0,\ 0)$.

Find the vertical asymptotes by equating the denominator to zero:

$x^2-9=(x+3)(x-3)=0$

Hence, the vertical asymptotes are the line $x=-3$ and $x=3$.

Draw $x=-3$ and $x=3$ using dotted lines on the xy-plane.

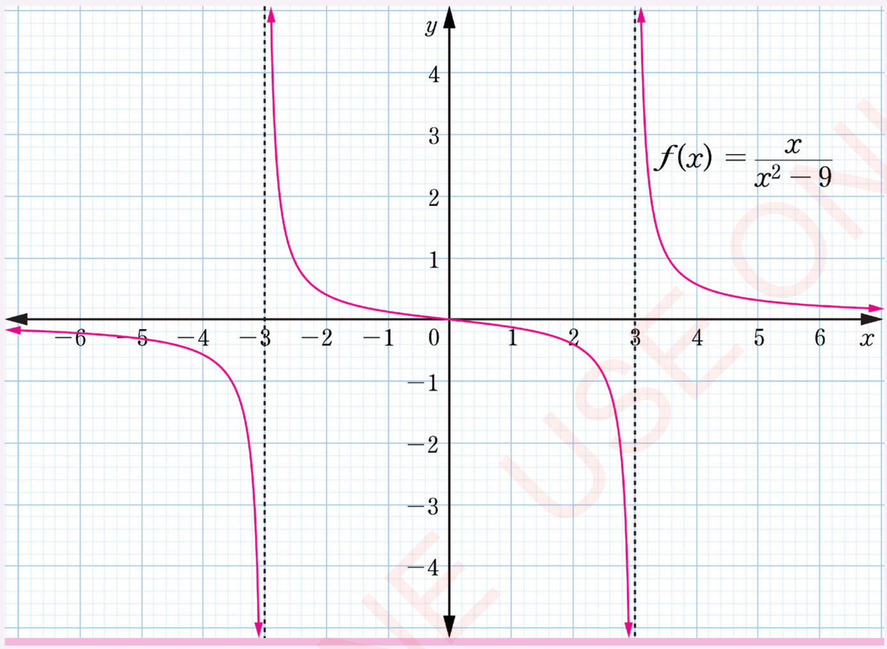
Find the horizontal asymptote by dividing the expression by the term with the highest power.

Since the degree of the numerator is 1 and that of the denominator is 2, divide each term by x^2.

That is, $f(x)=\frac{\frac{x}{x^2}}{\frac{x^2}{x^2}-\frac{9}{x^2}} \quad =\frac{\frac{1}{x}}{1-\frac{9}{x^2}}$

As $x \to \pm\infty$, $y \to 0$.

Hence, the horizontal asymptote is the line $y = 0$.
But, the line $y = 0$ cannot be a dotted line as it define the x-axis.
Locate the points on either side of the vertical asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.
The graph of $f(x) = \dfrac{x}{x^2 - 9}$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R}, x \neq \pm 3\}$
Range $= \{y : y \in \mathbb{R}\}$

Example 5.15

Sketch the graph of $f(x) = \dfrac{x^3 - 3x}{3(x^2 - 7)}$, and hence determine the domain and range.

Solution

Given $f(x) = \dfrac{x^3 - 3x}{3(x^2 - 7)}$.

Find the x and y-intercepts:
x-intercept; $y = 0$

So, $0 = \dfrac{x^3 - 3x}{3(x^2 - 7)}$

$x^3 - 3x = 0 \Rightarrow x\left(x^2 - 3\right) = 0$

$\Rightarrow x = 0$ or $x^2 - 3 = 0$

$\Rightarrow x^2 = 3$

$\Rightarrow x = \pm\sqrt{3}$

Thus, the x-intercepts are at the points $(0, 0)$ and $(\pm\sqrt{3}, 0)$

y-intercept; $x = 0$

That is, $y = \dfrac{0-0}{3(0-7)}$

$\Rightarrow y = 0$

Thus, the y-intercept is at the point (0, 0).

Find the vertical asymptotes by equating the denominator to zero, that is,

$3(x^2-7)=0$

$\Rightarrow x^2 = 7$

$\Rightarrow x = \pm\sqrt{7}$

Hence, the vertical asymptotes are the lines $x = -\sqrt{7}$ and $x = \sqrt{7}$.

Draw $x = -\sqrt{7}$ and $x = \sqrt{7}$ as dotted lines on the xy-plane.

Since the highest degree of the numerator is 3 and that of the denominator is 2, divide the numerator by the denominator to get the oblique asymptote.

That is,

$$\begin{array}{r|l} & \frac{1}{3}x \\ \hline 3x^2-21 & x^3-3x \\ & -\underline{x^3-7x} \\ & 4x \end{array}$$

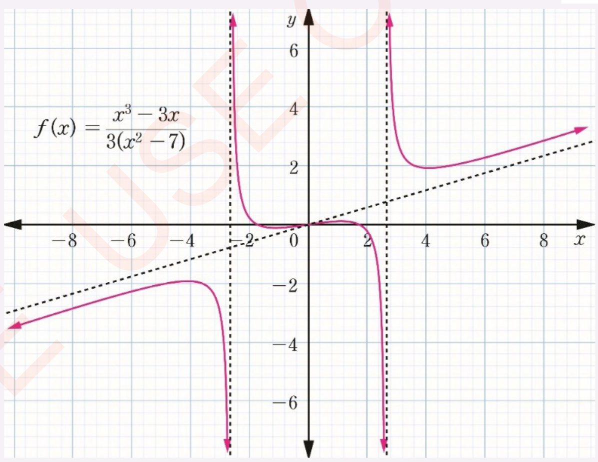
Thus,

$$f(x) = \frac{x^3-3x}{3(x^2-7)} = \frac{1}{3}x + \frac{4x}{3x^2-21}.$$

Hence, the oblique asymptote is the line $y = \frac{1}{3}x$.

Locate the points on either side of the vertical asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.

The graph of $f(x) = \dfrac{x^3-3x}{3(x^2-7)}$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R},\ x \neq \pm\sqrt{7}\}$

Range $= \{y : y \in \mathbb{R}\}$

Example 5.16

Sketch the graph of $f(x) = \dfrac{x^2-2x+1}{x^3+x^2-2x}$

and hence determine the domain and range.

Solution

Given $f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$.

Factorize the numerator and denominator:

$$f(x)=\frac{(x-1)(x-1)}{x(x+2)(x-1)}\rightarrow,$$

thus, $f(x)=\dfrac{x-1}{x(x+2)}$.

Hence, the hole is at $x=1$. Substituting $x=1$, gives $f(1)=0$, this implies that the hole is at $(1,0)$.

x-intercept; $y=0$

$$\Rightarrow 0=\frac{x-1}{x(x+2)}$$

$$\Rightarrow x-1=0$$

$$\Rightarrow x=1$$

Thus, the x-intercept is at the point (1, 0).

y-intercept; $x=0$

$$\Rightarrow\ y=\frac{0-1}{0(0+2)}$$

$$\Rightarrow\ y=\infty$$

Thus, the y-intercept is at infinity (does not exist).

The vertical asymptotes are obtained by equating the denominator to zero, that is, $x(x+2)=0$

$$\Rightarrow\ x=0\ \text{ or }\ x=-2$$

Hence, the vertical asymptotes are the lines $x=0$ and $x=-2$.

Draw $x=-2$ as dotted line on the graph.

But x = 0 cannot be a dotted line as it defines the y-axis. Since the degree of the numerator is 1, and the degree of the denominator is 2, the horizontal asymptotes is obtained as follows:

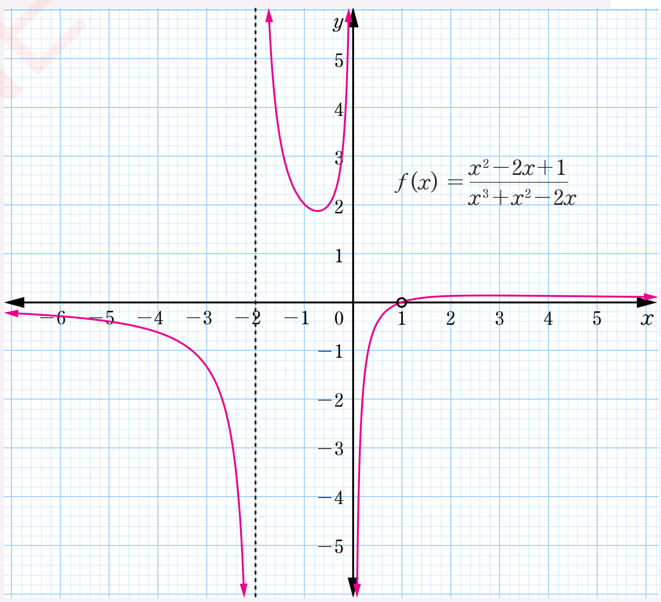
$$f(x)=\frac{\frac{x}{x^2}-\frac{1}{x^2}}{\frac{x^2}{x^2}+2\frac{x}{x^2}}=\frac{\frac{1}{x}-\frac{1}{x^2}}{1+\frac{2}{x}}.$$

As $x\rightarrow\pm\infty$, $y\rightarrow 0$.

Hence, the horizontal asymptote is the line $y=0$.

Locate the points on either side of the asymptotes, also select some values of x and determine the corresponding values of y, then use these values to sketch the graph.

The graph of $f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$ is plotted in the following figure:



Domain $= \{x : x \in \mathbb{R},\ x \neq -2,\ x \neq 0,\ x \neq 1\}$

Range $= \{y : y \in \mathbb{R}, y \geq 1.866,\ y \leq 0.134,\ y \neq 0\}$

Example 5.17

Sketch the graph of the function $f(x) = \dfrac{3x+3}{x(3-x)}$, and use it to show that it has no real values between $\dfrac{1}{3} < y < 3$.

Hence, state its domain and range.

Solution

Given $f(x) = \dfrac{3x+3}{x(3-x)}$.

Find the x and y-intercepts:

x-intercept; $y = 0$

$\Rightarrow\ 0 = \dfrac{3x+3}{x(3-x)}$

$\Rightarrow\ x = -1$

Thus, the x-intercept is at the point $(-1,\ 0)$.

y-intercept; $x = 0$, that is,

$y = \dfrac{3(0)+3}{0(3-0)}$

$\Rightarrow\ y = \infty$

Thus, the y-intercept is at infinity (does not exist).

The vertical asymptotes are obtained by equating the denominator to zero, that is,

$x(3-x) = 0$

$\Rightarrow x = 0$ or $x - 3 = 0$. This gives $x = 0$ or $x = 3$.

Hence, the vertical asymptotes are the lines $x = 0$ and $x = 3$.

Draw $x = 3$ as a dotted line on the graph.

Horizontal asymptote:

Since the degree of the numerator is 1 and that of the denominator is 2, divide each term by x^2.

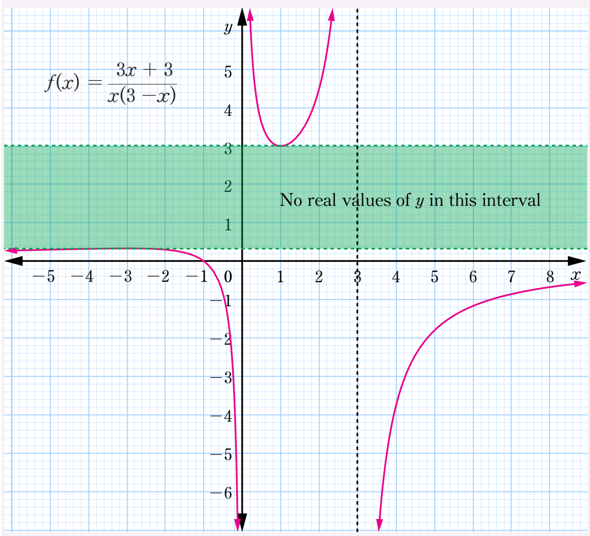
That is, $f(x) = \dfrac{3\dfrac{x}{x^2} + \dfrac{3}{x^2}}{3\dfrac{x}{x^2} - \dfrac{x^2}{x^2}} = \dfrac{\dfrac{3}{x} + \dfrac{3}{x^2}}{\dfrac{3}{x} - 1}$

As $x \to \pm\infty$, $y \to 0$.

Hence, the horizontal asymptote is the line $y = 0$.

Locate the points on either side of the vertical asymptotes, also select some value of x and determine the corresponding values of y. Use the selected values to sketch the graph.

The graph of $f(x) = \dfrac{3x+3}{x(3-x)}$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R},\ x \neq 0,\ x \neq 3\}$

Range $= \left\{y : x \in \mathbb{R},\ x \neq \frac{1}{3} < y < 3\right\}$

Exercise 5.5

Sketch the graph of each of the following functions and state the domain and range:

1. $f(x) = \dfrac{x}{(x-3)(x+2)}$
2. $f(x) = \dfrac{x^3 + 4x^2 + 5x + 2}{x^2 + x - 6}$
3. $f(x) = \dfrac{2x}{(x-3)(x+3)}$
4. $f(x) = \dfrac{x^2 - 4}{x^2 - 4x}$
5. $f(x) = \dfrac{x^3 + x}{x^2 - 9}$
6. $f(x) = \dfrac{x^3 + 2x^2 + 1}{(x+1)(x-1)}$
7. $f(x) = \dfrac{(x-1)(x^3 - 2x^2 - 2x + 1)}{x^3 - x^2 - x + 1}$
8. $f(x) = \dfrac{2x^3 + x^2 - 3}{x^2 - 9}$
9. $f(x) = \dfrac{x^2 - 2}{x^2 - 1}$
10. $f(x) = \dfrac{x^2 - 3x - 2}{x^2 - 4}$

Composite functions

A composite function is a function which is obtained after substituting one function into another function. Suppose $f(x)$ and $g(x)$ are two functions, then, the composite of $f(x)$ and $g(x)$ is given as $f[g(x)]$. The composite function $f[g(x)]$ is written in short as $(f \circ g)(x)$, and read as "f of g of x". The function $g(x)$ is the inner function and the function $f(x)$ is the outer function. Hence, $f[g(x)]$ can be read as "the function g is the inner function of the outer function f". The composite function is sometimes written as $f \circ g$.

Activity 5.2: Verifying whether a composite of two linear functions is also a function

Individually or in a group, perform the following tasks:

1. Write down any two linear functions.
2. Substitute one function into another for the functions obtained in task 1.
3. Verify whether the result obtained in task 2 is a function or not.

Steps to form a composite function

The following steps are used to form a composite function:

1. Decide which is the inner and which is the outer function.

2. Substitute the inner function into the variable x of the outer function.
3. Simplify the resulting composite function.

Example 5.18

Give the functions $g(x) = 2x - 2$ and $f(x) = x^2 + 5$, find each of the following:

(a) $(g \circ f)(x)$

(b) $(f \circ g)(x)$

Solution

Given $g(x) = 2x - 2$ and $f(x) = x^2 + 5$.

(a) To find $(g \circ f)(x)$, substitute $x^2 + 5$ as x in the function $g(x) = 2x - 2$.

That is, $(g \circ f)(x) = 2(x^2 + 5) - 2$

$$= 2x^2 + 10 - 2$$

$$= 2x^2 + 8$$

Therefore, $(g \circ f)(x) = 2x^2 + 8$.

(b) To find $(f \circ g)(x)$, substitute $g(x) = 2x - 2$ as x in the function $f(x) = x^2 + 5$.

That is $(f \circ g)(x) = (2x - 2)^2 + 5$

$$= 4x^2 - 8x + 9.$$

Therefore, $(f \circ g)(x) = 4x^2 - 8x + 9$.

Example 5.19

Given $f(x) = 2x + 7$, find $(f \circ f)(x)$.

Solution

Given $f(x) = 2x + 7$.

Substitute $2x + 7$ as x into the function $f(x) = 2x + 7$.

That is, $(f \circ f)(x) = 2(2x + 7) + 7$

$$= 4x + 14 + 7$$

$$= 4x + 21.$$

Therefore, $(f \circ f)(x) = 4x + 21$.

Example 5.20

Evaluate $f[g(6)]$ given that,

$f(x)=6x+2$ and $g(x)=x-2$.

Solution

Given $f(x)=6x+2$ and $g(x)=x-2$.

Substitute $x-2$ as x in the function $f(x)=6x+2$

That is, $f[g(x)]=6(x-2)+2$
$=6x-12+2$
$=6x-10$

Hence, $f[g(6)]=6(6)-10=36-10=26$

Therefore, $f[g(6)]=26$.

Example 5.21

Given

$f(x)=\sqrt{x-3}$ and $g(x)=\ln(1+x^2)$,

find $g(f(x))$.

Solution

Given that

$f(x)=\sqrt{x-3}$ and $g(x)=\ln(1+x^2)$.

Substitute $\sqrt{x-3}$ as x into the function

$g(x)=\ln(1+x^2)$ that is,

$$g(f(x)=\text{In}\left[1+(f(x))^2\right]$$

$$=\ln\left[1+\left(\sqrt{x-3}\right)^2\right]$$

$$=\ln|1+(x-3)|$$

$$=\ln|x-2|$$

Therefore, $g(f(x)=\text{In}|x-2|$.

Example 5.22

Given $f(x)=x^2+4$ and $g(x)=x-9$, then solve for x if $(f\circ g)(x)=(g\circ f)(x)$.

Solution

Given that $f(x)=x^2+4$ and $g(x)=x-9$.

Substitute $x-9$ as x into the function $f(x)=x^2+4$. That is,

$(f\circ g)(x)=(x-9)^2+4$
$=x^2-18x+85$.

Substitute x^2+4 as x into the function $g(x)=x-9$. That is,

$(g\circ f)(x)=(x^2+4)-9$
$=x^2-5$.

But $(f\circ g)(x)=(g\circ f)(x)$, thus,

$x^2-18x+85=x^2-5$

$\Rightarrow -18x+85=-5$

$\Rightarrow -18x=-90$

Thus, $x=5$.

Therefore, $x=5$.

Example 5.23

Find $(f\circ g)(x)$ and $(g\circ f)(x)$ given that $f=\{(2,5),(5,7),(9,0)\}$ and $g=\{(3,2),(4,5),(0,7)\}$.

Solution

Given $f=\{(2,5), (5,7), (9,0)\}$ and $g=\{(3, 2), (4, 5), (0, 7)\}$. The composite functions are given by:

$f[g(3)]=f[2]=5\rightarrow(3,5)$

$f[g(4)]=f[5]=7\rightarrow(4, 7)$

$f[g(0)]=f[7]$ does not exist.

$g[f(2)]=g[5]$ does not exist.

$g[f(5)]=g[7]$ does not exist.

$g[f(9)]=g[0]=7\rightarrow(9, 7)$

Therefore, $(f\circ g)(x)=\{(3, 5), (4, 7)\}$ and $(g\circ f)(x)=\{(9,7)\}$.

Example 5.24

If $f(x)=2x^2+4x+4$, and $(f\circ g)(x)=2x^2-8x+10$, find $g(x)$.

Solution

Given $f(x)=2x^2+4x+4$ and $(f\circ g)(x)=2x^2-8x+10$.

But $(f\circ g)(x)=f[g(x)]$

substitute $g(x)$ as x into the function $f(x)$ to obtain,

$2[g(x)]^2+4g(x)+4=2x^2-8x+10$

simplification gives,

$2[g(x)]^2+4g(x)-(2x^2-8x+6)=0$

This is a quadratic equation in $g(x)$.

Thus,

$$g(x)=\frac{-4\pm\sqrt{16+8(2x^2-8x+6)}}{4}$$

$$=-1\pm\sqrt{1+\frac{1}{2}(2x^2-8x+6)}$$

$$=-1\pm\sqrt{1+x^2-4x+3}$$

$$=-1\pm\sqrt{x^2-4x+4}$$

$$=-1\pm\sqrt{(x-2)^2}$$

$$=-1\pm(x-2)$$

Therefore,

$g(x)=x-3$ and $g(x)=1-x$.

Exercise 5.6

1. Given $f(x)=3x+5$ and $g(x)=4-x$, find $f(g(x))$ and $g(f(x))$ and use the results to verify whether or not the composition of the functions is commutative.

2. Find $f\left(\frac{1}{2}\right)$ if
   (a) $(f\circ g)(x)=\dfrac{x^4+x^2}{1+x^2}$ and $g(x)=1-x^2$.
   (b) $(f\circ g)(x)=3x^2-6x+11$ and $g(x)=x^2-2x+3$.

3. Given the functions $f=\{(-1, 1), (0, 3), (4, 6)\}$ and $g=\{(1, 1), (3, 5), (7, 9)\}$ find $(g\circ f)$ and determine its domain and range.

4. In each of the following pair of functions, find $(f\circ g)(x)$:
   (a) $f(x)=x^2-2,\ g(x)=x+3$
   (b) $f(x)=x+4,\ g(x)=x^2-11$
   (c) $f(x)=e^x-2,\ g(x)=x^3$
   (d) $f(x)=x^3,\ g(x)=\sqrt[3]{x}$

5. Show that if $f: A \rightarrow B$ and $g: B \rightarrow C$ are one to one, then $g \circ f: A \rightarrow C$ is also one to one.

6. Find $g(-4)$ if $f(x)=\dfrac{x+9}{3x}$ and $(f \circ g)(x)=\dfrac{2(x+5)}{3(2x+1)}$

7. If $f(x)=\sqrt{27-x^2}$ and $g(x)=x^2-2$, solve the equation $(g \circ f)(x)=0$.

8. Let $f(x)=3x-2$, $g(x)=4x$, and $h(x)=x^2+2$. Compute each of the following:

   (a) $f(g(3))$ (d) $f(g(h(3)))$
   (b) $f(h(6))$ (e) $h(g(f(5)))$
   (c) $(g \circ h)(12)$ (f) $h(x-1)$

9. Given that $h(x)=4x-13$ and $g(x)=2x^2-26x+9$, find:

   (a) $(g \circ h)(x)$

   (b) $(h \circ g)(x)$.

10. If $f=\{(3, 8), (2, 5), (4, -5), (9, 3)\}$ and $g=\{(7, 2), (-5, 3), (5, 7), (8,10), (1,7)\}$, find $f \circ g$ and $g \circ f$.

11. Functions f and g are sets of ordered pair $f=\{(-2, 1), (0, 3), (4, 5)\}$ and $g=\{(1, 1), (3, 3), (7, 9)\}$. Find $g \circ f$.

12. Find the composite function $f \circ g$ given that $f=\{(3, 6), (5,7), (9,0)\}$ and $g=\{(2, 3), (4,5), (6,7)\}$.

Graphs of composite functions

The procedures for drawing graph of composite functions are similar to the procedures used to draw other functions.

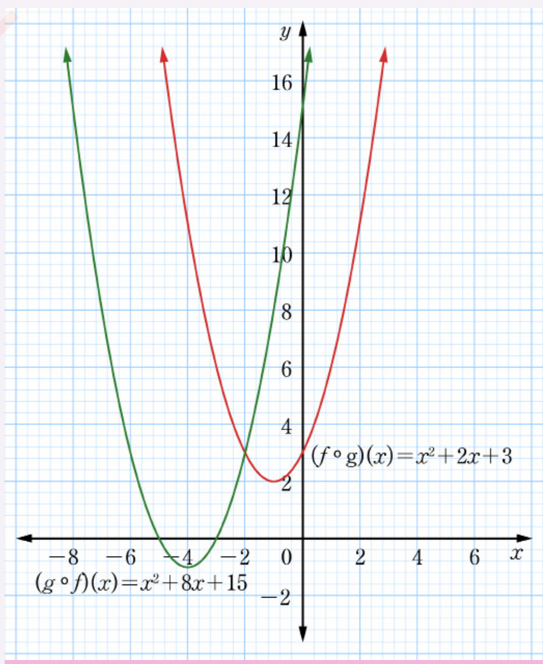
Example 5.25

Given $f(x)=x+3$ and $g(x)=x^2+2x$, sketch the graphs of $(f \circ g)(x)$ and $(g \circ f)(x)$ on the same xy-plane.

Solution

Given $f(x)=x+3$ and $g(x)=x^2+2x$.
Substitute x^2+2x for x in $f(x)=x+3$

and $x+3$ for x in $g(x)=x^2+2x$.
That is, $f(g(x))=x^2+2x+3$ and
$(g(f(x))=(x+3)^2+2(x+3)=x^2+8x+15$.
Hence, $(f \circ g)(x)=x^2+2x+3$ and
$(g \circ f)(x)=x^2+8x+15$
The graphs of $(f \circ g)(x)$ and $(g \circ f)(x)$ are plotted in the following figure:



Example 5.26

Given $f(x) = 3x - 1$ and $g(x) = \frac{4}{x-2}$, sketch separately the graphs of $(f \circ g)(x)$ and $(g \circ f)(x)$.

Solution

Given $f(x) = 3x - 1$ and

$g(x) = \frac{4}{x-2}$.

Then,

$$f(g(x)) = 3\left(\frac{4}{x-2}\right) - 1$$

$$= \frac{12}{x-2} - \frac{1}{1} = \frac{12 - x + 2}{x-2}$$

$$= \frac{14-x}{x-2}$$

Thus, $(f \circ g)(x) = \frac{14-x}{x-2}$

x-intercept; $y = 0$

So, $0 = \frac{14-x}{x-2}$

$\Rightarrow 0 = 14 - x$

$\Rightarrow x = 14$

Thus, the x-intercept is at the point (14, 0)

y-intercept; $x = 0$

$\Rightarrow y = \frac{14-0}{0-2}$

$\Rightarrow y = -7$

Thus, the y-intercept is at the point (0, –7)

Vertical asymptote:

Equate the denominator to zero, that is, $x - 2 = 0$

$\Rightarrow x = 2.$

Hence, the vertical asymptote is the line $x = 2$.

Sketch $x = 2$ as a dotted line on the xy-plane.

Horizontal asymptote:

Since the degree of numerator and denominator are equal.

Then, $(f \circ g)(x) = \dfrac{\frac{14}{x} - \frac{x}{x}}{\frac{x}{x} - \frac{2}{x}} = \dfrac{\frac{14}{x} - 1}{1 - \frac{2}{x}}$

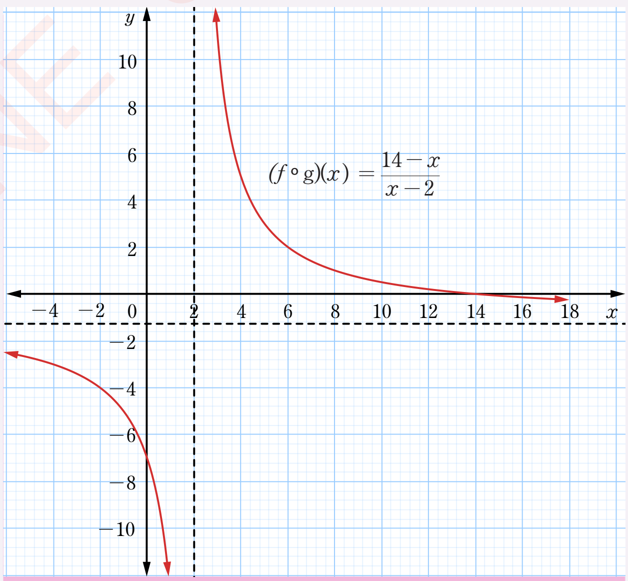
Thus, as $x \to \pm\infty$, $y \to -1$

Hence, the horizontal asymptote is the line $y = -1$.

Draw $y = -1$ as a dotted line on the xy-plane.

Locate the points on either side of the asymptotes, also, select some values of x and determine the corresponding values of y. Use the values to sketch the graph.

Therefore, the graph of $(f \circ g)(x) = \frac{14-x}{x-2}$ is as shown in the following figure:



Also, substitute $3x - 1$ for x in

$g(x) = \frac{4}{x-2}$, so that $g(f(x)) = \frac{4}{(3x-1)-2}$

$$= \frac{4}{3x-3}$$

Thus, $(g \circ f)(x) = \frac{4}{3x-3}$

x-intercept; $y = 0$

So, $0 = \frac{4}{3x-3}$, implies there is no x-intercept.

y-intercept; $x = 0$

So, $y = \frac{4}{3(0)-3}$

$\Rightarrow y = -\frac{4}{3}$

Thus, y-intercept is at the point $\left(0, -\frac{4}{3}\right)$

Vertical asymptote:

Equate the denominator to zero, that is, $3x - 3 = 0$

$$\Rightarrow x = 1$$

Hence, the vertical asymptote is the line $x = 1$.

Draw $x = 1$ as a dotted line on the xy-plane.

Horizontal asymptote:

Since the degree of the numerator is 0, and the degree of the denominator is 1, the horizontal asymptote is obtained as follows:

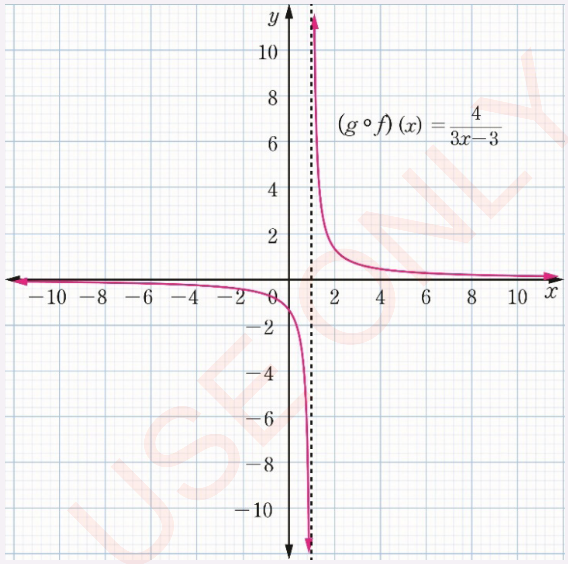
$$(g \circ f)(x) = \frac{\frac{4}{x}}{\frac{3x}{x} - \frac{3}{x}} = \frac{\frac{4}{x}}{3 - \frac{3}{x}}$$

Thus, as $x \to \pm\infty$, $y = 0$.

Hence, the horizontal asymptote is the line $y = 0$.

Plot the points on either side of the vertical asymptotes, also, select some values of x and determine the corresponding values of y. Use these values to sketch the graph.

Therefore, the graph of $(g \circ f)(x) = \frac{4}{3x-3}$ is shown as follows:



Exercise 5.7

1. The functions g and h are defined by $g(x) = x^2 + 1$ and $h(x) = x + 4$, where $x \in \mathbb{R}$. Sketch the graph of $(g \circ h)(x)$ and state its domain and range.
2. If $f(x) = x^2 + 2x + 3$, find two functions of $g(x)$, given that $(f \circ g)(x) = x^2 - 4x + 6$. Hence, sketch the graph of $(f \circ g)(x)$.
3. Given $f(x) = 2x^2 + 1$ and $g(x) = \frac{4x}{x^2 - 2}$, find:

   (a) $(f \circ g)(x)$

   (b) the asymptotes of $(f \circ g)(x)$

   Hence, sketch the graph of $(f \circ g)(x)$ and state its domain and range.

4. Given $f(x)=6x^2-1$ and $g(x)=\dfrac{6}{x-4}$, sketch the graph of $(g\circ f)(x)$.
5. Given that $(f\circ g)(x)=3x^2+10x+15$ and $f(x)=x^2+2x+1$.

   (a) Find $g(x)$ (b) Sketch the graph of $(f\circ g)(x)$
6. If $(f\circ g)(x)=\dfrac{x^2+2}{x^4+4}$ and $g(x)=x^2+1$, find $f(x)$.
7. Given that $f(x)=4x+1$ and $g(x)=6x+k$, find the value of k if $f(g(x))=g(f(x))$. Hence, sketch the graph of $g(f(x))(x)$.
8. A function is defined by $g: x\rightarrow x^2-10$ for $x\in\mathbb{R}$. Find the value of x for which $(g\circ g)(x)=26$. Hence, sketch the graph of $(g\circ g)(x)$.
9. If $f(x)=3x-2$ and $h(x)=\dfrac{1}{1+x}$, determine the intercepts and the asymptotes of $(f\circ h\circ h)(x)$. Hence, sketch the graph of the composite function.
10. Given the functions $f(x)=3x+4$ and $g(x)=x^2$,
    (a) find $(f\circ g)(x)$
    (b) find $(g\circ f)(x)$
    (c) Draw the graph of $(g\circ f)(x)$
11. Given $f(x)=x^2$, $g(x)=x-1$, and $h(x)=\dfrac{x}{2}$, find:
    (a) $(f\circ g\circ h)(x)$ and sketch its graph.
    (b) $(h\circ g\circ f)(4)$.
12. Find $(f\circ g)(x)$ and $(g\circ f)(x)$ for each of the following pair of functions and sketch their graphs on the same xy-plane:
    (a) $f(x)=x+6$ and $g(x)=x+3$
    (b) $f(x)=x^2+2$ and $g(x)=x$
    (c) $f(x)=3x+1$ and $g(x)=6-3x$

Graphs of exponential functions

An exponential function $f(x)$ with base a is defined by $f(x)=a^x$ or $y=a^x$, where $a>0$, $a\neq 1$, and $x\in\mathbb{R}$. To draw the graph of an exponential function, choose some values of the independent variable x and determine the corresponding values of the dependent variable y. Tabulate the values and plot them on the xy-plane.

Example 5.27

Draw the graph of $f(x) = 2^x$.

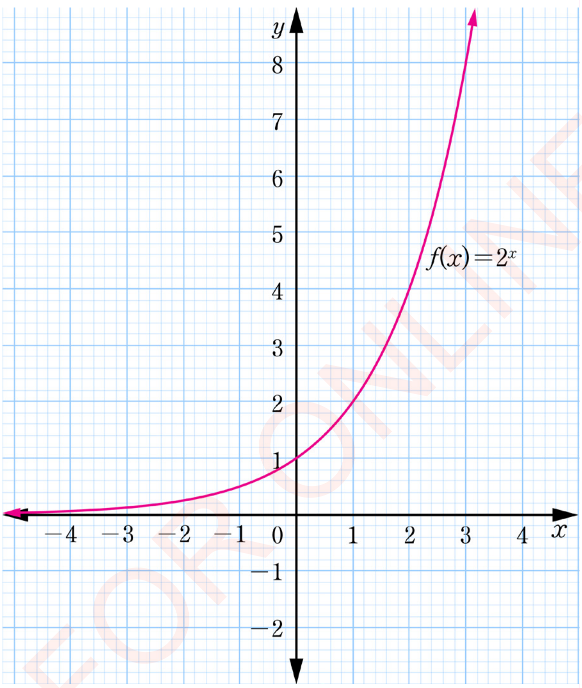
Solution

Given $f(x) = 2^x$.

The selected few values for graphing the function are tabulated as follows:

x	-3	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

The graph of $f(x) = 2^x$ is shown in the following figure:



Example 5.28

Draw the graph of $g(x) = \left(\frac{1}{2}\right)^x$ and, determine its domain and range.

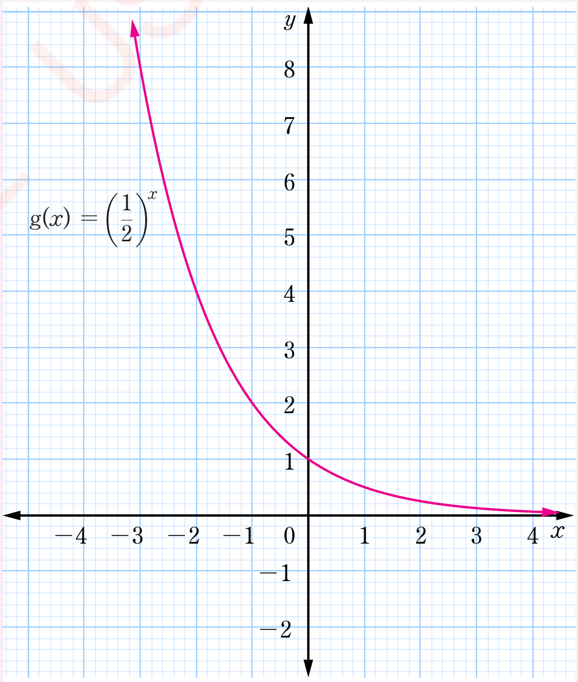
Solution

Given $g(x) = \left(\frac{1}{2}\right)^x$.

The selected few values for graphing the function are tabulated as follows:

x	-3	-2	-1	0	1	2	3
$g(x) = \left(\frac{1}{2}\right)^x$	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

The graph of $g(x) = \left(\frac{1}{2}\right)^x$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R},\ y > 0\}$

Example 5.29

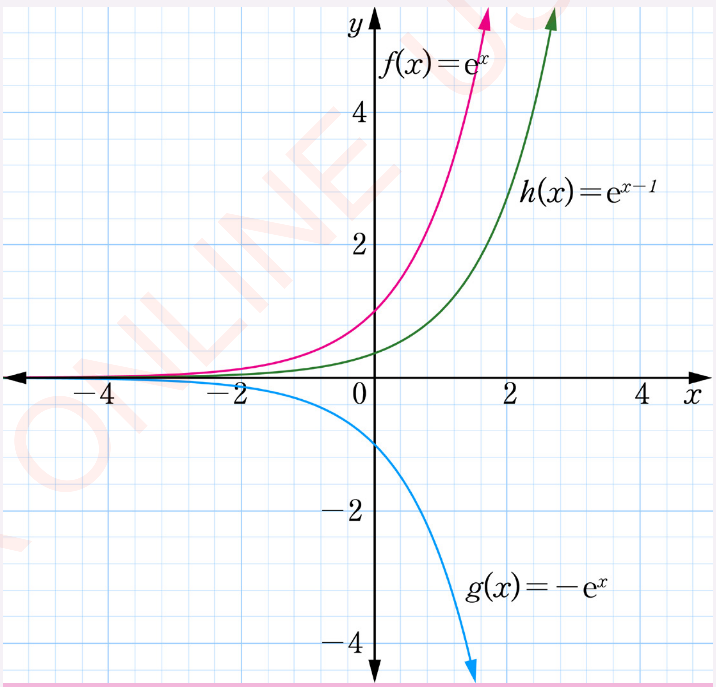
Draw the graphs of $f(x)=e^x$, $g(x)=-e^x$, and $h(x)=e^{x-1}$ on the same *xy*-plane

Solution

The selected few values for graphing function are tabulated as follows:

x	–3	–2	–1	0	1	2
$f(x)=e^x$	0.0498	0.1353	0.3679	1	2.7183	7.3891
$g(x)=-e^x$	−0.0498	−0.1353	−0.3679	−1	−2.7183	−7.3891
$h(x)=e^{x-1}$	0.0183	0.0498	0.1353	0.3679	1	2.7183

The graphs of $f(x)=e^x$, $g(x)=-e^x$, and $h(x)=e^{x-1}$, are shown in the following figure:



Exercise 5.8

1. Draw the graph of $f(x)=\left(\frac{1}{3}\right)^x$ and, determine its domain and range.
2. Graph each of the following function and state its domain and range.
   (a) $f(x)=2^{x+4}-4$
   (b) $f(x)=-e^{x+2}$
   (c) $f(x)=\left(\frac{1}{2}\right)^{x-2}-3$
   (d) $f(x)=-2^{x+3}+1$
   (e) $f(x)=e^{-x}+3$
3. Draw the graph of $y=4^x$, $y=5^x$, and $f(x)=\left(\frac{1}{3}\right)^x$ on the same axes.
4. Sketch the graph of $f(x)=a^x$ for the following values of a, on the same axes.
   (a) $a=3$ (d) $a=6$
   (b) $a=7$ (e) $a=\frac{1}{6}$
   (c) $a=2$
5. Graph each of the following exponential functions.
   (a) $f(x)=8^x$
   (b) $f(x)=e^{x-1}$
   (c) $f(x)=6^{x+1}$
   (d) $f(x)=2^{x+4}$
   (e) $f(x)=2^{x-3}-1$
6. Graph each of the following functions and state the domain and range.
   (a) $f(x)=\left(\frac{1}{7}\right)^{x-1}+7$
   (b) $f(x)=\left(\frac{1}{4}\right)^{x+2}-1$
   (c) $f(x)=5^{x+1}$
   (d) $f(x)=3^{x+2}+1$
7. Sketch the graph of $f(x)=e^{(x+3)\ln 2}$.

Graphs of logarithmic functions

If a is any positive real number not equal to 1, then a function defined as $f(x)=\log_a x,\ x>0$, is called a logarithmic function. A logarithmic function is the inverse of an exponential function, that is, if $y=a^x$, then $x=\log_a y,\ a>0,\ a\neq 1$.

The procedure for drawing graphs of logarithmic functions are similar to the procedure used for drawing exponential functions.

Example 5.30

Draw the graph of $f(x)=\log_2 x$ and state the domain and range.

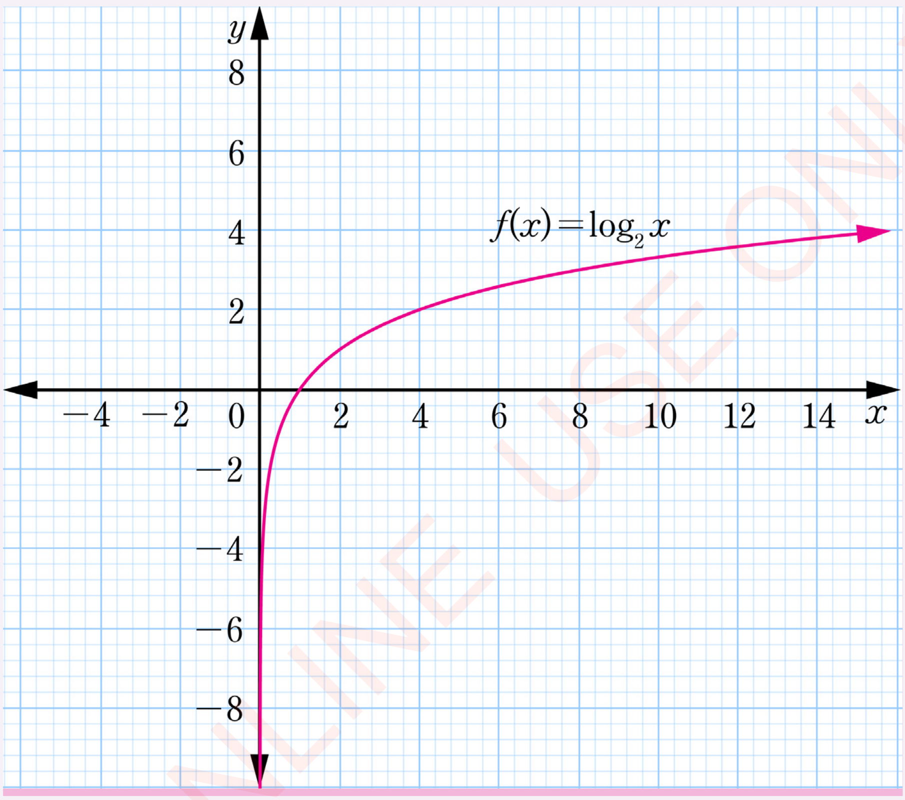
Solution

Given $f(x)=\log_2 x$.

The selected few values for graphing the function are tabulated as follows:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8
$f(x) = \log_2 x$	–3	–2	–1	0	1	2	3

The graph of $f(x) = \log_2 x$ is shown in the following figure:



Domain $= \{x : x \in \mathbb{R},\ x > 0\}$, Range $= \{y : y \in \mathbb{R}\}$.

Example 5.31

Draw the graph of $f(x) = \ln x$.

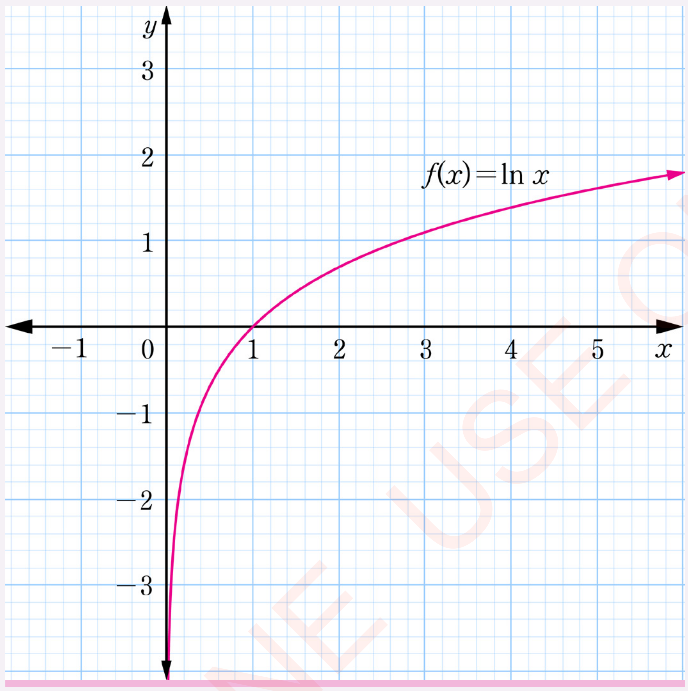
Solution

Given $f(x) = \ln x$.

The selected few values for graphing the function are tabulated as follows:

x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4
$f(x) = \ln x$	–2.08	–1.39	– 0.69	0	0.69	1.39

The graph of $f(x) = \ln x$ is shown in the following figure:



Example 5.32

Draw the graph of $f(x) = \log_{\frac{1}{2}} x$, hence state its domain and range.

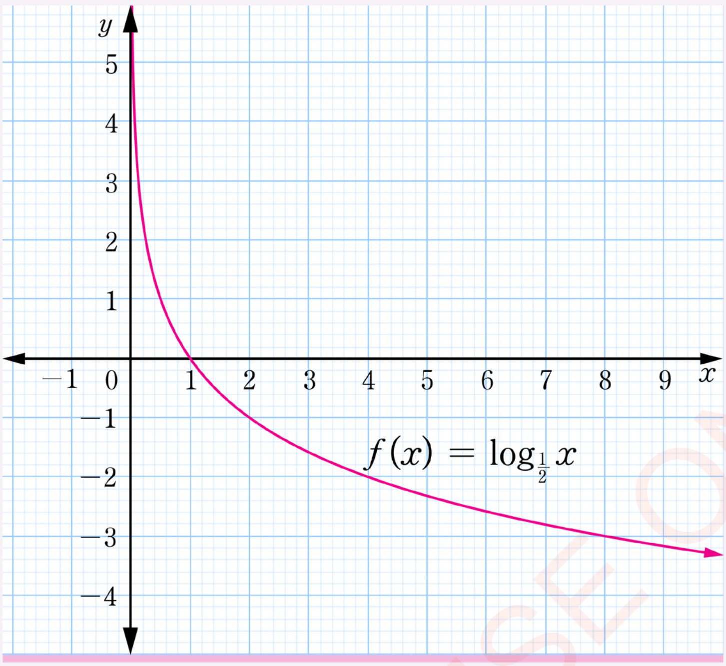
Solution

Given $f(x) = \log_{\frac{1}{2}} x$.

The selected few values for graphing the function are tabulated as follows:

x	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$
$f(x) = \log_{\frac{1}{2}} x$	–3	–2	–1	0	1	2	3

The graph of $f(x) = \log_{\frac{1}{2}} x$ is plotted in the following figure.



Therefore, the domain $= \{x : x \in \mathbb{R},\ x > 0\}$ and range $= \{y : y \in \mathbb{R}\}$.

Example 5.33

Draw the graph of $f(x) = \log_4 (x+2) + 1$, hence state its domain and range.

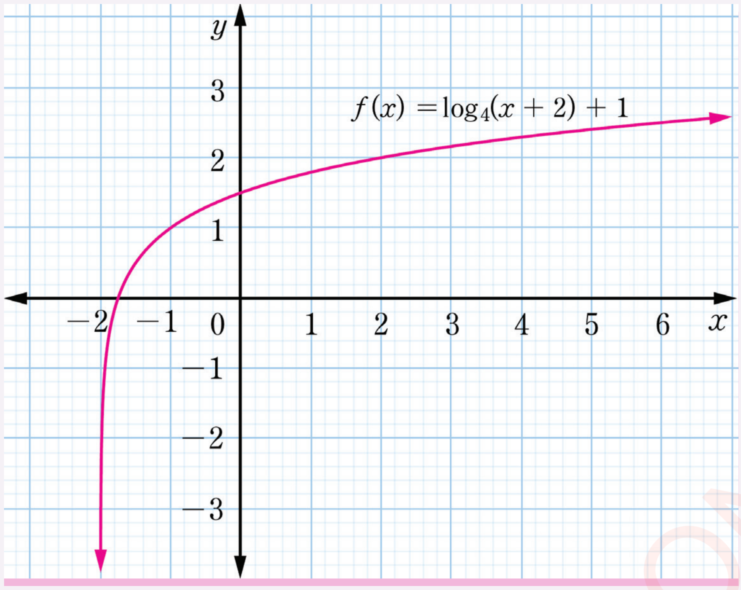
Solution

Given $f(x) = \log_4 (x+2) + 1$.

The selected few values for graphing the function are tabulated as follows:

x	–1.9961	–1.9844	–1.9375	–1.75	–1	2
$f(x) = \log_4 (x+2) + 1$	–3	–2	–1	0	1	2

The graph of $f(x) = \log_4 (x+2) + 1$ is shown in the following figure:



Therefore, the domain $= \{x : x \in \mathbb{R},\ x > -2\}$ and range $= \{y : y \in \mathbb{R}\}$.

Example 5.34

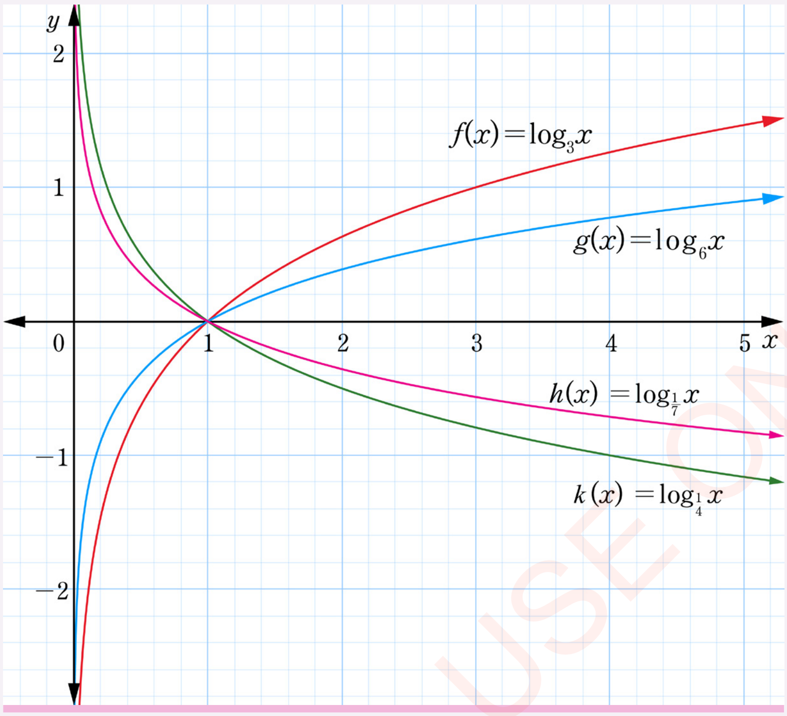
Draw the graphs of $f(x) = \log_3 x$, $g(x) = \log_6 x$, $h(x) = \log_{\frac{1}{7}} x$, and $k(x) = \log_{\frac{1}{4}} x$ on the same *xy*-plane.

Solution

The selected few values correct to one decimal point for graphing the functions are tabulated as follows:

x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27
$h(x) = \log_{\frac{1}{7}} x$	1.7	1.1	0.6	0	–0.6	–1.1	–1.7
$f(x) = \log_3 x$	–3	–2	–1	0	1	2	3
$g(x) = \log_6 x$	–1.8	–1.2	– 0.6	0	0.6	1.2	1.8
$k(x) = \log_{\frac{1}{4}} x$	2.4	1.6	0.8	0	–0.8	– 1.6	–2.4

The graphs of $f(x)=\log_3 x$, $g(x)=\log_6 x$, $h(x)=\log_{\frac{1}{7}} x$, and $k(x)=\log_{\frac{1}{4}} x$ are shown in the following figure:



Exercise 5.9

1. Draw the graph and identify the domain and range of each of the following:
   (a) $y=\log_6(x-1)-4$
   (b) $y=\log_6(3x-14)+1$
   (c) $y=-\log_5(x-1)+4$
   (d) $y=\log_5(x+2)+1$
   (e) $y=\log_2(x-2)-3$
2. Draw the graph of $f(x)=-2\log(x-3)+1$.
3. Draw the graph of $f(x)=-2\log_5(-(x-3))+1$.
4. Draw the graph of $f(x)=\log_9\left(\frac{15}{7}x\right)$, find its domain and range.
5. Draw the graphs of $f(x)=e^{2x}$ and $g(x)=\ln 2x$ on the same xy-plane.

6. Draw the graph of $f(x) = \log_2(x+1)$.
7. Draw the graph of $f(x) = 4^{x+2} - 6$. Hence, using the graph, find the
   (a) Domain and range.
   (b) Intercepts of $f(x)$.

Chapter summary

1. A function is a relation between a set of inputs and a set of outputs according to a certain rule.
2. A polynomial function is defined by $y = f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $n \in \mathbb{N}$, and $a_0, a_1, ..., a_n \in \mathbb{R}$.
3. A rational function is any function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where the numerator $p(x)$ and denominator $q(x)$ are polynomial functions and $q(x) \neq 0$.
4. A composite function is a function obtained by substituting one function into the other. If $f(x)$ and $g(x)$ are two functions, then the composite function $f[g(x)]$ is abbreviated as $(f \circ g)(x)$.
5. An exponential function $f(x)$ with base a is a function of the form $f(x) = a^x$ or $y = a^x$, where $a > 0$, $a \neq 1$, and $x \in \mathbb{R}$.
6. A logarithmic function is the inverse of an exponential function.

Revision exercise 5

1. Draw the graph of $f(x) = 2x - x^2$ and state its domain and range.
2. Draw the graph of each of the following:
   (a) $f(x) = -\dfrac{5}{3}x + 8$, and label the x-intercept.
   (b) $f(x) = \dfrac{1}{3}x + 1$, and $g(x) = 4$ on the same set of axes and determine the point where $f(x) = g(x)$.
   (c) $f(x) = \begin{cases} x+2, & x \in [-2, 2) \\ 2x-3, & x \in [1, 3] \end{cases}$

3. Draw the graph of each of the following:
   (a) $f(x)=-x^2+4x-3$.
   (b) $f(x)=x^2-4x+3$, and determine the domain and range.
   (c) $f(x)=-x^2$, $g(x)=-\frac{1}{3}x^2$, and $h(x)=-3x^2$ on the same axes.
4. Find the domain for which the functions $f(x)=x^2-3$ and $g(x)=3x-5$ are equal.
5. Draw the graph of each of the following functions and determine their domain and range:
   (a) $f(x)=(x+2)(x+4)(x-2)$
   (b) $f(x)=x^3-9x+8$ for $-4\le x\le 4$
   (c) $f(x)=x^3+3x^2+x-2$
6. Draw the graph of each of the following:
   (a) $f(x)=3x^4-7x^2$
   (b) $f(x)=x^4-2x^2$
   (c) $f(x)=x^4-2x^3-5x^2$
   (d) $f(x)=-(x^4-4x^3-x^2+10x+2)$
7. For each of the following rational functions, find the vertical asymptotes, horizontal asymptotes, oblique asymptotes, and holes (if any), then, sketch the graph and describe its behaviour near the asymptotes, hence, state its domain.
   (a) $f(x)=\dfrac{x}{x^2+x-14}$
   (b) $f(x)=\dfrac{x^3+2}{x^2-2}$
   (c) $f(x)=\dfrac{2x^2+6x}{x^2+4x+3}$
   (d) $f(x)=\dfrac{3}{3-x^2}$
   (e) $f(x)=\dfrac{x}{(x+1)(x-2)}$
   (f) $f(x)=\dfrac{5}{(x-1)^2(x+2)}$
   (g) $f(x)=\dfrac{1}{(x-1)(x+3)}$
   (h) $f(x)=\dfrac{x^3+2x^2-9x-18}{x^3+6x^2+5x-12}$
   (i) $f(x)=\dfrac{x^2-2x+1}{x^3+x^2-2x}$
   (j) $f(x)=\dfrac{12}{x^2+2x-3}$
   (k) $f(x)=\dfrac{3x-9}{x^2-x-2}$
   (l) $f(x)=\dfrac{2-3x}{x^2+3x-3}$
   (m) $f(x)=\dfrac{2x^3}{x^2-9}$

8. Given that $x = \{1, 2, 3, 4, 5\}$, $g(x) = \{(1, 0), (2, 1), (3, 3), (4, 6), (5, 8)\}$ and $f(x) = 2x - 4$, find:

   (a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$

9. Given that $f = \{(1, 7), (2, 9), (3, 12), (4, 15), (5, 19)\}$ and $g = \{(7,3), (-1,5), (9,4), (2,-3)\}$

   Find:

   (a) $f \circ g$ (b) $g \circ f$

10. Given that $f(x) = x^2 + 2x + 2$, $g(x) = x - 4$, and $h(x) = \cos x$, find each of the following:

    (a) $(f \circ g)(x)$

    (b) $(g \circ f)(x)$

    (c) $(g \circ f \circ h)(x)$

11. Given that $(g \circ f)(x) = 18x^2 + 2$ and $g(x) = 2x^2 + 5x + 8$, find $f(x)$.

12. If $f(x) = 4 - x$ and $g(x) = \frac{4}{x}$, show that $f \circ g \neq g \circ f$.

13. Given that

    $$f(x) = 2x - 2 \text{ and } g(x) = \frac{5}{x-2},$$

    find:

    (a) $(f \circ g)(x)$

    (b) $(g \circ f)(x)$

    (c) $(f \circ g)(3)$

    (d) $(f \circ g)(4)$

    (e) Sketch the graph of $(g \circ f)(x)$ and state its domain and range.

14. Given that

    $f(x) = \sqrt{20 - x^2}$ and $g(x) = x^2 - 4$, solve for x if $(f \circ g)(x) = 0$.

15. If $f(x) = x^2 - x$ and $g(x) = \frac{x-1}{x+2}$, find each of the following:

    (a) $(f \circ g)(x)$ (b) $(g \circ f)(-1)$

16. If $f(x) = \sqrt{x-4}$ and $g(x) = |x|$, find each of the following:

    (a) $(f \circ g)(8)$ (b) $(g \circ f)(20)$

17. Draw the graph of each of the following functions and determine the domain and range:

    (a) $f(x) = \left(\frac{5}{3}\right)^{-x}$

    (b) $f(x) = 4^x$

    (c) $f(x) = \left(\frac{1}{3}\right)^{x+1} - 1$

    (d) $f(x) = 2^{x-1} + 2$

    (e) $f(x) = \left(\frac{5}{3}\right)^{x}$

    (f) $f(x) = \left(\frac{1}{4}\right)^{x+2}$

    (g) $f(x) = \left(\frac{1}{4}\right)^{x-1} + 4$

    (h) $f(x) = e^{x-1}$

    (i) $f(x) = -\left(\frac{1}{6}\right)^{-x}$

18. Draw the graph of each of the following:

(a) $f(x) = -4\log(x-2)+1$

(b) $f(x) = \log\left(\dfrac{x+2}{x-3}\right)^2$.

19. (a) If $f(x) = \log_3 x$ and $g(x) = x-1$, graph $(f \circ g)(x)$ and find its domain and range.

(b) Draw the graph of $f(x) = (\log_6 x) - 2$.

20. (a) Describe a cubic function?

(b) Draw the graph of $f(x) = x^3 - 9x + 5$ and state the behaviour of the function for positive and negative large values of x.

Chapter Six

Algebra

Introduction

Algebra deals with the representation of numbers and quantities in formulae and equations by symbols and letters. In this chapter, you will learn about indices and logarithms, series, proofs by mathematical induction, roots of a polynomial function, remainder and factor theorems, inequalities, matrices, binomial theorem, and partial fractions. The competencies developed can be applied in various real-life situations such as in solving problems related to landscape designing, computer programming, real estate planning, business and finance management, geometry, budgeting, making a schedule of activities, cooking, shopping, and in many other fields.

Indices and logarithms

Indices and logarithms are inter-related areas of algebra in the sense that they all deal with powers of numbers. An index is the exponent for which a base is raised. For instance, if y, b, and x are any real numbers, such that $y = b^x$, then b^x is called a power with an exponent x (or index x) and a base b. The index x is called the logarithm of y to base b, and it is written as $x = \log_b y$. Fractional powers of positive numbers, in radical sign are called surds.

Indices

An index is a number which shows how many times a quantity has been multiplied by itself. This number is usually raised to another number or variable and is called the exponent. For example, given 10^2 and x^5, then 2 and 5 are indices of 10 and x, respectively. In algebra, the laws of indices are used to simplify the expressions involving indices.

There are several laws of indices which include multiplication, division, power of zero, brackets, power of products, negative power, and fractional power. These laws enable to simplify calculations or expressions involving powers of the same base.

Activity 6.1: Identifying the laws of indices

Individually or in a group, perform the following tasks:

1. List down laws of indices and prove any two laws.
2. Construct two questions for each law in task 1.
3. Use the laws to solve the respective questions in task 2.
4. Share your work with other students for more inputs.

Law of product of powers with the same base

When multiplying two terms of the same base, add their indices.

In general, $a^m \times a^n = a^{m+n}$, where a is any positive real number and m, n are positive integers.

Proof

$a^m = a \times a \times a \times ... \times a$ (m-times), and

$$a^n = a \times a \times a \times \cdots \times a \ (n\text{-times})$$

$$= \underbrace{(a \times a \times \cdots \times a)}_{m\text{-times}} \times \underbrace{(a \times a \times \cdots \times a)}_{n\text{-times}}$$

$$= a \times a \times a \times \cdots \times a \ (m+n\text{-times})$$

$$= a^{m+n}$$

Therefore, $a^m \times a^n = a^{m+n}$.

Law of quotient of powers

When dividing two terms of the same base, subtract their indices.

In general, $a^m \div a^n = a^{m-n}$.

Proof

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a \times a \times a \times \cdots \times a \ (m\text{-times})}{a \times a \times a \times \cdots \times a \ (n\text{-times})}$$

If $m > n$, the n factors in the denominator will cancel with n of the m factors in the numerator leaving $(m - n)$ factors, such that

$$a^m \div a^n = a \times a \times \cdots \times a \text{ to } (m-n) \text{ times}$$

$$= a^{m-n}$$

Therefore, $a^m \div a^n = a^{m-n}$.

Law of product of powers

If a quantity with a power is itself raised to another power, then the powers are multiplied together. That is, $(a^m)^n = a^{mn}$.

Proof

$$\left(a^m\right)^n = a^m \times a^m \times \cdots \times a^m \ (n\text{-times})$$

$$= a^{mn}$$

Therefore, $\left(a^m\right)^n = a^{mn}$.

Law of powers of product with the same exponent

If a product of two numbers is raised to a single power, then every factor of the product is raised to that power. That is, $(ab)^m = a^m b^m$, where a and b are any positive real numbers.

Proof

$$(ab)^m = (ab)\times(ab)\times(ab)\times\cdots(ab)(m\text{-times})$$

$$(ab)^m = \underbrace{a\times a\times\cdots\times a}_{m\text{-times}} \times \underbrace{b\times b\times\cdots\times b}_{m\text{-times}}$$

$$= a^m b^m$$

Therefore, $(ab)^m = a^m b^m$.

Law of negative power

A negative power can be written as a fraction.

In general, $a^{-n} = \dfrac{1}{a^n}$.

Law of power zero

If a power is divided by itself or if a quantity is raised to zero, then the result is 1.

In general, $\dfrac{a^n}{a^n} = a^{n-n}$

$$= a^0 = 1.$$

Law of fractional power

In case of a fractional power, the denominator stands for the type of the root, and the numerator stands for exponent (or index).

In general, $\left(\sqrt[m]{a}\right)^n = \left(a^{\frac{1}{m}}\right)^n = a^{\frac{n}{m}}$.

Example 6.1

Simplify $\dfrac{3^{2x}}{3^{2x-3}}$.

Solution

Since $a^m \div a^n = a^{m-n}$.

Thus, $\dfrac{3^{2x}}{3^{2x-3}} = 3^{2x-(2x-3)} = 3^3$

$= 27$

Therefore, $\dfrac{3^{2x}}{3^{2x-3}} = 27$.

Example 6.2

Solve for x if $4(3^x) - 3^x = 27$.

Solution

Since 3^x appears in both of the two terms of the equation, it can be factored out.

Thus, $4(3^x) - 3^x = 3^x(4-1)$

$\Rightarrow 3^x(4-1) = 27$

$\Rightarrow 3^x \times 3 = 27$

$\Rightarrow 3^{x+1} = 3^3$;

Comparing the exponents gives,

$x + 1 = 3$

$\Rightarrow x = 2$

Therefore, $x = 2$.

Example 6.3

Solve for x in the equation,

$\sqrt{x+15} = 15 - \sqrt{x-30}$.

Solution

Given $\sqrt{x+15} = 15 - \sqrt{x-30}$.

Squaring both sides gives,

$x + 15 = 225 - 30\sqrt{x-30} + x - 30$

$\Rightarrow 30\sqrt{x-30} = 180$

$\Rightarrow \sqrt{x-30} = 6$.

Again squaring both sides to get,

$x - 30 = 36$

$\Rightarrow x = 66$

Therefore, $x = 66$.

Example 6.4

Simplify $\dfrac{\sqrt{(x-4)^6} \times \sqrt{x-4}}{\sqrt{(x-4)^3}}$.

Solution

Given $\dfrac{\sqrt{(x-4)^6} \times \sqrt{x-4}}{\sqrt{(x-4)^3}}$.

$$\Rightarrow \frac{\sqrt{(x-4)^6} \times \sqrt{x-4}}{\sqrt{(x-4)^3}} = \frac{\sqrt{(x-4)^6 \times (x-4)}}{\sqrt{(x-4)^3}}$$

$$= \sqrt{(x-4)^6 \times (x-4)^1 (x-4)^{-3}}$$

$$= \sqrt{(x-4)^{6+1-3}}$$

$$= \sqrt{(x-4)^4}$$

$$= (x-4)^2$$

Therefore,

$$\frac{\sqrt{(x-4)^6} \times \sqrt{(x-4)}}{\sqrt{(x-4)^3}} = (x-4)^2.$$

Exercise 6.1

1. Solve for each of the variable in the following questions:
   (a) $3^5 = 3^{2n-1}$
   (b) $2^x = 4^{x+1}$
   (c) $81^u - 5(9^u) = 36$
   (d) $\left(\frac{1}{2}\right)^x = 2^{x-6}$
2. Find $\frac{p}{q}$, if $\frac{\sqrt{p}+2\sqrt{q}}{\sqrt{q}-2\sqrt{p}} = \frac{1}{2}$.
3. Solve the following system of simultaneous equations:
   $$\begin{cases} 3^{x+y} = 9 \\ 5^{2x-5y} = 625 \end{cases}$$
4. Solve for y if $\sqrt{y} = 11 - \sqrt{y-11}$.
5. Solve for x, if $2x^{-\frac{2}{5}} - 8x^{-\frac{1}{5}} + 4 = 0$.
6. Find the value of x, if $\sqrt[3]{3x-1} = 2$.
7. Solve for x, in $4^x\left(8^{x-3}\right) = \frac{1}{16^{2-x}}$.
8. Solve for x, if $6^{x+2} = 1296$.
9. Solve for x, if $\frac{4^{x+3}}{8^{10x}} = \frac{2^{10-2x}}{64^{3x}}$.
10. Solve for y, if $7^{y^2-2y-5} = \frac{1}{49}$.

Logarithms

The logarithm of a number to a given base is the index of the power to which the base must be raised to produce the number. It is simply a mirror image of an index. Thus, the logarithmic function to base a, where $a > 0$ is written as $y = \log_a x$ if and only if $x = a^y$.

Laws of logarithms

There are helpful laws for rewriting expressions involving logarithms in different ways but yet in an equivalent way. These laws are known as the laws of logarithms. The laws are applicable to any base, but the same base must be used throughout the calculation.

Activity 6.2: Recognizing the laws of logarithms

Individually or in a group, perform the following tasks:

1. List down the laws of logarithms you have learnt.
2. Prove the laws you have listed in task 1.
3. What have you observed from the tasks? Give comments.
4. List down the challenges you faced in tasks 1 and 2.
5. Share your work with other students for more inputs.

The following are four important laws of logarithms and their proofs.

Logarithm of a product

The logarithm of a product of two numbers with the same base is the sum of the logarithms of those numbers.

In general, $\log_b(xz) = \log_b x + \log_b z$, where x and z are positive numbers.

Proof

It is required to prove that

$\log_b(xz) = \log_b x + \log_b z.$

Consider the right-hand side:

Let $p = \log_b x$ and $q = \log_b z.$

Thus, $b^p = x$ ....(i)

and

$b^q = z$ ....(ii)

Multiplying equations (i) and (ii) gives,

$b^{p+q} = xz$ ....(iii)

Introducing logarithm to base b on both sides of equation (iii) gives,

$\log_b b^{p+q} = \log_b(xz)$

$\Rightarrow (p+q)\log_b b = \log_b(xz)$. But $\log_b b = 1$

Thus, $p+q = \log_b(xz)$.

Substituting the values of p and q, gives

$\log_b x + \log_b z = \log_b(xz)$

Therefore, $\log_b(xz) = \log_b x + \log_b z$.

Logarithm of a quotient

The logarithm of a quotient of two numbers with the same base is the difference of the logarithms of the numbers.

In general, $\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z$.

Proof

It is required to prove that

$\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z.$

Consider the right-hand side:

Let $p = \log_b x$ and $q = \log_b z$

Thus, $b^p = x$ .... (i)

and

$b^q = z$ ....(ii)

Dividing equation (i) by equation (ii) gives,

$$\frac{b^p}{b^q} = \frac{x}{z}$$

$\Rightarrow b^{p-q} = \frac{x}{z}$ ....(iii)

Introducing logarithm to base b on both sides of equation (iii) gives,

$\log_b b^{p-q} = \log_b\left(\frac{x}{z}\right)$

$\Rightarrow (p-q)\log_b b = \log_b\left(\frac{x}{z}\right)$

$\Rightarrow p-q = \log_b\left(\frac{x}{z}\right).$

But $p = \log_b x$ and $q = \log_b z.$

Substituting the values of p and q, gives

$\log_b x - \log_b z = \log_b\left(\frac{x}{z}\right).$

Therefore, $\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z.$

Logarithm of the n^{th} power

The logarithm of the n^{th} power of a positive quantity is n multiplied by the logarithm of the quantity. In general, $\log_b x^n = n\log_b x.$

Proof

It is required to prove that

$\log_b x^n = n\log_b x.$

Let $p = \log_b x$, so that

$x = b^p$ .... (i)

Raising to the power n on both sides of equation (i), gives

$x^n = b^{np}$ .... (ii)

Introducing logarithm to base b on both sides of equation (ii) gives,

$$\begin{aligned}\log_b x^n &= \log_b b^{np}\\ &= np\log_b b\\ &= np\end{aligned}$$

Thus, $\log_b x^n = n\log_b x$

Therefore, $\log_b x^n = n\log_b x$.

Logarithm of the m^{th} root

The logarithm of the m^{th} root of the positive quantity is $\frac{1}{m}$ multiplied by the logarithm of the quantity. In general,

$$\log_b\left(x^{\frac{1}{m}}\right) = \frac{1}{m}\log_b x.$$

Proof

It is required to prove that

$$\log_b\left(x^{\frac{1}{m}}\right) = \frac{1}{m}\log_b x$$

Let $p = \log_b x$, so that

$x = b^p$ .... (i)

Raising to the power $\frac{1}{m}$ on both sides of equation (i), gives

$x^{\frac{1}{m}} = b^{p\left(\frac{1}{m}\right)}$ .... (ii)

Introducing logarithm to base b on both sides of equation (ii) gives,

$$\log_b x^{\frac{1}{m}} = \log_b b^{\frac{p}{m}} \text{ where } \frac{p}{m} = p\left(\frac{1}{m}\right)$$

$$= \frac{p}{m}\log_b b$$

$$= \frac{p}{m}$$

Thus, $\log_b x^{\frac{1}{m}} = \frac{1}{m}\log_b x$

Therefore, $\log_b x^{\frac{1}{m}} = \frac{1}{m}\log_b x$.

If x and z are positive real numbers, and b is a real number greater than zero for $b \neq 1$, then the laws of logarithms can be summarized as follows:

(a) $\log_b(xz) = \log_b x + \log_b z$

(b) $\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z$

(c) $\log_b(x^a) = a\log_b x$

(d) $\log_b\left(x^{\frac{1}{m}}\right) = \frac{1}{m}\log_b x$

(e) $\log_b 1 = 0$

(f) $\log_b b = 1$

(g) $\log_b b^n = n$

Note that, if the logarithm of a quantity is written without indicating the base, then it is considered to be in base 10. For instance, $\log 2x = 1$ is the same as $\log_{10} 2x = 1$.

Example 6.5

Solve the following system of equations:

$$\begin{cases}\log(x+y) = 1\\ \log_2(2x-y) = 5\end{cases}$$

Solution

Given $\log(x+y) = 1$, then

$x + y = 10$ ....(i)

Also, $\log_2(2x-y) = 5$, then

$2x - y = 2^5$

$\Rightarrow 2x - y = 32$...........................(ii)

Solving equations (i) and (ii) simultaneously,

$$+\begin{cases} x+y=10 \\ 2x-y=32 \end{cases}$$

Thus, $3x=42 \Rightarrow x=14$

From equation (i), it gives

$y=10-14=-4$

Therefore, $(x,y)=(14,-4)$.

Example 6.6

Given that $\log_5 6 = 1.1133$ and $\log_5 3 = 0.6826$, find the value of $\log_5 54$ correct to five significant figures.

Solution

Let $\log_5 54 = \log_5(6\times 9)$

$$\begin{aligned} &= \log_5 6 + \log_5 9 \\ &= \log_5 6 + 2\log_5 3 \\ &= 1.1133 + 2(0.6826) \end{aligned}$$

Thus, $\log_5 54 = 2.4785$.

Therefore, $\log_5 54 = 2.4785$.

Example 6.7

Solve for x in the equation,

$\log_2(x-1)+\log_2(x+1)=3$.

Solution

Given $\log_2(x-1)+\log_2(x+1)=3$

$\Rightarrow \log_2[(x-1)(x+1)]=3$

$\Rightarrow \log_2(x^2-1)=3$

$\Rightarrow (x^2-1)=2^3$

$\Rightarrow x^2-1=8$

$\Rightarrow x^2=9$

Thus, $x=3$ or $x=-3$.

Since the logarithm of a negative number is not defined, then $x=3$ is the only solution.

Therefore, $x=3$.

Natural logarithms

There are two types of logarithms in extensive use. These are the common logarithms and the natural logarithms. Common logarithms are base 10 logarithms and they are usually written without showing the base. For example, $\log_{10} 7$ and $\log_{10} 2$ can be written as $\log 7$ and $\log 2$, respectively. Natural logarithms are logarithms with base e (where e = 2.71828...). Natural logarithms are written as "$\log_e$", but often "$\log_e$" is replaced by "ln". For example, $\log_e x$ can also be written as "$\ln x$". The natural logarithms also satisfy all laws of logarithms.

For instance;

(a) $\ln(xy)=\ln x+\ln y$

(b) $\ln\left(\frac{x}{y}\right)=\ln x-\ln y$

(c) $\ln x^n = n\ln x$

Example 6.8

Solve the following system of simultaneous equations:

$$\begin{cases} 2\ln y = 3\ln 2 + \ln x \\ y\ln 3 = x\ln 81 \end{cases}$$

Solution

Given $2\ln y = 3\ln 2 + \ln x$

$\Rightarrow \ln y^2 = \ln 8x$

$\Rightarrow y^2 = 8x$ .... (i)

Also, $y\ln 3 = x\ln 81$

$\ln 3^y = \ln 81^x$

$\Rightarrow 3^y = 81^x$

$\Rightarrow 3^y = 3^{4x}$

$\Rightarrow y = 4x$ .... (ii)

Substituting equation (ii) into equation (i) gives,

$(4x)^2 = 8x$

$\Rightarrow 16x^2 = 8x$

$\Rightarrow 8x(2x-1) = 0$

Hence, $x = 0$ or $x = \frac{1}{2}$. But x cannot be zero. Thus, $x = \frac{1}{2}$

From equation (ii), $y = 4x$, then $y = 4\left(\frac{1}{2}\right) = 2$

Therefore, $(x,\ y) = \left(\frac{1}{2},\ 2\right)$.

Example 6.9

Solve for y in $e^{2y^2-10y+19} = e^7$.

Solution

Introducing natural logarithm both sides gives,

$\ln e^{2y^2-10y+19} = \ln e^7$

$\left(2y^2 - 10y + 19\right)\ln e = 7\ln e$

$\Rightarrow 2y^2 - 10y + 19 = 7$, since $\ln e = 1$

$\Rightarrow 2y^2 - 10y + 12 = 0$

$\Rightarrow y = 3$ or $y = 2$

Therefore, $y = 3$ or $y = 2$.

Example 6.10

If $\ln 3 = 1.0986$, $\ln 7 = 1.9459$, and $\ln 11 = 2.3979$. Evaluate $\ln 14553$ correct to five significant figures.

Solution

$$\begin{aligned}\ln 14553 &= \ln(3\times3\times3\times7\times7\times11)\\ &= \ln(3^3\times7^2\times11)\\ &= 3\ln 3 + 2\ln 7 + \ln 11\\ &= 3(1.0986) + 2(1.9459) + 2.3979\end{aligned}$$

Thus, $\ln 14553 = 9.5855$

Therefore, $\ln 14553 = 9.5855$.

Conversion of logarithm from one base to another

One of the properties of logarithms as a powerful method of computation is its flexibility to change from one base to another. The change of base formula is used to write a logarithm of a number with a given base as the ratio of two logarithms each with the same base that is different from the base of the original logarithm.

Let b and c be positive numbers not equal to one. Suppose it is required to determine $\log_b x$, then it will be as follows:

If $p = \log_b x$, it implies that

$x = b^p$..(i)

Introducing logarithm to base c on both sides of equation (i) gives,

$\log_c x = p \log_c b$

Making p the subject of the equation gives,

$$p = \frac{\log_c x}{\log_c b}$$

But $p = \log_b x$

Thus, $\log_b x = \dfrac{\log_c x}{\log_c b}$.

Therefore, $\log_b x = \dfrac{\log_c x}{\log_c b}$.

This formula is used to convert a logarithm of a number from one base to another.

Note that: The reciprocal of logarithm can be expressed by;

$$\log_b a = \frac{1}{\log_a b}.$$

Example 6.11

Solve for m and n if

$\log_3 m = n = \log_9 (2m-1)$

Solution

Given $\log_3 m = n$

Thus, $m = 3^n$ ....(i)

and $\log_9 (2m-1) = n$

Thus, $2m-1 = 9^n$..................... (ii)

From equation (ii), it implies that

$2m-1 = 3^{2n}$ ....(iii)

But $3^n = m \Rightarrow m^2 = 3^{2n}$. .... (iv)

Comparing equations (iii) and (iv), it follows that;

$2m-1 = m^2$

$\Rightarrow m^2 - 2m + 1 = 0$

$\Rightarrow m = 1$

From equation (i) it gives;

$1 = 3^n$

$\Rightarrow 3^0 = 3^n$

Thus, $n = 0$.

Therefore, $m = 1$ and $n = 0$.

Example 6.12

Solve for k if $\log_k 9 + \log_3 k = 3$.

Solution

Given $\log_k 9 + \log_3 k = 3$.

$$\Rightarrow 2\log_k 3 + \frac{1}{\log_k 3} = 3$$

$$\Rightarrow 2(\log_k 3)^2 + 1 = 3\log_k 3$$

$$\Rightarrow 2(\log_k 3)^2 - 3\log_k 3 + 1 = 0,$$

Solving the quadratic equation gives;

$$\log_k 3 = 1 \text{ or } \log_k 3 = \frac{1}{2}$$

Thus, $k = 3$ or $k = 9$.

Therefore, $k = 3$ or $k = 9$.

Exercise 6.2

1. Solve for unknown variables in each of the following equations:

   (a) $\log_m 9+\log_{(m^2)} 3=\frac{5}{2}$

   (b) $\log_2 y=\log_4 (y+6)$

   (c) $\ln(\ln(3x))=0$

   (d) $\log_3 z+3\log_z 3=4$

   (e) $\log_3 4=\ln x+\ln 4$

   (f) $e^{4x-2}=e^x$

2. If $p^z=\left(\frac{p}{m}\right)^x=m^y$ where $p\neq 1$, show that $x=\frac{yz}{y-z}$.

3. Given that $\log_2 m+2\log_4 n=4$. Show that $mn=16$. Hence, solve for m and n in the system of simultaneous equations,
$$\begin{cases}\log_4(m+n)=1;\\ \log_2 m+2\log_4 n=4.\end{cases}$$

4. In each of the following solve for x giving answers to four decimal places:

   (a) $\log x=\log 2x^2-2$

   (b) $\ln x+\ln(x+1)=5$

5. Express $\log_{16}(fg)$ into $\log_4 f$ and $\log_4 g$. Hence, solve for f and g in the equations, $\log_{16}(fg)=3\frac{1}{2}$; $\frac{\log_4 f}{\log_4 g}=-8$.

6. If $\log_3 11=2.1827$ and $\log_3 5=1.4650$, determine each of the following:

   (a) $\log_3 \frac{11}{5}$ (c) $\log_3 \frac{1331}{625}$

   (b) $\log_3 275$

7. Find the solution of the following system of simultaneous equations:
$$\begin{cases}\log_2 x^2+\log_2 y^3=1\\ \log_2 x-\log_2 y^2=4\end{cases}$$

8. Find the values of p in the equation, $\log_3 p+3\log_p 3=4$.

9. Show that the values of q in the equation, $\log_3 q-4\log_q 3+3=0$ are $\frac{1}{8}$ and 3.

10. Given that $\log_8(z^2+z)=\log_8 12$, find the values of z.

Sequences and series

The itemized collection of elements in which repetitions are allowed is known as a sequence, while a series is the sum of all elements.

Sequence

A set of numbers or algebraic expressions for which a member can be obtained from a proceeding member by a definite rule is called a sequence. The numbers that make up the sequence are called terms of the sequence. The terms in a sequence are named by using their positions.

In general, a sequence is usually written as; $a_1, a_2, a_3, \ldots, a_n$, where a_1 is the first term, a_2 is the second term, a_3 is a third term, and a_n is the n^{th} term.

The n^{th} term is also called the general term of the sequence, and it represents the last term of the sequence. In this case, the subscript of each term represents the number of terms. A sequence is either finite or infinite depending upon the number of terms in a sequence. If the number of terms in a sequence is known, then the sequence is called a finite sequence, otherwise it is infinite. Generally, the sequence $a_1, a_2, a_3, \ldots$ is denoted by a_n . Other defined notation for a sequence can be used.

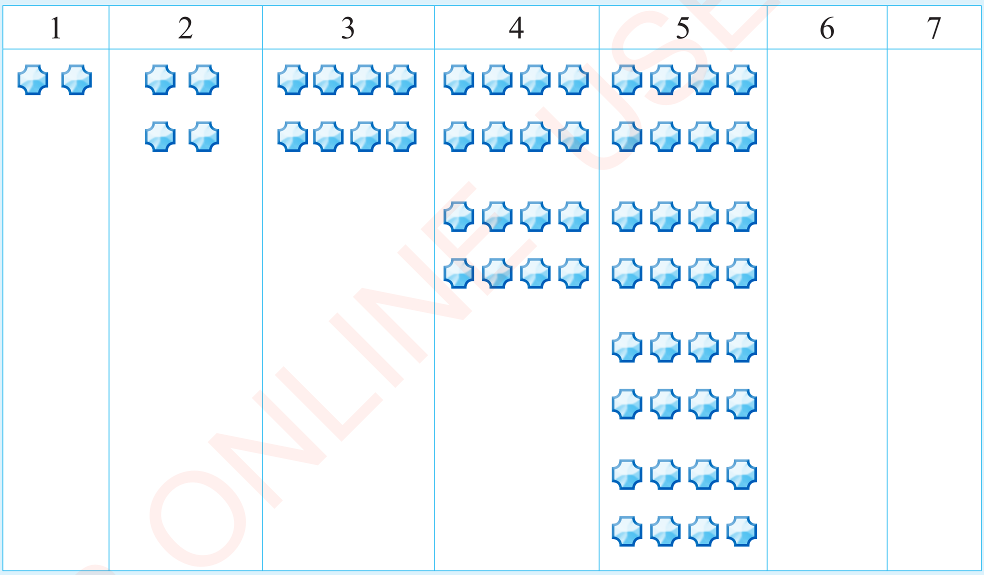
For example, in the sequence 1, 3, 5, 7, 9, 11, … the rule is that, 2 is added to obtain the next term.

By using symbols, it is written as $a_1 = 1$, $a_2 = a_1 + 2$, $a_3 = a_2 + 2, \ldots$

In general, $a_{r+1} = a_r + 2$, where $r = 1, 2, 3, \ldots$

Activity 6.3 Identifying a formula for a sequence

Study the pattern of the given objects and perform the tasks that follows individually or in a group:



1. Find the rule of the given pattern of the objects.
2. Suggest the number of objects in columns, 6 and 7 in task 1.
3. Write the formula for terms of the pattern.
4. Share your findings with your fellow students for more inputs.

Example 6.13

Find the first three terms of the following sequences:

(a) $a_r = \dfrac{1}{r+1}$ (b) $a_r = \dfrac{1}{2}r(r+1)$

Solution

(a) Given $a_r = \dfrac{1}{r+1}$

The first three terms can be obtained as follows;

$$a_1 = \frac{1}{1+1} = \frac{1}{2},$$

$$a_2 = \frac{1}{2+1} = \frac{1}{3}, \text{ and}$$

$$a_3 = \frac{1}{3+1} = \frac{1}{4}.$$

Therefore, the first three terms are $\dfrac{1}{2}, \dfrac{1}{3}$, and $\dfrac{1}{4}$.

(b) Given $a_r = \dfrac{1}{2}r(r+1)$.

The first three terms can be obtained as follows;

$$a_1 = \frac{1}{2}\times 1(1+1) = 1$$

$$a_2 = \frac{1}{2}\times 2(2+1) = 3, \text{ and}$$

$$a_3 = \frac{1}{2}\times 3(3+1) = 6$$

Therefore, the first three terms are 1, 3, and 6.

Series

When the terms of the sequence are linked together with signs of addition or subtraction, the resulting expression is known as series. A series is said to be finite if it is the sum of a finite number of terms. Otherwise, it is infinite. For instance, the series, $1-2x+3x^2-4x^3$ is finite, while the series, $1+2+3+4+5+\cdots$ is infinite.

Example 6.14

Find the next two terms of the series, $1+1+20+210+2300+\cdots$

Solution

By observing the given series, it can be seen that the next term is ten times the sum of the two consecutive previous terms.

The 6^{th} term $= 10(210+2300) = 25{,}100$

Similarly,

the 7^{th} term $= 10(2300+25100) = 274{,}000$

Therefore, the next two terms are 25,100 and 274,000.

Sigma notation

Consider the sum of the first 6 terms of the series $3+6+9+12+15+18$. This series can also be written as

3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6).

All terms of this series are of the form $3n$. Thus, $3n$ is the general term. The series can be defined as the sum of terms of the form $3n$, where n takes all integral values in order from 1 to 6 inclusive. The Greek letter $\sum$ called sigma is used to denote "the sum of terms". Therefore, the series can be defined more precisely as $\sum_{k=1}^{6} 3k$. This means that "the sum of all terms of the form $3k$, where k takes all integer values from 1 to 6 inclusive".

The symbol can be used to indicate the finite and infinite series.

For example, $\sum_{k=1}^{6} 3k = 3+6+9+12+15+18$ is a finite series and

$\sum_{k=1}^{n} 3k = 3+6+9+12+15+18+\cdots$ is an infinite series.

The sigma notation shortens the expression of series and enables to select a particular term of a series without writing down all the preceded terms. For example, given the series $\sum_{k=2}^{19} 3(k-1)$ then:

The first term is obtained when $k = 2 \Rightarrow\ 3(2-1) = 3$

The second term is obtained when $k = 3 \Rightarrow\ 3(3-1) = 6$

The tenth term is obtained when $k = 11 \Rightarrow 3(11-1) = 30$

The last term is obtained when $k = 19 \Rightarrow\ 3(19-1) = 54$.

Example 6.15

Write $\sum_{k=1}^{4} (-1)^{k+1}\ 3^{k-1}$ in expanded form.

Solution

$$\begin{aligned}\sum_{k=1}^{4} (-1)^{k+1}\ 3^{k-1} &= (-1)^2 3^0 + (-1)^3 3^1 + (-1)^4 3^2 + (-1)^5 3^3 \\ &= (1)(1) + (-1)(3) + (1)(9) + (-1)(27) \\ &= 1-3+9-27\end{aligned}$$

Therefore, $\sum_{k=1}^{4} (-1)^{k+1}\ 3^{k-1} = 1-3+9-27$.

Example 6.16

Evaluate $\sum_{k=5}^{12} 2^{4-k}$.

Solution

$$\sum_{k=5}^{12} 2^{4-k} = 2^{4-5} + 2^{4-6} + 2^{4-7} + 2^{4-8} + 2^{4-9} + 2^{4-10} + 2^{4-11} + 2^{4-12}$$

$$= 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4} + 2^{-5} + 2^{-6} + 2^{-7} + 2^{-8}$$

$$= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \frac{1}{256}$$

$$= \frac{255}{256}$$

Therefore, $\sum_{k=5}^{12} 2^{4-k} = \frac{255}{256}$.

Properties of sigma notation

If u is any constant that does not depend on summation of index, then:

1. $\sum_{k=n_0}^{n} ua_k = u\sum_{k=n_0}^{n} a_k$
2. $\sum_{k=n_0}^{n} (ua_k \pm ub_k) = u\sum_{k=n_0}^{n} a_k \pm u\sum_{k=n_0}^{n} b_k$
3. $\sum_{k=n_0}^{n} u = un$

Example 6.17

Evaluate $\sum_{k=1}^{8} [4 + (k-1)]$.

Solution

$$\sum_{k=1}^{8} [4 + (k-1)] = (4 + 0) + (4 + 1) + (4 + 2) + (4 + 3) + (4 + 4) + (4 + 5) + (4 + 6) + (4 + 7)$$

$$= 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11$$

$$= 60$$

Therefore, $\sum_{k=1}^{8} [4 + (k-1)] = 60$.

Example 6.18

Determine $\sum_{k=1}^{n} 1$.

Solution

$$\sum_{k=1}^{n} 1 = 1 + 1 + 1 + 1 + \cdots + 1.$$

(This process is repeated "n" times as 1 does not change with the change in k). Since the terms are not changing, we can take one term and multiply by the number of times.

$$\sum_{k=1}^{n} 1 = 1 \times n = n.$$

Therefore, $\sum_{k=1}^{n} 1 = n$.

Activity 6.4: Deducing a formula for the sum of the first n natural numbers

Individually or in a group, perform the following tasks:

1. Write the formula for the sum of the first n terms of an arithmetic progression.
2. Use the formula in task 1 to deduce the formula for the first n natural numbers.
3. What have you observed from tasks 1 and 2? Give comments.
4. Share your results with your fellow students for more inputs.

The sum of the first n natural numbers is $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Example 6.19

Prove that $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Solution

Let $S_n = \sum_{k=1}^{n} k$.

Expanding S_n gives:

$S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n$ ....(i)

The same sum can also be obtained by reversing order of the series, that is:

$$S_n = n+(n-1)+(n-2)+\cdots+3+2+1 \text{ ....(ii)}$$

Arrange equations (i) and (ii) so that the corresponding terms come together:

$$+\begin{cases} S_n = 1+2+3+\cdots+(n-2)+(n-1)+n \\ S_n = n+(n-1)+(n-2)+\cdots+3+2+1 \end{cases} \text{ ....(iii)}$$

Adding the two equations in the system (iii) gives;

$$2S_n = (n+1)+(n+1)+(n+1)+\cdots+(n+1)+(n+1)+(n+1)$$

$$= (n+1)n$$

Thus, $S_n = \frac{1}{2}(n+1)n$

Therefore, $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$.

Sum of squares of the first *n* natural numbers

Suppose k is a natural number and k^2 is its square. If n consecutive natural numbers are considered then the sum of their squares in expanded form is $1^2+2^2+3^2+4^2+5^2+6^2+\cdots+n^2$. In sigma notation, this is written as $\sum_{k=1}^{n} k^2$.

Now, suppose S_n is the required sum, then

$$S_n = \sum_{k=1}^{n} k^2 = 1^2+2^2+3^2+4^2+5^2+6^2+\cdots+n^2.$$

The value of k^2 can be obtained from the expansion of $(k-1)^3$ as follows;

$$(k-1)^3 = k^3-3k^2+3k-1$$

$$\Rightarrow k^3-(k-1)^3 = 3k^2-3k+1 \text{ ....(i)}$$

Substituting $k = 1, 2, 3, \ldots, (n-2), (n-1), n$ in equation (i), it implies that:

When $k=1$; $1^3-0^3 = 3(1)^2-3(1)+1$

When $k=2$; $2^3-1^3 = 3(2)^2-3(2)+1$

$\vdots \quad \vdots \quad \vdots \quad \vdots$

When $k=n-1$; $(n-1)^3-(n-2)^3 = 3(n-1)^2-3(n-1)+1$

When $k = n$; $\quad n^3 - (n-1)^3 = 3n^2 - 3n + 1$

Adding all the terms from both sides gives

$$\left(1^3 - 0^3\right) + \left(2^3 - 1^3\right) + \ldots + \left[(n-1)^3 - (n-2)^3\right] + \left[n^3 - (n-1)^3\right]$$

$$= 3\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - 3\left(1 + 2 + 3 + \cdots + n\right) + \left(1 + 1 + 1 + \cdots + 1;\ n \text{ times}\right)$$

$$\Rightarrow n^3 - 0^3 = 3\left(1^2 + 2^2 + 3^2 + \cdots + n^2\right) - 3\left(1 + 2 + 3 + \cdots + n\right) + \left(1 + 1 + 1 + \cdots + 1;\ n \text{ times}\right)$$

$$\Rightarrow n^3 = 3S_n - 3\left(\frac{n(n+1)}{2}\right) + n$$

$$\text{Thus, } 3S_n = n^3 + \frac{3}{2}n(n+1) - n$$

$$= n(n+1)(n-1) + \frac{3}{2}n(n+1)$$

$$= n(n+1)\left[(n-1) + \frac{3}{2}\right]$$

$$\Rightarrow 3S_n = \frac{n(n+1)(2n+1)}{2}$$

$$\text{Thus, } S_n = \frac{n(n+1)(2n+1)}{6}.$$

$$\text{Therefore, } \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

Example 6.20

Prove that $\sum_{k=1}^{n}\left(6k^2 - 8k + 2\right) = n(n-1)(2n+1)$. Hence, evaluate $\sum_{k=1}^{64}\left(6k^2 - 8k + 2\right)$.

Solution

$$\sum_{k=1}^{n}(6k^2 - 8k + 2) = 6\sum_{k=1}^{n}k^2 - 8\sum_{k=1}^{n}k + 2\sum_{k=1}^{n}1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

But $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ and $\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$

Substituting these values into equation (i) gives,

$$\sum_{k=1}^{n}\left(6k^2-8k+2\right)=6\left[\frac{1}{6}n(n+1)(2n+1)\right]-8\left[\frac{1}{2}n(n+1)\right]+2n$$

$$=n(n+1)(2n+1)-4n(n+1)+2n$$

$$=n\left[2n^2+3n+1-4n-4+2\right]$$

$$=n\left[2n^2-n-1\right]$$

$$=n(n-1)(2n+1)$$

Thus, $\sum_{k=1}^{n}\left(6k^2-8k+2\right)=n(n-1)(2n+1)$.

$$\Rightarrow \sum_{k=1}^{64}\left(6k^2-8k+2\right)=64(64-1)(2\times 64+1)$$

$$=64(63)(129)$$

$$=520{,}128$$

Therefore, $\sum_{k=1}^{n}\left(6k^2-8k+2\right)=n(n-1)(2n+1)$ and $\sum_{k=1}^{64}\left(6k^2-8k+2\right)=520{,}128$.

Activity 6.5: Deriving a formula for the sum of cubes of the first *n* natural numbers

Individually or in a group, perform the following tasks:

1. Write down the formula for $\sum_{k=1}^{n}k$, $\sum_{k=1}^{n}1$, and $\sum_{k=1}^{n}k^2$.
2. Find the expansion of $(k-1)^4$.
3. Use tasks 1 and 2 to deduce that the sum of the cubes of the first n natural numbers is given by the formula $\sum_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2$.
4. Have you observed any challenge from this activity? Give reasons.
5. Share your findings with other students for more inputs.

Example 6.21

Simplify $\sum_{k=1}^{n} k(k-1)(2k+3)$. Hence, evaluate $\sum_{k=1}^{100} k(k-1)(2k+3)$.

Solution

$$\sum_{k=1}^{n} k(k-1)(2k+3) = \sum_{k=1}^{n}\left(2k^3+k^2-3k\right)$$

$$= 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2 - 3\sum_{k=1}^{n} k$$

$$= 2\left[\frac{1}{4}n^2(n+1)^2\right] + \frac{1}{6}n(n+1)(2n+1) - 3\left[\frac{1}{2}n(n+1)\right]$$

$$= \frac{1}{2}n^2(n+1)^2 + \frac{1}{6}n(n+1)(2n+1) - \frac{3}{2}n(n+1)$$

$$= \frac{1}{6}n(n+1)(3n^2+5n-8)$$

$$= \frac{1}{6}n(n+1)(n-1)(3n+8)$$

Thus, $\sum_{k=1}^{n} k(k-1)(2k+3) = \frac{1}{6}n(n+1)(n-1)(3n+8)$, and

$$\sum_{k=1}^{100} k(k-1)(2k+3) = \frac{1}{6}(100)(101)(99)(308)$$

$$= 51,328,200.$$

Therefore, $\sum_{k=1}^{n} k(k-1)(2k+3) = \frac{1}{6}n(n+1)(n-1)(3n+8)$

and $\sum_{k=1}^{100} k(k-1)(2k+3) = 51,328,200.$

Exercise 6.3

1. Write each of the following series by using sigma notation:

(a) $8+11+14+17+20$

(b) $2+4+6+8+10+\cdots+28$

(c) $32+16+8+4+2+1+\frac{1}{2}$

(d) $1+4+9+16+\cdots+121$

(e) $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}$

(f) $-1+4-9+16-25+\cdots+n$

2. Write each of the following in expanded form:

(a) $\sum_{k=1}^{6}(k+3)$ (c) $\sum_{k=1}^{7}(k-3k^2)$ (e) $\sum_{k=0}^{9}2^{k-2}$

(b) $\sum_{k=2}^{4}(2k^3-k^2+8k+1)$ (d) $\sum_{k=2}^{8}(1-2^k)$

3. Evaluate each of the following series:

(a) $\sum_{k=0}^{5}(11+6k)$ (c) $\sum_{k=1}^{6}\left(\frac{2}{5}\right)k$ (e) $\sum_{k=0}^{18}(-1)^k$

(b) $\sum_{k=0}^{5}2^{k-1}$ (d) $\sum_{k=6}^{13}(57-k)$

4. Evaluate each of the following series:

(a) $\sum_{k=1}^{n}4k$ (b) $\sum_{k=1}^{n}(k^2+3k)$ (c) $\sum_{k=1}^{n}k(k+2)(k+3)$

5. Write each of the following series in sigma notation:

(a) $1\times1+2\times3+3\times5+4\times7$ (d) $-4\ -1\ +\ 2\ +\ 5+\cdots+17$

(b) $1-x+x^2-x^3+x^4$ (e) $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}$

(c) $y+2y^2+3y^3+4y^4+\cdots$ (f) $1\times4+4\times7+7\times10+10\times13$

6. Write down the first three terms in each of the following:

(a) $\sum_{k=2}^{n}2^k$ (d) $\sum_{k=1}^{8}\left(\frac{1}{k}-\frac{1}{k+1}\right)$

(b) $\sum_{k=2}^{n}k(k-1)$ (e) $\sum_{k=1}^{n}(-1)^{k+1}k^2$

(c) $\sum_{k=0}^{20}\frac{k+2}{(k+1)(2k+1)}$ (f) $\sum_{k=1}^{n}6k(4k-3)(2k+1)$

7. Simplify $\sum_{k=1}^{n}2k(k-4)(k+6)$. Hence, evaluate $\sum_{k=20}^{40}2k(k-4)(k+6)$.

8. Prove that $\sum_{k=1}^{n}(16k^2-10k+7)=\frac{n}{3}(16n^2+9n+14)$. Hence, evaluate $\sum_{k=1}^{208}(16k^2-10k+7)$.

9. Prove that $\sum_{k=1}^{n}\left(80k^3-220k^2+150\right)=\frac{1}{3}n(n+1)\left(60n^2-160n-85\right).$

   Hence, evaluate $\sum_{k=6}^{18}\left(80k^3-220k^2+150k\right).$

10. Show that the sum to n terms of the series

    $(1\times3\times6)+(2\times5\times9)+(3\times7\times12)+\cdots$ is $\frac{3}{2}n(n^3+4n^2+5n+2).$

Proofs by mathematical induction

Mathematical induction is a method that is used to establish if a given statement is true for all natural numbers. The method used in mathematical induction is to establish that the first statement in the infinity sequence of statements is true, and prove that if any one statement in the sequence is true, then so is the next one. This concludes that the statement are true for all natural numbers.

In general, proof by mathematical induction is an indirect method of proof which is used in case a direct method is either not possible or not convenient.

Activity 6.6: Verifying that the method of induction holds true on passing information through various people from the first to the last person

In a group, perform the following tasks:

1. Form a group of a convenient number of students.
2. Members of each group should stand in a straight line.
3. Let the first student in each line tell his/her secret to the one standing next. This process has to continue by each student to tell the same secret to the next student in the line, and stop at the last student.
4. What did you observe in each of the following:
   (a) Does the first student in your group know the secret?
   (b) Does everybody in the line know the secret?
   (c) Do you find the last student knowing the secret (if it was not altered on the way). In each case give reasons.
5. What have you observed during this activity? Give reasons.
6. Share your findings with other groups for more inputs.

From Activity 6.6, the following observations can be made:

Let P_n be the statement that "S_n students know the secret", and S_n be the n^{th} student in the line.

S_2 knows the secret because it was told by S_1. So, P_2 is true.

S_3 knows the secret because it was told by S_2. So, P_3 is true. etc.

S_n knows the secret because it was told by S_{n-1}. So, P_n is true as well.

Remark: If the secret is unaltered (remained the same), then the secret is induced from the first student to the last. At the end, all students in the line know the secret.

The method used is to establish that each person got the secret correctly and conclude that the secret was induced from one person to the last, hence proved by induction.

The proof by mathematical induction is done in three steps as follows:

1. Prove that the statement is true for an arbitrary natural number, say $n = 1$.
2. Assume the statement is true for $n = k$.
3. Using the assumption made in step 2, prove that the statement is true for $n = k+1$.

Generally, the principle of mathematical induction is stated as follows:

If n is a natural number and the statement P_k is true for $k = 1$ and also true for $k = n + 1$ then the statement P_k is true for all natural numbers.

Example 6.22

Prove by mathematical induction that if n is the element of natural numbers, then $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$.

Proof

Let P_n be the statement,

$$\sum_{r=1}^{n} r = \frac{1}{2}n(n+1).$$

$$P_n : 1+2+3+\cdots+n = \frac{1}{2}n(n+1)$$

For $n=1$, check if P_1 is true. Consider the left-hand side of P_n;

$$P_1 : \sum_{r=1}^{1} r = 1$$

Consider the right-hand side of P_n;

$$P_1 : \frac{1}{2}(1)(1+1) = 1$$

Since P_1 is the same in both sides, then P_n is true, for $n = 1$.

Suppose P_n is true for $n = k$.

$$\Rightarrow P_k : 1+2+3+\ldots+k = \frac{1}{2}k(k+1)$$

Now, check if P_n is true for $n = k+1$.

$$P_{k+1}: 1+2+3+\ldots+k+(k+1)=\frac{1}{2}k(k+1)+(k+1)$$

$$=\frac{k(k+1)+2(k+1)}{2}$$

$$=\frac{1}{2}(k+1)(k+2)$$

Thus, P_{k+1} is true. Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers.

Example 6.23

Use mathematical induction to prove that, $n^3 - n$ is divisible by 3 for all natural numbers.

Proof

Let P_n be the statement: $n^3 - n$ is divisible by 3

For $n = 1$, check if P_1 is true:

$P_1: \frac{n^3-n}{3}=\frac{1^3-1}{3}=0$, which is divisible by 3.

Thus, P_1 is true.

Suppose P_n is true for $n = k$, that is,

$P_k: \frac{k^3-k}{3}=m \Rightarrow k^3-k=3m$, where m is an integer.

$$\Rightarrow k^3 = k+3m \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Now, check if P_n is true for $n = k + 1$

$P_{k+1}: (k+1)^3-(k+1) = k^3+3k^2+3k+1-k-1$

$= k^3+3k^2+2k$

$= (3m+k)+3k^2+2k$, substituting the value of k^3 from equation (i)

$= 3k^2+3k+3m$

$= 3(k^2+k+m)$ is divisible by 3 since 3 is its factor.

Thus, P_{k+1} is true. Hence, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers.

Example 6.24

Prove by mathematical induction that, if n is a natural number greater than 1, then
$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}.$$

Proof

Let P_n be the statement,
$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}$$

For $n=2$, check if P_2 is true.

Consider the left-hand side of P_n;
$$P_2:\ \frac{2+3}{(1)(2)(3)}=\frac{5}{6}$$

Consider the right-hand side of P_n;
$$P_2:\ \frac{3}{2}-\frac{2+2}{2(3)}=\frac{5}{6}$$

Since P_2 is the same in both sides, then P_n is true, for = 2.

Suppose P_n is true for $n=k$.
$$\Rightarrow P_k:\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{k+3}{(k-1)k(k+1)}=\frac{3}{2}-\frac{k+2}{k(k+1)}.$$

Now, check if P_n is also true for $n=k+1$.
$$P_{k+1}:\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{k+3}{(k-1)k(k+1)}+\frac{k+4}{k(k+1)(k+2)}$$
$$=\frac{3}{2}-\frac{k+2}{k(k+1)}+\frac{k+4}{k(k+1)(k+2)}$$
$$=\frac{3}{2}-\left[\frac{k+2}{k(k+1)}-\frac{k+4}{k(k+1)(k+2)}\right]$$
$$=\frac{3}{2}-\frac{(k+2)(k+2)-(k+4)}{k(k+1)(k+2)}$$
$$=\frac{3}{2}-\frac{k^2+3k}{k(k+1)(k+2)}$$

$$=\frac{3}{2}-\frac{k(k+3)}{k(k+1)(k+2)}$$

$$=\frac{3}{2}-\frac{k+3}{(k+1)(k+2)}$$

Thus, P_k is true $\Rightarrow P_{k+1}$ is true.

Therefore, by mathematical induction, P_n is true for all natural numbers greater than one.

Exercise 6.4

Use mathematical induction to prove each of the following for every natural number n:

1. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

2. $\sum_{k=1}^{n} k(k+1) = \frac{n(n+1)(n+2)}{3}$

3. $\sum_{k=1}^{n} (k+1) = \frac{n(n+3)}{2}$

4. $\sum_{k=1}^{n} k(k+1)(k+2) = \frac{n(n+1)(n+2)(n+3)}{4}$

5. $2+4+6+\cdots+(4n-2)=2n^2$.

6. $1\times3+3\times5+5\times7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1)$

7. $\sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

8. $\frac{1}{1\times3}+\frac{1}{2\times3}+\frac{1}{3\times5}+\cdots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$

9. $\sum_{k=0}^{n} am^k = a\left(\frac{m^{n+1}-1}{m-1}\right)$

10. $\sum_{k=0}^{n} 2^k = 2^{n+1}-1$

11. $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$

12. $\sum_{k=1}^{n} k^3 = \left(\frac{1}{2}n(n+1)\right)^2$

13. $2^{3n} - 1$ is divisible by 7.

14. $n(n-1)(2n-1)$ is divisible by 6, for $n > 1$.

15. $7^{2n} + 2^{3n-3} \times 3^{n-1}$ is divisible by 25.

Roots of a polynomial function

Given a polynomial function, $p(x)$ the values of the independent variable x for which the polynomial is zero can be determined. This section discusses the general form of a polynomial function and how its roots can be obtained.

Polynomial function

A polynomial function $p(x)$ is well defined as an algebraic expression that takes the form $p(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$,

where $a_n, a_{n-1}, a_{n-2}, \ldots, a_1, a_0$ are real numbers and n is a natural number. The degree of a polynomial is the highest positive integer exponent of the variable with non-zero coefficient in the polynomial. Polynomials are named basing on their degrees. Thus, a polynomial of degree 1 is called linear, a degree 2 polynomial is called a quadratic, a degree 3 polynomial is called a cubic, a degree 4 polynomial is called a quartic, a degree 5 polynomial is called a quintic, and so on.

Roots of polynomials

The values of x for which the polynomial, $p(x)$ is zero are called roots or zeros of $p(x)$. The roots are called zeros because they are the values of x at which the polynomial equals to zero. The highest exponent of a variable gives the maximum number of roots the polynomial can have. If the highest exponent of a polynomial is 2, the polynomial can have at most two roots. If the highest exponent is 3, then the polynomial can have at most three roots, and so on. It is also important to note that $f(x) = 0$ is also a polynomial of undefined degree.

Example 6.25

Given that $p(x) = (x - 3)(x^2 - 1)$, find the roots of $p(x)$.

Solution

The roots of $p(x)$ are obtained when $p(x) = 0$.

$\Rightarrow (x - 3)(x^2 - 1) = 0$

$\Rightarrow (x - 3)(x + 1)(x - 1) = 0$

Thus, either $x - 3 = 0$, $x + 1 = 0$ or $x - 1 = 0$

$\Rightarrow x = 3$, $x = -1$ or $x = 1$

Therefore, the roots of $p(x)$ are -1, 1, or 3.

Relationships between roots and coefficients of a quadratic equation

The quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers, such that $a \neq 0$. The roots of a quadratic equation are obtained by solving the equation. Alternatively, the roots can be obtained through the coefficients of the equation. Therefore, there are relationships between coefficients of a quadratic equation and its roots.

Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$.

Thus, either $x - \alpha = 0$ or $x - \beta = 0$.

This suggests that, the equation $ax^2 + bx + c = 0$ is obtained by multiplying its factors, $(x - \alpha)$ and $(x - \beta)$. That is,

$(x - \alpha)(x - \beta) = 0.$

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$ .... (i)

Now, consider the equation,

$ax^2 + bx + c = 0$:

Divide it by a throughout to obtain,

$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ .... (ii)

Comparing coefficients of corresponding terms of equations (i) and (ii), gives:

$\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Hence, the sum of the roots of a quadratic equation is given by $\alpha + \beta = -\frac{b}{a}$ and the product of the roots is given by $\alpha\beta = \frac{c}{a}$.

Generally, if the roots of a quadratic equation are α and β, then the equation of the particular quadratic equation can be expressed in the form $x^2 - (\alpha + \beta)x + \alpha\beta = 0$. That is, $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$.

Example 6.26

If the roots of the equation $2x^2 + 3x - 2 = 0$ are α and β, find the values in each of the following:

(a) $\alpha + \beta$

(b) $\alpha\beta$

Solution

(a) Given $2x^2 + 3x - 2 = 0$. Compare it with the equation $ax^2 + bx + c = 0$.

$a = 2,\ b = 3,\ c = -2,\ \alpha + \beta = -\frac{b}{a}$.

Thus, $\alpha + \beta = -\frac{3}{2}$.

Therefore, $\alpha + \beta = -\frac{3}{2}$.

(b) $\alpha\beta = \frac{c}{a}$

$\Rightarrow \alpha\beta = -\frac{2}{2} = -1$

Therefore, $\alpha\beta = -1$.

Example 6.27

If α and β are the roots of the equation $2x^2 - 3x - 8 = 0$;

(a) Find the values of each of the following:

(i) $\alpha^2 + \beta^2$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(b) Write a quadratic equation with the roots $\frac{1}{\beta} + \frac{1}{\alpha}$ and $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

Solution

(a) Given $2x^2 - 3x - 8 = 0$,

then $\alpha + \beta = \frac{3}{2}$ and $\alpha\beta = -\frac{8}{2} = -4$

(i) Since $(\alpha + \beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$,

then $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$

$$= \left(\frac{3}{2}\right)^2 - 2(-4)$$

$$= \frac{41}{4}.$$

Therefore, $\alpha^2 + \beta^2 = \frac{41}{4}$.

(ii) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta}$

$$= \frac{3}{2} \div (-4)$$

$$= -\frac{3}{8}$$

Therefore, $\frac{1}{\alpha}+\frac{1}{\beta}=-\frac{3}{8}$.

(iii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{41}{4}\div(-4)$$

$$=-\frac{41}{16}$$

Therefore, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=-\frac{41}{16}$.

(b) The quadratic equation with the roots $\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)$ and $\left(\frac{1}{\beta}+\frac{1}{\alpha}\right)$ is given by;

$$x^2-(\text{sum of roots})+(\text{product of roots})=0$$

$$\Rightarrow x^2-\left[\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)+\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\right]x+\left[\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)\right]=0$$

$$\Rightarrow x^2-\left(\frac{-3}{8}+\frac{-41}{16}\right)x+\left(\frac{-3}{8}\right)\left(\frac{-41}{16}\right)=0$$

$$\Rightarrow x^2+\frac{47}{16}x+\frac{123}{128}=0$$

$$\Rightarrow 128x^2+376x+123=0$$

Therefore, the required equation is $128x^2+376x+123=0$.

Example 6.28

Find the value of m in the equation $x^2+mx+45=0$, given that the sum of the square of the roots is 54.

Solution

Given $x^2+mx+45=0$, then $\alpha+\beta=-m$ and $\alpha\beta=45$.

But $\alpha^2+\beta^2=54$

$$\Rightarrow(\alpha+\beta)^2=\alpha^2+\beta^2+2\alpha\beta$$

$$\Rightarrow(-m)^2=\alpha^2+\beta^2+2(45)$$

$\Rightarrow m^2 = \alpha^2 + \beta^2 + 90$

$\Rightarrow m^2 = 54 + 90$

$\Rightarrow m^2 = 144$

$\Rightarrow m = 12$ or -12

Therefore, $m = 12$ or $m = -12$.

Relationships between roots and coefficients of a cubic equation

Suppose α, β, and γ are the roots of the equation $ax^3 + bx^2 + cx + d = 0$, where $a \neq 0$, then $(x-\alpha)$, $(x-\beta)$, and $(x-\gamma)$ are its factors. Thus, the equation can be obtained by multiplying its factors. That is, $(x-\alpha)(x-\beta)(x-\gamma) = 0$.
On expanding, it gives:

$x^3 - (\gamma + \alpha + \beta)x^2 + (\beta\gamma + \gamma\alpha + \alpha\beta)x - \alpha\gamma\beta = 0$ .... (i)

Dividing equation $ax^3 + bx^2 + cx + d = 0$ by a throughout gives,

$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0$ ....(ii)

Comparing coefficients of corresponding terms of equations (i) and (ii) gives the following relationships:

$\alpha + \beta + \gamma = -\frac{b}{a}$, $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, and $\alpha\beta\gamma = -\frac{d}{a}$.

Therefore, the sum of the roots of a cubic equation is $\alpha + \beta + \gamma = -\frac{b}{a}$, the sum of the products of the roots in pairs is given by $\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$, and the product of the roots is $\alpha\beta\gamma = -\frac{d}{a}$.

Generally, if the roots of a cubic equation are α, β, and γ, then the equation of the particular cubic equation can be expressed in the form
$x^3 -$ (sum of roots)x^2 + (sum of product of roots in pairs) x – product of roots = 0.

Example 6.29

If α, β, and γ are the roots of the equation $2x^3 - 5x^2 + 4x - 1 = 0$, find the values of each of the following:

(a) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ (b) $\frac{1}{\alpha\beta\gamma}$ (c) $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}$

Hence, find a cubic equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}$, and $\frac{1}{\gamma}$.

Solution

Given $2x^3 - 5x^2 + 4x - 1 = 0$, then:

$$\alpha+\beta+\gamma=-\frac{b}{a}=\frac{5}{2},\ \alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}=2, \text{ and } \alpha\beta\gamma=-\frac{d}{a}=\frac{1}{2}.$$

(a) $$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}$$

$$=2\div\frac{1}{2}$$

$$=4$$

Therefore, $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=4$.

(b) $\frac{1}{\alpha\beta\gamma}=1\div\frac{1}{2}=2.$

Therefore, $\frac{1}{\alpha\beta\gamma}=2.$

(c) $$\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}=\frac{\alpha+\beta+\gamma}{\alpha\beta\gamma}$$

$$=\frac{5}{2}\div\frac{1}{2}$$

$$=5$$

Therefore, $\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}=5$.

The cubic equation is $x^3-\left(\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\right)x^2+\left(\frac{1}{\alpha\beta}+\frac{1}{\alpha\gamma}+\frac{1}{\beta\gamma}\right)x-\left(\frac{1}{\alpha\beta\gamma}\right)=0$

$\Rightarrow x^3-4x^2+5x-2=0$

Therefore, the cubic equation is $x^3-4x^2+5x-2=0$.

Example 6.30

The roots of the equation $x^3 + tx^2 + ux + 30 = 0$ are in the ratio 2:3:5. Find the values of t and u.

Solution

Let k be the factor of the roots, then

$\alpha = 2k$, $\beta = 3k$, and $\gamma = 5k$.

Given $x^3 + tx^2 + ux + 30 = 0$, then

$\alpha + \beta + \gamma = -t \Rightarrow 2k + 3k + 5k = -t$

$\Rightarrow 10k = -t$ ....(i)

Also, $\alpha\beta + \alpha\gamma + \beta\gamma = u \Rightarrow 6k^2 + 10k^2 + 15k^2 = u$

$\Rightarrow 31k^2 = u$ ....(ii)

Also, $\alpha\beta\gamma = -30 \Rightarrow 30k^3 = -30$

$\Rightarrow k = -1$

on substituting $k = -1$ in equation (i) gives,

$10(-1) = -1 \Rightarrow t = 10$.

From equation (ii),

$u = 31(-1)^2 = 31$.

Therefore, $t = 10$ and $u = 31$.

Exercise 6.5

1. If α and β are the roots of the equation $(1 - q)\, x^2 - 2x - 1 = 0$, find the values of $\alpha + \beta$ and $\alpha\,\beta$.
2. If α and β are the roots of the equation $2x^2 - 3x + 8 = 0$, find the values of
   (a) $\alpha^2\beta + \alpha\beta^2$ (b) $\alpha^3 + \beta^3$
3. If α and β are the roots of the equation $3x^2 + 5x - 1 = 0$, find the equations whose roots are:
   (a) $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ (b) $\alpha + \frac{1}{\beta}$ and $\beta + \frac{1}{\alpha}$
4. If $x^2 + mx + n = 0$ and $px^2 + 2mx - 3n = 0$ have common roots, show that $n(p+3)^2 = 5m^2(p-2)$.
5. If α and β are the roots of the equation $17x^2 - 34x = 18$, find the value of $\alpha - \beta$.

6. If α and β are the roots of the equation $x^2 + 3x - 3 = 0$, find the quadratic equation whose roots are $2(\alpha + \beta)$ and $\frac{1}{3\alpha\beta}$.
7. If α and β are the roots of the equation $2x^2 + 5x - 3 = 0$, find the equation whose roots are given by the following pairs of the roots:
   (a) $(3\alpha + \beta)$ and $(3\beta + \alpha)$
   (b) α^2 and β^2
   (c) $\frac{3}{\alpha}$ and $\frac{3}{\beta}$
   (d) $\frac{2\alpha}{\beta}$ and $\frac{2\beta}{\alpha}$
   (e) $(\alpha + 3)$ and $(\beta + 3)$
   (f) $(\alpha - 5)$ and $(\beta - 5)$
8. If the sum of squares of roots of the equation $kx^2 + tx + s = 0$ is 1, show that $t^2 = 2ks + k^2$.
9. If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $p:q$, prove that $ac(p+q)^2 = b^2pq$.
10. If α, β, and γ are the roots of the equation $2x^3 + 3x^2 - x - 8 = 0$, find the equation whose roots are α^2, β^2, and γ^2.
11. The equation $x^3 + 2x^2 - 5x + 1 = 0$ has roots α, β, and γ. Determine the equations with the following roots:
    (a) 2α, 2β, and 2γ (b) $\alpha + \beta$, $\alpha + \gamma$, and $\beta + \gamma$
12. The roots of the equation $x^3 - 5x^2 + x + 12 = 0$ are α, β, and γ. Calculate the value of $(\alpha + 2)(\beta + 2)(\gamma + 2)$, hence find the values in each of the following expressions:
    (a) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$
    (b) $\alpha^2 + \beta^2 + \gamma^2$
    (c) $\alpha^3 + \beta^3 + \alpha^3$
    (d) $\frac{1}{\alpha\beta\gamma}$
    (e) $\frac{1}{\alpha\beta} + \frac{1}{\alpha\gamma} + \frac{1}{\gamma\beta}$

Remainder theorem

Sometimes it is not easy to find factors of polynomials of degree three or higher or finding the remainder when polynomials are divided by an expression.

In arithmetic, numbers can be divided using long division method. By this method, the dividend equals the product of the divisor and quotient plus a remainder.

For example, dividing 627 by 50 gives 12 with a remainder 27. Thus, $627 = 50 \times 12 + 27$. This relationship is also applied to polynomials.

Consider the polynomial $p(x) = 2x^2 + 5x - 4$. It can be divided by any polynomial of degree less or equal to 2. Choosing to divide $p(x)$ by $x - 2$, then long division method gives,

$$\begin{array}{rl} & \quad 2x + 9 \\ \text{Divisor} \rightarrow x-2 \big) & 2x^2 + 5x - 4 \leftarrow \text{Dividend} \\ & \underline{-(2x^2 - 4x)} \\ & \qquad 9x - 4 \\ & \quad \underline{-(9x - 18)} \\ & \qquad\quad 14 \leftarrow \text{Remainder} \end{array}$$

From the long division, $2x^2 + 5x - 4$ is a dividend, $x - 2$ is a divisor, $2x + 9$ is a quotient, and 14 is a remainder. Thus, $2x^2 + 5x - 4 = (2x + 9)(x - 2) + 14$.

Generally, the remainder theorem states that, if $p(x)$ is a dividend, $q(x)$ is a quotient, $x - a$ is a divisor, and $r(x)$ is a remainder, then $p(x) = q(x) \times (x - a) + r(x)$. If $x = a$, then $p(a) = r(a)$. Therefore, the remainder theorem enables us to find the remainder when a polynomial is divided by an expression without actually carrying out the steps of the long division.

Example 6.31

Find the remainder when $p(x) = 3x^3 - 5x^2 + 2$ is divided by $x + 4$.

Solution

The divisor $x + 4$ can be rewritten as $x - (-4)$. Thus, the remainder is $p(-4)$.

But $p(-4) = 3(-4)^3 - 5(-4)^2 + 2$

$= 3 \times (-64) - 5 \times 16 + 2$

$= -270$

Therefore, when $p(x) = 3x^3 - 5x^2 + 2$ is divided by $x + 4$, the remainder is -270.

Example 6.32

If $hx^3 - 11x^2 + 4x + 12$ is divided by $x - 5$, the remainder is 132. Find the value of a constant h.

Solution

Let $p(x) = hx^3 - 11x^2 + 4x + 12$.

The divisor is $x - 5 \Rightarrow x - 5 = 0$.

$\Rightarrow p(5) = h(5)^3 - 11(5)^2 + 4(5) + 12$

$= 125h - 275 + 20 + 12$

$\Rightarrow p(5) = 125h - 243$. But $p(5) = 132$

$\Rightarrow 125h - 243 = 132$

$\Rightarrow 125h = 375$

Thus, $h = 3$.

Therefore, the value of h is 3.

Example 6.33

When the expression $x^5 + 4x^2 + cx + d$ is divided by $x^2 - 1$, the remainder is $2x + 3$. Find the values of c and d.

Solution

Let $p(x) = x^5 + 4x^2 + cx + d$

The divisor is $x^2 - 1 \Rightarrow x^2 - 1 = 0$

$x = -1$ or $x = 1$.

For $x = 1$, $p(1) = 1 + 4 + c + d$

But, $p(1) = 2(1) + 3 = 5$

$\Rightarrow 5 = 5 + c + d$

$\Rightarrow c + d = 0$ .... (i)

For $x = -1$, $p(-1) = -1 + 4 - c + d$

$\Rightarrow p(-1) = 3 - c + d$. But $p(-1) = 2(-1) + 3 = 1$.

$\Rightarrow 1 = 3 - c + d$

$\Rightarrow c - d = 2$ ....(ii)

Solving equations (i) and (ii) simultaneously gives $c = 1$ and $d = -1$.

Therefore, the values of c and d are 1 and -1, respectively.

Factor theorem

The factor theorem states that if a polynomial $p(x)$ is such that the remainder $p(a) = 0$, then $x - a$ is a factor of $p(x)$. This implies that $p(x) = (x - a) \times q(x)$. In other words, if $x - a$ is a factor of the polynomial $p(x)$, then $p(a) = 0$. Thus, the factor theorem is a special case of the remainder theorem in the sense that the factor theorem is deduced from the remainder theorem when $r(x) = 0$.

Example 6.34

Determine whether or not $x + 1$ is a factor of the polynomial $p(x) = x^4 - 3x^3 + 4x^2 - 8$.

Solution

Given $p(x) = x^4 - 3x^3 + 4x^2 - 8$.

Rewrite $x + 1$ as $x - (-1)$. This gives,

$p(-1) = (-1)^4 - 3(-1)^3 + 4(-1)^2 - 8 = 0$

Since $p(-1) = 0$, then $x + 1$ is a factor of $p(x)$.

Therefore, $x + 1$ is a factor of the polynomial $p(x)$.

Example 6.35

Find the factors of x^3+2x^2-x-2.

Solution

Let $p(x)=x^3+2x^2-x-2$

From the factor theorem, if $x-a$ is a factor of $p(x)$, then $p(a)=0$.

Choose values of a and substitute into $p(x)$ to check or inspect whether $p(a)=0$.

The factors of the constant term, -2 are $1, 2, -1, -2$. So, the values to choose must belong to the set $\{\pm 1, \pm 2\}$.

When $x=1$: $p(1)=1^3+2(1)^2-1-2=0$

Thus, $x-1$ is a factor of $p(x)$.

When $x=-1$: $p(-1)=(-1)^3+2(-1)^2-(-1)-2=0 \Rightarrow p(-1)=0$

Thus, $x+1$ is a factor of $p(x)$.

When $x=2$: $p(2)=2^3+2(2)^2-2-2=12 \Rightarrow p(2)\neq 0$.

Thus, $x-2$ is not a factor of $p(x)$.

When $x=-2$: $p(-2)=(-2)^3+2(-2)^2+2-2=0 \Rightarrow p(-2)=0$.

Thus, $x+2$ is a factor of $p(x)$.

Therefore, $p(x)=x^3+2x^2-x-2=(x+1)(x-1)(x+2)$.

Operations on polynomials

Mathematical operations such as addition, subtraction, multiplication, and division can be applied for polynomial functions. The following activity highlights some operations on polynomials.

Activity 6.7: Recognizing the use of polynomials to model blood flow

Individually or in a group, read the following information, then perform the tasks that follow.

Cardiac output is the amount of blood pumped through the heart. The output is measured by a technique called dye dilution. A doctor injects dye into a vein near the heart and measures the amount of dye in the arteries over time. The cardiac output of a particular patient can be approximated by the function $f(t)=0.0056t^3-0.22t^2+2.33t$, where t represents time in seconds after injection, $0\leq t\leq 23$ and $f(t)$ represents the concentration of dye (in milligrams per litre).

1. Evaluate $f(t)$ for $t = 0$ and $t = 3$.
2. Describe what do the values in task 1 represent.
3. Share your findings with other students.

Addition and subtraction of polynomials

Addition and subtraction of polynomials is simply done by adding or subtracting the terms of the same power. The exponent of variable in a polynomial is always a non-negative integer.

Example 6.36

Given $p(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15$ and $q(x) = x^4 + x^3 - 5x^2 - 6x - 10$, find the value of $p(x) + q(x)$.

Solution

Given $p(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15$ and $q(x) = x^4 + x^3 - 5x^2 - 6x - 10$.

$p(x) + q(x) = 3x^4 + 2x^3 - 10x^2 + 8x + 15 + x^4 + x^3 - 5x^2 - 6x - 10$

Combine like terms as follows:

$$\begin{aligned} p(x)+q(x) &= \left(3x^4 + x^4\right) + \left(2x^3 + x^3\right) + \left(-10x^2 - 5x^2\right) + (8x - 6x) + (15 - 10) \\ &= 4x^4 + 3x^3 - 15x^2 + 2x + 5 \end{aligned}$$

Therefore, $p(x) + q(x) = 4x^4 + 3x^3 - 15x^2 + 2x + 5$.

Example 6.37

Simplify the following and write your answer in descending powers of x.

$\left(3x^2 + 7 + x\right) + \left(14x^3 + 2 + x^2 - x\right)$

Solution

Given $\left(3x^2 + 7 + x\right) + \left(14x^3 + 2 + x^2 - x\right)$

Combine the like terms to get,

$14x^3 + \left(3x^2 + x^2\right) + (x - x) + (7 + 2) = 14x^3 + 4x^2 + 9$

Therefore, $\left(3x^2 + 7 + x\right) + \left(14x^3 + 2 + x^2 - x\right) = 14x^3 + 4x^2 + 9$.

Example 6.38

Find $\left(10x^5-4x^4+3x^3+x^2-10x+9\right)-\left(6x^5-10x^3+12x+19\right)$.

Solution

Insert the signs of the second polynomial, combine like terms, then simplify as follows.

$$\left(10x^5-4x^4+3x^3+x^2-10x+9\right)-\left(6x^5-10x^3+12x+19\right)$$

$$=\left(10x^5-4x^4+3x^3+x^2-10x+9\right)+\left(-6x^5+10x^3-12x-19\right)$$

$$=\left(10x^5-6x^5\right)+\left(-4x^4+0x^4\right)+\left(3x^3+10x^3\right)+\left(x^2+0x^2\right)+\left(-10x-12x\right)+\left(9-19\right)$$

$$=4x^5-4x^4+13x^3+x^2-22x-10.$$

Therefore, $\left(10x^5-4x^4+3x^3+x^2-10x+9\right)-\left(6x^5-10x^3+12x+19\right)=$
$4x^5-4x^4+13x^3+x^2-22x-10.$

Example 6.39

If $px^4-qx^3+rx^2+sx+t+2x^4-3x^3+2x-18=24x^4-16x^3+8x^2+10x-24$, find the values of p, q, r, s, and t.

Solution

Given $px^4-qx^3+rx^2+sx+t+2x^4-3x^3+2x-18=24x^4-16x^3+8x^2+10x-24$.

$$\Rightarrow px^4-qx^3+rx^2+sx+t=\left(24x^4-16x^3+8x^2+10x-24\right)-\left(2x^4-3x^3+2x-18\right)$$

$$=\left(24x^4-16x^3+8x^2+10x-24\right)+\left(-2x^4+3x^3-2x+18\right)$$

$$=\left(24x^4-2x^4\right)+\left(-16x^3+3x^3\right)+\left(8x^2\right)+\left(10x-2x\right)+\left(-24+18\right)$$

$$=22x^4-13x^3+8x^2+8x-6$$

$$\Rightarrow px^4-qx^3+rx^2+sx+t=22x^4-13x^3+8x^2+8x-6$$

Comparing the coefficients of like terms gives,
$p=22$, $q=13$, $r=8$, $s=8$, and $t=-6$.

Therefore, $p=22$, $q=13$, $r=8$, $s=8$, and $t=-6$.

Multiplication of polynomials

Multiplication of polynomial is the process of multiplying together two or more polynomials. This can be performed by applying the distributive property for multiplication to polynomials. To multiply two polynomials with each other, take each term of the first polynomial and multiply to the second polynomial.

Example 6.40

Find $(2x^2+x-3)\times(x^2-2x+5)$.

Solution

Multiply each term of the first polynomial by the second polynomial and then combine like terms as follows.

$$\begin{aligned}(2x^2+x-3)\times(x^2-2x+5) &= 2x^2(x^2-2x+5)+x(x^2-2x+5)+(-3)(x^2-2x+5)\\ &= 2x^4-4x^3+10x^2+x^3-2x^2+5x-3x^2+6x-15\\ &= 2x^4-4x^3+x^3+10x^2-2x^2-3x^2+5x+6x-15\\ &= 2x^4-3x^3+5x^2+11x-15.\end{aligned}$$

Therefore, $(2x^2+x-3)\times(x^2-2x+5)=2x^4-3x^3+5x^2+11x-15$.

Division of polynomials

Dividing polynomials is an arithmetic operation where a polynomial is divided by another polynomial with the same or lower degree as compared to the dividend. Division of polynomials can be done by using two ways namely long division method and synthetic division method.

Long division of polynomials

Division of polynomials can be done using a method similar to long division of numbers. The following are steps to follow when dividing polynomials by long division:

1. Rewrite the problem in long division form. Make sure that both polynomials are written in decreasing powers, filling in any missing term with a zero term.
2. Divide the first term of the dividend by the first term of the divisor. Place the quotient to its corresponding like term. Multiply the quotient by the divisor, place the product below its like terms then subtract.

3. Bring down the next term of the dividend.
4. Repeat steps 2 and 3 until the last term is obtained.
5. Write the final answer. If there is a remainder, place the remainder over the divisor and add it to the quotient.

Example 6.41

Divide $2x^4 - 3x^3 + 5x - 36$ by $x^2 + x + 2$.

Solution

Rewrite the problem in long division form and follow the steps discussed.

$$
\begin{array}{r}
2x^2 - 5x + 1 \\
x^2 + x + 2 \overline{\big) 2x^4 - 3x^3 + 0x^2 + 5x - 36} \\
-\underline{(2x^4 + 2x^3 + 4x^2)} \\
-5x^3 - 4x^2 + 5x \\
-\underline{\left(-5x^3 - 5x^2 - 10x\right)} \\
x^2 + 15x - 36 \\
-\underline{(x^2 + x + 2)} \\
14x - 38
\end{array}
$$

Thus, $2x^4 - 3x^3 + 5x - 36$ divided by $x^2 + x + 2$ gives $2x^2 - 5x + 1$ with a remainder $14x - 38$.

Therefore, $\dfrac{2x^4 - 3x^3 + 5x - 36}{x^2 + x + 2} = 2x^2 - 5x + 1 + \dfrac{14x - 38}{x^2 + x + 2}$.

Synthetic division method

Synthetic division is a simplified method for dividing a polynomial by another polynomial of degree one. It is performed by writing down only coefficients of several powers of the variable and changing sign of the constant term in the divisor so as to replace the usual subtraction by addition.

Example 6.42

Use synthetic division to divide $6x^3 + x - 1$ by $x + 2$.

Solution

Since the divisor is of the form $x-(-2)$, use the additive inverse of 2 which is -2.

Step 1: Write down all the coefficients of the dividend in a row, from left to right, and then place -2 in that same row, to the left of the leading coefficient of the dividend.

$$\begin{array}{r|rrrr} -2 & 6 & 0 & 1 & -1 \\ \hline \end{array}$$

Step 2: Bring down the leading coefficient of the dividend which is 6, and then multiply it by the divisor -2 to get -12. Write -12 to the corresponding coefficient 0.

$$\begin{array}{r|rrrr} -2 & 6 & 0 & 1 & -1 \\ & & -12 & & \\ \hline & 6 & & & \end{array}$$

Step 3: Add 0 and -12 to get -12, then write it to its corresponding term -12.

$$\begin{array}{r|rrrr} -2 & 6 & 0 & 1 & -1 \\ & & -12 & & \\ \hline & 6 & -12 & & \end{array}$$

Step 4: Apply Steps 2 and 3 to the result. That is, multiply -12 by -2 to get 24. Write 24 to the corresponding coefficient 1, then add 1 and 24 to get 25. Write 25 to the corresponding coefficient 24.

$$\begin{array}{r|rrrr} -2 & 6 & 0 & 1 & -1 \\ & & -12 & +24 & \\ \hline & 6 & -12 & +25 & \end{array}$$

Step 5: Apply Steps 2 and 3 to the result. That is, multiply 25 by -2 to get -50. Write -50 to the corresponding cefficcient -1, then add -50 and -1 to get -51. Write -51 to the corresponding coefficient -50.

$$\begin{array}{r|rrrr} -2 & 6 & 0 & 1 & -1 \\ & & -12 & +12 & -50 \\ \hline & 6 & -12 & +25 & \boxed{-51} \end{array}$$

Step 6: Read the coefficients of the quotient from the bottom row and write the quotient:

$q(x) = 6x^2 - 12x + 25$.

The remainder, $r(x)$ is -51, which is the last number in the bottom row.

Thus, $6x^3 + x - 1$ divided by $x + 2$ is equal to $6x^2 - 12x + 25$ remainder is -51.

Therefore, $\dfrac{6x^3 + x - 1}{x + 2} = 6x^2 - 12x + 25 + \left(\dfrac{-51}{x+2}\right)$.

Example 6.43

By using synthetic division, find the remainder when $p(x) = x^4 - 2x^3 - 15x + 2$ is divided by $x - 2$.

Solution

$$\begin{array}{r|rrrrr} 2 & 1 & -2 & 0 & -15 & 2 \\ & & +2 & +0 & +0 & -30 \\ \hline & 1 & +0 & +0 & -15 & \boxed{-28} \end{array}$$

Therefore, the remainder is -28.

Exercise 6.6

1. Given that $p(x) = -18x^5 + 14x^4 - 24x^3 - 24x + 70$ and $q(x) = -25x^5 + 72x^4 - 32x^3 + 12x^2 - 50$, find:
   (a) $p(x) + q(x)$ (b) $p(x) - q(x)$
2. If $x - 4$ and $x + 6$ are factors of the expression $x^3 + ax^2 + bx + c$ and it leaves a remainder of -18 when divided by $x + 3$, find the values of *a, b,* and *c*.
3. Use synthetic division to find the remainder when $2x^4 + 3x^3 - 2x + 5$ is divided by $x + 5$.
4. Find the constant *e*, *f*, and *g* such that when $y^5 - 7y^3 + 4y - 2$ is divided by $(y-1)(y+1)(y-3)$ the remainder is $ey^2 + fy + g$.
5. Expand each of the following polynomials:
   (a) $\left(x^2 - 2x - 3\right)^2$ (b) $\left(3x^2 - x + 1\right)\left(-x^2 + 2x - 1\right)$

6. Find the values of m, n, and p if $m+n(x-1)+p(x-1)(x-2)+2(x-1)(x-2)(x-3)$, is divided by $(x-1)$, $(x-2)$, and $(x-3)$, gives the remainders 10, 3, and 3, respectively.

7. By using synthetic division method, find the remainder and the quotient in each of the following:

   (a) $\dfrac{x^3-3x+1}{x-2}$ (d) $\dfrac{x^2-8x-1}{x+5}$

   (b) $\dfrac{x^4+x^3-8x^2+2x-1}{x+1}$ (e) $\dfrac{x^4-4x^2+2x+6}{x-4}$

   (c) $\dfrac{x^6-4x^4+2x-1}{x+2}$ (f) $\dfrac{2x^4+x^3+1}{x+1}$

8. Use long division method to find the result in each of the following:

   (a) $\dfrac{2x^5+8x^2-3x-2}{x^2-1}$ (c) $\dfrac{2x^4+x^3+1}{x+1}$

   (b) $\dfrac{1-x^2+x^4}{x^2+x+1}$ (d) $\dfrac{x^4-a^4}{x-a}$

9. Find the remainder in each of the following expressions:

   (a) $\dfrac{x^3+x^2+4x-4}{x-1}$ (c) $\dfrac{x^5+x^4-x-2}{x-3}$ (e) $\dfrac{16x^3+x^2-1}{x-2}$

   (b) $\dfrac{x^3-3x+5}{x+1}$ (d) $\dfrac{8x^4+32x^2-x-7}{x-4}$

10. If the remainder of $\dfrac{x^2-8x-t}{x+2}$ is -10, find the value of t and hence, factorize the polynomial.

11. Show that $x+1$ is a factor of $x^3-x^2-10x-8$.

Inequalities

It is common to compare two objects that are not of the same size. For instance, people say, "She must be older than you" or "No way, we cannot be equal" or "A goat is smaller than a cow". These statements compare two objects of different sizes or unequal sizes. In mathematics, a statement describing relationships like; "greater than, greater than or equal, less than, or less than or equal" between two numbers or algebraic expressions is called inequality.

In general, inequalities are statements that compare two values or sizes. The inequality symbols include: $\leq$ (less than or equal), $\geq$ (greater than or equal), $<$ (strictly less than), and $>$ (strictly greater than).

The inequality that involves variables can be solved and its solution is not a single value, thus it is a set of values that satisfies the inequality. The method of solving inequalities is very much like the method used for solving equations. The only difference is on multiplying or dividing both sides by a negative number. That is, when both sides of a given inequality are either multiplied or divided by a negative quantity, the sense of inequality must change ($>$ changes to $<$, $<$ changes to $>$, $\geq$ changes to $\leq$, and $\leq$ changes to $\geq$) .

Activity 6.8 Recognizing the solution of a quadratic inequality

Individually or in a group, perform the following tasks:

1. List down the steps of drawing the graph of the quadratic function.
2. Use the steps in task 1 to draw the graph of $f(x) = x^2 + 2x - 8$ on the interval $-6 \leq x \leq 4$.
3. Identify the solution of $f(x) \geq 0$ using the graph in task 2.
4. Identify the solution of $f(x) \leq 0$ using the graph in task 2.
5. What have you observed in tasks 3 and 4? Give reasons.
6. Share your findings with other students for more inputs.

Quadratic inequalities

A quadratic inequality is a polynomial of degree two that uses an inequality sign instead of an equal sign. There are several ways of solving quadratic inequalities which include graphical and algebraic methods. Quadratic inequalities can be solved graphically by first rewriting the inequality in standard form, with zero on one side. Sketch the quadratic function and determine where it is above or below the x-axis. Also, the inequality can be solved algebraically by making use of a sign chart. To make a sign chart, use the function and test values in each region bounded by the roots. The concern is to check if the function is positive or negative.

Example 6.44

Solve $x^2 + 2x \le 0$ graphically.

Solution

Given $x^2 + 2x \le 0$.

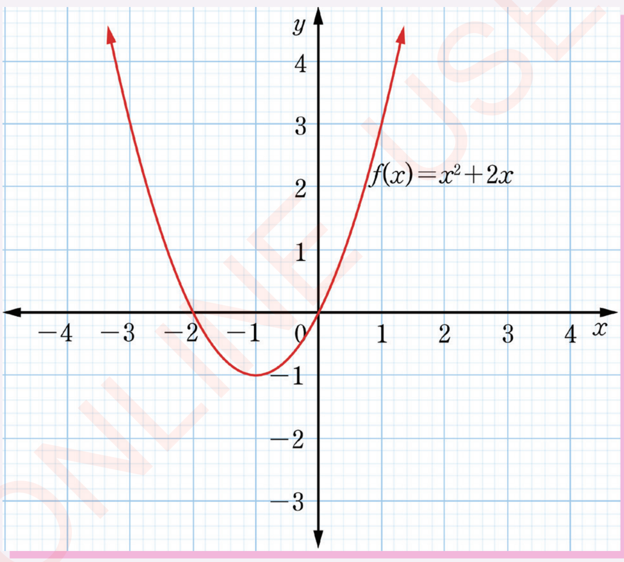
$\Rightarrow x(x+2) \le 0$

$\Rightarrow$Either $x \le 0$ and $x+2 \ge 0$ or $x \ge 0$ and $x+2 \le 0$

$\Rightarrow$Either $x \le 0$ and $-2 \le x$ or $x \ge 0$ and $x \le -2$

or $f(x) \le 0$, where $f(x) = x(x+2)$.

If the graph of $f(x)$ is sketched, then $f(x) \le 0$ where the graph is below the x-axis. The value of x corresponding to this portion of the graph satisfy $f(x) \le 0$. The points where $f(x) = 0$; that is $x = 0$ and $x = -2$ are part of the solution as shown in the following figure:



From the graph, it is observed hat the set of values of x which satisfy the given inequality is $-2 \le x \le 0$.

Therefore, the solution is $\{x : -2 \le x \le 0\}$.

Example 6.45

Solve $x^2 + x - 6 > 0$ algebraically.

Solution

Given $x^2 + x - 6 > 0$.

$\Rightarrow (x-2)(x+3) > 0$

$\Rightarrow$ Either $x - 2 > 0$ and $x + 3 > 0$ or $x - 2 < 0$ and $x + 3 < 0$

$\Rightarrow$ Either $x > 2$ and $x > -3$ or $x < 2$ and $x < -3$

Consider the signs of the factors in the following table of intervals:

	$x < -3$	$-3 < x < 2$	$x > 2$
$x - 2$	$-$ve	$-$ve	+ve
$x + 3$	$-$ve	+ve	+ve
$(x-2)(x+3)$	+ve	$-$ve	+ve

The solution for which $(x-2)(x+3) > 0$ is $x < -3$ or $x > 2$.

Therefore, the solution is $\{x : x < -3 \text{ or } x > 2\}$.

Example 6.46

Solve $-x^2 - x + 6 > 0$ graphically.

Solution

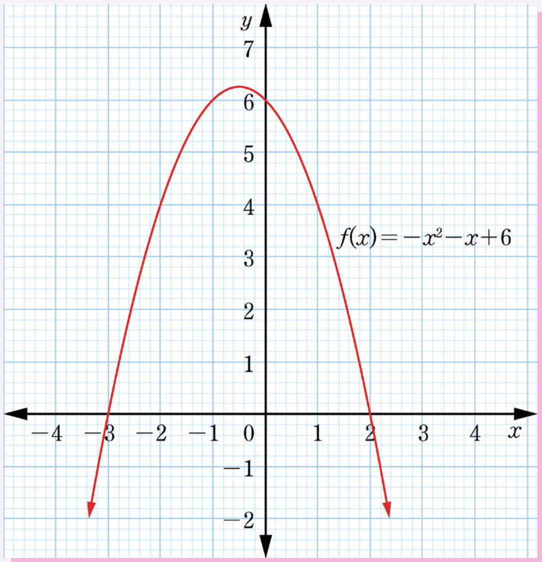
Given $-x^2 - x + 6 > 0$.

$\Rightarrow (x-2)(-x-3) > 0$

$\Rightarrow$ Either $(x-2) > 0$ and $(-x-3) > 0$ or $x - 2 < 0$ and $-x - 3 < 0$

$\Rightarrow$ Either $x > 2$ and $x < -3$ or $x < 2$ and $-3 < x$

If the graph of $f(x)$ is sketched, then $f(x) > 0$ where the graph is above the x-axis. The values of x corresponding to this portion of the graph satisfy $f(x) > 0$. The points where $f(x) = 0$; that is $x = 2$ and $x = -3$ are not part of this solution as shown in the following figure:



From the graph, it is observed that the set of values of x which satisfy the given inequality is $-3 < x < 2$.

Therefore, the solution is $\{x : -3 < x < 2\}$.

Rational inequalities

Rational inequalities are fractions that have a polynomial expression in the numerator, denominator or both. Generally, a rational inequality compares two expressions in the form $\frac{p(x)}{q(x)} < 0$; $\frac{p(x)}{q(x)} > 0$, $\frac{p(x)}{q(x)} \leq 0$ or $\frac{p(x)}{q(x)} \geq 0$, where $q(x) \neq 0$.

Note that; the properties of fractions can be applied to simplify rational inequalities, such as simplifying the expressions by cancelling common factors from the numerator and the denominator.

The following are steps in solving rational inequalities:

1. Rewrite the inequality to contain a zero on the right-hand side. Create a single fraction on the left-hand side.
2. Find the critical values of x that make the numerator equal to zero and that make the denominator equal to zero.
3. Construct a table of intervals in which the columns are separated by the critical values.

4. Select a test point in each interval and check to see if it satisfies the original inequality or the inequality set is less than 0.
5. Test the roots of the numerator for their possibility of inclusion to the overall solution. Include the roots to get a true statement if they are part of the overall solution.
6. Exclude the roots of the denominator in the overall solution since they make the rational expression undefined.

Example 6.47

Find the values of x for which $\frac{2x+1}{x} > 1$.

Solution

Given $\frac{2x+1}{x} > 1$.

Make the right-hand side equal to zero.

That is, $\frac{2x+1}{x} - 1 > 0$

$$\Rightarrow \frac{2x+1-x}{x} > 0$$

$$\Rightarrow \frac{x+1}{x} > 0$$

$$\Rightarrow f(x) = \frac{x+1}{x} > 0$$

The numerator of $f(x)$ is zero when $x = -1$, and the denominator is zero when $x = 0$. So the critical values are $x = -1$ and $x = 0$.

Construct a table of intervals in which the columns are separated by the critical values as follows:

	$x < -1$	$-1 < x < 0$	$x > 0$
$x+1$	$-$ve	$+$ve	$+$ve
x	$-$ve	$-$ve	$+$ve
$f(x) = \frac{x+1}{x}$	$+$ve	$-$ve	$+$ve

The solution for which $\frac{2x+1}{x} > 1$ is $x < -1$ or $x > 0$.

Therefore, the solution is $\{x : x < -1 \text{ or } x > 0\}$.

Example 6.48

Find the possible values of x for which $\dfrac{(x+1)(x-3)(x+4)}{x-2} \le 0.$

Solution

Given $f(x) = \dfrac{(x+1)(x-3)(x+4)}{x-2} \le 0.$

The numerator of $f(x)$ is zero when $(x+1)(x-3)(x+4) = 0$, thus $x = -1$, $x = 3$, and $x = -4$. The denominator $f(x)$ is zero when $x-2=0 \Rightarrow x=2$. So, the critical values are $x=-1$, $x=3$, $x=-4$, and $x=2$.

Construct a table of intervals in which the columns are separated by the critical values as follows:

	$x<-4$	$-4<x<-1$	$-1<x<2$	$2<x<3$	$x>3$
$x+4$	−ve	+ve	+ve	+ve	+ve
$x+1$	−ve	−ve	+ve	+ve	+ve
$x-2$	−ve	−ve	−ve	+ve	+ve
$x-3$	−ve	−ve	−ve	−ve	+ve
$f(x)$	+ve	−ve	+ve	−ve	+ve

The solution for which $f(x)<0$ is $-4<x<-1$ or $2<x<3$. The critical values $x=-4$, $x=-1$, and $x=3$ satisfy the overall solution and the critical value of the denominator does not satisfy the solution as it makes the rational expression undefined.

Therefore, the solution is $\{x: -4 \le x \le -1 \text{ or } 2 < x \le 3\}$.

Absolute value inequalities

The symbol $|\;|$ is the sign of the absolute value. The absolute value of a real number x is written as $|x|$, where $|x| = \begin{cases} x \text{ if } x \ge 0 \\ -x \text{ if } x < 0 \end{cases}$

The solution set of $|x| = a$ is the union of the solution sets of the two equations. Similarly, an equation that has the sign of an absolute value is equivalent to two equations without the sign of the absolute value. An absolute value inequality is an expression with absolute functions as well as inequality signs. For example, the expression $|p| < c$ is an absolute value inequality containing "a less than" symbol. This expression is defined as follows:

Definition 1

If $|p| < c$, then $-c < p < c$.

If $|p| \leq c$, then $-c \leq p \leq c$, where p is any algebraic expression.

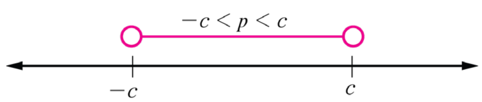
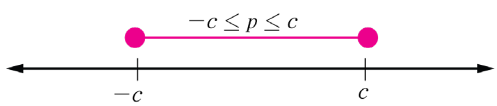
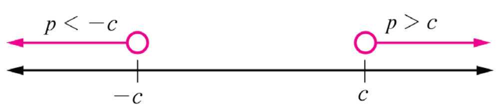
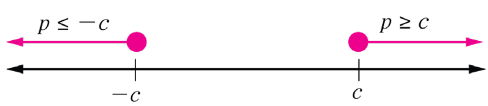
Note that, the value of c cannot be negative because if $c < 0$, then $|p| < c$ becomes $|p| < 0$, which is impossible. Thus, the definition of $|p|$ requires that $|p|$ is not negative.

Definition 2

If $|p| > c$, then $p > c$ or $p < -c$.

If $|p| \geq c$, then $p \geq c$ or $p \leq -c$, where p is any algebraic expression.

Solutions of absolute value inequalities can be presented algebraically or graphically. The general solutions of such graphs are as follows:

Type of inequality	Equivalent inequality to be solved	Nature of the graph
$\|p\| < c$	$-c < p < c$	
$\|p\| \leq c$	$-c \leq p \leq c$	
$\|p\| > c$	$p > c$ or $p < -c$	
$\|p\| \geq c$	$p \geq c$ or $p \leq -c$	

Example 6.49

Solve $|2x| < 10$ and plot its solution on the number line.

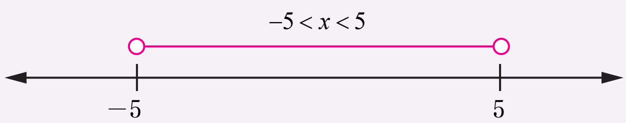
Solution

By definition, $|\, p \,| < c$ is equivalent to $-c < p < c$.

This implies that $-10 < 2x < 10$.

Dividing by 2 each term gives $-5 < x < 5$.

This solution is plotted on the number line as follows:



Therefore, the solution is $\{x \in \mathbb{R} : -5 < x < 5\}$.

Example 6.50

Solve $|3x - 5| \geq 2$ and plot its solution set on the number line.

Solution

Generally, the solution of $|\, p \,| \geq c$ is given by $p \geq c$ or $p \leq -c$.

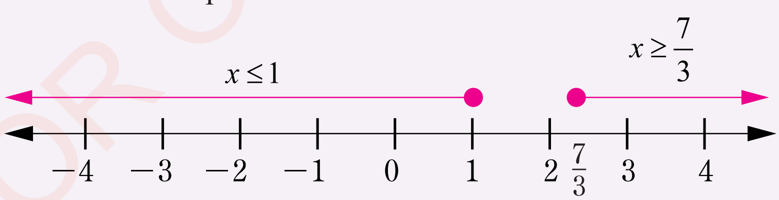
Thus, the solution of $|3x - 5| \geq 2$ is given by $3x - 5 \geq 2$ or $3x - 5 \leq -2$.

$\Rightarrow 3x \geq 7$ or $3x \leq 3$

Dividing by 3 both sides gives:

$$x \geq \frac{7}{3} \quad \text{or} \quad x \leq 1$$

This solution is plotted on the number line as follows:



Therefore, the solution is $\left\{x \in \mathbb{R} : x \leq 1,\ x \geq \frac{7}{3}\right\}$.

Example 6.51

Solve $\left|\frac{2-3x}{2}\right| \leq 6$ and plot its solution set on the number line.

Solution

Since $|p| \leq c$ is equivalent to $-c \leq p \leq c$.

This implies that the solution of $\left|\frac{2-3x}{2}\right| \leq 6$ is given by,

$$-6 \leq \frac{2-3x}{2} \leq 6$$

Multiplying by 2 in each term gives,

$$-12 \leq 2-3x \leq 12$$

Subtracting 2 from each term gives,

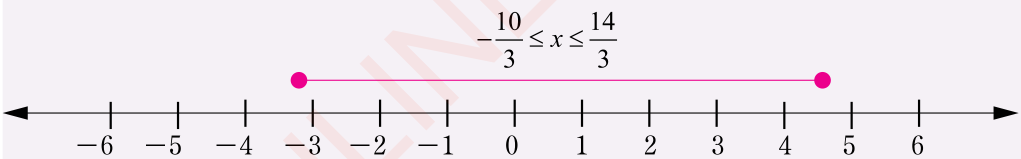
$$-14 \leq -3x \leq 10$$

Divide by -3 each term and change the sense of inequality signs:

$$\Rightarrow \frac{14}{3} \geq x \geq -\frac{10}{3}$$

$$\Rightarrow -\frac{10}{3} \leq x \leq \frac{14}{3}$$

This solution is shown on the number line as follows:



Therefore, the solution set is $\left\{x \in \mathbb{R}: -\frac{10}{3} \leq x \leq \frac{14}{3}\right\}$.

Exercise 6.7

Solve questions 1 – 19.

1. $|3-2x| < |4+x|$
2. $\left|\frac{x+3}{x-4}\right| > 2$
3. $|2x+2| > 5$
4. $|2x-1| + 4 < 2$
5. $|3x-1| - x \geq 0$
6. $|x-3| + |x+1| > 4$

7. $\frac{3x-1}{x-1}<2$

8. $\frac{x}{x^2-1}\geq 0$

9. $\frac{(x+1)(x-1)}{x}\leq 0$

10. $x^2-8x+15\geq 0$

11. $x^2-5x\leq 0$

12. $\frac{(x+1)(x+5)}{(x+2)(2x+3)}>0$

13. $\frac{2x^2+x-5}{2x^2+x-3}\leq 1$

14. $\log_x\left(\frac{x+3}{x-1}\right)>1$

15. $\frac{x^2+2}{x^2-1}<-2$

16. $\frac{x-2}{2x-3}<\frac{x}{3x-2}$

17. $\frac{x+3}{x-2}<\frac{4}{3x-3}$

18. $\log\left(\frac{1-x}{1+2x}\right)<1$

19. $|8x-4|+10>4$

20. Show that $0\leq\frac{(x+1)^2}{x^2+x+1}\leq\frac{4}{3}$ for all real values of x.

Matrices

A set of numbers arranged in a rectangular array having m rows and n columns and enclosed by a square bracket [ ] or round bracket ( ) is called a matrix. Matrices are denoted by capital letters A, B, C or any other letter, and described by using rows and columns with rows appearing first. A matrix $m\times n$ (read as m by n) is expressed as follows:

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$$

The entry a_{ij} is the element in the i^{th} row and j^{th} column of the matrix. For example, a_{12} is the element in the first row and the second column. The order or dimension of a matrix is the ordered pair $m\times n$ of which m is the number of rows and n is the number of columns of a matrix. If a matrix has 3 rows and 2 columns, then its order is written as 3×2.

If the number of rows and columns are equal, the matrix is called a square matrix, and if the number of rows and columns are not equal, the matrix is just called a rectangular matrix.

For example, $A=\begin{pmatrix} 1 & 1 & 3 \\ 2 & 5 & 6 \\ 7 & 7 & 5 \end{pmatrix}$ is a square matrix and $B=\begin{bmatrix} -4 & -2 \\ -8 & -1 \\ -7 & -9 \end{bmatrix}$ is a rectangular matrix.

Activity 6.9: Recognizing the number of rows and columns in the matrix

Individually or in a group, read carefully the following paragraph and then perform the tasks that follow.

An electronic company produces laptops, mobile phones, and desktop computers. Each laptop requires 10 units of gold, 8 units of copper, and 12 units of iron. Each mobile phone requires 9 units of gold, 7 units of copper, and 5 units of iron. Each desktop computer requires 15, 11, and 14 units of gold, copper, and iron, respectively.

1. Present this information in matrix form.
2. Identify the number of rows and columns of the matrix in task 1.
3. Identify the order of the matrix in task 1.
4. What are the similarities and differences between the number of rows and columns identified in task 2?
5. Share your findings with your fellow students for more inputs.

Types of matrices

There are several types of matrices. The following are some types of matrices:

Row and column matrices

A matrix consisting of a single row is called a row matrix or a row vector, whereas a matrix having a single column is called a column matrix or a column vector. For example, $D = \begin{bmatrix} 1 & 4 & 8 \end{bmatrix}$ and $E = \begin{bmatrix} -6 \\ -8 \\ -10 \end{bmatrix}$ are row and column matrices, respectively.

Null or zero matrix

A matrix in which each element is 0 is called a null or zero matrix. For example, the matrix

$$C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is called a zero matrix.

Square matrix

A matrix having the same number of rows and columns is called a square matrix. For example,

$$C = \begin{bmatrix} 2 & 6 & 10 \\ -2 & -5 & 12 \\ -4 & -8 & -15 \end{bmatrix}$$

is a square matrix. Any matrix of order $n \times n$ is a square matrix.

Scalar matrix

A square matrix whose elements in the leading diagonal are equal or the same, and the rest of the elements are zero is called a scalar matrix.

For example,

$$K = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix}$$

is a scalar matrix.

Diagonal matrix

A square matrix in which all elements are zero except those in the main or leading diagonal is called a diagonal matrix.

For example, $G = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -6 \end{bmatrix}$

is a diagonal matrix of order 3×3.

Identity matrix or unit matrix

A square matrix whose elements in the leading diagonal are one and the rest of the elements are zero is called an identity or unit matrix. For example,

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is an identity matrix of order 3×3.

Operations on matrices

The operations on matrices include addition, subtraction, and multiplication.

Addition and subtraction of matrices

Two or more matrices are added or subtracted if and only if they have the same order.

Multiplication of a matrix

Two matrices can be multiplied if the number of columns in the first matrix is equal to the number of rows in the second matrix. The dimension of the resulting matrix will have the number of rows of the first matrix and the number of columns of the second matrix. For example, if matrix A is of order $m \times p$ and matrix B is of order $q \times n$, then multiplication of the two matrices, AB is possible if and only if $p = q$, and the resulting matrix will be of order $m \times n$. The steps for multiplying a matrix by another matrix are as follows:

Let matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ and matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}.$$

To obtain the matrix AB, use the following steps:

1. Multiply the elements of the first row in A by the elements of the first column in B,

   that is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}$ .... 1st element is obtained.

2. Multiply the elements of the first row in A by the elements of the second column in B, that is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}$ .... 2^{nd} element is obtained.

3. Multiply the elements of the first row in A by the elements of the third column in B, that is $\begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix} \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33}$ .... 3^{rd} element is obtained.

4. Multiply the elements of the second row in A by the elements of the first column in B, that is $\begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}$ .... 4^{th} element is obtained.

5. Multiply the elements of the second row in A by the elements of the second column in B, that is $\begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}$ .... 5^{th} element is obtained.

6. Multiply the elements of the second row in A by the elements of the third column in B, that is $\begin{pmatrix} a_{21} & a_{22} & a_{23} \end{pmatrix} \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33}$ .... 6^{th} element is obtained.

7. Multiply the elements of the third row in A by the elements of the first column in B, that is $\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \end{pmatrix} = a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31}$ .... 7^{th} element is obtained.

8. Multiply the elements of the third row in A by the elements of the second column in B, that is $\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \end{pmatrix} = a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32}$ .... 8^{th} element is obtained.

9. Multiply the elements of the third row in A by the elements of the third column in B, that is $\begin{pmatrix} a_{31} & a_{32} & a_{33} \end{pmatrix} \times \begin{pmatrix} b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33}$ .... 9^{th} element is obtained.

$$\text{Thus, } AB = \begin{pmatrix} 1^{\text{st}} \text{ element} & 2^{\text{nd}} \text{ element} & 3^{\text{rd}} \text{ element} \\ 4^{\text{th}} \text{ element} & 5^{\text{th}} \text{ element} & 6^{\text{th}} \text{ element} \\ 7^{\text{th}} \text{ element} & 8^{\text{th}} \text{ element} & 9^{\text{th}} \text{ element} \end{pmatrix}$$

Therefore,

$$AB = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{pmatrix}$$

Example 6.52

If $G = \begin{pmatrix} -2 & -8 & 9 \\ -6 & -5 & 7 \\ 12 & 10 & -3 \end{pmatrix}$ and $H = \begin{pmatrix} 4 & -4 & -10 \\ 13 & -9 & -11 \\ 3 & 6 & -15 \end{pmatrix}$, find GH.

Solution

$$GH = \begin{pmatrix} -2 & -8 & 9 \\ -6 & -5 & 7 \\ 12 & 10 & -3 \end{pmatrix} \times \begin{pmatrix} 4 & -4 & -10 \\ 13 & -9 & -11 \\ 3 & 6 & -15 \end{pmatrix} =$$

$$\begin{pmatrix} (-2\times4)+(-8\times13)+(9\times3) & (-2\times-4)+(-8\times-9)+(9\times6) & (-2\times-10)+(-8\times-11)+(9\times-15) \\ (-6\times4)+(-5\times13)+(7\times3) & (-6\times-4)+(-5\times-9)+(7\times6) & (-6\times-10)+(-5\times-11)+(7\times-15) \\ (12\times4)+(10\times13)+(-3\times3) & (12\times-4)+(10\times-9)+(-3\times6) & (12\times-10)+(10\times-11)+(-3\times-15) \end{pmatrix}$$

$$= \begin{pmatrix} -85 & 134 & -27 \\ -68 & 111 & 10 \\ 169 & -156 & -185 \end{pmatrix}$$

$$\text{Therefore, } GH = \begin{pmatrix} -85 & 134 & -27 \\ -68 & 111 & 10 \\ 169 & -156 & -185 \end{pmatrix}.$$

Example 6.53

Given that $E=\begin{pmatrix}4 & a & -6\\ -8 & c & -4\\ 10 & 3 & b\end{pmatrix}$, $F=\begin{pmatrix}12\\ -15\\ -10\end{pmatrix}$, and $EF=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}$. Find the values of a, b, and c.

Solution

$$EF=\begin{pmatrix}4 & a & -6\\ -8 & c & -4\\ 10 & 3 & b\end{pmatrix}\times\begin{pmatrix}12\\ -15\\ -10\end{pmatrix}=\begin{pmatrix}4\times12+a\times(-15)+(-6)\times(-10)\\ -8\times12+c\times(-15)+(-4)\times(-10)\\ 10\times12+3\times(-15)+b\times(-10)\end{pmatrix}$$

$$=\begin{pmatrix}108-15a\\ -56-15c\\ 75-10b\end{pmatrix} \text{ since, } EF=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}, \text{ then}$$

$$\begin{pmatrix}108-15a\\ -56-15c\\ 75-10b\end{pmatrix}=\begin{pmatrix}-72\\ 109\\ -80\end{pmatrix}$$

Equating the corresponding elements gives,

$108-15a=-72 \Rightarrow a=12,$

$-56-15c=109 \Rightarrow c=-11,$

$75-10b=-80 \Rightarrow b=15.5.$

Therefore, $a=12$, $b=15.5$, and $c=-11$.

Example 6.54

Given $M=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}$, verify that MI=IM = M, where I is a multiplicative identity matrix.

Solution

$$MI=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}\times\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}-12 & 20 & 40\\ 14 & -8 & -2\\ 26 & 15 & 6\end{pmatrix}$$

Also, $\text{IM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} -12 & 20 & 40 \\ 14 & -8 & -2 \\ 26 & 15 & 6 \end{pmatrix} = \begin{pmatrix} -12 & 20 & 40 \\ 14 & -8 & -2 \\ 26 & 15 & 6 \end{pmatrix}$

Therefore, MI=IM = M.

Transpose of a matrix

Let A be an $m \times n$ matrix, then the $n \times m$ matrix obtained after interchanging the rows and columns of matrix A is called the transpose of matrix A and it is denoted by A^{T}. For example, if $\text{A} = \begin{pmatrix} 2 & 8 & -7 \\ 4 & -2 & 9 \\ 6 & 1 & 5 \end{pmatrix}$, then $\text{A}^{\text{T}} = \begin{pmatrix} 2 & 4 & 6 \\ 8 & -2 & 1 \\ -7 & 9 & 5 \end{pmatrix}$.

Example 6.55

Given $\text{L} = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix}$ and $\text{M} = \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$, verify that $(\text{LM})^{\text{T}} = \text{M}^{\text{T}}\text{L}^{\text{T}}$.

Solution

Consider the left-hand side

$$\text{LM} = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix} \times \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} -40+30-12 & -10-45+0 & -70-30+108 \\ 32-8-3 & 8+12+0 & 56+8+27 \\ 24-20+1 & 6+30+0 & 42+20-9 \end{pmatrix}$$

$$= \begin{pmatrix} -22 & -55 & 8 \\ 21 & 20 & 91 \\ 5 & 36 & 53 \end{pmatrix}$$

Thus, $(LM)^T = \begin{pmatrix} -22 & 21 & 5 \\ -55 & 20 & 36 \\ 8 & 91 & 53 \end{pmatrix}$

Similarly, consider the right-hand side:

Since $M = \begin{pmatrix} 4 & 1 & 7 \\ -2 & 3 & 2 \\ -1 & 0 & 9 \end{pmatrix}$, then $M^T = \begin{pmatrix} 4 & -2 & -1 \\ 1 & 3 & 0 \\ 7 & 2 & 9 \end{pmatrix}$

Also, $L = \begin{pmatrix} -10 & -15 & 12 \\ 8 & 4 & 3 \\ 6 & 10 & -1 \end{pmatrix}$, $L^T = \begin{pmatrix} -10 & 8 & 6 \\ -15 & 4 & 10 \\ 12 & 3 & -1 \end{pmatrix}$.

$$\Rightarrow M^TL^T = \begin{pmatrix} 4 & -2 & -1 \\ 1 & 3 & 0 \\ 7 & 2 & 9 \end{pmatrix} \times \begin{pmatrix} -10 & 8 & 6 \\ -15 & 4 & 10 \\ 12 & 3 & -1 \end{pmatrix}$$

Thus, $M^TL^T = \begin{pmatrix} -22 & 21 & 5 \\ -55 & 20 & 36 \\ 8 & 91 & 53 \end{pmatrix}$

Therefore, $(ML)^T = M^TL^T$.

Exercise 6.8

1. If $N = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, verify that $N^2 - 4N - 5I = M$, where I is an identity matrix and M is a null matrix.

2. Given $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 9 \\ 1 & -3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} -2 & 5 & 8 \\ -6 & 10 & 12 \\ 13 & 4 & 7 \end{pmatrix}$, find AB.

3. Given that $J = \begin{bmatrix} 8 & 7 & 2 \\ 10 & 6 & 4 \\ 12 & 14 & 21 \end{bmatrix}$ and $K = \begin{bmatrix} -4 & 5 & -10 \\ -6 & 4 & -12 \\ -8 & 10 & -18 \end{bmatrix}$, show that $JK \neq KJ$.

4. If $L = \begin{pmatrix} 80 & 50 & 40 \\ 20 & 30 & -50 \\ -90 & -70 & -95 \end{pmatrix}$ and $M = \begin{pmatrix} -19 \\ -57 \\ -38 \end{pmatrix}$, evaluate each of the following:

   (a) LM (b) ML (c) M^2

5. Given that $P = \begin{pmatrix} c & 15 & -20 \\ -18 & 12 & d \\ 14 & 20 & 25 \end{pmatrix}$, $Q = \begin{pmatrix} -4 \\ -6 \\ e \end{pmatrix}$, and $PQ = \begin{pmatrix} -70 \\ 36 \\ 49 \end{pmatrix}$ find the values of c, d, and e.

6. Find $\begin{pmatrix} -2 & 12 & 0 \\ -4 & 8 & -2 \\ -5 & 4 & -6 \end{pmatrix} \times \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 1 & -2 \end{pmatrix} \times \begin{pmatrix} 10 \\ -10 \\ 20 \end{pmatrix}$.

7. If $R = \begin{pmatrix} -14 & 40 & -8 \\ -16 & 20 & -6 \\ -21 & 30 & -4 \end{pmatrix}$, show that $RI = IR = R$, where I is an identity matrix.

8. Given $P = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{pmatrix}$ and $Q = \begin{pmatrix} -2 & 1 & -4 \\ -1 & 2 & -5 \\ -3 & 3 & -6 \end{pmatrix}$, find:

   (a) $(PQ)^T$

   (b) $Q^T P^T$. Hence show that $(PQ)^T = Q^T P^T$.

9. The distribution of number of grades A, B, and C, and grade points from the departments of Geography, Chemistry, and Biology are represented by matrices M and N, respectively as follows:

$$M = \begin{array}{r} \\ \text{Geography} \\ \text{Chemistry} \\ \text{Biology} \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 45 & 60 & 50 \\ 35 & 55 & 40 \\ 48 & 50 & 36 \end{bmatrix} \end{array} \quad \text{and} \quad N = \begin{array}{r} A \\ B \\ C \end{array} \begin{bmatrix} 20 \\ 18 \\ 12 \end{bmatrix}$$

Find the matrix MN and interpret it.

10. Three students buy soft drinks of type D_1, D_2, and D_3. The first student buys 8, 12, and 16 bottles of D_1, D_2, and D_3, respectively. The second student buys 5, 9, and 10 bottles of D_1, D_2, and D_3, respectively. The third student buys 15, 18, and 12 bottles of D_1, D_2, and D_3, respectively.
    (a) Represent the information in matrix form.
    (b) If the cost of each bottle of D_1, D_2, and D_3 is 500 Tanzanian shillings, using matrix operations find the total amount of money spent by each student individually.

Determinant of a 3 × 3 matrix

A real number that represents the magnitude of square matrix is called a determinant of a matrix. The determinant of a matrix A is denoted by det(A) or $|A|$. It can be calculated provided that A is a square matrix. If $|A| = 0$, then the matrix A is called singular matrix, and if $|A| \neq 0$, then the matrix is called non-singular matrix.

Finding determinants of 3×3 matrices

The following steps can be used in determining the determinant of a 3×3 matrix:

1. Choose any row or any column in the given matrix.
2. Compute the cofactors of each element in the chosen row or column.
3. The determinant is equal to the sum of the products of the elements and their respective cofactors.

Consider matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

Choose the first row with elements a_{11}, a_{12}, and a_{13}. According to the patterns of signs $(+ \text{ or } -)$ for calculating cofactors:

cofactor of $a_{11} = +\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$, cofactor of $a_{12} = -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$, and cofactor of $a_{13} = +\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Therefore, $|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$.

Example 6.56

Given $D = \begin{pmatrix} 19 & -20 & -24 \\ 36 & 18 & -16 \\ -10 & 15 & 14 \end{pmatrix}$, evaluate $|D|$.

Solution

Using the first row of the matrix D:

$$|D| = \begin{vmatrix} 19 & -20 & -24 \\ 36 & 18 & -16 \\ -10 & 15 & 14 \end{vmatrix}$$

$$= 19\begin{vmatrix} 18 & -16 \\ 15 & 14 \end{vmatrix} - (-20)\begin{vmatrix} 36 & -16 \\ -10 & 14 \end{vmatrix} + (-24)\begin{vmatrix} 36 & 18 \\ -10 & 15 \end{vmatrix}$$

$$= 9348 + 6880 - 17280$$

$$= -1052$$

Therefore, $|D| = -1052$.

Example 6.57

Given $W = \begin{pmatrix} 2 & 4 & -3 \\ -1 & -2 & 1 \\ 5 & 3 & -4 \end{pmatrix}$ and $V = \begin{pmatrix} -5 & -2 & 6 \\ -6 & 5 & 1 \\ 2 & 3 & -3 \end{pmatrix}$, evaluate:

(a) $\det\left((WV)^T\right)$ (b) $\det\left(V^T W^T\right)$.

Hence, show that $\det\left((WV)^T\right) = \det\left(V^T W^T\right)$.

Solution

(a) $WV = \begin{pmatrix} 2 & 4 & -3 \\ -1 & -2 & 1 \\ 5 & 3 & -4 \end{pmatrix} \times \begin{pmatrix} -5 & -2 & 6 \\ -6 & 5 & 1 \\ 2 & 3 & -3 \end{pmatrix}$

$$=\begin{pmatrix} -40 & 7 & 25 \\ 19 & -5 & -11 \\ -51 & -7 & 45 \end{pmatrix}$$

$$\Rightarrow (WV)^T = \begin{pmatrix} -40 & 19 & -51 \\ 7 & -5 & -7 \\ 25 & -11 & 45 \end{pmatrix}$$

Using the first column of matrix $(WV)^T$:

$$\det\left((WV)^T\right) = -40\begin{vmatrix} -5 & -7 \\ -11 & 45 \end{vmatrix} - 7\begin{vmatrix} 19 & -51 \\ -11 & 45 \end{vmatrix} + 25\begin{vmatrix} 19 & -51 \\ -5 & -7 \end{vmatrix}$$

$$= 12080 - 2068 - 9700$$

$$= 322$$

Therefore, $\det\left((WV)^T\right) = 322$.

(b) $V^T = \begin{pmatrix} -5 & -6 & 2 \\ -2 & 5 & 3 \\ 6 & 1 & -3 \end{pmatrix}$ and $W^T = \begin{pmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \\ -3 & 1 & -4 \end{pmatrix}$

$$\Rightarrow V^TW^T = \begin{pmatrix} -5 & -6 & 2 \\ -2 & 5 & 3 \\ 6 & 1 & -3 \end{pmatrix} \times \begin{pmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \\ -3 & 1 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} -40 & 19 & -51 \\ 7 & -5 & -7 \\ 25 & -11 & 45 \end{pmatrix}$$

Using the third column of matrix W^TV^T:

$$\det\left(V^TW^T\right) = +(-51)\begin{vmatrix} 7 & -5 \\ 25 & -11 \end{vmatrix} - (-7)\begin{vmatrix} -40 & 19 \\ 25 & -11 \end{vmatrix} + 45\begin{vmatrix} -40 & 19 \\ 7 & -5 \end{vmatrix}$$

$$= -2448 - 245 + 3015$$

$$= 322$$

$\det\left(V^TW^T\right) = 322$.

Therefore, $\det\left((WV)^T\right) = 322 = \det\left(V^TW^T\right)$.

Minors of a matrix

The minor of an element in a matrix is the determinant of a sub-matrix formed by deleting the rows and columns in a given element. Each element in a 3×3 matrix has its own minor. The minors may be denoted by M_{ij}, where i and j are the deleted rows and columns, respectively.

The steps for obtaining the minors are as follows:

Let matrix $\mathrm{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, and M_{ij} be the minors of A.

For M_{11}, delete the first row and first column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$\mathrm{M}_{11}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}=(a_{22}\times a_{33})-(a_{23}\times a_{32})$$

For M_{12}, delete the first row and second column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$\mathrm{M}_{12}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}=(a_{21}\times a_{33})-(a_{23}\times a_{31})$$

For M_{13}, delete the first row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$\mathrm{M}_{13}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}=(a_{21}\times a_{32})-(a_{22}\times a_{31})$$

For M_{21}, delete the second row and the first column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$\mathrm{M}_{21}=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}=\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}=(a_{12}\times a_{33})-(a_{13}\times a_{32})$$

For M_{22}, delete the second row and the second column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$M_{22} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} = (a_{11} \times a_{33}) - (a_{13} \times a_{31})$$

For M_{23}, delete the second row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = (a_{11} \times a_{32}) - (a_{12} \times a_{31})$$

For M_{31}, delete the third row and the first column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$M_{31} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = (a_{12} \times a_{23}) - (a_{13} \times a_{22})$$

For M_{32}, delete the third row and the second column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$M_{32} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = (a_{11} \times a_{23}) - (a_{13} \times a_{21})$$

For M_{33}, delete the third row and the third column of A, then compute the determinant of the resulting 2×2 matrix as follows:

$$M_{33} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11} \times a_{22}) - (a_{12} \times a_{21})$$

Therefore, $M_{ij} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$ is a matrix of minors.

Cofactors of a 3 × 3 matrix

A cofactor of an element a_{ij} is obtained after multiplying by $(-1)^{i+j}$ to the minor M_{ij}. The matrix of cofactors of a matrix A is abbreviated as cof (A). Generally, if

$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then $\text{cof}(a_{ij}) = (-1)^{i+j} M_{ij}$.

The cofactors of a matrix A are obtained as follows:

The cofactor of $a_{11} = (-1)^{1+1} \times M_{11}$

The cofactor of $a_{12} = (-1)^{1+2} \times M_{12}$

The cofactor of $a_{13} = (-1)^{1+3} \times M_{13}$

The cofactor of $a_{21} = (-1)^{2+1} \times M_{21}$

The cofactor of $a_{22} = (-1)^{2+2} \times M_{22}$

The cofactor of $a_{23} = (-1)^{2+3} \times M_{23}$

The cofactor of $a_{31} = (-1)^{3+1} \times M_{31}$

The cofactor of $a_{32} = (-1)^{3+2} \times M_{32}$

The cofactor of $a_{33} = (-1)^{3+3} \times M_{33}$

The following pattern of signs may be useful in calculating cofactors of a 3×3 matrix.

That is, $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

Generally, the matrix of cofactors of a matrix A is given by,

$\text{cof}(A) = \left[(-1)^{i+j} M_{ij}\right]$.

For a 3×3 matrix A, the matrix of cofactors is,

$$\text{cof}(A) = \begin{bmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{bmatrix}.$$

Example 6.58

Given $M = \begin{pmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{pmatrix}$, find:

(a) The minors, M_{13} and M_{22}.

(b) The cofactors of the elements a_{13} and a_{22}.

Solution

(a) $M_{13} = \begin{vmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{vmatrix} = \begin{vmatrix} -12 & 28 \\ 15 & -39 \end{vmatrix}$

$= (-12 \times -39) - (28 \times 15)$

$= 48$

$M_{22} = \begin{vmatrix} -8 & 56 & -72 \\ -12 & 28 & 66 \\ 15 & -39 & 95 \end{vmatrix} = \begin{vmatrix} -8 & -72 \\ 15 & 95 \end{vmatrix}$

$= (-8 \times 95) - (-72 \times 15)$

$= 320$

Therefore, the minors, M_{13} and M_{22} are 48 and 320, respectively.

(b) Cofactor of $a_{13} = (-1)^{1+3} \times M_{13}$

$= 1 \times 48$

$= 48$

cofactor of $a_{22} = (-1)^{2+2} \times M_{22}$

$= (1) \times 320$

$= 320$

Therefore, the cofactors of the elements a_{13} and a_{22} are 48 and 320, respectively.

Exercise 6.9

1. Calculate each of the following:

(a) $\begin{vmatrix} 5 & 9 & 6 \\ 3 & 7 & 8 \\ 21 & 55 & 6 \end{vmatrix}$ (b) $\begin{vmatrix} -2 & 4 & 1 \\ 5 & 10 & 3 \\ 3 & -6 & -11 \end{vmatrix}$

2. Find the matrix of the minors in each of the following:

(a) $D = \begin{pmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{pmatrix}$ (b) $E = \begin{pmatrix} 1 & 2 & 5 \\ 2 & 4 & 5 \\ 3 & 1 & 3 \end{pmatrix}$

3. Find the matrix of cofactors in each of the following:

(a) $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{pmatrix}$ (b) $\begin{pmatrix} 2 & 0 & 6 \\ 1 & 5 & 0 \\ 4 & -1 & 3 \end{pmatrix}$

4. Use cofactors to evaluate each of the following:

(a) $\begin{vmatrix} 4 & -7 & 6 \\ -2 & 4 & 0 \\ 5 & 7 & -4 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & -2 & 6 \\ 2 & 1 & -3 \\ 3 & 0 & 5 \end{vmatrix}$

5. If $J = \begin{pmatrix} 3 & -3 & 0 \\ 6 & 3 & 9 \\ 12 & 3 & 24 \end{pmatrix}$ and $K = \begin{pmatrix} 2 & 3 & 0 \\ 6 & -9 & 3 \\ 3 & 3 & -3 \end{pmatrix}$, show that $\left|(JK)^T\right| = \left|J^T K^T\right|$.

6. Verify each of the following:

(a) $\begin{vmatrix} a+b & a & a \\ a & a+b & a \\ a & a & a+b \end{vmatrix} = b^2(3a+b)$

(b) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

7. In each of the following, show that:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

(b) $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix} = xyz\left(1+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$

(c) $\begin{vmatrix} x-y-z & 2x & 2x \\ 2y & y-z-x & 2y \\ 2z & 2z & z-x-y \end{vmatrix} = (x+y+z)^3$

8. Given $K = \begin{pmatrix} -18 & -16 & 39 \\ -42 & 15 & 41 \\ 32 & 28 & 60 \end{pmatrix}$ and $L = \begin{pmatrix} -5 & -10 & 14 \\ -6 & -11 & 7 \\ -4 & 15 & 2 \end{pmatrix}$, show that

   (a) $\det(KL) = \det(LK)$ (b) $\det\left((KL)^T\right) = \det\left(L^T K^T\right)$

9. Find the values each of the unknown elements in each of the following matrices:

   (a) $D = \begin{pmatrix} -2 & 1 & 1 \\ 3 & 2 & 2 \\ 1 & m & 4 \end{pmatrix}$ if D is a singular matrix

   (b) $E = \begin{pmatrix} 8 & -2 & r \\ 2 & -3 & -2 \\ 6 & 3 & 8 \end{pmatrix}$ if $|E| = -328$

10. Given that matrix $C = \begin{pmatrix} -1 & 2 & 3 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, find:

   (a) The matrix from the expression $C^2 - 2C + I$, where I is an identity matrix.

   (b) $\det(C^2 - 2C + I)$

Solving systems of linear simultaneous equations using Cramer's rule

Consider the following system of linear equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = d_1 \\ a_{21}x + a_{22}y + a_{23}z = d_2 \\ a_{31}x + a_{32}y + a_{33}z = d_3 \end{cases}$$

Write the system of equations in matrix form as follows:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a coefficient matrix of the system of linear equations.

By Cramer's rule, the values of x, y, and z are obtained by using the following formulae:

$x=\frac{|A_x|}{|A|}$, $y=\frac{|A_y|}{|A|}$, and $z=\frac{|A_z|}{|A|}$ where $|A|$ is the determinant of the coefficient matrix.

That is, $|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$.

$|A_x|$ is the determinant of matrix A, with the first column entries replaced by the values d_1, d_2, and d_3. That is,

$$|A_x|=\begin{vmatrix} d_1 & a_{12} & a_{13} \\ d_2 & a_{22} & a_{23} \\ d_3 & a_{32} & a_{33} \end{vmatrix},$$

$|A_y|$ is the determinant of matrix A, with the second column entries replaced by the values d_1, d_2, and d_3. That is,

$$|A_y|=\begin{vmatrix} a_{11} & d_1 & a_{13} \\ a_{21} & d_2 & a_{23} \\ a_{31} & d_3 & a_{33} \end{vmatrix}, \text{ and}$$

$|A_z|$ is the determinant of matrix A, where the third column entries are replaced by the values d_1, d_2, and d_3. That is,

$$|A_z|=\begin{vmatrix} a_{11} & a_{12} & d_1 \\ a_{21} & a_{22} & d_2 \\ a_{31} & a_{32} & d_3 \end{vmatrix}.$$

Example 6.59

Solve the following system of linear equations using Cramer's rule.

$$\begin{cases} x+y+z=3 \\ x-3y+2z=0 \\ 5x-y-3z=1 \end{cases}$$

Solution

The matrix form of the given system of equations is $\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

$$\text{Let A}=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{pmatrix}$$

$$\Rightarrow |A|=\begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 2 \\ 5 & -1 & -3 \end{vmatrix}$$

$$=1\begin{vmatrix} -3 & 2 \\ -1 & -3 \end{vmatrix}-1\begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix}+1\begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix}$$

$$=11+13+14$$

$$=38$$

$$\Rightarrow |A_x|=\begin{vmatrix} 3 & 1 & 1 \\ 0 & -3 & 2 \\ 1 & -1 & -3 \end{vmatrix}$$

$$=3\begin{vmatrix} -3 & 2 \\ -1 & -3 \end{vmatrix}-1\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix}+1\begin{vmatrix} 0 & -3 \\ 1 & -1 \end{vmatrix}$$

$$=33+2+3$$

$$=38$$

$$\Rightarrow |A_y|=\begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 2 \\ 5 & 1 & -3 \end{vmatrix}$$

$$=1\begin{vmatrix} 0 & 2 \\ 1 & -3 \end{vmatrix}-3\begin{vmatrix} 1 & 2 \\ 5 & -3 \end{vmatrix}+1\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix}$$

$$=-2+39+1$$

$$=38$$

$$\Rightarrow |A_z|=\begin{vmatrix} 1 & 1 & 3 \\ 1 & -3 & 0 \\ 5 & -1 & 1 \end{vmatrix}$$

$$=1\begin{vmatrix} -3 & 0 \\ -1 & 1 \end{vmatrix}-1\begin{vmatrix} 1 & 0 \\ 5 & 1 \end{vmatrix}+3\begin{vmatrix} 1 & -3 \\ 5 & -1 \end{vmatrix}$$

$$=-3-1+42$$

$$=38$$

By using cramer's rule;

$$\Rightarrow x=\frac{|A_x|}{|A|}=\frac{38}{38}$$

$$\Rightarrow x=1$$

$$\Rightarrow y=\frac{|A_y|}{|A|}=\frac{38}{38}$$

$$\Rightarrow y=1$$

$$\Rightarrow z=\frac{|A_z|}{|A|}=\frac{38}{38}$$

$$\Rightarrow z=1$$

Therefore, the solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Adjoint of a matrix

Adjoint matrix of a matrix A is a transpose of the matrix of cofactors of matrix A. Therefore, adjoint matrix of A is a given by: $\text{Adj}(A)=(\text{cof}(A))^{T}$.

But $\operatorname{cof}(A)=\left((-1)^{i+j}M_{ij}\right)$, where M_{ij} are minors of A.

Therefore, $\operatorname{Adj}(A)=\left((-1)^{i+j}M_{ij}\right)^T$.

Inverse of a 3×3 matrix

The inverse of a non-singular matrix A is denoted by A^{-1}. It can only be defined if the given matrix is a square matrix. The inverse of a matrix exists if and only if it is non-singular.

Properties of inverse of matrices

Let A and B be matrices whose inverses are A^{-1} and B^{-1}, respectively. The following properties hold:

(a) $AA^{-1}=A^{-1}A=I$, where, I is the identity matrix.

(b) $\left(A^{-1}\right)^{-1}=A$

(c) $(AB)^{-1}=B^{-1}A^{-1}$

(d) When $|A|=0$, then A is singular matrix. Hence, $\frac{1}{|A|}$ is undefined and therefore singular matrices have no inverse.

If A is a 3×3 matrix, then A^{-1} is defined as $A^{-1}=\frac{\operatorname{Adj}(A)}{|A|}$, where Adj(A) and $|A|$ are the adjoint and determinant of matrix A, respectively.

Example 6.60

Given $A=\begin{pmatrix}1&2&1\\1&1&1\\2&5&3\end{pmatrix}$, find the following:

(a) det (A)
(b) Minors of A
(c) Cofactors of A
(d) adjoint of A
(e) A^{-1}

Solution

(a) Given $A=\begin{pmatrix}1&2&1\\1&1&1\\2&5&3\end{pmatrix}$

$$\Rightarrow \det(A)=1\begin{vmatrix}1&1\\5&3\end{vmatrix}-2\begin{vmatrix}1&1\\2&3\end{vmatrix}+1\begin{vmatrix}1&1\\2&5\end{vmatrix}$$

$$= 1(-2)-2(1)+1(3)$$
$$=-2-2+3$$
$$=-1$$

Therefore, $\det(A)=-1$.

(b) Let $M=\begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}$ be a matrix of minors of A.

Given $A=\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 5 & 3 \end{pmatrix}$, then the following are the minors of A:

$$M_{11}=\begin{vmatrix} 1 & 1 \\ 5 & 3 \end{vmatrix}=-2,\ M_{12}=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1,\ M_{13}=\begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix}=3,$$

$$M_{21}=\begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix}=1,\ M_{22}=\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}=1,\ M_{23}=\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}=1,$$

$$M_{31}=\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix}=1,\ M_{32}=\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=0,\ M_{33}=\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}=-1.$$

Therefore, the matrix of the minors of $A=\begin{pmatrix} -2 & 1 & 3 \\ 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.

(c) The matrix of cofactors is obtained by multiplying the matrix of minors by the signs $(-1)^{i+j}$, as shown in the following pattern.

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Thus, the matrix of cofactors is $\text{cof}(A)=\left((-1)^{i+j}M_{ij}\right)=\begin{pmatrix} -2 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

Therefore, $\text{cof}(A)=\begin{pmatrix} -2 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}$.

(d) $\text{Adj}(A)=[\text{Cof}(A)]^T$

$$\Rightarrow \text{Adj(A)} = \begin{pmatrix} -2 & -1 & 3 \\ -1 & 1 & -1 \\ 1 & 0 & -1 \end{pmatrix}^{T}$$

$$= \begin{pmatrix} -2 & -1 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$$

Therefore, adjoint of $\text{A} = \begin{pmatrix} -2 & -1 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix}$.

(e) The inverse of matrix A is determined by dividing each element in the adjoint matrix by the determinant of A. That is,

$$\text{A}^{-1} = \frac{\text{Adj}(\text{A})}{|\text{A}|} = -\frac{1}{1}\begin{pmatrix} -2 & -1 & 1 \\ -1 & 1 & 0 \\ 3 & -1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

Therefore, $\text{A}^{-1} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$.

Solution of systems of linear equations using inverse matrix method

Consider the following system of linear equations

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = d_1 \\ a_{21}x + a_{22}y + a_{23}z = d_2 \\ a_{31}x + a_{32}y + a_{33}z = d_3 \end{cases} \qquad (6.1)$$

The system of linear equations (6.1) can be written in matrix form as;

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} \qquad (6.2)$$

Thus, the equations (6.2)becomes;

$$\text{A}\underline{x} = \underline{d} \qquad (6.3)$$

where $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $\underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$,

and $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

Pre-multiply the equations (6.3) by A^{-1} on both sides to obtain;

$A^{-1}A\underline{x} = A^{-1}\underline{d}$.

But $A^{-1}A = I$, where I is an identity matrix.

Thus, $I\underline{x} = A^{-1}\underline{d}$

$\Rightarrow \underline{x} = A^{-1}\underline{d}$

Therefore, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

Example 6. 61

Solve the following system of linear equations by using the inverse matrix method.

$$\begin{cases} x + y + z = 3 \\ x + 2y + 3z = 4 \\ x + 4y + 9z = 6 \end{cases}$$

Solution

The matrix form of this equation is,

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$, then

$$|A| = 1\begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix}$$

$= 6 - 6 + 2$

$= 2$

The cofactors of A are as follows:

$C_{11} = \begin{vmatrix} 2 & 3 \\ 4 & 9 \end{vmatrix} = 6$

$C_{12} = -\begin{vmatrix} 1 & 3 \\ 1 & 9 \end{vmatrix} = -6$

$C_{13} = \begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2$

$C_{21} = -\begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -5$

$C_{22} = \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 8$

$C_{23} = -\begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -3$

$C_{31} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$

$C_{32} = -\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} = -2$

$C_{33} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1$

Thus, $\operatorname{cof}(A) = \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}$

But, $\operatorname{Adj}(A) = (\operatorname{cof}(A))^T$.

$$\Rightarrow \text{Adj(A)} = \begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

$$\text{A}^{-1} = \frac{\text{Adj}(\text{A})}{|\text{A}|} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}$$

Now, from the equation $\underline{x} = \text{A}^{-1}\underline{d}$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

Therefore, $x = 2$, $y = 1$, and $z = 0$.

Exercise 6.10

1. Let R be a matrix such that,

$$\text{R} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -3 & 2 \\ 3 & 1 & -1 \end{pmatrix}$$

   (a) Find the inverse of R

   (b) Use the inverse obtained in (a) to solve the following system of equations:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 2z = 14 + 3y \\ 3x + y = z - 2 \end{cases}$$

2. Solve the following system of equations by using determinant method:

   (a) $\begin{cases} 2x + y + z = 11 \\ 3y - z = -1 \\ 2z = 8 \end{cases}$

   (b) $\begin{cases} 4x + 6y + 4z = 28 \\ 4x + 2y - z = 15 \\ x + y + 3z = 8 \end{cases}$

   (c) $\begin{cases} x + 2y - z = -3 \\ 2x - 4y + z = -7 \\ 2x + 2y - 3z = 4 \end{cases}$

   (d) $\begin{cases} x + y + z = 4 \\ 2x + y - z = -3 \\ -x + 3y + 4z = 19 \end{cases}$

   (e) $\begin{cases} 3x + 2y + z = 5 \\ 2x + y - z = 2 \\ 2x + 2y + 2z = 0 \end{cases}$

3. (a) Find the adjoint of the matrix,

$$\begin{pmatrix} 3 & -2 & -2 \\ 2 & 3 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

   (b) Use the adjoint obtained in (a) to solve the following system of linear equations.

$$\begin{cases} 3x = 1 + 2y + 2z \\ 2x + 3y = z + 13 \\ x + 3z = y - 8 \end{cases}$$

4. If $D = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 5 & 2 \\ 1 & -1 & 1 \end{pmatrix}$ and $E = \begin{pmatrix} -1 & 2 & 0 \\ 1 & 3 & 2 \\ 2 & 0 & 1 \end{pmatrix}$, find $D^{-1}E$.

5. Use the inverse method to solve the following system of equations.

$$\begin{cases} 4a - b + 5c = 8 \\ 5a + 7b - 3c = 42 \\ 3a + 4b + c = 27 \end{cases}$$

6. The following system of linear equations represent the currents flow in a unbalanced three-plane, star-connected electrical network. Use Crammer's rule to determine the values of I_1, I_2, and I_3.

$$\begin{cases} 2I_1 - 5I_2 + 3I_3 = 14 \\ 9I_1 + 3I_2 - 4I_3 = 13 \\ 7I_1 + 3I_2 - 2I_3 = 3 \end{cases}$$

7. The height, h of an object thrown into the air is known to be given by a quadratic function of time (t) of the form, $h = kt^2 + mt + n$. If the object is at $h = \frac{23}{4}$ metres, $t = \frac{1}{2}$ sec; at $h = 7$ metres, $t = 1$ sec; and $h = 2$ metres, $t = 2$ sec. Find the values of k, m, and n by using Cramer's rule.

8. The daily cost of hospital, H is a linear function of the number of in-patients, I and out-patients, P plus a fixed cost q, given by $H = q + rP + sI$. The following data represents the three days cost for in-patients.

Day	**Cost in Tshs, *H***	**Number of in-patients, *I***	**Number of out-patients, *P***
1	55,600	320	80
2	53,800	280	72
3	56,800	320	96

By using inverse matrix method find the values of q, r, and s by setting up a linear system of equations.

9. A manufacturer produces three types of products: simple, medium, and complex. Each product is first processed in machine M_1, then in machine M_2, and lastly sent to machine M_3 for finishing. The following table shows the number of minutes and the total time available.

Machine / Product	M_1	M_2	M_3
Simple	180	90	270
Medium	135	180	90
Complex	90	270	135
Total time	2,700	4,050	3,150

Calculate the number of units of three types of the products produced by constructing a matrix equation and then solve it by the inverse matrix method.

10. In a market survey, three commodities namely groundnuts, beans, and rice were considered. In finding out the index number some fixed weights were assigned to three varieties V_1, V_2, and V_3 in each of the commodities. The following table provides the information regarding the consumption of three commodities according to three varieties and the total weight received by the commodity.

Varieties / Commodities	V_1	V_2	V_3	Total weight
Groundnuts	324	540	648	2,916
Beans	216	432	540	2,268
Rice	108	216	324	1,188

Find the weights assigned to the three varieties by using inverse matrix method, given that the weights assigned to a commodity are equal to the sum of weights of the various varieties multiplied by the corresponding consumption.

Binomial theorem

The binomial expression $(a + b)$ can be raised to different powers. When $(a + b)$ is raised to any power different from 1, the need of having its expanded form arises. These expansions range from simple expansion when such powers are small to complex expansion when the powers are large fraction or negative numbers. The pattern of expansion may give a rule of obtaining the coefficients of terms in an expansion as it was discovered by a French Mathematician Blaise Pascal.

Activity 6.10: Identifying coefficients of a binomial expression

Individually or in a group, perform the following tasks:

1. Write the expansion of $(a+b)^1$, $(a+b)^2$, and $(a+b)^3$
2. Expand $(a+b)^n$ for $n = 4$, $n = 5$, and $n = 6$.
3. Study how every coefficient in the following table is obtained and then fill in the exponent in the expanded form and the binomial coefficients of the last three rows.

Binomial expression	Exponent	Expanded form	Binomial coefficients						
$(a+b)^0$	0	1				1			
$(a+b)^1$	1	$a+b$			1		1		
$(a+b)^2$	2	$a^2+2ab+b^2$		1		2		1	
$(a+b)^3$	3	$a^3+3a^2b+3ab^2+b^3$	1		3		3		1
$(a+b)^4$	4								
$(a+b)^5$	5								
$(a+b)^6$	6								

4. Is it possible to identify the binomial coefficients for $(a+b)^{20}$? Give reasons.
5. What did you observe in tasks 1, 2, and 3?
6. Share your results with other students for more inputs.

The following can be deduced from Activity 6.10.

1. The exponent of a is higher on the first term from left and decreases by 1 as you move to the next terms.
2. The exponent of b is zero on the first term from left and increases by 1 as you move to the next terms.
3. The sum of the exponents in each term is n, where n is the value of the power.
4. The coefficients are symmetric since they increase from the beginning to the line of symmetry and then decrease to the end.
5. If the coefficients are detached from the expansions, then a triangle is formed. The resulting triangle is called Pascal's triangle.

Figure 6.1 shows a Pascal's triangle for the binomial expression $(a+b)^n$, where $n = 0, 1, 2, 3,$ and 4.

$(a+b)^0$:					1				
$(a+b)^1$:				1		1			
$(a+b)^2$:			1		2		1		
$(a+b)^3$:		1		3		3		1	
$(a+b)^4$:	1		4		6		4		1

Figure 6.1: *Pascal's triangle*

Two things can be observed from Pascal's triangle as seen in Figure 6.1:

(i) The rows start with 1 and end with 1

(ii) Each coefficient, except the first and the last, is the sum of 2 coefficients to the left and to the right of the row directly above it.

Example 6.62

Expand $(2x+3)^3$ in descending powers of x.

Solution

Given $(2x+3)^3$, where $a = 2x$, $b = 3$, and $n = 3$.

Applying the Pascal's triangle to expand:

$$\Rightarrow (2x+3)^3 = 1(2x)^3(3)^0 + 3(2x)^2(3)^1 + 3(2x)^1(3)^2 + 1(2x)^0(3)^3$$

$$= 8x^3 + 36x^2 + 54x + 27.$$

Therefore, $(2x+3)^3 = 8x^3 + 36x^2 + 54x + 27.$

Example 6.63

Simplify $\left(2+\sqrt{3}\right)^4 - \left(2-\sqrt{3}\right)^4$ leaving your answer in surd form.

Solution

Apply the pascal's triangle to expand:

$$\Rightarrow\left(2+\sqrt{3}\right)^4=1(2)^4\left(\sqrt{3}\right)^0+4(2)^3\left(\sqrt{3}\right)^1+6(2)^2\left(\sqrt{3}\right)^2+4(2)^1\left(\sqrt{3}\right)^3+1(2)^0\left(\sqrt{3}\right)^4$$

$$=16+32\sqrt{3}+72+24\sqrt{3}+9$$

$$=97+56\sqrt{3}$$

and $\left(2-\sqrt{3}\right)^4=1(2)^4\left(-\sqrt{3}\right)^0+4(2)^3\left(-\sqrt{3}\right)^1+$

$$6(2)^2\left(\sqrt{3}\right)^2+4(2)^1\left(-\sqrt{3}\right)^3+1(2)^0\left(-\sqrt{3}\right)^4$$

$$=16-32\sqrt{3}+72-24\sqrt{3}+9$$

$$=97-56\sqrt{3}$$

$$\Rightarrow\left(2+\sqrt{3}\right)^4-\left(2-\sqrt{3}\right)^4=\left(97+56\sqrt{3}\right)-\left(97-56\sqrt{3}\right)$$

$$=112\sqrt{3}$$

Therefore, $\left(2+\sqrt{3}\right)^4-\left(2-\sqrt{3}\right)^4=112\sqrt{3}$.

Example 6.64

Find the expansion of $\left(1+\frac{1}{4}x\right)^4$. If $x=0.1$, evaluate the value of $(1.025)^4$, correctly to four decimal places.

Solution

Let

$$\left(1+\frac{1}{4}x\right)^4=1(1)^4\left(\frac{1}{4}x\right)^0+4(1)^3\left(\frac{1}{4}x\right)^1+6(1)^2\left(\frac{1}{4}x\right)^2+4(1)^1\left(\frac{1}{4}x\right)^3+1(1)^0\left(\frac{1}{4}x\right)^4$$

$$=1+x+\frac{3}{8}x^2+\frac{1}{16}x^3+\frac{1}{256}x^4$$

Thus, $\left(1+\frac{1}{4}x\right)^4=1+x+\frac{3}{8}x^2+\frac{1}{16}x^3+\frac{1}{256}x^4$.

Substituting $x=0.1$ in the equation gives;

$$\left(1+\frac{1}{4}(0.1)\right)^4 = 1+0.1+\frac{3}{8}(0.1)^2+\frac{1}{16}(0.1)^3+\frac{1}{256}(0.1)^4$$

$$\Rightarrow (1.025)^4 = 1.103812891$$

Therefore, $(1.025)^4 \approx 1.1038$.

Binomial expansion

The Pascal's triangle becomes tedious when the value of the power becomes large. The easy way to expand the expression of higher power is to apply the binomial theorem. Actually, binomial theorem is obtained after studying the expansions of the individual expressions. It is the general expansion that can be used to obtain all expansions. The binomial theorem states that,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \ldots + na\,b^{n-1} + b^n, \qquad (6.4)$$

where n is a positive integer.

Sometimes, the binomial theorem is presented with factorials. To understand how factorials are involved in the expression, consider the expansion of $(x + 3)^4$ by using the binomial theorem (6.4) as shown in the following example.

Example 6.65

Expand $(x+3)^4$ in descending powers of x.

Solution

Let $(x + 3)^4 = \dfrac{x^4\times 3^0}{1} + \dfrac{4}{1}x^3\times 3^1 + \dfrac{4\times 3}{1\times 2}x^2\times 3^2 + \dfrac{4\times 3\times 2}{1\times 2\times 3}x^1\times 3^3 + \dfrac{4\times 3\times 2\times 1}{1\times 2\times 3\times 4}x^0\times 3^4$

$$= x^4 + 12x^3 + 54x^2 + 108x + 81$$

Therefore, $(x+3)^4 = x^4 + 12x^3 + 54x^2 + 108x + 81$.

From Example 6.65, it can be noted that, the denominators of the terms are in a particular order. The denominators of the 5th, 4th, and 3rd terms are the factorials of 4, 3, and 2, respectively. This is due to the definition of the factorial which states that; if n is any positive integer, then $n! = n(n-1)(n-2)(n-3)\cdots(3)(2)(1)$, where $n!$ is the factorial of the positive integer n.

The 1st and 2nd terms have denominators equal to 1. By following the trend from the 3rd term to the 5th term, it is possible to suggest the denominators of the 1st and 2nd terms in factorial. Since it has been observed that from the 5th term towards the 3rd term, there is a decrease of the denominator by 1. Hence, the denominators of the 2nd and 1st terms are 1! and 0!, respectively.

To incorporate the factorial, the binomial theorem can now be modified to,

$$(a+b)^n = \frac{n!}{0!(n-0)!}a^n + \frac{n!}{1!(n-1)!}a^{n-1}b + \frac{n!}{2!(n-2)!}a^{n-2}b^2 + \cdots + \frac{n!}{n!(n-n)!}b^n.$$

The expression can further be simplified to;

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \cdots + \binom{n}{n}b^n,$$

where $\binom{n}{r} = \frac{n!}{r!(n-r)} = {}^nC_r$.

Generally,

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots + b^n$$

Example 6.66

Expand $(x+1)^3$ by using binomial theorem.

Solution

From binomial theorem:

$$(x+1)^3 = \frac{3!}{0!3!}x^3 + \frac{3!}{2!}x^2(1) + \frac{3!}{2!}x^1(1)^2 + \frac{3!}{3!(0)!}(1)^3$$

$$= x^3 + 3x^2 + \frac{3\times 2!}{2!}x + (1)^3$$

$$= x^3 + 3x^2 + 3x + 1$$

Therefore, $(x+1)^3 = x^3 + 3x^2 + 3x + 1$.

Example 6.67

Obtain the expansion of $\left(1+x-3x^2\right)^{10}$, as far as the term in x^3.

Solution

The expression $\left(1+x-3x^2\right)^{10}$ can be written as $\left[1+\left(x-3x^2\right)\right]^{10}$, which may be expanded by using the binomial theorem as follows:

$$\left[1+\left(x-3x^2\right)\right]^{10}=1+10\left(x-3x^2\right)+\frac{10\times 9}{2!}\left(x-3x^2\right)^2+\frac{10\times 9\times 8}{3!}\left(x-3x^2\right)^3+\cdots$$

$$=1+10x-30x^2+45x^2-270x^3+120x^3-\cdots$$

$$=1+10x+15x^2-150x^3+\cdots$$

Therefore, $\left(1+x-3x^2\right)^{10}=1+10x+15x^2-150x^3+\cdots$

Example 6.68

If x^3 and higher powers can be neglected, show that

$$(1-x)^5\left(2+\frac{1}{2}x\right)^{10}\approx 5(205+511x+578x+\cdots)$$

Solution

Consider the binomial expansion of $(1-x)^5$ and $\left(2+\frac{1}{2}x\right)^{10}$.

$$\Rightarrow (1-x)^5=1+5(-x)^1+\frac{5\times 4}{2\times 1}(-x)^2+\cdots$$

$$=1-5x+10x^2+\cdots$$

Also, $\left(2+\frac{1}{2}x\right)^{10}=2^{10}+10(2)^9\left(\frac{1}{2}x\right)+\frac{10(9)(2)^8\left(\frac{1}{2}x\right)^2}{1\times 2}+\cdots$

$$=2^{10}+5(2)^9x+90(2)^5x^2+\cdots$$

$$\Rightarrow (1-x)^5\left(2+\frac{1}{2}x\right)^{10}=1-5x+10x^2+2^{10}+5(2)^9x+90(2)^5x^2+\cdots$$

$$=1025+2555x+2890x^2+\cdots$$

$$= 5\left(205 + 511x + 578x^2 + \cdots\right)$$

Therefore, $(1-x)^5\left(2+\frac{1}{2}x\right)^{10} \approx 5\left(205 + 511x + 578x^2 + \cdots\right)$.

Exercise 6.11

1. Use Pascal's triangle to expand each of the following:

   (a) $(2x+3z)^5$ (b) $(a-b)^7$ (c) $\left(2x+\frac{2}{x}\right)^5$

2. Evaluate each of the following, leave the answer in surd form where appropriate:

   (a) $\left(\sqrt{2}+1\right)^6-\left(\sqrt{2}-1\right)^6$ (c) $\left(\sqrt{6}+\sqrt{2}\right)^3-\left(\sqrt{6}-\sqrt{2}\right)^3$

   (b) $\left(2+\sqrt{5}\right)^7+\left(2-\sqrt{5}\right)^7$ (d) $\left(\sqrt{2}+\sqrt{3}\right)^4+\left(\sqrt{2}-\sqrt{3}\right)^4$

3. Expand $\left(1+x+x^2\right)^{10}$ as a series in ascending powers of x up to and including the term in x^3. Hence, evaluate $(1.0101)^{10}$ correct to three decimal places.

4. Use binomial theorem to evaluate each of the following correct to five significant figures:

   (a) $(1.009)^8$ (b) $(2.045)^{10}$ (c) $(2.098)^{12}$

5. The expansion of $(1+y)^{10}\left(1+my+ny^2\right)$ in ascending powers of y begins with the terms $1+y+y^2$. Find the numerical values of m and n.

6. In the expansion of $\left(1-2a+ka^2\right)^4$ as a series of powers of a, the coefficient of a^3 is zero. Verify that;

   (a) $k=-1\frac{1}{3}$ (b) The coefficient of a^4 is $-37\frac{1}{3}$

7. Find the expansion of $(2+y)^5$ in ascending powers of y. Taking the first three terms of the expansion, put $y=0.001$ and find the value of $(2.001)^5$ correct to five decimal places.

8. Find the first four terms of the expansion of $\left(2+\frac{1}{4}y\right)^{10}$ in ascending powers of y. Hence find the value of $(2.025)^{10}$ correctly to four significant figures.

9. Obtain the expansion of $(c-2)^2(1-c)^6$ in ascending powers of c as far as the term in c^4.

10. Find the expansion of $\left(3a-\frac{1}{2}b\right)^4$ by using the binomial theorem. Hence, obtain the value of $(29.5)^4$ correct to six significance figures.

Binomial expansion for fractional and negative indices

The binomial expansion is applicable for a case when n is a positive integer. For the case when n is negative or fraction, the modified binomial theorem becomes:

$$(1+x)^n = \frac{1}{0!}+\frac{n}{1!}x+\frac{n(n-1)}{2!}x^2+\frac{n(n-1)(n-2)}{3!}x^3+\frac{n(n-1)(n-2)(n-3)}{4!}x^4+\cdots \quad (6.5)$$

The modified binomial theorem is used in the case when n is negative or a fraction. The series of equation (6.5) does not terminate and it is only convergent with the limit of its sum as $(1+x)^n$ when x is between -1 and 1. This interval is also known as limits of x for which the expansion is valid.

Note that, when n is a positive integer, the series of equation (6.6) terminates at the term in x^n and its sum is $(1+x)^n$ for all values of x.

Example 6.69

Find the first four terms in the expansion of each of the following binomial expressions. State the range of values of x for which the expansion is valid.

(a) $(1+2x)^{-1}$

(b) $\sqrt[3]{2+x}$

Solution

(a) $(1+2x)^{-1} = 1+(-1)(2x)+\frac{(-1)(-2)}{2!}(2x)^2+\frac{(-1)(-2)(-3)}{3!}(2x)^3+\cdots$

$$= 1-2x+\frac{8}{2\times1}x^2-\frac{(6)8}{3\times2\times1}x^3+\cdots$$

$$= 1-2x+4x^2-8x^3+\cdots$$

Therefore, $(1+2x)^{-1} = 1-2x+4x^2-8x^3+\cdots$

This expansion converges in the interval,

$|2x|<1 \Rightarrow -1<2x<1$

$$\Rightarrow -\frac{1}{2}<x<\frac{1}{2} \text{ or } |x|<\frac{1}{2}$$

(b) $\sqrt[3]{2+x} = 2^{\frac{1}{3}}\left[1+\frac{x}{2}\right]^{\frac{1}{3}}$

$$= \sqrt[3]{2}\left[1+\frac{1}{3}\left(\frac{x}{2}\right)+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{x}{2}\right)^2}{2!}+\frac{\left(\frac{1}{3}\right)\left(\frac{-2}{3}\right)\left(\frac{-5}{3}\right)\left(\frac{x}{2}\right)^3}{3!}+\cdots\right]$$

$$= \sqrt[3]{2}\left[1+\frac{x}{6}-\frac{1}{36}x^2+\frac{5}{648}x^3+\cdots\right]$$

Therefore, $\sqrt[3]{2+x} = \sqrt[3]{2}\left(1+\frac{x}{6}-\frac{1}{36}x^2+\frac{5}{648}x^3+\cdots\right)$.

This expansion converges in the range,

$\left|\frac{x}{2}\right|<1 \Rightarrow -1<\frac{x}{2}<1$

$$\Rightarrow -2<x<2 \text{ or } |x|<2.$$

Example 6.70

If x is small in comparison with unity such that x^3 and higher powers can be neglected, show that $\frac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}} \approx 1-\frac{3}{2}x-\frac{33}{8}x^2$.

Solution

Given $\frac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}} = (1-4x)^{\frac{1}{2}}(1+3x)^{\frac{1}{3}}(1+x)^{-\frac{1}{2}}$

$$=\left[1+\frac{1}{2}(-4x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-4x)^2}{2!}+\cdots\right]\times\left[1+\frac{1}{3}(3x)+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)(3x)^2}{2!}+\cdots\right]\times$$

$$\left[1-\frac{1}{2}(x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^2}{2!}+\cdots\right]$$

$$=\left(1-2x-2x^2+\cdots\right)\left(1+x-x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$=\left(1-x-5x^2+\cdots\right)\left(1-\frac{x}{2}+\frac{3}{8}x^2+\cdots\right)$$

$$=1-\frac{3}{2}x-\frac{33}{8}x^2-\cdots$$

Therefore, $\dfrac{\sqrt{1-4x}\sqrt[3]{1+3x}}{\sqrt{1+x}}\approx 1-\dfrac{3}{2}x-\dfrac{33}{8}x^2.$

Example 6.71

Expand $\sqrt{1+x}$ in ascending powers of x as far as the term in x^2. Hence, estimate $\sqrt{30}$ correct to four significant figures.

Solution

Given $\sqrt{1+x} = (1+x)^{\frac{1}{2}}$.

From $(1+x)^n = 1+nx+n\left(\dfrac{n-1}{2!}\right)x^2+\dfrac{n(n-1)(n-2)}{3!}x^3+\cdots$

$$\Rightarrow (1+x)^{\frac{1}{2}} = 1+\frac{1}{2}x+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2+\cdots$$

$$=1+\frac{1}{2}x-\frac{1}{8}x^2+\cdots$$

Therefore, $\sqrt{1+x}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2.$

$$\text{Let } \sqrt{30} = \sqrt{25+5}$$
$$= \sqrt{25\left(1+\frac{1}{5}\right)}$$
$$= \sqrt{25}\ \sqrt{\left(1+\frac{1}{5}\right)}$$
$$= 5\left(1+\frac{1}{5}\right)^{\frac{1}{2}}$$

Let $x = \frac{1}{5}$, then $\left(1+\frac{1}{5}\right)^{\frac{1}{2}} = (1+x)^{\frac{1}{2}}$.

$$\Rightarrow \sqrt{30} = 5\left[1+\frac{1}{2}\left(\frac{1}{5}\right)-\frac{1}{8}\left(\frac{1}{25}\right)+\cdots\right]$$
$$= 5\left[1+(0.5)(0.2)-(0.125)(0.04)+\cdots\right]$$
$$\approx 5[1.095]$$
$$\approx 5.475$$

Therefore, the value of $\sqrt{30} \approx 5.475$.

Example 6.72

If x is small such that its cube and higher powers can be neglected, show that $\sqrt{\frac{1-x}{1+x}} \approx 1-x+\frac{1}{2}x^2$. Also, if $x = \frac{1}{8}$, show that $\sqrt{7} \approx 2\frac{83}{128}$.

Solution

Let $\sqrt{\frac{1-x}{1+x}} = (1-x)^{\frac{1}{2}}(1+x)^{-\frac{1}{2}}$

$$= \left[1+\tfrac{1}{2}(-x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)(-x)^2}{2!}+\cdots\right]\left[1+\left(-\frac{1}{2}\right)(x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x)^2}{2!}+\cdots\right]$$

$$=\left(1-\frac{1}{2}x-\frac{1}{8}x^2+\cdots\right)\left(1-\frac{1}{2}x+\frac{3}{8}x^2+\cdots\right)$$

$$=1-\frac{1}{2}x+\frac{3}{8}x^2-\frac{1}{2}x+\frac{1}{4}x^2-\frac{1}{8}x^2+\cdots$$

$$=1-x+\frac{1}{2}x^2+\cdots$$

Hence, $\sqrt{\frac{1-x}{1+x}}\approx 1-x+\frac{1}{2}x^2$

Given $x=\frac{1}{8}$, then $\sqrt{\frac{1-\frac{1}{8}}{1+\frac{1}{8}}}\approx 1-\frac{1}{8}+\frac{1}{2}\left(\frac{1}{8}\right)^2$.

$$\Rightarrow \sqrt{\frac{7}{9}}\approx 1-\frac{1}{8}+\frac{1}{128}$$

$$\Rightarrow \frac{\sqrt{7}}{3}\approx\frac{113}{128}$$

$$\Rightarrow \sqrt{7}\approx\frac{113}{128}\times 3\approx\frac{339}{128}$$

Therefore, $\sqrt{7}\approx 2\frac{83}{128}$.

Exercise 6.12

1. Use binomial theorem to show that $\frac{1}{(2+3y)^4}=\frac{1}{16}\left(1-6y+\frac{45}{2}y^2-\frac{135}{2}y^3+\cdots\right)$

2. If x is very small compared to unity, show that $\frac{\left(\sqrt{1+x}\right)\sqrt[3]{(1-x)^2}}{(1+x)\sqrt{(1+x)}}\approx 1-\frac{5}{3}x+\cdots$

3. Express $\frac{\sqrt{1+2x}}{\sqrt[3]{1-3x}}$ as a power series, as far as the term involves x^2. State the range of values of x for which the series is convergent.

4. Using binomial theorem, show that $\frac{1}{(2+3y)^6}=\frac{1}{64}\left(1-9y+\frac{189}{4}y^2-\cdots\right)$. For what values of y is the expansion valid?

5. Use the binomial theorem to expand $\frac{(1+a)\sqrt[3]{(1-3a)^2}}{\sqrt{1+a^2}}$ in ascending powers of a as far as the term in a^2. State the interval of a for which the series is valid.

6. Obtain the first four terms of the expansion of $(1-16y)^{\frac{1}{4}}$. By substituting $y=\frac{1}{10,000}$, evaluate $\sqrt[4]{39}$ correct to three decimal places.

7. When t is small, use binomial theorem to show that $\frac{1}{(1+t)^2\sqrt{1-t}}=1-\frac{3}{2}t+\cdots$

8. Expand $\sqrt[3]{1+t}$ up to the term in t^2. Hence, by putting $t=\frac{1}{8}$, calculate $\sqrt[3]{9}$ correct to three decimal places.

9. If x is small such that x^3 and higher powers of x are neglected, show that $(2x+3)(1-2x)^{10}\approx 3-58x+500x^2$.

10. Use binomial theorem to expand $\frac{1+2n}{(n-1)(n+2)^2}$ so that n^3 and higher powers may be neglected. State the interval of values of n for which the expansion is valid.

The general term in the binomial expansion

Consider the binomial expansion (6.5), that is

$$(a+b)^n=a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\cdots+b^n.$$

This equation can be written as:

$$(a+b)^n={}^nC_0a^nb^0+{}^nC_1a^{n-1}b+{}^nC_2a^{n-2}b^2+\cdots+{}^nC_ra^{n-r}b^r+\cdots+{}^nC_na^0b^n \qquad (6.6)$$

From the expansion (6.6), it can be observed that:

1st term $={}^nC_0a^nb^0$, the 2nd term $={}^nC_1a^{n-1}b$, the 3rd term $={}^nC_2a^{n-2}b^2$, the 4th term $={}^nC_3a^{n-3}b^3$.

Hence, the $(r+1)^{\text{th}}$ term, is given by ${}^{n}C_{r}a^{n-r}b^{r}$.

Therefore, if T_{r+1} denotes the $(r+1)^{\text{th}}$ term, then $T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$.

Note that, the formulae for the general term of binomial expansion are:

For $(a+b)^n \Rightarrow T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$

For $(a-b)^n \Rightarrow T_{r+1} = (-1)^r\ {}^{n}C_{r}a^{n-r}b^{r}$

For $(1+x)^n \Rightarrow T_{r+1} = {}^{n}C_{r}x^{r}$

For $(1-x)^n \Rightarrow T_{r+1} = (-1)^r\ {}^{n}C_{r}x^{r}$.

Example 6.73

Find the fifth term in the expansion of $\left(\frac{2}{x^2}-\frac{x^3}{4}\right)^8$.

Solution

From $\left(\frac{2}{x^2}-\frac{x^3}{4}\right)^8$, $n=8$, $a=\frac{2}{x^2}$, $b=-\frac{x^3}{4}$ and $r=5$

$\Rightarrow T_{r+1} = {}^{n}C_{r}a^{n-r}b^{r}$

$$\Rightarrow T_6 = {}^{8}C_5\left(\frac{2}{x^2}\right)^{8-5}\left(-\frac{x^3}{4}\right)^5 = {}^{8}C_5\left(\frac{2}{x^2}\right)^3\left(-\frac{x^3}{4}\right)^5$$

$$= -\frac{7}{16}x^9$$

Therefore, the fifth term is $-\frac{7}{16}x^9$.

Example 6.74

Find the term independent of x in each of the following:

(a) $\left(x^2-\frac{1}{x^2}\right)^{14}$ (b) $\left(\frac{5}{2}x^2-\frac{1}{5x}\right)^6$

Solution

(a) Let $(r+1)^{\text{th}}$ term be independent of x in the expansion.

Now, $T_{r+1} = {}^{14}C_r\left(x^2\right)^{14-r}\left(-\frac{1}{x^2}\right)^r = {}^{14}C_r x^{28-4r}(-1)^r$

For this term to be independent of x, set $28-4r=0 \Rightarrow r=7$

So, $(7+1)^{\text{th}} = 8^{\text{th}}$ term is independent of x.

Putting $T_{7+1} = T_8 = {}^{14}C_7 x^{28-28}(-1)^7 = (-1)\,{}^{14}C_7$

$= -3432$

Therefore, the term independent of x is -3432.

(b) Let $(r+1)^{\text{th}}$ term be independent of x in the given expression.

Now, $T_{r+1} = {}^{6}C_r\left(\frac{5}{2}x^2\right)^{6-r}\left(-\frac{1}{5x}\right)^r = {}^{6}C_r\left(\frac{5}{2}\right)^{6-r}\left(x^2\right)^{6-r}(-1)^r\frac{1}{5^r}x^{-r}$

$= {}^{6}C_r\left(\frac{5}{2}\right)^{6-r}(x)^{12-2r}(-1)^r\frac{1}{5^r}x^{-r}$

$= {}^{6}C_r\left(\frac{5}{2}\right)^{6-r}(x)^{12-3r}(-1)^r\frac{1}{5^r}$

For this term to be independent of x, set $12 - 3r = 0$. That is,
$12-3r=0 \Rightarrow r=4$

$\Rightarrow T_5 = {}^{6}C_4\left(\frac{5}{2}\right)^2 x^0(-1)^4\frac{1}{5^4} = \frac{375}{4\times 625} = \frac{3}{20}$.

Therefore, the term independent of x is $\frac{3}{20}$.

Example 6.75

Find the coefficient of the term involving x^{10} in the expansion of $\left(x^6+\frac{1}{x^4}\right)^{15}$.

Solution

$T_{r+1} = {}^{15}C_r\left(x^6\right)^{15-r}\left(\frac{1}{x^4}\right)^r = {}^{15}C_r x^{90-10r}$

Find the value of r where the term containing x^{10} occurs.

$\Rightarrow x^{90-10r} = x^{10}$

Thus, $90 - 10r = 10 \Rightarrow r = 8$

$\Rightarrow T_9 = {}^{15}C_8 x^{90-80} = 6,435x^{10}$

Therefore, the coefficient of the term involving x^{10} is $6,435$.

Middle terms in a binomial expansion

The binomial expansion of $(a+b)^n$ has $(n+1)$ terms. Therefore, if n is even, then the middle term is the $\left(\frac{n}{2}+1\right)^{\text{th}}$ term, and if n is odd, the middle terms are $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$.

Example 6.76

Find the middle term in the expansion of $\left(4+\frac{1}{3}x\right)^{10}$.

Solution

Here, $n = 10$, which is an even number. So, $\left(\frac{10}{2}+1\right)^{\text{th}}$ term = 6^{th} term is the middle term.

Hence, $T_6 = T_{5+1} = {}^{10}C_5 (4)^5 \left(\frac{x}{3}\right)^5 = \frac{258,048}{243}x^5$

Therefore, the middle term is $\frac{258,048}{243}x^5$.

Example 6.77

Find the middle term in the expansion of $\left(2-\frac{x^4}{2}\right)^5$.

Solution

From $\left(2-\frac{x^4}{2}\right)^5$, $n = 5$, which is an odd number.

So, $\left(\frac{5+1}{2}\right)^{\text{th}}$ and $\left(\frac{5+3}{2}\right)^{\text{th}}$, that is, 3^{rd} and 4^{th} terms are middle terms.

Now, $T_3 = T_{2+1} = {}^5C_2(2)^2\left(-\dfrac{x^4}{2}\right)^3$

$$= \frac{5\times4\times3!}{2!3!}\times2^2\times\left(-\frac{x^{12}}{8}\right)$$

$$= -5x^{12}$$

$$T_4 = T_{3+1} = {}^5C_3(2)^3\left(-\frac{x^4}{2}\right)^2$$

$$= \frac{5\times4\times3!}{3!2!}\times8\times\left(\frac{x^8}{4}\right)$$

$$= 20x^8$$

Therefore, the middle terms are $-5x^{12}$ and $20x^8$.

Exercise 6.13

1. Show that, there is no term independent of x in the expansion of $\left(2x^2-\dfrac{1}{4x}\right)^{11}$, $x\neq0$.
2. If the coefficient of y^7 and y^8 in the expansion of $\left(3+\dfrac{1}{2}y\right)^n$ are equal, find the value of n.
3. Find the coefficient of z^5 in the expansion of $(1-z)^3(1+3z)^4$.
4. In the expansion of $(1+t)^{20}$, the coefficient of the r^{th} term and that of the $(r+1)^{\text{th}}$ term are in the ratio 1:2. Find the value of r.
5. Find the general term in the binomial expansion of each of the following:

   (a) $\left(\dfrac{4y}{5}-\dfrac{5}{2y}\right)^9$, $y\neq0$

   (b) $\left(2y+\dfrac{1}{y}\right)^5$, $y\neq0$

   (c) $\left(y^2-\dfrac{1}{y}\right)^{12}$, $y\neq0$

   (d) $(t^2-s)^6$
6. Show that the 13^{th} term in the expansion of $\left(9y-\dfrac{y^{-\frac{1}{2}}}{3}\right)^{18}$, $y\neq0$ is 18,564.
7. Verify that the middle terms in the expansion of $\left(3-\dfrac{y^3}{6}\right)^7$ are $-\dfrac{105}{8}y^9$ and $\dfrac{35}{48}y^{12}$.

8. Find the values of p, q, and n in the expansion of $(p+q)^n$ if the first three terms of the expansion are 729, 7290, and 30375, respectively.
9. Find the term independent of y in the expansion of each of the following:

(a) $\left(\frac{3y^2}{2}-\frac{1}{3y}\right)^{15}$

(b) $\left(\sqrt{y}-\frac{3}{y^2}\right)^{10}$

(c) $\left(1+y+2y^3\right)\left(\frac{3}{2}y^2-\frac{1}{3y}\right)^9$

(d) $\left(y-\frac{1}{y}\right)^{14}$

10. Find the coefficient of the terms indicated in the expansions of the following:

(a) $\left(2m+\frac{1}{m}\right)^7$, term in m^{-5}

(b) $\left(p-\frac{2}{p}\right)^8$, term in p^6

Partial fractions

In arithmetic, it is possible to decompose a fraction like $\frac{3}{4}$ into a sum of smaller fractions like $\frac{3}{4}=\frac{1}{2}+\frac{1}{4}$. Similarly, a rational expression can be decomposed into simpler parts known as partial fractions. For example, $\frac{2x+1}{(x-1)(x+2)}$ can be decomposed as $\frac{2x+1}{(x-1)(x+2)}=\frac{1}{x-1}+\frac{1}{x+2}$. The fractions $\frac{1}{x-1}$ and $\frac{1}{x+2}$ are called partial fractions of $\frac{2x+1}{(x-1)(x+2)}$. The process of taking a rational expression and decomposing it into simpler rational expressions that can be added or subtracted to get the original rational expression is called partial fraction decomposition.

Decomposition of fraction

The process of splitting fractions into its partial fractions is done by the method of undetermined coefficients. The decomposition techniques differ depending on the nature of the expression on the denominator.

Decomposition of fractions whose denominators consist of non-repeated linear factors

If the denominator of a fraction is the product of linear factors or it is an expression which can be split into linear factors, then the fraction is split into terms of the form, $\frac{A}{a_1x+b_1}+\frac{B}{a_2x+b_2}+\frac{C}{a_3x+b_3}+\cdots$

Observe that, the number of partial fractions to be determined is normally shown by the factors in the denominator. The denominators of the partial fractions are the factors. However, their corresponding numerators are not known. In this case, the assumption is that, one constant for each denominator will be the numerator, then the values of constants can be computed.

Example 6.78

Find the partial fractions of $\frac{6x}{(x-1)(x+2)}$.

Solution

There will be two partial fractions, and thus choose constants A and B to the numerators. The expression is rewritten as:

$$\frac{6x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

Writing under common denominators gives;

$$\frac{6x}{(x-1)(x+2)} = \frac{A(x+2) + B(x-1)}{(x-1)(x+2)}$$

Since the denominators of left-hand side and right-hand side are equal, then the numerators are also equal. That is,

$6x = A(x+2) + B(x-1)$.

To solve for the values of A or B, substitute any value in x.

However, for simplicity choose x in such a way that either A or B cancels wherever possible.

Setting $x=1 \Rightarrow 6\times 1 = A(1+2) + B(1-1)$

$\Rightarrow 6 = 3A + 0$

$\Rightarrow A=2$

Setting $x=-2$,

$\Rightarrow 6\times -2 = A(-2+2) + B(-2-1)$

$\Rightarrow -12 = 0 - 3B$

$\Rightarrow B = 4$

Thus, $\frac{6x}{(x-1)(x+2)} = \frac{2}{(x-1)} + \frac{4}{(x+2)}$

Therefore,

$$\frac{6x}{(x-1)(x+2)} = \frac{2}{x-1} + \frac{4}{x+2}.$$

Decomposition of fractions whose denominators consist of non–repeated irreducible quadratic factors

If the denominator is a quadratic term which cannot be factorized, then the fraction is split into $\frac{Ax+B}{ax^2+bx+c}$.

However, the method is useful when the denominator is a product of either two quadratic terms which cannot be split further or it is a product of linear factors and a quadratic term which cannot be split into linear factors.

Example 6. 79

Decompose $\dfrac{x+2}{(x-1)(x^2+1)}$ into its partial fractions.

Solution

Let $\dfrac{x+2}{(x-1)(x^2+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+1}$

Writing under common denominators gives,

$$\frac{x+2}{(x-1)(x^2+1)} = \frac{A(x^2+1)+(x-1)(Bx+C)}{(x-1)(x^2+1)}$$

Equating numerators gives,

$x+2 = A(x^2+1)+(x-1)(Bx+C)$

$\Rightarrow x+2 = (A+B)x^2 + (-B+C)x + A - C$

Equating the coefficients of x^2 terms gives,

$0 = A + B$ ....(i)

Equating the coefficients of x terms gives,

$1 = -B+C$ ....(ii)

Equating the constant terms gives,

$2 = A - C$ ....(iii)

Subtracting equation (i) from equation (iii) gives,

$2 = -B - C$ ....(iv)

Adding equations (ii) and (iv) gives, $3 = -2B \Rightarrow B = -\dfrac{3}{2}$

Subtracting equation (ii) from equation (iv) gives, $1 = -2C \Rightarrow C = -\dfrac{1}{2}$.

Substituting $C = -\dfrac{1}{2}$ in equation (iii) gives, $2 = A + \dfrac{1}{2} \Rightarrow A = \dfrac{3}{2}$.

$$\Rightarrow \frac{x+2}{(x-1)(x^2+1)} = \frac{\frac{3}{2}}{x-1} + \frac{-\frac{3}{2}x - \frac{1}{2}}{x^2+1}$$

Therefore, $\dfrac{x+2}{(x-1)(x^2+1)} = \dfrac{3}{2(x-1)} - \dfrac{3x+1}{2(x^2+1)}$.

Decomposition of fractions whose denominator consist of a repeated linear or quadratic factors

If the denominator is of the form $(cx + a)^n$, the number of fractions will be equal to n in the following order: $\frac{B_1}{cx+a}, \frac{B_2}{(cx+a)^2}, \frac{B_3}{(cx+a)^3}, \ldots, \frac{B_n}{(cx+a)^n}$

If the denominator is of the form $\left(ax^2+bx+c\right)^n$ the number of fractions will be in the following order:

$$\frac{B_1}{ax^2+bx+c}, \frac{B_2}{\left(ax^2+bx+c\right)^2}, \frac{B_3}{\left(ax^2+bx+c\right)^3}, \ldots, \frac{B_n}{\left(ax^2+bx+c\right)^n}$$

where $B_1, B_2, B_3, \ldots, B_n$ are constants.

Example 6.80

Decompose $\frac{2x+8}{(x-1)^3}$ into partial fractions.

Solution

Let $\frac{2x+8}{(x-1)^3} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3}$

Writing under common denominators gives,

$$\frac{2x+8}{(x-1)^3} = \frac{A(x-1)^2 + B(x-1) + C}{(x-1)^3}$$

Equating numerators gives:

$2x + 8 = A(x-1)^2 + B(x-1) + C$

Setting $x = 1$, $10 = 0 + 0 + C \Rightarrow C = 10$

Setting $x = 2$, $12 = A + B + 10 \Rightarrow A + B = 2$ ....(i)

Setting $x = 0$, $8 = A - B + 10 \Rightarrow A - B = -2$ ....(ii)

Solving equations (i) and (ii) simultaneous gives:

$A = 0$ and $B = 2$

Therefore, $\frac{2x+8}{(x-1)^3} = \frac{2}{(x-1)^2} + \frac{10}{(x-1)^3}$.

Decomposition of fractions when the degree of the numerator is equal or higher than the degree of the denominator

When the degree of the numerator is equal or greater than the degree of the denominator, divide the numerator by the denominator and then proceed depending on the type of fraction to be decomposed.

Example 6.81

Decompose $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}$ into partial fractions.

Solution

Since the numerator has degree greater than the denominator, divide the numerator by the denominator:

$$\begin{array}{r l} & 2x \\ x^3-2x^2+3x-6 & \overline{)\,2x^4-4x^3-42} \\ & \underline{-(2x^4-4x^3+6x^2-12x)} \\ & \qquad -6x^2+12x-42 \end{array}$$

Thus, $\dfrac{2x^4-4x^3-42}{(x-2)(x^2+3)}=2x+\dfrac{-6x^2+12x-42}{(x-2)(x^2+3)}$

$$\Rightarrow \frac{2x^4-4x^3-42}{(x-2)(x^2+3)}=2x+\frac{\mathrm{A}}{x-2}+\frac{\mathrm{B}x+\mathrm{C}}{x^2+3}$$

$$\Rightarrow \frac{-6x^2+12x-42}{(x-2)(x^2+3)}=\frac{\mathrm{A}}{x-2}+\frac{\mathrm{B}x+\mathrm{C}}{x^2+3}$$

Writing under common denominators gives,

$$\frac{-6x^2+12x-42}{(x-2)(x^2+3)}=\frac{\mathrm{A}(x^2+3)+(x-2)(\mathrm{B}x+\mathrm{C})}{(x-2)(x^2+3)}$$

Equating numerators gives,

$$-6x^2+12x-42=\mathrm{A}(x^2+3)+(x-2)(\mathrm{B}x+\mathrm{C})$$

Setting $x=2 \Rightarrow -42=7\mathrm{A} \Leftrightarrow \mathrm{A}=-6$

Setting $x=0 \Rightarrow -42=3\mathrm{A}-2\mathrm{C}$ ....(i)

Substituting $\mathrm{A}=-6$ in equation (i) gives $\mathrm{C}=12$.

Setting $x=1 \Rightarrow -36 = 4A - B - C$ ....(ii)

Substitute $A = -6$ and $C = 12$ in equation (ii):

$-36 = 4(-6) - B - 12$

$\Rightarrow -36 = -B - 36$

$\Rightarrow B = 0.$

Thus, $\dfrac{-6x^2 + 12x - 42}{(x-2)(x^2+3)} = -\dfrac{6}{x-2} + \dfrac{12}{x^2+3}$

Therefore, $\dfrac{2x^4 - 4x^3 - 42}{(x-2)(x^2+3)} = 2x - \dfrac{6}{x-2} + \dfrac{12}{x^2+3}.$

Example 6.82

Decompose $\dfrac{2x^3 - x + 4}{x^3 + 4x}$ into partial fractions.

Solution

Since the numerator and the denominator have the same degree, divide the numerator by the denominator as follows:

$$\begin{array}{r|l} & \quad\quad\quad\quad\quad\quad 2 \\ \hline x^3 + 4x & 2x^3 + 0x^2 - x + 4 \\ & -(2x^3 + 0x^2 + 8x + 0) \\ \hline & \quad\quad\quad\quad -9x + 4 \end{array}$$

Thus, $\dfrac{2x^3 - x + 4}{x^3 + 4x} = 2 + \dfrac{-9x + 4}{x(x^2+4)}.$

The fraction $\dfrac{-9x+4}{x(x^2+4)}$ can be split further as follows:

$$\frac{-9x+4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 4}$$

Now, write the equation under common denominators as follows.

$$\frac{-9x+4}{x(x^2+4)} = \frac{A(x^2+4) + x(Bx + C)}{x(x^2+4)}$$

It follows that, $-9x + 4 = A(x^2+4) + x(Bx + C).$

Setting $x=0$, $4=4A \Leftrightarrow A=1$

Setting $x=1$, $-5=5+B+C \Leftrightarrow B+C=-10$ ....(i)

Setting $x=-1$, $13=5+B-C \Leftrightarrow B-C=8$ ....(ii)

Solving equations (i) and (ii) simultaneously gives,

$B=-1$ and $C=-9$.

Thus, $\dfrac{-9x+4}{x\left(x^2+4\right)}=\dfrac{1}{x}-\dfrac{x+9}{x^2+4}$

Therefore, $\dfrac{2x^3-x+4}{x^3+4x}=2+\dfrac{1}{x}-\dfrac{x+9}{x^2+4}$.

Exercise 6.14

Decompose each of the following expressions into partial fractions:

1. $\dfrac{2x}{(x+1)(x+2)}$
2. $\dfrac{10-2x}{(x-3)\ (x-1)}$
3. $\dfrac{x^2-14x-10}{x^3-4x^2+x+6}$
4. $\dfrac{(x-1)(x-2)}{(2x-1)\left(x^2+3\right)}$
5. $\dfrac{\left(x^2-x+7\right)(x-2)}{x\left(x^2-x+1\right)}$
6. $\dfrac{3x^2+7x+5}{\left(x^2+2\right)(x+1)}$
7. $\dfrac{y^4+y^3-19y^2-44y-21}{(y+3)(y+2)(y+1)}$
8. $\dfrac{2x^2-10x+10}{(x-1)^2(x-2)(x-3)}$
9. $\dfrac{(x-3)^2}{x^3+1}$
10. $\dfrac{4t^2-28}{t^4-t^2-6}$

Summation of series using partial fractions

Some series may be summed using partial fractions. The series $\sum u_r$ is called a telescopic series if u_r can be expressed as a sum of terms which appear in the expression for u_{r+k}, where $k>0$, such that some of the terms of the series cancel on addition, leaving only a few terms which can be easily added. The following two examples explains on how to use partial fractions in finding the summation of the series.

Example 6.83

Express $\frac{n+3}{(n-1)n(n+1)}$ into partial fractions and deduce that

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}.$$

Hence, find the sum to infinity of this series.

Solution

Let $\frac{n+3}{(n-1)n(n+1)}=\frac{A}{n-1}+\frac{B}{n}+\frac{C}{n+1}$

$$\Rightarrow \frac{n+3}{(n-1)n(n+1)}=\frac{An(n+1)+B(n-1)(n+1)+Cn(n-1)}{(n-1)n(n+1)}$$

Equating numerators gives;

$$n+3=An(n+1)+B(n-1)(n+1)+Cn(n-1)$$

Setting $n=1 \Rightarrow 4=2A \Leftrightarrow A=2$

Setting $n=0 \Rightarrow 3=-B \Leftrightarrow B=-3$

Setting $n=-1 \Rightarrow 2=2C \Leftrightarrow C=1$

Thus, $\frac{n+3}{(n-1)n(n+1)}=\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$.

Deducing the sum of the series is easily done by setting the working in columns form:

Setting $n=2$; $\quad \frac{5}{1\times2\times3}=2-\frac{3}{2}+\frac{1}{3}$

Setting $n=3$; $\quad \frac{6}{2\times3\times4}=1-1+\frac{1}{4}$

Setting $n=4$; $\quad \frac{7}{3\times4\times5}=\frac{2}{3}-\frac{3}{4}+\frac{1}{5}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

Setting $n=n-2$; $\quad \frac{n+1}{(n-3)(n-2)(n-1)}=\frac{2}{n-3}-\frac{3}{n-2}+\frac{1}{n-1}$

Setting $n=n-1$; $\quad \frac{n+2}{(n-2)(n-1)(n)}=\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}$

Setting $n=n$; $\quad \frac{n+3}{(n-1)n(n+1)}=\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$

On adding, it implies that

$$\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=2-\frac{3}{2}+1+\frac{1}{n}-\frac{3}{n}+\frac{1}{n+1}$$

$$=\frac{3}{2}+\frac{n+1-3(n+1)+n}{n(n+1)}$$

$$=\frac{3}{2}+\frac{-n-2}{n(n+1)}$$

$$=\frac{3}{2}-\frac{n+2}{n(n+1)}$$

Therefore, $\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots+\frac{n+3}{(n-1)n(n+1)}=\frac{3}{2}-\frac{n+2}{n(n+1)}$

Note that; as $n\to\infty$, $\frac{n+2}{n(n+1)}\to 0$. This means $\frac{3}{2}-\frac{n+2}{n(n+1)}\to\frac{3}{2}$ as $n\to\infty$

Therefore, the infinite series $\frac{5}{1\times2\times3}+\frac{6}{2\times3\times4}+\frac{7}{3\times4\times5}+\cdots$ is convergent and its sum to infinity is $\frac{3}{2}$.

Example 6.84

Use partial fractions to find

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(4n^2-1)(2n+3)}.$$

Hence, determine the sum of this series when n tends to infinity.

Solution

Let $\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{A}{2n-1}+\frac{B}{2n+1}+\frac{C}{2n+3}$

$$\Rightarrow \frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{A(2n+1)(2n+3)+B(2n-1)(2n+3)+C(2n-1)(2n+1)}{(2n-1)(2n+1)(2n+3)}$$

Equating numerators gives;

$$n+1 = A(2n+1)(2n+3) + B(2n-1)(2n+3) + C(2n-1)(2n+1)$$

Setting $n = \frac{1}{2} \Rightarrow \frac{3}{2} = 8A \Leftrightarrow A = \frac{3}{16}$

Setting $n = -\frac{1}{2} \Rightarrow \frac{1}{2} = -4B \Leftrightarrow B = -\frac{1}{8}$

Setting $n = -\frac{3}{2} \Rightarrow -\frac{1}{2} = 8C \Leftrightarrow C = -\frac{1}{16}$

Thus, $\frac{n+1}{(2n-1)(2n+1)(2n+3)} = \frac{3}{16(2n-1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$

Setting $n=1$, $\frac{2}{1\times3\times5} = \frac{3}{16} - \frac{1}{24} - \frac{1}{80}$

Setting $n=2$, $\frac{3}{3\times5\times7} = \frac{3}{48} - \frac{1}{40} - \frac{1}{112}$

Setting $n=3$, $\frac{4}{5\times7\times9} = \frac{3}{80} - \frac{1}{56} - \frac{1}{144}$

$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$

Setting $n = n-2$, $\frac{n-1}{(2n-5)(2n-3)(2n-1)} = \frac{3}{16(2n-5)} - \frac{1}{8(2n-3)} - \frac{1}{16(2n-1)}$

Setting $n = n-1$, $\frac{n}{(2n-3)(2n-1)(2n+1)} = \frac{3}{16(2n-3)} - \frac{1}{8(2n-1)} - \frac{1}{16(2n+1)}$

Setting $n = n$, $\frac{n+1}{(2n-1)(2n+1)(2n+3)} = \frac{3}{16(2n-1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$

On adding , $\frac{2}{1\times3\times5} + \frac{3}{3\times5\times7} + \ldots = \frac{3}{16} - \frac{1}{24} + \frac{3}{48} - \frac{1}{16(2n+1)} - \frac{1}{8(2n+1)} - \frac{1}{16(2n+3)}$

$$= \frac{5}{24} - \frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right)$$

Therefore, $\frac{2}{1\times3\times5} + \frac{3}{3\times5\times7} + \frac{4}{5\times7\times9} + \cdots + \frac{n+1}{(4n^2-1)(2n+3)} = \frac{5}{24} - \frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right).$

Note that; as $n \to \infty$, $\frac{1}{8}\left(\frac{4n+5}{(2n+1)(2n+3)}\right) \to 0$.

Therefore, the infinity series $\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots$ is convergent, and its sum to infinity is $\frac{5}{24}$.

Exercise 6.15

1. Express $\frac{2}{n(n+1)(n+2)}$ in partial fractions and deduce that $\frac{1}{1\times2\times3}+\frac{1}{2\times3\times4}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$. Hence, find the sum to infinity.

2. Use partial fractions to show that $\frac{1}{1\times2}+\frac{1}{2\times3}+\frac{1}{3\times4}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.

3. Find the sum of the series $\sum_{k=1}^{n}\frac{3k+2}{k(k+1)(k+2)}$, then evaluate the sum to infinity.

4. Decompose $\frac{2}{4n^2-1}$ into partial fractions, hence find $\sum_{k=1}^{n}\frac{2}{4k^2-1}$.

5. Evaluate $\sum_{k=1}^{n}\frac{4k}{(2k-1)(2k+1)(2k+3)}$ by using partial fractions.

6. Express $\frac{2}{n(n+2)}$ in partial fractions and deduce that
$$\frac{1}{1\times3}+\frac{1}{2\times4}+\frac{1}{3\times5}+\cdots+\frac{1}{n(n+2)}=\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$$

7. Express $\frac{1}{(2k-1)(2k+3)}$ into partial fractions. By multiplying by $\frac{1}{2r+1}$, or otherwise prove that $\sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)(2k+3)}=\frac{1}{12}-\frac{1}{4(2n+1)(2n+3)}$

8. Use partial fractions to show that
$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{n(5n+7)}{6(2n+1)(2n+3)}$$

9. Find the sum of the series $\frac{1}{2\times4}+\frac{1}{3\times5}+\frac{1}{4\times6}+\cdots+\frac{1}{(n+1)(n+3)}$.

10. Find the sum of the first n terms in each of the following series:

(a) $\frac{1}{3\times4\times5}+\frac{2}{4\times5\times6}+\frac{3}{5\times6\times7}+\cdots$

(b) $\frac{1}{1\times4}+\frac{1}{2\times5}+\frac{1}{3\times6}+\cdots$

(c) $\frac{1}{3\times6}+\frac{1}{6\times9}+\frac{1}{9\times12}+\cdots$

Chapter summary

1. Laws of indices include the following:

(a) $a^m \times a^n = a^{m+n}$

(b) $a^m \div a^n = a^{m-n}$

(c) $a^{-n} = \frac{1}{a^n}$

(d) $a^0 = 1$

(e) $a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n$

2. Laws of logarithms include the following:

(a) $\log_b(xz) = \log_b x + \log_b z$

(b) $\log_b\left(\frac{x}{z}\right) = \log_b x - \log_b z$

(c) $\log_b(x^a) = a\log_b x$

(d) $\log_b(x^{\frac{1}{m}}) = \frac{1}{m}\log_b x$

3. Natural logarithms satisfy all laws of common logarithms that is,

(a) $\ln xy = \ln x + \ln y$

(b) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

(c) $\ln x^n = n\ln x$

(d) $\ln x^{\frac{1}{m}} = \frac{1}{m}\ln x$

4. If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then

$$(x-\alpha)(x-\beta)=0$$

$$\Rightarrow x^2-(\alpha+\beta)x+\alpha\beta=0$$

$$\Rightarrow \alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}$$

5. If α, β, and γ are roots of a cubic equation $ax^3+bx^2+cx+d=0$, then

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\alpha\gamma+\beta\gamma)x-\alpha\beta\gamma=0$$

$$\Rightarrow \alpha+\beta+\gamma=-\frac{b}{a},\quad \alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}, \text{ and } \alpha\beta\gamma=-\frac{d}{a}.$$

6. The remainder theorem states that, if $p(x)$ is a dividend, $q(x)$ is a quotient, $x-a$ is a divisor, and $r(x)$ is a remainder, then $p(x)=q(x)\times(x-a)+r(x)$. When $x=a$, gives $p(a)=r(a)$. This theorem is used to calculate the remainder of the division of any polynomial by a linear polynomial without carrying out the steps of the division process.

7. The formulae for the sum of the first n, n^2, and n^3 natural numbers are:

(a) $\sum_{k=1}^{n} k=\frac{1}{2}n(n+1)$

(b) $\sum_{k=1}^{n} k^2=\frac{1}{6}n(n+1)(2n+1)$

(c) $\sum_{k=1}^{n} k^3=\frac{1}{4}n^2(n+1)^2$

(d) $\sum_{k=1}^{n} 1=n$

8. If A is a 3×3 matrix, then:

(a) $\text{Adj}(\text{A})=\left[\text{cof}(\text{A})\right]^{\text{T}}$

(b) $\text{A}^{-1}=\frac{\text{Adj}(\text{A})}{|\text{A}|}$ if $|\text{A}|\neq 0$.

9. If A is a 3×3 matrix and $|\text{A}|\neq 0$, then by Cramer's rule,

$x=\frac{|\text{A}_x|}{|\text{A}|}$, $y=\frac{|\text{A}_y|}{|\text{A}|}$, $z=\frac{|\text{A}_z|}{|\text{A}|}$ and by inverse matrix, $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\text{A}^{-1}\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$.

10. The formula for the general term of binomial expansion is $T_{r+1}={}^{n}\text{C}_{r}a^{n-r}b^{r}$.

11. The binomial theorem for fractional and negation indices is given by

$$(1+x)^n=1+nx+\frac{n(n-1)x^2}{2!}+\frac{n(n-1)(n-2)x^3}{3!}+\dots, \text{ where } |x|<1.$$

12. The absolute value of a real number x is written as $|x|$, and it is defined by;

$$|x|=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

13. The following definitions apply for absolute value inequality expressions:

If $|p|<c$, then $-c<p<c$, where, $c\geq 0$.

If $|p|\leq c$, then $-c\leq p\leq c$.

If $|p|>c$, then $p>c$ or $p<-c$.

If $|p|\geq c$, then $p\geq c$ or $p\leq -c$,

where p is any algebraic expression.

Revision exercise 6

1. The roots of the equation $ax^2+bx+c=0$ are α and β. The roots of the equation $a^2x^2+b^2x+c^2=0$ are 2α and 2β. Show that the equation whose roots are $n\alpha$ and $n\beta$ is $x^2+2nx+4n^2=0$.
2. If one root of the equation $mx^2+nx+t=0$ is twice the other, show that $\frac{2n^2}{9t}=m$.
3. The roots of the equation $x^3+3x^2+5x+7=0$ are α, β, and γ. Find the equation whose roots are:
   (a) 3α, 3β, and 3γ (b) α^2, β^2, and γ^2 (c) $\alpha+3$, $\beta+3$, and $\gamma+3$.
4. Use the remainder theorem to find the three factors of x^4+3x^2-4 and hence write $\frac{2x^3-x^2-7x-14}{x^4+3x^2-4}$ into partial fractions.
5. If $y=a+bx^n$ is satisfied by the following table of values,

x	1	2	4
y	7	10	15

   Show that $n=\log_2\left(\frac{5}{3}\right)$, and hence deduce the values of a and b.
6. Express $\frac{7+x}{1+x+x^2+x^3}$ into partial fractions. Assuming that $-1<x<1$, obtain an expression for $\frac{7+x}{1+x+x^2+x^3}$, give your answer in the form $a+bx+cx^2+dx^3+\cdots$ Hence, find the value of the coefficients as far as the term in x^5 inclusive.
7. If $u^v=v^{2u}$ and $v^2=u^3$, find the possible values of u and v.
8. Express $\frac{2x^2-3x+2}{(x-2)^8}$ in partial fractions.
9. Use the principal of mathematical induction to prove that the sum of cubes of any three consecutive natural numbers is divisible by 9.
10. Show that $ab+5(a-b)=1$, if $a=\log_{12}18$ and $b=\log_{24}54$.

11. Prove that $\dfrac{\log_b a}{\log_{bc} a} = 1 + \log_b c$.

12. Simplify the following expression, to the most simplified form. $\dfrac{3(2^{n+1}) - 4(2^{n-1})}{2^{n+1} - 2^n}$

13. (a) Find the value of x and y in the following system of equations,
$\log x - 3\log y = 1, \quad xy = 160$.

(b) Solve for x if $75\left(2^{\log_5\left(\frac{x}{3}\right)}\right) = 4x$.

14. Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 5 & 4 & 3 \\ 10 & 5 & 1 \end{bmatrix}$.

15. Solve the following system of equations by using the matrix method:
$$\begin{cases} x + 2y + z = 10 \\ 2x + y + z = 20 \\ x + 3y + z = 30 \end{cases}$$

16. Solve the following system of equations by using the matrix method:
$$\begin{cases} x + y + z = 3 \\ 5x + 4y + 3z = 11 \\ 10x + 5y + z = 11\frac{1}{2} \end{cases}$$

17. Write down the first four terms of the expansion of each of the following and state the values of x where the expansion is valid:

(a) $(1+x)^{-3}$ (b) $(1-x^2)^{\frac{2}{3}}$

18. Find the 5th term in the expansion of $(1-5x)^{\frac{7}{2}}$.

19. If $|x| < 1$, show that $(1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2 + \dfrac{n(n+1)(n+2)}{3!}x^3 + \cdots$

20. Express $\dfrac{7x+3}{(3x-1)(x+1)^2}$ into partial fractions and then use binomial theorem to find the coefficient of x^n in ascending powers of x.

21. Use binomial expansion to evaluate each of the following:

(a) $\sqrt{1.01}$ (b) $\sqrt[3]{0.98}$

22. In the binomial expansion of $(1+x)^n$, where n is a positive integer, the coefficient of x^4 is $\frac{3}{2}$ times the sum of the coefficients of x^2 and x^3. Find the value of n and determine these three coefficients.

23. Prove that $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1+b_1 & b_1 & c_1 \\ a_2+b_2 & b_2 & c_2 \\ a_3+b_3 & b_3 & c_3 \end{vmatrix}$

24. Expand each of the following:

(a) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & c+a & a+b \end{vmatrix}$ (b) $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

25. Prove that $\log\left(\frac{m}{n}\right)+\log\left(\frac{n}{z}\right)+\log\left(\frac{z}{m}\right)=0.$

26. Show that $3\sum_{k=1}^{n}k^2+3\sum_{k=1}^{n}k+\sum_{k=1}^{n}1=(n+1)^3-1.$

27. Prove each of the following statements by mathematical induction:

(a) $2+6+12+\cdots+n(n+1)=\frac{1}{3}n(n+1)(n+2).$

(b) $\frac{1}{3}+\frac{1}{8}+\frac{1}{15}+\cdots+\frac{1}{n(n+2)}=\frac{n(3n+5)}{4(n+1)(n+2)}$

28. Prove that $\begin{vmatrix} x^2 & 2xy & y^2 \\ y^2 & x^2 & 2xy \\ 2xy & y^2 & x^2 \end{vmatrix}=\left(x^3+y^3\right)^2.$

29. Write down the first four terms of the expansion of $(1-x)^{-2}$ in ascending powers of x, hence deduce that $\sum_{n=0}^{\infty}(a+bn)x^n=\frac{a+(b-a)x}{(1-x)^2}$.

30. Use Cramer's rule to solve the following system of equations:
$2x-5y+2z=14,\ \ 9x+3y-4z=13,\ \ 7x+3y-2z=3.$

31. Evaluate each of the following:

(a) $\begin{pmatrix} 2 & 3 & 1 \end{pmatrix}\times\begin{pmatrix} 3 & 2 \\ 4 & 7 \\ 1 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 1 & -3 & -1 \end{pmatrix}\times\begin{pmatrix} -1 & -2 & 1 \\ -1 & 2 & 3 \\ -1 & -2 & 2 \end{pmatrix}$

32. If $A=\begin{bmatrix}1&2&2\\2&1&2\\2&2&1\end{bmatrix}$, verify that $A^2-4A-5I=0$.

33. Write the first three terms in the expression of $\frac{2+x}{(3-2x)^2}$ in ascending powers of x. Also, state the condition under which the expression is valid.

34. Use partial fractions to show that

$$\frac{2}{1\times3\times5}+\frac{3}{3\times5\times7}+\frac{4}{5\times7\times9}+\cdots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{5n^2+7n}{6(4n^2+8n+3)}$$

35. Prove by mathematical induction that:

$$1\times3+3\times5+5\times7+\cdots+(2n-1)(2n+1)=\frac{1}{3}n(4n^2+6n-1).$$

36. Find the set of values of k for which the equation $f(x)=0$, if $f(x)=3x^2-kx+3$ has:

(a) Repeated roots (b) Distinct real roots

37. Solve each of the following inequalities:

(a) $|3x+2|>|2x-3|$ (b) $\frac{(x+2)(x-5)}{(x-3)(x-2)}>1$

38. Find $\sum_{k=1}^{n}3(k^2+3k+2)$, hence evaluate $\sum_{k=15}^{65}3(k^2+3k+2)$.

39. A botanist requires 12, 6, and 7 units of copper, zinc, and calcium, respectively for his secondary school. A concentrated product contains 3, 1, and 1 units of copper, zinc, and calcium per carton, respectively. A diluted product contains 1, 0, and 2 units of copper, zinc, and calcium per carton, respectively. A dry product contains 1, 1, and 1 units of copper, zinc, and calcium per carton, respectively.

(a) Formulate and express a system of equations in matrix form.

(b) By using the inverse matrix method, find how many of each should be purchased in order to meet the requirements.

40. Prove that the sum to n terms of the series $1\times3^2+4\times4^2+7\times5^2+10\times6^2+\cdots$ is $\frac{1}{12}n(n+1)(9n^2+49n+44)-8n$.

Chapter Seven

Trigonometry

Introduction

The word trigonometry comes from the Greek words "trigonon" which means triangle and "metron" which means to measure. Trigonometry deals with determination of measures of sides and angles of triangles by means of relevant trigonometric functions. The trigonometric functions are sine, cosine, tangent, secant, cosecant, and cotangent of an angle. In this chapter you will learn about trigonometric ratios, trigonometric identities, compound angle formulae, double angle formulae, trigonometric equations, factor formulae, radians and small angles, trigonometric functions, and inverse of trigonometric functions. The competencies developed can be applied in various real life situations such as in solving problems related to astronomy, navigation, architecture, oceanography, and in creation of maps. Generally, trigonometry has great practical importance to builders, architects, surveyors, engineers, and users in many other fields.

Trigonometric ratios

Trigonometric ratios are values of trigonometric functions based on the ratio of sides of a right-angled triangle.

Activity 7.1: Deducing the formulae of trigonometric ratios

Individually or in a group, perform the following tasks:

1. Construct a right-angled triangle in the *xy*-plane.
2. Deduce the formulae for trigonometric ratios of sine, cosine and tangent of an acute angle α.
3. Use the derived formulae in task 2 to deduce the formulae for secant, cosecant, and cotangent of an angle α.
4. Give a suggestion of the formulae deduced in tasks 2 and 3.
5. What have you observed in task 4? Give comments.
6. Share your findings with your fellow students for more inputs.

The ratios of sides of a right-angled triangle with respect to any of its acute angles are known as the trigonometric ratios of that particular angle. The three sides of the right-angled triangle are hypotenuse (the longest side), opposite (side across from the angle), and adjacent (base of the angle). Consider the right-angled triangle OAB in Figure 7.1.

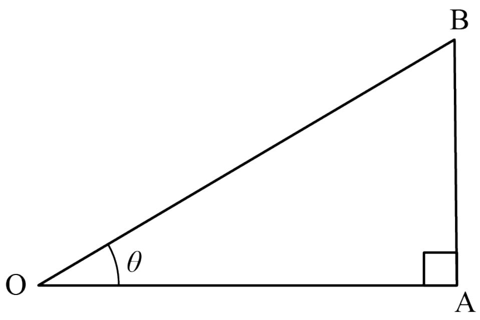


Figure 7.1: *A right-angled triangle OAB*

From Figure 7.1, the hypotenuse of the right-angled triangle is $\overline{OB}$, $\overline{OA}$ is the adjacent side to angle θ, and $\overline{AB}$ is the opposite side to angle θ. The three basic trigonometric ratios with their abbreviations are defined as follows:

$$\sin\theta = \frac{\text{Length of opposite}}{\text{Length of hypotenuse}} = \frac{\text{Side }\overline{AB}}{\text{Side }\overline{OB}}$$

$$\cos\theta = \frac{\text{Length of adjacent}}{\text{Length of hypotenuse}} = \frac{\text{Side }\overline{OA}}{\text{Side }\overline{OB}}$$

$$\tan\theta = \frac{\text{Length of opposite}}{\text{Length of adjacent}} = \frac{\text{Side }\overline{AB}}{\text{Side }\overline{OA}}$$

Now, using Pythagoras' theorem, the following relation is obtained;

$$\left(\overline{OA}\right)^2 + \left(\overline{AB}\right)^2 = \left(\overline{OB}\right)^2.$$

The reciprocal of the trigonometric functions of sine, cosine, and tangent are cosecant, secant, and cotangent, respectively. These are defined from Figure 7.1 as follows:

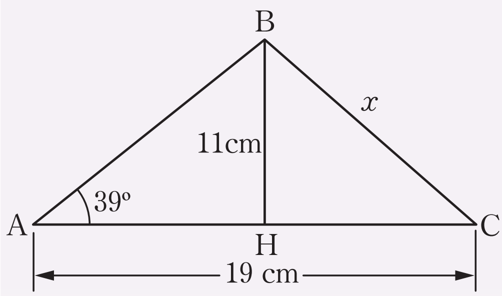
$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{Side }\overline{OB}}{\text{Side }\overline{AB}}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{Side }\overline{OB}}{\text{Side }\overline{OA}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{Side }\overline{OA}}{\text{Side }\overline{AB}}$$

Example 7.1

In the following figure, if side BH is perpendicular to side $\overline{AC}$, find the value of x correct to three significant figures.



Solution

From the figure, $\tan 39° = \frac{\overline{BH}}{\overline{AH}}$

$$\tan 39° = \frac{11}{\overline{AH}} \Rightarrow \overline{AH} = \frac{11}{\tan 39°}$$

$$\overline{HC} = 19 - \overline{AH}$$

$$= 19 - \frac{11}{\tan 39°}$$

Using Pythagoras' theorem, for the right-angled triangle HBC, it gives;

$$\left(\overline{BH}\right)^2 + \left(\overline{HC}\right)^2 = \left(\overline{BC}\right)^2$$

$$\Rightarrow 11^2 + \left(\overline{HC}\right)^2 = x^2$$

Substitute $\overline{HC}$ to find the value of x, so that;

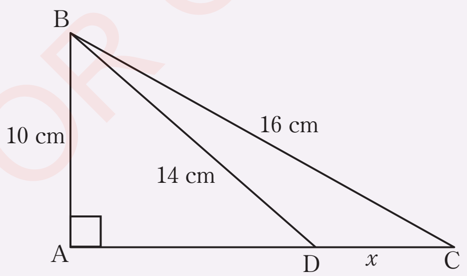
$$x = \sqrt{11^2 + \left(19 - \frac{11}{\tan 39°}\right)^2}$$

$= 12.3$.

Therefore, $x = 12.3$ cm.

Example 7.2

In the following figure, A is a right angle. Find the value of x correct to two decimal places.



Solution

Since A is a right angle, both triangles ABC and ABD in the figure, are right angled-triangles.

Using Pythagoras' theorem, the value of x can be found as follows:

From a triangle BAD;

$$14^2 = 10^2 + \left(\overline{AD}\right)^2 \Rightarrow \overline{AD} = \sqrt{14^2 - 10^2}$$

Also, from a triangle BAC;

$$16^2 = 10^2 + \left(\overline{AC}\right)^2 \Rightarrow \overline{AC} = \sqrt{16^2 - 10^2}$$

Thus, $x = \overline{AC} - \overline{AD}$

$$= \sqrt{16^2 - 10^2} - \sqrt{14^2 - 10^2}$$

$$= \sqrt{156} - \sqrt{96}$$

$$= 2.69$$

Therefore, $x = 2.69$ cm.

Example 7.3

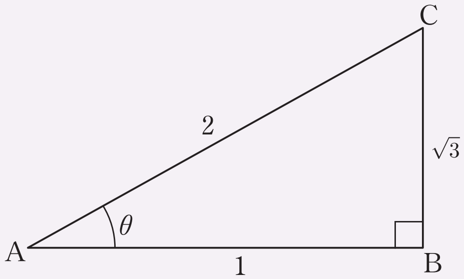
If $\tan\theta = \sqrt{3}$, where θ is an acute angle, find the value of each of the following leaving your answer in surd form.

(a) $\sec\theta$ (b) $\cot\theta$ (c) $\operatorname{cosec}\theta$

Solution

Given $\tan\theta = \sqrt{3} \Rightarrow \tan\theta = \frac{\sqrt{3}}{1}$

The values corresponding to the sides of the corresponding right-angled triangle are as shown in following right-angle triangle ABC:



Using the Pythagoras' theorem,

$\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$

$\Rightarrow 1^2 + \sqrt{3}^2 = \overline{AC}^2$

$\Rightarrow 4 = \overline{AC}^2$

$\Rightarrow \overline{AC} = 2$ units.

Thus, it follows that;

(a) $\sec\theta = \dfrac{1}{\cos\theta}$

But $\cos\theta = \dfrac{1}{2} \Rightarrow \sec\theta = 2$

Therefore, $\sec\theta = 2$.

(b) $\cot\theta = \dfrac{1}{\tan\theta}$

But $\tan\theta = \sqrt{3} \Rightarrow \cot\theta = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$

Therefore, $\cot\theta = \dfrac{\sqrt{3}}{3}$.

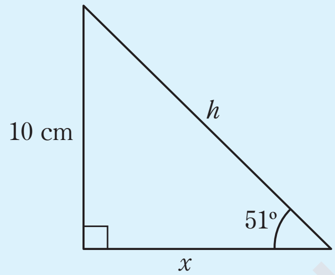
(c) $\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

But $\sin\theta = \dfrac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec}\theta = \dfrac{2\sqrt{3}}{3}$

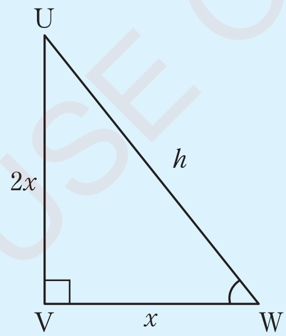
Therefore, $\operatorname{cosec}\theta = \dfrac{2\sqrt{3}}{3}$.

Exercise 7.1

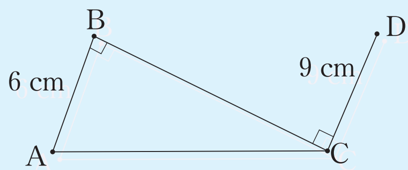
1. Find the values of x and h in the following figure.



2. Find the lengths of all sides of the following right-angled triangle with an area of 400 square units.



3. In the following figure, $\overline{AB}$ and $\overline{CD}$ are both perpendicular to $\overline{BC}$ and the size of angle ACB is 31°. Find the length of the line segment $\overline{BD}$, correct to 3 significant figures.



4. The area of a right-angled triangle JKL is 50 square centimetres. One of its angles is 45°. Find the lengths of the sides of the triangle.

5. In a right-angled triangle DEF, $\tan \mathrm{D} = \frac{119}{120}$. Find the values of $\operatorname{cosec} \mathrm{D}$ and $\sec \mathrm{D}$.

6. If $\sec \mathrm{B} = \frac{13}{5}$, where B is an acute angle, find the value of each of the following;
   (a) $\operatorname{cosec} \mathrm{B}$ (b) $\tan \mathrm{B}$

7. In a right-angled triangle LMN with angle L equal to 90°, find angle M and N so that $\sin \mathrm{M} = \cos \mathrm{M}$.

8. A rectangle has dimensions 10 cm by 5 cm. Determine the measures of the angles at the point where the diagonals intersect and find the length of the diagonal.

9. The lengths of sides PQ and QR of a scalene triangle PQR are 12 cm and 8 cm respectively. If the size of angle R is 59°, find the length of side PR.

10. Maria is riding vertically in a hot air balloon directly over a point P on the ground. Maria spots a parked car on the ground at an angle of depression of 30°, then, the balloon rises 50 metres. If the angle of depression to the car is 35°. How far is the car from point P?

11. If the shadow of a building increases by 10 metres when the angle of elevation of the sun rays decreases from 70° to 60°, what is the height of the building?

Trigonometric identities

Consider the right-angled triangle shown in Figure 7.2.

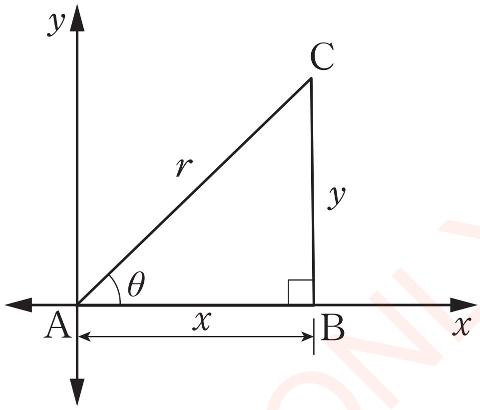


Figure 7.2: *A right-angled triangle in the xy-plane*

Using Pythagoras' theorem;

$$\left(\overline{\mathrm{AB}}\right)^2 + \left(\overline{\mathrm{BC}}\right)^2 = \left(\overline{\mathrm{AC}}\right)^2$$

Since, $\overline{\mathrm{AC}} = r$, $\overline{\mathrm{AB}} = x$, and $\overline{\mathrm{BC}} = y$ then,

$$x^2 + y^2 = r^2$$

Divide throughout by r^2

$$\Rightarrow \frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$

$$\Rightarrow \left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

But $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$.

Substituting the values for $\cos\theta$ and $\sin\theta$ into $\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$ gives:

$$\cos^2\theta + \sin^2\theta = 1 \qquad (7.1)$$

Two similar identities can be deduced from equation (7.1) as follows;
Divide by $\sin^2\theta$ both sides of equation (7.1) as follows:

$$\frac{\cos^2\theta}{\sin^2\theta}+\frac{\sin^2\theta}{\sin^2\theta}=\frac{1}{\sin^2\theta}$$

where as, $\dfrac{\cos^2\theta}{\sin^2\theta}=\cot^2\theta$

$$\Rightarrow \cot^2\theta+1=\operatorname{cosec}^2\theta \qquad (7.2)$$

Again, divide by $\cos^2\theta$ both sides of equation (7.1) as follows:

$$\frac{\cos^2\theta}{\cos^2\theta}+\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{\cos^2\theta}$$

where as, $\dfrac{\sin^2\theta}{\cos^2\theta}=\tan^2\theta$

$$\Rightarrow 1+\tan^2\theta=\sec^2\theta \qquad (7.3)$$

Equations (7.1), (7.2), and (7.3) are also known as Pythagorean identities as they express the Pythagorean theorem in terms of trigonometric functions.

Example 7.4

Simplify $\sin\theta+\cos\theta\cot\theta$.

Solution

$$\begin{aligned}\sin\theta+\cos\theta\cot\theta&=\sin\theta+\cos\theta\left(\frac{\cos\theta}{\sin\theta}\right)\\&=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta},\text{ but }\sin^2\theta+\cos^2\theta=1\\&=\frac{1}{\sin\theta}\\&=\operatorname{cosec}\theta\end{aligned}$$

Therefore, $\sin\theta+\cos\theta\cot\theta=\operatorname{cosec}\theta$.

Example 7.5

If $x = \sin\theta$, then show that

$$\frac{x}{\sqrt{1-x^2}} = \tan\theta.$$

Solution

Given $x = \sin\theta$

$$\text{Thus, } \frac{x}{\sqrt{1-x^2}} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$

$$= \frac{\sin\theta}{\sqrt{\cos^2\theta}}, \text{ since } 1-\sin^2\theta = \cos^2\theta$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

Therefore, if $x = \sin\theta$, then

$$\frac{x}{\sqrt{1-x^2}} = \tan\theta.$$

Example 7.6

Solve the equation $2\sin^2\theta - 1 = \cos\theta$, for values of θ between $0°$ and $360°$.

Solution

Given $2\sin^2\theta - 1 = \cos\theta$

$\Rightarrow 2\sin^2\theta - \cos\theta - 1 = 0$

But $\sin^2\theta = 1-\cos^2\theta$, thus,

$\Rightarrow 2(1-\cos^2\theta) - \cos\theta - 1 = 0$

$\Rightarrow 2 - 2\cos^2\theta - \cos\theta - 1 = 0$

$\Rightarrow -2\cos^2\theta - \cos\theta + 1 = 0$

Factorizing the resulted quadratic equation, gives;

$(-2\cos\theta + 1)(\cos\theta + 1) = 0$

Hence

$\cos\theta = \frac{1}{2}$ or $\cos\theta = -1$

If $\cos\theta = \frac{1}{2}$, $\theta = 60°, 300°$.

If $\cos\theta = -1$, $\theta = 180°$

Therefore, the values of θ between $0°$ and $360°$ are $60°, 180°$, and $300°$.

Example 7.7

Solve the equation

$4\cos\theta - 3\sec\theta = 2\tan\theta$, for

$0° \le \theta \le 180°$.

Solution

Given $4\cos\theta - 3\sec\theta = 2\tan\theta$

$\Rightarrow 2\tan\theta - 4\cos\theta + 3\sec\theta = 0$

$$\Rightarrow 2\left(\frac{\sin\theta}{\cos\theta}\right) - 4\cos\theta + \frac{3}{\cos\theta} = 0$$

Multiplying both sides by $\cos\theta$ gives;

$2\sin\theta - 4\cos^2\theta + 3 = 0$

But $\cos^2\theta = 1-\sin^2\theta$

$\Rightarrow 2\sin\theta - 4(1-\sin^2\theta) + 3 = 0$

$\Rightarrow 4\sin^2\theta + 2\sin\theta - 1 = 0.$

The general formula of solving a quadratic equation in $\sin\theta$ is

$$\sin\theta = \frac{-b \pm \sqrt{b^2-4ac}}{2a},$$

where $a = 4$, $b = 2$, and $c = -1$. Hence,

$$\sin\theta = \frac{-2 \pm \sqrt{4+16}}{8}$$

So that, $\sin\theta = 0.3090$ or $\sin\theta = -0.8090$

If $\sin\ \theta = 0.3090$ then $\theta = \sin^{-1}(0.3090) = 18°$ or $162°$.

Also, if $\sin\ \theta = -0.8090$, then $\theta = \sin^{-1}(-0.8090) = -54°$ or $-126°$.

Therefore, the values of θ are $18°$ and $162°$.

Example 7.8

Prove the identity $\operatorname{cosec}^4 x - \operatorname{cosec}^2 x = \cot^4 x + \cot^2 x$.

Solution

Given $\operatorname{cosec}^4 x - \operatorname{cosec}^2 x = \cot^4 x + \cot^2 x$

Proving from the right-hand side,

$$\begin{aligned}\cot^4 x + \cot^2 x &= \cot^2 x(\cot^2 x + 1)\\ &= (\operatorname{cosec}^2 x - 1)(\operatorname{cosec}^2 x - 1 + 1), \text{ since } \operatorname{cosec}^2 x = 1 + \cot^2 x\\ &= (\operatorname{cosec}^2 x - 1)(\operatorname{cosec}^2 x)\\ &= \operatorname{cosec}^4 x - \operatorname{cosec}^2 x\end{aligned}$$

Therefore, $\cot^4 x + \cot^2 x = \operatorname{cosec}^4 x - \operatorname{cosec}^2 x$.

Example 7.9

Verify the identity $\tan x + \cot x = \sec x \operatorname{cosec} x$.

Solution

Given $\tan x + \cot x = \sec x \operatorname{cosec} x$

From left hand-side,

$$\begin{aligned}\tan x + \cot x &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}\\ &= \frac{1}{\cos x \sin x}\end{aligned}$$

$$\tan x + \cot x = \frac{1}{\cos x} \times \frac{1}{\sin x}$$
$$= \sec x \operatorname{cosec} x.$$

Therefore, $\tan x + \cot x = \sec x \operatorname{cosec} x$.

Example 7.10

Eliminate θ from the following parametric equations.

$x = 2 + 4\cos\theta$

$y = 3 + 5\sin\theta$

Solution

Given

$x = 2 + 4\cos\theta$ and $y = 3 + 5\sin\theta$.

Write the given equations in terms of $\cos\theta$ and $\sin\theta$. That is,

$$\cos\theta = \frac{x-2}{4} \text{ and } \sin\theta = \frac{y-3}{5}$$

Square on both sides of each equation to obtain:

$$\cos^2\theta = \left(\frac{x-2}{4}\right)^2 \text{ and}$$

$$\sin^2\theta = \left(\frac{y-3}{5}\right)^2$$

Since, $\sin^2\theta + \cos^2\theta = 1$ then,

$$\left(\frac{y-3}{5}\right)^2 + \left(\frac{x-2}{4}\right)^2 = 1.$$

Therefore, $\left(\frac{y-3}{5}\right)^2 + \left(\frac{x-2}{4}\right)^2 = 1.$

Exercise 7.2

1. Simplify each of the following expressions:
   (a) $\sqrt{(1-\sin y)(1+\sin y)}$
   (b) $\sec\theta - \sec\theta\sin^2\theta$
   (c) $\frac{1}{\sec^2 x} + \frac{1}{\operatorname{cosec}^2 x}$
   (d) $\tan\theta + \frac{\cos\theta}{1+\sin\theta}$

2. Verify each of the following trigonometric identities:
   (a) $\frac{\sec x}{\cot x + \tan x} = \sin x$
   (b) $\frac{\sin x}{1+\cos x} = \frac{1-\cos x}{\sin x}$
   (c) $\sqrt{\frac{\sec x - \tan x}{\sec x + \tan x}} = \frac{1}{\sec x + \tan x}$

3. If $p = 2\sin x$ and $3q = \cos x$, show that $p^2 + 36q^2 = 4$.

4. Eliminate θ in each of the following pair of equations:
   (a) $x = \sin\theta,\ y = \cos\theta$
   (b) $x = 2 + \sin\theta,\ \cos\theta = 1 + y$
   (c) $x = 3\sin\theta,\ y = \operatorname{cosec}\theta$

5. Solve each of the following equations for all values of θ between $0°$ and $360°$:
   (a) $3\cot\theta = \tan\theta$
   (b) $2\tan\theta = 5\operatorname{cosec}\theta + \cot\theta$.
   (c) $\operatorname{cosec}^2\theta = 3\cot\theta - 1$.

6. Prove each of the following identities:
   (a) $\cos^4 x - \sin^4 x + 1 = 2\cos^2 x$
   (b) $(1 + \cot\theta - \operatorname{cosec}\theta)(1 + \tan\theta + \sec\theta) = 2$
   (c) $\dfrac{\tan x - \sin x}{\sin^3 x} = \dfrac{\sec x}{1 + \cos x}$
   (d) $\dfrac{\tan y + \sec y - 1}{\tan y - \sec y + 1} = \tan y + \sec y$
   (e) $\sec A\,(1 - \sin A)(\sec A + \tan A) = 1.$

7. If $\cot^2\theta + 3\operatorname{cosec}^2\theta = 7$, show $\tan\theta = \pm 1$.

8. Show that, $\sqrt{\dfrac{1 - \cos x}{1 + \cos x}} = \operatorname{cosec} x - \cot x$.

9. If $m = 4\sec 2\theta$ and $n = 8\tan 2\theta$, show that $4m^2 = 64 + n^2$.

10. If $x = a(1 - \operatorname{cosec}\theta)$ an $y = a(\sec\theta + \tan\theta)$, prove that $xy^2 + a^2(2a - x) = 0$.

Compound angle formulae

The compound angle formula can be used to find the sine, cosine, and tangent of the sum and difference of angles. The basic operations on sums and differences of trigonometric functions can be computed using the concept of compound angles.

Consider the triangle in Figure 7.3.

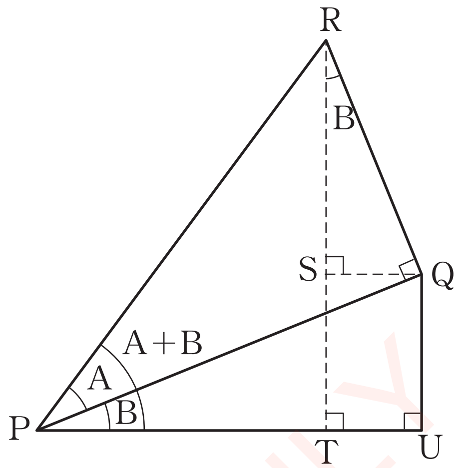


Figure 7.3: *Illustrative sketch for deriving compound angle formulae*

From Figure 7.3, angles A and B are acute such that $A + B < 90°$.

Consider the right-angled triangle PRT,

$$\sin(A+B) = \frac{\overline{RT}}{\overline{PR}} = \frac{\overline{RS} + \overline{ST}}{\overline{PR}} = \frac{\overline{RS} + \overline{QU}}{\overline{PR}},$$

since $\overline{ST} = \overline{QU}$. Thus,

$$\begin{aligned}\text{Sin}(A+B) &= \frac{\overline{RS}}{\overline{PR}} + \frac{\overline{QU}}{\overline{PR}}\\ &= \frac{\overline{QR}}{\overline{QR}} \times \frac{\overline{RS}}{\overline{PR}} + \frac{\overline{PQ}}{\overline{PQ}} \times \frac{\overline{QU}}{\overline{PR}}\\ &= \frac{\overline{QR}}{\overline{PR}} \times \frac{\overline{RS}}{\overline{QR}} + \frac{\overline{PQ}}{\overline{PR}} \times \frac{\overline{QU}}{\overline{PQ}}\end{aligned}$$

But $\dfrac{\overline{QR}}{\overline{PR}} = \sin A$, $\dfrac{\overline{RS}}{\overline{QR}} = \cos B$,

$\dfrac{\overline{PQ}}{\overline{PR}} = \cos A$, and $\dfrac{\overline{QU}}{\overline{PQ}} = \sin B$

Thus, $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Therefore, the compound angle formula for finding the sine of the sum of two angles is given by;

$$\sin(A+B) = \sin A\cos B + \cos A\sin B \quad (7.4)$$

Suppose B is replaced by –B. Then, substituting it into equation (7.4), gives;
$\sin(A+(-B)) = \sin A\cos(-B) + \cos A\sin(-B)$.
Note that, $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$, and $\tan(-x) = -\tan x$ because $\cos x$ is an even function while $\sin x$ and $\tan x$ are odd functions.
Thus, $\sin(A-B) = \sin A\cos B - \cos A\sin B$.

Therefore, the compound angle formula for finding the sine of the difference of two angles is given by;

$\sin(A-B) = \sin A\cos B - \cos A\sin B$.
From Figure 7.3, considering the right-angled triangle PRT

$$\cos(A+B) = \frac{\overline{PT}}{\overline{PR}} = \frac{\overline{PU}-\overline{TU}}{\overline{PR}}, \text{ since } \overline{PU} = \overline{PT}+\overline{TU}$$

$$\cos(A+B) = \frac{\overline{PU}-\overline{SQ}}{\overline{PR}} = \frac{\overline{PU}}{\overline{PR}} - \frac{\overline{SQ}}{\overline{PR}}, \text{ since } \overline{TU} = \overline{SQ}$$

$$= \frac{\overline{PQ}}{\overline{PQ}}\times\frac{\overline{PU}}{\overline{PR}} - \frac{\overline{QR}}{\overline{QR}}\times\frac{\overline{SQ}}{\overline{PR}}$$

$$= \frac{\overline{PQ}}{\overline{PR}}\times\frac{\overline{PU}}{\overline{PQ}} - \frac{\overline{QR}}{\overline{PR}}\times\frac{\overline{SQ}}{\overline{QR}}$$

But $\frac{\overline{PQ}}{\overline{PR}} = \cos A$, $\frac{\overline{PU}}{\overline{PQ}} = \cos B$, $\frac{\overline{QR}}{\overline{PR}} = \sin A$, and $\frac{\overline{SQ}}{\overline{QR}} = \sin B$.

Therefore, the compound angle formula for finding the cosine of the sum of two angles is given by;
$\cos(A+B) = \cos A\cos B - \sin A\sin B$. (7.5)
Suppose B is replaced by –B then substituting into equation (7.5), gives;
$\cos(A+(-B)) = \cos A\cos(-B) - \sin A\sin(-B)$.
Thus, $\cos(A-B) = \cos A\cos B + \sin A\sin B$.
Therefore, the compound angle formula for finding the cosine of the difference of two angles is given by;
$\cos(A-B) = \cos A\cos B + \sin A\sin B$

The derivation of the compound angle formula for $\tan(A+B)$ can be done as follows;

From trigonometric identities $\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)}$. Substituting equations

(7.4) and (7.5) for $\sin(A+B)$ and $\cos(A+B)$ gives,

$$\Rightarrow \tan(A+B) = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}$$

Divide by $\cos A\cos B$ both the numerator and denominator to get;

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}}$$

$$= \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \left(\dfrac{\sin A}{\cos A}\right)\left(\dfrac{\sin B}{\cos B}\right)}$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

Therefore, the compound angle formula for finding the tangent of the sum of two angles is written as;

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \qquad (7.6)$$

Similarly, if B is replaced by –B then substituting it into equation (7.6) results to;

$$\tan(A+(-B)) = \frac{\tan A + \tan(-B)}{1 - \tan A\tan(-B)}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

Therefore, the compound angle formula for the tangent of the difference of two angles is written as,

Thus, $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$.

In summary, the compound angle formulae are written as;

(i) $\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B$

(ii) $\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B$

(iii) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$

Example 7.11

Prove that

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$.

Solution

Given

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$

Consider the left-hand side of the given identity,

$\cos(x+y)\cos(x-y)$

$= (\cos x\cos y - \sin x\sin y)\times$
$(\cos x\cos y + \sin x\sin y)$

$= \cos^2 x\cos^2 y + \cos x\cos y\sin x\sin y -$
$\cos x\cos y\sin x\sin y - \sin^2 x\sin^2 y$

$= \cos^2 x\cos^2 y - \sin^2 x\sin^2 y$

$= \cos^2 x(1-\sin^2 y) - (1-\cos^2 x)\sin^2 y$

$= \cos^2 x - \cos^2 x\sin^2 y - \sin^2 y$

$+ \cos^2 x\sin^2 y = \cos^2 x - \sin^2 y$

Therefore,

$\cos(x+y)\cos(x-y) = \cos^2 x - \sin^2 y$.

Example 7.12

Simplify $\dfrac{\tan(A+B) - \tan A}{1 + \tan(A+B)\tan A}$.

Solution

By using the compound angle formula,

$$\frac{\tan(A+B) - \tan A}{1 + \tan(A+B)\tan A} = \tan((A+B) - A)$$

$= \tan(A - A + B)$

$= \tan B$

Therefore, $\dfrac{\tan(A+B) - \tan A}{1 + \tan(A+B)\tan A} = \tan B.$

Example 7.13

Simplify $(\sin x \cos y - \cos x \sin y)^2 + (\cos x \cos y + \sin x \sin y)^2$.

Solution

$(\sin x \cos y - \cos x \sin y)^2$

$+ (\cos x \cos y + \sin x \sin y)^2$

$= \sin^2(x - y) + \cos^2(x - y)$

$= 1$

Therefore, $(\sin x \cos y - \cos x \sin y)^2 + (\cos x \cos y + \sin x \sin y)^2 = 1$.

Example 7.14

If $A + B + C = 180°$, show that $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Solution

Given $A + B + C = 180°$.

$\Rightarrow A + B = 180° - C$

Applying tangent on both sides gives;

$\tan(A + B) = \tan(180° - C)$

By using the compound angle formula gives;

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\tan 180° - \tan C}{1 + \tan 180° \tan C}$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{0 - \tan C}{1 + 0 \times \tan C}$$

since $\tan 180° = 0$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

Cross multiplication gives,

$\tan A + \tan B = (1 - \tan A \tan B)(-\tan C)$

Thus,

$\tan A + \tan B = -\tan C + \tan A \tan B \tan C$

Therefore,

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Example 7.15

Solve the equation $2\sin x = \cos(x + 60°)$ for values of x between $0°$ and $360°$.

Solution

Given $2\sin x = \cos(x + 60°)$

$= \cos x \cos 60° - \sin x \sin 60°$

$$\Rightarrow 2\sin x = \frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x$$

Divide by $\cos x$ on both sides to get;

$$2\tan x = \frac{1}{2} - \frac{\sqrt{3}}{2}\tan x$$

Collecting like terms gives;

$$2\tan x + \frac{\sqrt{3}}{2}\tan x = \frac{1}{2}$$

$$\Rightarrow \left(2 + \frac{\sqrt{3}}{2}\right)\tan x = \frac{1}{2}$$

$$\Rightarrow \left(\frac{4 + \sqrt{3}}{2}\right)\tan x = \frac{1}{2}$$

$$\Rightarrow \left(4 + \sqrt{3}\right)\tan x = 1$$

$$\Rightarrow \tan x = \frac{1}{4+\sqrt{3}}$$

$$\Rightarrow x = \tan^{-1}\left(\frac{1}{4+\sqrt{3}}\right)$$

$$\Rightarrow x = 9.9° \text{ and } 189.9°$$

Therefore, the values of x are $9.9°$ and $189.9°$.

Example 7.16

If $\tan(x+45°) = 2$, find the value of $\tan x$.

Solution

Given $\tan(x+45°) = 2$.

$$\Rightarrow \frac{\tan x + \tan 45°}{1 - \tan x \tan 45°} = 2$$

$$\frac{\tan x + 1}{1 - \tan x} = 2, \text{ since } \tan 45° = 1$$

$$\Rightarrow \tan x + 1 = 2(1 - \tan x)$$

$$\Rightarrow \tan x + 1 = 2 - 2\tan x$$

$$\Rightarrow 3\tan x = 1$$

$$\Rightarrow \tan x = \frac{1}{3}$$

Therefore, $\tan x = \frac{1}{3}$.

Exercise 7.3

1. Find the values of $\sin(x+y)$, $\cos(x+y)$, and $\tan(x+y)$ for each of the following:
   (a) $\sin x = \frac{3}{5}$, $\sin y = \frac{5}{13}$ where x and y lie in the first quadrant.
   (b) $\cos x = -\frac{12}{13}$, $\cot y = \frac{24}{7}$ where x lies in the second quadrant and y lies in the third quadrant.

2. Find the values of $\sin(A-B)$, cos (A–B), and tan (A–B) given:
   (a) $\cos A = -\frac{12}{13}$, $\cot B = \frac{24}{7}$, where A lies in the second quadrant and B lies in the first quadrant.
   (b) $\sin A = \frac{3}{5}$, $\sin B = -\frac{5}{13}$, where A lies in the first quadrant and B lies in the fourth quadrant.

3. If $\tan x - \tan y = m$ and $\cot y - \cot x = n$, prove that $\cot(x-y) = \frac{1}{m} + \frac{1}{n}$.

4. Prove each of the following identities:
   (a) $\tan(45° - \theta) = \frac{1-\tan\theta}{1+\tan\theta}$
   (b) $\frac{\sin(x+y)}{\cos(x-y)} = \frac{\tan x + \tan y}{1 + \tan x \tan y}$
   (c) $\frac{\tan(\alpha-\beta)}{\cot(\alpha+\beta)} = \frac{\tan^2\alpha + \tan^2\beta}{1 - \tan^2\alpha\tan^2\beta}$
   (d) $\cot(\alpha+\beta) = \frac{\cot\alpha\cot\beta - 1}{\cot\alpha + \cot\beta}$

5. Given that A, B, and C are angles of a triangle, prove that:
   (a) $\cot\left(\frac{B+C}{2}\right)=\tan\frac{A}{2}$
   (b) $\tan B\tan C=2$, if $\cos A=\cos B\cos C$
   (c) $\sin A\sin(A+2C)+\sin B\sin(B+2A)+\sin C\sin(C+2B)=0$
6. Solve each of the following equations for values of x between $0°$ and $360°$
   (a) $\cos(x+45°)=\cos x$
   (b) $\sin(x-30°)=\frac{1}{2}\cos x$
   (c) $3\sin(x+10°)=4\cos(x-10°)$
7. Given that $\sin(x+\beta)=2\cos(x-\beta)$. Prove that, $\tan x=\frac{2-\tan\beta}{1-2\tan\beta}$.
8. Evaluate each of the following expressions (Leave your answers in surd form).
   (a) $\sin 47°\cos 13°+\cos 47°\sin 13°$
   (b) $\frac{\tan 75°-\tan 15°}{1+\tan 75°\tan 15°}$
   (c) $\sin 75°\cos 300°+\cos 1470°\sin(-1020°)$.
9. If $\cot A\cot B=2$, show that $\frac{\cos(A-B)}{\cos(A+B)}=3$.
10. Given that $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$. If $B=45°$, show that,
   $\tan(A-B)=\frac{\tan^2 A-2\tan A+1}{\tan^2 A-1}$.

Double angle formulae

Double angle formulae are trigonometric identities which express trigonometric functions of 2θ in terms of trigonometric functions of θ. They are special cases of compound angle formulae.

The compound angle formulae under addition of two angles can be used to deduce the double angle formula.

Considering equation (7.4), $\sin(A+B)=\sin A\cos B+\cos A\sin B$

Replacing B by A, the double angle formula for sine of an angle is obtain as follows:

$\sin(A+A)=\sin A\cos A+\cos A\sin A$

Thus, $\sin 2A = 2\sin A\cos A$

Therefore, the double angle formula for sine is $\sin 2A = 2\sin A\cos A$.

From equation (7.5), $\cos(A+B) = \cos A\cos B - \sin A\sin B$.

Replacing B by A, gives the double angle formula for cosine of an angle, that is,

$\cos(A+A) = \cos A\cos A - \sin A\sin A$

$\cos 2A = \cos^2 A - \sin^2 A$, where $\sin^2 A = 1-\cos^2 A$

$$= \cos^2 A - (1-\cos^2 A)$$

$$= 2\cos^2 A - 1$$

Therefore, the double angle formula for cosine is,

$\cos 2A = 2\cos^2 A - 1$

But $\cos^2 A = 1-\sin^2 A$, thus,

$\cos 2A = \cos^2 A - \sin^2 A$

$$= 1-\sin^2 A - \sin^2 A$$

$$= 1-2\sin^2 A$$

Therefore, the double angle formula for cosine involving sine is,

$\cos 2A = 1-2\sin^2 A$.

Considering equation (7.6),

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}.$$

Replacing B by A, gives the double angle formula for tangent of an angle is, that is.

$$\tan(A+A) = \frac{\tan A + \tan A}{1-\tan A\tan A}$$

$$= \frac{2\tan A}{1-\tan^2 A}$$

Therefore, the double angle formula for tangent of an angle is,

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}.$$

The double angle formulae can be summarized as follows;

(i) $\sin 2A = 2\sin A\cos A$

(ii) $\cos 2A = 1 - 2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A$

(iii) $\tan 2A = \dfrac{2\tan A}{1-\tan^2 A}$

Example 7.17

Express $\sin 3\alpha$ in terms of $\sin\alpha$.

Solution

Let $\sin 3\alpha = \sin(2\alpha + \alpha)$

$$\begin{aligned}
&= \sin 2\alpha\cos\alpha + \cos 2\alpha\sin\alpha \\
&= (2\sin\alpha\cos\alpha)\cos\alpha + (1-2\sin^2\alpha)\sin\alpha \\
&= 2\sin\alpha\cos^2\alpha + \sin\alpha - 2\sin^3\alpha \\
&= 2\sin\alpha(1-\sin^2\alpha) + \sin\alpha - 2\sin^3\alpha \\
&= 2\sin\alpha - 2\sin^3\alpha + \sin\alpha - 2\sin^3\alpha \\
&= 3\sin\alpha - 4\sin^3\alpha
\end{aligned}$$

Therefore, $\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha$.

Example 7.18

Prove that $\cos 2x = \cos^4 x - \sin^4 x$.

Solution

Proving from the right-hand side.

$\cos^4 x - \sin^4 x = (\cos^2 x - \sin^2 x)(\cos^2 x + \sin^2 x)$, the difference of two squares

$$\begin{aligned}
&= \cos^2 x - \sin^2 x, \text{since } \cos^2 x + \sin^2 x = 1 \\
&= \cos 2x
\end{aligned}$$

Therefore, $\cos 2x = \cos^4 x - \sin^4 x$.

Example 7.19

Prove that $2\tan 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x}$.

Solution

Consider the right-hand side of the given equation;

$$\frac{\cos x + \sin x}{\cos x - \sin x} - \frac{\cos x - \sin x}{\cos x + \sin x} = \frac{(\cos x + \sin x)^2 - (\cos x - \sin x)^2}{(\cos x + \sin x)(\cos x - \sin x)}$$

$$= \frac{(\cos^2 x + 2\cos x \sin x + \sin^2 x) - (\cos^2 x - 2\cos x \sin x + \sin^2 x)}{\cos^2 x - \sin^2 x}$$

$$= \frac{4\cos x \sin x}{\cos^2 x - \sin^2 x}$$

$$= \frac{2(2\cos x \sin x)}{\cos 2x} = \frac{2\sin 2x}{\cos 2x}$$

$$= 2\tan 2x$$

Therefore, $2\tan 2x = \dfrac{\cos x + \sin x}{\cos x - \sin x} - \dfrac{\cos x - \sin x}{\cos x + \sin x}$.

Example 7.20

Eliminate λ from the following equations:

$x = 3 + 4\sin\lambda$ and $y = 5\cos 4\lambda$

Solution

Given,

$x = 3 + 4\sin\lambda$ and $y = 5\cos 4\lambda$

Consider, $x = 3 + 4\sin\lambda$

Make $\sin\lambda$ the subject to obtain,

$$\sin\lambda = \frac{x-3}{4}$$

From, $y = 5\cos 4\lambda$, write $\cos 4\lambda$ in terms of $\sin\lambda$, that is,

$$\begin{aligned}\cos 4\lambda &= 1 - 2\sin^2 2\lambda \\ &= 1 - 2(2\sin\lambda\cos\lambda)^2 \\ &= 1 - 2(4\sin^2\lambda\cos^2\lambda) \\ &= 1 - 8\sin^2\lambda(1 - \sin^2\lambda)\end{aligned}$$

Thus, $y = 5(1 - 8\sin^2\lambda(1 - \sin^2\lambda))$

But, $\sin\lambda = \dfrac{x-3}{4}$ then,

$$y = 5\left(1 - 8\left(\frac{x-3}{4}\right)^2\left(1 - \left(\frac{x-3}{4}\right)^2\right)\right)$$

Therefore,

$$y = 5 - 40\left(\frac{x-3}{4}\right)^2\left(1 - \left(\frac{x-3}{4}\right)^2\right).$$

Example 7.21

Simplify $\dfrac{\sin 2\theta}{1+\cos 2\theta}$.

Solution

$$\frac{\sin 2\theta}{1+\cos 2\theta} = \frac{2\sin\theta\cos\theta}{1+\left(2\cos^2\theta-1\right)}$$

$$= \frac{2\sin\theta\cos\theta}{2\cos^2\theta}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \tan\theta$$

Therefore, $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$.

Example 7.22

Solve the equation

$2\sin 2x + 2\sin x + 2\cos x + 1 = 0$

for $0 \le x \le 2\pi$.

Solution

Given,

$2\sin 2x + 2\sin x + 2\cos x + 1 = 0$,

but, $\sin 2x = 2\sin x\cos x$

$\Rightarrow 2\left(2\sin x\cos x\right) + 2\sin x + 2\cos x + 1 = 0$

$\Rightarrow 4\sin x\cos x + 2\sin x + 2\cos x + 1 = 0$

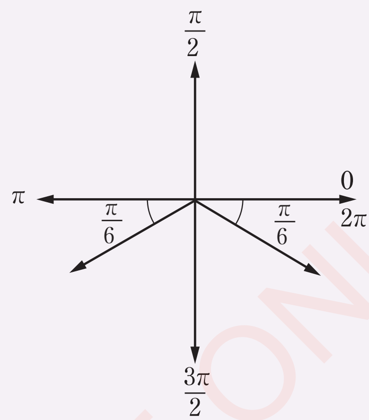
$\Rightarrow 2\sin x(2\cos x + 1) + 2\cos x + 1 = 0$

$\Rightarrow (2\cos x + 1)(2\sin x + 1) = 0$

$\Rightarrow 2\cos x + 1 = 0$ or $2\sin x + 1 = 0$

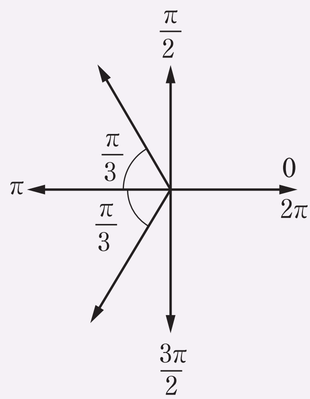
$\Rightarrow \cos x = -\dfrac{1}{2}$ or $\sin x = -\dfrac{1}{2}$

The following sketch shows the values of x satisfying $\sin x = -\dfrac{1}{2}$ in the third and fourth quadrants.



From the sketch, $x = -\dfrac{\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \ldots$

The following sketch shows the values of x satisfying $\cos x = -\dfrac{1}{2}$ in the second and third quadrants.



From the sketch, $x = -\dfrac{2\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \ldots$

Therefore, the values of x are $\dfrac{2\pi}{3}, \dfrac{7\pi}{6}, \dfrac{4\pi}{3}$, and $\dfrac{11\pi}{6}$, for $0 \le x \le 2\pi$.

Half angle formulae

Half angle formulae are special trigonometric identities which are derived from the double angle formulae. These are useful in evaluating trigonometrical ratios of half angles and deriving other identities which involve half angles.

Consider the double angle formulae for sine and cosine of an angle.

$\sin 2A = 2\sin A\cos A$ and

$\cos 2A = 1-2\sin^2 A = 2\cos^2 A - 1 = \cos^2 A - \sin^2 A$

Then, the half angle formulae will be obtained by dividing the angle A by 2 to obtain;

(i) $\sin\frac{2A}{2} = 2\sin\frac{A}{2}\cos\frac{A}{2} \Rightarrow \sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos\frac{2A}{2} = 1-2\sin^2\frac{A}{2} \Rightarrow \cos A = 1-2\sin^2\frac{A}{2}$

(iii) $\cos\frac{2A}{2} = 2\cos^2\frac{A}{2}-1 \Rightarrow \cos A = 2\cos^2\frac{A}{2}-1$

(iv) $\cos\frac{2A}{2} = \cos^2\frac{A}{2}-\sin^2\frac{A}{2} \Rightarrow \cos A = \cos^2\frac{A}{2}-\sin^2\frac{A}{2}$

Similarly, the half angle formulae for tangent of an angle will be obtained from its double angle formula as follows;

$$\tan 2A = \frac{2\tan A}{1-\tan^2 A}$$

Divide the angle A by 2 gives,

$$\tan\frac{2A}{2} = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}} \Rightarrow \tan A = \frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$$

Hence, the half angle formula can be summarized as follows;

(i) $\sin A = 2\sin\frac{A}{2}\cos\frac{A}{2}$

(ii) $\cos A = 1-2\sin^2\frac{A}{2} = 2\cos^2\frac{A}{2}-1 = \cos^2\frac{A}{2}-\sin^2\frac{A}{2}$

(iii) $\tan A = \dfrac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}$

Example 7.23

Simplify $\cos^2\left(\frac{45}{2}\right)-\sin^2\left(\frac{45}{2}\right)$ using half angle formula leaving your answer in surd form.

Solution

Given $\cos^2\left(\frac{45}{2}\right)-\sin^2\left(\frac{45}{2}\right)$.

Using, $\cos A=\cos^2\frac{A}{2}-\sin^2\frac{A}{2}$

Comparing the two expressions,

$$\cos^2\frac{A}{2}-\sin^2\frac{A}{2}=\cos^2\left(\frac{45}{2}\right)-\sin^2\left(\frac{45}{2}\right)$$

It can be noticed that,

$$\frac{A}{2}=\left(\frac{45}{2}\right)\Rightarrow A=2\times\left(\frac{45}{2}\right)=45$$

$$\cos^2\left(\frac{45}{2}\right)-\sin^2\left(\frac{45}{2}\right)=\cos 45=\frac{\sqrt{2}}{2}$$

Therefore, $\cos^2\left(\frac{45}{2}\right)-\sin^2\left(\frac{45}{2}\right)=\frac{\sqrt{2}}{2}$.

Example 7.24

Prove that $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$.

Solution

To prove that $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$,

Consider the right-hand side of the equation,

$$\Rightarrow\frac{\sin\theta}{1+\cos\theta}=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1+\left(2\cos^2\frac{\theta}{2}-1\right)}$$

$$=\frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}$$

$$=\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}=\tan\frac{\theta}{2}$$

Therefore, $\tan\frac{\theta}{2}=\frac{\sin\theta}{1+\cos\theta}$.

Example 7.25

Show that $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Solution

To show that $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Consider the left-hand side of the equation.

$$\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\frac{(1+\cos A)+\sin A}{(1-\cos A)+\sin A}$$

$$=\frac{2\cos^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}{2\sin^2\frac{A}{2}+2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{2\cos\frac{A}{2}\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)}{2\sin\frac{A}{2}\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)}$$

$$=\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}}=\cot\frac{A}{2}$$

Therefore, $\frac{1+\sin A+\cos A}{1+\sin A-\cos A}=\cot\frac{A}{2}$.

Exercise 7.4

1. Prove each of the following identies

(a) $\cot\theta = \sqrt{\dfrac{1+\cos 2\theta}{1-\cos 2\theta}}$

(b) $\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

(c) $\cos 6\theta = 1-2\sin^2 3\theta$

(d) $\sin^2\dfrac{1}{2}\theta = \dfrac{1}{2}(1-\cos\theta)$

(e) $\tan 4\theta = \dfrac{4\tan\theta(1-\tan\theta)(1+\tan\theta)}{1-6\tan^2\theta+\tan^4\theta}$

(f) $\sin\theta = \dfrac{2\tan\dfrac{\theta}{2}}{1+\tan^2\dfrac{\theta}{2}}$

(g) $\cot 15° = \sqrt{\dfrac{2+\sqrt{3}}{2-\sqrt{3}}}$

2. Prove each of the following identities:

(a) $1-\dfrac{1}{2}\sin 2x = \dfrac{\sin^3 x+\cos^3 x}{\sin x+\cos x}$

(b) $\cos x = \dfrac{1-\tan^2\dfrac{1}{2}x}{1+\tan^2\dfrac{1}{2}x}$

(c) $\dfrac{1+\sin C+\cos C}{1+\sin C-\cos C} = \cot\dfrac{C}{2}$

(d) $\dfrac{1-\sin 2A}{\cos 2A} = \dfrac{1-\tan A}{1+\tan A}$

3. Simplify each of the following expressions:

(a) $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A}$

(b) $\dfrac{1+\cos 2\theta}{\sin 2\theta}$

(c) $\left(\sin\dfrac{1}{2}x-\cos\dfrac{1}{2}x\right)^2$

(d) $\operatorname{cosec} 2x+\cot 2x$

4. If $2A+B=45°$ show that

$\tan B = \dfrac{1-2\tan A-\tan^2 A}{1+2\tan A-\tan^2 A}$

5. Eliminate θ in each of the following pair of equations:

(a) $x+1=\cos 2\theta,\ y=\sin\theta$

(b) $x=\cos 2\theta,\ y=\cos\theta-1$

(c) $x=\tan\theta,\ y=\tan 4\theta$

(d) $x=4-3\sin 2\theta,\ y=7-9\cos 2\theta$

6. Solve each of the following equations for $0° \le x \le 360°$.

(a) $2\sin x\cos x = \cos 2x$

(b) $3\cot 2x+\cot x-1=0$

(c) $2\cos 2x+13\sin x=5$

(d) $4\sin x(7\cos x-3)=5\sin 2x$

7. In a triangle ABC, prove each of the following:

   (a) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$

   (b) $\cos A + \cos B + \cos C - 1 = 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$

   (c) If $\cos(A+B) = \cos A + \cos B$, then $\cos(A+B) = \cos A + \cos B + \cos C = 0$.

8. Find the values of $\tan 2\beta$, $\sin 2\beta$, and $\sec 2\beta$, for each of the following cases when β is in the fourth quadrant:

   (a) $\sin\beta = -\frac{24}{25}$ (b) $\cot\beta = -\frac{120}{119}$

9. If $t = \tan\lambda$, show that $\tan 4\lambda = \frac{4t - 4t^3}{t^4 - 6t^2 + 1}$. Hence verify that, $t^4 + 4t^3 - 6t^2 - 4t + 1 = 0$ when $\tan\lambda = 1$.

10. Evaluate each of the following and leave your answers in surd form:

    (a) $\frac{2\tan 75°}{1 - \tan^2 75°}$ (c) $2\sin 165°\cos 165°$

    (b) $\frac{1 - \tan^2 15°}{1 + \tan^2 15°}$ (d) $\sin 7.5°$

11. Without using mathematical tables or scientific calculators evaluate each of the following:

    (a) $\frac{2\tan 22.5°}{1 - \tan^2 22.5°}$ (b) $1 - 2\sin^2 67.5°$ (c) $\sec 22.5° \operatorname{cosec} 22.5°$

12. Prove each of the following trigonometric identities:

    (a) $\sqrt{\frac{1+\sin\theta}{1-\sin\theta}} = \tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right)$ (c) $\frac{1+\cos\theta}{\sin\theta} = \cot\frac{\theta}{2}$

    (b) $\left(\cos\frac{B}{2} + \sin\frac{B}{2}\right)^2 = 1 + \sin B$ (d) $\tan\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$

Trigonometric equations of the form $a\cos\theta + b\sin\theta = c$

Trigonometric equations of the form $a\cos\theta + b\sin\theta = c$ where a, b, and c are real numbers can be solved by using t-formulae or by expressing it in the form of $R\cos(\theta \pm \alpha) = c$ or $R\sin(\theta \pm \alpha) = c$.

Using t-formulae in solving equations of the form *acosθ + bsinθ = c*

Recall the double angle formulae,

(i) $\sin 2x = 2\sin x\cos x$

(ii) $\cos 2x = \cos^2 x - \sin^2 x$

Both equations for $\sin 2x$ and $\cos 2x$ can be expressed in terms of $\tan\theta$.

From,

$$\sin 2x = 2\sin x\cos x$$

$$= 2\sin x\cos x \times \frac{\cos x}{\cos x}$$

$$= \frac{2\sin x}{\cos x}\cos^2 x$$

$$= 2\tan x\frac{1}{\sec^2 x}, \text{ since } \sec x = \frac{1}{\cos x}$$

$$= \frac{2\tan x}{1+\tan^2 x} \text{ since } \sec^2 x = 1+\tan^2 x$$

Therefore, $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x}$. (7.7)

Now, let $\theta = 2x$ then $x = \dfrac{\theta}{2}$, equation (7.7) becomes,

$$\sin\theta = \frac{2\tan\frac{1}{2}\theta}{1+\tan^2\frac{1}{2}\theta}$$

$$\Rightarrow \sin\theta = \frac{2t}{1+t^2}, \text{ where } t = \tan\frac{1}{2}\theta$$

This is called the t – formula for $\sin\theta$.

Activity 7.2: Deducing the *t*-formulae for cos*θ* and tan*θ*

Individually or in a group, perform the following tasks:

1. Express $\cos 2x$ and $\tan 2x$ in terms of $\tan x$.
2. Use the concept applied in the derivation of t – formula for $\sin\theta$ to deduce the corresponding *t*-formulae for $\cos\theta$ and $\tan\theta$.
3. Give a suggestion for the *t*-formulae for $\cos\theta$ and $\tan\theta$.
4. What have you observed in task 3? Give comments.

The relationship between the tangent, sine, and cosine of an angle in terms of tangent of half angle can be obtained using a right-angled triangle as shown in Figure 7.4.

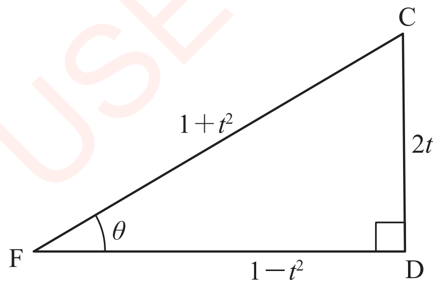


Figure 7.4: A *right-angled triangle DCF*

From Figure 7.4, the *t*-formulae of the trigonometric ratios are;

(a) $\sin\theta = \dfrac{2t}{1+t^2}$

(b) $\cos\theta = \dfrac{1-t^2}{1+t^2}$

(c) $\tan\theta = \dfrac{2t}{1-t^2}$

Example 7.26

Solve the equation $5\cos\theta - 2\sin\theta = 2$ for $0° \leq \theta \leq 360°$ by using the substitution $t = \tan\frac{1}{2}\theta$.

Solution

Solving $5\cos\theta - 2\sin\theta = 2$.

From the t-formulae, $\cos\theta = \dfrac{1-t^2}{1+t^2}$ and $\sin\theta = \dfrac{2t}{1+t^2}$, where $t = \tan\frac{1}{2}\theta$.

$$\Rightarrow 5\left(\frac{1-t^2}{1+t^2}\right) - 2\left(\frac{2t}{1+t^2}\right) = 2$$

$$\Rightarrow 5\left(1-t^2\right) - 4t = 2\left(1+t^2\right)$$

$$\Rightarrow 5 - 5t^2 - 4t = 2 + 2t^2$$

$$\Rightarrow 7t^2 + 4t - 3 = 0$$

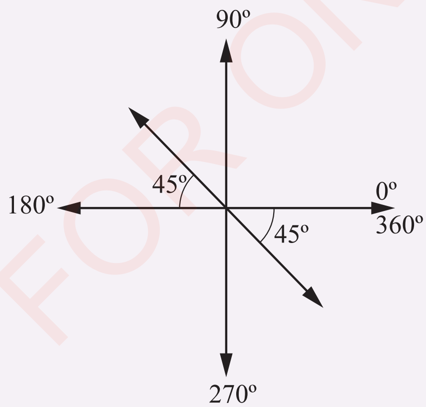
$$\Rightarrow t = -1 \text{ or } t = \frac{3}{7}$$

Hence, $\tan\frac{1}{2}\theta = -1$ or $\tan\frac{1}{2}\theta = \frac{3}{7}$

Now, $\tan\frac{1}{2}\theta = -1 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(-1)$

$$= -45°$$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = -1$ in the second and fourth quadrants:



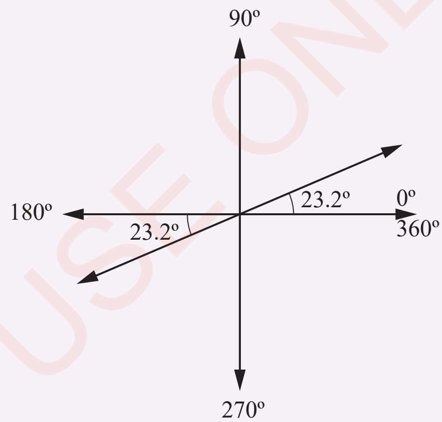
$$\frac{\theta}{2} = -45°, 135°, \ldots$$

$$\theta = -90°, 270°, \ldots$$

Now, $\tan\frac{1}{2}\theta = \frac{3}{7} \Rightarrow \frac{1}{2}\theta = \tan^{-1}\left(\frac{3}{7}\right)$

$$\Rightarrow \frac{1}{2}\theta = 23.2°$$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = \frac{3}{7}$ in the first and third quadrants:



$$\frac{\theta}{2} = -156.8°, 23.2°, \ldots$$

$$\theta = -313.6°, 46.4°, \ldots$$

Therefore, the required solutions are $46.4°$ and $270°$, for $0 \leq \theta \leq 360°$.

Example 7.27

Solve the equation

$15\cos 2x + 2\sin 2x = 10$ for

$-180° \leq \theta \leq 180°$.

Solution

Given $15\cos 2x + 2\sin 2x = 10$.

Substituting $\cos 2x = \frac{1-t^2}{1+t^2}$ and $\sin 2x = \frac{2t}{1+t^2}$, where $t = \tan x$ gives;

$$15\left(\frac{1-t^2}{1+t^2}\right) + 2\left(\frac{2t}{1+t^2}\right) = 10$$

Multiplying by $1+t^2$ on both sides,

$$15(1-t^2) + 4t = 10(1+t^2)$$

$$\Rightarrow 15 - 15t^2 + 4t = 10 + 10t^2$$

$$\Rightarrow 25t^2 - 4t - 5 = 0$$

Solving the quadratic equation to obtain,

$t = 0.5343$ or -0.3743.

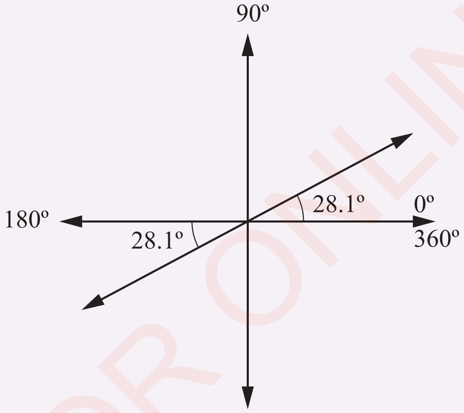
Hence,

$\tan x = 0.5343$ or $\tan x = -0.3743$.

Now, $\tan x = 0.5343 \Rightarrow x = \tan^{-1}(0.5343)$

$$x = 28.1°$$

The following sketch illustrates the values of x satisfying $\tan x = 0.5343$ in the first and third quadrants:

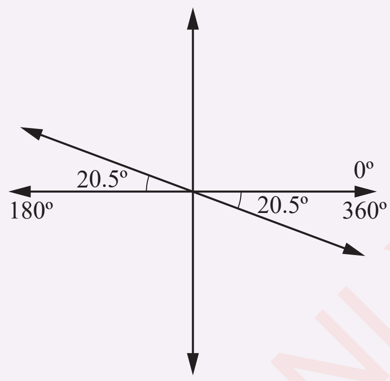


$x = -151.9°, 28.1°, \ldots$

Now, $\tan x = -0.3743 \Rightarrow x = \tan^{-1}(-0.3743)$

$$x = -20.5°$$

The following sketch illustrates the values of x satisfying $\tan x = -0.3743$ in the second and fourth quadrants:



$x = -20.5°, 159.5°, \ldots$

Therefore, the solution are $-151.90°$, $-20.5°$, $28.1°$, and $159.5°$, for $-180° \le \theta \le 180°$.

Example 7.28

Solve the equation $3\cos\theta + 5\sin\theta = 2$ for $-360° \le \theta \le 360°$.

Solution

Substituting $t = \tan\frac{1}{2}\theta$, where $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\sin\theta = \frac{2t}{1+t^2}$ to get;

$$3\left(\frac{1-t^2}{1+t^2}\right) + 5\left(\frac{2t}{1+t^2}\right) = 2$$

Multiply both sides by $1+t^2$ to get;

$$3(1-t^2) + 10t = 2(1+t^2)$$

$$\Rightarrow 3 - 3t^2 + 10t = 2 + 2t^2$$

$$\Rightarrow 5t^2 - 10t - 1 = 0$$

Solving the quadratic equation gives;

$t = -0.0954$ or 2.0954

Hence, $\tan\frac{1}{2}\theta = -0.0954$ or

$\tan\frac{1}{2}\theta = 2.0954.$

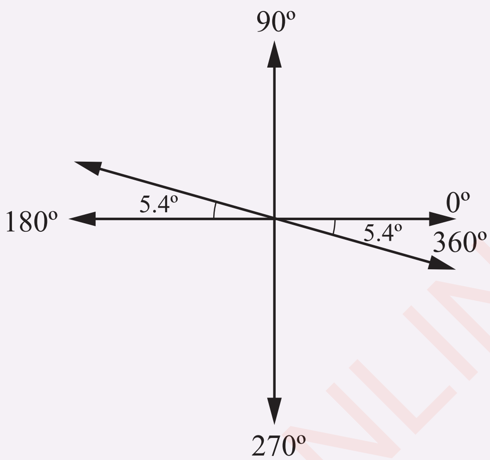
Now,

$\tan\frac{1}{2}\theta = -0.0954 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(-0.0954)$

$\Rightarrow \frac{1}{2}\theta = -5.4°$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = -0.0954$ in the second and fourth quadrants:

$\frac{1}{2}\theta = \tan^{-1}(-0.0954) = -5.4°$



$\frac{1}{2}\theta = -5.4°, 174.6°, ...$

$\theta = -10.8°, 349.2°, ...$

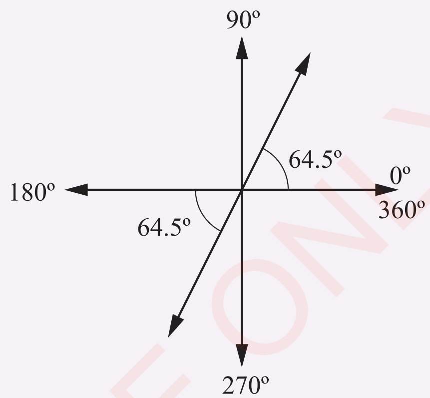
Now,

$\tan\frac{1}{2}\theta = 2.0954 \Rightarrow \frac{1}{2}\theta = \tan^{-1}(2.0954)$

$\frac{1}{2}\theta = 64.5°$

The following sketch illustrates the values of θ satisfying $\tan\frac{1}{2}\theta = 2.0954$ in the first and third quadrants:

$\frac{1}{2}\theta = \tan^{-1}(2.0954) = 64.5°$



$\frac{1}{2}\theta = -115.5°, 64.5°, ...$

$\theta = -231°, 129°, ...$

Therefore, the required solutions are $\theta = -231°, -10.8°, 129°$, and $349.2°$.

Expressing $a\cos\theta + b\sin\theta$ in the form $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$

The technique of expressing $a\cos\theta + b\sin\theta$ in the form of $R\cos(\theta \pm \alpha)$ or $R\sin(\theta \pm \alpha)$ is useful in finding maximum and minimum values of trigonometric equations of the form $a\cos\theta + b\sin\theta = c$.

Expressing $a\cos\theta + b\sin\theta$ in the form of $R\cos(\theta - \alpha)$

Using the compound angle formula for $R\cos(\theta - \alpha)$ gives;

$R\cos(\theta - \alpha) \equiv a\cos\theta + b\sin\theta \equiv R\cos\theta\cos\alpha + R\sin\theta\sin\alpha.$

Comparing the coefficients of $\cos\theta$ gives;

$a = \mathrm{R}\cos\alpha$ .... (i)

Comparing the coefficients of $\sin\theta$ gives;

$b = \mathrm{R}\sin\alpha$ .... (ii)

Squaring and adding equations (i) and (ii), gives;

$$\begin{aligned} a^2 + b^2 &= \mathrm{R}^2\cos^2\alpha + \mathrm{R}^2\sin^2\alpha \\ &= \mathrm{R}^2(\cos^2\alpha + \sin^2\alpha) \\ &= \mathrm{R}^2 \end{aligned}$$

Now, $\mathrm{R} = \sqrt{a^2 + b^2}$; where the value of R is taken as being positive.

The value of α is obtained by dividing equation (i) by (ii) as follows;

$$\frac{\mathrm{R}\sin\alpha}{\mathrm{R}\cos\alpha} = \frac{b}{a}$$

Thus, $\tan\alpha = \frac{b}{a} \Rightarrow \alpha = \tan^{-1}\left(\frac{b}{a}\right)$.

Therefore, $a\cos\theta + b\sin\theta = \sqrt{a^2 + b^2}\cos\left(\theta - \tan^{-1}\left(\frac{b}{a}\right)\right)$.

Note that, both $\cos\alpha = \frac{a}{\mathrm{R}}$ and $\sin\alpha = \frac{b}{\mathrm{R}}$ give the same value of α. If they do not, use the value of α which is not between $0°$ and $90°$.

Example 7.29

Express $3\cos\theta - 4\sin\theta$ in the form of $\mathrm{R}\cos(\theta - \alpha)$.

Solution

Let $3\cos\theta - 4\sin\theta = \mathrm{R}\cos(\theta - \alpha)$

$$= \mathrm{R}\cos\theta\cos\alpha + \mathrm{R}\sin\theta\sin\alpha$$

Compare the coefficients of $\sin\theta$ and $\cos\theta$ to obtain the following pair of equations;

$$\begin{cases} \mathrm{R}\cos\alpha = 3 \\ \mathrm{R}\sin\alpha = -4 \end{cases}$$

But $\mathrm{R} = \sqrt{a^2 + b^2} = \sqrt{3^2 + (-4)^2} = 5$

Thus, $\cos\alpha = \frac{3}{5}$ and $\sin\alpha = -\frac{4}{5}$

$\Rightarrow \alpha = 53.1°$ and $\alpha = -53.1°$

Hence, $\alpha = -53.1°$ as it is not between $0°$ and $90°$.

Therefore,
$3\cos\theta - 4\sin\theta = 5\cos(\theta + 53.1°)$.

Example 7.30

Show that $5\cos\theta - 12\sin\theta = 5$ can be expressed in the form $13\cos(\theta + 67.4°) = 5$.

Solution

Given $5\cos\theta - 12\sin\theta = 5$.
Consider the left-hand side, then using
$5\cos\theta - 12\sin\theta = R\cos(\theta + \alpha)$

$\Rightarrow 5\cos\theta - 12\sin\theta$
$= R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$.
Compare the coefficients of $\sin\theta$ and $\cos\theta$ to obtain;
$R\cos\alpha = 5$ .... (i)
$R\sin\alpha = 12$ ....(ii)
But,

$$R = \sqrt{a^2 + b^2} = \sqrt{5^2 + (-12)^2} = 13$$

Dividing equation (ii) by equation (i), gives;

$$\tan\alpha = \frac{12}{5} \Rightarrow \alpha = 67.4°$$

Therefore,
$5\cos\theta - 12\sin\theta = 13\cos(\theta + 67.4°) = 5$.

Expressing $a\cos\theta + b\sin\theta$ in the form $R\sin(\theta + \alpha)$

Using compound angle formula for $R\sin(\theta + \alpha)$ gives;
$a\cos\theta + b\sin\theta = R\cos\theta\sin\alpha + R\sin\theta\cos\alpha$

Comparing the coefficients of $\sin\theta$ gives:

$b = R\cos\alpha$.....................................(i)

Comparing the coefficients of $\cos\theta$ gives:

$a = R\sin\alpha$ .... (ii)

Squaring and adding equations (i) and (ii), gives;

$$a^2 + b^2 = R^2\sin^2\alpha + R^2\cos^2\alpha$$

$\Rightarrow a^2 + b^2 = R^2(\sin^2\alpha + \cos^2\alpha)$,
Thus, $a^2 + b^2 = R^2$,

$\Rightarrow R = \sqrt{a^2 + b^2}$.

The value of α is obtained by dividing equation (ii) by equation (i) as follows;

$$\frac{R\sin\alpha}{R\cos\alpha} = \frac{a}{b}$$

Thus, $\tan\alpha = \dfrac{a}{b} \Rightarrow \alpha = \tan^{-1}\left(\dfrac{a}{b}\right)$.

Therefore, $a\cos\theta + b\sin\theta$

$$= \sqrt{a^2 + b^2}\sin\left(\theta + \tan^{-1}\left(\frac{a}{b}\right)\right).$$

Example 7.31

Express $7\cos\theta + 24\sin\theta$ in the form of $\mathrm{R}\sin(\theta+\alpha)$.

Solution

Given $7\cos\theta + 24\sin\theta$, then

$\mathrm{R} = \sqrt{a^2+b^2} = \sqrt{7^2+24^2} = 25$ and

$\tan\alpha = \frac{a}{b}$, where $a = 7$ and $b = 24$.

$\tan\alpha = \frac{7}{24} \Rightarrow \alpha = 16.3°$

Therefore,

$7\cos\theta + 24\sin\theta = 25\sin(\theta + 16.3°)$.

Note that, In order to avoid the problem of obtaining two different values of α, select the one among, $\mathrm{R}\sin(\theta+\alpha)$, $\mathrm{R}\sin(\theta-\alpha)$, $\mathrm{R}\cos(\theta+\alpha)$ and $\mathrm{R}\cos(\theta-\alpha)$ that the same sign as the expression $a\cos\theta + b\sin\theta$.

Example 7.32

Find the maximum and minimum values of $24\sin\theta - 7\cos\theta$, by stating the values of θ for which the maximum and minimum values are attained.

Solution

Given

$24\sin\theta - 7\cos\theta \Rightarrow 24\sin\theta - 7\cos\theta$

$= R\sin(\theta-\alpha)$.

$\Rightarrow \mathrm{R} = \sqrt{a^2+b^2} = \sqrt{24^2+(-7)^2} = 25$

Thus, $\tan\alpha = \frac{b}{a} = \frac{7}{24}$

$\alpha = \tan^{-1}\left(\frac{7}{24}\right) = 16.3°$

Hence,

$24\sin\theta - 7\cos\theta = 25\sin(\theta - 16.3°)$.

Note that; $\sin\theta$ has maximum value of $+1$ and minimum value of -1, that is, $-1 \le \sin\theta \le +1$.

Similarly, $-1 \le \sin(\theta - 16.3°) \le +1$ multiplying throughout by 25 gives,

$$-25 \le 25\sin(\theta - 16.3°) \le +25$$

At maximum value $\sin(\theta - 16.3°) = 1$ and minimum value $\sin(\theta - 16.3°) = -1$, then

$\Rightarrow \sin(\theta - 16.3°) = 1$ and

$\sin(\theta - 16.3°) = -1$

$\Rightarrow \theta - 16.3° = \sin^{-1}1$

$= 90°$

Thus, $\theta = 106.3°$

$\Rightarrow \theta - 16.3° = \sin^{-1}(-1)$

$= 270°$

Thus, $\theta = 286.3°$

Therefore, the maximum and minimum values of $24\sin\theta - 7\cos\theta$ are 25 and -25 which occur at $\theta = 106.3°$ and $286.3°$, respectively.

Example 7.33

Show that, $13\cos\lambda + 7\sin\lambda$ can be expressed in the form $\sqrt{218}\cos(\lambda-\alpha)$,where $\tan\alpha = \frac{7}{13}$. Hence, find the maximum and minimum values of the function, giving the corresponding values of λ for which the maximum and minimum values occur.

Solution

Given $13\cos\lambda + 7\sin\lambda$

$\Rightarrow 13\cos\lambda + 7\sin\lambda = R\cos(\lambda-\alpha)$

$13\cos\lambda + 7\sin\lambda = R(\cos\lambda\cos\alpha + \sin\lambda\sin\alpha)$

Comparing the coefficients of $\sin\lambda$ gives;

$7 = \text{R}\sin\alpha$ .... (i)

Comparing the coefficients of $\cos\lambda$ gives;

$13 = \text{R}\cos\alpha$ .... (ii)

Squaring and adding equations (i) and (ii) gives;

$R^2\cos^2\alpha + R^2\sin^2\alpha = 13^2 + 7^2$

$\Rightarrow R^2(\cos^2\alpha + \sin^2\alpha) = 169 + 49$

$\Rightarrow R^2 = 218 \Rightarrow R = \sqrt{218}$

Diving equation (i) by equation (ii), gives;

$\tan\alpha = \frac{7}{13} \Rightarrow \alpha = \tan^{-1}\left(\frac{7}{13}\right) = 28.3°$

Hence, $13\cos\lambda + 7\sin\lambda = \sqrt{218}\cos(\lambda - 28.3°)$

At maximum and minimum values $\cos(\lambda-28.3°)=1$ and $\cos(\lambda-28.3°)=-1$, respectively.

$\Rightarrow \sqrt{218}\cos(\lambda-28.3°) = \sqrt{218}$ and $\sqrt{218}\cos(\lambda-28.3°) = -\sqrt{218}$

$\Rightarrow \cos(\lambda-28.3°) = 1$ and $\cos(\lambda-28.3°) = -1$

$\Rightarrow \lambda - 28.3° = \cos^{-1}1$ and $\lambda - 28.3° = \cos^{-1}(-1)$

$\Rightarrow \lambda - 28.3° = 0$ and $\lambda - 28.3° = 180°$

$\Rightarrow \lambda = 28.3°$ and $\lambda = 208.3°$

Therefore, the maximum and minimum values of the function are $\sqrt{218}$ and $-\sqrt{218}$ and are attained at $\lambda = 28.3°$ and $208.3°$, respectively.

Example 7.34

Solve the equation $4\cos 2x + 3\sin 2x = \frac{5\sqrt{2}}{2}$, for the values in the interval $-180° \leq x \leq 180°$.

Solution

Solving $4\cos 2x + 3\sin 2x = \frac{5\sqrt{2}}{2}$.

Let $4\cos 2x + 3\sin 2x = R\sin(2x+\alpha)$, then

$4\cos 2x + 3\sin 2x = R\sin 2x\cos\alpha + R\cos 2x\sin\alpha$.

Comparing coefficients of $\sin 2x$ gives;

$R\cos\alpha = 3$ .... (i)

Comparing coefficients of $\cos 2x$ gives;

$R\sin\alpha = 4$ .... (ii)

Dividing equation (ii) by equation (i) results to;

$$\tan\alpha = \frac{4}{3} \Rightarrow \alpha = \tan^{-1}\left(\frac{4}{3}\right) = 53.1°$$

$$\Rightarrow R = \sqrt{a^2 + b^2} = \sqrt{4^2 + 3^2} = 5$$

Hence, $5\sin(2x + 53.1°) = \frac{5\sqrt{2}}{2}$

$$\Rightarrow \sin(2x + 53.1°) = \frac{\sqrt{2}}{2}$$

Applying $\sin^{-1}$ on both sides, gives

$$2x + 53.1° = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$\Rightarrow 2x + 53.1° = 45°$$

$$\Rightarrow 2x = 45° - 53.1°,\ 135° - 53.1°,\ 405° - 53.1°,\ -225° - 53.1°$$

$$\Rightarrow 2x = -8.1°,\ 81.9°,\ 351.9°,\ -278.1°$$

$$\Rightarrow x = -4.05°,\ 40.95°,\ 175.95°,\ \text{and } -139.05°$$

Therefore, the required solution are $x = -4.1°, -139.05°, 40.95°$, and $175.75°$, for $-180° \leq x \leq 180°$.

Example 7.35

Solve the equation $3\cos x + \sin x = 2$ for $0° \le x \le 360°$.

Solution

Let $3\cos x + \sin x = R\cos(x - \alpha)$ then,

$R = \sqrt{a^2 + b^2} = \sqrt{3^2 + 1^2} = \sqrt{10}$

$\tan\alpha = \frac{b}{a} = \frac{1}{3}$

$\alpha = \tan^{-1}\frac{1}{3}$

$\alpha = 18.43°$

The equation $3\cos x + \sin x = 2$ can be written as,

$\sqrt{10}\cos(x - 18.43°) = 2$

$\Rightarrow \cos(x - 18.43°) = \frac{2}{\sqrt{10}}$

$\Rightarrow x - 18.43° = \cos^{-1}\frac{2}{\sqrt{10}}$

$\Rightarrow x - 18.43 = 50.77°$ or $309.23°$

$\Rightarrow x = 69.2°$ or $327.7°$

Therefore, the solutions are $x = 69.2°$ and $327.7°$.

Exercise 7.5

1. Use the t-formulae to solve each of the following equations for $0° \le x \le 360°$:
   (a) $2\cos x + 3\sin x - 2 = 0$
   (b) $7\cos x + \sin x - 5 = 0$
   (c) $3\cos x - 4\sin x + 1 = 0$
   (d) $3\cos x + 4\sin x = 2$
   (e) $2\tan x + 2 = \sec x$
   (f) $2\cos x + 7\sin x = -3$
   (g) $32\cos x + 36\sin x = 29$
   (h) $7\cot x - 2\operatorname{cosec} x = 6$
   (i) $12\cos x - 5\sin x = -3$
   (j) $1 + 7\tan x = 5\sec x$

2. Find the values of each of the following equations for $-180° \le \theta \le 180°$.
   (a) $\sqrt{3}\cos\theta + \sin\theta = 1$
   (b) $3\tan\theta = 4 + 2\sec\theta$
   (c) $5\cos\theta - 12\sin\theta = 6$
   (d) $2\sin 2\theta + 15\cos 2\theta = 10$
   (e) $\cos\theta + \sin\theta = 0.5$
   (f) $\cos\theta(\cos\theta + \sin\theta) = 1$
   (g) $\cos\theta = 2 + 7\sin\theta$
   (h) $3\cos 4\theta - 2\sin 4\theta = 3$
   (i) $2\sin\theta + 7\cos\theta - 4 = 0$
   (j) $\operatorname{cosec}\theta - \sin\theta = 0.5$

3. Find the maximum and minimum values of the following expressions, stating the values of angle θ from 0° to 360° inclusive, for which the maximum and minimum values occur.
   (a) $\cos\theta + \sin\theta$
   (b) $\cos(\theta + 30°) - \cos\theta$
   (c) $4\sin\theta - 3\cos\theta$
   (d) $\sqrt{2}\cos(\theta + 60°) + 7\sin\theta$
   (e) $3\sin\theta + \cos\theta$
   (f) $3\sin\theta + 4\cos\theta$
   (g) $8\cos\theta - 15\sin\theta$
   (h) $3\sin\theta + 5\cos\theta$

(i) $6\sin\theta - \cos\theta$

(j) $\dfrac{1}{\cot\theta + \tan\theta}$

4. Show that $3\cos\theta + 2\sin\theta$ can be expressed in the form $\sqrt{13}\cos(\theta - \alpha)$, where $\tan\alpha = \frac{2}{3}$. Hence, find the maximum and minimum values of $3\cos\theta + 2\sin\theta$ by calculating the corresponding values of θ in the range $-180° \leq \theta \leq 180°$.

5. Express $\cos\theta + 2\sin\theta$ in the form $R\sin(\theta + \alpha)$, where α is an acute angle. For what values of θ in the range $-180° \leq \theta \leq 180°$ does the expression $\cos\theta + 2\sin\theta$ is at maximum and minimum? State the maximum and minimum values of $\cos\theta + 2\sin\theta$.

General solutions

A general solution is a solution that contains all possible solutions of a given trigonometric equation.

Consider the graph of the straight lines $y = a$, where $-1 \leq a \leq 1$ and $y = f(\theta) = \cos\theta$ as shown in Figure 7.5

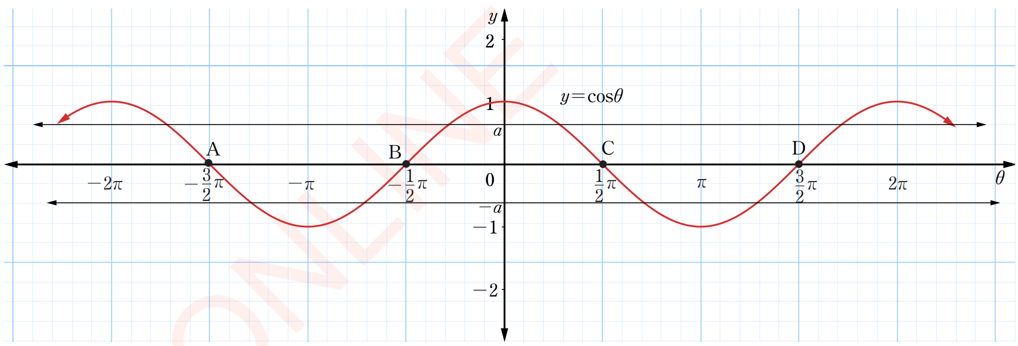


Figure 7.5: *Graph of* $f(\theta) = \cos\theta$

The principal value of the equation $\alpha = \cos\theta$ is $= \cos^{-1} a$ (which is the intercept of the line $y = a$ and the curve $y = \cos\theta$ as shown in Figure 7.5). Since the cosine function is periodic with period of 360°, then other solutions of the equation $\cos\theta = a$ corresponding to the principal value are obtained by adding, or subtracting a multiple of 360° to it (the points of intersection in the θ–axis, that is, A, B, C, and D for $-2\pi \leq \theta \leq 2\pi$ as illustrated in Figure 7. 5). If α is the principal value of θ for which $\cos\theta = a$, then $-\alpha$ is also a solution, and is not obtained by adding or subtracting a multiple of $360°$. All the remaining solutions are obtained by adding or subtracting multiples of $360°$ to or from α. The general solution of the equation $\cos\theta = a$ where $-1 \le a \le 1$ is given as; $\theta = 360°n \pm \alpha$ where $\alpha = \cos^{-1} a$ is a principal value and n is an integer, positive or negative.

Activity 7.3: Finding the general solution of a trigonometric equation

Learning resources: Graph papers, ruler, pencil, and scientific calculator.
Individually or in a group, perform the following tasks:

1. Draw the graph of $y = \sin\theta$ and $y = \tan\theta$ for $-720° \le \theta \le 720°$.
2. Use the graph of $y = \sin\theta$ and $y = \tan\theta$ to find the general solution for the equations $\sin\theta = a$ for $-1 \le a \le 1$ and $\tan\theta = a$ for $-\infty < a < \infty$. In each case, assume α to be the principal value, and n to be an integer, then, express the general solution in terms of radian.
3. Give a suggestion between the formulas obtained in task 2.

Therefore, the general solutions for the equations of sine, cosine and tangent can be summarized as follows:

In radians:

(i) If $\sin\theta = \sin\alpha$, then $\theta = \pi n + (-1)^n \alpha$

(ii) If $\cos\theta = \cos\alpha$, then $\theta = 2\pi n \pm \alpha$

(iii) If $\tan\theta = \tan\alpha$, then $\theta = \pi n + \alpha$

where $n = 0, 1, 2, \ldots$

In degrees:

(i) If $\sin\theta = \sin\alpha$, then $\theta = 180°n + (-1)^n \alpha$

(ii) If $\tan\theta = \tan\alpha$, then $\theta = 180°n + \alpha$

(iii) If $\cos\theta = \cos\alpha$, then $\theta = 360°n \pm \alpha$

Example 7.36

Find the general solution of the equation $\cos x + \sin x = 1$ giving your answer in terms of π.

Solution

Equate $\cos x + \sin x = \mathrm{R}\cos(x - \alpha)$

$\Rightarrow \mathrm{R} = \sqrt{a^2 + b^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$

But $\alpha = \tan^{-1}\frac{b}{a} = \tan^{-1}1 = \frac{\pi}{4}$.

Then,

$$\cos x + \sin x = \sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = 1$$

$$\sqrt{2}\cos\left(x - \frac{\pi}{4}\right) = 1$$

$$\Rightarrow \cos\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x - \frac{\pi}{4} = \cos^{-1}\frac{1}{\sqrt{2}}$$

$$\Rightarrow x - \frac{\pi}{4} = \frac{\pi}{4}$$

Using the general solution of cosine,

$\theta = 2\pi n \pm \alpha$

But $\theta = x - \frac{\pi}{4}$ and $\alpha = \frac{\pi}{4}$

$$\Rightarrow x - \frac{\pi}{4} = 2\pi n \pm \frac{\pi}{4}$$

$$\Rightarrow x = 2\pi n \pm \frac{\pi}{4} + \frac{\pi}{4}$$

Therefore, $x = 2\pi n + \frac{\pi}{2}$ and $x = 2\pi n$ where n is an integer.

Example 7.37

Find the general solution of the equation $\cot x + \operatorname{cosec} x = \sqrt{3}$ giving the answer in terms of π.

Solution

Given $\cot x + \operatorname{cosec} x = \sqrt{3}$.

$$\Rightarrow \frac{\cos x}{\sin x} + \frac{1}{\sin x} = \sqrt{3}$$

Multiplying by $\sin x$ both sides gives;

$\cos x + 1 = \sqrt{3}\sin x$

Re-arranging the equation to obtain;

$\sqrt{3}\sin x - \cos x = 1$

Using $\sqrt{3}\sin x - \cos x = \mathrm{R}\sin(x - \alpha)$.

$\Rightarrow \sqrt{3}\sin x - \cos x =$

$\mathrm{R}\cos\alpha\sin x - \mathrm{R}\sin\alpha\cos x$

But $\mathrm{R} = \sqrt{a^2 + b^2} = \sqrt{(\sqrt{3})^2 + (-1)^2} = 2$

Comparing coefficients of $\sin x$ gives;

$\mathrm{R}\cos\alpha = \sqrt{3}$ …………………… (i)

Comparing coefficients of $\cos x$ gives;

$\mathrm{R}\sin\alpha = 1$ …………………….. (ii)

Dividing equation (ii) by equation (i) gives;

$$\tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}.$$

Then,

$$\sqrt{3}\sin x - \cos x = 2\sin\left(x - \frac{\pi}{6}\right) = 1$$

$$\Rightarrow 2\sin\left(x - \frac{\pi}{6}\right) = 1$$

$$\Rightarrow \sin\left(x - \frac{\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow x - \frac{\pi}{6} = \sin^{-1}\frac{1}{2}$$

$$\Rightarrow x - \frac{\pi}{6} = \frac{\pi}{6}$$

By using the general solution for sine

$\theta = \pi n + (-1)^n\alpha$

But $\theta = x - \frac{\pi}{6}$ and $\alpha = \frac{\pi}{6}$

Thus, $x-\frac{\pi}{6}=\pi n+(-1)^n\frac{\pi}{6}$

$\Rightarrow x=\pi n+(-1)^n\frac{\pi}{6}+\frac{\pi}{6}$

$\Rightarrow x=\pi n+\left((-1)^n+1\right)\frac{\pi}{6}$

Therefore, the general solution is $x=\pi n+\left((-1)^n+1\right)\frac{\pi}{6}$, where n is an integer.

Example 7.38

Use the t-formulae to find the general solution of the equation $2\cos x-\sin x=1$ giving the answers in radians.

Solution

Given $2\cos x-\sin x=1$.

From the t-formulae;

$\sin x=\frac{2t}{1+t^2}$, $\cos x=\frac{1-t^2}{1+t^2}$,

where $t=\tan\frac{1}{2}x$

Then, $2\left(\frac{1-t^2}{1+t^2}\right)-\frac{2t}{1+t^2}=1$

Multiplying both sides by $1+t^2$ to obtain;

$2(1-t^2)-2t=1+t^2$

$\Rightarrow 3t^2+2t-1=0$

$\Rightarrow t=-1$ or $t=\frac{1}{3}$

Thus, $\tan\frac{1}{2}x=-1$ or $\tan\frac{1}{2}x=\frac{1}{3}$,

since $t=\tan\frac{1}{2}x$

Case 1:

$\tan\frac{1}{2}x=-1$

$\Rightarrow \frac{1}{2}x=\tan^{-1}(-1)=-\frac{\pi}{4}$

Using the general solution of tangent,

$\theta=n\pi+\alpha$

But, $\theta=\frac{1}{2}x$ and $\alpha=-\frac{\pi}{4}$

Then, $\frac{1}{2}x=\pi n-\frac{\pi}{4}$

$\Rightarrow x=2\pi n-\frac{\pi}{2}$

Case 2:

$\tan\frac{1}{2}x=\frac{1}{3}$

$\Rightarrow \frac{1}{2}x=\tan^{-1}\frac{1}{3}=0.32$

The general solution of tangent, is given by;

$\theta=\pi n+\alpha$

But, $\theta=\frac{1}{2}x$ and $\alpha=0.32$

Then, $\frac{1}{2}x=\pi n+0.32$

$\Rightarrow x=2\pi n+0.64$

Therefore, the general solution of the equation $2\cos x-\sin x=1$ is $x=2\pi n-\frac{\pi}{2}$ and $x=2\pi n+0.64$ where n is an integer.

Exercise 7.6

Find the general solution of each of the following equations (leave the answer in radians):

1. $2\sin 2\theta = 1$
2. $\tan 5\theta = 1$
3. $\sec\theta = \sqrt{2}$
4. $3\cos\theta + 5\sin\theta = 2$
5. $2\sin 2\theta + 7\cos 2\theta = -7.23$
6. $3\sin^2 x + \cos x \sin x = 3\cos^2 x + 2$
7. $\cos 2\theta + \sin\theta = 0$
8. $8\sin\theta + 15\cos\theta = 6$
9. $3\cos 2\theta + 5\sin\theta\cos\theta = 2$
10. $\sec\theta - \tan\theta = \sqrt{3}$

Factor formulae

In algebra, factors are commonly used to solve equations and simplify expressions. Similarly, in trigonometry, factors are used to factorize the sum and difference of two terms. Moreover, factors are used to express a product as a sum or difference of two terms. The factor formulae are applied in solving equations, simplifying expressions, and in proving trigonometry identities. Factor formulae are deduced from the compound angle formulae as follows:

From compound angle formulae for sine,

$$\begin{cases}\sin(A+B) = \sin A\cos B + \cos A\sin B \\ \sin(A-B) = \sin A\cos B - \cos A\sin B\end{cases}$$

Adding the two equations gives;

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B \quad \text{(i)}$$

Subtracting the two equations gives;

$$\sin(A+B) - \sin(A-B) = 2\cos A\sin B \quad \text{(ii)}$$

Again, from the compound angle formulae for cosine,

$$\begin{cases}\cos(A+B) = \cos A\cos B - \sin A\sin B \\ \cos(A-B) = \cos A\cos B + \sin A\sin B\end{cases}$$

Adding the two equations gives;

$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B \quad \text{(iii)}$$

Subtracting the two equations gives;

$$\cos(A+B) - \cos(A-B) = -2\sin A\sin B \quad \text{(iv)}$$

The left-hand side of the identities (i) to (iv) are the factors.

Let $P = A + B$ and $Q = A - B$

Adding P and Q gives,

$P + Q = 2A$

Thus, $A = \dfrac{P+Q}{2}$

Subtracting Q from P gives,

$P - Q = 2B$

Thus, $B = \dfrac{P-Q}{2}$

Hence, substituting $A = \dfrac{P+Q}{2}$ and $B = \dfrac{P-Q}{2}$ in the equations (i), (ii), (iii), and (iv);

$$\sin P + \sin Q = 2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \qquad (7.8)$$

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \qquad (7.9)$$

$$\cos P + \cos Q = 2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right) \qquad (7.10)$$

$$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right) \qquad (7.11)$$

Equations (7.8) to (7.11) are known as factor formulae. They express the sum or difference as a product.

Example 7.39

Prove that $\sin\theta + \sin 2\theta + \sin 3\theta = \sin 2\theta\left(2\cos\theta + 1\right)$.

Solution

Consider the left-hand side;

$$\begin{aligned}\sin\theta + \sin 2\theta + \sin 3\theta &= \sin 2\theta + (\sin 3\theta + \sin\theta)\\ &= \sin 2\theta + 2\sin\left(\frac{3\theta+\theta}{2}\right)\cos\left(\frac{3\theta-\theta}{2}\right)\\ &= \sin 2\theta + 2\sin 2\theta\cos\theta\\ &= \sin 2\theta(1 + 2\cos\theta)\end{aligned}$$

Therefore, $\sin\theta + \sin 2\theta + \sin 3\theta = \sin 2\theta(2\cos\theta + 1)$.

Example 7.40

In any triangle ABC, prove that, $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Solution

Consider the left-hand side;

$$\sin A+\sin B+\sin C=2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}$$

But $A+B=180°-C\Rightarrow\frac{A+B}{2}=90°-\frac{C}{2}$

$$\begin{aligned}
\Rightarrow \sin A+\sin B+\sin C &= 2\sin\left(90°-\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}\\
&=2\cos\left(\frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right)+2\sin\frac{C}{2}\cos\frac{C}{2}\\
&=2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\sin\frac{C}{2}\right]\\
&=2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\sin\left(90°-\frac{A+B}{2}\right)\right]\\
&=2\cos\left(\frac{C}{2}\right)\left[\cos\left(\frac{A-B}{2}\right)+\cos\left(\frac{A+B}{2}\right)\right]\\
&=2\cos\left(\frac{C}{2}\right)\left[2\cos\left(\frac{\frac{A+B}{2}+\frac{A-B}{2}}{2}\right)\cos\left(\frac{\frac{A+B}{2}-\frac{A-B}{2}}{2}\right)\right]\\
&=2\cos\left(\frac{C}{2}\right)\left[2\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\right]\\
&=4\cos\left(\frac{A}{2}\right)\cos\left(\frac{B}{2}\right)\cos\left(\frac{C}{2}\right).
\end{aligned}$$

Therefore, $\sin A+\sin B+\sin C=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

Example 7.41

If $\sin x + \sin y = a$ and $\cos x + \cos y = b$, show that $\cos^2 \frac{1}{2}(x-y) = \frac{1}{4}\left(a^2 + b^2\right)$.

Solution

Given $\sin x + \sin y = a$ and $\cos x + \cos y = b$.

$$\sin x + \sin y = 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$

$$\Rightarrow 2\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = a \quad \text{.................................. (i)}$$

$$\cos x + \cos y = 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)$$

$$\Rightarrow 2\cos\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y) = b \quad \text{.................................. (ii)}$$

Squaring equations (i) and (ii) gives;

$$4\sin^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = a^2 \quad \text{.................................. (iii)}$$

$$4\cos^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = b^2 \quad \text{.................................. (iv)}$$

Adding equations (iii) and (iv) result to,

$$4\sin^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) + 4\cos^2\frac{1}{2}(x+y)\cos^2\frac{1}{2}(x-y) = a^2 + b^2$$

$$\Rightarrow 4\cos^2\frac{1}{2}(x-y)\left[\sin^2\frac{1}{2}(x+y) + \cos^2\frac{1}{2}(x+y)\right] = a^2 + b^2$$

But $\sin^2\frac{1}{2}(x+y) + \cos^2\frac{1}{2}(x+y) = 1$

Thus, $4\cos^2\frac{1}{2}(x-y) = a^2 + b^2 \Rightarrow \cos^2\frac{1}{2}(x-y) = \frac{1}{4}\left(a^2 + b^2\right)$.

Therefore, $\cos^2\frac{1}{2}(x-y) = \frac{1}{4}(a^2 + b^2)$.

Example 7.42

Simplify $\dfrac{\sin\theta+\sin3\theta+\sin5\theta}{\cos\theta+\cos3\theta+\cos5\theta}$.

Solution

$$\frac{\sin\theta+\sin3\theta+\sin5\theta}{\cos\theta+\cos3\theta+\cos5\theta}=\frac{(\sin\theta+\sin5\theta)+\sin3\theta}{(\cos\theta+\cos5\theta)+\cos3\theta}$$

$$=\frac{2\sin\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)+\sin3\theta}{2\cos\left(\frac{\theta+5\theta}{2}\right)\cos\left(\frac{\theta-5\theta}{2}\right)+\cos3\theta}$$

$$=\frac{2\sin3\theta\cos(-2\theta)+\sin3\theta}{2\cos3\theta\cos(-2\theta)+\cos3\theta},\text{ but }\cos(-2\theta)=\cos2\theta$$

$$=\frac{\sin3\theta(2\cos2\theta+1)}{\cos3\theta(2\cos2\theta+1)}=\frac{\sin3\theta}{\cos3\theta}$$

$$=\tan3\theta$$

Therefore, $\dfrac{\sin\theta+\sin3\theta+\sin5\theta}{\cos\theta+\cos3\theta+\cos5\theta}=\tan3\theta.$

Example 7.43

Simplify $\tan\frac{1}{2}(\mathrm{A}-\mathrm{B})+\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})$.

Solution

$$\tan\frac{1}{2}(\mathrm{A}-\mathrm{B})+\tan\frac{1}{2}(\mathrm{A}+\mathrm{B})$$

$$=\frac{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}+\frac{\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})}$$

$$=\frac{\sin\frac{1}{2}(\mathrm{A}-\mathrm{B})\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})+\sin\frac{1}{2}(\mathrm{A}+\mathrm{B})\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}{\cos\frac{1}{2}(\mathrm{A}+\mathrm{B})\cos\frac{1}{2}(\mathrm{A}-\mathrm{B})}$$

$$=\frac{\frac{1}{2}(\sin A-\sin B)+\frac{1}{2}(\sin A+\sin B)}{\frac{1}{2}(\cos A+\cos B)}$$

$$=\frac{\sin A-\sin B+\sin A+\sin B}{\cos A+\cos B}$$

$$=\frac{2\sin A}{\cos A+\cos B}$$

Therefore, $\tan\frac{1}{2}(A-B)+\tan\frac{1}{2}(A+B)=\frac{2\sin A}{\cos A+\cos B}$.

Example 7.44

Solve the equation $\cos 5x+\cos x=\cos 3x$ for $0\le x\le \pi$ giving the answer in terms of π.

Solution

Given $\cos 5x+\cos x=\cos 3x$. Using $\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$

That is, $2\cos\left(\frac{5x+x}{2}\right)\cos\left(\frac{5x-x}{2}\right)=\cos 3x$

$\Rightarrow 2\cos 3x\cos 2x=\cos 3x$

$\Rightarrow 2\cos 3x\cos 2x-\cos 3x=0$

$\Rightarrow \cos 3x(2\cos 2x-1)=0$

Thus,

$\cos 3x=0$ or $2\cos 2x=1$

Applying $\cos^{-1}$ on both sides, gives;

$3x=\cos^{-1}0$

$\Rightarrow 3x=\frac{1}{2}\pi,\frac{3}{2}\pi,\frac{5}{2}\pi,\ldots$

Thus, $x=\frac{1}{6}\pi,\frac{1}{2}\pi,\ \frac{5}{6}\pi,\ldots$

Applying $\cos^{-1}$ on both sides, gives;

$2x=\cos^{-1}\frac{1}{2}$

$\Rightarrow 2x=\frac{1}{3}\pi,\frac{5}{3}\pi,\ldots$

Thus, $x=\frac{1}{6}\pi,\ \frac{5}{6}\pi,\ldots$

Therefore, the values of x are $\frac{1}{6}\pi,\frac{1}{2}\pi$, and $\frac{5}{6}\pi$.

Exercise 7.7

1. Express each of the following as the product:
   (a) $\sin 50° + \sin 40°$
   (b) $\sin 70° - \sin 20°$
   (c) $\cos 55° + \cos 25°$
   (d) $\cos 35° - \cos 75°$

2. Express each of the following as sums or differences:
   (a) $\sin 55° \sin 40°$
   (b) $\cos 110° \sin 55°$
   (c) $\sin 40° \cos 30°$
   (d) $\cos 50° \cos 35°$

3. Find the general solution of each of the following equations, expressing the answer in multiples of π.
   (a) $\sin 4\theta + \sin 2\theta + \sin 6\theta = 0$
   (b) $\cos x + \cos 2x + \cos 3x = 0$
   (c) $\sin 7x + \sin x + \sin 4x = 0$
   (d) $\cos 7\theta - \cos \theta = 2\sin 3\theta$
   (e) $\sin 3x + \sin 5x = \sin 4x$
   (f) $\sin 7x - \sin 5x = \sqrt{2}\cos 6x$

4. Prove each of the following identities:
   (a) $\cos 130° + \cos 110° + \cos 10° = 0$
   (b) $\cos^3 x \sin^2 x = \frac{1}{16}(2\cos x - \cos 3x - \cos 5x)$
   (c) $\frac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = \cot\tfrac{1}{2}(\alpha - \beta)\cot\tfrac{1}{2}(\alpha + \beta)$
   (d) $\frac{\tan\frac{1}{2}(A - B)}{\tan\frac{1}{2}(A + B)} = \frac{\sin A - \sin B}{\sin A + \sin B}$

5. In each of the following expressions, show that:
   (a) $\frac{\sin 75° - \sin 15°}{\cos 75° + \cos 15°} = \frac{\sqrt{3}}{3}$
   (b) $2\sin 82\tfrac{1}{2}° \cos 37\tfrac{1}{2}° = \frac{\sqrt{3} + \sqrt{2}}{2}$
   (c) $\cos 465° + \cos 165° = -\frac{\sqrt{6}}{2}$
   (d) $2\sin 45° \cos 15° = \frac{\sqrt{3} + 1}{2}$

6. If $\tan(p+q)=a$ and $\tan(p-q)=b$, express $\dfrac{\sin 2p+\sin 2q}{\sin 2p-\sin 2q}$ in terms of a and b. Hence, show that $\tan 2q=\dfrac{a-b}{1+ab}$ and use this result to obtain an expression for $\tan(p+3q)$ in terms of a and b.
7. By expressing $2\sin 3\theta \sin\frac{1}{2}\theta$ and other similar expressions as the difference of two sines, prove the identity $(2\cos 3\theta+2\cos 2\theta+2\cos\theta+1)\sin\frac{1}{2}\theta=\sin\frac{7}{2}\theta$. Express $\cos 3\theta$ and $\cos 2\theta$ in terms of $\cos\theta$ and deduce the identity

   $(8\cos^3\theta+4\cos^2\theta-4\cos\theta-1)\sin\frac{1}{2}\theta=\sin\frac{7}{2}\theta$, hence show that $\cos\frac{2}{7}\pi, \cos\frac{4}{7}\pi$ and $\cos\frac{6}{7}\pi$, are roots of the equation $8x^3+4x^2-4x-1=0$.
8. Solve each of the following equations for $0°\le\theta\le 180°$.
   (a) $\cos 4\theta+\cos 6\theta+\cos 2\theta=0$
   (b) $\cos(2\theta+40°)+\cos(2\theta-60°)=0$
   (c) $\cos 4\theta+\cos 2\theta=\cos\theta$
   (d) $\sin 5\theta+\sin 3\theta=\sin(-4\theta)$
9. If $p=\cos x+\cos 2x+\cos 3x$ and $q=\sin x+\sin 2x+\sin 3x$, prove that:
   (a) $p=q\cot 2x$
   (b) $p^2+q^2=3+4\cos x+2\cos 2x$
10. Solve the equation $\sin 2y+\cos 2y=\sin y+\cos y$ for $0\le y\le 2\pi$.
11. If $f(\lambda)=\dfrac{\sin 4\lambda+\sin 3\lambda+\sin 2\lambda+\sin\lambda}{\cos 4\lambda+\cos 3\lambda+\cos 2\lambda+\cos\lambda}$, then;
    (a) Simplify $f(\lambda)$
    (b) Solve the equation if $f(\lambda)=1$ for $-180°\le\lambda\le 180°$
12. Simplify each of the following expressions:
    (a) $\dfrac{\sin 10A+\sin 9A+\sin 8A+\sin 7A}{\cos 10A+\cos 9A+\cos 8A+\cos 7A}$
    (b) $\dfrac{\sin\left(x+\frac{2\pi}{3}\right)-\sin\left(\frac{2\pi}{3}-x\right)}{\cos\left(x+\frac{\pi}{4}\right)-\cos\left(\frac{\pi}{4}-x\right)}$

Radians and small angles

Radian is a unit of an angle which is defined as the size of the central angle subtended by an arc of length l equal in length to the radius r of a circle.

Converting degrees to radians

Consider Figure 7.6

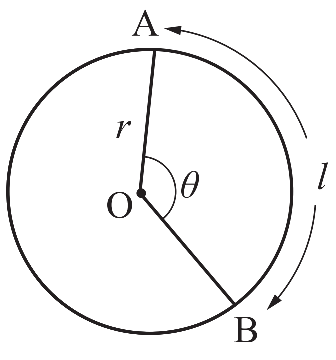


Figure 7.6: *Sketch describing an arc of a circle.*

From Figure 7.6, an arc of length equal to the circumference of a circle subtends a central angle of one complete revolution. Since the circumference, $C = 2\pi r$ subtends an angle of $360°$, while the arc of length l subtends a central angle θ, where θ is an angle in degrees, then

$$\frac{\text{Length of an arc AB}}{\text{Circumference of a circle}}$$

$$= \frac{\text{Measure of the central angle}}{\text{Total measurement in a circle}}$$

$$\Rightarrow \frac{l}{2\pi r} = \frac{\theta}{360°}$$

Thus, $l = \frac{\theta}{360°} \times 2\pi r = \frac{\pi r \theta}{180°}$.

Again, $\frac{l}{r} = \frac{\pi\theta}{180°}$.

But $\frac{l}{r} = s$ where s is an angle in radian.

Therefore, $s = \frac{\pi\theta}{180°}$.

Note that; the radian measure has no dimension. It is a dimensionless number. Radians can be converted into degrees by multiplying by the factor $\frac{180°}{\pi}$, also, degrees can be converted into radians by multiplying by the factor $\frac{\pi}{180°}$.

Example 7.45

Convert each of the following into radians giving the answers in multiples of π:

(a) $60°$ (b) $2970°$ (c) $1°$

Solution

(a) From $s = \frac{\pi\theta}{180°}$.

But $\theta = 60°$, thus,

$$s = \frac{\pi \times 60°}{180°} = \frac{\pi}{3}.$$

Therefore, $60° = \frac{\pi}{3}$ radians.

(b) From $s = \frac{\pi\theta}{180°}$

But $\theta = 2970°$, thus,

$$s = \frac{\pi \times 2970°}{180°} = 16.5\pi.$$

Therefore, $2970° = 16.5\pi$ radians.

(c) From $s = \frac{\pi\theta}{180°}$.

But $\theta = 1°$

$$s=\frac{\pi\times 1^{\circ}}{180^{\circ}}$$

$$s=\frac{1}{180}\pi$$

Therefore, $1^{\circ}=\frac{1}{180}\pi$ radians.

Example 7.46

Convert each of the following into degrees:

(a) $\frac{2\pi}{3}$ radians (c) 1 radian

(b) $\frac{37\pi}{5}$ radians

Solution

(a) From $s=\frac{\pi\theta}{180^{\circ}}$, then $\theta=\frac{180^{\circ}}{\pi}s$.

Thus, $\theta=\frac{180^{\circ}}{\pi}\times\frac{2\pi}{3}=120^{\circ}$.

Therefore, $\frac{2\pi}{3}$ radians $=120^{\circ}$.

(b) From $s=\frac{\pi\theta}{180^{\circ}}$, then $\theta=\frac{180^{\circ}}{\pi}s$

Thus, $\theta=\frac{180^{\circ}}{\pi}\times\frac{37\pi}{5}=1332^{\circ}$.

Therefore, $\frac{37\pi}{5}$ radians $=1332^{\circ}$.

(c) From $s=\frac{\pi\theta}{180^{\circ}}$, then $\theta=\frac{180^{\circ}}{\pi}s$

Thus, $\theta=\frac{180^{\circ}\times 1}{\pi}\approx 57.296^{\circ}$

Therefore, 1 radian $\approx 57.296^{\circ}$.

Example 7.47

Find in radians an interior angle of a regular nonagon.

Solution

The sum of the 9 exterior angles $=360^{\circ}$.

Each exterior angle $=\frac{360^{\circ}}{9}=40^{\circ}$

Each interior angle $=180^{\circ}-40^{\circ}=140^{\circ}$

From, $s=\frac{\pi\theta}{180^{\circ}}=\frac{\pi\times 140^{\circ}}{180^{\circ}}=\frac{7\pi}{9}$ radians.

Therefore, the interior angle of a regular nonagon is $\frac{7\pi}{9}$ radians.

Example 7.48

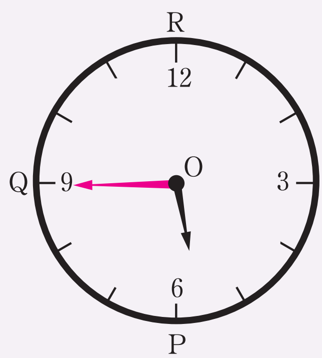
Find the angle between the minute- hand and the hour-hand of a clock, when time is 5:45 p.m. Give your answers

(a) in degrees

(b) in radians

Solution

(a) In the following figure, the hour-hand and the minute - hand are along the arcs RP and RQ, respectively.



Angle POQ = (angle subtended by the minute-hand in 45 minutes) – (angle subtended by the hour- hand in 45 minutes)

$$=\left(\frac{360}{60}\times 45\right)^{\circ}-\left(\frac{360}{12}\times\frac{23}{4}\right)^{\circ}$$

$$=270^{\circ}-172.5^{\circ}$$

$$=97.5^{\circ}$$

Therefore, the required angle is 97.5°.

Altenatively,

$$\text{Angle POQ}=\left(\frac{\text{Position of a minute-hand}-\text{Position of an hour-hand}}{12}\right)\times 360^{\circ}$$

$$=\left(\frac{9-5.75}{12}\right)\times 360^{\circ}$$

$$=97.5^{\circ}$$

(b) Let, $s=\dfrac{\pi\theta}{180^{\circ}}$

$$=\frac{\pi\times 97.5^{\circ}}{180^{\circ}}=\frac{13\pi}{24}\text{ radians}$$

Therefore, the required angle is $\dfrac{13\pi}{24}$ radians.

Exercise 7.8

1. Convert each of the following angles into radians, giving the answer in terms of π.

   (a) 30° (c) 90° (e) 40° (g) 315°

   (b) 240° (d) –135° (f) 270° (h) 405°

2. Convert each of the following angles into degrees.

   (a) $\dfrac{\pi}{3}$ (b) $\dfrac{5\pi}{3}$ (c) $\dfrac{7\pi}{6}$ (d) $\dfrac{-2\pi}{3}$

   (e) 3π (f) $\dfrac{7\pi}{5}$ (g) $\dfrac{17\pi}{8}$ (h) 2π

3. Solve each the following for $0\le x\le 2\pi$, giving the answer in multiple of π.

   (a) $\cos x=\dfrac{\sqrt{3}}{2}$ (c) $\sin x=\dfrac{1}{\sqrt{2}}$ (e) $\cos x=-\dfrac{1}{\sqrt{2}}$

   (b) $\tan x=1$ (d) $\sin x=-\dfrac{1}{2}$ (f) $\tan x=\sqrt{3}$

4. Convert each of the following agles into degrees, giving the answer correct to 2 decimal places:
   (a) 0.49 radians
   (b) 1.72 radians
   (c) -2.36 radians
   (d) -0.85 radians
   (e) 4.39 radians
   (f) -5.08 radians

5. Convert each of the following angles into radians, giving the answer into 3 significant figures:
   (a) $65.4°$
   (b) $32°45'$
   (c) $84°32'25''$

6. The difference between two angle is $60°$ and their sum is $\frac{5\pi}{6}$ radians. Find the value of each angle, giving the answer in radians.

7. Find the angle in radians between the minute- hand and the hour-hand of the clock at:
   (a) 3:45 p.m
   (b) 7:20 a.m
   (c) 11:50 p.m

8. Find the interior angles in radian of a regular:
   (a) heptagon
   (b) hexagon
   (c) octagon

9. The difference between two angles is $\frac{5\pi}{4}$ and their sum is $\frac{11\pi}{4}$. Find the value of each angle in degrees.

10. Find the degree measures corresponding to each of the following radians:
   (a) $-\frac{5\pi}{14}$
   (b) $\frac{21\pi}{14}$
   (c) -2
   (d) $7\frac{2}{3}$

Approximating small angles

If the measured angle is small and is measured in radians, the concept of small-angle approximations can be used to approximate the values of the principal trigonometric functions. The geometry of small angles has some unique properties.

Activity 7.4: Recognizing geometrical properties of small angles

Learning resources: Graph papers, scientific calculator, ruler, and pencil.
Individually or in a group, perform the following tasks:

1. Construct a table of values for $y=\sin\theta$, $y=\theta$, and $y=\cos\theta$ for convenient values of θ in radians.
2. Draw on the same plane the graphs for $y=\sin\theta$, $y=\theta$, and $y=\cos\theta$
3. State the possible conclusions suggested by the graphs drawn in task 2.
4. What have you observed in task 1 about the values of θ (radians) for $\sin\theta$ and $\cos\theta$? Give comments.
5. Deduce the relationship between $\cos\theta$ and θ versus $\sin\theta$ and θ.
6. Share your findings with your fellow students for more inputs.

In Figure 7.7, the chord $\overline{AB}$ subtends an angle θ in radians at the centre O of a circle with radius r and the tangent at A meets $\overline{OB}$ at C.

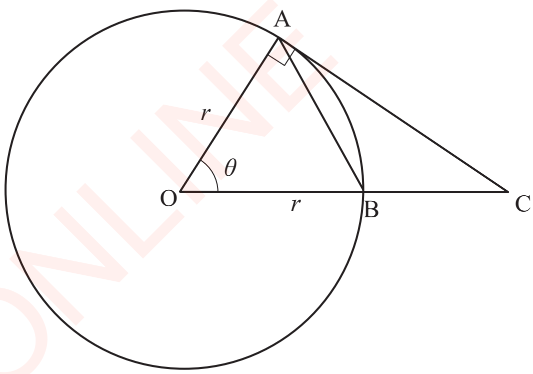


Figure 7.7: *Sketch for small angle formula derivation*

Figure 7.7, shows a sector of a circle, AOB centred at O with radius r. $\overline{OB}$ is extended to C where $\overline{CA}$ is a tangent to circle at A. From Figure 7.7 it can be deduced that:

Area of triangle AOB < Area of sector AOB < Area of right-angled triangle AOC

But area of triangle AOB $=\frac{1}{2}\overline{OA}\times\overline{OB}\sin\theta=\frac{1}{2}r^2\sin\theta$.

Area of sector AOB $= \frac{1}{2}r^2\theta$ and area of right-angled triangle AOC $= \frac{1}{2}\overline{AC} \times r$, but $\overline{AC} = r\tan\theta$.

Area of right-angled triangle AOC $= \frac{1}{2}r^2 \tan\theta$, thus,

$$\frac{1}{2}r^2 \sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r^2 \tan\theta$$

Dividing each term by $\frac{1}{2}r^2$, gives;

$\sin\theta < \theta < \tan\theta$ where $0 < \theta < \frac{\pi}{2}$, dividing each term by $\sin\theta$, gives;

$$\frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{\tan\theta}{\sin\theta}.$$

$$\Rightarrow 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta} \text{ since, } \tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Then, as $\theta \to 0$, $\cos\theta \to 1$ and $\frac{1}{\cos\theta} \to 1$, thus,

$$1 < \frac{\theta}{\sin\theta} < 1$$

Thus, $\frac{\theta}{\sin\theta} \to 1$ as $\theta \to 0$.

Therefore, for small values of θ, $\sin\theta \approx \theta$.

An approximation for $\cos\theta$ is obtained from the identity

$\cos\theta = 1 - 2\sin^2\frac{1}{2}\theta$ where for small angles $\sin\frac{1}{2}\theta \approx \frac{1}{2}\theta$. Thus,

$$\cos\theta \approx 1 - 2\left(\frac{1}{2}\theta\right)^2$$

$$\Rightarrow \cos\theta \approx 1 - 2\left(\frac{1}{2}\theta\right)^2$$

Therefore, for small angles $\cos\theta \approx 1 - \frac{1}{2}\theta^2$.

The approximation of $\tan\theta$ in terms of θ is derived from the fact that

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ as $\theta \to 0$, $\cos\theta \to 1$, and $\sin\theta \to \theta$

Therefore, for small values of θ, $\tan\theta \approx \theta$.

Example 7.49

Approximate the value of each of the following functions when θ is small:

(a) $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta}$ (b) $\dfrac{1-\cos 2\theta}{\tan 2\theta \sin\theta}$ (c) $\dfrac{3\tan\theta - \theta}{\sin 2\theta}$.

Solution

(a) Given $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta}$.

As $\theta \to 0$, $\sin 4\theta \approx 4\theta$, $\tan 2\theta \approx 2\theta$

Hence, $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta} \approx \dfrac{4\theta - 2\theta}{3\theta} \approx \dfrac{2}{3}$.

Therefore, for small values of θ, $\dfrac{\sin 4\theta - \tan 2\theta}{3\theta} \approx \dfrac{2}{3}$.

(b) Given $\dfrac{1-\cos 2\theta}{\tan 2\theta \sin\theta}$.

As $\theta \to 0$, $\cos 2\theta \approx 1 - \frac{1}{2}(2\theta)^2 \approx 1 - 2\theta^2$, $\sin\theta \approx \theta$, and $\tan 2\theta \approx 2\theta$

Hence,
$$\frac{1-\cos 2\theta}{\tan 2\theta \sin\theta} \approx \frac{1-\left(1-2\theta^2\right)}{2\theta \times \theta}$$
$$\approx \frac{2\theta^2}{2\theta^2}$$
$$\approx 1$$

Therefore, for small values of θ, $\dfrac{1-\cos 2\theta}{\tan 2\theta \sin\theta} \approx 1$.

(c) $\dfrac{3\tan\theta - \theta}{\sin 2\theta}$.

As $\theta \to 0$, $\sin 2\theta \approx 2\theta$ and $\tan\theta \approx \theta$

Hence, $\dfrac{3\tan\theta - \theta}{\sin 2\theta} \approx \dfrac{3\theta - \theta}{2\theta} \approx 1$

Therefore, for small values of θ, $\dfrac{3\tan\theta - \theta}{\sin 2\theta} \approx 1$.

Example 7.50

Find an approximation of $\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta}$, when θ is small.

Solution

Given $\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta}$.

By using the factor formula,

$$\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta}=\frac{-2\sin\left(\frac{\alpha+\theta+\alpha}{2}\right)\sin\left(\frac{\alpha+\theta-\alpha}{2}\right)}{\theta}$$

$$=\frac{-2\sin\left(\alpha+\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)}{\theta}$$

As $\theta\to 0$, $\sin\left(\frac{\theta}{2}\right)\approx\frac{\theta}{2}$, and $\sin\left(\alpha+\frac{\theta}{2}\right)\approx\sin\alpha$.

$$\Rightarrow\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta}\approx\frac{-2\sin\alpha}{\theta}\times\frac{\theta}{2}$$

$$\approx-\sin\alpha$$

Therefore, $\frac{\cos(\alpha+\theta)-\cos\alpha}{\theta}\approx-\sin\alpha$, when θ is small.

Example 7.51

Show that, if θ is small $\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2$

Solution

Required to show that $\sin\left(\theta+\frac{\pi}{6}\right)\approx\frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2$, thus,

$$\sin\left(\theta+\frac{\pi}{6}\right)=\sin\frac{\pi}{6}\cos\theta+\sin\theta\cos\frac{\pi}{6}$$

$$=\frac{1}{2}\cos\theta+\frac{\sqrt{3}}{2}\sin\theta$$

Thus, as $\theta \to 0$, $\sin\theta \approx \theta$

and $\cos\theta \approx 1-\frac{1}{2}\theta^2$. Then,

$$\sin\left(\theta+\frac{\pi}{6}\right) \approx \frac{1}{2}(1-\frac{1}{2}\theta^2)+\frac{\sqrt{3}}{2}\theta$$

$$\approx \frac{1}{2}-\frac{1}{4}\theta^2+\frac{\sqrt{3}}{2}\theta.$$

Therefore,

$$\sin\left(\theta+\frac{\pi}{6}\right) \approx \frac{1}{2}+\frac{\sqrt{3}}{2}\theta-\frac{1}{4}\theta^2.$$

Exercise 7.9

1. Approximate each of the following expressions for small values of θ:
   (a) $\dfrac{21+7\tan\theta-20\cos\theta}{1+\sin 2\theta}$
   (b) $\dfrac{\sin 3\theta+\tan 5\theta}{2\theta}$
   (c) $\dfrac{1+\sin\theta}{5+3\tan\theta-4\cos\theta}$
   (d) $\dfrac{\sin 3\theta+\tan\theta}{\cos 2\theta}$
   (e) $\sin(\theta+45°)$
   (f) $\dfrac{\cos\theta-1}{\sin\theta}$
2. Suppose that x is small, use the small angle approximation formulae to show that $4\cos 4x+\cos^2 2x \approx 5-36x^2+4x^4$ hence, find the approximation of $4\cos 4x+\cos^2 2x$ when $x=3$ radians.
3. If θ is small, show that $\tan 3\theta\cos 2\theta$ can be approximated by $3\theta-6\theta^3$ hence, approximate the value of $\tan 0.3\cos 0.2$.
4. If θ is small, prove that $\dfrac{\tan(\alpha+\theta)-\tan\alpha}{\theta} \approx 1+\tan^2\alpha$.
5. If θ is small, show that $\cos\left(\dfrac{2\pi}{3}-\theta\right) \approx \dfrac{1}{4}(\theta^2+2\sqrt{3}\theta-2)$.
6. Find an approximate value of $\dfrac{1-\cos 4\theta+\sin\theta-\sin\theta\cos 4\theta}{1+\sin\theta}$ when θ is small.
7. Simplify the expression, $\dfrac{\sin(3\alpha+2\theta)\sin 2\theta}{3\theta}$ when θ is small.
8. Use the identity $\sin^2 x+\cos^2 x=1$, the binomial theorem, and the approximation $\sin x \approx x$ to verify that $\cos x \approx 1-\frac{1}{2}x^2$ where x is small. Hence, deduce that $\cos 4° \approx 0.9976$.
9. If θ is a small angle in radians, find the limit of $\dfrac{\sin 3\theta+\tan 6\theta}{\cot 2\theta+1+3\theta}$ as θ tends to zero.
10. Evaluate each of the following when θ is small:
    (a) $\dfrac{1-\sin\theta-7\sin^2\theta+3\sin^3\theta}{1-3\sin\theta}$
    (b) $\tan(\alpha+\theta)$

Domain and range of trigonometric functions

The domain of a given function is the set of all possible inputs (x-values) to the functions, while the range is the set of all possible outputs (y-values) of the function.

The graph of sine and cosine functions behave like waves which oscillate between the amplitudes -1 and 1, inclusive with periodic unit of 2π.

Thus, the domain of $f(x)=\sin x$ is the set of all real numbers, that is $\{x: x\in\mathbb{R}\}$, while the range is $\{f(x): -1\le f(x)\le 1\}$.

Similarly, the domain of $f(x)=\cos x$ is the set of all real numbers, while the range is $\{f(x): -1\le f(x)\le 1\}$. Moreover, for the function $f(x)=\tan x=\dfrac{\sin x}{\cos x}$, then the domain of $f(x)=\tan x$ is the set of all real values except the values where $\cos x$ is zero. Therefore, the domain of $f(x)=\tan x$ is $\left\{x: x\in\mathbb{R},\ x\ne\dfrac{\pi}{2}+\pi n\right\}$, for all integer numbers n, and the range is $\{f(x): f(x)\in\mathbb{R}\}$.

Example 7.52

Give the domain and range of $f(x)=2\sin x$.

Solution

Given $f(x)=2\sin x$

Domain of $f(x)=2\sin x$ is a set of all real numbers, since the sine function is always defined over the entire set of real numbers. Thus, domain $=\{x: x\in\mathbb{R}\}$.

The range of $f(x)=2\sin x$ is determined by using the following procedures:

Since, $-1\le\sin x\le 1$, multiplying by 2 throughout gives $-2\le 2\sin x\le 2$ thus, the range of $f(x)=2\sin x$ is $\{f(x): -2\le f(x)\le 2\}$.

Therefore, the domain $=\{x: x\in\mathbb{R}\}$ and range $=\{f(x): -2\le f(x)\le 2\}$.

Example 7.53

Determine the domain and range of $g(x)=\dfrac{2}{3}\cos\dfrac{x}{2}$, for $-2\pi\le x\le 2\pi$.

Solution

Given $g(x)=\dfrac{2}{3}\cos\dfrac{x}{2}$, for $-2\pi\le x\le 2\pi$.

The domain is the set from, $-2\pi\le x\le 2\pi$

Range: Since, $-1\le\cos\frac{x}{2}\le 1$, thus multiplying $\frac{2}{3}$ throughout the inequalities gives, $-\dfrac{2}{3}\le\dfrac{2}{3}\cos\dfrac{x}{2}\le\dfrac{2}{3}$, thus,

$$\text{Range}=\left\{g(x): -\frac{2}{3}\le g(x)\le\frac{2}{3}\right\}$$

Therefore, the domain $=\{x: -2\pi\le x\le 2\pi\}$ and the range $=\left\{g(x): -\dfrac{2}{3}\le g(x)\le\dfrac{2}{3}\right\}$.

Exercise 7.10

Give the domain and range of each of the following functions:

1. $f(x) = \sec x$
2. $f(x) = \tan 2x$
3. $g(x) = \cot x$
4. $h(x) = \operatorname{cosec} x$
5. $g(x) = \sec \frac{x}{2}$
6. $f(x) = -2\cos x$, for $-\pi \le x \le \pi$
7. $g(x) = -4\tan 2x$
8. $h(x) = \frac{1}{2}\sec x$
9. $f(x) = 6\sin \frac{x}{4}$, for $0 \le x \le \pi$
10. $f(x) = \tan x$, *for* $-2\pi \le x \le 2\pi$

The graphs of sine and cosine functions

The sine and cosine functions are periodic functions with a period of 2π and are defined for all real values of θ. For $\theta \ge 0$, the first period of the function $y = \sin\theta$ is between 0 and 2π radians, and the second period is between 2π and 4π radians and so on. In general the n^{th} period is between $2(n-1)\pi$ and $2\pi n$ radians.

Similarly, for $y = \cos\theta$ the n^{th} period is between $\frac{\pi}{2} + 2(n-1)\pi$ and $\frac{\pi}{2} + 2\pi n$.

Activity 7.5: Drawing graphs of $f(\theta) = \sin\theta$

Learning resources: Graph paper, pencil, scientific calculator, and ruler.
Individually or in a group, perform the following tasks:

1. Construct a table of values of $f(\theta) = \sin\theta$ for $-2\pi \le \theta \le 2\pi$.
2. Draw the graph of $f(\theta) = \sin\theta$ on the graph paper.
3. Identify the properties of the graph drawn in task 2.
4. Share your results with your fellow students.

Example 7.54

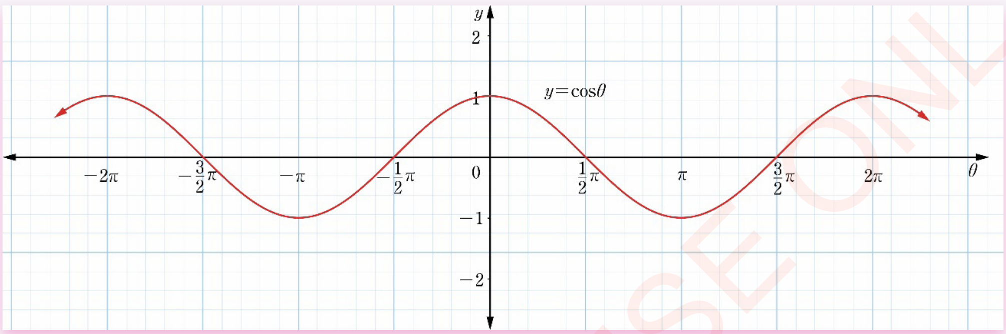
Draw the graph of $f(\theta) = \cos\theta$ for $-2\pi \le \theta \le 2\pi$.

Solution

In order to draw the graph of $f(\theta)$ prepare a table of values as follows;

θ	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$f(\theta)=\cos\theta$	1	0	-1	0	1	0	-1	0	1

Then, use the values in the table to draw the following graph



From the graphs in Example 7.54, the properties of the cosine function are summarized as:

1. The minimum and maximum values are -1 and 1, respectively.
2. The cosine function has a period of 2π radians.
3. The cosine is an even function, that is $f(-x)=\cos(-x)=f(x)$.
4. The amplitudes of the function is are -1 and 1.

Graph of the tangent function

The graph of $f(\theta)=\tan\theta$ is symmetrical about the origin (0, 0), since it is an odd function. The function $f(\theta)=\tan\theta$ is a periodic function with a period of π and it is defined at all points except where $\theta=\frac{\pi}{2}+n\pi$, n is any integer. The lines $\theta=\frac{\pi}{2}+n\pi$ are vertical asymptotes since $f(\theta)=\tan\theta$ tends to $\pm\infty$ as $\theta\to\frac{\pi}{2}+n\pi$.

Example 7. 55

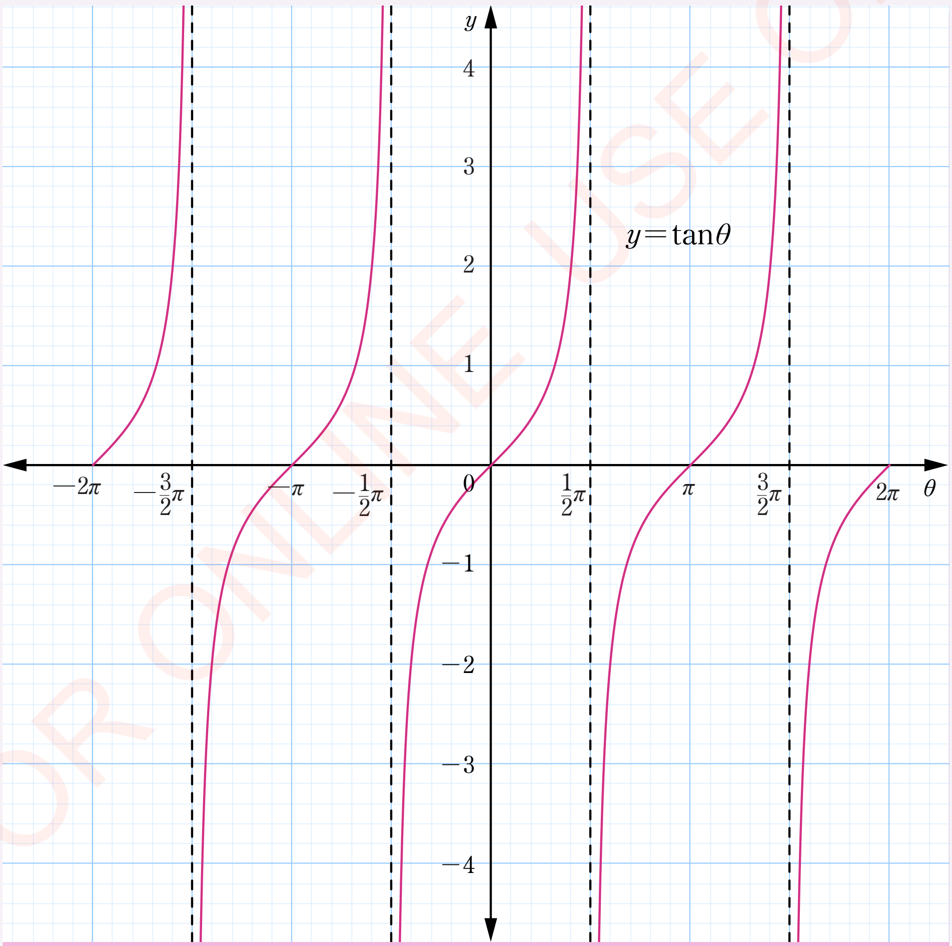
Draw the graph of $f(\theta) = \tan\theta$.

Solution

In order to draw the graph of $f(\theta)$ prepare a table of values as follows;

θ	-2π	$-\frac{3}{2}\pi$	$-\pi$	$-\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	π	$\frac{3}{2}\pi$	2π
$f(\theta) = \tan\theta$	0	$-\infty$	0	$-\infty$	0	∞	0	∞	0

Then, use the table of values to draw the following graph.



Inverse trigonometric functions

The inverse trigonometric functions are inverses of sine, cosine, tangent, cotangent, secant, and cosecant. In trigonometry, inverse functions are used to find the angle of any trigonometric ratios. The inverse of sine, cosine, and tangent are defined as follows;

(i) $\theta = \sin^{-1} x$ or $\theta = \arcsin x$ if and only if $x = \sin\theta$ and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

(ii) $\theta = \cos^{-1} x$ or $\theta = \arccos x$ if and only if $x = \cos\theta$ and $0 \le \theta \le \pi$.

(iii) $\theta = \tan^{-1} x$ or $\theta = \arctan x$ if and only if $x = \tan\theta$ and $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

The inverses of other trigonometric functions can be defined using the inverse of sine, cosine and tangent. For instance, if $y = \sec\theta$ then $\theta = \sec^{-1} y$ can be expressed in terms $\cos^{-1} y$ as follows,

Given $y = \sec\theta$

$\Rightarrow \theta = \sec^{-1} y$ .... (i)

Also, $y = \dfrac{1}{\cos\theta}$

$\Rightarrow \cos\theta = \dfrac{1}{y}$

$\theta = \cos^{-1}\left(\dfrac{1}{y}\right)$ ....(ii)

Equate equations (i) and (ii) to obtain;

$\theta = \sec^{-1} y = \cos^{-1}\dfrac{1}{y}$, for $y \ge 1$ or $y \le 1$

Therefore, $\sec^{-1} y = \cos^{-1}\dfrac{1}{y}$.

Using a similar approach the following results can be derived;

(i) If $y = \operatorname{cosec}\theta$ then $\theta = \operatorname{cosec}^{-1} y = \sin^{-1}\dfrac{1}{y}$, for $y \ge 1$ or $y \le 1$.

(ii) If $y = \cot\theta$ then $\theta = \cot^{-1} y = \tan^{-1}\dfrac{1}{y}$, for $0 < y < \pi$.

Example 7.56

Show that $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x$

Solution

From the left-hand side,

Let $x=\sin\theta$ then

$$\sin^{-1}\left(2x\sqrt{1-x^2}\right)=\sin^{-1}\left(2\sin\theta\sqrt{1-\sin^2\theta}\right)$$

$$=\sin^{-1}\left(2\sin\theta\sqrt{\cos^2\theta}\right)$$

$$=\sin^{-1}(2\sin\theta\cos\theta)$$

$$=\sin^{-1}(\sin 2\theta)=2\theta,\text{ but } \theta=\sin^{-1}x$$

$$=2\sin^{-1}x$$

Therefore, $\sin^{-1}\left(2x\sqrt{1-x^2}\right)=2\sin^{-1}x.$

Example 7.57

If $\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)=\frac{\pi}{4}$, then find the positive value of x correct to four significant figures.

Solution

Given $\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)=\frac{\pi}{4}$.

Applying tangent on both sides gives, $\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)-\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)=\tan\frac{\pi}{4}$

let $\mathrm{A}=\tan^{-1}\left(\frac{1-4x}{1+6x}\right)$ and $\mathrm{B}=\tan^{-1}\left(\frac{1+2x}{1-3x}\right)$, and using

$$\tan(\mathrm{A}-\mathrm{B})=\frac{\tan\mathrm{A}-\tan\mathrm{B}}{1+\tan\mathrm{A}\tan\mathrm{B}}.$$

$$\Rightarrow \frac{\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)\right)-\tan\left(\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)}{1+\tan\left(\tan^{-1}\left(\frac{1-4x}{1+6x}\right)\right)\tan\left(\tan^{-1}\left(\frac{1+2x}{1-3x}\right)\right)}=1$$

$$\Rightarrow \frac{\left(\frac{1-4x}{1+6x}\right)-\left(\frac{1+2x}{1-3x}\right)}{1+\left(\frac{1-4x}{1+6x}\right)\left(\frac{1+2x}{1-3x}\right)}=1\text{, since }\tan\left(\tan^{-1}B\right)=B.$$

$$\Rightarrow \frac{1-7x+12x^2-1-8x-12x^2}{1+3x-18x^2+1-2x-8x^2}=1$$

$$\Rightarrow 26x^2-16x-2=0$$

Either $x=0.7219$ or $x=-0.1066$

Therefore, the positive value of x is 0.7219.

Example 7.58

Express $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$ in its simplest form.

Solution

To simplify $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$.

From,

$$\cos x=\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right),\ \cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)=1,$$

and $\sin x=2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$ the given expression can be written as,

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)=\tan^{-1}\left(\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$=\tan^{-1}\left[\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)}{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}\right]$$

$$= \tan^{-1}\left(\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right)$$

Dividing by $\cos\frac{x}{2}$ to both the numerator and denominator to obtain;

$$\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \tan^{-1}\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$$

$$= \frac{\pi}{4}+\frac{x}{2}$$

Therefore, $\tan^{-1}\left(\frac{\cos x}{1-\sin x}\right) = \frac{\pi}{4}+\frac{x}{2}$.

Example 7.59

If $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$, prove that, $x^2+y^2+z^2+2xyz=1$.

Solution

Given $\sin^{-1}x+\sin^{-1}y+\sin^{-1}z=\frac{\pi}{2}$.

Let, $\mathrm{A}=\sin^{-1}x \Rightarrow x=\sin\mathrm{A}$

$\mathrm{B}=\sin^{-1}y \Rightarrow y=\sin\mathrm{B}$

$\mathrm{C}=\sin^{-1}z \Rightarrow z=\sin\mathrm{C}$

But $\mathrm{A}+\mathrm{B}+\mathrm{C}=\frac{\pi}{2} \Rightarrow \mathrm{A}+\mathrm{B}=\frac{\pi}{2}-\mathrm{C}$

Applying cosine on both sides of the equation, $\mathrm{A}+\mathrm{B}=\frac{\pi}{2}-\mathrm{C}$ gives;

$$\cos(\mathrm{A}+\mathrm{B})=\cos\left(\frac{\pi}{2}-\mathrm{C}\right)$$

Using compound angle formula gives,

$\cos A \cos B - \sin A \sin B = \sin C$

But $\cos A = \sqrt{1-x^2}$ and $\cos B = \sqrt{1-y^2}$

Thus, $\sqrt{1-x^2} \times \sqrt{1-y^2} - xy = z$

$\Rightarrow \sqrt{1-x^2} \times \sqrt{1-y^2} = z + xy$

Squaring both sides,

$$\left(\sqrt{1-x^2} \times \sqrt{1-y^2}\right)^2 = (z+xy)^2$$

$\Rightarrow (1-x^2)(1-y^2) = (z+xy)(z+xy)$

$\Rightarrow 1 - x^2 - y^2 + x^2y^2 = z^2 + 2xyz + x^2y^2$

Simplify the equation to obtain,

$x^2 + y^2 + z^2 + 2xyz = 1$

Therefore, $x^2 + y^2 + z^2 + 2xyz = 1$.

Example 7.60

Solve the equation $\cos^{-1} x + \cos^{-1}\left(x\sqrt{3}\right) = \frac{\pi}{2}$.

Solution

Given $\cos^{-1} x + \cos^{-1}\left(x\sqrt{3}\right) = \frac{\pi}{2}$.

Let $A = \cos^{-1} x \Rightarrow \cos A = x$ and $B = \cos^{-1}\left(x\sqrt{3}\right) \Rightarrow \cos B = \left(x\sqrt{3}\right)$

But $\sin A = \sqrt{1-\cos^2 A}$

$\Rightarrow \sin A = \sqrt{1-x^2}$

Also, $\sin B = \sqrt{1-\cos^2 B}$

$\Rightarrow \sin B = \sqrt{1-3x^2}$

Now, $A + B = \frac{\pi}{2}$

Apply cosine on both sides of the equation $A + B = \frac{\pi}{2}$ to obtain:

$\cos(A+B)=\cos\frac{\pi}{2}$

Thus, $\cos A\cos B-\sin A\sin B=0$

Substituting the values gives;

$\left(x^2\sqrt{3}\right)-\left(\sqrt{1-x^2}\right)\left(\sqrt{1-3x^2}\right)=0$

$\Rightarrow \sqrt{1-4x^2+3x^4}=x^2\sqrt{3}$

Squaring both sides, gives;

$1-4x^2+3x^4=3x^4$

Simplify the equation to obtain;

$4x^2-1=0$

Hence, $x=\pm\frac{1}{2}$

But $x=-\frac{1}{2}$ does not satisfy the given equation.

Therefore, the value of x is $\frac{1}{2}$.

Exercise 7.11

1. Evaluate each of the following expressions without using a non-programmable scientific calculator:

   (a) $\sin\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right)+\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right)$

   (b) $\cos\left(\sin^{-1}\frac{1}{2}-\sin^{-1}\frac{\sqrt{2}}{2}\right)$

   (c) $\cot\left(\frac{\pi}{2}-2\cot^{-1}\sqrt{3}\right)$

   (d) $\cos\left(\cos^{-1}\left(\frac{3}{5}\right)-\sin^{-1}\left(\frac{4}{5}\right)\right)$

2. Without using a scientific calculator, find the value of each of the following:

   (a) $\tan x$, if $x=\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{7}{24}$

   (b) $\sin x$, if $x=\sin^{-1}\frac{4}{5}-\cos^{-1}\frac{8}{17}$

   (c) $\cos x$, if $x=\cos^{-1}\frac{24}{25}+\sin^{-1}\frac{15}{17}$

   (d) $\sin x$, if $x=\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{5}{13}$

3. Prove each of the following trigonometric equations:

(a) $\frac{1}{4}\pi - \tan^{-1}\frac{2}{3} = \tan^{-1}\frac{1}{5}$

(b) $\tan^{-1} p + \tan^{-1} q = \tan^{-1}\left(\frac{p+q}{1-pq}\right)$

(c) $\cos\left(2\tan^{-1} x\right) = \frac{1-x^2}{1+x^2}$

(d) $3\tan^{-1} 2 - \tan^{-1}\frac{2}{11} = \pi$

(e) $2\tan^{-1} 2 + \tan^{-1} 3 = \pi + \tan^{-1}\frac{1}{3}$

(f) $\frac{\pi}{4} + \tan^{-1} x = \tan^{-1}\left(\frac{1+x}{1-x}\right)$

(g) $\cot^{-1}\left(\frac{\sqrt{1+\sin\theta}-\sqrt{1-\sin\theta}}{\sqrt{1+\sin\theta}+\sqrt{1-\sin\theta}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}\theta$

(h) $\tan^{-1}\left(\frac{\cos\alpha}{1+\sin\alpha}\right) = \frac{\pi}{4} - \frac{\alpha}{2}$

4. Simplify each of the following expressions:

(a) $\cos\left(\sin^{-1} x\right)$

(b) $\sin\left(\tan^{-1} x\right)$

5. Solve each of the following equations:

(a) $\arcsin x + \arctan x = \frac{\pi}{2}$

(b) $\sin\left(2\cos^{-1} x\right) = \sqrt{1-x^2}$

(c) $\arctan 2x + \arctan x = \frac{\pi}{4}$

(d) $2\sin^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(x\sqrt{2}\right) = \frac{\pi}{2}$

(e) $\cos^{-1} x + \cos^{-1}\left(x\sqrt{8}\right) = \frac{\pi}{2}$

(f) $2\sin^{-1}\left(x\sqrt{6}\right) + \sin^{-1}(4x) = \frac{\pi}{2}$

(g) $\tan^{-1}\left(\frac{z-2}{z-4}\right) + \tan^{-1}\left(\frac{z+2}{z+4}\right) = \frac{\pi}{4}$

(h) $\arccos x + \arctan x = \frac{\pi}{2}$

(i) $\tan^{-1}\left(\frac{2y}{1-y^2}\right) = \pi - \cot^{-1}\left(\frac{1-y^2}{2y}\right)$

(j) $\cos\left(\tan^{-1} x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

6. If $\sin^{-1} y = 2\cos^{-1} x$, show that $y^2 = 4x^2\left(1-x^2\right)$.

7. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

8. For each of the following expressions, show that:

(a) $\cos^{-1}\left(\frac{63}{65}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \sin^{-1}\left(\frac{3}{5}\right)$

(b) $2\sin^{-1}\left(\sqrt{x-a+\frac{1}{2}}\right) = \cos^{-1}(2a-2x)$

(c) $\sin^{-1} y + \cos^{-1} y = \frac{\pi}{2} = \cot^{-1} y + \tan^{-1} y$

9. If U, V, and W are such that $(U-W)(V-W)=1+W^2$, then verify that, $\tan^{-1}\left(\frac{1}{U}\right)+\tan^{-1}\left(\frac{1}{V}\right)=\tan^{-1}\left(\frac{1}{W}\right)$.
10. Evaluate $\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{3}\right)-\tan^{-1}\left(\frac{13}{9}\right)$ without using a non-programmable scientific calculator.

Graphs of inverse trigonometric functions

The graphs of inverse trigonometric functions differ from those of trigonometric functions, the roles of y and θ are interchanged. For example, the graph of $y=\sin^{-1}\theta$ is a sine curve drawn on the y-axis instead of the x-axis. The domains of the inverse of trigonometric functions are restricted.

Activity 7.6: Drawing the graph of y = sin^{-1} x

Learning resources: Graph papers, pencil, scientific calculator, and ruler.
Individually or in a group, perform the following tasks:

1. Construct a table of values of $y=\sin^{-1}x$ for $-1\le x\le 1$.
2. Draw the graph of $y=\sin^{-1}x$ on the xy-plane.
3. Identify the domain and range of $y=\sin^{-1}x$ from the graph drawn in task 2.
4. On a different xy-plane draw the graph of $y=\sin x$ for $-\frac{\pi}{2}\le x\le\frac{\pi}{2}$.
5. Give the suggestions on the graphs drawn in tasks 2 and 4.
6. What have you observed from the tasks? Give comments.

Example 7.61

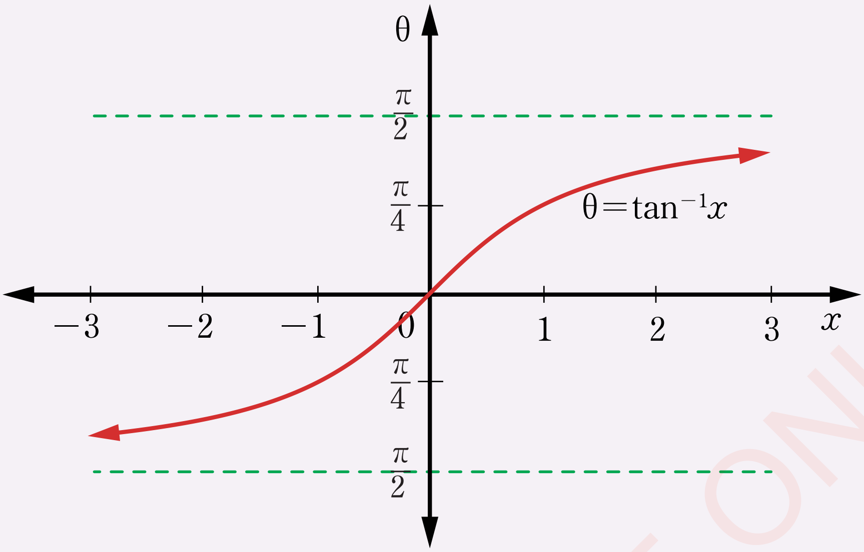
Draw the graph of $\theta=\tan^{-1}x$ for $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and identify its domain and range.

Solution

In order to draw the graph of $\theta=\tan^{-1}x$ prepare a table of values as follows;

x	$-\infty$	-1	0	1	∞
$\theta=\tan^{-1}x$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$

Use the table of values to draw the following graph.



From the graph,

Domain $= \{x : x \in \mathbb{R}\}$

Range $= \left\{\theta : -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right\}$

Example 7.62

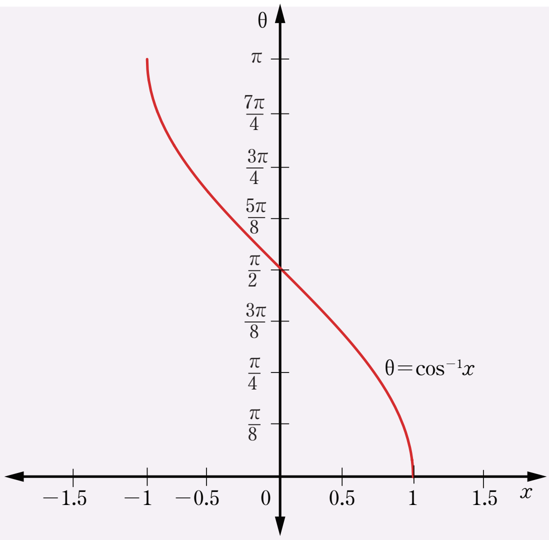
Draw the graph of $\theta = \cos^{-1} x$ for $-1 < x < 1$ and identify its domain and range.

Solution

In order to draw the graph of $\theta = \cos^{-1} x$ prepare a table of values as follows;

x	-1	0	1
$\theta = \cos^{-1} x$	π	$\frac{\pi}{2}$	0

Use the table of values to draw the following graph of $\theta = \cos^{-1} x$.



From the graph

Domain $= \{x \in \mathbb{R} : -1 \le x \le 1\}$

Range $= \{\theta \in \mathbb{R} : 0 \le \theta \le \pi\}$

Exercise 7.12

1. Draw the graph of each of the following functions:
   (a) $h(x) = \sin\left(\cos^{-1} x\right)$
   (b) $f(x) = \cos^{-1}(\tan x)$
   (c) $g(x) = \cot^{-1} x$
   (d) $g(x) = \sec^{-1} x$
   (e) $g(x) = \operatorname{cosec}^{-1} x$
   (f) $h(x) = \cos\left(\sin^{-1} \dfrac{x}{2}\right)$
2. State the domain and range of each of the following functions:
   (a) $y = \sin^{-1} x$
   (b) $y = \cot^{-1} x$
   (c) $y = \tan^{-1} 2x$
   (d) $y = \cos^{-1} x$
3. (a) To define an arccotangent function, first restrict cotangent to an interval on which it is one to one and takes on all real numbers. What is the interval for such restrictions?
   (b) Using the interval found in part (a) define $\cot x$ and $\cot^{-1} x$ functions, hence draw the graphs of both functions on the same axes.
4. Simplify

$$x = \frac{\sin(y+\pi) + \sin(y-\pi)}{\sin\dfrac{\pi}{6}}$$ and

   draw the graph of the simplified equation.
5. Given $x = 2\cos^2 y - 1$, express y in terms of x and draw the graph of the resulting function.

Chapter summary

1. The trigonometric ratios are $\sin\theta$, $\cos\theta$, and $\tan\theta$. Their corresponding reciprocals are $\operatorname{cosec}\theta$, $\sec\theta$, and $\cot\theta$, respectively.
2. The trigonometric identities (Pythagorean identities) are:
   (a) $\cos^2\theta + \sin^2\theta = 1$
   (b) $1 + \cot^2\theta = \operatorname{cosec}^2\theta$
   (c) $1 + \tan^2\theta = \sec^2\theta$

3. The compound angle formulae are:
   (a) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$
   (b) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$
   (c) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$

4. The double angle formulae are:
   (a) $\sin 2A = 2\sin A \cos A$
   (b) $\cos 2A = \cos^2 A - \sin^2 A$
   (c) $\cos 2A = 2\cos^2 A - 1$
   (d) $\cos 2A = 1 - 2\sin^2 A$
   (e) $\tan 2A = \dfrac{2\tan A}{1 - \tan^2 A}$

5. The *t*-formulae are:
   (a) $\sin\theta = \dfrac{2t}{1+t^2}$
   (b) $\cos\theta = \dfrac{1-t^2}{1+t^2}$
   (c) $\tan\theta = \dfrac{2t}{1-t^2}$ where $t = \tan\dfrac{1}{2}\theta$.

6. The factor formulae are given by:
   (a) $\sin P + \sin Q = 2\sin\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$
   (b) $\sin P - \sin Q = 2\cos\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$
   (c) $\cos P + \cos Q = 2\cos\left(\dfrac{P+Q}{2}\right)\cos\left(\dfrac{P-Q}{2}\right)$
   (d) $\cos P - \cos Q = -2\sin\left(\dfrac{P+Q}{2}\right)\sin\left(\dfrac{P-Q}{2}\right)$.

7. For a small angle θ the approximation of sine, cosine, and tangent are:
   (a) $\sin\theta \approx \theta$
   (b) $\cos\theta \approx 1 - \dfrac{1}{2}\theta^2$
   (c) $\tan\theta \approx \theta$

Revision exercise 7

1. If $\cos A = \frac{21}{24}$ and $\sin B = -\frac{8}{17}$, where A and B are angles in the fourth and third quadrants, respectively, evaluate each of the following:
   (a) $\tan(A-B)$ (b) $\cos(A-B)$ (c) $\sin(A+B)$

2. Eliminate θ in each of the following pairs of equations:
   (a) $x = 9+4\cos\theta,\ y = 7+15\sin\theta$
   (b) $x = 5+3\tan\theta,\ y = 4+\tan 2\theta$
   (c) $x = 1+\cos 2\theta,\ y = \sin\theta$
   (d) $x = 4\sec\theta,\ y = \cos 3\theta$

3. Simplify each of the following expressions:
   (a) $\tan(45°+\theta)-\tan(45°-\theta)$ (c) $\frac{\cos 5x-\cos 3x}{\sin 5x+\sin 3x}$
   (b) $\sin^3\theta+\sin^3(120°+\theta)+\sin^3(240°+\theta)$ (d) $\frac{\sin 5y-\sin y}{\sin 10y-\sin 6y}$

4. Verify each of the following identities:
   (a) $\frac{\sin 4x-\sin 2x}{\cos 4x+\cos 2x} = \tan x$ (b) $\frac{\sin x-\sin 3x}{\sin^2 x-\cos^2 x} = 2\sin x$

5. Convert each of the following radian angles into degrees:
   (a) $\frac{23\pi}{9}$ (b) $\frac{51\pi}{29}$ (c) 7.5 (d) $-\frac{5\pi}{6}$

6. Find in radians the angle between the minute-hand and the hour-hand of a clock at:
   (a) 4:55 p.m (b) 5:48 p.m

7. Convert each of the following into radians:
   (a) $-540°$ (b) $-1050°$ (c) $120°30'45''$

8. For each of the following expressions find its approximation when θ is a small angle
   (a) $\frac{\sin^2(3\theta)+2\theta}{\tan\theta}$ (b) $\frac{1-\cos 4\theta}{2\theta\sin 3\theta}$ (c) $\frac{\cos 3\theta-1}{\theta\sin 4\theta}$ (d) $\tan 3\theta\sin 2\theta$

9. Prove each of the following identities:
   (a) $\frac{\sin\theta-\cos\theta+1}{\sin\theta+\cos\theta-1} = \frac{1}{\sec\theta-\tan\theta}$ (b) $\frac{1-\sec x}{\sec x} = \frac{\sin^2 x}{1-\cos x}$

(c) $\dfrac{\tan\theta}{1-\cot\theta}+\dfrac{\cot\theta}{1-\tan\theta}=1+\sec\theta\operatorname{cosec}\theta$

(d) $\dfrac{\tan\theta+\tan 7\theta}{\tan 3\theta+\tan 5\theta}=\dfrac{\cos 3\theta\cos 5\theta}{\cos\theta\cos 7\theta}$

(e) $\left(\dfrac{\cos\theta-\cos 3\theta}{\sin 5\theta-\sin\theta}\right)\left(\dfrac{\sin 8\theta+\sin 2\theta}{\cos 4\theta-\cos 6\theta}\right)=1$

(f) $\cos\left(\dfrac{1}{4}\pi+\theta\right)\cos\left(\dfrac{1}{4}\pi+\theta\right)=\dfrac{1}{2}\cos 2\theta$

(g) $\sqrt{\dfrac{1+\sin x}{1-\sin x}}=\sec x+\tan x$

(h) $(\operatorname{cosec} x-\sin x)(\sec x-\cos x)=\dfrac{1}{\tan x+\cot x}$

(i) $\dfrac{\sin x-2\sin^3 x}{2\cos^3 x-\cos x}=\tan x$

(j) $\sqrt{\dfrac{1+\sin\theta}{1-\sin\theta}}+\sqrt{\dfrac{1-\sin\theta}{1+\sin\theta}}=2\sec\theta$

10. Use t-formula to solve the following equations for $-180°\le\theta\le 180°$

(a) $5\sin 2\theta-12\cos 2\theta=1$ (b) $2\sin\theta+7\cos\theta=-4$

11. Find the general solution of each of the following equations:

(a) $\sin x-\sin 2x=\sin 4x-\sin 3x$

(b) $\cos\theta+\sin\theta=1$

(c) $\cos\theta-\sin 4\theta=0$

(d) $3\cos\theta+\sin\theta=2$

(e) $\cos ax+\cos bx=0$, where a and b are constants

(f) $6\cos\theta-4\sin\theta=7$

(g) $\sin\theta+\sin 3\theta+\sin 5\theta=0$

(h) $\tan 3\theta=\cot\theta+\cot 2\theta$

12. Find the solution of each of the following equations for the angle 0° and 360°, inclusive.

(a) $\begin{cases}\tan x+\tan y=4\\ \tan 2x+\tan 2y=0\end{cases}$ (b) $\begin{cases}\cos x+\cos y=\cos\dfrac{7\pi}{12}\\ \sin x+\sin y=\sin\dfrac{7\pi}{12}\end{cases}$

13. If $\sin\theta = \frac{4}{5}$, find the value of $\frac{5\sec\theta + 4\operatorname{cosec}\theta + 3\tan\theta}{4\cot\theta + 3\sec\theta + 5\sin\theta}$.

14. Prove the identity $\cos 4\theta + 4\cos 2\theta = 8\cos^4\theta - 3$, hence, solve the equation $\cos 4\theta + 4\cos 2\theta = 2$ for θ between $0°$ and $360°$.

15. Show that $\operatorname{cosec}2\theta + \cot 2\theta = \cot\theta$, hence solve the equation $\operatorname{cosec}2\theta + \cot 2\theta = 2$, for $0° \le \theta \le 360°$.

16. Solve each of the following equations by using $t-$formulae for $0° \le x \le 360°$:

(a) $3\sin x + 4\cos x = 2$

(b) $2\cos x - \sin x = 1$

(c) $2\cos x + 3\sin x = 2$

(d) $4\cos x\sin x + 15\cos 2x = 10$

(e) $3\tan x - 2\sec x = 4$

(f) $\sqrt{3}\cos x + \sin x = 1$

17. Find the values of R and α in each of the following equation:

(a) $5\cos\theta + 12\sin\theta = \text{R}\cos(\theta - \alpha)$

(b) $3\sin\theta - 4\cos\theta = \text{R}\sin(\theta - \alpha)$

(c) $6\sin 3\theta + 8\cos 3\theta = \text{R}\sin(3\theta + \alpha)$

(d) $\cos 2\theta + \sin 2\theta = \text{R}\cos(2\theta - \alpha)$

18. Find the maximum and minimum values of each of the following expressions, and their corresponding values of θ for $0° \le \theta \le 360°$, hence give the values of θ for which the maximum and minimum occur.

(a) $\cos\theta + \sin\theta$

(b) $3\sqrt{2}\cos(\theta + 45°) + 7\sin\theta$

(c) $8\cos\theta - 15\sin\theta$

(d) $\sin\theta - 6\cos\theta$

19. Express $3\cos x + 4\sin x$ in the form $R\sin(x + \alpha)$ where α is an acute angle. Hence, find the maximum and minimum values of the expression and their corresponding values of x for $-180° \le x \le 180°$ at which the maximum and minimum occur.

20. If θ is small, simplify each of the following:

(a) $\frac{\cos(\theta + \alpha) - \cos(\theta - \alpha)}{4\theta}$ (b) $\frac{(1 - \cos 2\theta)(1 + \tan\theta)}{3\tan\theta\sin\theta}$ (c) $\tan\left(\frac{\pi}{3} + \theta\right)$

21. Express $5\sin^2 x - 3\sin x\cos x + \cos^2 x$ in the form $a + b\cos(2x - \alpha)$ where a, b, α are independent of x. Hence or otherwise, find the maximum and minimum value of $5\sin^2 x - 3\sin x\cos x + \cos^2 x$ as x varies.

22. Express $\tan(45° + x) - \tan x = 2$ it in the form $\tan^2 x + 2\tan x - 1 = 0$. Hence, solve the equation $\tan(45° + x) - \tan x = 2$, giving all solutions in the interval $0° < x < 180°$.

23. If $a\cos^2 x + b\sin^2 x = c$, show that $\tan^2 x = \dfrac{c-a}{b-c}$. Hence, deduce that $\tan^2 x = \dfrac{1}{3}$ given that $6\cos^2 x + 2\sin^2 x = 5$.

24. If $t = \tan\dfrac{1}{2}\theta$, find the values of t which satisfy the equation $(p+2)\sin\theta + (2p-1)\cos\theta = 2p+1$ where p is a non-zero constant. Hence, find the angles which satisfy the equation when $p = \sqrt{3}$, for $-180° < \theta < 180°$.

25. Express each of the following in factor form:

(a) $\sin 16\theta - \sin 7\theta$

(b) $\sin 13\theta - \sin(-9\theta)$

(c) $\cos\left(-\dfrac{27}{5}\theta\right) - \cos\left(\dfrac{16}{5}\theta\right)$

(d) $\cos\left(\dfrac{5\theta}{2}\right) + \cos\left(\dfrac{9\theta}{2}\right)$

26. If $\sin x + \sin 2x = a$ and $\cos x + \cos 2x = b$, prove that $(a^2 + b^2)(a^2 + b^2 - 3) = 2b$

27. Prove each of the following:

(a) $\sin^{-1} x + \sin^{-1} y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$

(b) $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$

(c) $\cos\tan^{-1}\left(\sin\cot^{-1} x\right) = \left(\dfrac{x^2+1}{x^2+2}\right)^{\frac{1}{2}}$

(d) $\tan^{-1} x + \tan^{-1} y = \dfrac{1}{2}\sin^{-1}\left[\dfrac{2(x+y)(1-xy)}{(1+x^2)(1+y^2)}\right]$

(e) $\cot^{-1}\dfrac{1}{3} = \cot^{-1} 3 + \cos^{-1}\dfrac{3}{5}$

28. Find the values of x which satisfy the equation

$$\tan^{-1}\left(\frac{1-x}{1+x}\right) - \tan^{-1}\left(\frac{1-6x}{1+6x}\right) = \frac{5\pi}{4}.$$

29. If the equation $m\cos 2\theta + n\sin 2\theta = p$ has θ_1 and θ_2 as its root, prove that

$$\tan\theta_1 + \tan\theta_2 = \frac{2n}{p+m}.$$

30. If $p' = p\cos\theta + q\sin\theta$ and $q' = p\sin\theta - q\cos\theta$, show that $p'^2 + q'^2 = p^2 + q^2$.

31. Show that $\dfrac{1+\sin x - \cos x}{1+\sin x + \cos x} + \dfrac{1+\sin x + \cos x}{1+\sin x - \cos x} = \dfrac{2}{\sin x}$.

32. Prove that $\dfrac{\sin(x+3y) + \sin(3x+y)}{\sin 2x + \sin 2y} = 2\cos(x+y)$.

33. Show that $\tan\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right) = \sqrt{y}$ given that $\sin\theta = \dfrac{1-y}{1+y}$.

34. Show that the general solution of $\tan\left(3x - \dfrac{\pi}{4}\right) = \tan x$ is $x = \dfrac{(4n+1)\pi}{8}$, where n is an integer.

35. Prove that $\cos x - \sin x = \sqrt{2}\cos\left(x + \dfrac{\pi}{4}\right) = -\sqrt{2}\sin\left(x - \dfrac{\pi}{4}\right)$. Hence, evaluate $\cos x - \sin x = 1$, for $-2\pi \le x \le 2\pi$.

36. Express $3\sin 6\theta + 5\cos 6\theta$ in the form $R\cos(6\theta - \alpha)$. Hence, or otherwise solve the equation $3\sin 6\theta + 5\cos 6\theta = 5$, for $0° \le \theta \le 360°$.

37. Given that $t = \tan x$, write an expression for $\tan 2x$ in terms of t. Hence, or otherwise, find the general solution in radians, of the equation $\tan x + \tan 2x = 0$.

38. Prove each of the following identities:

(a) $\dfrac{1+\sin\theta - \cos\theta}{1+\sin\theta + \cos\theta} = \tan\dfrac{\theta}{2}$

(b) $\dfrac{\cos\theta + \cos 2\theta + \cos 3\theta + \cos 4\theta}{\sin\theta + \sin 2\theta + \sin 3\theta + \sin 4\theta} = \cot\dfrac{5}{2}\theta$

39. If $\sin x = \dfrac{1}{4}$ and x is in the second quadrant, find each of the following equations (leaving the answers in surd form):

(a) $\sin\dfrac{x}{2}$ (b) $\cos\dfrac{x}{2}$ (c) $\tan\dfrac{x}{2}$

40. Use the compound angle formula to show that $\cot(\lambda - \beta) = \dfrac{1 + \cot\lambda\cot\beta}{\cot\beta - \cot\lambda}$. Hence, deduce that, if $\cot\lambda = 0.5$, $\cot\beta = 2$, and $\cot\gamma = 3$, then $\cot(\lambda + \beta + \gamma) = 3$.

Chapter Eight

Linear programming

Introduction

Most of the production companies and industries focus on optimization of profit in such a way that operational cost are minimized in order to maximize profit. Linear programming is a mathematical technique for finding optimal solutions to problems that can be expressed as linear equations and inequalities. In this chapter, you will learn about formulation of linear programming problems, graphical solutions, and transportation problems. The competencies developed will help you to decide, allocate, select, schedule, and evaluate resources in the possible way for the purpose of optimizing the available resources especially in the fields of agriculture, business, engineering, energy, manufacturing, and transportation.

Formulation of linear programming problems

Formulation of a linear programming problem involves the interpretation of the verbal description of the problem into algebraic equations or inequalities aiming at optimizing the scarce resources available. Formulation of a linear programming problem requires the decision variables, objective function, and constraints.

Decision variables

The decision variables are unknown quantities that decide the output. They are set of quantities that need to be determined in order to solve the problem. They represent the ultimate solution. To solve any linear programming problem, the decision variables need to be identified. Decision variables are presented by letters. For instance, let x represents the number of products of type A manufactured per month, and y represents the number of products of type B manufactured per month.

Objective function

An objective function is a linear function whose value is to be either minimized or maximized subject to the constraints defined over the set of feasible solutions. For instance, in maximization problems, the objective function is written as: Maximize $z = ax + by$, while for minimization problems it is written as: Minimize $z = ax + by$.

Constraints

Constraints are inequalities or equations which connect the decision variables under certain restrictions or limitations. Usually, constraints limit the values of the decision variables due to availability of resources. For instance, $ax + by \leq c$ for a maximization problem and $ax + by \geq c$ for a minimization problem. The constants a and b are proportional contributions of each decision variable to both the objective function and constraints.

Non-negativity constraints

The decision variables should always take non-negative values for all linear programming problems. That is, values for decision variables should be greater than or equal to zero.

Example 8.1

Two products, P_1 and P_2 require machines and labours in order to be produced from a manufacturing industry. Product P_1 requires 1.5 machine hours and 2.5 labour hours, while product P_2 requires 2.5 machine hours and 1.5 labour hours. The available machine hours is 300 and labour hours is 240 per month. The profit for P_1 is 1,600 Tanzanian shillings and for P_2 is 1,280 Tanzanian shillings per month. Formulate a linear programming problem for maximization of profit.

Solution

The given information are summarized as shown in the following table:

Products	Resources	
	Machine	**Labour**
P_1	1.5	2.5
P_2	2.5	1.5
Available hours	300	240

Identify the decision variables as follows:

Let: x represents the number of products of type P_1 manufactured per month,

y represents the number of products of type P_2 manufactured per month.

But the profit on the production of the two products has to be maximized. Thus, the objective function is given by;

Maximize $z = 1{,}600x + 1{,}280y$

The constraints are:

$$1.5x + 2.5y \leq 300$$

$$2.5x + 1.5y \leq 240$$

$$x \geq 0,\ y \geq 0$$

Therefore, the linear programming problem is;

Maximize z = 1,600x + 1,280y

Subject to: $1.5x + 2.5y \leq 300$

$$2.5x + 1.5y \leq 240$$

$$x \geq 0,\ y \geq 0$$

Example 8.2

The labour cost for two professional tailors, A and B are Tshs 80,000 and Tshs 100,000 per day, respectively. Tailor A can stitch 6 shirts and 4 pairs of trousers per day, while tailor B can stitch 10 shirts and 4 pairs of trousers per day. The tailors intend to produce at least 60 shirts and 32 pairs of trousers. Formulate a linear programming problem which minimizes the labour cost involved.

Solution

The given information are summarized as shown in the following table:

	Tailor A	Tailor B	Minimum requirements
Shirts	6	10	60
Pair of trousers	4	4	32
Labour cost per day (Tshs)	80,000	100,000	

Let: x be the number of days tailor A works,

y be the number of days tailor B works.

Thus, the objective function is given by;

Minimize $z = 80,000x + 100,000y$

The constraints are:

$$6x + 10y \geq 60$$

$$4x + 4y \geq 32$$

$$x \geq 0,\ y \geq 0$$

Therefore, the linear programming problem is;

Minimize $z = 80,000x + 100,000y$

Subject to: $6x + 10y \geq 60$

$4x + 4y \geq 32$

$x \geq 0,\ y \geq 0$

Example 8.3

Antonia wishes to mix two types of drinks, D_1 and D_2 in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Drink D_1 costs Tshs 8,880 per litre and drink D_2 costs Tshs 11,840 per litre. Drink D_1 contains 3 units of vitamin A per kilogram and 5 units of vitamin B per kilogram, while drink D_2 contains 4 units of vitamin A per kilogram and 2 units of vitamin B per kilogram. Formulate a linear programming problem to minimize the cost of the mixture.

Solution

The given information are summarized as shown in the following table:

Vitamin	**Units of Vitamin per kg**		**Minimum requirements (Units)**
	D_1	D_2	
A	3	4	8
B	5	2	11
Cost (Tshs) per litre	8,880	11,840	

Let: x be the number of units of mixture of drink D_1,

y be the number of units of mixture of drink D_2.

Thus, the objective function is given by;

Minimize $z = 8,880x + 11,840y$

The constraints are:

$3x + 4y \geq 8$

$5x + 2y \geq 11$

$x \geq 0,\ y \geq 0$

Therefore, the linear programming problem is;

Minimize $z = 8{,}880x + 11{,}840y$

Subject to: $3x + 4y \geq 8$

$5x + 2y \geq 11$

$x \geq 0,\ y \geq 0$

Exercise 8.1

1. A paint factory makes two varieties of paints with standard quality P_1 and one of high quality P_2. In order to manufacture these paints only two ingredients, namely; dye and pitch are needed. P_1 requires 2 units of dye and 3 units of pitch for each unit made, and it is sold at a profit of Tshs 2,000 per unit. P_2 requires 4 units of dye and 2 units of pitch for each unit made, and it is sold at a profit of Tshs 2,500 per unit. The factory has stocks of 12 units of dye and 10 units of pitch. Formulate a linear programming problem to maximize the profit.
2. A company workshop manufactures chairs and tables. Each table requires 4 hours of labour from the construction department and 2 hours of labour from the finishing department. Each chair requires 3 hours of construction and one hour of finishing. During the current week, 140 hours of construction time and 100 hours of finishing time are available. Each table produced gives a profit of Tshs 3,250 and each chair gives a profit of Tshs 3,165. Formulate this problem as a linear programming problem to maximize the profit.
3. A workshop prints two circuits of types C_1 and C_2. Type C_1 requires 20 resistors, 10 transistors, and 20 capacitors. Circuit of type C_2 requires 10 resistors, 10 transistors, and 30 capacitors. The workshop has a stock of 200 resistors, 120 transistors, and 150 capacitors. The profit on each type C_1 circuit is Tshs 4,450 and Tshs 3,570 on each type C_2 circuit. Formulate the linear programming problem for maximizing profit.
4. A small company manufactures two types of garden chairs. Type A requires 2 hours of machine time and 5 hours of craftsman time. Type B requires 3 hours of machine time and 5 hours of craftsman time. Each day there are 30 hours of machine time and 60 hours of craftsman time. The profit on each type A chair is Tshs 2,330 and on each type B chair is Tshs 1,890. Formulate an appropriate linear programming problem for maximizing profit.

5. Mr. Juma produces two packages of fruit. Package A contains 20 peaches, 15 apples, and 10 pears. Package B contains 10 peaches, 30 apples, and 12 pears. He has 40,000 peaches, 60,000 apples, and 27,000 pears available for packaging. He earns the profit of Tshs 5,000 for selling package A and Tshs 7,000 for selling package B. Formulate a profit maximization linear programming problem.

6. A chef wishes to mix type I and type II foods, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B, and 8 units of vitamin C. The vitamin contents of one kilogram for each type of food is given in the following table:

Food	Vitamin in kilograms		
	A	B	C
Type I	1	2	3
Type II	2	2	1

If one kilogram of type I food costs Tshs 2,500 and one kilogram of type II food costs Tshs 2,850, formulate a linear programming problem to minimize the cost.

7. A patient needs 5 mg, 20 mg, and 15 mg of vitamins A, B, and C per day, respectively from a mango and an orange. A mango has 0.5 mg of vitamin A, 2 mg of vitamin B, and 3 mg of vitamin C. An orange has 1 mg of vitamin A, 2 mg of vitamin B, and 3 mg vitamin C. The costs of the mango and the orange are Tshs 400 and Tshs 150, respectively. Formulate a cost minimization linear programming problem.

8. A dietitian wants to combine two types of ingredients, I_1 and I_2, so that the mixture's vitamin content includes at least 6 units of vitamin A and 8 units of vitamin B. Ingredient I_1 contains 2 units/kg of vitamin A and 3 units/kg of vitamin B, while ingredient I_2 contains 3 units/kg of vitamin A and 4 units/kg of vitamin B. Ingredient I_1 and I_2 cost Tshs 8,000 per kilogram and Tshs 7,500 per kilogram, respectively. Formulate a linear programming problem to minimize the cost of mixture.

9. A manufacturer has 90, 80, and 60 running metres of plywood, pine, and birch, respectively. Product A requires 2, 2, and 4 running metres of plywood, pine, and birch, respectively, and product B requires 4, 5, and 1 running metres of plywood, pine, and birch, respectively. If product A is sold at Tshs 10,000 and product B is sold at Tshs 13,000. Formulate a linear programming problem for finding the optimal growth of the product.

Graphical solution

The constraints of the problem can be solved graphically. The solution is obtained by treating the inequalities as linear equations but the set of inequalities will be satisfied by the obtained region. The following terminologies are useful when graphing linear programming problems.

Optimal problem

Optimal problem is a problem where a particular objective function is maximized or minimized subject to constraints over the set of feasible solutions. It is the problem of finding the most desirable solution from the feasible solutions. An optimal problem may involve maximization of profit of production, or minimization of cost from the available resources.

Feasible region

A feasible region is the set of all possible feasible solutions. The feasible region includes also the boundary lines. If the feasible region is enclosed by a polygon, it is said to be bounded otherwise, it is unbounded.

Feasible solution

A feasible solution is a solution that satisfies all constraints. Points within and on the boundary of the feasible region represent feasible solutions of the constraints.

Optimal solution

An optimal solution is any solution in the feasible region that gives the optimal value. It is the largest objective function value for a maximization problem and smallest objective function value for a minimization problem.

Optimal value

An optimal value is the quantity from the optimal solution that maximizes or minimizes the objective function of the linear programming problem.

Optimal point

An optimal point is a point where the objective function attains its optimal value.

Steps for solving linear programming problems graphically

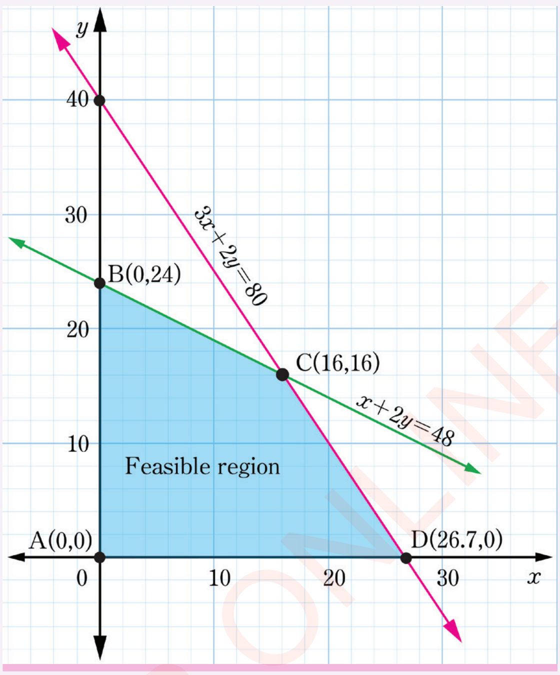
The following are steps of graphing linear programming problems:

1. Formulate the linear programming problem.
2. Replace an inequality symbol with an equal sign to form an equation of the boundary line of the graph.
3. Draw the straight line that is the boundary line. Use dotted lines if an inequality sign $<$ or $>$ is included. Use a solid line if an inequality sign $\leq$ or $\geq$ is included.
4. Identify the feasible region by testing any convenient point (coordinates) which does not lie on the boundary. Substitute the coordinates of the point in the inequality. If the inequality is satisfied at the tested point, then the point lies in the feasible region, otherwise the point does not lie in the feasible region.

5. Identify the coordinates of an optimum or corner points.
6. Evaluate the objective function at the optimum points to obtain the optimal value which may either be maximum or minimum as per required linear programming problem.

Example 8.4

Determine the maximum value of $z = 4x + 3y$ of the feasible region represented by the linear programming problem shown in the following figure.



Solution

The shaded region ABCD represents the feasible region. Hence, the maximum of z must occur at the corner points of the feasible region. The following table shows corner points and their corresponding values of the objective function:

Corner points of the feasible region	Value of objective function $z = 4x + 3y$
A(0, 0)	$z = 4(0) + 3(0) = 0$
B(0, 24)	$z = 4(0) + 3(24) = 72$
C(16, 16)	$z = 4(16) + 3(16) = 112$
D(26.7, 0)	$z = 4(26.7) + 3(0) = 106.8$

Therefore, the maximum value of z is 112 and it is obtained at point (16, 16).

Example 8.5

Solve graphically the following linear programming problem.

Minimize $z = 2x + 5y$

Subject to: $3x + 2y \leq 6$

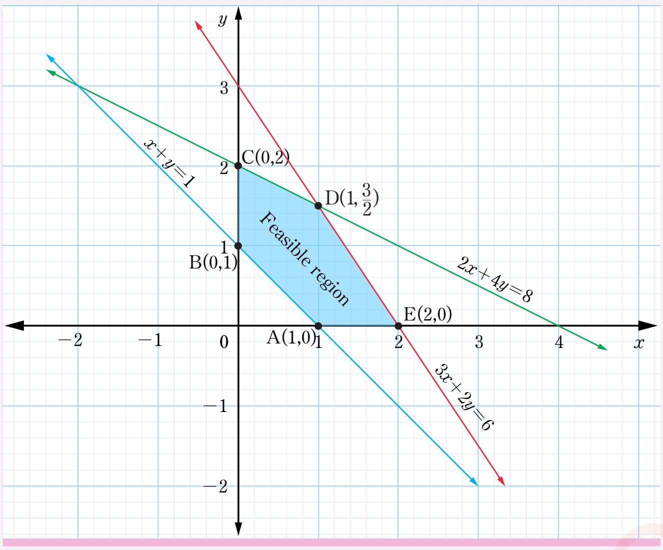
$2x + 4y \leq 8$

$x + y \geq 1$

$x \geq 0,\ y \geq 0$

Solution

The graph of the given constraints is as follows:



The value of objective function at each of the extreme points is shown in the following table:

Corner points of the feasible region	Value of the objective function $z = 2x + 5y$
A(1, 0)	2
B(0, 1)	5
C(0, 2)	10
D$\left(1, \dfrac{3}{2}\right)$	$9\dfrac{1}{2}$
E(2, 0)	4

Since, at point A (1, 0) the objective function has the smallest value of 2. Therefore, the minimum value is 2.

Example 8.6

A manufacturer has 24, 36, and 18 tonnes of wood, plastics, and steel, respectively. Product A requires 1, 3, and 2 tonnes of wood, plastic, and steel, respectively. Product B requires 3, 4, and 1 tonnes of wood, plastic, and steel, respectively. If product A is sold for Tshs 400,000 and product B for Tshs 600,000, how many products of each type should be manufactured to obtain the maximum gross income?

Solution

The given information are summarized as shown in the following table:

	Wood	**Plastic**	**Steel**	Profit (Tshs)
Product A	1	3	2	400,0000
Product B	3	4	1	600,0000
Maximum requirements	24	36	18	

Let: x be the number of units of product A

y be the number of units of product B.

Thus, the objective function is given by;

Maximize $z = 400{,}000x + 600{,}000y$

Subject to:

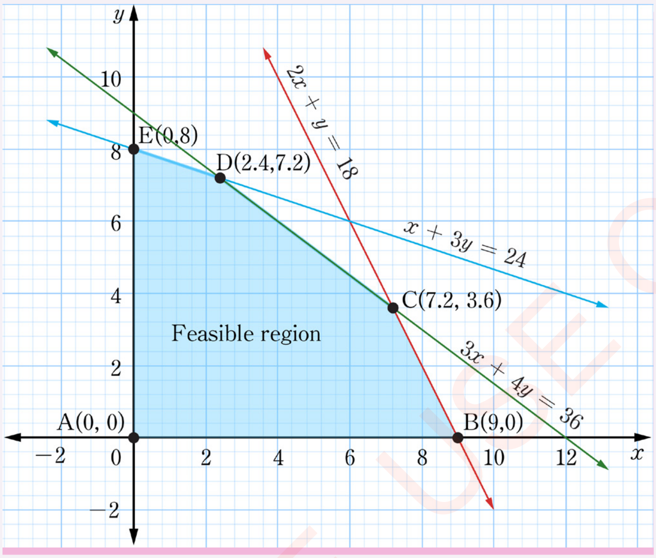
$x + 3y \leq 24$

$3x + 4y \leq 36$

$2x + y \leq 18$

$x \geq 0,\ y \geq 0$

The graph of the constraints is shown in the following figure.



Corner points and values of the objective function are shown in the following table:

Corner points	**Value of the objective function** $z = 400{,}000x + 600000y$
A (0, 0)	$z = 400{,}000(0) + 600{,}000(0) = 0$
B (9, 0)	$z = 400{,}000(9) + 600{,}000(0) = 3{,}600{,}000$
C (7.2, 3.6)	$z = 400{,}000(7.2) + 600{,}000(3.6) = 5{,}040{,}000$
D (2.4, 7.2)	$z = 400{,}000(2.4) + 600{,}000(7.2) = 5{,}280{,}000$
E (0, 8)	$z = 400{,}000(0) + 600{,}000(8) = 4{,}800{,}000$

The maximum value of the objective function is at point D(2.4, 7.2) which is Tshs 5,280,000.

Therefore, in order to obtain maximum gross income the manufacturer should make 2 units of product A and 7 units of product B.

Example 8.7

John requires 10, 12, and 12 units of chemicals of types A, B, and C, respectively for his farm. A liquid product contains 5, 2, and 1 units of A, B, and C, respectively per litre, while a powder product contains 1, 2, and 4 units of A, B, and C, respectively per carton. If a litre costs Tshs 3,000 and a carton costs Tshs 2,000, how many of each should he purchase so as to minimize the cost but meet the requirements?

Solution

The given information are interpreted and summarized as shown in the following table:

Product	Chemicals		
	A	B	C
Liquid	5	2	1
Powder	1	2	4
Minimum requirements	10	12	12

Let x be the number of litres of liquid product and y be the number of cartons of powder product to be purchased. Thus, the objective function is given by,

Minimize $z = 3{,}000x + 2{,}000y$

Subject to:

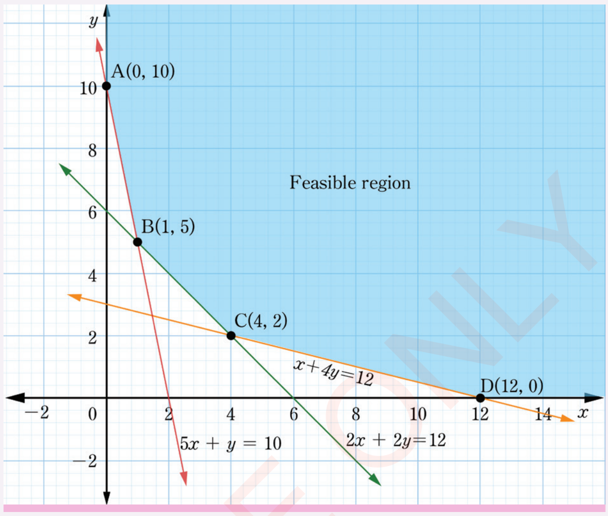
$5x + y \geq 10$

$2x + 2y \geq 12$

$x + 4y \geq 12$

$x \geq 0,\ y \geq 0$

The graph of the constraints is shown in the following figure:



The corner points and values of the objective function are shown in the following table:

Corner points of the feasible region	Value of the objective function $z = 3{,}000x + 2{,}000y$
A(0, 10)	$z = 3{,}000(0) + 2{,}000(10) = 20{,}000$
B(1, 5)	$z = 3{,}000(1) + 2{,}000(5) = 13{,}000$
C(4, 2)	$z = 3{,}000(4) + 2{,}000(2) = 16{,}000$
D(12, 0)	$z = 3{,}000(12) + 2{,}000(0) = 36{,}000$

The minimum value of the objective function is at point B(1, 5) which is Tshs 13,000.

Therefore, in order to minimize the cost while meeting the requirements, John requires to purchase 1 litre of liquid product and 5 cartons of powder product.

Exercise 8.2

1. Find the solution of each of the following linear programming problems graphically:

(a) Maximize $z = 2x + 3y$

Subject to: $x + 2y \leq 10$

$3x + y \leq 15$

$x \geq 0,\ y \geq 0$

(b) Minimize $z = 4x + 5y$

Subject to: $4x + y \geq 8$

$2x + 3y \geq 6$

$x \geq 0,\ y \geq 0$

(c) Maximize $z = 5x + 3y$

Subject to: $5x + 3y \leq 15$

$2x + 5y \leq 10$

$x \geq 0,\ y \geq 0$

(d) Minimize $z = x + 2y$

Subject to: $2x + y \geq 4$

$x + 2y \geq 6$

$x \geq 0,\ y \geq 0$

(e) Maximize $z = 4x + 9y$

Subject to: $x + 5y \leq 20$

$2x + 3y \leq 14$

$x \geq 0,\ y \geq 0$

(f) Minimize $z = x + 2y$

Subject to: $x + 2y \geq 10$

$2x - y \leq 0$

$2x + y \geq 20$

$x \geq 0,\ y \geq 0$

(g) Maximize $z = 3x + 5y$

Subject to: $x + y \leq 30$

$2x + y \leq 36$

$x \geq 0,\ y \geq 0$

(h) Minimize $z = 5x + 10y$

Subject to: $x + 2y \leq 12$

$x + y \geq 6$

$x - 2y \geq 0$

$x \geq 0,\ y \geq 0$

2. A lightweight mountain tents manufacturing company produces a standard and an expedition model. Each standard tent requires 1 hour of cutting and 3 hours of assembling. Each expedition tent requires 2 hours of cutting and 4 hours assembling. The maximum labour hours available per day in the cutting department and the assembling departments are 32 and 84, respectively. If the company makes a profit of Tshs 10,950 on each standard tent and Tshs 12,750 on each expedition tent, use the graphical method to determine the number of tents of each type that should be manufactured each day to maximize the daily profit.

3. A firm makes two products, P_1 and P_2, and has production capacity of 18 tonnes per day. Product P_1 requires a production capacity of 2 tonnes per day, while product P_2

requires a production capacity of 1 tonne per day. Each tonne of P_1 and P_2 requires 60 hours of machine work. The maximum hours available are 720. If the profit per tonne for P_1 is Tshs 14,670 and for P_2 is Tshs 13,280, find optimal solution by graphical method.

4. A farmer can buy two types of plant food, F_1 and F_2. Each cubic metre of F_1 contains 30 kg of phosphoric acid, 20 kg of nitrogen, and 15 kg of potash. Each cubic metre of F_2 contains 6 kg of phosphoric acid, 18 kg of nitrogen, and 24 kg of potash. The minimum monthly requirements are 120 kg of phosphoric acid, 180 kg of nitrogen, and 288 kg of potash. If food F_1 costs Tshs 55,000 per cubic metre and food F_2 costs Tshs 70,000 per cubic metre. Use the graphical method to solve the linear programming problem to minimize the cost.

5. A firm produces two products, P and Q. The daily total production limit is 600 units. The firm requires at least 300 total units that must be produced every day. Machine hours consumption per unit is 6 for P and 2 for Q. At least 1,200 machine hours must be used daily. Manufacturing costs per unit are Tshs 1,500 for P and Tshs 850 for Q. Find an optimal solution for the linear programming problem.

6. A carpenter makes two products, tables and chairs. Processing of these products is done on machines A and B. A chair requires 2 hours on machine A and 4 hours on machine B. A table requires 4 hours on machine A and 5 hours on machine B. There are 16 hours per day available on machine A and 23 hours per day on machine B. Profits gained by the carpenter from a chair and a table are Tshs 1,570 and Tshs 1,860, respectively. What should be the daily production of each of the two products in order to maximize the profit?

7. An agricultural company has 180 tonnes of Nitrogen fertilizers, 160 tonnes of phosphate, and 220 tonnes of potash. It will be able to sell 3:2:4 mixtures of these substances at a profit of Tshs 8,980 per tonne and 2:2:2 mixtures at a profit of Tshs 9,880 per tonne respectively. Use the graphical method to determine the number of units of these two mixtures that should be prepared so as to maximize profit.

8. A company manufactures two products, X and Y. Each product has to be processed in three departments: Welding, assembling, and painting departments. Each unit of X takes 2 hours in the welding department, 3 hours in assembling, and 1 hour in painting. The corresponding processing hours for a unit of Y are 3, 2, and 1 hours, respectively. The labour hours available in a month are 1,500 for the welding department, 1,500 in assembling and 550 in painting. The contribution to profit and fixed overheads are Tshs 19,660 for product X and Tshs 21,430 for product Y. Solve this problem graphically to obtain the optimal solution for the maximum contribution for each product.

9. A wheat and barley farmer has 168 hectare of ploughed land, and a capital of Tshs 4,000,000. It costs Tshs 28,000 to sow one hectare of wheat and Tshs 20,000 to sow one hectare of barley. Suppose that his profit is Tshs 160,000 per hectare of wheat and Tshs 110,000 per hectare of barley. Use the graphical method to find the optimal number of hectares of wheat and barley that must be ploughed in order to maximize profit.

Transportation problems

In most cases, products or goods produced are transported from the storage location (source) to the consumer (destination) at a minimum cost in order to attain the desired profit. Since transportation costs are not controllable, then minimizing total cost requires making the best product routing decisions in distribution processes. The transportation problem is a type of linear programming problem designed to minimize the cost of distributing a product from sources to the destinations.

Note that, a transportation problem is considered as a balanced one, that is, the total supply from the sources to each destination is equal to the total demands, otherwise the transportation problem is unbalanced. Unbalanced transportation has no feasible region.

There are several kinds of transportation methods which all serve the purpose of minimization of cost. In this section, the transportation problems which involve two sources (S_1 and S_2) and two destinations (D_1 and D_2) are studied. Also, transportation problems which involve two sources (S_1 and S_2) and three destinations (D_1, D_2 , and D_3) are studied as illustrated in Figure 8.1(*a*) and Figure 8.1(*b*), respectively.

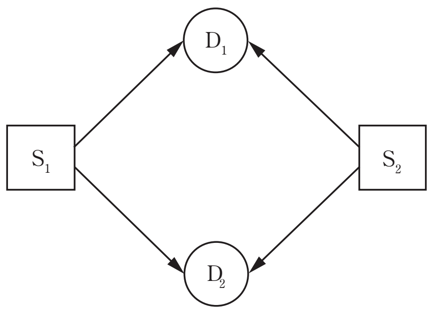


Figure 8.1 (a): *Two sources and two destinations*

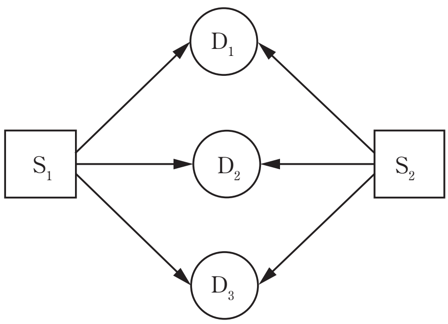


Figure 8.1 (b): *Two sources and three destinations*

Formulation of transportation problems

Formulation of a transportation problem depends on the number of destinations from the given sources. Consider the transportation problem which involves two sources having k products in source P and h products in source Q and two destinations which require a products in destination A and b products in destination B as shown in Figure 8.2.

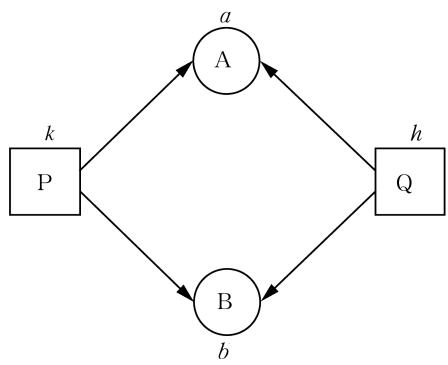


Figure 8.2: *Two sources and two destinations*

From Figure 8.2, the decision variables will be defined as follows:

Let: x be the number of products to be transported from source P to destination A.

y be the number of products to be transported from source P to destination B.

Since, there are already products transported from source P to destinations A and B, the decision variables will deduct the products transported before as follows:

Let: $(a-x)$ be the number of products to be transported from source Q to destination A,

$(b-y)$ be the number of products to be transported from source Q to destination B.

Similarly, if the transportation problem involves two sources which has k products in source P and h products in source Q and three destinations which require a products in destination A, b products in destination B, and c products in destination C as shown in Figure 8.3.

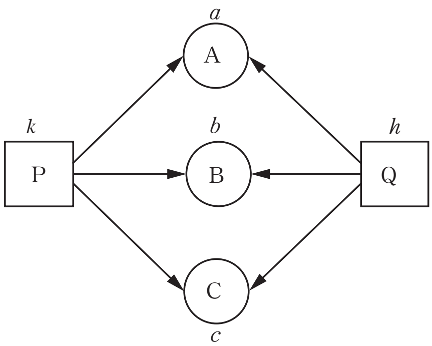


Figure 8.3: *Two sources and three destinations*

From Figure 8.3, the decision variables will be as follows:

Let: x be the number of products to be transported from source P to destination A,

y be the number of products to be transported from source P to destination B,

z be the number of products to be transported from source P to destination C.

Since there are already products transported from source P to destinations A, B, and C, the decision variables will deduct the products transported before as follows;

Let: $(a-x)$ be the number of products to be transported from source Q to destination A,

$(b-y)$ be the number of products to be transported from source Q to destination B,

$(c-z)$ be the number of products to be transported from source Q to destination C.

After deciding the variables, other procedures of writing objective functions and constraints on the Cartesian coordinate system is the same as that of linear programming problems.

Example 8.8

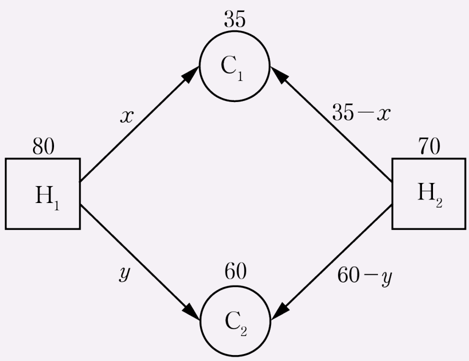
A manufacturer of certain products has two warehouses H_1 and H_2 for storing her products. She stores 80 units of the products in H_1 and 70 units in H_2. The customers C_1 and C_2 placed orders for 35 and 60 units in H_1 and H_2, respectively. The transportation cost in Tanzanian shillings for each unit is as shown in the following table:

To / From	Transport Costs	
Warehouses	C_1	C_2
H_1	80	120
H_2	100	130

Formulate a linear programming problem for minimizing the total transportation cost.

Solution

The given transportation problem involves two sources and two destinations as illustrated pictorially in the following figure:



The decision variables are as follows:

Let: x be the number of products to be transported from warehouse H_1 to customer C_1,
y be the number of products to be transported from warehouse H_1 to customer C_2,
$(35-x)$ be the number of products to be transported from warehouse H_2 to customer C_1,
$(60-y)$ be the number of products to be transported from warehouse H_2 to customer C_2,

Thus, the objective function is formulated as;

Minimize $z = 80x + 120y + 100(35-x) + 130(60-y)$
$$z = -20x - 10y + 11,300$$

The constraints are formulated as follows:

$x \leq 35$
$y \leq 60$
$x + y \leq 80$
$35 - x + 60 - y \leq 70 \Rightarrow x + y \geq 25$
$35 - x \leq 35 \Rightarrow x \geq 0$
$60 - y \leq 60 \Rightarrow y \geq 0$

Therefore, the formulated linear programming problem is to;

Minimize $z = -20x - 10y + 11,300$
Subject to : $x \leq 35$
$y \leq 60$
$x + y \leq 80$
$x + y \geq 25$
$x \geq 0,\ y \geq 0$

Example 8.9

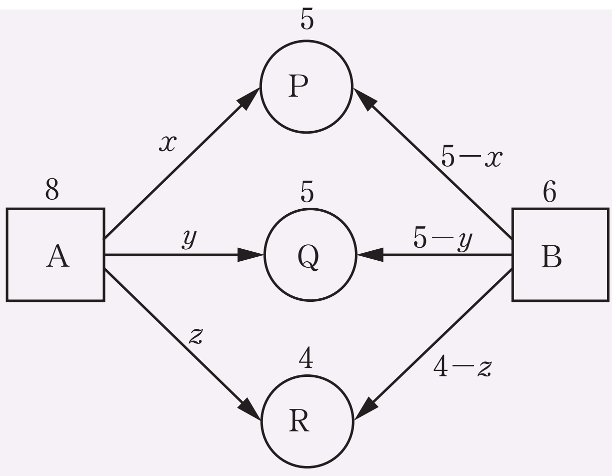
A medical company has factories at two locations, A and B. From these locations, the supply is assigned to each of the three agencies situated at P, Q, and R. The monthly requirements of the agencies are 5, 5, and 4 packets of medicine, respectively. While the production capacities of the factories A and B are 8 and 6 units per month, respectively. The transportation cost per packet from the factories is shown in the following table

To \ From	A	B
P	15	10
Q	10	12
R	15	10

Formulate a linear programming problem for minimizing the transportation cost.

Solution

The transportation problem involves two sources and three destinations as illustrated pictorially in the following figure:



The decision variables will be as follows;

Let x be the number of packets to be transported from factory A to agency P,
y be the number of packets to be transported from factory A to agency Q,
z be the number of packets to be transported from factory A to agency R,
$(5-x)$ be the number of packets to be transported from factory B to agency P,
$(5-y)$ be the number of packets to be transported from factory B to agency Q,
$(4-z)$ be the number of packets to be transported from factory B to agency R.

Thus, the objective function is given by,

Minimize $f = 15x + 10y + 15z + 10(5-x) + 12(5-y) + 10(4-z)$
$= 5x - 2y + 5z + 150$

The constraints are formulated as follows:

$x \leq 5$

$y \leq 5$

$z \geq 0$ and $4 - z \geq 0$

but $x + y + z = 8$

$z = 8 - (x + y)$

$8 - (x + y) \geq 0 \Rightarrow 8 \geq x + y$

Thus, $x + y \leq 8$

Also, $4 - z \geq 0$

Thus, $4 - (8 - (x + y)) \geq 0$

Simplification gives,

$-4 + x + y \geq 0$

Thus, $x + y \geq 4$

Also, $5 - x \leq 5 \Rightarrow x \geq 0$

$5 - y \geq 5 \Rightarrow y \geq 0$

Therefore, the formulated linear programming problem is to:

Minimize

$f = 5x - 2y + 5[8 - (x + y)] + 150$

$= 190 - 7y$

Subject to : $x \leq 5$

$y \leq 5$

$x + y \geq 4$

$x + y \leq 8$

$x \geq 0,\ y \geq 0$

Exercise 8.3

1. Two timber production centres, A and B, are capable of manufacturing 120 and 100 pieces of timber each week, respectively. Three customers U, V, and W placed weekly orders of 90, 70, and 60 pieces of timber from the timber production centres, respectively. The transportation costs in Tanzanian shillings are shown in the following table:

To \ From	A	B
U	5,000	3,500
V	4,500	3,000
W	4,000	5,000

Formulate a linear programming problem for minimization of the cost of transportation.

2. The daily meat requirements of three hotels L, M, and N is 70 kg, 40 kg, and 60 kg, respectively. Two meat suppliers, X and Y have the capacities of supplying 90 kg and 80 kg of meat, respectively. The costs of transportation in Tanzanian shillings are given in the following table:

To \ From	X	Y
L	3,000	2,000
M	2,000	3,000
N	4,000	4,000

Formulate a linear programming problem to minimize the cost of transportation.

3. Two suppliers of sweets, F and G, are capable of supplying not more than 40 kg and 35 kg of sweets, respectively. The supplier is required to supply sweets to three day care school, X, Y, and Z. The daily school requirements are 30 kg, 25 kg, and 20 kg for schools X, Y, and Z, respectively. The distance in kilometres from the suppliers to the respective hotels are tabulated in the following table:

To \ From	F	G
X	6,000	5,000
Y	8,500	7,000
Z	9,000	5,500

If the transportation costs are proportional to the distance travelled per each kilometre, formulate a linear programming problem to minimize the transportation cost for the sweet supplier.

4. Two strawberry producers, A and B, with 50 tonnes and 45 tonnes of strawberries, respectively, are requested to supply strawberries to three food processing industries, namely; P, Q, and R, demanding 30 tonnes, 25 tonnes, and 40 tonnes of strawberries, respectively. The transportation costs in Tanzanian shillings from the sources to the demanding sites are tabulated as follows:

To \ From	A	B
P	30,000	25,000
Q	45,000	30,000
R	20,000	40,000

Formulate a linear programming problem representing the given transportation problem.

5. Nana and Nina are poultry farmers who can supply a maximum of 80 and 65 trays of eggs every month, respectively. Three hotels, STAR, MOON, and CLOUD, each places a monthly order of 40, 55, and 50 trays of eggs, respectively. The transportation costs of eggs in Tanzanian shillings from the poultry farms to the hotels are tabulated in the following table:

To \ From	Nana	Nina
STAR	7,500	7,500
MOON	9,000	9,500
CLOUD	12,000	12,500

Prepare a linear programming problem representing the given information.

6. A certain electric bulb manufacturing company has two factories located at cities, F_1 and F_2, and three retail centres located at C_1, C_2, and C_3. The monthly demand at the retail centres are 8, 5, and 2, electric bulbs, respectively, while the monthly supply at the factories are 6 and 9, electric bulbs respectively. The cost of transportation in Tanzanian shillings of one electrical bulb between each factory and each retail centre is shown in the following table.

From \ To	C_1	C_2	C_3
F_1	50	50	30
F_2	60	40	10

Formulate a linear programming problem from the given information.

7. A sand dealer has two quarries Q_1 and Q_2 which produce 3,000 tonnes and 1,500 tonnes of sand per day, respectively. Three road constructors, A, B, and C require each day 2,000 tonnes,

1,500 tonnes, and 1,000 tonnes of sand, respectively. The distances in kilometres from the quarries to each site of the road constructors are shown in the following table.

From \ To	A	B	C
Q_1	7	4	2
Q_2	3	2	2

If the transportation costs are proportional to the distance travelled per each kilometre formulate a linear programming problem for minimization of the cost.

8. Two sugar production factories located in Morogoro and Kagera have production capacities of 8 and 6 tonnes, respectively. The products from the factories are delivered to depots situated in Mwanza, Dodoma, and Arusha. The weekly requirements of the depots are 5, 5, and 4 tonnes, respectively. Formulate a linear programming problem to minimize costs if the transportation cost in Tanzanian shillings per tonne is as shown in the following table:

From \ To	Mwanza	Dodoma	Arusha
Morogoro	16,000	10,000	15,000
Kagera	10,000	12,000	10,000

9. Mr. Bakari has two warehouses, W_1 and W_2 that contain 900 tonnes and 600 tonnes of grained maize, respectively. He is planning to supply flour to three customers, T_1, T_2, and T_3, which are in need of 500 tonnes, 600 tonnes, and 400 tonnes respectively. Formulate a linear programming problem given that the transportation costs in Tanzanian shillings per tonne of wheat flour from each warehouse to each customer are as shown in the following table:

From \ To	T_1	T_2	T_3
W_1	600	300	400
W_2	400	200	600

Solving transportation problems graphically

Transportation problems can be solved graphically similar to the way linear programming problems are solved. After a transportation problem is formulated, a graph is drawn to identify the feasible region and corner points. The corner points are then used to obtain the optimal solution of the given transportation problem.

Example 8.10

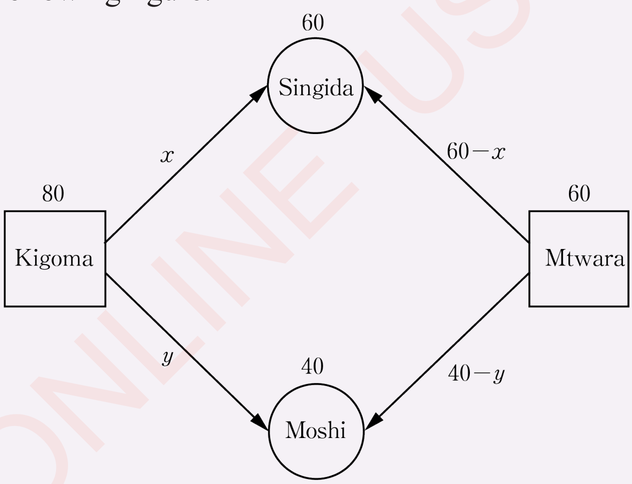
Mariam has two warehouses, one in Kigoma and one in Mtwara. She wants to transport tonnes of goods to Singida and Moshi. She needs to transport 60 tonnes and 40 tonnes of goods to Singida and Moshi, respectively. The warehouses at Kigoma and Mtwara contain 80 and 60 tonnes of goods, respectively. The transportation costs in Tanzanian shillings per tonne of goods are shown in the following table:

From / To	Kigoma	Mtwara
Singida	180,000	240,000
Moshi	120,000	100,000

How should the transportation of tonnes of goods be done at a minimum cost?

Solution

The transportation problem involves two sources and two destinations as illustrated pictorially in the following figure:



The decision variables are defined as follows;

Let: x be the number of tonnes of goods to be transported from Kigoma to Singida, and

y be the number of tonnes of goods to be transported from Kigoma to Moshi, then

$(60-x)$ is the number of tonnes of goods to be transported from Mtwara to Singida,

$(40-y)$ is the number of tonnes of goods to be transported from Mtwara to Moshi.

Thus, the objective function is formulated as:

Minimize $z = 180,000x + 120,000y + 240,000(60 - x) + 100,000(40 - y)$

$$z = 18,400,000 - 60,000x + 20,000y$$

The constraints are formulated as follows:

$x \leq 60$

$y \leq 40$

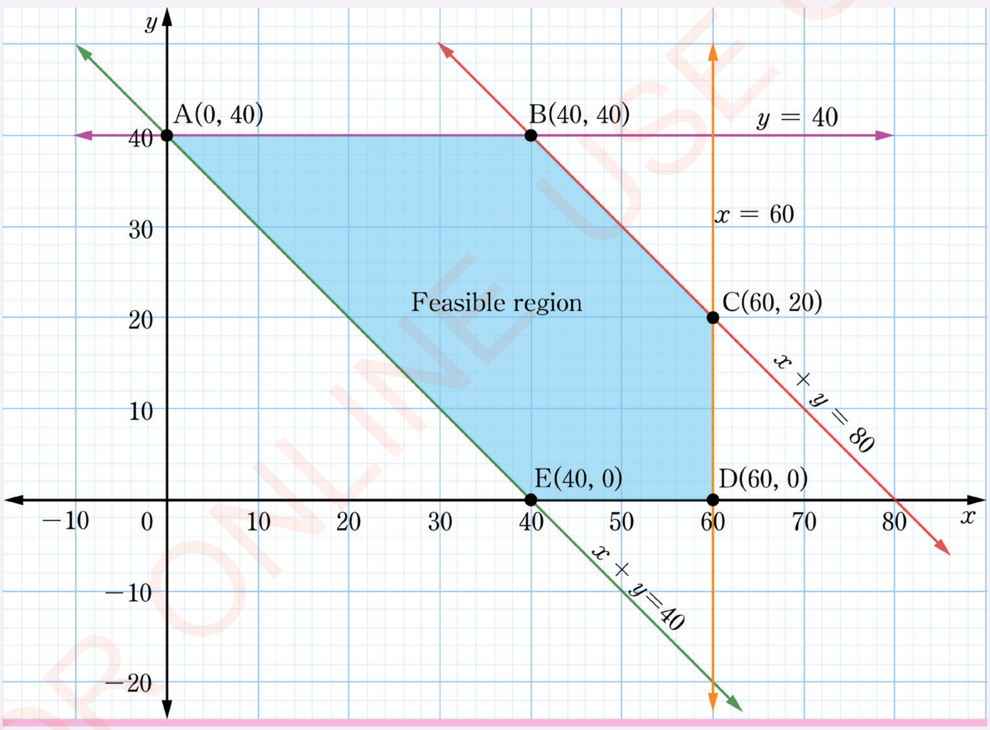
$x + y \leq 80$

$60 - x + 40 - y \leq 60 \Rightarrow x + y \geq 40$

$60 - x \leq 60 \Rightarrow x \geq 0$

$40 - y \leq 40 \Rightarrow y \geq 0$

The graph of the constraints is as shown in the following figure:



The corner points and values of the objective function are shown in the following table:

Corner points	Value of the objective function $z = 18,400,000 - 60,000x + 20,000y$
A(0, 40)	$z = 18,400,000 - 60,000(0) + 20,000(40) = 19,200,000$
B(40, 40)	$z = 18,400,000 - 60,000(40) + 20,000(40) = 16,800,000$
C(60, 20)	$z = 18,400,000 - 60,000(60) + 20,000(20) = 15,200,000$
D(60, 0)	$z = 18,400,000 - 60,000(60) + 20,000(0) = 14,800,000$
E(40, 0)	$z = 18,400,000 - 60,000(40) + 20,000(0) = 16,000,000$

The minimum cost is at point D(60, 0) which is 14,800,000 Tanzanian shillings.

Therefore, in order for Mariam to minimize cost, the transportation should be as follows:
60 tonnes of goods to be transported from Kigoma to Singida,
0 tonnes of goods to be transported from Kigoma to Moshi,
0 tonnes of goods to be transported from Mtwara to Singida, and
40 tonnes of goods to be transported from Mtwara to Moshi.

Example 8.11

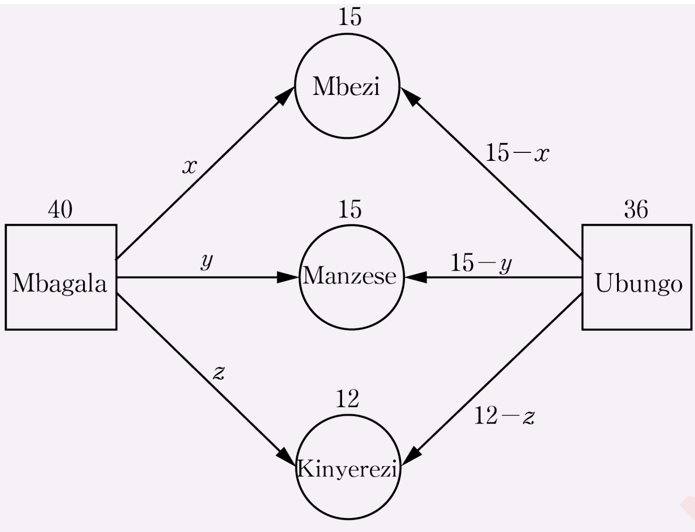
A supplier of rice has stores at Mbagala and Ubungo. Three customers located at Mbezi, Manzese, and Kinyerezi order sacks of rice weekly. The customers weekly requirements are 15, 15, and 12 sacks of rice, respectively. The stores' output capacities at Mbagala and Ubungo are 40 and 36 sacks of rice, respectively. The cost of transportation per kilogram in Tanzanian shillings are shown in the following table:

From / To	**Mbagala**	**Ubungo**
Mbezi	160	100
Manzese	100	120
Kinyerezi	150	100

How many sacks should be transported from each store to each customer at the minimum cost? Find the minimum transportation cost.

Solution

The transportation problem involves two sources and three destinations as illustrated pictorially in the following figure:



The decision variables are defined as follows;

Let: x be the number of sacks of rice to be transported from Mbagala to Mbezi,
y be the number of sacks of rice to be transported from Mbagala to Manzese,
z be the number of sacks of rice to be transported from Mbagala to Kinyerezi,

$(15-x)$ is the number of sacks of rice to be transported from Ubungo to Mbezi,
$(15-y)$ is the number of sacks of rice to be transported from Ubungo to Manzese,
$(12-z)$ is the number of sacks of rice to be transported from Ubungo to Kinyerezi.

Thus, the objective function is given by;

Minimize $w = 160x + 100y + 150z + 100(15-x) + 120(15-y) + 100(12-z)$
$= 60x - 20y + 50z + 4,500$

Subject to the following constraints:

$x \leq 15$
$y \leq 15$
$z \leq 12$
$15 - x + 15 - y + 12 - z \leq 36$
$15 - x \leq 15$
$15 - y \leq 15$
$12 - z \leq 12$
$x \geq 0,\ y \geq 0$

But $x + y + z = 40 \Rightarrow z = 40 - (x + y)$

Therefore, the linear programming problem is:

Minimize $w = 60x - 20y + 50(40 - (x + y)) + 4{,}500$

$$w = 10x - 70y + 6{,}500$$

Subject to : $x \le 15$

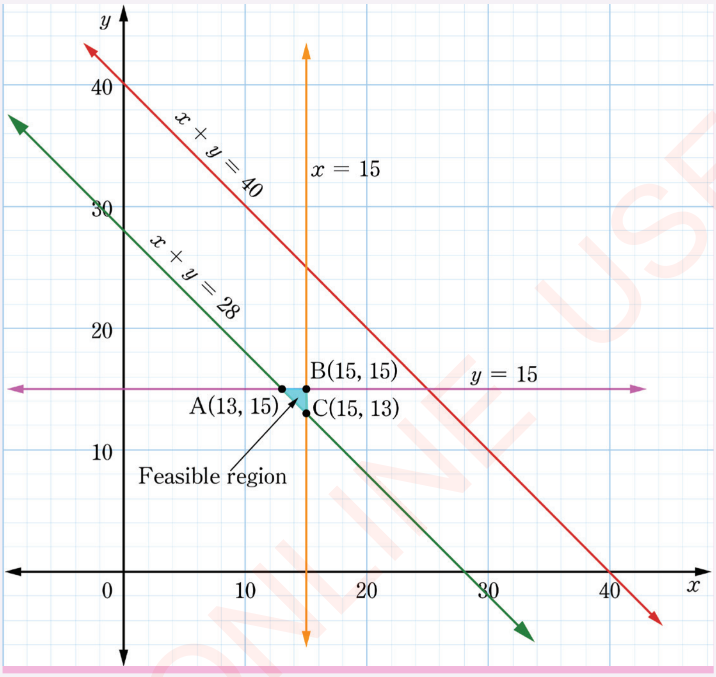
$$y \le 15$$

$$x + y \ge 28$$

$$x + y \le 40$$

$$x \ge 0,\ y \ge 0$$

The graph of the formulated linear programming problem is as shown in the following figure:



The corner points and values of the objective function are shown in following table:

Corner points of the feasible region	Value of the objective function $w = 10x - 70y + 6{,}500$
A(13, 15)	$w = 10(13) - 70(15) + 6{,}500 = 5{,}580$
B(15, 15)	$w = 10(15) - 70(15) + 6{,}500 = 5{,}600$
C(15, 13)	$w = 10(15) - 70(13) + 6{,}500 = 5{,}740$

The minimum value is at point A(13, 15), which is Tshs 5, 580.

Therefore, in order for the supplier to minimize cost, the transportation should be as follows;
13 sacks of rice to be transported from Mbagala to Mbezi,
15 sacks of rice to be transported from Mbagala to Manzese,
12 sacks of rice to be transported from Mbagala to Kinyerezi,
2 sacks of rice to be transported from Ubungo to Mbezi,
0 sacks of rice to be transported from Ubungo to Manzese, and
0 sacks of rice to be transported from Ubungo to Kinyerezi.
The minimum transportation cost is Tshs 5,580.

Exercise 8.4

1. A cement dealer has two depots D_1 and D_2 which hold 120 and 40 tonnes, respectively. He has two customers, C_1 and C_2 who have ordered 80 and 50 tonnes of cement, respectively. Their distances in kilometres are shown in the following table:

From / To	D_1	D_2
C_1	20	40
C_2	15	30

If the transportation costs are proportional to the distance travelled per kilometre, how should the cement dealer supply the cement to the two customers to minimize the total transportation cost?

2. A manufacturer has a warehouse at Misungwi which contains 25 units of his products. She has another warehouse at Ngudu which contains 30 units. She has to supply to the shops in Mabuki and Misasi with 20 and 15 units, respectively. The transportation costs in Tanzanian shillings per each unit are tabulated as follows;

From / To	Misungwi	Ngudu
Mabuki	150	220
Misasi	130	180

How should the manufacturer supply her products to minimize the transportation costs?

3. Almasi has made 900 and 600 bricks at his houses H_1 and H_2, respectively. He plans to build new houses at three sites P, Q, and R. He expects to use 500 bricks at P, 600 bricks at Q, and 400 bricks at R. The transportation costs in Tanzanian shilling per brick from each of his house to each of the three sites are shown in the following table:

To \ From	H_1	H_2
P	60	40
Q	30	20
R	40	60

(a) Determine the number of bricks that he will transport to each site at a minimum cost.

(b) What will be the overall minimum cost?

4. Two factories are located at two places M_1 and M_2. From these locations, a certain product is to be delivered to each of the three destinations, A, B, and C. The weekly requirements of the destinations are respectively 5, 5, and 4 units of the products, while the production capacities of the factories at M_1 and M_2 are respectively 8 and 6 units. The costs of transportation in Tanzanian shillings per packet are given in the following table:

To \ From	M_1	M_2
A	160	100
B	100	120
C	150	100

(a) How many units of products should be transported from each factory to each destination in order to minimize transportation cost?

(b) What will be the minimum transportation cost?.

5. Mrs. Mlunda has two depots, A and B with a stock of 30,000 and 20,000 bricks, respectively. She receives orders from three customers. Mr. Malaga requires 15,000 bricks, Mr. Falulu requires 20,000 bricks and Ms. Tina requires 15,000 bricks. The costs of transportation per brick in Tanzanian shillings are tabulated in the following table:

To \ From	A	B
Mr. Malaga	40	20
Mr. Falulu	20	60
Ms. Tina	30	40

How should Mrs. Mlunda fulfil the orders so as to minimize the cost of transportation?

6. A soft drink dealer has two sources, C_1 and C_2 with capacities of 50,000 and 30,000 crates, respectively. The dealer wishes to supply drinks to three shops, S_1, S_2, and S_3, whose requirements are 35,000, 20,000, and 25,000 crates, respectively. The costs of transportation per crate in Tanzanian shillings from the sources to the shops are as summarized in the following table:

From / To	C_1	C_2
S_1	70	30
S_2	60	40
S_3	30	20

How should the delivery be scheduled at a minimum cost of transportation? Find the minimum cost.

7. A business person has maize storages in two wards, W_1 and W_2 with capacities of 100 tonnes and 50 tonnes, respectively. The storages supply maize to three centres, C_1, C_2, and C_3 whose requirements are 60, 50, and 40 tonnes, respectively. The transportation costs in Tanzanian shillings per tonne from each ward to each centre are shown in the following table:

To / From	C_1	C_2	C_3
W_1	60,000	30,000	25,000
W_2	40,000	20,000	30,000

(a) How should the supply be made to minimize the cost of transportation?

(b) Determine the minimum cost of each supply. What is the overall minimum cost?

8. Three jogging clubs, C_1, C_2, and C_3 need 20, 15, and 20 sets of equipments, respectively. The clubs are sponsored by two benefactors B_1 and B_2 who have only 25 and 30 set of equipments, respectively. The transportation costs per set of equipments in Tanzania shillings are shown in the following table:

Benefactors	**Jogging clubs**		
	C_1	C_2	C_3
B_1	300	200	100
B_2	250	100	150

(a) Determine how the benefactors should supply the equipments to each of the jogging clubs at a minimum cost.

(b) Find the minimum cost.

9. A packaging company has two market locations, L_1 and L_2 to sell coconuts. The company can supply 80,000 bags from L_1 and 70,000 bags from L_2. The retail traders, Msigwa and Asha can place orders for 35,000 and 60,000 bags, respectively. The transportation costs in Tanzania shillings per bag from each market location to each of the retail traders are as shown in the following table:

From \ To	Msigwa	Asha
L_1	8,000	12,000
L_2	10,000	13,000

How many bags of coconuts should the entrepreneur deliver to each retail trader from each market location in order to minimize the total cost of transportation?

Chapter summary

1. Linear programming is a mathematical technique for finding optimal solutions to problems that can be expressed as linear equations and inequalities.
2. The transportation problem is a type of linear programming problem designed to minimize the cost of distributing products from different sources to different destinations.
3. Decision variables are the unknown quantities that decide the output of the linear programming problem.
4. An objective function is a linear function whose value is to be either minimized or maximized subject to some constraints defined over the set of feasible solutions.
5. Constraints are inequalities or equations which connect the decision variables under certain restrictions or limitations on the decision variables.
6. A feasible region is a set of all possible solutions of the linear programming problem.
7. An optimal solution is the value at a point where the objective function reaches its optimal value.
8. A feasible solution is solution that satisfies all constraints of the linear programming problem.
9. An optimal value is the quantity from the optimal solution that maximizes or minimizes the objective function of the linear programming problem.

Revision exercise 8

1. Two departments of inspection and evaluation produce two products, Alpha and Beta. Alpha requires 2 hours per unit in the inspection department and 4 hours per unit in the evaluation department. Beta requires 3 hours per unit in the inspection department and 2 hours per unit in the department evaluation. There are 60 and 80 hours per week available in the inspection and evaluation departments, respectively. The profit per unit for Alpha and Beta are Tshs 4,000 and Tshs 6,000, respectively.
   (a) Formulate the linear programming problem which maximizes the total profit of the product.
   (b) Use the graphical method to determine the recommended product mix.
   (c) Find the greatest number of the Beta product that can be produced if the company wants to spend the time available in the inspection and evaluation departments.

2. A manufacturer produces two different models, X and Y of the same product. Model X contributes Tshs 50,000 per unit, while model Y contributes Tshs 30,000 per unit toward the total profit. Raw materials r_1 and r_2 are required for production. At least 18 kg of r_1 and 12 kg of r_2 must be used daily. Also, at most 34 hours of labour are to be utilized. A quantity of 2 kg of r_1 are needed for model X and 1 kg of r_1 for model Y. For each X and Y, 1 kg of r_2 is required. It takes 3 hours to manufacture model X and 2 hours to manufacture model Y. How many units of each model should be produced to maximize the profit? What is the overall maximum profit?

3. A metal company plans to purchase at least 200 kg of scrap containing high and low quality metals, X and Y. The scrap to be purchased must contain at least 100 quintals of type X and not more than 35 kg of type Y. The company can purchase the scrap from two supplies, A and B. The percentage of X and Y metals in terms of the weight in the scrap supplied by A and B is given in the following table:

Metals	A	B
X	20%	75%
Y	10%	20%

The price of the type A scrap is Tshs 20,000 per kg and Tshs 40,000 per kg of type B scrap. Find the number of scraps from each supplier the company should buy to minimize cost. What is the minimum cost?

4. A factory manufactures two types of machines, A and B. To manufacture a machine of type A requires 20 kg of alloy metal, and type B requires 50 kg of alloy metal. Machines, A and B require 40 minutes and 50 minutes in the assembling department, respectively and the assembling department has only 600 minutes. Furthermore, machines A and B require 18 minutes and 15 minutes, respectively to be painted. The painting department is restricted not to use more than 240 minutes in a day. The company earns a profit of Tshs 10,000 and Tshs 20,000 for types A and B machines, respectively.
   (a) How many of each machine should be produced for maximum profit?
   (b) Find the maximum profit.

5. A manufacturer has 24, 37, and 18 tonnes of wood, plastics, and steel, respectively for making two products, A and B. Product A requires 1, 3, and 2 tonnes of wood, plastic, and steel, respectively. Product B requires 3, 4, and 1 tonnes of wood, plastic and steel, respectively. If product A is sold at Tshs 400,000 and B is sold at Tshs 600,000, how many products of each should the manufacturer make to obtain the maximum gross income?

6. For each cup of coffee a student drinks before examination, he will answer two extra questions correctly. For each cup of tea he drinks, he will answer one question correctly. A cup of coffee has 10 grams of sugar and 100 grams of caffeine and a cup of tea has 20 grams of sugar and 25 grams of caffeine. If the student drinks more than 100 grams of sugar, he will suffer from diabetes and shocks, and will fail his examinations. If the student drinks more than 300 grams of caffeine, he will suffer heart palpitation and fail his examinations. How many cups of tea and coffee should the student drink to maximize his correct answers?

7. Mr. Makungu plans to sow maize and beans in his farm. He estimates to use 4 and 6 men per hectare of maize and beans, respectively from 26 men available. Sowing maize and beans costs Tshs 1,200 and Tshs 800 per hectare, respectively. If he is prepared to spend up to Tshs

4,800, find the greatest possible area he can sow.

8. A carpenter's workshop in a certain college makes tables and chairs which are processed through assembling and finishing sections. The college uses 48 hours available in assembling and 36 hours in finishing sections. A table requires 3 hours in assembling and 3 hours in finishing sections, while a chair requires 4 hours in assembling and 2 hours in finishing sections. If a chair makes a profit of Tshs 4,000 and Tshs 5,000 for a table, how many chairs and tables can be made to maximize the profit?

9. A company makes two types of toys, a cheetah toy and a cat toy. The cheetah toy requires Tshs 2,000 for materials and 4 hours of work. The cat toy requires Tshs 2,500 for materials and 3 hours of work. The company has Tshs 40,000 for materials and has to spend 64 hours of work to make at least 7 toys of each type.
   (a) Determine the number of ways the company can produce the toys.
   (b) Which among the ways in (a) gives the greatest balance in purchasing materials for the toys.
   (c) Find the number of toys the company can produce, if the company wants to spent all the time available.
   (d) Find the number of toys of each type the company should make to maximize the profit if the cheetah and cat toys make a profit of Tshs 12,000 and Tshs 16,000, respectively.

10. Two friends are planning an exercise program to keep their bodies fit. They would want to spend up to 1.5 hours per day on aerobics and flexibility exercises. They would prefer to do more aerobics than flexibility, but could not manage more than 60 minutes of aerobics. They discovered that aerobics use 8 calories per minute and flexibility exercises use 3 calories per minute. They wish to lose the maximum number of calories. How should the planning program be done in order to have maximum loss of calories? What is the maximum losses of calories?

11. A special diet for laboratory animals is to contain at least 300 units of vitamins, 110 units of minerals, and 560 calories. There are two mixtures, A and B of diet available. One gram of mixture A contains 3 units of vitamins, 1 unit of minerals, and 4 calories. One gram of mixture

B contains 2 units of vitamins, 1 unit of minerals, and 7 calories. Mixture A costs Tshs 40 per gram and mixture B costs Tshs 60 per gram.

(a) How many grams of each mixture should be used to satisfy the requirement of the diet at a minimum cost?

(b) Find the cost of the cheapest diet.

12. A poultry farmer wishes to produce a chicken feed which is a blend of two inputs labelled P and Q. Each unit of input P and Q costs Tshs 400 and Tshs 100, respectively. The following table shows the percentage of ingredients in each blend input and the minimum daily requirements:

Ingredients	**Proportion of input per unit of blend (%)**		Minimum requirements (grams)
	P	Q	
Protein	15	60	180
Carbohydrates	10	10	50
Salt	15	30	120

How can the farmer produce a chicken feed which meets the minimum requirements at the least cost?

13. An oil company has two depots, N and Q with capacities, 7,000 litres and 4,000 litres of oil, respectively. The company is to supply oil to three petrol pumps, D, E, and F whose requirements are 4,500 litres, 3,000 litres, and 3,500 litres, respectively. The distances between the depots and the petrol pumps in kilometres are given in the following table:

From / To	N	Q
D	7	3
E	6	4
F	3	2

Assuming the transportation cost is 1 Tanzanian shilling per kilometre, how should the delivery be scheduled in order to minimize the cost? Determine the minimum cost of the delivery.

14. A certain factory has two warehouses, W_1 and W_2 each containing shilling 900 crates of soda to be supplied to three regions, R_1, R_2, and R_3. There are 600 crates of soda to be transported to R_1, 500 crates of soda to be transported to R_2 and 700 crates to be transported to R_3. The transport costs per crate to each region are shown in the following table:

From \ To	R_1	R_2	R_3
W_1	60/=	30/=	40/=
W_2	40/=	20/=	60/=

How should the factory supply crates of soda to the three regions at a minimum cost?

15. One kind of cake requires 100 g of flour and 30 g of fat, and another kind of cake requires 300 g of flour and 20 g of fat. Find the maximum number of cakes to be made from 6 kg of flour and 0.75 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

16. There are two types of fertilizers, F_1 and F_2. Fertilizer F_1 consists of 10% nitrogen and phosphoric acid each and F_2 consists of 10% of nitrogen and 5% of phosphoric acid. After testing the soil condition, a farmer finds that she needs at least 14 kg of Nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Tshs 6 per kg and F_2 costs Tshs 5 per kg, how much of each type of fertilizer should be used so that nutrient requirements are met at minimum cost?

17. Ms. Matlida has three machines I, II, and III installed in her factory. Machines, I and II are capable of being operated for at least 5 hours a day, while machine III can operate for not more than 14 hours a day. She produces only two items M and N each requiring the use of all three machines. The number of hours required for producing 1 unit of items M and N on the three machines are given in the following table:

Items	**Number of hours required on machines**		
	I	II	III
M	1	2	1
N	2	1	1

She makes a profit of Tshs 600 and Tshs 400 on items M and N, respectively.

(a) How many of each item should she produce so as to maximize her profit, assuming that she can sell all the items that she produced?

(b) Determine the maximum profit.

18. Bangulo company have factories at towns A and B, which supply at warehouses, P_1 and P_2. The weekly factory capacities are 160 and 140 units, respectively and warehouses requirements are 70 and 120 units, respectively. The cost of transportation of 1 unit from A to P_1 is Tshs 160 and from A to P_2 is Tshs 240. Similarly, the cost of transportation from B to P_1 is Tshs 200 and from B to P_2 is Tshs 260. How should the company make supplies to the warehouses at the minimum cost?

19. Mr. Nyahuye have two storage deposits. He stores 200 tonnes of rice at deposit 1 and 300 tonnes of rice at deposit 2. The rice has to be sent to three marketing centres A, B, and C. The demands at A, B, and C are 150, 150, and 200 tonnes of rice, respectively. The transportation costs in Tanzanian shillings per tonne from the deposit to each market centre are shown in the following table:

Deposits	**Market centres**		
	A	B	C
Deposit 1	50	100	70
Deposit 2	80	150	40

How many tonnes of rice should be sent from the deposits to each marketing centre so that the transportation cost is minimum?

20. Kisarawe company has factories at P and Q which supply tonnes of cement to regions A and B. The weekly factory capacities are 160 and 140 tonnes of cement, respectively and the regions require 70 and 120 tonnes of cement, respectively. The cost of transportation of one tonne of cement from factory P to region A is Tshs 160 and the cost from factory P to region B is Tshs 240. Similarly, the transportation cost from factory Q to region A is Tshs 200 and from factory Q to region B is Tshs 260.

(a) Find the objective function to be minimized by the company so as to supply tonnes of cement to each region.

(b) Find the constraints associated with the transportation problem.

(c) Does the problem balance or not and why?

Chapter Nine

Differentiation

Introduction

The concept of differentiation refers to the method of finding the derivative of a continuous function. It is equivalent to finding the slope or gradient of the continuous function. For a straight line, the slope remains the same all the way along the line. The slope of a curve changes continuously along the curve. In this case, the gradient of the curve is considered at the point of tangent to the curve. In this chapter, you will learn about derivatives, differentiation of functions, applications of differentiation in real life problems, Taylor's and Maclaurin's series, and introduction to partial derivatives of functions. The competencies developed are applicable in various real life situations such as in business, science, engineering, economics, building and constructions, population modeling, dynamical systems, among many other applications.

Derivatives

The derivative of a function is defined as the rate of change of a function with respect to a given variable. Given the graph of the function, the derivative can be interpreted as the slope of the graph of the function, or more precisely as the slope of the tangent line at a point. In the coordinate plane, slope of a straight line is given by the ratio of the change in the y coordinate to the change in the x coordinate. In Figure 9.1, if (x_0, y_0) and (x_1, y_1) are two points on the line, then the slope is given by the ratio $\frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$.

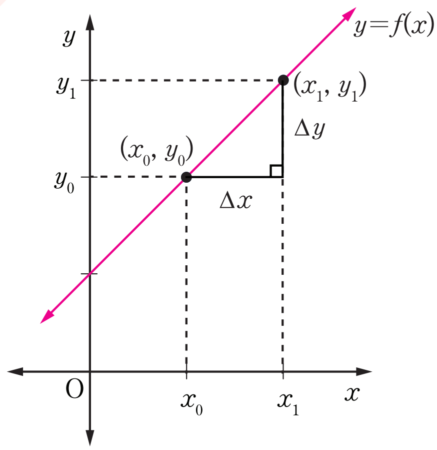


Figure 9.1: *Graph of a straight line*

Let h be a small increase in x_0, and $x_1 = x_0 + h$, then $h = x_1 - x_0$.

If $y = f(x)$, then $y_0 = f(x_0)$ and $y_1 = f(x_1) = f(x_0+h)$.

Thus, in terms of function notation, slope is given by $\frac{f(x_1)-f(x_0)}{x_1-x_0}$. This change in notation is useful for advancing from the idea of the slope of the line to the more general concept of the derivative of a function.

Given a point (x, y) on the graph of f, the process of finding a function f' (read as "f prime") whose expression is $f'(x)$ gives the slope of the graph at the point (x, y). The expression $f'(x)$ is called the derivative of the function, f at a point x. The process of finding the derivative of a function is called differentiation.

Notations of the derivative

The derivative of a function $y = f(x)$ with respect to the variable x is denoted either by y' or $f'(x)$ or $\frac{dy}{dx}$, and is known as the first derivative. The notation $\frac{dy}{dx}$ is read as "$Dy\ Dx$".

Generally, $\frac{dy}{dx}$ means the rate at which y changes when x changes. It defines the gradient of $y = f(x)$.

Note that, derivative is the rate of change of one quantity with respect to another quantity. For example, $\frac{dy}{dz}$ is the rate of change of y with respect to z. The second derivative of a function of $y = f(x)$ is obtained by differentiating the first derivative, and it is denoted either by y'' or $f''(x)$ or $\frac{d^2y}{dx^2}$.

That is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) \text{ or } f''(x) = \frac{d}{dx}(f'(x)).$$

The third derivative is denoted either by y''' or $f'''(x)$ or $\frac{d^3y}{dx^3}$. That is,

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \text{ or } f'''(x) = \frac{d}{dx}(f''(x)).$$

Note that, the derivative of a constant function is zero. That is, $\frac{d}{dx}(c) = 0$, where c is any constant. This is because, in a constant function there is no change in the values of the function.

Differentiation of a function from first principles

The slope of a secant line is used to derive the formula for differentiation of functions from first principles. Figure 9.2 shows a curve of the function $y = f(x)$ with a secant line passing through the points Q and P, where h is a small increase in the value of x coordinate at point Q.

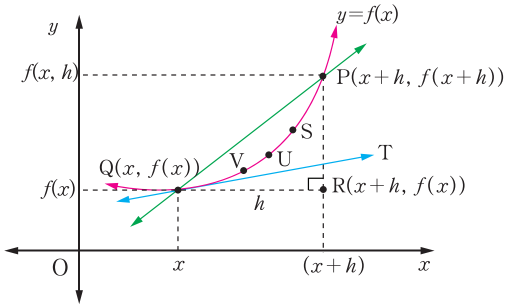


Figure 9.2: *Illustration of the gradient of a curve $y = f(x)$ between the points $P(x+h, f(x+h))$ and $Q(x, f(x))$*

The slope of the secant line

$$\overline{PQ} = \frac{\overline{PR}}{\overline{RQ}} = \frac{f(x+h)-f(x)}{(x+h)-x}$$

$$= \frac{f(x+h)-f(x)}{h}$$

The concept of limit as applied in derivatives

The limit of a function $f(x)$ is the value of $f(x)$ as x approaches a certain value. For example, $\lim_{x \to a} f(x) = L$ means that, as x approaches a, the value of $f(x)$ is L. The notation of limit is used to describe the behaviour of graphs as the variable approaches a certain point. The slope of a tangent line to the function can be defined using the concept of limits. The following activity illustrates the concept of limits.

Activity 9.1: Application of limits in derivatives of functions

Individually or in a group, perform the following tasks:

1. Sketch the graph of the curve $f(x)$ of your choice on the xy-plane.
2. Locate a point Q $(x_0, f(x_0))$ of your choice on the curve.
3. Draw a tangent line to the curve at point Q.
4. Pick a point P$(x_0 + h, f(x_0+h))$, near $Q(x, f(x))$, where h is a small change in x. Then, draw a secant line to join the points Q and P.
5. Find the gradient of the secant line $\overline{PQ}$ in task 4.
6. Discuss how you can use the gradient of a secant line to approximate the slope of the tangent line.
7. What did you observe from the tasks?

In Figure 9.2, if h is small, the gradient of the secant line is an approximation of the gradient of the tangent line. Therefore, it can be concluded that the gradient of the tangent line is the limit of the gradient of the secant line as h becomes smaller and smaller (means as h approaches zero).
If point P is moved nearer to Q, say to points S, U, V,…, then the chords $\overline{QS}, \overline{QU}, \overline{QV}, \ldots$ will be getting closer and closer to the tangent line drawn at point Q. The action of moving point P towards point Q decreases the value of h. Thus, as h approaches zero, (that is, $h \to 0$), the line

$\overline{PQ}$ approaches the tangent line at point Q. Hence, the gradient of the chord $\overline{PQ}$ approaches the gradient of the tangent at point Q. The gradient of the tangent line at point Q can then be said to be the limiting value of the gradient of the chord $\overline{PQ}$ as $h \to 0$.

This can be written as, gradient of the curve at $Q = \lim\limits_{h\to 0}\dfrac{f(x_0+h)-f(x_0)}{h}$. Thus, the gradient of the curve at any point Q(x, $f(x)$) on the curve is the gradient of the tangent line at the same point Q. Therefore, the gradient of the curve is $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$, provided that this limit exists.

The gradient of the curve is called the gradient function or derivative function because it is derived from the original function. Thus, the formula for differentiation from first pricniples of $f(x)$ is given by:

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$. This means that the function is differentiated with respect to x.

The formula

$f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ is used for differentiation of a function from first principles, and $\dfrac{dy}{dx}$ or $f'(x)$ is called the derivative of $y = f(x)$.

Activity 9.2: Recognizing the derivative of a function from first principles

Individually or in a group, write down any two polynomial functions $f(x)$ and $g(x)$ of degrees 2 and 3, respectively. Then, perform the following tasks:

1. Write the general formula for differentiating a function from first principles.
2. Use the formula in task 1 to find the derivative of each of the polynomial functions.
3. Use the results in task 2 to deduce the general formula for differentiating the polynomial $f(x) = x^n$, where n is a positive integer
4. What have you observed from task 3?
5. Share your findings with other students through discussion.

Example 9.1

Differentiate from first principles the function $f(x) = 7x + 6$.

Solution

Given $f(x) = 7x + 6$, then from first principles;

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $f(x+h) = 7(x+h)+6$.

Thus,

$$f'(x) = \lim_{h\to 0} \frac{7(x+h)+6-(7x+6)}{h}$$

$$= \lim_{h\to 0} \frac{7x+7h+6-7x-6}{h}$$

$$= \lim_{h\to 0} \frac{7h}{h}$$

$$= \lim_{h\to 0} 7 = 7$$

Therefore, $f'(x) = 7$.

Example 9.2

Find $f'(x)$ from first principles if, $f(x) = x^2 - 6x + 1$.

Solution

Given $f(x) = x^2 - 6x + 1$, then from first principles;

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

But $f(x+h) = (x+h)^2 - 6(x+h) + 1$.

Thus, $f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - 6(x+h) + 1 - (x^2 - 6x + 1)}{h}$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 6x - 6h + 1 - x^2 + 6x - 1}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} \frac{2xh + h^2 - 6h}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0} (2x + h - 6) = 2x - 6$$

Therefore, $f'(x) = 2x - 6$.

Example 9.3

Find the gradient function of the function $f(x) = x^3 + 7$ using first principles, hence evaluate the gradient at the point $x = 2$.

Solution

Given $f(x)=x^3+7$, then from first principles;

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$\Rightarrow f(x+h)=(x+h)^3+7$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{(x+h)^3+7-\left(x^3+7\right)}{h}$$

$$=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3+7-x^3-7}{h}$$

$$=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}$$

$$=\lim_{h\to 0}\left(3x^2+3xh+h^2\right)$$

$$\Rightarrow f'(x)=3x^2$$

Thus, $f'(2)=3(2)^2=12$.

Therefore, $f'(x)=3x^2$ and $f'(2)=12$.

Example 9.4

Differentiate from first principles the function $f(x)=\dfrac{1}{\sqrt{x}}$.

Solution

Given $f(x)=\dfrac{1}{\sqrt{x}}$, then from first principles;

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

But $f(x+h)=\dfrac{1}{\sqrt{x+h}}$

Therefore, $f'(x)=\lim_{h\to 0}\dfrac{\dfrac{1}{\sqrt{x+h}}-\dfrac{1}{\sqrt{x}}}{h}$

$$=\lim_{h\to 0}\left(\frac{\sqrt{x}-\sqrt{x+h}}{\sqrt{x}\left(\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

Rationalizing the numerator of the expression gives

$$f'(x)=\lim_{h\to 0}\left(\frac{\sqrt{x}-\sqrt{x+h}\left(\sqrt{x}+\sqrt{x+h}\right)}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{\left(\sqrt{x}-\sqrt{x+h}\right)\times\left(\sqrt{x}+\sqrt{x+h}\right)}{\sqrt{x}\left(\sqrt{x+h}\right)\times\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{x-(x+h)}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{-h}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)\times\frac{1}{h}$$

$$=\lim_{h\to 0}\left(\frac{-1}{\sqrt{x}\left(\sqrt{x+h}\right)\left(\sqrt{x}+\sqrt{x+h}\right)}\right)$$

Now, as $h\to 0$ gives,

$$f'(x)=\left(\frac{-1}{\sqrt{x}\left(\sqrt{x}\right)\left(\sqrt{x}+\sqrt{x}\right)}\right)$$

$$\Rightarrow f'(x)=\frac{-1}{2x\sqrt{x}}$$

Therefore, $f'(x)=\dfrac{-1}{2x\sqrt{x}}$.

Exercise 9.1

1. From first principles of differentiation, find the derivative of each of the following functions:
   (a) $f(x)=5x+6$
   (b) $f(t)=t^3-t^2$
   (c) $f(x)=-3x+5$
   (d) $f(x)=2x^2-3x+5$
   (e) $f(v)=2v-500$
   (f) $f(x)=\dfrac{1}{3-x}$
   (g) $f(t)=kt^5$
   (h) $f(x)=\sqrt{2x}$
2. Using first principles of differentiation, find the gradient function of each of the following functions and hence compute the gradient of each curve $y=f(x)$ at the given points.

(a) $y = 5x\,;\, x = 1$ (b) $y = x^3 - x^2 + 4;\, x = 2$ (c) $y = x - \frac{1}{4};\, x = 2$

(d) $y = x^2 + 6;\, x = 3$ (e) $y = \frac{1}{x^2};\, x = 4$

3. If $f(x) = 5x - 2x^2$, find $f'(x)$ using first principles of differentiation and evaluate $f'(3)$ and $f'(-1)$.
4. For each of the following curves, find the y-coordinate for the given value of x and use first principles to find the gradient.

   (a) $y = 5x^3; x = 6$ (b) $y = x(x^2 - 3);\, x = 2$ (c) $y = x(x-2)^2;\, x = 1$
5. From first principles of differentiation, show that:

   (a) $\frac{d}{dx}\left(\frac{1}{x^2}\right) = -\frac{2}{x^3}$ (b) $\frac{d}{dx}\left(\frac{1}{x^3}\right) = -\frac{3}{x^4}$
6. Use first principles to show that $\frac{d}{dx}\left(x^2\sqrt{1-x}\right) = \frac{\sqrt{1-x}}{2}\left(\frac{4x - 5x^2}{1-x}\right)$.
7. From first principles, show that $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{2x}{\left(1+x^2\right)^2}$. Hence, verify that $\frac{d}{dx}\left(\frac{1}{1+x^2}\right) = -\frac{16}{25}$ at $x = \frac{1}{2}$.

Differentiation of a function

The process of finding the derivative of a function or rate of change of one variable with respect to another variable is called differentiation. It is important to note that, not all functions are differentiable. The function $f(x)$ is said to be differentiable at a point x if $\lim\limits_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists. The functions considered in this chapter are differentiable for all values of x.

Derivatives of polynomial functions

The derivative of the polynomial function $f(x) = x^n$, where n is a positive integer can be determined from first principles.

Given $f(x) = x^n$, then from first principles,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

But $f(x+h)=(x+h)^n$

Thus, $f'(x)=\lim_{h\to 0}\dfrac{(x+h)^n-x^n}{h}$

$$=\lim_{h\to 0}\frac{\left(x\left(1+\frac{h}{x}\right)\right)^n-x^n}{h}$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\left(x^n\left(1+\frac{h}{x}\right)^n\right)-x^n}{h} \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\text{(i)}$$

On expanding $\left(1+\frac{h}{x}\right)^n$ by using binomial theorem, gives

$$\left(1+\frac{h}{x}\right)^n=1+n\frac{h}{x}+\frac{n(n-1)}{2!}\left(\frac{h}{x}\right)^2+\frac{n(n-1)(n-2)}{3!}\left(\frac{h}{x}\right)^3+\cdots \quad \dots\dots\dots\dots \text{(ii)}$$

Substituting equation (ii) into equation (i), it gives;

$$f'(x)=\lim_{h\to 0}\frac{\left[x^n+nhx^{n-1}+\frac{n(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots\right]-x^n}{h}$$

$$=\lim_{h\to 0}\frac{\left[x^n+nhx^{n-1}+\frac{n(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots-x^n\right]}{h}$$

$$=\lim_{h\to 0}\frac{\left[nhx^{n-1}+n\frac{(n-1)}{2!}(h)^2x^{n-2}+\frac{n(n-1)(n-2)}{3!}(h)^3x^{n-3}+\cdots\right]}{h}$$

$$=\lim_{h\to 0}\left[nx^{n-1}+n\frac{(n-1)}{2!}hx^{n-2}+\frac{n(n-1)(n-2)}{3!}h^2x^{n-3}+\cdots\right]$$

$$=\left[nx^{n-1}+n\frac{(n-1)}{2!}(0)x^{n-2}+\frac{n(n-1)(n-2)}{3!}(0)^2x^{n-3}+\cdots\right] \text{ as } h\to 0$$

Thus, $f'(x)=nx^{n-1}$.

Therefore, $\dfrac{d}{dx}(x^n)=f'(x)=nx^{n-1}$.

Note that, this formula is true even for negative values of n and fractional values such as $n=\dfrac{p}{q}$, where p and q are integers.

Example 9.5

Differentiate $y = x^5$ with respect to x.

Solution

From $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, then

$$\Rightarrow \frac{d}{dx}\left(x^5\right) = 5x^{5-1}$$
$$= 5x^4$$

Therefore, $\frac{d}{dx}\left(x^5\right) = 5x^4$.

Example 9.6

Differentiate the function
$f(x) = (x-2)(x+3)$ with respect to x at the point $x = 1$.

Solution

Given $f(x) = (x-2)(x+3)$

$\Rightarrow f(x) = x^2 + x - 6$

Using the rule, $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$, then

$\Rightarrow f'(x) = 2x^{2-1} + x^{1-1} - 0$

$\Rightarrow f'(x) = 2x + 1$

At the point $x = 1$, $f'(1) = 2(1) + 1 = 3$

Therefore, at the point $x = 1$, $f'(x) = 3$.

Properties of derivatives

The following are properties of derivatives for polynomial functions:

1. If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.
2. $\frac{d}{dx}\left[f(x) \pm g(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right] = f'(x) \pm g'(x)$
3. If k is any scalar, then $\frac{d}{dx}\left[kf(x)\right] = k\frac{d}{dx}\left[f(x)\right] = k\,f'(x)$.

Example 9.7

Find the derivative of the function $f(x)=\dfrac{4x^3+2x^2}{\sqrt[3]{x}}$.

Solution

Since the derivative of sum of two functions is the sum of their derivatives, then

$$f(x)=\frac{4x^3+2x^2}{\sqrt[3]{x}}=\frac{4x^3}{\sqrt[3]{x}}+\frac{2x^2}{\sqrt[3]{x}}$$

$$=\frac{4x^3}{x^{\frac{1}{3}}}+\frac{2x^2}{x^{\frac{1}{3}}}=4x^{\frac{8}{3}}+2x^{\frac{5}{3}}$$

Using $\dfrac{d}{dx}\left(x^n\right)=nx^{n-1}$

$$\Rightarrow f'(x)=4\left(\frac{8}{3}\right)x^{\frac{8}{3}-1}+2\left(\frac{5}{3}\right)x^{\frac{5}{3}-1}=\frac{32}{3}x^{\frac{5}{3}}+\frac{10}{3}x^{\frac{2}{3}}$$

Therefore, $f'(x)=\dfrac{32}{3}x^{\frac{5}{3}}+\dfrac{10}{3}x^{\frac{2}{3}}$.

Exercise 9.2

1. Find the derivative of each of the following functions with respect to x at the given point.

   (a) $f(x)=2x\left(x^2-4\right);\ x=1$

   (b) $f(x)=3x^4-2x^3+x^2+9;\ x=0$

   (c) $f(x)=\dfrac{6x^5+2x^3}{x^2};\ x=-1$

   (d) $f(x)=x^4-\dfrac{1}{3}x^3+\dfrac{1}{4}x^2+9;\ x=-2$

2. Differentiate each of the following functions with respect to x:

   (a) $f(x)=\dfrac{1}{4}ax^2-3bx+c$ (b) $f(x)=\dfrac{1}{5}\left(\dfrac{x^2+4}{\sqrt[5]{x}}\right)$ (c) $f(x)=\dfrac{\sqrt{x}+x^2-4x}{\sqrt{x}}$

3. Find the derivative of each of the following functions with respect to x:

   (a) $f(x)=2(x-1)(x-5)$

   (b) $g(x)=\dfrac{3(x-1)(x-2)}{4}$

   (c) $h(x)=\dfrac{x^3-2x^2}{3x}$

4. If the gradient of the curve $f(x)=(x-3)(x^2+a)$ at the point $x=1$ is -1. Determine the value of a.

5. Find the coordinates of the point on the curve $y=\frac{2}{x^2}$ at which its gradient is $\frac{1}{2}$.

6. If $g(x)=2x^3-8x+5$, find the values of x for which $g'(x)=0$.

7. Find the values of t for which the gradient of the curve $x=2t^3+3t^2$ is zero. Hence, determine the gradient at $t=2$.

8. Determine the gradient of the function $h(t)=9t^4-7t^3+8t^2-\frac{8}{t}+\frac{10}{t^3}$ at $t=1$.

9. Differentiate $2x^2+5+4x^{-2}$ with respect to x and then find the value of the derivative when $x=4$.

10. If $y(x+2)^2=-100$, find $\frac{dy}{dx}$ and the gradient of the curve at $x=-1$.

Derivative of product of polynomials

Let y be the product of two functions u and v of an independent variable x. That is, $y=uv$. The derivative of the product of functions can be obtained as follows.

Let $y=u(x)v(x)$. If δy, δu and δv denote small increments in y, u, and v, respectively, then

$$y+\delta y=(u+\delta u)(v+\delta v)$$

$$\Rightarrow y+\delta y=uv+v\delta u+u\delta v+\delta u\delta v$$

But $y=uv$

$$\Rightarrow \delta y=v\delta u+u\delta v+\delta u\delta v$$

Dividing by δx on both sides gives,

$$\frac{\delta y}{\delta x}=v\frac{\delta u}{\delta x}+u\frac{\delta v}{\delta x}+\frac{\delta u}{\delta x}\delta v$$

Now, as $\delta x\to 0$, $\frac{\delta y}{\delta x}=\frac{dy}{dx}$, $\frac{\delta v}{\delta x}=\frac{dv}{dx}$, $\frac{\delta u}{\delta x}=\frac{du}{dx}$, and $\delta v\to 0$,

Therefore, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}+\frac{du}{dx}(0)=v\frac{dv}{dx}+u\frac{dv}{dx}$

The result, $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$ is called the product rule of differentiation.

Therefore, the product rule for differentiation is given by $\frac{dy}{dx}=v\frac{du}{dx}+u\frac{dv}{dx}$.

Example 9.8

Use the product rule to differentiate the following function with respect to x.
$y=(x^3-4x^2+6x)(2x^4-19x+5)$.

Solution

Let $u=x^3-4x^2+6x$ and $v=2x^4-19x+5$

$\Rightarrow \dfrac{du}{dx}=3x^2-8x+6$ and $\dfrac{dv}{dx}=8x^3-19$

From, $\dfrac{dy}{dx}=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx}=\left(2x^4-19x+5\right)\left(3x^2-8x+6\right)+\left(x^3-4x^2+6x\right)\left(8x^3-19\right)$$

$$=14x^6-48x^5+60x^4-76x^3+243x^2-268x+30$$

Therefore, $\dfrac{dy}{dx}=14x^6-48x^5+60x^4-76x^3+243x^2-268x+30$.

Example 9.9

Differentiate with respect to x the function
$f(x)=(x^3-2)(x+4)^2$, then find $f'(x)$ at the point $x=1$.

Solution

The function $f(x)=(x^3-2)(x+4)^2$ is a product of two functions.

Using $f'(x)=v\dfrac{du}{dx}+u\dfrac{dv}{dx}$

Let $v=\left(x+4\right)^2$ and $u=x^3-2$

$\Rightarrow \dfrac{dv}{dx}=2(x+4)$ and $\dfrac{du}{dx}=3x^2$

$$\Rightarrow f'(x)=\left(x+4\right)^2\left(3x^2\right)+\left(x^3-2\right)\left(2x+8\right)$$

$$\Rightarrow f'(x)=5x^4+32x^3+48x^2-4x-16$$

At the point $x=1$,

$$f'(1) = 5(1)^4 + 32(1)^3 + 48(1)^2 - 4(1) - 16$$
$$= 65$$

Therefore, $f'(x) = 5x^4 + 32x^3 + 48x^2 - 4x - 16$ and $f'(x)$ at $x = 1$ is 65.

Exercise 9.3

1. Differentiate each of the following functions with respect to the given variables:
   (a) $f(x) = (1+x)^2(x^2-1)$
   (b) $f(s) = (6s^2 - 4s)(s^2 + 4s + 8)$
   (c) $f(y) = \left(6 - 3y^2 + \frac{1}{y^2}\right)\left(\frac{1}{y} - 4y + 8y^2\right)$
   (d) $f(t) = (7t^2 - 4t + 9)(2t^3 - 2t + 1)$
2. Find the x-coordinate of the point where the gradient of the curve $y = (x^2 - 2)\sqrt{3+x}$ is zero.
3. Evaluate the derivative of the curve $y = x^3(4-x)^{\frac{1}{2}}$ at the point $x = 1$.
4. Find the derivative of each of the following functions with respect to x:
   (a) $f(x) = (x^2 + 4x + 1)\sqrt{x-3}$
   (b) $g(x) = 2x^6(2+x)^5$
   (c) $h(x) = x^{-3}(1+x^2)^{\frac{1}{2}}$
   (d) $k(x) = (1-x^3)(2 - x^5 + x^6)$
5. Differentiate $g(x) = (x^2+1)(x+4)^{-2}$ with respect to x.
6. If $f(x) = 2x^{\frac{3}{2}}(2+\sqrt{x})(-1+\sqrt{x})$, show that $f'(x) = 4x + 5x\sqrt{x} - 6\sqrt{x}$.
7. The equation of a curve is $y = (x-3)(x+4)$. Find the gradient of the curve:
   (a) at the point where the curve crosses the y-axis.
   (b) at each of the points where the curve crosses the x-axis.
8. Use the product rule to find $f'(r)$, where $f(r) = (1+r^2)^2(1-r^2)$.
9. Differentiate $z = (y-2)^2(2y+3)$ with respect to y.
10. Given that $f(u) = (u+1)^2(u-1)^4$, find $f'(u)$ and $f'(-5)$.

The derivative of quotient of two functions

If u and v are two differentiable functions of x and $y=\frac{u}{v}$, then the derivative of y can be determined as follows:

Given $y=\frac{u}{v}$, then it implies that

$y+\delta y=\frac{u+\delta u}{v+\delta v}$, where δy, δu, and δv denote small increments in y, u, and v, respectively.

Thus, $\delta y=\frac{u+\delta u}{v+\delta v}-\frac{u}{v}$

$$\Rightarrow \delta y=\frac{(vu+v\delta u)-(uv+u\delta v)}{v(v+\delta v)}$$

$$\Rightarrow \delta y=\frac{v\delta u-u\delta v)}{v(v+\delta v)}$$

Divide by δx both sides to get

$$\frac{\delta y}{\delta x}=\frac{v\frac{\delta u}{\delta x}-u\frac{\delta v}{\delta x}}{v(v+\delta v)}$$

Taking the limit as $\delta x\to 0$ and $\delta v\to 0$ gives

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{\lim_{\delta x\to 0}\left(v\frac{\delta u}{\delta x}-\frac{\delta v}{\delta x}\right)}{\lim_{\delta x\to 0}\left(v^2+v\delta v\right)}$$

$$=\frac{\lim_{\delta x\to 0}v\frac{\delta u}{\delta x}-\lim_{\delta x\to 0}u\frac{\delta v}{\delta x}}{\lim_{\delta x\to 0}\left(v^2+v\delta v\right)}$$

$$=\frac{v\lim_{\delta x\to 0}\frac{\delta u}{\delta x}-u\lim_{\delta x\to 0}\frac{\delta v}{\delta x}}{\lim_{\delta x\to 0}v^2+v\lim_{\delta x\to 0}\delta v}$$

But $\lim_{\delta x\to 0}\frac{\delta y}{\delta x}=\frac{dy}{dx}$, $\lim_{\delta x\to 0}\frac{\delta u}{\delta x}=\frac{du}{dx}$, $\lim_{\delta x\to 0}\frac{\delta v}{\delta x}=\frac{dv}{dx}$, $\lim_{\delta v\to 0}v^2=v^2$, and $\lim_{\delta v\to 0}\delta v=0$

Thus, $\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ is called the quotient rule of differentiation.

Therefore, the quotient rule of differentiation is given by;

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}.$$

Example 9.10

Find the derivative of $y=\frac{x^2+4}{x^2-4}$.

Solution

Given $y=\frac{x^2+4}{x^2-4}$.

Let $u=x^2+4$ and $v=x^2-4$

$$\Rightarrow \frac{du}{dx}=2x \quad \text{and} \quad \frac{dv}{dx}=2x$$

Using quotient rule, it implies that

$$\frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Thus,

$$\frac{dy}{dx}=\frac{(x^2-4)(2x)-(x^2+4)(2x)}{(x^2-4)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-16x}{(x^2-4)^2}$$

Therefore, $\frac{dy}{dx}=\frac{-16x}{(x^2-4)^2}$.

Example 9.11

If $y=\frac{5x^3+2x^2}{x+1}$, find $\frac{dy}{dx}$.

Solution

Given $y=\frac{5x^3+2x^2}{x+1}$.

Let $u=5x^3+2x^2$ and $v=x+1$

$\Rightarrow \frac{du}{dx}=15x^2+4x$ and $\frac{dv}{dx}=1$

Using the quotient rule,

$$\Rightarrow \frac{dy}{dx}=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Thus,

$$\frac{dy}{dx}=\frac{(x+1)(15x^2+4x)-(5x^3+2x^2)(1)}{(x+1)^2}$$

$$=\frac{15x^3+4x^2+15x^2+4x-5x^3-2x^2}{(x+1)^2}$$

$$=\frac{10x^3+17x^2+4x}{(x+1)^2}$$

Therefore, $\frac{dy}{dx}=\frac{10x^3+17x^2+4x}{(x+1)^2}$.

Exercise 9.4

1. Find $\frac{dy}{dx}$ in each of the following:

   (a) $y=\frac{2x+4}{3x-2}$ (c) $y=\frac{2x+1}{x^2-2}$

   (b) $y=\left(\frac{x+1}{x-1}\right)^2$

2. Differentiate the following functions with respect to x. For each case, find $f'(2)$ correct to 3 decimal places.

   (a) $f(x)=\sqrt{\left(\frac{1+x}{3+x}\right)}$

   (b) $f(x)=\frac{\sqrt{x}}{\sqrt{x+3}}$

   (c) $f(x)=\frac{2x^3-x^2}{\sqrt{x+3}}$

3. Find the derivative of each of the following:

   (a) $s=\frac{t^{-1}+2t^2}{3t^2}$

   (b) $p=\frac{q^3-2q^2+3q}{q^2}$

   (c) $u=\frac{v+2v^2}{3v}$

4. Given that $t=\frac{3z^2+z-6}{2z}$, find $\frac{dt}{dz}$ at $z=2$.

5. If $g(x)=x-1+\frac{1}{x+1}$, where $x\neq-1$, find the values of x for which $g'(x)=0$.

6. Show that the gradient of the curve $y=\frac{2x}{x^2-5}$ at the point $x=2$ is -18.

7. If $y = \frac{x}{x+5}$, prove that $x\frac{dy}{dx} = y(1-y)$.
8. Verify that, the derivative of $g(y) = \frac{1+y^2}{1-y^2}$ is $g'(y) = \frac{4y}{\left(1-y^2\right)^2}$.
9. Determine $\frac{d}{ds}\left(\frac{(s-1)(s-2)}{(s-3)(s-4)}\right)$, then find the gradient when $s = 4.5$.
10. If $z^2\left(1+w^2\right) = 1-w^2$, show that $\left(\frac{dz}{dw}\right)^2 = \frac{1-z^4}{1+w^4}$.

The chain rule

The chain rule is used to find the derivative of a composite function.
Consider the following functions:

(i) $f(x) = (4x-1)^3$ is a cubic of a linear function $g(x) = (4x-1)$.

(ii) $n(x) = \sqrt{\left(x^4-3\right)}$ is a square root of the quartic function $m(x) = \left(x^4-3\right)$.

(iii) $h(x) = \left(x^2-2\right)^5$ is a quintic of a quadratic function $g(x) = \left(x^2-2\right)$.

If y is a function of u, say $y = f(u)$, where u itself is a function of x, say $u = g(x)$, then y is called a composite function of x.
Given $y = f(u)$ then,
$y + \delta y = f(u + \delta u)$, where δy and δu denote small increments in y and u, respectively.

$$\delta y = f(u+\delta u) - f(u)$$

Dividing by δx both sides gives,

$$\frac{\delta y}{\delta x} = \frac{f(u+\delta u)-f(u)}{\delta x}$$

$$\Rightarrow \frac{\delta y}{\delta x} = \frac{f(u+\delta u)-f(u)}{\delta u} \times \frac{\delta u}{\delta x}$$

Introducing limits both sides gives,

$$\lim_{\delta x\to 0}\frac{\delta y}{\delta x} = \lim_{\delta u\to 0}\frac{f(u+\delta u)-f(u)}{\delta u} \times \lim_{\delta x\to 0}\frac{\delta u}{\delta x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{d}{\delta u}[f(u)] \times \frac{du}{dx}$$

Thus, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$, which is the chain rule of differentiation.

Therefore, the chain rule of differentiation is given by; $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Example 9.12

Given that $y = \left(2x^2+4\right)^6$, find $\frac{dy}{dx}$.

Solution

Given $y = \left(2x^2+4\right)^6$.

Let $u = 2x^2 + 4$, then $y = u^6$.

$$\Rightarrow \frac{du}{dx} = 4x \text{ and } \frac{dy}{du} = 6u^5$$

Using the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\Rightarrow \frac{dy}{dx} = 4x \times 6u^5$$

$$\Rightarrow \frac{dy}{dx} = 24xu^5. \text{ But } u = 2x^2 + 4.$$

Therefore, $\frac{dy}{dx} = 24x\left(2x^2+4\right)^5$.

Example 9.13

If $y = 6(x^3 - 3x + 2)^2$ and $u = (x^3 - 3x + 2)$, express y in terms of u. Hence, find $\frac{dy}{dx}$ at $x = 2$.

Solution

Given $y = 6(x^3 - 3x + 2)^2$ and $u = (x^3 - 3x + 2)$

$$\Rightarrow y = 6u^2$$

$$\Rightarrow \frac{dy}{du} = 12u \text{ and } \frac{du}{dx} = 3x^2 - 3$$

Using the chain rule,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Thus, $\frac{dy}{dx} = 12u \times \left(3x^2 - 3\right)$

$$\Rightarrow \frac{dy}{dx} = 12\left(x^3 - 3x + 2\right)\left(3x^2 - 3\right)$$

At $x = 2$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{x=2} = 12\left((2)^3 - 3(2) + 2\right)\left(3(2)^2 - 3\right) = 432.$$

Therefore, $\frac{dy}{dx} = 432$, at $x = 2$.

Example 9.14

Find the derivative of $\left(x^4 - 3x^2 + 4\right)^{\frac{2}{3}}$ at $x = 1$.

Solution

Given $y = \left(x^4 - 3x^2 + 4\right)^{\frac{2}{3}}$.

Let $u = (x^4 - 3x^2 + 4) \Rightarrow y = u^{\frac{2}{3}}$

$$\Rightarrow \frac{dy}{du} = \frac{2}{3}u^{-\frac{1}{3}} \text{ and } \frac{du}{dx} = 4x^3 - 6x$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{2}{3}u^{-\frac{1}{3}} \times \left(4x^3 - 6x\right)$$

$$\frac{dy}{dx} = \frac{2}{3}\left(x^4 - 3x^2 + 4\right)^{-\frac{1}{3}} \times \left(4x^3 - 6x\right)$$

At $x = 1$,

$$\left[\frac{dy}{dx}\right]_{x=1} = \frac{2}{3}\left((1)^4 - 3(1)^2 + 4\right)^{-\frac{1}{3}} \times \left(4(1)^3 - 6(1)\right)$$

$$= -1.058$$

Therefore, $\left[\frac{dy}{dx}\right]_{x=1} = -1.058$.

Exercise 9.5

1. Using the chain rule, find the derivative of each of the following functions with respect to x:

   (a) $f(x)=\sqrt{x^4-2}$ (b) $f(x)=\dfrac{1}{x^3+1}$

2. Differentiate each of the following using the chain rule:

   (a) $y=\left(x^2+1\right)^{16}$ (c) $y=\left(3x^2-4\right)^3$

   (b) $y=\left(z^3+4z^2-3z-3\right)^{-6}$ (d) $y=\sqrt{x^2+5}$

3. Differentiate each of the following functions using the chain rule:

   (a) $f(t)=\sqrt{t^2-6t+7}$ (c) $g(z)=\dfrac{1}{\sqrt{3-z^3}}$ (e) $y=\left(t^3-\sqrt{t}\right)^{2.9}$

   (b) $f(t)=(3-t)^{21}$ (d) $z=\left(x+\dfrac{1}{x}\right)^{\frac{2}{5}}$

4. Use the chain rule of differentiation to prove that,

   $$\frac{d}{dx}\left[\frac{1}{\sqrt[3]{6x^5-7x^3+9}}\right]=-x^2\left(10x^2-7\right)\left(6x^2-7x^3+9\right)^{-\frac{4}{3}}.$$

5. Given that $f(t)=\dfrac{1}{\sqrt[3]{2t^4+3t^3-5t+6}}$, find the following:

   (a) $f'(t)$

   (b) $f'(-1)$

6. Find $\dfrac{d}{dr}\left(\sqrt{r^2-\dfrac{1}{r^2}}\right)$ using the chain rule.

7. Determine $\dfrac{d}{d\theta}(4-2\theta)^5$.

8. Find the derivative of $x=\left(1-\dfrac{2}{\sqrt{u}}\right)^{\frac{1}{3}}$ using the chain rule.

9. Evaluate $\dfrac{d}{dy}\left(y+1-\dfrac{1}{y}\right)^3$ at the point $y=-2$.

10. Verify that $\dfrac{d}{dt}\left[\dfrac{\sqrt{1+t^2}-t}{\sqrt{1+t^2}+t}\right]=\dfrac{-2}{\sqrt{1+t^2}\left(\sqrt{1+t^2}+t\right)^2}$.

Differentiation of implicit functions

An implicit function is a function written in terms of both dependent and independent variables. It is the function of the form $f(x, y, \ldots)$ whose variables are connected by an equation which does not express one variable in terms of the other variables. For example, $x^2y^3 = 8$ is an implicit function because each variable in the equation $x^2y^3 = 8$ is not expressed as the function of the other. Also, the equation $x = y^2$ does not define y as the function of x, since each positive value of x has two values of y.

In this case, $x = y^2$ can be differentiated with respect to y as follows.

$$\frac{d(x)}{dy} = \frac{d}{dy}(y^2)$$

$$\Rightarrow \frac{dy}{dx} = 2y$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

Generally, $\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$, where δx and δy are small increments in x and y, respectively.

Now, as $\delta x \to 0$, $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ and $\frac{\delta x}{\delta y} = \frac{dx}{dy}$

Therefore, $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

Implicit differentiation is a method for finding the slope of a curve when the equation of the curve is not written in explicit form, $g(x, y) = 0$. The chain rule and the product rule are useful in differentiating implicit functions.

To differentiate a function of y with respect to x requires to differentiate the function with respect to y, and then multiply by $\frac{dy}{dx}$ to complete the chain rule. That is,

$$\frac{d}{dx}\left(f(y)\right) = \frac{d}{dy}\left(f(y)\right) \times \frac{dy}{dx}.$$

Suppose z is in terms of y, and y is in terms of x. It can be written as; $z = z(y)$ and $y = y(x)$. Then using the chain rule, it gives $\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx}$

If an expression has a term in the form $x^m y^n$, then

$$\frac{d}{dx}\left(x^m y^n\right) = x^m \frac{d}{dx}\left(y^n\right) + y^n \frac{d}{dx}\left(x^m\right).$$

$$\Rightarrow \frac{d}{dx}\left(x^m y^n\right) = nx^m y^{n-1}\frac{dy}{dx} + mx^{m-1}y^n.$$

Example 9.15

Differentiate $z = y^2$ with respect to x.

Solution

Given $z = y^2 \Rightarrow \frac{dz}{dy} = 2y$

Using the chain rule,

$$\frac{dz}{dx} = \frac{dz}{dy} \times \frac{dy}{dx},$$

$$\Rightarrow \frac{dz}{dx} = 2y \times \frac{dy}{dx}$$

Therefore, $\frac{dz}{dx} = 2y\frac{dy}{dx}$.

Example 9.16

Find $\frac{dy}{dx}$ for the curve $3x^2 - 4y^2 = 12$ at the point (4, 3).

Solution

Given the curve $3x^2 - 4y^2 = 12$.

Differentiate with respect to x:

$$\frac{d}{dx}\left(3x^2 - 4y^2\right) = \frac{d}{dx}(12)$$

$$\Rightarrow \frac{d}{dx}\left(3x^2\right) - \frac{d}{dx}\left(4y^2\right) = 0$$

$$\Rightarrow 6x - 8y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{3x}{4y}$$

At the point (4, 3),

$$\frac{dy}{dx} = \frac{3 \times 4}{4 \times 3}$$

$$= 1$$

Therefore, $\frac{dy}{dx}$ at (4, 3) is 1.

Example 9.17

If $y^2 = \frac{x-6}{2x+3}$, show that $\left(\frac{dy}{dx}\right)^2 = \frac{225}{4(x-6)(2x+3)^3}$.

Solution

Given $y^2 = \frac{x-6}{2x+3}$

$$2y\frac{dy}{dx} = \frac{(2x+3)(1) - (x-6)(2)}{(2x+3)^2}$$

$$2y\frac{dy}{dx} = \frac{15}{(2x+3)^2}$$

Squaring both sides gives

$$4y^2\left(\frac{dy}{dx}\right)^2 = \frac{225}{(2x+3)^4}.$$

But $y^2 = \frac{x-6}{2x+3}$

$$\left(\frac{dy}{dx}\right)^2 = \frac{225}{4\left(\frac{x-6}{2x+3}\right)(2x+3)^4}$$

$$= \frac{225}{4(x-6)(2x+3)^3}$$

Therefore, $\left(\frac{dy}{dx}\right)^2 = \frac{225}{4(x-6)(2x+3)^3}$.

Exercise 9.6

1. Differentiate each of the following functions with respect to x:
   (a) $\sqrt{y^6}$ (b) $x + y$
2. By implicit differentiation of $x^2 - y^2 = 1$, show that $\frac{dy}{dx} = \frac{x}{y}$.
3. Given $y^2 = \frac{x-2}{x+3}$, find $\frac{dy}{dx}$.
4. Differentiate $(3x+6)^4 = 5y^3$ with respect to x.
5. If $x^2 + y^2 = 4x$, show that $\frac{dy}{dx} = \frac{2-x}{y}$.
6. If $\frac{1}{x} + \frac{1}{y} = y$, show that $-\frac{1}{x^2} = \left(y + \frac{1}{y^2}\right)\frac{dy}{dx}$.

7. Find the gradient of the curve $x^2 + y^2 = 9$ at the point $(1, -2\sqrt{2})$.

8. Find $\frac{dy}{dx}$ for the curve $x^2 - 3y^2 = 9$ at the point (6, 3).

9. If $x^2 - 3y^2 = 9$ find y'.

10. Given that $x^2 + z^2 = 25$, determine the values of $\frac{dz}{dx}$ when $x = -4$.

Further implicit differentiation of functions

implicit differentiation can also be done without rearranging the terms in the expression of the function. In this case, each term is differentiated as a function of x by applying the chain rule and the product rule if needed.

Example 9.18

Using implicit differentiation, find $\frac{dy}{dx}$ for $y^2 + x^3 - y^3 + 5 = 4y$.

Solution

Differentiate each term with respect to x as follows;

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) + \frac{d}{dx}(5) = \frac{d}{dx}(4y)$$

$$\frac{d}{dy}(y^2) \times \frac{dy}{dx} + \frac{d}{dx}(x^3) - \frac{d}{dy}(y^3) \times \frac{dy}{dx} + \frac{d}{dx}(5) = \frac{d}{dy}(4y) \times \frac{dy}{dx}$$

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} + 0 = 4\frac{dy}{dx}$$

Collect together all the terms containing $\frac{dy}{dx}$ yields:

$$\frac{dy}{dx} = \frac{3x^2}{3y^2 - 2y + 4}$$

Therefore, $\frac{dy}{dx} = \frac{3x^2}{3y^2 - 2y + 4}$.

Example 9.19

If $x^5 + 4xy^3 - 3y^5 = 2$, find $\frac{dy}{dx}$.

Solution

Differentiate each term with respect to x as follows;

$$\frac{d}{dx}\left(x^5\right) + \frac{d}{dx}\left(4xy^3\right) - \frac{d}{dx}\left(3y^5\right) = \frac{d}{dx}(2)$$

For the term $4xy^3$ product rule is used as follows:

$$5x^4 + 4\left(x\frac{d}{dy}\left(y^3\right) \times \frac{dy}{dx} + y^3\frac{dx}{dx}\right) - 3\frac{d}{dy}\left(y^5\right)\frac{dy}{dx} = 0$$

$$\Rightarrow 5x^4 + 4\left(3xy^2\frac{dy}{dx} + y^3\right) - 15y^4\frac{dy}{dx} = 0$$

Collect all terms containing $\frac{dy}{dx}$:

$$\Rightarrow \left(12xy^2 - 15y^4\right)\frac{dy}{dx} = -\left(5x^4 + 4y^3\right)$$

$$\Rightarrow \frac{dy}{dx} = -\frac{5x^4 + 4y^3}{12xy^2 - 15y^4}$$

Therefore, $\frac{dy}{dx} = -\frac{5x^4 + 4y^3}{12xy^2 - 15y^4}$.

Example 9.20

Find the gradient of the curve $x^2y - 2xy^2 + y^2 = 1$ at the point (2, 1).

Solution

Differentiate each term with respect to x as follows:

$$\frac{d}{dx}\left(x^2y\right) - \frac{d}{dx}\left(2xy^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(1)$$

$$\Rightarrow 2xy + x^2\frac{dy}{dx} - \left(2y^2 + 4xy\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

Collect together all terms containing $\frac{dy}{dx}$:

$$\Rightarrow (x^2 - 4xy + 2y)\frac{dy}{dx} = 2y^2 - 2xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y^2 - 2xy}{x^2 - 4xy + 2y}$$

At the point $(2,1)$,

$$\left[\frac{dy}{dx}\right]_{(2,1)} = \frac{2(1)^2 - 2(2)(1)}{(2)^2 - 4(2)(1) + 2(1)} = 1.$$

Therefore, the gradient of the curve at the point $(2,1)$ is 1.

Exercise 9.7

1. Find $\frac{dy}{dx}$ for each of the following:
   (a) $y^2(1+x^2) = 3 - x^2$
   (b) $2xy + y^2 = x + y$
   (c) $4x^2y^3 - 6xy^2 + 4xy - \frac{1}{yx} = 40$
   (d) $x^2 + xy^2 + y^3 = 2$
2. If $x^2y^3 = x - 6$, find $\frac{dy}{dx}$.
3. Determine the gradient of the graph of $3(x^2+y^2)^2 = 100xy$ at the point (3, 1).
4. Given that $x\sqrt{1+y} + y\sqrt{1+x} = 0$, prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.
5. Find y', given $x^2y - xy^2 + x^2 + y^2 = 0$.
6. Find $\frac{dz}{dx}$, if $x^2 + z^2 - 6xz + 3x - 2z + 15 = 0$.
7. Given the implicit function $x^2 + xy = 2$, show that $\frac{dy}{dx} = -\frac{2x+y}{x}$.
8. Show that the gradient to the curve $x^3 + 3xy + y^3 = 5$ at the point (1, 1) is -1.
9. Given that $x^2y - xy^2 + x^2 + y^2 = 0$, find $\frac{dy}{dx}$.
10. If $x^3 + y^3 = 3axy$, where a is an arbitrary constant, find $\frac{dy}{dx}$.
11. Given $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = 6$, show that $\frac{dy}{dx} = \frac{3y - \sqrt{xy}}{\sqrt{xy} - 3x}$.
12. If $\sqrt{1-p^2} + \sqrt{1-q^2} = k(p-q)$, show that $\frac{dq}{dp} = \frac{\sqrt{1-q^2}}{1-p^2}$.
13. Differentiate each of the following with respect to x:
    (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 0$
    (b) $y + \sqrt{xy} = x^2$
    (c) $(x^2+y^2)^2 = xy$

Derivatives of trigonometric functions

The trigonometric functions involve sine x, cosine x, tangent x, secant x, cosecant x, and cotangent x and their units are in radians.

Derivatives of trigonometric functions can be derived using first principles of differentiation.

Example 9.21

Use first principles to find the derivative of each of the following:

(a) $f(x)=\sin x$ (b) $y=\cos x$

Solution

(a) From first principles of differentiation,

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\text{, where } f(x)=\sin x \text{ and } f(x+h)=\sin(x+h)$$

$$\Rightarrow f'(x)=\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}$$

Using the compound angle formula, it implies that

$$f'(x)=\lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}$$

$$=\lim_{h\to 0}\frac{\sin x(\cos h-1)+\cos x\sin h}{h}$$

$$=\lim_{h\to 0}\sin x\frac{(\cos h-1)}{h}+\lim_{h\to 0}\cos x\frac{\sin h}{h}$$

$$=\sin x\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)+\cos x\lim_{h\to 0}\frac{\sin h}{h}$$

But, $\lim_{h\to 0}\frac{\sin h}{h}=1$ and $\lim_{h\to 0}\left(\frac{\cos h-1}{h}\right)=0$

$$\Rightarrow f'(x)=0\times\sin x+1\times\cos x$$

$$=\cos x$$

Therefore, $f'(x)=\cos x$.

(b) From the first principles of differentiation:

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{y(x+h)-y(x)}{h}\text{, where } y(x+h)=\cos(x+h) \text{ and } y(x)=\cos x.$$

$$\Rightarrow\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

Using the compound angle formula, it implies that

$$\frac{dy}{dx}=\lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}$$

$$=\lim_{h\to 0}\frac{\cos x(\cos h-1)-\sin x\sin h}{h}$$

$$=\cos x\lim_{h\to 0}\frac{(\cos h-1)}{h}-\sin x\lim_{h\to 0}\frac{\sin h}{h}$$

But $\lim_{h\to 0}\frac{(\cos h-1)}{h}=0$ and $\lim_{h\to 0}\frac{\sin h}{h}=1$

Thus, $\frac{dy}{dx}=0\times\cos x-1\times\sin x$

$$=-\sin x$$

Therefore, $\frac{dy}{dx}=-\sin x$.

Example 9.22

Find the derivative of $\sin^3\theta$ with respect to θ.

Solution

Let $y=\sin^3\theta$

Also, let $u=\sin\theta\Rightarrow y=u^3$.

$\Rightarrow\frac{du}{d\theta}=\cos\theta$ and $\frac{dy}{du}=3u^2$

Using the chain rule,

$$\frac{dy}{d\theta}=\frac{du}{d\theta}\times\frac{dy}{du}$$

$$\Rightarrow\frac{dy}{d\theta}=\cos\theta\times 3u^2$$

$$=\cos\theta\times 3\sin^2\theta$$

Therefore, $\frac{dy}{d\theta}=3\cos\theta\sin^2\theta$.

Example 9.23

Find the gradient of the curve $y=5\sin x-x^2$ at the point $x=-\pi$.

Solution

Differentiate with respect to x as follows:

$$\frac{dy}{dx}=\frac{d}{dx}(5\sin x-x^2)$$

$$=5\cos x-2x$$

At $x=-\pi$,

$$\left[\frac{dy}{dx}\right]_{x=-\pi}=5\cos(-\pi)-2(-\pi)$$

$$=5(-1)-2(-\pi)$$

$$=2\pi-5.$$

Therefore, the gradient of the curve at $x=-\pi$ is $2\pi-5$.

Example 9.24

Using first principles, show that $f'(\theta) = \sec\theta\tan\theta$, if $f(\theta) = \sec\theta$.

Solution

Let $f(\theta) = \sec\theta = \dfrac{1}{\cos\theta}$.

From first principles,

$$f'(\theta) = \lim_{h\to 0}\frac{f(\theta+h) - f(\theta)}{h}$$

$$\Rightarrow f'(\theta) = \lim_{h\to 0}\frac{\dfrac{1}{\cos(\theta+h)} - \dfrac{1}{\cos(\theta)}}{h}$$

$$= \lim_{h\to 0}\left[\frac{\cos\theta - \cos(\theta+h)}{h\cos\theta\cos(\theta+h)}\right]$$

Using the factor formula, it implies that

$$f'(\theta) = \lim_{h\to 0}\frac{-2\sin\left(\theta+\dfrac{h}{2}\right)\sin\left(-\dfrac{h}{2}\right)}{h\cos\theta\cos(\theta+h)}$$

$$= \lim_{h\to 0}\frac{-2\sin\left(\theta+\dfrac{h}{2}\right)}{\cos\theta\cos(\theta+h)} \times \lim_{h\to 0}\frac{\sin\left(-\dfrac{h}{2}\right)}{h}$$

$$= \frac{-2\sin\theta}{\cos\theta\cos\theta} \times \left(-\frac{1}{2}\right)$$

$$\Rightarrow f'(\theta) = \sec\theta\tan\theta.$$

Therefore, $f'(\theta) = \sec\theta\tan\theta$.

Example 9.25

Differentiate $y = \cos\left(\dfrac{1}{x}\right)$ with respect to x.

Solution

Given $y = \cos\left(\dfrac{1}{x}\right)$.

Let $u = \dfrac{1}{x} = x^{-1} \Rightarrow y = \cos u$

Thus, $\dfrac{dy}{du} = -\sin u$ and $\dfrac{du}{dx} = -x^{-2}$

Using chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\sin u \times \left(-x^{-2}\right)$$

Thus, $\dfrac{dy}{dx} = x^{-2}\sin\left(\dfrac{1}{x}\right) = \dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)$

Therefore, $\dfrac{dy}{dx} = \dfrac{1}{x^2}\sin\left(\dfrac{1}{x}\right)$.

Example 9.26

If $f(x) = \dfrac{\cos x}{x}$, show by using first principles that $f'(x) = -\dfrac{x\sin x + \cos x}{x^2}$.

Solution

Given

$$f(x) = \frac{\cos x}{x} \Rightarrow f(x+h) = \frac{\cos(x+h)}{x+h}$$

Using first principles

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h\to 0}\frac{\dfrac{\cos(x+h)}{x+h} - \dfrac{\cos x}{x}}{h}$$

$$=\lim_{h\to 0}\frac{x\cos(x+h)-(x+h)\cos x}{h(x+h)x}$$

$$=\lim_{h\to 0}\frac{x[\cos(x+h)-\cos x]-h\cos x}{hx(x+h)}$$

$$=\lim_{h\to 0}\frac{x\left[-2\sin\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)\right]-h\cos x}{hx(x+h)}$$

$$=-\lim_{h\to 0}\frac{\sin\left(x+\frac{h}{2}\right)}{(x+h)}\times\lim_{h\to 0}\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}-\lim_{h\to 0}\frac{\cos x}{x(x+h)}$$

$$=-\frac{\sin x}{x}-\frac{\cos x}{x\times x}$$

$$=-\frac{\sin x}{x}-\frac{\cos x}{x^2}$$

$$\Rightarrow f'(x)=-\frac{x\sin x+\cos x}{x^2}$$

Therefore, $f'(x)=-\dfrac{x\sin x+\cos x}{x^2}$.

Exercise 9.8

1. Differentiate each of the following functions with respect to x:

(a) $\sin 2x$ (e) $\cos\left(1-x^2\right)$ (i) $\tan 4x$

(b) $x^2\sin x$ (f) $\dfrac{\cos x}{x^3}$ (j) $\sin^2(3x^2-1)$

(c) $\sin\left(\dfrac{2}{x}\right)$ (g) $\sin x\left(1+\cos x\right)$ (k) $\dfrac{\sin x}{1+\cos x}$

(d) $6\cos 2x$ (h) $\tan\sqrt{x+2}$ (l) $\cos\left(\dfrac{1-\sin x}{1+\cos x}\right)$

2. Find the gradient of each of the following curves at the given points:

(a) $y=2\sin x-x^2; x=-\pi$ (b) $y=-4\cos x; x=\dfrac{1}{2}\pi$

3. If $y=\sqrt{\dfrac{1+\sin x}{1-\sin x}}$, show that $\dfrac{dy}{dx}=\dfrac{1}{1-\sin x}$.

4. Show that $\dfrac{d}{dx}\left(\dfrac{\tan x}{1-\tan^2 x}\right)^{\frac{1}{2}}=\dfrac{\sec^4 x}{2\sqrt{\tan x}\left(1-\tan^2 x\right)^{\frac{3}{2}}}$.

5. Differentiate $f(x)=\dfrac{\sin x}{x}$ with respect to x.

6. If $g(x)=3\cot x+5\operatorname{cosec} x$, show that $g'(x)=-3\operatorname{cosec}^2 x-5\operatorname{cosec} x\cot x$.

7. Find $\dfrac{d}{d\theta}\left(\sin\sqrt{\theta^2-1}\right)$.

8. Prove that $\dfrac{d}{dx}\left(\tan\sqrt{6x^3+2}\right)=\dfrac{9x^2}{6x^3+2}\sec^2\sqrt{6x^3+2}$.

9. Given that $x=\dfrac{1+\sin\theta+\cos\theta}{1-\sin\theta+\cos\theta}$, show that $x'=(1-\sin\theta)^{-1}$.

10. Find $\dfrac{dy}{dx}$ of each of the following:

(a) $xy^2+\cos 2y=4$

(b) $y^4+x^3+\cos\left(x+y^2\right)=0$

(c) $y=\dfrac{1-\cos 2x}{\sin 2x}$

(d) $y=\dfrac{1+\tan x}{1-\tan x}$

11. From first principles, show that:

(a) $\dfrac{d}{d\theta}(1+2\cos 4\theta)=-8\sin 4\theta$

(b) $\dfrac{d}{dx}(\tan x)=\sec^2 x$

(c) $\dfrac{d}{dx}(\cot x)=-\operatorname{cosec}^2 x$

(d) $\dfrac{d}{dx}(\cos^2 x)=-\sin 2x$

(e) $\dfrac{d}{dx}\left(\dfrac{\operatorname{cosec} 2\theta}{2\theta}\right)=-\dfrac{\operatorname{cosec} 2\theta\left(2\theta\cot 2\theta+1\right)}{2\theta^2}$

Derivatives of inverse trigonometric functions

The inverse trigonometric functions include $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$. This section discusses how to determine the derivatives of inverse trigonometric functions.

Activity 9.3: Recognizing derivatives of inverse trigonometric functions

Individually or in a group, perform this activity using the following tasks:

1. Think and write down any inverse of trigonometric function.
2. Write the formula for differentiation from first principles.
3. By showing all steps clearly, differentiate the function in task 1 from first principles.
4. What have you observed from task 3? .
5. Think and use any other alternative method of performing task 3.
6. Give an opinion between the methods in tasks 3 and 5.

Derivatives of some inverses of trigonometric functions are derived as follows:

(a) Consider the function $y = \sin^{-1} x$.

Then, $x = \sin y$.

Thus, $\frac{dx}{dy} = \cos y \Rightarrow \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos y}.$$

Since $\sin^2 y + \cos^2 y = 1$, then

$$\cos y = \sqrt{1-\sin^2 y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}}$$

But $\sin y = x$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

Therefore, $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$.

Example 9.27

Differentiate $y = \sin^{-1}(2x-3)$ with respect to x.

Solution

Given $y = \sin^{-1}(2x-3)$.

Let $u = 2x-3 \Rightarrow y = \sin^{-1} u$

$$\Rightarrow \frac{du}{dx} = 2 \text{ and } \frac{dy}{du} = \frac{1}{\sqrt{1-u^2}}$$

Using the chain rule,

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.$$

Substituting the expressions for $\frac{dy}{du}$ and $\frac{du}{dx}$, gives

$\frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}} \times 2$. But $u = 2x-3$.

Thus, $\frac{dy}{dx}=\frac{1}{\sqrt{1-(2x-3)^2}}\times 2$.

Therefore, $\frac{dy}{dx}=\frac{1}{\sqrt{3x-x^2-2}}$.

Example 9.28

Find the derivative of each of the following functions with respect to x:

(a) $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

(b) $y=\cos^{-1}x$

Solution

(a) Given $y=\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

$$\Rightarrow \sin y=\left(\frac{1-x^2}{1+x^2}\right)$$

$$\frac{d}{dx}(\sin y)=\frac{d}{dx}\left(\frac{1-x^2}{1+x^2}\right)$$

$$=\frac{-2x\left(1+x^2\right)-2x\left(1-x^2\right)}{\left(1+x^2\right)^2}$$

$$\Rightarrow \cos y\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2\cos y}$$

But $\sin^2 y+\cos^2 y=1$

$$\Rightarrow \cos y=\sqrt{1-\sin^2 y}$$

$$\Rightarrow \cos y=\sqrt{1-\left(\frac{1-x^2}{1+x^2}\right)^2}$$

$$\Rightarrow \cos y=\sqrt{\frac{4x^2}{\left(1+x^2\right)^2}}=\frac{2x}{1+x^2}$$

Thus, $\frac{dy}{dx}=\frac{-4x}{\left(1+x^2\right)^2\times\frac{2x}{1+x^2}}$

$$\Rightarrow \frac{dy}{dx}=\frac{-2}{\left(1+x^2\right)}$$

Therefore,

$$\frac{d}{dx}\left[\sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]=\frac{-2}{\left(1+x^2\right)}.$$

(b) Given $y=\cos^{-1}x$.

$\Rightarrow x=\cos y$, which implies that

$$\frac{dx}{dy}=-\sin y$$

Using the relation $\frac{dy}{dx}=\frac{1}{\frac{dx}{dy}}$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\sin y}$$

But

$$\sin^2 y+\cos^2 y=1\Rightarrow \sin y=\sqrt{1-\cos^2 y}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{\sqrt{1-\cos^2 y}}=-\frac{1}{\sqrt{1-x^2}}$$

Therefore,

$$\frac{d}{dx}(\cos^{-1}x)=-\frac{1}{\sqrt{1-x^2}}.$$

Example 9.29

If $y = \cos^{-1}(3x-1)$, find $\frac{dy}{dx}$.

Solution

Given $y = \cos^{-1}(3x-1)$.

$\Rightarrow 3x-1 = \cos y$

Thus, $\frac{d}{dx}(3x-1) = \frac{d}{dx}(\cos y)$

$\Rightarrow 3 = -\sin y \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{3}{-\sin y}$

But

$\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \sqrt{1-\cos^2 y}$

Thus, $\frac{dy}{dx} = -\frac{3}{\sqrt{1-\cos^2 y}}$

$= -\frac{3}{\sqrt{1-(3x-1)^2}}$

$= -\frac{3}{\sqrt{1-(9x^2-6x+1)}}$

$\Rightarrow \frac{dy}{dx} = -\frac{3}{\sqrt{-9x^2+6x}}$

Therefore, $\frac{dy}{dx} = -\frac{3}{\sqrt{3x(2-3x)}}$.

Example 9.30

(a) Given $y = \cos^{-1}\left(3x-5x^3\right)$, show that

$$\frac{dy}{dx} = \frac{3\left(5x^2-1\right)}{\sqrt{1-9x^2+30x^4-25x^6}}.$$

(b) If $y = \tan^{-1} x$, show that

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}.$$

Solution

(a) Given $y = \cos^{-1}\left(3x-5x^3\right)$.

$\Rightarrow \frac{d}{dx}(\cos y) = \frac{d}{dx}\left(3x-5x^3\right)$

$\Rightarrow -\sin y \frac{dy}{dx} = 3-15x^2$

$\Rightarrow \frac{dy}{dx} = -\frac{3-15x^2}{\sin y}$

But

$\sin^2 y + \cos^2 y = 1 \Rightarrow \sin y = \sqrt{1-\cos^2 y}$

$\Rightarrow \frac{dy}{dx} = -\frac{3-15x^2}{\sqrt{1-\cos^2 y}}$

$\Rightarrow \frac{dy}{dx} = -\frac{\left(3-15x^2\right)}{\sqrt{1-\left(3x-5x^3\right)^2}}$

Therefore,

$$\frac{dy}{dx} = \frac{3\left(5x^2-1\right)}{\sqrt{1-9x^2+30x^4-25x^6}}.$$

(b) Given $y = \tan^{-1} x$, Then,

$x = \tan y$

$\Rightarrow \frac{dx}{dy} = \sec^2 y$

Using the relation $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{\sec^2 y}$

But $\sec^2 y = 1 + \tan^2 y$.

Thus, $\frac{dy}{dx} = \frac{1}{1+\tan^2 y}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}$$

Therefore, $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$.

Example 9.31

Differentiate $y = \tan^{-1}\left(\frac{1}{x^2}\right)$ with respect to x.

Solution

Given $y = \tan^{-1}\left(\frac{1}{x^2}\right)$.

$$\Rightarrow \frac{1}{x^2} = \tan y$$

$$\Rightarrow \frac{d}{dx}\left(\frac{1}{x^2}\right) = \frac{d}{dx}(\tan y)$$

$$\Rightarrow -2x^{-3} = \sec^2 y \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{x^3 \sec^2 y}$$

But $\sec^2 y = 1 + \tan^2 y$.

Thus, $\frac{dy}{dx} = \frac{-2}{x^3\left(1+\tan^2 y\right)}$

$$\Rightarrow \frac{dy}{dx} = \frac{-2x}{x^4+1}$$

Therefore, $\frac{dy}{dx} = -\frac{2x}{x^4+1}$.

Example 9.32

If $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$, show that $\frac{dy}{dx} = \frac{1}{2}$.

Solution

Given $y = \tan^{-1}\left(\frac{\cos x}{1-\sin x}\right)$.

$$\Rightarrow \tan y = \frac{\cos x}{1-\sin x}$$

$$\Rightarrow \frac{d}{dx}(\tan y) = \frac{d}{dx}\left(\frac{\cos x}{1-\sin x}\right)$$

$$\Rightarrow \sec^2 y \frac{dy}{dx} = \frac{-\sin x(1-\sin x) - \cos x(-\cos x)}{(1-\sin x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin x + \sin^2 x + \cos^2 x}{(1-\sin x)^2 \sec^2 y}$$

$$= \frac{1-\sin x}{(1-\sin x)^2 \sec^2 y}$$

But

$$\sec^2 y = 1 + \tan^2 y \Rightarrow \sec^2 y = 1 + \left(\frac{\cos x}{1-\sin x}\right)^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\sin x}{(1-\sin x)^2\left(1+\left(\frac{\cos x}{1-\sin x}\right)^2\right)}$$

Upon rearranging, it gives

$$\frac{dy}{dx} = \frac{1-\sin x}{2-2\sin x} = \frac{1-\sin x}{2(1-\sin x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}$$

Therefore, $\frac{dy}{dx} = \frac{1}{2}$.

Exercise 9.9

1. If $\sqrt{y} = \tan^{-1} x$, show that $\frac{d}{dx}\left[\left(1+x^2\right)\frac{dy}{dx}\right]\left(1+x^2\right) = 2$.

2. Differentiate $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ with respect to x.
3. If $y=\sin^{-1}(3\theta-4\theta^3)$, show that $\sqrt{1-\theta^2}\frac{dy}{d\theta}=3$.
4. Differentiate $y=\frac{x}{\sqrt{a^2-x^2}}-\sin^{-1}\left(\frac{x}{a}\right)$ with respect to x.
5. Find the first derivative of $y^3\sin x+y+6=\tan^{-1}x$.
6. Given that $y=\tan^{-1}\left(\frac{1-x}{1+x}\right)$, find $\frac{dy}{dx}$.
7. Given that x and y satisfy the equation, $\tan^{-1}x+\tan^{-1}y+\tan^{-1}xy=\frac{11}{12}\pi$.
   Show that if $x=1$, then $\frac{dy}{dx}=-1-\frac{1}{2}\sqrt{3}$.
8. Show that $\frac{d}{dx}\left(\sin(\tan^{-1}x)\right)=\left(1+x^2\right)^{-\frac{3}{2}}$.
9. If $y=x^2\sin^{-1}x$, find $\frac{dy}{dx}$.
10. Differentiate each of the following with respect to x:
    (a) $y=\sin^{-1}5x$ (b) $y=\cos^{-1}x\sqrt{2}$

Computer packages in differentiating polynomial and trigonometric functions

Computer packages can be applied to differentiate polynomial and trigonometric functions. In this section, Maple and MATLAB packages are discussed.

(a) Maple 18

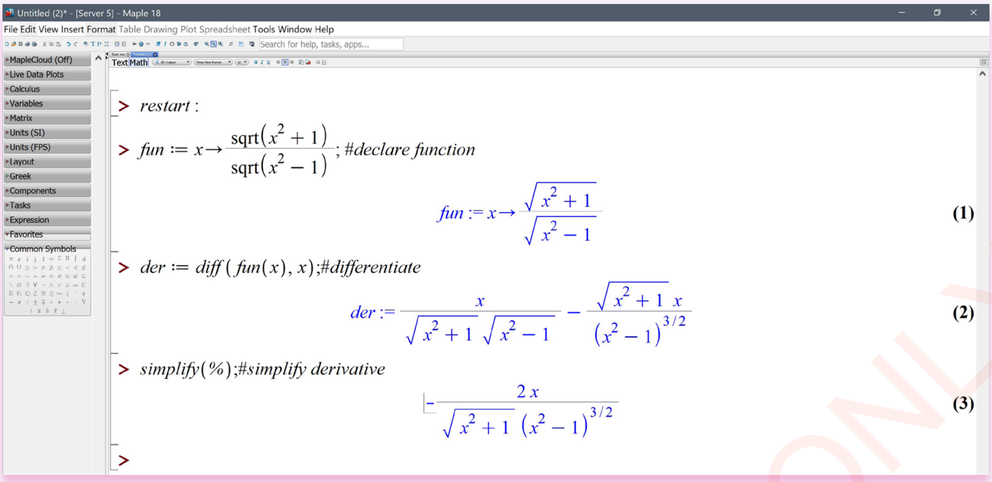
Maple is a computing software which uses the function "*diff*" that allows differentiation of a function. The "*diff*" function works similar to that of the function D in Mathematica. The function "*diff*" in Maple 18 software has two arguments;

1. The function to be differentiated.
2. The independent variable to be considered in differentiation.

Example 9.33

Using Maple 18 software, differentiate $\frac{\sqrt{x^2+1}}{\sqrt{x^2-1}}$ with respect to x.

Solution

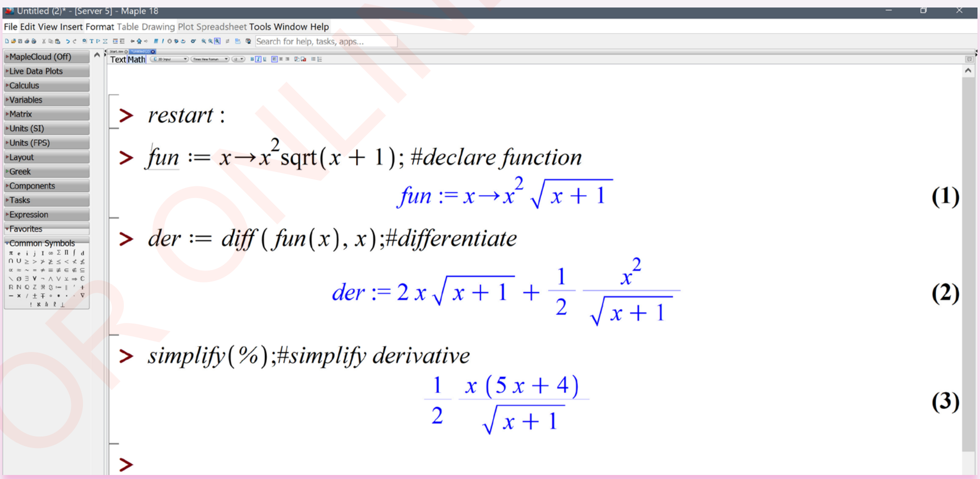


Therefore, $\dfrac{d}{dx}\left(\dfrac{\sqrt{x^2+1}}{\sqrt{x^2-1}}\right) = -\dfrac{2x}{\sqrt{x^2+1}\left(x^2-1\right)^{\frac{3}{2}}}$.

Example 9.34

Using Maple 18 software, differentiate $x^2\sqrt{x+1}$ with respect to x.

Solution

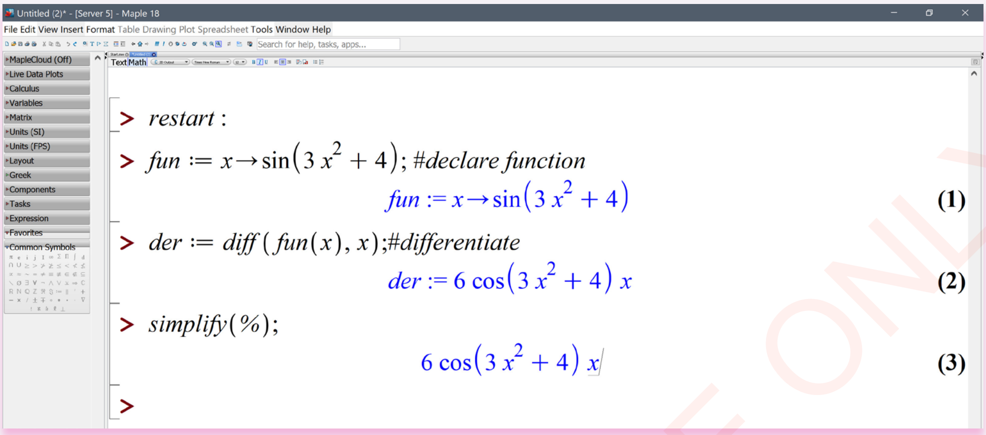


Therefore, $\dfrac{d}{dx}\left(x^2\sqrt{x+1}\right) = \dfrac{1}{2}\dfrac{x(5x+4)}{\sqrt{x+1}}$.

Example 9.35

Using Maple 18 software, find the derivative of $\sin\left(3x^2+4\right)$.

Solution



Therefore, $\dfrac{d}{dx}\left(\sin\left(3x^2+4\right)\right)=6x\cos\left(3x^2+4\right)$.

(b) MATLAB

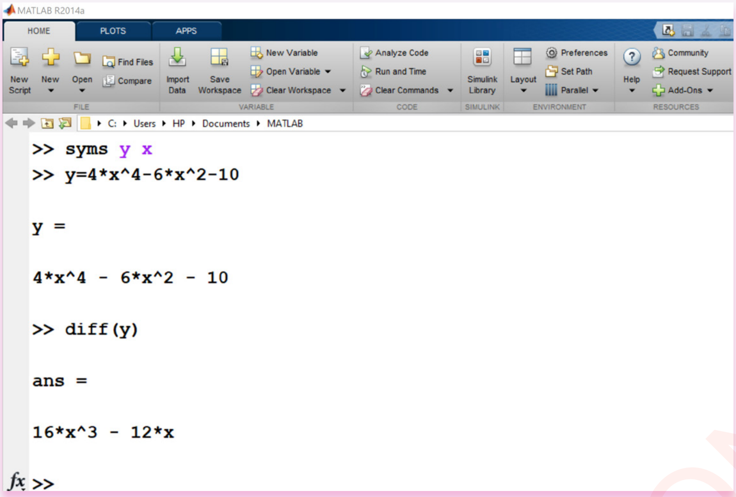
MATLAB is another software that can be used to find the derivative of a function. It involves declaring variables, defining the function whose derivative is needed and finally the "diff" function is applied to find the derivative.

Differentiation using MATLAB R2014a software involves finding the rate of change of a quantity with respect to the other, such as, to find the rate at which y changes with respect to x.

Example 9.36

Using MATLAB R2014a, differentiate $y=4x^4-6x^2-10$ with respect to x.

Solution

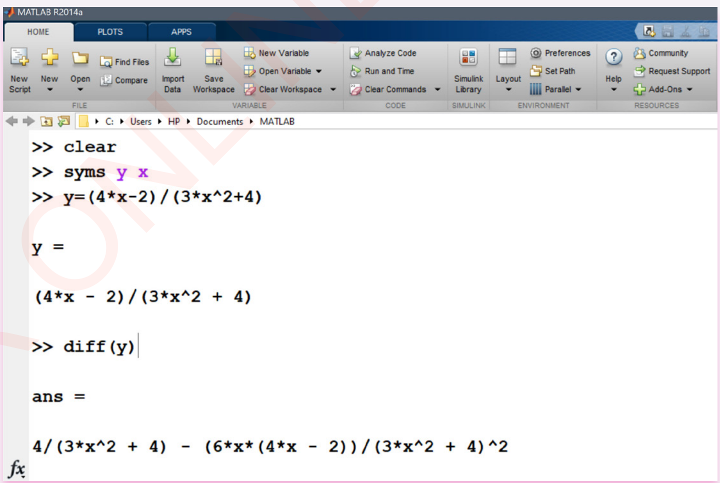


Therefore, $\dfrac{d}{dx}\left(4x^4-6x^2-10\right)=16x^3-12x.$

Example 9.37

Use MATLAB R2014a software to differentiate the function $y=\dfrac{4x-2}{3x^2+4}$ with respect to x.

Solution

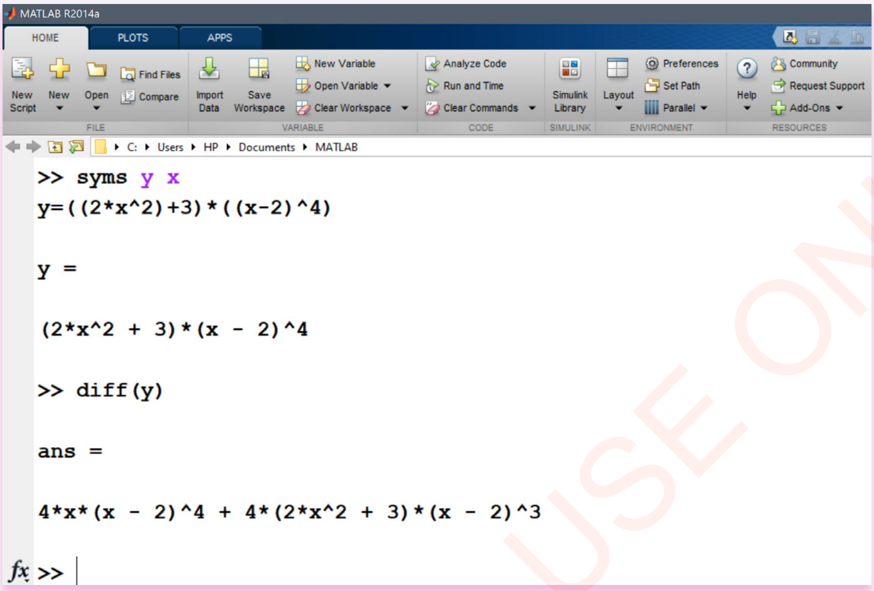


Therefore, $\dfrac{dy}{dx}=\dfrac{4}{3x^2+4}-\dfrac{6x(4x-2)}{\left(3x^2+4\right)^2}.$

Example 9.38

Use MATLAB R2014a software to differentiate the function $y=\left(2x^2+3\right)\left(x-2\right)^4$ with respect to x.

Solution



Therefore, $\dfrac{dy}{dx}=4x\left(x-2\right)^4+4\left(2x^2+3\right)\left(x-2\right)^3$.

Exercise 9.10

1. Differentiate each of the following functions with respect to x using Maple 18 software.

   (a) $y=4x^2\left(x-8\right)^3$ (c) $y=\dfrac{3+x}{x^2-4}$ (e) $y=\sin\left(7x+4\right)$

   (b) $y=\tan 2x+\tan^2 5x$ (d) $y=\cos\left(x^3+2x+1\right)$

2. Differentiate each of the following functions with respect to x using MATLAB R2014a:

   (a) $y=\sin 2x\cos 4x$

   (b) $y=\sin^{-1}2x+\cos^{-1}\left(x^2-1\right)$

   (c) $y=\sec\left(3x+6\right)$

   (d) $y=\dfrac{\sqrt[5]{2x+1}}{2\left(x-8\right)^{\frac{2}{3}}}$

   (e) $y=\cot\left(\operatorname{cosec}2x\right)$

Derivatives of logarithmic functions

Logarithmic functions are functions of the form $y = \log_b x$, where $x > 0$, $b > 0$, $b \neq 1$ It is an inverse of the exponential function. Thus, any exponential function can be written in logarithmic form. The derivative of logarithmic functions to base e (natural logarithm) can easily be found. If the base is different from e, it should first be converted to base e, then the appropriate technique of differentiation is applied.

Activity 9.4: Recognizing derivatives of logarithmic functions

Individually or in a group, perform the following tasks:

1. Write down a natural logarithmic function of any independent variable.
2. Determine the derivative of the function in task 1 from first principles.
3. Determine the derivative of the function in task 1 by using any alternative method.
4. Give an opinion between the answers obtained in tasks 2 and 3.
5. What have you observed from this activity? Give reasons.

Example 9. 39

Differentiate $y = \log_e x$ with respect to x.

Solution

Given $y = \log_e x$, then,

$$e^y = x$$

$$\Rightarrow \frac{d}{dx}(e^y) = \frac{d}{dx}(x)$$

$$\Rightarrow e^y \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$$

Therefore, $\frac{dy}{dx} = \frac{1}{x}$.

Generally, $\frac{d}{dx}\left(\ln f(x)\right) = \frac{f'(x)}{f(x)}$.

Example 9.40

If $y = \log_e (x^3 - 2)$, show that

$$\frac{dy}{dx} = \frac{3x^2}{\left(x^3 - 2\right)}.$$

Solution

Given $y = \log_e \left(x^3 - 2\right)$.

Let $u = x^3 - 2$ and $y = \log_e u$

Thus, $\frac{du}{dx} = 3x^2$ and $\frac{dy}{du} = \frac{1}{u}$.

Using the chain rule of differentiation,

$$\frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 \times \frac{1}{u}$$

But $u = x^3 - 2$

$$\Rightarrow \frac{dy}{dx} = 3x^2 \times \frac{1}{x^3 - 2}$$

Therefore, $\dfrac{dy}{dx} = \dfrac{3x^2}{x^3 - 2}$.

Example 9.41

Show that

$$\frac{d}{dx}\left(x^{\sin x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

Solution

Let $y = x^{\sin x}$.

Introducing natural logarithms both sides gives

$$\ln y = \ln x^{\sin x}$$

$$\Rightarrow \ln y = \sin x \ln x$$

$$\Rightarrow \frac{d}{dx}(\ln y) = \frac{d}{dx}(\sin x \ln x).$$

Implicit differentiation gives

$$\frac{1}{y}\frac{dy}{dx} = \cos x \ln x + \frac{1}{x}\sin x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\cos x \ln x + \frac{1}{x}\sin x\right)$$

But $y = x^{\sin x}$

$$\Rightarrow \frac{dy}{dx} = x^{\sin x}\left(\cos x \ln x + \frac{1}{x}\sin x\right)$$

Therefore,

$$\frac{d}{dx}\left(x^{\sin x}\right) = x^{\sin x}\left(\cos x \ln x + \frac{\sin x}{x}\right).$$

Example 9.42

Given $y = \log_e\left(\sqrt{\dfrac{1-x^2}{1+x^2}}\right)$, evaluate y' at $x = \dfrac{1}{2}$.

Solution

Given $y = \log_e\left(\sqrt{\dfrac{1-x^2}{1+x^2}}\right)$.

$$\Rightarrow y = \log_e\left(\frac{1-x^2}{1+x^2}\right)^{\frac{1}{2}}$$

$$= \frac{1}{2}\log_e\left(\frac{1-x^2}{1+x^2}\right)$$

$$= \frac{1}{2}\left[\log_e\left(1-x^2\right) - \log_e\left(1+x^2\right)\right]$$

$$\Rightarrow y' = \frac{1}{2}\left[\frac{-2x}{\left(1-x^2\right)} - \frac{2x}{\left(1+x^2\right)}\right]$$

$$\Rightarrow y' = \frac{-2x}{\left(1-x^2\right)\left(1+x^2\right)}$$

At $x = \dfrac{1}{2}$,

$$\left[\frac{dy}{dx}\right]_{x=\frac{1}{2}} = \frac{-2\left(\frac{1}{2}\right)}{\left(1-\left(\frac{1}{2}\right)^2\right)\left(1+\left(\frac{1}{2}\right)^2\right)}$$

Therefore, y' at $x = \dfrac{1}{2}$ is $\dfrac{-16}{15}$.

Example 9.43

Find $\frac{dy}{dx}$ if $y = e^{x^2}\sqrt{\sin x}$.

Solution

Given $y = e^{x^2}\sqrt{\sin x}$.

Introducing natural logarithms both sides:

$$\Rightarrow \ln y = \ln e^{x^2}\sqrt{\sin x}$$

$$= \ln e^{x^2} + \ln(\sin x)^{\frac{1}{2}}$$

$$= x^2 \ln e + \frac{1}{2}\ln \sin x$$

$$\Rightarrow \ln y = x^2 + \frac{1}{2}\ln \sin x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}\left(x^2 + \frac{1}{2}\ln \sin x\right)$$

Implicit differentiation gives,

$$\frac{1}{y}\frac{dy}{dx} = 2x + \frac{1}{2}\frac{\cos x}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = y\left(2x + \frac{1}{2}\cot x\right)$$

But $y = e^{x^2}\sqrt{\sin x}$

$$\Rightarrow \frac{dy}{dx} = e^{x^2}\sqrt{\sin x}\left(2x + \frac{1}{2}\cot x\right)$$

Therefore, $\frac{dy}{dx} = \left(2x + \frac{1}{2}\cot x\right)e^{x^2}\sqrt{\sin x}$.

Example 9.44

Prove that the derivative of

$y = \ln(x^4 \sin^2 x)$ is $y' = \frac{4}{x} + 2\cot x$.

Solution

Given $y = \ln\left(x^4 \sin^2 x\right)$.

$$\Rightarrow y = \ln x^4 + \ln \sin^2 x$$

$$\Rightarrow y = 4\ln x + 2\ln \sin x$$

Taking derivative on both sides:

$$y' = 4 \times \frac{1}{x} + 2 \times \frac{1}{\sin x} \times \cos x$$

$$\Rightarrow y' = \frac{4}{x} + 2\cot x$$

Therefore, $y' = \frac{4}{x} + 2\cot x$.

Example 9.45

Find y' if $y = \log_a x^2$.

Solution

Given $y = \log_a x^2$.

Convert the base of the logarithmic function to base e.

$$\Rightarrow y = \frac{\ln x^2}{\ln a}$$

$$\Rightarrow y = \frac{2\ln x}{\ln a}$$

Now, differentiation with respect to x gives

$$y' = \frac{2}{x \ln a}.$$

Therefore, $y' = \frac{2}{x \ln a}$.

Exercise 9.11

1. Differentiate each of the following functions with respect to x:
   (a) $y=\ln(5x-4)$
   (b) $y=x\ln x-x$
   (c) $y=x(\sin(\ln x)-\cos(\ln x))$

2. Find the derivative of $y=\ln\left(\frac{x+3}{x+5}\right)$ with respect to x.

3. Given that $y=\ln\left[\left(x^2+3\right)\left(x^3+2\right)\right]$, find the expression for $\frac{dy}{dx}$.

4. Differentiate each of the following functions with respect to x.
   (a) $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$
   (b) $f(x)=\ln\left(\frac{x^4}{(3x-5)^2}\right)$
   (c) $f(x)=(\sin x)^x$
   (d) $f(x)=\ln\left[\left(x^2+2\right)^3\left(1-x^3\right)^4\right]$

5. Find the derivative of each of the following functions with respect to x:
   (a) $y=(\sin x)^{\tan x}$
   (b) $y=\ln\left[\frac{\left(x^2+3\right)\sqrt{\cot x}}{\tan^3 x}\right]$
   (c) $y=\ln\left(x\sqrt{x^2-4}\right)$
   (d) $y=x^x$

6. Given that $f(x)=\ln(\ln\tan x)$, find $f'(x)$.

7. Differentiate $x=\ln\left(\frac{\sin\theta+\cos\theta}{\sin\theta-\cos\theta}\right)$ with respect to θ.

8. Show that the derivative of the function $f(x)=\frac{1}{5}x^5\left(\ln x-\frac{1}{5}\right)$ is $f'(x)=x^4\ln x$.

9. Given $f(x)=\ln\frac{x^2\sqrt{x^2-2}}{x+1}$, prove that
$$f'(x)=\frac{2}{x}+\frac{x}{x^2-2}-\frac{1}{x+1}.$$

10. If $x^y=y^x$, show that $\frac{dy}{dx}=\frac{y(x\ln y-y)}{x(y\ln y-x)}$.

11. Find the derivative of the function $y=\log_2(3x+1)$ at $x=1$.

12. If $y=\ln\sqrt{\frac{x-1}{x+1}}$, show that $\frac{dy}{dx}=\frac{1}{x^2-1}$. Hence, find $\frac{dy}{dx}$ when $x=\sqrt{3}$.

Derivatives of exponential functions

Activity 9.5: Recognizing derivative of an exponential function

Individually or in a group, perform the following tasks:

1. Determine the derivative of $f(x)=e^{x}$ from first principles.
2. Determine the derivative in task 1 using any alternative method.
3. Give your opinion in answers obtained in tasks 1 and 2.
4. What have you observed from this activity? Give reasons.
5. Share your results with other students for more inputs.

Consider the following series expansion of the exponential function.

$$e^{x}=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\frac{x^{4}}{4!}+\cdots$$

The derivatives of the left-hand side and of the right-hand side are determined by the rule for derivative of the sum. That is, the derivative of the sum equals the sum of the derivatives of each of the term.

That is,

$$\frac{d}{dx}\left(e^{x}\right)=\frac{d}{dx}(1)+\frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{x^{2}}{2!}\right)+\frac{d}{dx}\left(\frac{x^{3}}{3!}\right)+\frac{d}{dx}\left(\frac{x^{4}}{4!}\right)+\cdots$$

$$\Rightarrow\frac{d}{dx}\left(e^{x}\right)=0+1+\frac{2}{2\times1}x+\frac{3}{3\times2\times1}x^{2}+\frac{4}{4\times3\times2\times1}x^{3}+\cdots$$

$$\Rightarrow\frac{d}{dx}\left(e^{x}\right)=1+x+\frac{1}{2\times1}x^{2}+\frac{1}{3\times2\times1}x^{3}+\cdots$$

$$\Rightarrow\frac{d}{dx}\left(e^{x}\right)=1+x+\frac{x^{2}}{2!}+\frac{x^{3}}{3!}+\cdots$$

The expression for the derivative is the same as the original function.

Therefore, $\frac{d}{dx}\left(e^{x}\right)=e^{x}$. Generally, $\frac{d}{dx}\left(e^{f(x)}\right)=e^{f(x)}\frac{d}{dx}f(x)$.

Note that, e^{x} is its own derivative, and in fact, is the only function which is its own derivative.

Example 9.46

If $f(x)=e^{ax}$, where a is a constant, find $f'(x)$.

Solution

Let $u=ax$ and $y=e^{u}$.

$\Rightarrow \frac{du}{dx} = a$ and $\frac{dy}{du} = e^u$.

Using the chain rule of differentiation,

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(e^u\right) \times \frac{d}{dx}(ax)$$

$$= e^u \times a$$

$$= ae^{ax}$$

Therefore, $f'(x) = ae^{ax}$.

Example 9.47

Find $\frac{d}{dx}(e^{x^3+5x})$.

Solution

Given $\frac{d}{dx}(e^{x^3+5x})$.

Let $y = e^{x^3+5x}$ and $u = x^3 + 5x$

$$\Rightarrow y = e^u$$

$$\Rightarrow \frac{dy}{du} = e^u \text{ and } \frac{du}{dx} = 3x^2 + 5,$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = e^u(3x^2 + 5)$$

$$\Rightarrow \frac{dy}{dx} = \left(3x^2 + 5\right)e^u$$

Therefore, $\frac{d}{dx}(e^{x^3+5x}) = (3x^2 + 5)e^{x^3+5x}$.

Example 9.48

Find the derivative of $y = \frac{e^{2x}}{1+e^{-x}}$ with respect to x.

Solution

Given $y = \frac{e^{2x}}{1+e^{-x}}$.

Let $u = e^{2x}$ and $v = 1 + e^{-x}$

$$\Rightarrow \frac{du}{dx} = 2e^{2x} \text{ and } \frac{dv}{dx} = -e^{-x}$$

Using quotient rule of differentiation:

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(1+e^{-x}\right)\left(2e^{2x}\right) - \left(-e^{-x}\right)\left(e^{2x}\right)}{\left(1+e^{-x}\right)^2}$$

$$= \frac{2e^{2x} + 3e^{x}}{\left(1+e^{-x}\right)^2}.$$

Therefore, $\frac{dy}{dx} = \frac{e^x\left(2e^x + 3\right)}{\left(1+e^{-x}\right)^2}$.

Example 9.49

Find y' given that $y = e^{-x}\ln x$.

Solution

Given $y = e^{-x}\ln x$.

Let $u = e^{-x}$ and $v = \ln x$

$$\Rightarrow \frac{du}{dx} = -e^{-x} \text{ and } \frac{dv}{dx} = \frac{1}{x}$$

Using the product rule of differentiation:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$\Rightarrow y' = \ln x\left(-e^{-x}\right) + e^{-x}\left(\frac{1}{x}\right)$$

$$\Rightarrow y' = -e^{-x}\ln x + \frac{e^{-x}}{x}.$$

Therefore, $y' = -e^{-x}\ln x + \dfrac{e^{-x}}{x}$.

Exercise 9.12

1. Find the derivative with respect to x of each of the following functions:
   (a) $f(x) = e^{x^2}$
   (b) $g(x) = e^{x}\left(1-6x^2\right)$
   (c) $h(x) = e^{x}\ln x$
   (d) $f(x) = e^{\sqrt{x^2-1}}$
2. Differentiate $y = e^{\cos x}\sin x$ with respect to x.
3. If $y = a^{x^2}$, show that $y' = 2xa^{x^2}\ln a$.
4. If $y = e^{-2x}\sin 3x$, show that $y' = e^{-2x}\left(3\cos 3x - 2\sin 3x\right)$.
5. Given that $f(x) = x^2e^{x}$, find $f'(x)$.
6. Find the derivative of each of the following:
   (a) $y = e^{\sin^2 4x}$
   (b) $y = 3^{2x}$
   (c) $y = x\left(3^{x}\right)$
   (d) $x = e^{\tan\theta}$
   (e) $x = t^3 - e^{-3t^2} + \ln e^{\sec t}$
   (f) $v = e^{u^2+2u-8}$
7. Show that
   $$\frac{d}{dx}\left(\frac{e^{2x}+e^{-2x}}{e^{2x}-e^{-2x}}\right) = \frac{-8}{\left(e^{2x}-e^{-2x}\right)^2}.$$
8. If $\dfrac{x}{x-y} = \ln\left(\dfrac{a}{x-y}\right)$, show that $\dfrac{dy}{dx} = 2 - \dfrac{x}{y}$.
9. If $x^{y} = e^{x-y}$, show that $\dfrac{dy}{dx} = \dfrac{x-y}{x\left(1+\ln x\right)}$.
10. Differentiate $y = \dfrac{x^3\ln 2x}{e^{x}\sin x}$ with respect to x.

The second derivative of a function

In general, the derivative of a function $f(x)$ is also the function of x. Suppose $g(x) = f'(x)$. If $g(x)$ is differentiable, then $g'(x)$ is called the second derivative of $f(x)$, and it is denoted by $\dfrac{d^2f}{dx^2}$. Thus,

$$g'(x) = \frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2}.$$

The second derivative of a variable y with respect to x can be written as $\dfrac{d^2y}{dx^2}$.

Example 9.50

If $y = x^2\cos 8x$, find $\dfrac{d^2y}{dx^2}$ at $x = 0$.

Solution

Given $y = x^2\cos 8x$.

Let $u = x^2$ and $v = \cos 8x$

$$\Rightarrow \frac{du}{dx} = 2x \text{ and } \frac{dv}{dx} = -8\sin 8x$$

Using the product rule of differentiation:

$$\Rightarrow \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

Thus, $\frac{dy}{dx} = 2x\cos 8x - 8x^2 \sin 8x$.

Again, differentiate $\frac{dy}{dx}$ to obtain $\frac{d^2y}{dx^2}$, that is,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\cos 8x - 8x^2 \sin 8x\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos 8x - 16x\sin 8x - 16x\sin 8x - 64x^2 \cos 8x$$

$$\Rightarrow \frac{d^2y}{dx^2} = 2\cos 8x - 32x\sin 8x - 64x^2 \cos 8x$$

At $x = 0$,

$$\Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=0} = 2\cos(8)\times 0 - 32\times 0\times \sin(8)\times 0 - 64\times (0)^2 \times \cos(8)\times 0 = 2$$

Therefore, $\left[\frac{d^2y}{dx^2}\right]_{x=0} = 2$.

Example 9.51

If $y = \frac{\sin x}{x}$, prove that $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + y = 0$.

Solution

Let $u = \sin x$ and $v = x$.

$$\Rightarrow \frac{du}{dx} = \cos x \text{ and } \frac{dv}{dx} = 1$$

Using the quotient rule of differentiation:

$$\Rightarrow \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x\cos x - \sin x}{x^2}.$$

Multiplying by x^2 both sides gives;

$$x^2\frac{dy}{dx}=x\cos x-\sin x.$$

Differentiate again with respect to x gives

$$2x\frac{dy}{dx}+x^2\frac{d^2y}{dx^2}=-x\sin x$$

Divide by x^2 both sides to obtain,

$$\frac{2}{x}\frac{dy}{dx}+\frac{d^2y}{dx^2}=-\frac{\sin x}{x}$$

But $y=\dfrac{\sin x}{x}$

$$\Rightarrow \frac{2}{x}\frac{dy}{dx}+\frac{d^2y}{dx^2}=-y$$

Therefore, $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\dfrac{dy}{dx}+y=0.$

Derivatives of parametric functions

Parametric functions are functions which have a pair of equations in which two variables x and y are related by two equations of the form $x=f(t)$ and $y=g(t)$, where x and y are functions of a parameter t. Using the chain rule, $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}$. But $\dfrac{dt}{dx}=\dfrac{1}{\left(\dfrac{dx}{dt}\right)}$.

Therefore, the useful formula for finding the first derivatives of parametric functions is given by

$$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{1}{\frac{dx}{dt}}=\frac{dy}{dt}\times\frac{dt}{dx}.$$

The second derivative of a parametric function is obtained by differentiating $\dfrac{dy}{dx}$ with respect to t followed by dividing the result by the derivatives of x with respect to t. That is, $\dfrac{d^2y}{dx^2}=\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.

Since $\dfrac{dy}{dx}$ is a function of t, then $\dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\times\dfrac{dt}{dx}$. This formula is useful in finding second derivatives of parametric functions.

Example 9.52

If $x=\sin t$ and $y=\cos 2t$, show that $\dfrac{d^2y}{dx^2}=-4$.

Solution

Given $x=\sin t$ and $y=\cos 2t$.

$$\Rightarrow \frac{dx}{dt}=\cos t \text{ and}$$

$$\frac{dy}{dt}=-2\sin 2t=-4\sin t\cos t$$

But $\dfrac{dy}{dx}=\dfrac{dy}{dt}\times\dfrac{dt}{dx}$

Thus, $\dfrac{dy}{dx}=(-4\sin t\cos t)\times\dfrac{1}{\cos t}$

$$\Rightarrow \frac{dy}{dx}=-4\sin t$$

Again, $\dfrac{d^2y}{dx^2}=\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)\times\dfrac{dt}{dx}$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{dt}(-4\sin t)\times\frac{1}{\cos t}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-4\cos t\times\frac{1}{\cos t}$$

$$\Rightarrow \frac{d^2y}{dx^2}=-4.$$

Therefore, $\frac{d^2y}{dx^2}=-4.$

Example 9.53

Given that $2x=\theta+\frac{1}{\theta}$ and $2y=\theta-\frac{1}{\theta}$, show that:

(a) $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$ (b) $\frac{d^2y}{dx^2}=-\frac{8\theta^3}{\left(\theta^2-1\right)^3}$

Solution

(a) Given $2x=\theta+\frac{1}{\theta}$ and $2y=\theta-\frac{1}{\theta}$.

$$\Rightarrow 2\frac{dx}{d\theta}=1-\frac{1}{\theta^2} \text{ and } 2\frac{dy}{d\theta}=1+\frac{1}{\theta^2}$$

Thus, $\frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}$

$$=\frac{1+\theta^{-2}}{2}\times\frac{2}{1-\theta^{-2}}$$

$$\frac{dy}{dx}=\frac{1+\theta^{-2}}{1-\theta^{-2}}$$

$$\Rightarrow \frac{dy}{dx}=\frac{1+\theta^{-2}}{1-\theta^{-2}}\times\frac{\theta^2}{\theta^2}$$

Therefore, $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$.

(b) From $\frac{dy}{dx}=\frac{\theta^2+1}{\theta^2-1}$,

$$\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\times\frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{\theta^2+1}{\theta^2-1}\right)\times\frac{2}{1-\theta^{-2}}$$

$$=\frac{2\theta^2\left(\theta^2-1\right)-2\theta\left(\theta^2+1\right)}{\left(\theta^2-1\right)^2}\times\frac{2\theta^2}{\theta^2-1}$$

$$=\frac{2\theta^3-2\theta^3-2\theta}{\left(\theta^2-1\right)^3}\times 2\theta^2$$

$$=-\frac{8\theta^3}{\left(\theta^2-1\right)^3}$$

Therefore, $\frac{d^2y}{dx^2}=-\frac{8\theta^3}{\left(\theta^2-1\right)^3}$.

Example 9.54

If $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Give your answer in terms of half angle of θ.

Solution

Given $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

$$\Rightarrow \frac{dx}{d\theta}=a(1+\cos\theta) \text{ and } \frac{dy}{d\theta}=a\sin\theta$$

$$\Rightarrow \frac{dy}{dx}=\frac{dy}{d\theta}\times\frac{d\theta}{dx}, \text{ but } \frac{d\theta}{dx}=\frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow \frac{dy}{dx}=a\sin\theta\times\frac{1}{a(1+\cos\theta)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin\theta}{1+\cos\theta}$$

Using half angles formulae for sine and cosine, it gives;

$$\frac{dy}{dx}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\cos^2\frac{1}{2}\theta}$$

$$\Rightarrow \frac{dy}{dx}=\tan\tfrac{1}{2}\theta$$

Also, $\frac{d^2y}{dx^2}=\frac{d}{d\theta}\left(\frac{dy}{dx}\right)\times\frac{d\theta}{dx}$

$$=\frac{d}{d\theta}\left(\tan\tfrac{1}{2}\theta\right)\times\frac{1}{a\left(1+\cos\theta\right)}$$

$$=\frac{1}{2}\sec^2\tfrac{1}{2}\theta\times\frac{1}{a\left(2\cos^2\tfrac{1}{2}\theta\right)}$$

$$=\frac{\sec^2\tfrac{1}{2}\theta}{4a\cos^2\tfrac{1}{2}\theta}$$

$$=\frac{1}{4a}\sec^4\tfrac{1}{2}\theta.$$

Therefore, $\frac{d^2y}{dx^2}=\frac{1}{4a}\sec^4\tfrac{1}{2}\theta.$

Exercise 9.13

1. If $x=\sin 3\theta$ and $y=\cos 3\theta\sin\theta$, prove that
$$\frac{d^2x}{d\theta^2}-\frac{d^2y}{d\theta^2}-10y=3x\left(2\cos\theta-3\right).$$

2. Given that $y=\frac{\sin x}{x^2}$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence, verify that
$$x^4\frac{d^2y}{dx^2}+4x^3\frac{dy}{dx}+\left(x^2+2\right)\sin x=0$$

3. If $x=\frac{t+2}{2t+1}$ and $y=\frac{2t+3}{t}$, find:
   (a) $\frac{dy}{dx}$
   (b) $\frac{d^2y}{dx^2}$
   (c) $\frac{d^2y}{dx^2}$, when $x=0$

4. If $x=e^t(\cos t+\sin t)$ and $y=e^t(\cos t-\sin t)$, find:
   (a) $\frac{dy}{dx}$
   (b) $\frac{d^2y}{dx^2}$
   (c) $\frac{d^2y}{dx^2}$, when $t=0$

5. Show that $\frac{d^2y}{dx^2}-2\frac{dy}{dx}+5y=0$, given that $y=e^x\cos 2x$.

6. The equation of a curve is given parametrically by the equations $x=\frac{t^2}{1+t^3}$ and $y=\frac{t^3}{1+t^3}$, find:
   (a) $\frac{dy}{dx}$, when $t=2$
   (b) $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2},\frac{1}{2}\right)$.

7. A curve is defined parametrically as $x=2t^2+5t+1$ and $y=t+6$. Find its gradient at the point (1, 6).

8. Verify that $\frac{dy}{dx}=1$, if $x=\tan^{-1}\left(\frac{2t}{1-t^2}\right)$ and $y=\sin^{-1}\left(\frac{2t}{1+t^2}\right)$.

9. Find $\frac{d^2y}{dx^2}$ if $y=\frac{\ln x}{x}$. Hence, evaluate $\frac{d^2y}{dx^2}$ when $x=2$ (leave your answer in logarithmic form).

10. Given that $x=\frac{1}{2}\left(\sin^{-1}\theta\right)^2$, find the value of $\left(1-\theta^2\right)\frac{d^2x}{d\theta^2}-\theta\frac{dx}{d\theta}$.

11. Given $y=ae^{nx}+be^{-nx}$, show that $\frac{d^2y}{dx^2}=n^2y$.

Applications of differentiation

Differentiation is useful in solving real life problems particularly in physical sciences, computer sciences, engineering, economics, and in many other areas. For instance, in ecology, derivatives are used in predictions of the population of species in a habitat after a certain period of time. In seismology, it is used to determine earthquakes. In business, the concept of derivative is used in calculating profit and loss.

Solving problems involving small changes

Small changes of quantities are related to the concept of differentiation. If δx is small, then $\frac{\delta y}{\delta x}\approx\frac{dy}{dx}$, which implies, $\delta y\approx\frac{dy}{dx}\delta x$. This notation can be written as $\Delta y\approx\frac{dy}{dx}\Delta x$.

If y is a function of x, then the change in y corresponding to a given small change in x can be determined.

Note that, the symbol δ can also be written as Δ. The notation δy is read as "delta y".

Example 9.55

If $y=2x^2-3x$, find an approximate change in y when x increases from 7 to 7.02.

Solution

Given $y=2x^2-3x$.

$$\Rightarrow\frac{dy}{dx}=4x-3$$

But $\Delta y\approx\frac{dy}{dx}\Delta x$

$\Rightarrow\Delta y\approx(4x-3)\Delta x$, where

$\Delta x=7.02-7=0.02$

$$\Rightarrow\Delta y\approx(4\times7-3)(0.02)$$

$$\Rightarrow\Delta y\approx0.5$$

Therefore, the approximate change in y is 0.5.

Example 9.56

The side of a square is 10 cm. Find the increase in the area of the square when its side expands by 0.01cm.

Solution

Let A denote the area of the square and l denote the length of side.

$$\Rightarrow A=l^2 \text{ and } \frac{dA}{dl}=2l$$

Using small changes;

$$\frac{\Delta A}{\Delta l}\approx\frac{dA}{dl}$$

$$\Rightarrow\Delta A\approx\frac{dA}{dl}\Delta l$$

$\Rightarrow \Delta A \approx 2l\Delta l$

But $l = 10$ cm and $\Delta l = 0.01$

Thus,

$\Delta A \approx 2\times 10 \text{ cm}\times 0.01 \text{ cm} \approx 0.2 \text{ cm}^2$

Therefore, the increase in area is 0.2 cm^2.

Example 9.57

Use the technique of small changes to find an approximate value of $\sqrt{25.08}$.

Solution

Given $\sqrt{25.08}$.

Let $y = \sqrt{x}$ and $(x, y) = (25, 5)$

$\Rightarrow (x+\delta x,\ y+\delta y) = (25+0.08,\ 5+\delta y)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

But $\delta y \approx \frac{dy}{dx}\delta x$

$$\Rightarrow \delta y \approx \frac{1}{2\sqrt{25}}(0.08)$$

$\Rightarrow \delta y \approx 0.008$

Thus, $y + \delta y = 5 + 0.008 \approx 5.008$

Therefore, $\sqrt{25.08} \approx 5.008$

Example 9.58

If $R = kr^n$, where k is a constant and an error of $y\%$ is made in measuring the radius r. Prove that the resulting error in R is $ny\%$.

Solution

Given $R = kr^n$.

$$\Rightarrow \frac{dR}{dr} = knr^{n-1} = \frac{knr^n}{r}$$

$$\Rightarrow \frac{dR}{dr} = kr^n \frac{n}{r} = \frac{Rn}{r}$$

But $\Delta R = \frac{dR}{dr}\Delta r$

$$\Rightarrow \Delta R = \frac{Rn}{r}\Delta r$$

$$\Rightarrow \frac{Rn}{r} = n\left(\frac{\Delta r}{r}\right). \text{ But } \frac{\Delta R}{r} = y\%$$

$$\Rightarrow \frac{\Delta R}{R} = n\left(\frac{\Delta r}{r}\right)$$

$$\Rightarrow \frac{\Delta R}{R} = ny\%$$

Therefore, the resulting error in R is $ny\%$.

Exercise 9.14

1. Find an approximate increase in y when x increases from 8 to 8.01, given $y = x^2 + 2x$.
2. Find an approximate value of each of the following:

   (a) $\sqrt[3]{27.27}$ (c) $\sqrt[4]{64.96}$

   (b) $(5.03)^3$
3. Find an approximate change in the volume of a cube of side x cm caused by increasing the sides by 1%.
4. Approximate the decrease in volume and surface area if an ice sphere of radius 10 m shrinks to radius 9.8 m.

5. Find the approximate increase in the volume of a sphere its radius increases from 10 cm to 10.1 cm.
6. If $y = x + \frac{1}{x}$, find an approximate increase in y when x changes from 2 to 2.02.
7. The solid circular cylinder has a base radius 5 cm and height 12.5 cm. If the radius of the base increases to 5.04 cm, find an approximate increase in surface area of the cylinder. (Leave π in your answer).
8. An error of $2\frac{1}{2}\%$ is made in the measurement of the area of a circle. Find the percentage error of results in the circumference.
9. As x increases, prove that the area of a circle of radius x and the area of a square of side x increase by the same percentage, provided that the increase in x is small.
10. Let y centimetres be the length of a pendulum and t seconds the time of one complete swing. It is known that $y = kt^2$, where k is a constant. If the length of the pendulum is increased by $y\%$, y being small, find the corresponding percentage increase in time of swing.

Solving problems involving rates of change

The rate of change is defined as the change of one quantity with respect to another. The derivative of a function representing the position of a particle along a line at a time t is the velocity at that time, and the derivative of the velocity which is the second derivative of the position function represents the acceleration of the particle at time t.

Understanding the nature of change and the rate at which it takes place enable the expert to make important predictions and decisions about the atmospheric pressure, humidity, wind patterns, temperature, and many others. Also, it is useful in determination of rates of change of quantities such as volumes, areas, motion, and patterns. The chain rule can be used when dealing with problems involving rates of changes.

Distance, velocity, and acceleration

The concept of rates of change can be used in motion of objects to find velocity and acceleration when the displacement function is known.

Let $s(t)$ be the position of a particle at time t. Then, $s'(t)$ represents the instantaneous velocity and $a(t) = v'(t) = s''(t)$ represents the instantaneous acceleration of the particle at time t.

Acceleration, $\frac{d^2s}{dt^2}$ may be written in another form by using the fact that $\frac{dv}{dt}=\frac{dv}{ds}\times\frac{ds}{dt}=\frac{dv}{ds}\times v.$

Therefore, $a=\frac{dv}{dt}=\frac{d^2s}{dt^2}=v\frac{dv}{ds}.$

Example 9.59

The position of a particle on a line is given by $s(t)=t^3-2t^2+6t+5$, where t is measured in seconds and s is in metres. Find the velocity and acceleration of the particle at the end of 2 seconds.

Solution

Given $s(t)=t^3-2t^2+6t+5$.

$\Rightarrow v=s'(t)=3t^2-4t+6$

At $t=2\,\text{sec}$,

$v=s'(2)=3(2)^2-4(2)+6$

$\Rightarrow v=10$ m/s

$a=v'(t)=6t-4$

Also, at $t=2\,\text{sec}$,

$\Rightarrow a=6(2)-4=8\text{ m/s}^2$.

Therefore, at the end of 2 seconds, the velocity is 10 m/s and acceleration is 8 m/s^2.

Example 9.60

A particle moves along a straight line $\overline{\text{OB}}$ so that it is s metres from O in t seconds, where $s=t(2t-3)(t-4)$. Deduce the expressions for its velocity and acceleration in terms of t, hence describe the following motion at $t=2$ seconds:

(a) The position of the particle.

(b) The direction of the particle with reference to point B.

(c) Its speed.

(d) State whether the speed is increasing or decreasing.

(e) The rate of change of the speed.

Solution

Given

$s=t(2t-3)(t-4)=2t^3-11t^2+12t.$

$\Rightarrow s'(t)=6t^2-22t+12$

Also, $a=v'(t)=12t-22$.

At time $t=2$ seconds,

(a) $s=2(2(2)-3)(2-4)=-4$. The particle is 4 m from O on $\overline{\text{BO}}$ $(s=-4\text{m})$.

(b) The particle is moving away from B in the opposite direction.

(c) Its speed:

$s'(t)=v=6t^2-22t+12$

$\Rightarrow v=6(2)^2-22(2)+12$

$=-8.$

Since speed is the magnitude of velocity, then its speed is 8 m/s.

(d) Its speed is decreasing.

(e) Rate of change of speed,

$a = v'(t) = 12t - 22$

$$\Rightarrow \left[\frac{dv}{dt}\right]_{t=2} = 12(2) - 22$$

$$= 2\text{m}/\text{s}^2.$$

Therefore, the rate of change of speed is $2\ \text{m/s}^2$.

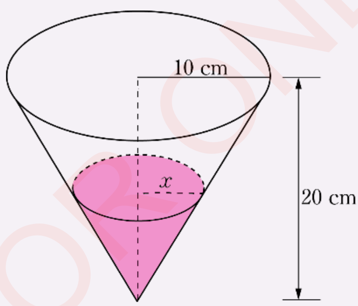
Example 9.61

A liquid is running out of a conical funnel at the rate of $5\ \text{cm}^3/\text{s}$. The radius of the funnel is 10 cm and its height is 20 cm. How fast is the liquid level dropping when the liquid is 10 cm deep? (Leave your answer in π).

Solution

Let h be the depth, r the radius, and V the volume of the liquid at time t.

Thus, $\frac{dV}{dt} = -5\ \text{cm}^3/\text{s}$ (liquid is decreasing). Consider the following diagram:



Given $\frac{r}{h} = \frac{10}{20}$, then $r = \frac{h}{2}$.

Volume of the cone is $V = \frac{1}{3}\pi r^2 h$.

Upon substitution, it gives

$$V = \frac{1}{12}\pi h^3$$

$$\Rightarrow \frac{dV}{dh} = \frac{\pi}{4}h^2$$

Apply chain rule of differentiation:

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$$

$$\Rightarrow -5 = \frac{\pi}{4}h^2 \times \frac{dh}{dt}$$

But $h = 10$

$$\Rightarrow \frac{dh}{dt} = -\frac{20}{100\pi}$$

$$= -\frac{1}{5\pi}$$

Therefore, the liquid level is dropping at the rate of $\frac{1}{5\pi}\text{cm}^3/\text{s}$.

Example 9.62

The volume of a spherical balloon increases at a constant rate of $1.5\ \text{cm}^3/\text{s}$ when the balloon is blown up. If the volume of the balloon is $62\ \text{cm}^3$, find the rate of increase of its radius.

Solution

Let r be the radius of the balloon, V the volume, and t the time in seconds.

Since $V = \frac{4}{3}\pi r^3$, then $\frac{dV}{dr} = 4\pi r^2$.

Using the chain rule of differentiation:

$$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}.$$

Thus, $1.5 = 4\pi r^2 \times \frac{dr}{dt}$

$$\Rightarrow \frac{dr}{dt} = \frac{1.5}{4\pi r^2}$$

But $V = \frac{4}{3}\pi r^3$

$$\Rightarrow 62 = \frac{4}{3}\pi r^3$$

$$\Rightarrow r = \left(\frac{62 \times 3}{4\pi}\right)^{\frac{1}{3}}$$

$$= 2.4553 \text{ cm}$$

$$\Rightarrow \frac{dr}{dt} = \frac{1.5}{4\pi(2.4553)^2}$$

$$= 0.02 \text{ cm/s.}$$

Therefore, the rate of increase of the radius is 0.02 cm/s.

Exercise 9.15

1. The surface area of a sphere is $4\pi r^2$, where r is its radius. If the radius increases at the rate of $0.1\text{cm}^2/\text{s}$, find the rate of change of area in cm^2/s when the radius of the sphere is 40 cm.
2. The formula $s(t) = -4.9t^2 + 49t + 20$ represents the height in metres of an object after it is thrown vertically upward from a point 15 m above the ground at a velocity of 49 m/s. How high above the ground will the object reach?
3. A hollow circular cone is held downward beneath the water tap leaking at the rate of $2 \text{ cm}^3/\text{s}$. Find the rate of rise of water level when the depth of the cone is 6 cm, given that its height is $18\,\text{cm}$ and radius is $12\,\text{cm}$.
4. The radius of a spherical bottom increases at the rate of 6 cm/min. Find the rate of increase of volume when the radius of the bottom is 4 cm.
5. At what rate is the area increasing when the radius is 4 cm, given the radius of the spherical balloon increases at a rate of 6 cm/sec?
6. The area of a circle is increasing at the rate of $3 \text{ cm}^2/\text{s}$. Find the rate of change of the circumference of the circle when its radius is 2 cm.
7. A container in the form of right circular cone of height 10 cm and base radius 1 cm is receiving drops of liquid from a leaking tap at the rate of $0.1 \text{ cm}^3/\text{s}$. Find the rate at which the surface area of the liquid is increasing when the liquid is half way up the cone.
8. An inverted right circular cone of vertical angle $120°$ is collecting water from a tap at the rate of $18\pi \text{ cm}^3/\text{min}$. Find the water level after 12 minutes and the rate of increase of water level in the cone.

9. If the volume of a sphere increases at a rate of 6 cm^3/s, find the rate of increase in surface area of the sphere at the instant when the radius is 4 cm.
10. A particle moves along the x-axis and its position is given by $s(t) = 5 + 4\sin 2t + 3\cos 2t$. Prove that its velocity is zero when $s = 10$ and its acceleration is $20 - 4s$.
11. If $s(t) = 3t + t^3$ is a displacement function in metres, calculate the velocity and acceleration after 2 seconds.
12. A particle moves back and forth along a horizontal line defined by the position function $s(t) = t^3 - 12t^2 + 36t - 30$, for $t \geq 0$.
    (a) Determine the velocity and acceleration functions.
    (b) When is the velocity zero?
    (c) When is the acceleration zero?
    (d) At what interval is the velocity negative? Explain its physical meaning.

Turning points and points of inflexion of a curve

Turning points are points on a curve where its graph changes direction. The following activity illustrates the concept of turning points and point of inflexion.

Activity 9.6: Identification of turning points and points of inflexion

Individually or in a group, perform the following tasks:

1. Draw the graphs of curves of your choice. The curves should be of degrees two and three.
2. Identify the turning points and points of inflexion of the graphs in task 1.
3. Identify the values of the turning points and points of inflexion found in task 2.
4. Use the results in task 3 to determine the following:
   (a) Maximum points
   (b) Minimum points
   (c) Points of inflexion
5. Share your results with other students for more inputs.

A turning point is a point at which values of the function change from decreasing to increasing or from increasing to decreasing. It is a point at which the gradient of the curve zero. If a curve is defined by $y = f(x)$, then a turning point is determined when $f'(x) = 0$. In other words, the turning point is obtained when the derivative of the curve is zero.

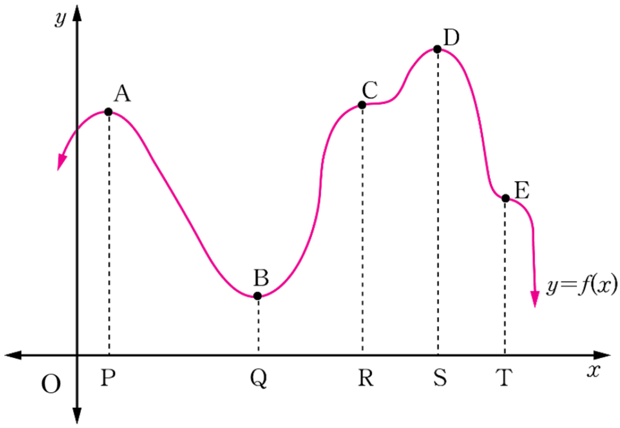


Figure 9.3: *Turning points and points of inflexion*

Figure 9.3 illustrates the concepts of turning points and points of inflexion. Points A and D are maximum points of the curve, point B is a minimum point and points C and E are points of inflexion. The function has maximum values $\overline{PA}$ and $\overline{SD}$ when $x = \overline{OP}$ and $x = \overline{OS}$, respectively. Also, the function has a minimum value $\overline{QB}$ when $x = \overline{OQ}$. The values of points of inflexion are $\overline{RC}$ and $\overline{TE}$ when $x = \overline{OR}$ and $x = \overline{OT}$, respectively.

Note that, O is the origin.

The following are steps for determining the turning points:

1. Differentiate the given function to obtain $f'(x)$.
2. Equate the expression for $f'(x)$ to zero.
3. Solve $f'(x) = 0$, for the x-coordinate(s) of the turning point(s)
4. Find the x values $(x_1, x_2, x_3, \ldots)$ which satisfy the equation $f'(x) = 0$.
5. Calculate the corresponding values of y using $y = f(x)$.

Example 9.63

Given the function defined by the equation $y = x^2 - 4x + 5$, find the coordinates of the turning points.

Solution

Given $y = x^2 - 4x + 5$.

$$\Rightarrow \frac{dy}{dx} = 2x - 4$$

Solve the equation $\frac{dy}{dx} = 0$:

$\Rightarrow 2x - 4 = 0$

$\Rightarrow x = 2$

Calculate y-coordinate from $y = f(x)$:

$\Rightarrow y = (2)^2 - 4(2) + 5$

$\quad = 1$

Therefore, the turning point is $(2, 1)$.

Example 9.64

Determine the turning points of the curve $y = 4x^3 + 3x^2 - 60x - 12$.

Solution

Given $y = 4x^3 + 3x^2 - 60x - 12$.

$$\Rightarrow \frac{dy}{dx} = 12x^2 + 6x - 60$$

At the turning points, $\frac{dy}{dx} = 0$.

$\Rightarrow 12x^2 + 6x - 60 = 0$

Upon solving the equation, $x = 2$ and $x = -2.5$.
Calculate the y-coordinates from $y = f(x)$:

When $x = 2 \Rightarrow y = 4(2)^3 + 3(2)^2 - 60(2) - 12 = -88$.
When $x = -2.5$

$\Rightarrow y = 4(-2.5)^3 + 3(-2.5)^2 - 60(-2.5) - 12 = 94.25$.

Therefore, the turning points are $(2, -88)$ and $(-2.5, 94.25)$.

Example 9.65

Determine the turning points of the curve $f(x) = 7 + 24x - 9x^2 - 2x^3$.

Solution

Given $f(x)=7+24x-9x^2-2x^3$.

$\Rightarrow f'(x)=24-18x-6x^2$.

At the turning points, $f'(x)=0$.

Solving $f'(x)=0$ for x

$\Rightarrow 24-18x-6x^2=0$.

Upon solving, $x=-4$ and $x=1$.

Determine the corresponding values of y.

When $x=-4$: $f(-4)=7+24(-4)-9(-4)^2-2(-4)^3=-105$.

When $x=1$: $f(1)=7+24(1)-9(1)^2-2(1)^3=20$

Therefore, the turning points are $(-4,-105)$ and $(1,20)$.

Exercise 9.16

1. Determine the turning points of the curve $y=4x^3+3x^2-6x-2$.
2. Find the turning points of the curve $y=x^2(x-6)$.
3. Determine the turning points of the function $y=2x^3-12x^2-30x-10$.
4. For each of the following curves, find the coordinates of the turning points:

   (a) $y=x^2-2x-8$ (b) $y=3x-x^2$ (c) $y=x+\frac{1}{x}$
5. Find the coordinates of the turning points on each of the following curves:

   (a) $y=x^3+x^2-x+3$ (c) $y=2x^2-8x$

   (b) $y=8+4x-2x^2-x^3$ (d) $y=(1-x)(x-5)$
6. The curve $f(x)=x^3-kx^2+x-3$ has two turning points. If one point occurs at $x=1$, find the value of k and the coordinates of the turning points.
7. For the curve $y=x^3-9$, find its turning points.
8. Show that the turning points of the curve $f(t)=t^3-\frac{1}{2}t^2-2t+4$ are $(1, 2.5)$ and $\left(-\frac{2}{3}, 4\frac{22}{27}\right)$.
9. Verify that the turning point of the curve $y=2x-e^x$ is $(0.6931, -0.6136)$.
10. Determine the equation of the cubic curve whose turning points are $(1, 3)$ and $(-1, 7)$.

Real life problems involving maximum and minimum values

The turning points and points of inflexion of the functions (curves) have numerous applications in real life situations. For instance, stationary points (maximum, minimum and points of inflexion) are useful in sketching the curves. In economics, the stationary points are used to determine the equilibrium prices, break-even points, marginal costs and marginal revenues. The following are steps for classifying stationary points.

1. Given $y = f(x)$, determine $\frac{dy}{dx}$ (that is, $y' = f'(x)$).
2. Let $\frac{dy}{dx} = 0$ and solve for the values of x.
3. Substitute the values of x into the given equation, $y = f(x)$ in order to obtain the corresponding y-coordinate values. This establishes the coordinates of the stationary points.

To determine the nature of stationary points: Either,

4. Find $\frac{d^2y}{dx^2}$ and substitute into it the values of x in order to obtain the nature of the stationary point as follows:
   (a) If the value of $\frac{d^2y}{dx^2}$ at the point is positive, then the point is a minimum.
   (b) If the value of $\frac{d^2y}{dx^2}$ at the point is negative, then the point is a maximum
   (c) If the value of $\frac{d^2y}{dx^2}$ at the point is zero, then the point is a point of inflexion or maximum or minimum. To verify this, use the variation signs of the first derivatives.

Or

5. Determine the sign of the gradient of the curve just before and just after the stationary points. If the sign of the gradient of the curve changes from positive to negative, then the point is a maximum. If the sign of the gradient changes from negative to positive, the point is a minimum. When the gradient sign does not change, then the point is a point of inflexion.

Example 9.66

Identify the nature of the stationary points of the function $f(x) = 3x^4 - 8x^3 + 6x^2 - 5$. Hence, sketch its graph.

Solution

Given $f(x) = 3x^4 - 8x^3 + 6x^2 - 5$.

$\Rightarrow f'(x) = 12x^3 - 24x^2 + 12x$.

At stationary points, $f'(x) = 0$

$\Rightarrow 12x^3 - 24x^2 + 12x = 0$

$\Rightarrow 12x(x-1)(x-1) = 0$

$\Rightarrow x = 0$ and $x = 1$.

When, $x=0 \Rightarrow f(x)=-5$.

When, $x=1$, $f(x)=-4$

The stationary points are $(0,-5)$, and $(1,-4)$

Classify the stationary points by finding the second derivative. That is,

$$f''(x)=\frac{d}{dx}\left(12x^3-24x^2+12x\right)$$

$$\Rightarrow f''(x)=36x^2-48x+12.$$

When $x=0 \Rightarrow f''(0)=12>0$. Thus, the point $(0,-5)$ is a minimum point.

When $x=1 \Rightarrow f''(0)=0$.

The nature of the point is determined by variation of signs of the gradient of the curve. That is, $f'(x)=12x^3-24x^2+12x$.

If x is slightly less than 1, say $x=0.9$, then

$$f'(0.9)=12(0.9)^3-24(0.9)^2+12(0.9)$$

$$=0.108>0$$

Also, if If x is slightly greater than 1, say $x=1.1$, then

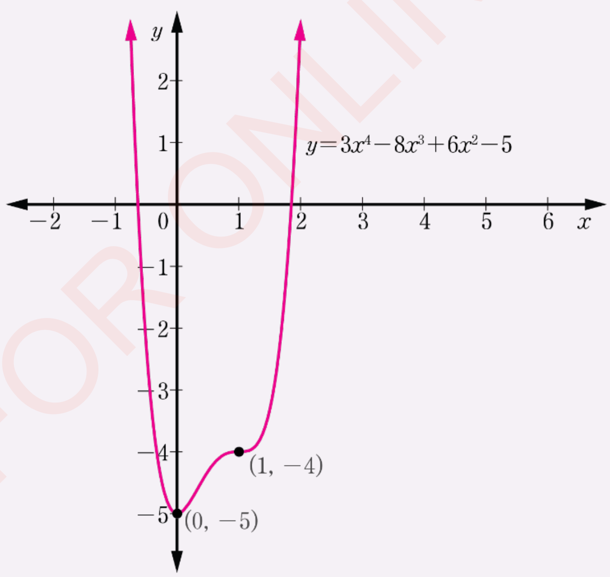
$$f'(1.1)=12(1.1)^3-24(1.1)^2+12(1.1)$$

$$=0.132>0$$

Since the gradient does not change, then the point $(1,-4)$ is an inflexion point.

Therefore, $(0,-5)$ is a minimum point and $(1,-4)$ is an inflexion point.

The following is the graph of the curve $f(x)=3x^4-8x^3+6x^2-5$.



Example 9.67

A gardener has 400 m of fencing material and wishes to build a rectangular field completely enclosed by the fence.

(a) Find the dimensions of the field that the gardener should make so as to maximize the area.

(b) Find the maximum area to be enclosed.

Solution

(a) Let x be the length of the field and y the width as shown in the following figure.

Area = length × width

$\Rightarrow$ Area, A = xy

But the fencing does not exceed 400 m. So the field must have a maximum perimeter of 400 m

$\Rightarrow x+x+y+y=400$

$\Rightarrow 2x+2y=400$

$\Rightarrow y=200-x$.

Substituting $y=200-x$ into the equation $A=xy$ gives,

$A=x(200-x)=200x-x^2$

$$\Rightarrow \frac{dA}{dx}=200-2x$$

At the turning points, $\frac{dA}{dx}=0$.

Thus, $200-2x=0 \Rightarrow x=\frac{200}{2}=100$

$\Rightarrow y=200-100=100$.

Therefore, the dimensions of the field should be 100 m by 100 m.

(b) When $x=100$,

$$\left[\frac{d^2A}{dx^2}\right]_{x=100}=-2<0 \Rightarrow (100,100)$$

is a maximum point.

The area is maximum when

$x=100\,\text{m}$ and $y=100\,\text{m}$

Thus, $A=xy=100\,\text{m}\times 100\,\text{m}$

$\Rightarrow A=10,000\ \text{m}^2$

Therefore, the maximum area is $10,000\ \text{m}^2$.

Example 9.68

Find the coordinates and the nature of turning points of the curve $y=x^3+3x^2-9x+6$. Hence, sketch the curve.

Solution

Given $y=x^3+3x^2-9x+6$.

The gradient of the curve is given by

$y'=3x^2+6x-9$

At turning points, $y'=0$.

$\Rightarrow 0=3x^2+6x-9$

$\Rightarrow (x-1)(x+3)=0$

$\Rightarrow x = 1$ and $x = -3$.

The corresponding y coordinates are obtained as follows:

When $x = 1 \Rightarrow y = (1)^3 + 3(1)^2 - 9(1) + 6 = 1$.

When $x = -3 \Rightarrow y = (-3)^3 + 3(-3)^2 - 9(-3) + 6 = 33$.

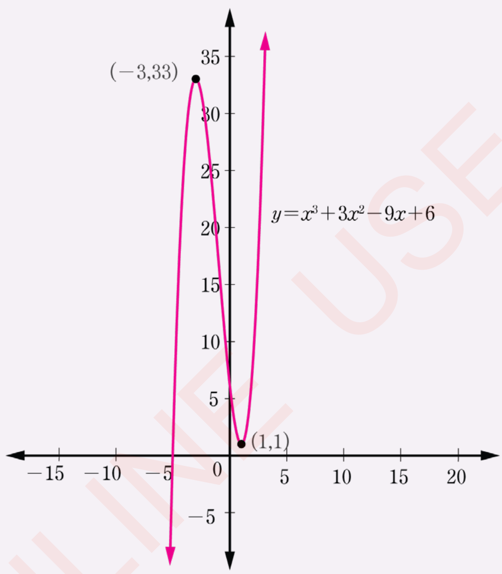
Thus, the turning points are $(1,1)$ and $(-3, 33)$.

To test the nature of the turning points, use the second derivative test that is,

$y'' = 6x + 6$ when $x = 1 \Rightarrow y''(1) = 6(1) + 6 = 12 > 0 \Rightarrow (1,1)$ is a minimum point.

$\Rightarrow y'' = 6x + 6$ when $x = -3 \Rightarrow y''(-3) = 6(-3) + 6 = -12 < 0 \Rightarrow (-3, 33)$ is a maximum point.

Therefore, $(1,1)$ is a minimum point and $(-3, 33)$ is a maximum point.



Example 9.69

Find the maximum and minimum values of the curve $y = 3x^2 + 3 - x^3$ by examining the variation signs of its gradient.

Solution

Given $y = 3x^2 + 3 - x^3 \Rightarrow \frac{dy}{dx} = 6x - 3x^2$.

For turning points, $\frac{dy}{dx} = 0$.

Thus, $6x - 3x^2 = 0 \Rightarrow x = 0$ and $x = 2$.

When $x=0 \Rightarrow y=3(0)^2+3-(0)^3=3$, thus $(x,y)=(0,3)$.
When $x=2 \Rightarrow y=3(2)^2+3-(2)^3=7$, thus $(x,y)=(2,7)$.
Hence, $(0,3)$ and $(2,7)$ are the coordinates of the turning points.
Next, examine the nature of the turning points using variation signs of the gradient.

Considering the point $(0,3)$:
If x is slightly less than 0, let say $x=-0.9$, then

$$\left[\frac{dy}{dx}\right]_{x=-0.9}=6(-0.9)-3(-0.9)^2=-7.8<0.$$

If x is slightly greater than 0, let say $x=0.1$, then

$$\left[\frac{dy}{dx}\right]_{x=0.1}=6(0.1)-3(0.1)^2=0.57>0.$$

Since the gradient changes from negative to positive, the point $(0,3)$ is a minimum point and the minimum value is 3.
Considering the point $(2,7)$:
If x is slightly less than 2, let say $x=1.9$, then,

$$\left[\frac{dy}{dx}\right]_{x=1.9}=6(1.9)-3(1.9)^2=0.57>0.$$

If x is slightly greater than 2, let say $x=2.1$, then

$$\left[\frac{dy}{dx}\right]_{x=2.1}=6(2.1)-3(2.1)^2=-0.63<0.$$

Since the gradient changes from positive to negative, then the point $(2,7)$ is a maximum point.

Therefore, the maximum value is 7.

Example 9.70

Find and classify the nature of the turning points for the function $y=x^4-4x^3+16x-16$.

Solution

Given $y = x^4 - 4x^3 + 16x - 16$

$$\frac{dy}{dx} = 4x^3 - 12x^2 + 16$$

For turning points, $\frac{dy}{dx} = 0$.

Hence, $4x^3 - 12x^2 + 16 = 0$

$\Rightarrow x^3 - 3x^2 + 4 = 0$

$\Rightarrow x^3 - 3x^2 + 4 = (x+1)(x^2 - 4x + 4) = 0$

$\Rightarrow (x+1)(x-2)^2 = 0$

$\Rightarrow x = -1$ and $x = 2$

When $x = -1 \Rightarrow y = (-1)^4 - 4(-1)^3 + 16(-1) - 16 = -27$

Thus, $(x, y) = (-1, -27)$.

When $x = 2 \Rightarrow y = (2)^4 - 4(2)^3 + 16(2) - 16 = 0$

Thus, $(x, y) = (2, 0)$.

The turning points are $(-1, -27)$ and $(2, 0)$.

Consider the point $(-1, -27)$:

If x is slightly less than -1, let say $x = -2$, then

$$\left[\frac{dy}{dx}\right]_{x=-2} = 4(-2)^3 - 12(-2)^2 + 16 = -64 < 0$$

If x is slightly greater than -1, let say $x = 0$, then

$$\left[\frac{dy}{dx}\right]_{x=0} = 4(0)^3 - 12(0)^2 + 16 = 16 > 0.$$

Since the gradient changes from negative to positive, then $(-1, -27)$ is a minimum point.

Consider the point $(2,0)$:

If x is slightly less than 2, let say $x = 1$, then

$$\left[\frac{dy}{dx}\right]_{x=1} = 4(1)^3 - 12(1)^2 + 16 = 8 > 0.$$

If x is slightly greater than 2, let say $x = 3$, then

$$\left[\frac{dy}{dx}\right]_{x=3} = 4(3)^3 - 12(3)^2 + 16 = 16 > 0.$$

Since the gradient does not change sign, then $(2,0)$ is the inflexion point.

Exercise 9.17

1. An open rectangular prism is to be constructed from a square base and four vertical rectangular shapes in order to have the capacity of 32 m^3. Find the least area of the cuboid to be used in the construction of the rectangular prism.
2. A rectangular sheet of cardboard has length 12 cm and width 7.5 cm. Equal squares of side x cm are cut from each corner. The flaps are then folded to make an open box in the form of a cuboid. If the volume of the box is V cm^3:
   (a) Show that $V = 4x^3 - 39x^2 + 90x$.
   (b) Find the value of x which gives the maximum value of V.
   (c) What is the maximum value of V.
3. The product of two positive numbers is 100. Find their least possible sum.
4. Find the coordinates of the turning points of the function $\dfrac{15+10x}{\sqrt{x^2+1}}$ and classify them.
5. Find the values of x for which the curve $(x-2)(x-3)^2$ has turning points and determine the maximum and minimum turning points of the curve.
6. For each of the following curves, find the coordinates of the turning points, classify the turning points, and determine where the curves cross the axes.
   (a) $y = (x-2)^2(x-1)$ (d) $y = 5x^4 - 12x^5$
   (b) $y = x^4 - 4x^3$ (e) $y = x^2 - 4$
   (c) $y = 36x - 3x^2 - 2x^3$ (f) $y = 4x^3 - 2x^2 - x - 1$
7. A rectangle of length x centimetres has a constant area of 12 cm^2. Find the perimeter of the rectangle in terms of x and the least possible perimeter of the rectangle.
8. A lidless box with square ends is to be made from a sheet of metal. Determine the least area of the metal for which its volume is 3.5 m^3.
9. A closed cylindrical container has a surface area of 400 cm^2. Determine the dimensions for maximum volume.

10. Determine the area of the largest fence of a rectangular ground that can be enclosed by 100 m of fencing, if part of an existing straight wall is used as one side.

Taylor's and Maclaurin's series

Taylor's series is an expression of a function $f(x)$ near a point $x = a$ which consists of an infinite sum of terms involving derivatives of the function at the point $x = a$, where $a \neq 0$. If we allow $a = 0$, the infinite series is then known as Maclaurin's series.

(a) Taylor's theorem

Consider the polynomial in power of $(x-a)$ given by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots + c_n(x-a)^n.$$

The successive derivatives of $f(x)$ will be as follows;

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots + nc_n(x-a)^{n-1}$$

$$f''(x) = 2\times c_2 + 3\times 2c_3(x-a) + 4\times 3c_4(x-a)^2 + \cdots + n(n-1)c_n(x-a)^{n-2}$$

$$f''(x) = 3\times 2c_3 + 4\times 3\times 2c_4(x-a) + \cdots + n(n-1)(n-2)c_n(x-a)^{n-3}$$, and so on.

The constants $c_0, c_1, c_2, \ldots, c_n$ are calculated by substituting $x = a$ in the function $f(x)$ and to the respective derivatives.

Putting $x = a$ gives,

$$c_0 = f(a),\ c_1 = f'(a),\ c_2 = \frac{f''(a)}{2!},\ c_3 = \frac{f'''(a)}{3!},\ c_4 = \frac{f^{iv}(a)}{4!},\ \ldots,\ c_n = \frac{f^n(a)}{n!}.$$

Substituting these coefficients in the polynomial gives the Taylor's theorem.
Thus, the Taylor's theorem states that,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f'''(a)}{n!}(x-a)^n$$

If $x = a + h \Rightarrow h = x - a$, then Taylor's theorem becomes

$$f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \cdots + \frac{f'''(a)}{n!}h^n.$$

Taylor's theorem is useful in expanding the function $f(x)$ about $x = a$.

Example 9.71

Use Taylor's theorem to expand $\sin\left(\frac{\pi}{3}+h\right)$ in ascending powers of h as far as the term in h^4.

Solution

Let $f(x)=\sin x=\sin\left(\frac{\pi}{3}+h\right)$ and $f\left(\frac{\pi}{3}\right)=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$.

$\Rightarrow f'(x)=\cos x$ and $f'\left(\frac{\pi}{3}\right)=\cos\left(\frac{\pi}{3}\right)=\frac{1}{2}$

$\Rightarrow f''(x)=-\sin x$ and $f''\left(\frac{\pi}{3}\right)=-\sin\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

$\Rightarrow f'''(x)=-\cos x$ and $f'''\left(\frac{\pi}{3}\right)=-\cos\left(\frac{\pi}{3}\right)=-\frac{1}{2}$

$\Rightarrow f^{(iv)}(x)=\sin x$ and $f^{(iv)}\left(\frac{\pi}{3}\right)=\sin\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{2}$

By Taylor's theorem,

$$f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\frac{f^{iv}(a)}{4!}h^4+\cdots+\frac{f^n(a)}{n!}$$

Thus, $\sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h+\frac{\left(-\frac{\sqrt{3}}{2}\right)}{2!}h^2+\frac{\left(-\frac{1}{2}\right)}{2!}h^3+\frac{\left(\frac{\sqrt{3}}{2}\right)}{4!}h^4+\cdots$

$$\Rightarrow \sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2-\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$$

Therefore, $\sin\left(\frac{\pi}{3}+h\right)=\frac{\sqrt{3}}{2}+\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2-\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$

Example 9.72

Use Taylor's theorem to expand $\cos\left(\frac{\pi}{6}+x\right)$ in ascending powers of h as far as the term in h^4. Hence, find the value of $\cos 31°$ correct to four significant figures, taking $1°$ as 0.01745 radians and $\sqrt{3}$ as 1.7321.

Solution

Let $f(x)=\cos x=\cos\left(\frac{\pi}{6}+h\right)$ and $f\left(\frac{\pi}{6}\right)=\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$

$\Rightarrow f'(x)=-\sin x$ and $f'\left(\frac{\pi}{6}\right)=-\sin\left(\frac{\pi}{6}\right)=-\frac{1}{2}$

$\Rightarrow f''(x)=-\cos x$ and $f''\left(\frac{\pi}{6}\right)=-\cos\left(\frac{\pi}{6}\right)=-\frac{\sqrt{3}}{2}$

$\Rightarrow f'''(x)=\sin x$ and $f'''\left(\frac{\pi}{6}\right)=\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}$

$\Rightarrow f^{(iv)}(x)=\cos x$ and $f^{(iv)}\left(\frac{\pi}{6}\right)=\cos\left(\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$

By Taylor's theorem,

$$f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\frac{f^{iv}(a)}{4!}h^4+\cdots$$

Thus, $\cos\left(\frac{\pi}{6}+h\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}h+\frac{\left(-\frac{\sqrt{3}}{2}\right)}{2!}h^2+\frac{\left(-\frac{1}{2}\right)}{3!}h^3+\frac{\left(\frac{\sqrt{3}}{2}\right)}{4!}h^4+\cdots$

$$\Rightarrow \cos\left(\frac{\pi}{6}+h\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2+\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$$

For the value of $\cos 31°=\cos(30°+1°)$, it implies that

$$\cos 31°=\frac{\sqrt{3}}{2}-\frac{1}{2}(0.01745)-\frac{\sqrt{3}}{4}(0.01745)^2+\frac{1}{12}(0.01745)^3+\frac{\sqrt{3}}{48}(0.01745)^4+\cdots$$

$$=0.8572.$$

Therefore, $\cos\left(\frac{\pi}{6}+h\right)=\frac{\sqrt{3}}{2}-\frac{1}{2}h-\frac{\sqrt{3}}{4}h^2+\frac{1}{12}h^3+\frac{\sqrt{3}}{48}h^4+\cdots$ and

$\cos 31°=0.8572.$

Example 9.73

Use Taylor's theorem to expand $\tan\left(\frac{\pi}{4}+h\right)$ in ascending powers of h as far as the term in h^3.

Let $f(x)=\tan x=\tan\left(\frac{\pi}{4}+h\right)$ and $f\left(\frac{\pi}{4}\right)=\tan\frac{\pi}{4}=1.$

$\Rightarrow f'(x)=\sec^2 x$ and $f'\left(\frac{\pi}{4}\right)=\sec^2\left(\frac{\pi}{4}\right)=2$

$\Rightarrow f''(x)=2\sec^2 x\tan x$ and $f''\left(\frac{\pi}{4}\right)=2\sec^2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right)=4$

$\Rightarrow f'''(x)=4\sec^2 x\tan^2 x+2\sec^4 x$ and

$$f'''\left(\frac{\pi}{4}\right)=4\sec^2\left(\frac{\pi}{4}\right)\tan^2\left(\frac{\pi}{4}\right)+2\sec^4\left(\frac{\pi}{4}\right)=16$$

By Taylor's theorem,

$$\Rightarrow f(x)=f(a+h)=f(a)+f'(a)h+\frac{f''(a)}{2!}h^2+\frac{f'''(a)}{3!}h^3+\frac{f^{iv}(a)}{4!}h^4+\cdots$$

Thus, $\tan\left(\frac{\pi}{4}+h\right)=1+2h+\frac{4}{2!}h^2+\frac{16}{3!}h^3+\cdots$

$$=1+2h+2h^2+\frac{8}{3}h^3+\cdots$$

Therefore, $\tan\left(\frac{\pi}{4}+h\right)=1+2h+2h^2+\frac{8}{3}h^3+\cdots$

Maclaurin's series

Considering a relationship $x=a+h$, where a is a constant, and x and h are variables, there is a special case given by $a=0$ when $x=h$. In this case, Taylor's theorem can be reduced to:

$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$ which is called Maclaurin's series. Maclaurin's series is a special case of Taylor's theorem for $a=0$.

Example 9.74

Use Maclaurin's series to expand $\ln(1+x)$ in ascending powers of x as far as the term in x^4.

Solution

Given $f(x)=\ln(1+x)$.

$\Rightarrow f(0)=\ln(1)=0$

$\Rightarrow f'(x)=(1+x)^{-1}$ and $f'(0)=1$

$\Rightarrow f''(x)=-(1+x)^{-2}$ and $f''(0)=-1$

$\Rightarrow f'''(x)=2(1+x)^{-3}$ and $f'''(0)=2$

$\Rightarrow f^{iv}(x)=-6(1+x)^{-4}$ and $f^{(iv)}(0)=-6$

By Maclaurin's series,

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$$

Thus, $\ln(1+x)=0+1\times x+\frac{(-1)}{2!}x^2+\frac{2}{3!}x^3+\frac{(-6)}{4!}x^3+\cdots$

$$\Rightarrow \ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Therefore, $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

Example 9.75

Find the Maclaurin's series for e^{5x} as far as the term in x^3.

Solution

Given $f(x)=e^{5x}$.

$\Rightarrow f(0)=1$

$\Rightarrow f'(x)=5e^{5x}$ and $f'(0)=5e^{5(0)}=5$

$\Rightarrow f''(x)=5^2e^{5x}$ and $f''(0)=5^2e^{5(0)}=5^2$

$\Rightarrow f'''(x)=5^3e^{5x}$ and $f'''(0)=5^3e^{5(0)}=5^3$.

By Maclaurin' series,

$$\Rightarrow f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{iv}(0)}{4!}x^4+\cdots$$

Thus, $e^{5x}=1+5x+\dfrac{25}{2}x^2+\dfrac{125}{6}x^3+\cdots$

Therefore, $e^{5x}=1+5x+\dfrac{25}{2}x^2+\dfrac{125}{6}x^3+\cdots$

Exercise 9.18

1. Show that the first three terms in the Maclaurin's expansion of $\log_e(1+e^x)$ are $\log_e 2+\dfrac{1}{2}x+\dfrac{1}{8}x^2$. Show that there is no term in x^3.
2. Using Taylor's theorem, obtain a series expansion of $\sin(\theta+x)$, where θ is an acute angle measured in radians such that $\sin\theta=\dfrac{4}{5}$ (Give the first four non-zero terms).
3. Apply Taylor's theorem to expand the following:
   (a) $\ln(1+4x)$ in ascending powers of $4x$ as far as the term in x^4 and deduce its corresponding Maclaurin's series.
   (b) $\operatorname{cosec} x$ in ascending powers of $\left(x-\dfrac{\pi}{2}\right)$ as far as the term in $\left(x-\dfrac{\pi}{2}\right)^4$.
   (c) $\sec x$ in ascending powers of $\left(x+\dfrac{\pi}{4}\right)$ as far as the $\left(x+\dfrac{\pi}{4}\right)^4$ term.
   (d) $\cos\left(\dfrac{\pi}{3}+x\right)$ in ascending powers of x as far as the term in x^4.
4. Prove that $\log_e\left(\dfrac{x+1}{x}\right)=2\left[\dfrac{1}{2x+1}+\dfrac{1}{3(2x+1)^3}+\dfrac{1}{5(2x+1)^5}+\cdots\right]$ and state the range of values of x for which the series is valid.
5. Find the Maclaurin's series of the following functions up to the term containing x^4.
   (a) $\cos x$ (b) $\sin x$ (c) a^x (d) $\sqrt{1+4x}$
6. Produce the power series for $\cos^2 2\theta$ as far as the term in θ^6.

7. Using Maclaurin's series, develop a series for $e^{\frac{3}{2}x}$ as far as the fourth term.
8. Show that the first three non-zero terms in the Maclaurin series for sin(sin$x$) is $x-\frac{x^3}{3}+\frac{x^5}{10}+\cdots$
9. Use Taylor's theorem to expand $\sin\left(\frac{\pi}{2}+h\right)$ in ascending powers of h as far as the term in h^4.
10. If x is the angle of radian measure of an angle which is so small that x^3 and higher powers of x can be neglected, use Taylor's theorem to show that, $\sin\left(\frac{\pi}{2}+x\right)=\frac{1}{2}+\frac{\sqrt{3}}{2}x-\frac{1}{4}x^2+\cdots$

Introduction to partial derivatives

The function $z=f(x,y)$ depends on two variables x and y, where x and y are independent of each other. Thus, f depends on x and y. The derivative of f with respect to x or y is called a partial derivative.

Identifying functions of two variables

The definition of a function of two variables is very similar to the definition for a function of one variable. The main difference is that, instead of mapping values of one variable to values of another variable, the ordered pairs of variables are mapped to another variable.

(a) $f(x,y)$ means f is a function of x and y

(b) $h(q, r)$ means h is a function of q and r.

Let $z=f(x,y)$ be a function of two independent variables, x and y. The partial derivatives of z are obtained as follows:

If x varies while y remains constant, then z is a function of x. The partial derivative of z with respect to x is given by, $\frac{\partial z}{\partial x}=f_x(x, y)=\lim\limits_{h\to 0}\frac{f(x+h,y)-f(x)}{h}$.

If y varies while x remains constant, then z is a function of y. The partial derivative of z with respect to y is given by;

$$\frac{\partial z}{\partial y}=f_y(x, y)=\lim_{h\to 0}\frac{f(x,h+y)-f(x,y)}{h}.$$

Other notations used for partial derivatives include

$$\frac{\partial^2 z}{\partial x^2} = z_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right),\ \frac{\partial^2 z}{\partial y^2} = z_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right), \text{ and } \frac{\partial^2 z}{\partial x \partial y} = z_{xy} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right),$$

where z_{xx} is the second derivative of z with respect to x, z_{yy} is the second derivative of z with respect to y, and z_{xy} is a mixed second derivative of z with respect to x and y.

Example 9.76

Find the first partial derivatives of $z = x^2 + 3xy + y^2$.

Solution

The partial derivatives are given by:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^2 + 3xy + y^2\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 2x + 3y.$$

Similarly,

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^2 + 3xy + y^2\right)$$

$$\Rightarrow \frac{\partial z}{\partial y} = 3x + 2y.$$

Example 9.77

Find the first partial derivatives of $z = \sin 3x \cos 4y$.

Solution

Given $z = \sin 3x \cos 4y$.

$$\Rightarrow \frac{\partial z}{\partial x} = 3\cos 3x \cos 4y \text{ and } \frac{\partial z}{\partial y} = -4\sin 3x \sin 4y.$$

Example 9.78

Find $\frac{\partial^2 z}{\partial x^2}$ if $z = e^{x^3 + y^2}$.

Solution

Given $z = e^{x^3+y^2}$.

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(e^{x^3+y^2}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = 3x^2 e^{x^3+y^2}.$$

Thus, $\dfrac{\partial^2 z}{\partial x^2} = \dfrac{\partial}{\partial x}\left(3x^2 e^{x^3+y^2}\right)$

$$\Rightarrow \frac{\partial^2 z}{\partial x^2} = \left(9x^4 + 6x\right)e^{x^2+y^3}.$$

Exercise 9.19

1. If $p = \dfrac{kT}{V}$, find the first partial derivatives of p, where k is a constant.
2. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if $z = x^2 y^3$.
3. Find the first partial derivatives of the function $z = \dfrac{x}{y^2} - \dfrac{y}{x^2}$ with respect to x and y.
4. Find $\dfrac{\partial z}{\partial x}$ for each of the following functions:

   (a) $z = xy\cos(xy)$

   (b) $z = \dfrac{x-y}{x+y}$

   (c) $z = (3x+y)^2$
5. For each of the following functions, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

   (a) $f(x, y) = x^3 y + x^2 y^2$

   (b) $f(x, y) = x^2 + 3xy$

   (c) $f(x, y) = x^4 y^3 + 8x^2 y + y^4 + 5x$

   (d) $f(x, y) = \dfrac{x^2}{y} + \dfrac{y^2}{x}$

6. Given $z = x^2 \sin(x-2y)$, find:

   (a) $\frac{\partial^2 z}{\partial x^2}$ (c) $\frac{\partial^2 z}{\partial x \partial y}$

   (b) $\frac{\partial^2 z}{\partial y^2}$ (d) $\frac{\partial^2 z}{\partial y \partial x}$

7. Find $\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$ when $z = \cos^{-1}\left(\frac{x}{y}\right)$.

8. If $z = \sqrt{\frac{3x}{y}}$, show that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, and hence evaluate $\frac{\partial^2 z}{\partial x^2}$ at $\left(\frac{1}{2}, 3\right)$.

9. Verify that if $z = \frac{x}{y} \ln y$, then $\frac{\partial z}{\partial y} = x \frac{\partial^2 z}{\partial y \partial x}$. Find the value of $\frac{\partial^2 z}{\partial y^2}$ when $x = -3$ and $y = 1$.

10. The time of oscillation t of a pendulum is given by $t = 2\pi\sqrt{\frac{l}{g}}$, where l is the length and g is the acceleration due to gravity. Determine:

    (a) $\frac{\partial t}{\partial l}$ (b) $\frac{\partial t}{\partial g}$

Chapter summary

1. Differentiation from first principles is done using the formula, $\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, provided this limit exists.
2. The derivative of a polynomial term is given by $\frac{d}{dx}\left(x^n\right) = nx^{n-1}$.
3. The derivative of the sum or difference of two or more polynomial functions is given by:
   $$\frac{d}{dx}\left[f(x) \pm g(x) \pm h(x)\right] = \frac{d}{dx}\left[f(x)\right] \pm \frac{d}{dx}\left[g(x)\right] \pm \frac{d}{dx}\left[h(x)\right]$$
4. If y = u(x) v (x), then the product rule of differentiation is given by $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$ and the quotient rule for differentiation of $y = \frac{u(x)}{v(x)}$ is given by, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

5. To find the nature of turning point, using the second derivative:
   (a) If $f''(x) > 0$, then the turning point is minimum
   (b) If $f''(x) < 0$, then the turning point is maximum
   (c) If $f''(x) = 0$, then the point is an inflexion, minimum or maximum point.

6. Taylor's theorem is given by;

$$f(x) = f(a+h) = f(a) + f'(a)h + \frac{f''(a)}{2!}h^2 + \frac{f'''(a)}{3!}h^3 + \frac{f^{iv}(a)}{4!}h^4 + \cdots + \frac{f^n(a)h^n}{n!}$$

7. Maclaurin's theorem is a special case of Taylor's theorem and it is given by;

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{iv}(0)}{4!}x^4 + \cdots$$

8. The first partial derivatives of $z = f(x, y)$ are given by;

$$z_x = f_x(x, y) = \frac{\partial z}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$z_y = f_y(x, y) = \frac{\partial z}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Revision exercise 9

1. Differentiate each of the following functions from first principles:

   (a) $\frac{\sin x}{x}$ (b) $\cos^2 x$ (c) $x \sin x$ (d) $\cos(5x+2)$

2. Given that $y = \tan\left(x(2x+3)^{-1}\right)$, show that $y' = \left((2x+3)\sec^2 x - 2\tan x\right)(2x+3)^{-1}$.

3. Find the derivative of $5\sin x - 4\cos x + 8$.

4. Differentiate $(5 - 4\cos x)(1 - 2\tan x)$.

5. Find $\frac{dy}{dx}$ of each of the following:

   (a) $2xy^3 + 3x^2y = 7$

   (b) $\tan^{-1}\left(\frac{y}{x} + 3\right) = \ln\left(x^2 + y^2 + 2\right)$

   (c) $y^2 \ln x - 3 = 1 - x^2 \ln x$

6. Differentiate each of the following functions:

(a) xe^x (b) $\frac{x}{e^x}$ (c) $e^x\left(x^2-3\right)$ (d) $\frac{\ln x}{x^3}$ (e) $5e^{1-x}$

7. Find the derivative of each of the following functions:

(a) x^2e^{4x} (b) $\sqrt{e^x-x}$ (c) $(x+2)^2\ln 4x^2$ (d) $\frac{1+4x^2}{1+x^2}$

8. If x^5 and higher terms are neglected, show that

$\log_e\left(x+\sqrt{1+x^2}\right)=x-\frac{1}{6}x^3$.

9. Expand each of the following:

(a) $(2+x)^2e^{-x}$ in ascending powers of x as far as the term in x^3.

(a) $\log_e y$ at $y=1$ in ascending powers of $(y-1)$ as far as the term in $(y-1)^3$.

10. If $y=e^x\tan x$, show that $\frac{d^2y}{dx^2}-2(1+\tan x)\frac{dy}{dx}+(1+2\tan x)y=0$.

11. Given $y=e^{4x}\cos 3x$, show that $\frac{d^2y}{dx^2}-8\frac{dy}{dx}+25y=0$.

12. If $x>1$, show that $\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right)=\frac{1}{x}+\frac{1}{3x^3}x+\frac{1}{5x^5}+\cdots$, hence find $\log_e 2$.

13. Use Taylor's theorem to find the expansion of each of the following:

(a) $\sin(x+h)$

(b) e^x

(c) $\sin\left(\frac{\pi}{6}+h\right)$ in ascending powers of h as far as the term in h^4

(d) $\tan\left(\frac{\pi}{3}+h\right)$ in ascending powers of h as far as the term in h^4

(e) $\tan\left(\frac{\pi}{6}+x\right)$ in ascending powers of x as far as the term in x^4

14. Use Taylor's theorem to express $e^{\sin x}$ in ascending powers of x as far as the x^4 term.

15. Expand $\frac{\cos x}{\sqrt{1-x}}$ in ascending powers of x as far as the x^4 term by considering the product of the expansion of $\cos x$ and $(1-x)^{-\frac{1}{2}}$.

16. Given that $f(x) = \ln(3+x)$:

    (a) Find $f(0)$ and $f'(0)$, hence show that $f''(0) = -\frac{1}{9}$.

    (b) Write down the first three terms of the Maclaurin's series for $f(x)$, where $-3 < x \leq 3$.

17. Find the coordinates of the turning points of the curve $y = x - e^x$ and sketch its graph.

18. The equation of a curve is given by $y = 2x^2 - 3x - 2$, find:
    (a) the gradient at the point where $x = 0$.
    (b) the coordinates of the points where the curve crosses the x-axis.
    (c) the gradient at each of the points in (b).

19. Oil is dropping onto a surface at the rate of $\frac{1}{10}\pi\,\text{cm}^3/\text{s}$ and forms a circular film which may be considered to have a uniform depth of 0.1 cm. Find the rate at which the radius of the circular film is increasing when the radius is 7 cm.

20. The radius of a circle is to be measured and its area calculated. If the radius is measured to 0.001 m and the area must be accurate to $0.1\ \text{m}^2$, find the maximum radius.

21. Suppose the volume of a sphere is increasing at a constant rate of $7\ \text{cm}^3/\text{s}$.
    (a) Find in terms of π the rate at which the radius of the sphere is increasing at the instant when the radius is 5 cm.
    (b) Show that at any instant, the rate of increase of the surface area of the sphere is inversely proportional to the radius of the sphere.

22. Show that $\sin\left(\frac{1}{6}\pi + x\right) = \frac{1}{2} + \frac{1}{2}\sqrt{3x - \frac{1}{4}x^2}$, where x is measured in radians.

23. Show that $\frac{1}{2}\log_e\left(\frac{x+1}{x-1}\right) = \frac{1}{x} + \frac{1}{3x^2}x + \frac{1}{5x^5} + \cdots$ and use it to calculate $\log_e 2$ to two decimal places.

24. Differentiate each of the following functions:

    (a) $\ln\left(\frac{2+\cos x}{3-\sin x}\right)$

    (b) $\arctan\left(\frac{1-x}{1+x}\right)$

    (c) $\ln(\sec 2x + \tan 2x)$

    (d) $\frac{1}{x^2}\sqrt{1+x^3}$

25. Let $g(x) = \tan x$.

    (a) Show that $g'''(x) = 2\sec^4 x + 4\tan^2 x \sec^2 x$.

    (b) Obtain the Maclaurin expansion for $g(x)$ up to the term in x^3.

26. Give the first three non-zero terms of Maclaurin's series for each of the following functions:

    (a) $\cos 3x$ (b) $\sin 2x$

27. Given $f(x) = e^{2x}$, obtain the Maclaurin expansion of $f(x)$ up to the term in x^3.

28. Find the partial derivatives of each of the following:

    (a) $z = \sin(2x + 3y)$ (b) $z = \dfrac{1}{x^2 + y^2}$ (c) $z = x^3 + y^2 + xy^3$

29. Find the partial derivatives of z with respect to the independent variables x and y for each of the following:

    (a) $x^2 + y^2 + z^2 = 25$ (d) $z = \dfrac{x^3}{y^2} - \dfrac{y^3}{x}$

    (b) $z = e^{x^2 + xy}$ (e) $z = 2x^2 - 6xy + y^2$

    (c) $z = x^3 + 2xy + y^4$ (f) $z = xy$

30. A waste paper basket consists of open circular top. If the volume of the basket is to be 20 cm^3, find the radius of its base when the material used is minimum.

31. A wire of length 100 cm is to be cut into two parts. One portion is bent into the shape of a circle and the other into a square and then fixed on a tray to occupy a minimum area. Find the length of the two parts.

32. A solid is formed by a cylinder of radius r and height h, together with two hemispheres of radius r attached at each end. If the volume v of the solid is constant, but r is increasing at the rate of $\dfrac{1}{2\pi}$ metres per minute, how fast must h be changing when r and h are both 10 metres.

33. The position function of a stone thrown from a cliff is given by $s(t) = 10t - 16t^2$ metres (below the cliff) after t seconds. Find the average velocity of the stone between $t = 1$ second and $t = 5$ seconds.

34. A particle moving in a straight line has its displacement governed by the equation $s(t) = -2t^2 + 10t - 1,\ t \geq 0$, where s is in metres and t in seconds. Find the velocity and acceleration of the particle at time t.

35. Given $z = x^3y^2 - yx^{-2} + y^{-1}$, find each of the following:

    (a) $\dfrac{\partial z}{\partial x}$, when $x = 1$ and $y = 2$

    (b) $\dfrac{\partial z}{\partial x}$ at the point $(-1, -1)$

36. If $z = \dfrac{x - y}{x + y}$, find each of the following:

    (a) $\dfrac{\partial^2 z}{\partial x \partial y}$ (b) $\dfrac{\partial^2 z}{\partial y \partial x}$

37. Prove that the power series for
$$\cos 2\theta = 1 - 2\theta^2 + \frac{2}{3}\theta^4 - \frac{4}{45}\theta^6 + \cdots$$

38. Use Maclaurin's series to prove that the expansion of $(3 + 2t)^4 = 81 + 216t^2 + 96t^3 + 16t^4$.

39. The time, T of a swing of a pendulum is given by $T = k\sqrt{l}$, where k is a constant. Determine the percentage change in time of the swing if the length l of the pendulum changes from 32.1 cm to 32.0 cm.

40. Determine the value of $\dfrac{d^2y}{dx^2}$, correct to 4 significant figures, at $\theta = \dfrac{\pi}{6}$ radians for the cardioid formed by $x = 5(2\theta - \cos 2\theta)$ and $y = 5(2\sin\theta - \sin 2\theta)$.

Chapter Ten

Integration

Introduction

Integration is one of the main operations in calculus and it is an important concept in Mathematics. Integration is the inverse process of differentiation. In this chapter, you will learn about inverse process of differentiation, integration of functions, and applications of integration. The competencies developed has many real-life applications such as determining the total cost and total revenue of goods produced, finding displacement, velocity and acceleration of moving bodies, moment of inertia of vehicles, and the rate of a chemical reaction. It is also used in building constructions, in graphical representations where three-dimensional models are demonstrated in analysis of the spread of infectious diseases, among many other applications.

Inverse process of differentiation

Suppose $f(x)$ is a known function, then its derivative $f'(x)$ with respect to x can be obtained. The inverse process gives the original function $f(x)$ from that derivative. This inverse process of differentiation is called integration. For instance, the derivative of $f(x)=x^3$ with respect to x gives $f'(x)=3x^2$.

Therefore, the integral of $3x^2$ is x^3.

Anti-derivative and integral notation

The opposite of the derivative of a function is called the anti-derivative. The anti-derivative of a function is the integration which is represented by the integral notation, $\int$. When this notation includes a function $f(x)$ it is written as $\int f(x)\,dx$ and read as, "the integral of $f(x)$ with respect to the variable x". In this case, the function $f(x)$ is called an integrand, and the change in a variable is represented by a symbol dx.

Note that, the derivative of a constant is always zero. This implies that, differentiation makes the function lose the constant. For instance, the derivatives of $y=x^3+1$ and $y=x^3+2$ are both equal to $3x^2$. Thus, the integration of $3x^2$ gives x^3. This means that, the reverse of $3x^2$ cannot give back the functions, $y=x^3+1$ and $y=x^3+2$. This is because during differentiation the constant becomes 0 as it does not change. Therefore, to show the presence of a constant, an arbitrary constant is added after the completion of the integration process. That is, $\int 3x^2 dx = x^3 + c,$ where c is an arbitrary constant.

Example 10.1

Find the anti-derivative of each of the following:

(a) $\frac{d}{dx}\left(x^4\right) = 4x^3$ (b) $\frac{d}{dx}\left(2x^3\right) = 6x^2$

Solution

(a) If $\frac{d}{dx}\left(x^4\right) = 4x^3$, it means that,

$$d\left(x^4\right) = 4\left(x^3\right)dx$$

$$\Rightarrow 4\left(x^3\right)dx = d\left(x^4\right)$$

Introduce the integral sign on both sides of the equation to obtain:

$$\int 4x^3 dx = \int d\left(x^4\right)$$

$$= x^4 + c$$

Therefore, $\int 4x^3 dx = x^4 + c.$

(b) $\frac{d}{dx}\left(2x^3\right) = 6x^2$, it means that,

$$6(x^2)dx = d(2x^3)$$

Introduce the integral sign on both sides of the equation to obtain

$$\int 6x^2 dx = \int d\left(2x^3\right)$$

$$= 2x^3 + c$$

Therefore, $\int 6x^2 dx = 2x^3 + c.$

Table 10.1 shows some useful standard integrals.

Table 10.1: *Some useful standard integrals*

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$; where $n \neq -1$	$\int \cot x\, dx = \ln\lvert\sin x\rvert + c$
$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$	$\int \sec^2 x\, dx = \tan x + c$
$\int e^x dx = e^x + c$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
$\int \sin x\, dx = -\cos x + c$	$\int \sec x\, dx = \ln\lvert\sec x + \tan x\rvert + c$
$\int \cos x\, dx = \sin x + c$	$\int \operatorname{cosec} x\, dx = \ln\lvert\operatorname{cosec} x - \cot x\rvert + c$
$\int \tan x\, dx = \ln\lvert\sec x\rvert + c$	$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

Types of integrals

There are two types of integrals, namely; indefinite and definite integrals.

Indefinite integrals

An indefinite integral of a function $f(x)$ is a differentiable function $F(x)$ such that its derivative gives the original function $f(x)$. An indefinite integral of function $f(x)$ is expressed as $\int f(x)\,dx = F(x) + c$, where c is an arbitrary constant of integration.

Example 10.2

Find $\int 2x^2 dx$.

Solution

Given $\int 2x^2 dx$.

$$\Rightarrow \int 2x^2 dx = 2\int x^2 dx$$

$$= 2\left(\frac{x^3}{3}\right) + c.$$

Therefore, $\int 2x^2\, dx = \frac{2x^3}{3} + c$.

Example 10.3

Find $\int 12\cos\theta\, d\theta$.

Solution

Given $\int 12\cos\theta\, d\theta$.

$$\Rightarrow \int 12\cos\theta\, d\theta = 12\int \cos\theta\, d\theta$$
$$= 12\sin\theta + c$$

Therefore, $\int 12\cos\theta\, d\theta = 12\sin\theta + c$.

Exercise 10.1

1. Find the anti derivative of each of the following:
   (a) $\frac{d}{dx}\left(x^6\right) = 6x^5$
   (b) $\frac{d}{dx}\left(2x^{\frac{1}{2}}\right) = x^{\frac{-1}{2}}$
   (c) $\frac{d}{dx}\left(3x^{-7}\right) = -21x^{-8}$
   (d) $\frac{d}{dx}\left(\sqrt{x}\right) = \frac{1}{2\sqrt{x}}$
   (e) $\frac{d}{dx}\left(\sqrt[3]{x^2}\right) = \frac{2}{3}x^{\frac{-1}{3}}$
   (f) $\frac{d}{dx}\left(\frac{3}{2}x^{-2}\right) = -3x^{-3}$
2. Determine each of the following:
   (a) $\int x^2 dx$
   (b) $\int 5e^x dx$
   (c) $\int \frac{1}{4x} dx$
   (d) $\int 2\cos\theta d\theta$
   (e) $\int \frac{1}{2}\sin\theta d\theta$
   (f) $\int 4\tan\theta d\theta$
3. Find the integral of each of the following functions:
   (a) $f(x) = \frac{3}{x^2}$
   (b) $g(x) = \frac{1}{\sqrt{x}}$
   (c) $f(t) = 3\sqrt[4]{t}$
   (d) $h(x) = -2e^x$
   (e) $f(\theta) = 10\sin\theta$
   (f) $f(x) = 2\sqrt[5]{x^3}$

Integration of simple expressions

Expressions which involve mathematical operations on simple polynomials, fractions, and exponential functions are referred to as simple expressions. Simple expressions can be integrated by treating each term separately using the integration rules. The rules of mathematical operations such as scalar multiplication, additional, and subtraction of integral functions are also used, that is;

(i) $\int af(x)dx = a\int f(x)dx$

(ii) $\int \left(af(x) \pm bg(x)\right)dx$
$= a\int f(x)\,dx \pm b\int g(x)dx$, where a and b are constants.

Note that;

(i) $\int 0dx = c$

(ii) $\int adx = ax + b$, where a and b are constants.

Example 10.4

Integrate $x^3 + 2x - 1$ with respect to x.

Solution

Required to find $\int (x^3 + 2x - 1)\, dx$.

$$\Rightarrow \int (x^3 + 2x - 1)\, dx = \int x^3 dx + 2\int x\, dx - 1\int x^0 dx$$

$$= \frac{x^{3+1}}{3+1} + \frac{2x^{1+1}}{1+1} - 1\frac{x^{0+1}}{0+1} + c$$

$$= \frac{x^4}{4} + \frac{2x^2}{2} - x + c$$

Therefore, $\int \left(x^3 + 2x - 1\right) dx = \frac{x^4}{4} + x^2 - x + c.$

Example 10.5

Integrate $\sqrt{x^3} + \frac{2}{x^2}$ with respect to x.

Solution

Given $\int \left(\sqrt{x^3} + \frac{2}{x^2}\right) dx.$

$$\Rightarrow \int \left(\sqrt{x^3} + \frac{2}{x^2}\right) dx = \int \sqrt{x^3}\, dx + \int \frac{2}{x^2} dx$$

$$= \int x^{\frac{3}{2}} dx + \int 2x^{-2} dx$$

$$= \frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} + \frac{2x^{-2+1}}{-2+1} + c$$

$$= \frac{2x^{\frac{5}{2}}}{5} - 2x^{-1} + c$$

$$= \frac{2x^{\frac{5}{2}}}{5} - \frac{2}{x} + c$$

Therefore, $\int \left(\sqrt{x^3} + \frac{2}{x^2}\right) dx = \frac{2\sqrt{x^5}}{5} - \frac{2}{x} + c.$

Example 10.6

Integrate $3 + \frac{5x^3 - 2x}{3x^4}$ with respect to x.

Solution

Given $\int\left(3+\frac{5x^3-2x}{3x^4}\right)dx.$

$$\Rightarrow \int\left(3+\frac{5x^3-2x}{3x^4}\right)dx = \int 3dx + \int\frac{5x^3-2x}{3x^4}dx$$

$$= 3\int x^0 dx + \int\left(\frac{5x^3}{3x^4}-\frac{2x}{3x^4}\right)dx$$

$$= 3x + \frac{5}{3}\int\frac{1}{x}dx - \frac{2}{3}\int x^{-3}dx$$

$$= 3x + \frac{5}{3}\ln|x| + \frac{1}{3}x^{-2} + c;\ \text{since}\ \frac{d}{dx}(\ln x) = \frac{1}{x}.$$

Therefore, $\int\left(3+\frac{5x^3-2x}{3x^4}\right)dx = 3x + \frac{5}{3}\ln|x| + \frac{1}{3}x^{-2} + c.$

Example 10.7

Find $\int\left(\frac{3}{2}\cos\theta + \frac{5}{3}\sin\theta\right)d\theta.$

Solution

Given $\int\left(\frac{3}{2}\cos\theta + \frac{5}{3}\sin\theta\right)d\theta.$

$$\Rightarrow \int\left(\frac{3}{2}\cos\theta + \frac{5}{3}\sin\theta\right)d\theta = \int\frac{3}{2}\cos\theta d\theta + \int\frac{5}{3}\sin\theta d\theta$$

$$= \frac{3}{2}\int\cos\theta d\theta + \frac{5}{3}\int\sin\theta d\theta$$

$$= \frac{3}{2}\sin\theta - \frac{5}{3}\cos\theta + c$$

Therefore, $\int\left(\frac{3}{2}\cos\theta + \frac{5}{3}\sin\theta\right)d\theta = \frac{3}{2}\sin\theta - \frac{5}{3}\cos\theta + c.$

Exercise 10.2

Evaluate each the following integrals:

1. $\int\left(5x^3+8x^2-3x+5\right)dx$

2. $\int(-6x^3+9x^2+4x-3)\,dx$

3. $\int\left(x^{\frac{3}{2}}+2x+3\right)dx$

4. $\int\left(12t^{\frac{3}{4}}-9t^{\frac{5}{2}}\right)dt$

5. $\int\left(\frac{1}{3}x^3+5x^2-7x+2\right)dx$

6. $\int\left(t^{-3}-2t^4+\frac{7}{t}+1\right)dt$

7. $\int\frac{(1-x)^3}{\sqrt{x}}\,dx$

8. $\int\left(\frac{3}{x^2}+\frac{\cos x}{2}\right)dx$

9. $\int\left(\sqrt{x}+\frac{1}{3\sqrt{x}}\right)dx$

10. $\int\left(\frac{8}{x}-\frac{5}{x^2}+\frac{6}{x^3}\right)dx$

11. $\int\frac{t^2+5t^6-4}{t^3}\,dt$

12. $\int\left(\frac{1}{x}+\frac{\sin x}{5}-\frac{\cos x}{3}\right)dx$

13. $\int(6-2x)\,dx$

14. $\int\left(x^2+4x+1\right)dx$

15. $\int\left(x^4-2x-3\sqrt{x}\right)dx$

16. $\int\left(4t^3-3t^2-2t+1\right)dt$

17. $\int\left(7e^x+4e^x\right)dx$

18. $\int\frac{y^3+2}{y^2}\,dy$

19. $\int(2\cos\theta+2\sin\theta)\,d\theta$

20. $\int\frac{4\sin x}{3\tan x}\,dx$

21. $\int\left(4x^3+7x^2-\frac{3}{2}\cos x\right)dx$

22. $\int\frac{x^4-x^3+\sqrt[3]{x}-1}{x^2}\,dx$

23. $\int\frac{t^2+5t^6-4}{t^3}\,dt$

24. $\int\left(\frac{1}{r^2}-\frac{1}{r^4}+3r\right)dr$

Techniques of integration

There are several techniques of integration depending on the given function. The most useful techniques of integration include integration by inspection, by substitution, by parts, and by partial fractions.

Integration by substitution method

This is a method which is used in reversing the chain rule. It is sometimes called chain rule backward. The following are steps to be followed when integrating by substitution method:

Step 1: Choose a new variable for which the given function is to be reduced.

Step 2: Solve for dx of the given integral, where the function is integrated with respect to x.

Step 3: Rewrite the integral in terms of the new variable.

Step 4: Integrate the resulting function with respect to the new variable.

Step 5: Substitute back the value of the new variable to get the function in terms of x.

Consider the following cases where the integration by substitution method is applicable:

(a) ***Integrals of the form*** $\int (ax \pm b)^n \, dx$, where "a" and "b" are constants.

Let $u = ax \pm b$, then $du = adx$.

$$\Rightarrow dx = \frac{du}{a}.$$

$$\text{Thus, } \int (ax \pm b)^n \, dx = \int u^n \times \frac{du}{a}$$

$$= \frac{1}{a}\int u^n du$$

$$= \frac{1}{a}\left(\frac{u^{n+1}}{n+1}\right) + c$$

But $u = ax \pm b$.

Therefore,

$$\int (ax \pm b)^n \, dx = \frac{1}{a}\left(\frac{(ax \pm b)^{n+1}}{n+1}\right) + c.$$

(b) ***Integrals of the form*** $\int \sqrt[m]{(ax \pm b)^n}\, dx$, where "$a$" and "$b$" are constants.

Let $u = ax \pm b$, then $du = adx$.

$$\Rightarrow dx = \frac{du}{a}.$$

$$\text{Thus, } \int \sqrt[m]{(ax \pm b)^n}\, dx = \int (u)^{\frac{n}{m}} \times \frac{du}{a}$$

$$= \frac{1}{a}\int u^{\frac{n}{m}} du$$

$$= \frac{1}{a}\left(\frac{u^{\frac{n}{m}+1}}{\frac{n}{m}+1}\right) + c$$

But $u = ax \pm b$.

Therefore,

$$\int \sqrt[m]{(ax \pm b)^n}\, dx = \frac{1}{a}\left(\frac{(ax \pm b)^{\frac{n}{m}+1}}{\frac{n}{m}+1}\right) + c.$$

Example 10.8

Find each of the following:

(a) $\int (2x+3)^3 \, dx$

(b) $\int \sqrt[3]{(1-2x)^7}\, dx$

Solution

(a) Given $\int (2x+3)^3 \, dx$.

Let $u = 2x+3$, then $du = 2\,dx$.

$$\Rightarrow dx = \frac{du}{2}.$$

$$\int (2x+3)^3 \, dx = \int u^3 \times \frac{du}{2}$$

$$= \frac{1}{2}\int u^3 du$$

$$= \frac{1}{2}\left(\frac{u^{3+1}}{3+1}\right) + c = \frac{1}{2}\left(\frac{u^4}{4}\right) + c$$

But $u = 2x+3$.

Therefore,

$$\int (2x+3)^3 \, dx = \frac{1}{8}(2x+3)^4 + c.$$

(b) Given $\int \sqrt[3]{(1-2x)^7}\, dx$.

Let $u = 1-2x$, then $du = -2dx$

$$\Rightarrow \ dx = -\frac{du}{2} \text{ which gives}$$

$$\int \sqrt[3]{(1-2x)^7}\,dx = \int (u)^{\frac{7}{3}} \times -\frac{du}{2}$$

$$= -\frac{1}{2}\int u^{\frac{7}{3}}du$$

$$= -\frac{1}{2}\left(\frac{u^{\frac{10}{3}}}{\frac{10}{3}}\right) = -\frac{3}{20}u^{\frac{10}{3}} + c$$

But $u = 1-2x$.

Therefore,

$$\int \sqrt[3]{(1-2x)^7}\,dx = -\frac{3}{20}(1-2x)^{\frac{10}{3}} + c.$$

(c) *Integrals of the form* $\int \frac{dx}{ax \pm b}$,

where "a" and "b" are constants.
Let $u = ax \pm b$, then $du = adx$.

$$\Rightarrow \ dx = \frac{du}{a}.$$

Thus, $\int \frac{dx}{ax \pm b} = \int \frac{1}{u} \times \frac{du}{a}$

$$= \frac{1}{a}\int \frac{du}{u}$$

$$= \frac{1}{a}\ln|u| + c$$

But $u = ax \pm b$.

Therefore, $\int \frac{dx}{ax \pm b} = \frac{1}{a}\ln|ax \pm b| + c.$

Example 10.9

Find each of the following:

(a) $\int \frac{4dx}{2x+3}$ (b) $\int \frac{dx}{5-7x}$

Solution

(a) Given $\int \frac{4\,dx}{2x+3}$.

Let $u = 2x+3$, then $du = 2dx$.

$$\Rightarrow \ dx = \frac{du}{2}.$$

$$\Rightarrow \int \frac{4dx}{2x+3} = 4\int \frac{1}{u} \times \frac{du}{2}$$

$$= \frac{4}{2}\int \frac{du}{u}$$

$$= 2\ln|u| + c.$$

But $u = 2x+3$.

Therefore, $\int \frac{4\,dx}{2x+3} = 2\ln|2x+3| + c.$

(b) Given $\int \frac{dx}{5-7x}$.

Let $u = 5-7x$, then $du = -7dx$.

$$\Rightarrow \ dx = -\frac{du}{7}.$$

Thus, $\int \frac{dx}{5-7x} = \int \frac{1}{u} \times \left(-\frac{du}{7}\right)$

$$= -\frac{1}{7}\int \frac{du}{u}$$

$$= -\frac{1}{7}\ln|u| + c$$

But $u = 5-7x$.
Therefore,

$$\int \frac{dx}{5-7x} = -\frac{1}{7}\ln|5-7x| + c.$$

(d) *Integrals of the form* $\int \sin(ax \pm b)\,dx$
where "a" and "b" are constants. Let
$u = ax \pm b$, then $du = adx$,

$$\Rightarrow \ dx = \frac{du}{a}.$$

$$\text{Thus,} \int \sin(ax \pm b)\, dx = \int \sin u \times \frac{du}{a}$$

$$= \frac{1}{a}\int \sin u du$$

$$= -\frac{1}{a}\cos u + c$$

But $u = ax \pm b$.

Therefore,

$$\int \sin(ax \pm b)\, dx = -\frac{1}{a}\cos(ax \pm b) + c.$$

Example 10.10

Find $\int 5\sin(8x+2)\, dx$.

Solution

Given $\int 5\sin(8x+2)dx$.

Let $u = 8x+2$, then $du = 8dx$.

$$\Rightarrow \ dx = \frac{du}{8}.$$

$$\Rightarrow \int 5\sin(8x+2)\, dx = \int 5\sin u \times \frac{du}{8}$$

$$= \frac{5}{8}\int \sin u du$$

$$= -\frac{5}{8}\cos u + c$$

But $u = 8x+2$.

Therefore,

$$\int 5\sin(8x+2)\, dx = -\frac{5}{8}\cos(8x+2) + c.$$

(e) *Integrals of the form* $\int e^{ax \pm b} dx$,

where "a" and "b" are constants.

Let $u = ax \pm b$, then $du = adx$.

Thus, $dx = \dfrac{du}{a}$.

$$\Rightarrow \int e^{ax \pm b} dx = \int e^u \times \frac{du}{a}$$

$$= \frac{1}{a}\int e^u du$$

$$= \frac{1}{a}e^u + c.$$

But $u = ax \pm b$.

Therefore, $\displaystyle\int e^{ax \pm b} dx = \frac{1}{a}e^{ax \pm b} + c.$

Example 10.11

Determine $\int 5e^{5x+2} dx$.

Solution

Given $\int 5e^{5x+2} dx$.

Let $u = 5x+2$, then $du = 5dx$.

$$\Rightarrow \ dx = \frac{du}{5}.$$

$$\Rightarrow \int 5e^{5x+2} dx = \int 5e^u \times \frac{du}{5}$$

$$= \frac{5}{5}\int e^u du$$

$$= e^u + c$$

But $u = 5x+2$.

Therefore, $\displaystyle\int 5e^{5x+2} dx = e^{5x+2} + c.$

Exercise 10.3

Find each of the following:

1. $\int (6x-9)^8 dx$
2. $\int (1-2x)^{\frac{3}{2}} dx$

3. $\int (3t-1)^{-3}\,dt$
4. $\int 3(1-x)^{-1}\,dx$
5. $\int 4e^{\frac{x}{3}}\,dx$
6. $\int \frac{1}{4-3t}\,dt$
7. $\int 6\cos(1-3x)\,dx$
8. $\int \frac{dx}{5-7x}$
9. $\int \tan(2\theta+1)\,d\theta$
10. $\int \frac{dx}{2x-1}$
11. $\int \frac{2}{7}e^{3-4x}\,dx$
12. $\int 5\sin\left(\frac{1}{2}x-1\right)dx$
13. $\int e^{2x+2}\,dx$
14. $\int \cos(ax\pm b)\,dx$
15. $\int \tan(ax\pm b)\,dx$

Integration by inspection method

This method is applicable when integrating functions such as polynomials, trigonometric, logarithmic, and exponentials. Consider the following cases where integration by inspection method is applicable.

(a) ***Integrals of the form*** $\int f(x)g(x)dx$

Let $u = g(x)$ such that

$\frac{du}{dx} = f(x) \Rightarrow du = f(x)dx$

Thus, $dx = \frac{du}{f(x)}$.

$$\Rightarrow \int f(x)g(x)dx = \int f(x)u \times \frac{du}{f(x)}$$

$$= \int u\,du$$

$$= \frac{1}{2}u^2 + c$$

But $u = g(x)$.

Therefore, $\int f(x)\,g(x)dx = \frac{1}{2}[g(x)]^2 + c$.

(b) ***Integrals of the form*** $\int \frac{f(x)}{g(x)}dx$

Let $u = g(x)$ such that

$\frac{du}{dx} = f(x) \Rightarrow du = f(x)dx$.

Thus, $dx = \frac{du}{f(x)}$.

$$\Rightarrow \int \frac{f(x)}{g(x)}dx = \int \frac{f(x)}{u} \times \frac{du}{f(x)}$$

$$= \int \frac{du}{u}$$

$$= \ln|u| + c$$

But $u = g(x)$.

Therefore, $\int \frac{f(x)}{g(x)}dx = \ln|g(x)| + c$.

Example 10.12

Find each of the following:

(a) $\int (2x-1)(x^2-x+1)\,dx$

(b) $\int \frac{2x}{1+x^2}dx$

Solution

(a) Given $\int (2x-1)(x^2-x+1)\,dx$.

By inspection, it can be seen that the derivative of x^2-x+1 is equal to $2x-1$.

Now, let $u=x^2-x+1$, then $\frac{du}{dx}=2x-1 \Rightarrow du=(2x-1)dx$.

Substitute $x^2-x+1=u$ and $dx=\frac{du}{(2x-1)}$ into the given integral as follows,

$$\int (2x-1)(x^2-x+1)\,dx = \int (2x-1)u \times \frac{du}{(2x-1)}$$
$$= \int u\,du$$
$$= \frac{u^2}{2}+c$$

But $u=x^2-x+1$.

Therefore, $\int (2x-1)(x^2-x+1)\,dx = \frac{1}{2}(x^2-x+1)^2+c$.

(b) Given $\int \frac{2x}{1+x^2}dx$.

Let $u=1+x^2$, then $\frac{du}{dx}=2x \Rightarrow du=2xdx$.

Substitute $1+x^2=u$ and $dx=\frac{du}{2x}$ into the given integral as follows,

$$\int \frac{2x}{1+x^2}dx = \int \frac{2x}{u}\times\frac{du}{2x}$$
$$= \int \frac{du}{u}$$
$$= \ln|u|+c$$

But $u=1+x^2$.

Therefore, $\int \frac{2x}{1+x^2}dx = \ln\left|1+x^2\right|+c$.

Example 10.13

Find $\int xe^{x^2}dx$.

Solution

Given $\int xe^{x^2}dx$.

Let $u = x^2$, then

$\frac{du}{dx} = 2x \Leftrightarrow du = 2xdx$.

Substitute $x^2 = u$ and $dx = \frac{du}{2x}$ into the given integral as follows,

$$\int xe^{x^2}dx = \int xe^{u}\frac{du}{2x}$$

$$= \frac{1}{2}\int e^{u}\,du$$

$$= \frac{1}{2}e^{u} + c.$$

But $u = x^2$.

Thus, $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c.$

Therefore, $\int xe^{x^2}dx = \frac{1}{2}e^{x^2} + c.$

Exercise 10.4

Find each of the following:

1. $\int (x^3 + x^2 - 3)(3x^2 + 2x)\,dx$
2. $\int e^{-t}(1 + e^{-t})dt$
3. $\int 8t^2(3t^3 - 1)dt$
4. $\int \frac{e^x}{1+e^x}dx$
5. $\int \frac{1+\ln x}{x\ln x}dx$
6. $-\int \sin\theta\cos\theta d\theta$
7. $\int \frac{1+\tan x}{1-\tan x}dx$
8. $\int x^{\frac{3}{2}}\left(\frac{3}{2}x^{\frac{1}{2}} + 3\right)dx$
9. $\int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}d\theta$
10. $\int \frac{1+\ln x}{x}dx$
11. $\int \frac{4+18x-18x^2}{3+4x+9x^2-6x^3}dx$
12. $\int \frac{\ln(x+2)}{2x+4}dx$
13. $\int \frac{x^{\frac{1}{2}}}{1+x^{\frac{3}{2}}}dx$
14. $\int \frac{\sqrt{x}}{1+\sqrt{x}}dx$
15. $\int x\sqrt{x^2-1}\,dx$
16. $\int (x+2)e^{x^2+4x}dx$
17. $\int xe^{x^2}\cos(3e^{x^2})dx$
18. $\int \frac{1-e^t}{1+e^t}dt$
19. $\int \frac{1}{x^2\ln x}\left(1+\frac{1}{\ln x}\right)dx$
20. $\int \frac{1}{x(\ln x)^2}dx$
21. $\int \frac{e^{\tan^{-1}x}}{1+x^2}dx$
22. $\int \frac{\sec^2(x+1)}{\tan(x+1)}dx$
23. Show that
$$\int \frac{2\sin 2x}{1+\cos x}dx = 4\ln(1+\cos x) - 4\cos x - 4 + c,$$
where c is an arbitrary constant.

24. Show that $\int \frac{e^{3x}}{1+e^{x}}dx = \frac{1}{2}e^{2x} - e^{x} + \ln(1+e^{x}) - \frac{3}{2} + c$, where c is an arbitrary constant.

Integration by parts

If $u(x)$ and $v(x)$ are any two differentiable functions, then the derivative of the product of the two functions is given by;

$$\frac{d}{dx}[u(x)v(x)] = v(x)\frac{d}{dx}u(x) + u(x)\frac{d}{dx}v(x).$$

Integrating both sides with respect to x gives,

$$\int \frac{d}{dx}[u(x)v(x)]\,dx = \int v(x)\frac{d}{dx}u(x)dx + \int u(x)\frac{d}{dx}v(x)\,dx.$$

Applying the definition of indefinite integral gives,

$$u(x)v(x) = \int v(x)\frac{d}{dx}u(x)dx + \int u(x)\frac{d}{dx}v(x)\,dx.$$

Thus, $\int u(x)\frac{d}{dx}v(x)\,dx = u(x)v(x) - \int v(x)\frac{d}{dx}u(x)dx$.

Since du and dv are differentials of a function of one variable, then the formula for the integration by parts is given by,

$$\int u(x)\frac{d}{dx}v(x)dx = u(x)v(x) - \int v(x)\frac{d}{dx}u(x)dx.$$

If $\frac{d}{dx}v(x) = v'(x)$ and $\frac{d}{dx}u(x) = u'(x)$, then the formula for integration by parts can also be written as

$$\int uv'dx = uv - \int vu'dx \text{ or simply } \int udv = uv - \int v\,du.$$

Generally, if $u(x)$ and $v(x)$ are any two differentiable functions of x, then $\int udv = uv - \int vdu$, which is a formula for integration by parts.

Note that, the integrand is in a form of a product of two functions, where the left part of the integrand is considered as a first function and its right part is considered as a second function. However, the first function is chosen in such a way that its derivative can easily be integrated. Generally, the order of preference can be chosen in such a way that the first function as the function which comes first in the word ILATE, where I stands for inverse trigonometric functions, L stands for logarithmic functions, A stands for algebraic functions, T stands for trigonometric functions, and E stands for exponential functions.

Consider the following cases where the method of integration by parts is applicable.

(a) ***Integrals of the form***

$\int x^n \sin(ax)\,dx$ or $\int x^n \cos(ax)\,dx$,

where "a" is a constants. In this case, let the polynomial part be the function u and the trigonometric part be dv.

Example 10.14

Find $\int x\sin x dx$.

Solution

Given $\int x\sin x dx$.

Let $u = x$, then $\frac{du}{dx} = 1$.

Also, let $\frac{dv}{dx} = \sin x \Rightarrow dv = \sin x dx$.

Introduce the integral sign on both sides and integrating gives,

$\int dv = \int \sin x dx \Rightarrow v = -\cos x$

Apply the formula for integrating by parts:

$\int u dv = uv - \int v du$,

$$\Rightarrow \int x\sin x\, dx = -x\cos x - \int(-\cos x)\times 1 dx$$

$$= -x\cos x + \int \cos x dx$$

$$= -x\cos x + \sin x + c$$

Therefore,

$\int x\sin x\, dx = -x\cos x + \sin x + c.$

Example 10.15

Find $\int x^2 \cos(4x-1)dx$.

Solution

Given $\int x^2 \cos(4x-1)dx$.

The integral can be obtained by integration by parts as follows:

Let $u = x^2$, then $du = 2xdx$.

Also,

let $\frac{dv}{dx} = \cos(4x-1) \Rightarrow dv = \cos(4x-1)dx$.

Introduce the integral sign on both sides and integrating gives,

$$\int dv = \int \cos(4x-1)dx$$

$$\Rightarrow v = \frac{1}{4}\sin(4x-1).$$

From the formula $\int u dv = uv - \int v du$, it implies that

$$\int x^2\cos(4x-1)dx$$

$$= \frac{x^2}{4}\sin(4x-1) - \int \frac{2x}{4}\sin(4x-1)dx$$

$$= \frac{x^2}{4}\sin(4x-1) - \frac{1}{2}\int x\sin(4x-1)dx$$

But, $\int x\sin(4x-1)dx$ requires further integration and it is integrated by parts as follows,

Let $u = x$, then $\frac{du}{dx} = 1$.

Also,

let $\frac{dv}{dx} = \sin(4x-1) \Rightarrow dv = \sin(4x-1)dx$

Introducing the integral sign on both sides and integrating gives,

$$\int dv = \int \sin(4x-1)dx$$

$$\Rightarrow v = -\frac{1}{4}\cos(4x-1)\cdot$$

Apply the formula for integrating by parts, that is,

$$\int x\sin(4x-1)dx = -\frac{x}{4}\cos(4x-1) - \int -\frac{1}{4}\cos(4x-1)dx$$

$$= -\frac{x}{4}\cos(4x-1) + \frac{1}{4}\int \cos(4x-1)dx$$

$$= -\frac{x}{4}\cos(4x-1) + \frac{1}{4}\left[\frac{1}{4}\sin(4x-1)\right]$$

Substitute this result in the integral to get,

$$\int x^2\cos(4x-1)dx = \frac{x^2}{4}\sin(4x-1) - \frac{1}{2}\left[-\frac{x}{4}\cos(4x-1) + \frac{1}{16}\sin(4x-1)\right] + c\,.$$

Therefore, $\int x^2\cos(4x-1)dx = \frac{x^2}{4}\sin(4x-1) + \frac{x}{8}\cos(4x-1) - \frac{1}{32}\sin(4x-1) + c.$

(b) ***Integrals of the form*** $\int x^n \ln ax\,dx$, where "a" is a constants

In this case, let the logarithmic part be the function u, and differentiate it. The polynomial part in the integral is dv which will be integrated.

Example 10.16

Determine each of the following:

(a) $\int x^n \ln x dx$

(b) $\int x^2 \ln 2x\, dx$

Solution

(a) Given $\int x^n \ln x dx$.

Let $u = \ln x$, then $\frac{du}{dx} = \frac{1}{x}$.

Also, let $\frac{dv}{dx} = x^n \Rightarrow dv = x^n dx$.

Integrate both sides:

$$\int dv = \int x^n dx \Leftrightarrow v = \frac{x^{n+1}}{n+1}$$

From $\int u dv = uv - \int v du$, it implies that

$$\int x^n \ln x dx = \frac{x^{n+1}}{n+1}\ln x - \int \frac{x^{n+1}}{n+1}\times\frac{1}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int \frac{x^{n+1}}{x}dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\int x^n dx$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{1}{n+1}\times\frac{x^{n+1}}{n+1} + c$$

$$= \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c$$

Therefore,

$$\int x^n \ln x dx = \frac{x^{n+1}}{n+1}\ln x - \frac{x^{n+1}}{(n+1)^2} + c.$$

(b) Given $\int x^2 \ln 2x dx$.

Let $u = \ln 2x$, then

$$\frac{du}{dx}=\frac{2}{2x}\Rightarrow du=\frac{1}{x}dx$$

Also,

let $\frac{dv}{dx}=x^2\Rightarrow dv=x^2dx$.

Integrate both sides:

$$\int dv=\int x^2dx\Rightarrow v=\frac{x^3}{3}.$$

From $\int udv=uv-\int vdu$, it gives

$$\int x^2\ln 2xdx=\frac{x^3}{3}\ln 2x-\int\frac{x^3}{3}\times\frac{1}{x}dx$$

$$=\frac{x^3}{3}\ln 2x-\frac{1}{3}\int x^2dx$$

$$=\frac{x^3}{3}\ln 2x-\frac{x^3}{9}+c.$$

Therefore,

$$\int x^2\ln 2xdx=\frac{x^3}{3}\ln 2x-\frac{x^3}{9}+c.$$

Example 10.17

Find $\int \ln xdx$.

Solution

Given $\int \ln xdx=\int 1\times\ln xdx$.

Let $u=\ln x\Rightarrow\frac{du}{dx}=\frac{1}{x}\Leftrightarrow du=\frac{1}{x}dx$

Also, let $\frac{dv}{dx}=1\Rightarrow dv=dx\Leftrightarrow v=x$.

Apply the formula $\int udv=uv-\int vdu$.

Now, the integral becomes,

$$\int\ln xdx=x\ln x-\int x\times\frac{1}{x}dx$$

$$=x\ln x-\int dx$$

$$=x\ln x-x+c$$

Therefore, $\int\ln xdx=x\ln x-x+c$.

(c) ***Integrals of the form***

$\int x^n\sin(ax)\,dx$ or $\int x^n\cos(ax)\,dx$

or $\int x^ne^{ax}\,dx$, where "a" is a constant.

A tabular method can be used to integrate problems involving repeated application of integration by parts.

Example 10.18

Find $\int x^3\sin(10x-19)\,dx$.

Solution

Given $\int x^3\sin(10x-19)\,dx$.

Let $u=x^3$, and $dv=\sin(10x-19)\,dx$

Prepare a table consisting of three columns and six rows as follows.

Alternate signs	u and its derivatives	v' and its anti-derivatives
$+$	x^3	$\sin(10x-19)$
$-$	$3x^2$	$-\frac{1}{10}\cos(10x-19)$
$+$	$6x$	$-\frac{1}{100}\sin(10x-19)$
$-$	6	$\frac{1}{1,000}\cos(10x-19)$
$+$	0	$\frac{1}{10,000}\sin(10x-19)$

Differentiate until the derivative of the function is zero.

The solution is given by adding the signed products of the diagonal entries.

That is, $\int x^3\sin(10x-19)\,dx = -\frac{x^3}{10}\cos(10x-19)+\frac{3x^2}{100}\sin(10x-19)+\frac{6x}{1,000}\cos(10x-19)-\frac{6}{10,000}\sin(10x-19)+c$

Therefore,

$$\int x^3\sin(10x-19)dx = -\frac{x^3}{10}\cos(10x-19)+\frac{3x^2}{100}\sin(10x-19)+\frac{3x}{500}\cos(10x-19)-\frac{3}{5,000}\sin(10x-19)+c.$$

Example 10.19

Show that $\int x^4e^{12x}dx = \frac{e^{12x}}{10,368}\left(864x^4-288x^3+72x^2-12x+1\right)+c$, where c is constant of integration.

Solution

Given $\int x^4e^{12x}dx.$

Let $u = x^4$ and $dv = e^{12x}dx$

Prepare a table consisting of three columns and seven rows as follows.

Alternate signs	u and its derivatives	v' and its antiderivatives
$+$	x^4	e^{12x}
$-$	$4x^3$	$\frac{1}{12}e^{12x}$
$+$	$12x^2$	$\frac{1}{144}e^{12x}$
$-$	$24x$	$\frac{1}{1,728}e^{12x}$
$+$	24	$\frac{1}{20,736}e^{12x}$
$-$	0	$\frac{1}{248,832}e^{12x}$

The solution is given by adding the signed products of the diagonal entries. That is,

$$\int x^4e^{12x}dx = \frac{x^4}{12}e^{12x} - \frac{4x^3}{144}e^{12x} + \frac{12x^2}{1,728}e^{12x} - \frac{24x}{20,736}e^{12x} + \frac{24}{248,832}e^{12x} + c$$

$$= \frac{x^4}{12}e^{12x} - \frac{x^3}{36}e^{12x} + \frac{x^2}{144}e^{12x} - \frac{x}{864}e^{12x} + \frac{1}{10,368}e^{12x} + c$$

$$= \frac{864x^4e^{12x} - 288x^3e^{12x} + 72x^2e^{12x} - 12xe^{12x} + e^{12x}}{10,368} + c$$

Therefore, $\int x^4e^{12x}dx = \frac{e^{12x}}{10,368}\left(864x^4 - 288x^3 + 72x^2 - 12x + 1\right) + c.$

Examples 10.20

Verify that $\int (t+2)^3 \cos t\, dt = (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c.$

Solution

Let $u = (t+2)^3$ and $dv = \cos t\, dt$

Prepare a table consisting of three columns and six rows as follows.

Alternate signs	u and its derivatives	v' and its anti-derivatives
$+$	$(t+2)^3$	$\cos t$
$-$	$3(t+2)^2$	$\sin t$
$+$	$6(t+2)^1$	$-\cos t$
$-$	6	$-\sin t$
$+$	0	$\cos t$

The solution is given by adding the signed products of the diagonal entries. That is,

$$\int (t+2)^3 \cos t\, dt = (t+2)^3 \sin t + 3(t+2)^2 \cos t - 6(t+2)\sin t - 6\cos t + c$$

$$= (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c$$

Therefore, $\int (t+2)^3 \cos t\, dt = (t+2)\sin t\left[(t+2)^2 - 6\right] + 3\cos t\left[(t+2)^2 - 2\right] + c.$

(d) ***Integrals of the form*** $\int e^{ax}\sin(bx)\,dx$ **or** $\int e^{ax}\cos(bx)\,dx$**,** where "a" and "b" are constants.

In this case, each part can be either u or dv. This is due to the fact that, neither e^{ax} nor $\sin bx$ decreases in power during differentiation. Therefore, to get the integral of this type, perform the integration two times and then combine the results.

Example 10.21

Find $\int e^x \cos x\, dx.$

Solution

Given $\int e^x \cos x dx.$

Let $u = e^x$ or $u = \cos x$; either way will work.

Let $u = e^x$.

Thus, $\frac{du}{dx} = e^x \Rightarrow du = e^x dx.$

Also, let $dv = \cos x dx \Rightarrow v = \sin x$.

Apply the formula $\int u dv = uv - \int v du.$

This gives

$\int e^x \cos x dx = e^x \sin x - \int e^x \sin x dx$ ... (i)

The $\int e^x \sin x\, dx$ requires further integration by parts.

Use the same type of substitutions for the next integration by parts.

Let $u = e^x$, then $\frac{du}{dx} = e^x \Rightarrow du = e^x dx.$

Also, let $\frac{dv}{dx} = \sin x \Rightarrow v = -\cos x$.

Thus, $\int e^x \sin x \, dx = -e^x \cos x - \left[\int - e^x \cos x \, dx\right]$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx \text{ .... (ii)}$$

Substitute the integral $\int e^x \sin x \, dx$ into equation (i):

$$\int e^x \cos x \, dx = e^x \sin x - \left(-e^x \cos x + \int e^x \cos x \, dx\right)$$

$$\Rightarrow \int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

Collect like terms of the integral to get;

$$\int e^x \cos x \, dx + \int e^x \cos x \, dx = e^x \sin x + e^x \cos x$$

$$\Rightarrow 2\int e^x \cos x \, dx = e^x (\sin x + \cos x)$$

Therefore, $\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + c$.

Note that;

1. $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx) + c$.
2. $\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx) + c$.

(e) *Integrals of the form* $\int x^n \sin^{-1} x \, dx$, $\int x^n \cos^{-1} x \, dx$ *or* $\int x^n \tan^{-1} x \, dx$.

In this case, let the inverse of trigonometric part be the function u and then differentiate. Let the polynomial part of the integral be dv, then integrate it.

Example 10.22

Find each of the following:

(a) $\int \sin^{-1} x \, dx$

(b) $\int \cos^{-1} x \, dx$

Solution

(a) Given $\int \sin^{-1} x dx = \int x^0 \sin^{-1} x\, dx$.

From the formula $\int u dv = uv - \int v du$.

Let $u = \sin^{-1} x$, then $du = \frac{1}{\sqrt{1-x^2}} dx$

Also, let $dv = 1dx \Rightarrow v = x$

Substituting into the formula gives

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int x \times \frac{dx}{\sqrt{1-x^2}}$$

$$= x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}}$$

But $\int \frac{x dx}{\sqrt{1-x^2}}$ requires further integration,

So, let $t = \sqrt{1-x^2} \Rightarrow t^2 = 1 - x^2$

$\Rightarrow 2tdt = -2xdx$

$\Rightarrow tdt = -xdx$

Thus, $\int \frac{x dx}{\sqrt{1-x^2}} = -\int \frac{tdt}{t}$

$$= -\int dt$$

$$= -t.$$

Substituting back $t = \sqrt{1-x^2}$ gives

$$\int \frac{x dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$$

Therefore,

$$\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + c.$$

(b) Given $\int \cos^{-1} x\, dx$.

Let $u = \cos^{-1} x$, then $\frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}}$.

Also, let $v' = x^0 \Rightarrow dv = x^0 dx$.

Integrating both sides gives, $v = x$

$$\int \cos^{-1} x dx = x \cos^{-1} x - \int x \times \frac{-1}{\sqrt{1-x^2}} dx$$

$$= x \cos^{-1} x + \int \frac{x dx}{\sqrt{1-x^2}}$$

But $\int \frac{x dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$

Therefore,

$$\int \cos^{-1} x\, dx = x \cos^{-1} x - \sqrt{1-x^2} + c.$$

Reduction formula

A reduction formula is regarded as an important method of integration. It helps to solve complex integration problems by relying on recurrence relations and it is used when an expression cannot be integrated directly. It can also be used for powers of elementary functions, trigonometrical functions, logarithmic functions, and exponential functions, as well as products of two or more complex functions. The formula enables to reduce the degree of the integrand and calculate the integrals in a finite number of steps. Consider the problem of finding $I_n = \int x^n e^{-x} dx$ for $n \geq 0$. The process of integrating by parts reduces the power of x by 1, each time. This process will give a formula for I_n. The integral is integrated as follows.

Let $u = x^n$ and $dv = e^{-x} dx$

$\Rightarrow du = nx^{n-1}dx$ and $v = \int e^{-x}dx = -e^{-x}$

Applying the integration by parts formula $\int udv = uv - \int vdu$ gives
$I_n = -x^n e^{-x} + n\int x^{n-1}e^{-x}dx$

But $\int x^{n-1}e^{-x}dx = I_{n-1}$

Thus, $I_n = -x^n e^{-x} + nI_{n-1}$.

Therefore, $I_n = -x^n e^{-x} + nI_{n-1}$ is called a reduction formula for $I_n = \int x^n e^{-x}dx$ for $n \geq 1$.

Example 10.23

Use the reduction formula for $I_n = \int x^n e^{-x}dx$ to evaluate $\int x^4 e^{-x}dx$.

Solution

Given $\int x^4 e^{-x}dx$. In this case, the reduction formula for $I_n = \int x^n e^{-x}dx$ can be used.

$I_n = -x^n e^{-x} + nI_{n-1}, n \geq 1$

Thus, $I_0 = -x^0 e^{-x} + 0 = -e^{-x}$

$I_1 = -x^1 e^{-x} + I_0 = -e^{-x}(x+1)$

$I_2 = -x^2 e^{-x} + 2I_1 = -e^{-x}(x^2 + 2x + 2)$

$I_3 = -x^3 e^{-x} + 3I_2 = -e^{-x}(x^3 + 3x^2 + 6x + 6)$

$I_4 = -x^4 e^{-x} + 4I_3 = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + c$

Therefore, $\int x^4 e^{-x}dx = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + c$.

Example 10.24

If $I_n = \int \sec^n x\, dx$, show that

(a) $I_n = \dfrac{1}{n-1}\sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$

(b) $\int \sec^6 x\, dx = \dfrac{1}{5}\sec^4 x \tan x + \dfrac{4}{15}\sec^2 x \tan x + \dfrac{8}{15}\tan x + D.$

Solution

(a) $I_n = \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx$

Let $u = \sec^{n-2} x$ and $dv = \sec^2 x\, dx$

$\Rightarrow du = (n-2)\sec^{n-2} x \tan x\, dx$ and $v = \tan x$

Applying the integration by parts formula, $\int u dv = uv - \int v du$, gives

$$I_n = \sec^{n-2} x \tan x - \int \tan x (n-2)\sec^{n-2} x \tan x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x \tan^2 x\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^{n-2} x(\sec^2 x - 1)\, dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int \sec^n x\, dx + (n-2)\int \sec^{n-2} x dx$$

But $\int \sec^n x\, dx = I_n$ and $\int \sec^{n-2} x\, dx = I_{n-2}$

Thus, $I_n = \sec^{n-2} x \tan x - (n-2)I_n + (n-2)I_{n-2}$

Collecting like terms gives,

$$I_n + (n-2)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

$$\Rightarrow (n-1)I_n = \sec^{n-2} x \tan x + (n-2)I_{n-2}$$

Therefore, $I_n = \dfrac{1}{n-1}\sec^{n-2} x \tan x + \dfrac{n-2}{n-1} I_{n-2}$.

(b) $I_6 = \dfrac{1}{6-1}\sec^{6-2} x \tan x + \dfrac{6-2}{6-1} I_{6-2}$

$$= \frac{1}{5}\sec^4 x \tan x + \frac{4}{5} I_4$$

$$= \frac{1}{5}\sec^4 x \tan x + \frac{4}{5}\left[\frac{1}{3}\sec^2 x \tan x + \frac{2}{3} I_2\right]$$

$= \dfrac{1}{5}\sec^4 x \tan x + \dfrac{4}{15}\sec^2 x \tan x + \dfrac{8}{15} I_2$. But $I_2 = \int \sec^2 x\, dx = \tan x + c$.

$$\Rightarrow I_6 = \frac{1}{5}\sec^4 x \tan x + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}[\tan x + c].$$

$\Rightarrow I_6 = \dfrac{1}{5}\sec^4 x \tan x + \dfrac{4}{15}\sec^2 x \tan x + \dfrac{8}{15}\tan x + D$, where $D = \dfrac{8}{15}C$.

Therefore, $\int \sec^6 x\, dx = \dfrac{1}{5}\sec^4 x \tan x + \dfrac{4}{15}\sec^2 x \tan x + \dfrac{8}{15}\tan x + D$.

Example 10.25

Derive a reduction formula for I_n in terms of I_{n-2}, where $I_n = \int \cos^n x\,dx$. Hence, show that $\int \cos^5 x\,dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + A$.

Solution

Given $I_n = \int \cos^n x\,dx = \int \cos^{n-1} x\cos x\,dx.$

Let $u = \cos^{n-1} x$ and $dv = \cos x\,dx$

$\Rightarrow du = (n-1)\cos^{n-2} x(-\sin x)dx$ and $v = \sin x$

Applying integration by parts formula,

$\int udv = uv - \int v\,du$ gives;

$$I_n = \cos^{n-1} x\sin x + \int (n-1)\cos^{n-2} x\sin x\sin x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\left(1-\cos^2 x\right)dx$$

$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

$$= \cos^{n-1} x\sin x + (n-1)I_{n-2} - (n-1)I_n$$

Collecting like terms gives:

$$I_n + (n-1)I_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

$$\Rightarrow nI_n = \cos^{n-1} x\sin x + (n-1)I_{n-2}$$

Now, $I_n = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}I_{n-2}$

$$\Rightarrow I_5 = \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}I_3$$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{5}\left(\frac{1}{3}\cos^2 x\sin x + \frac{2}{3}I_1\right)$$

$$= \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}I_1$$

But $I_1 = \int \cos x\,dx = \sin x + c.$

Thus, $I_5 = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + \text{A}$, where $\text{A} = \frac{8}{15}c$

Therefore, $\int \cos^5 x\,dx = \frac{1}{5}\cos^4 x\sin x + \frac{4}{15}\cos^2 x\sin x + \frac{8}{15}\sin x + A.$

Exercise 10.5

Evaluate each of the following integrals:

1. $\int x^2 e^{2x} dx$
2. $\int \tan^{-1} 3x\, dx$
3. $\int \cos^{-1}(0.5x) dx$
4. $\int x^3 \sin x\, dx$
5. $\int x \sin x \cos x dx$
6. $\int x^2 \cos x\, dx$
7. $\int x^3 \ln x\, dx$
8. $\int (x-1)\ln(2x) dx$
9. $\int \ln 3x\, dx$
10. $\int e^{2x} \cos x\, dx$
11. $\int e^{-x} \sin 2x\, dx$
12. $\int e^{\frac{1}{2}x} \cos xdx$
13. $\int e^{4x} \sin 2x\, dx$
14. $\int e^{2x} \cos 3x\, dx$
15. $\int \theta \sec^2 \theta\, d\theta$
16. $\int x^2 e^{-3x} dx$
17. $\int \sin^{-1} 2x\, dx$
18. $\int x^2 \ln x\, dx$
19. $\int \frac{\ln x}{\sqrt{x}} dx$
20. $\int \frac{\ln x}{x^7} dx$
21. $\int (x+1)^2 \ln(3x) dx$
22. Obtain a reduction formula for $I_n = \int (\ln x)^n$, hence use it to evaluate I_4.
23. Obtain a reduction formula for $\int x^n e^x dx$ and use it to find $\int x^4 e^x dx$.
24. Show that
$$\int x \tan^{-1} x\, dx = \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + c.$$
25. Derive a reduction formula for I_n in terms of I_{n-1}, when $I_n = \int x^3 (\ln x)^n dx$.
Hence, verify that
$$I_3 = \frac{x^4}{128}\left[32(\ln x)^3 - 24(\ln x)^2 + 12\ln x - 3\right] + c$$
26. If $I_n = \int \tan^n \theta\, d\theta$, show that $I_n = \frac{1}{n-1}\tan^{n-1}\theta - I_{n-2}$.
27. Prove that, if $I_n = \int x^n (1+x^3)^7 dx$, then
$$I_n = \frac{1}{n+22}\left[x^{n-2}(1+x^3)^8 - (n-2)I_{n-3}\right].$$
Hence, determine
$$\int x^5 (1+x^3)^7 dx.$$
28. Show that
$$\int x^n \ln x\, dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\ln x - 1].$$

Integration using partial fractions

When an integrand is a proper rational function, it is resolved into partial fractions before integrating. The partial fractions are integrated separately. Table 10.2 shows some forms of rational functions, their partial fractions, and corresponding integrals.

Table 10.2: Forms of rational functions, partial fractions, and integrals.

Rational function	Partial fraction	Integral
$\frac{px+q}{(x-a)(x-b)}$, where $a \neq b$	$\frac{A}{x-a}+\frac{B}{x-b}$	$\int\left(\frac{A}{x-a}+\frac{B}{x-b}\right)dx$
$\frac{1}{x^2-a^2}$	$\frac{A}{x-a}+\frac{B}{x+a}$	$\int\left(\frac{A}{x-a}+\frac{B}{x+a}\right)dx$
$\frac{px+q}{(x-a)^2}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}$	$\int\left(\frac{A}{x-a}+\frac{B}{(x-a)^2}\right)dx$
$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	$\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}$	$\int\left(\frac{A}{x-a}+\frac{B}{x-b}+\frac{C}{x-c}\right)dx$
$\frac{px^2+qx+r}{(x-a)^2(x-c)}$	$\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-c}$	$\int\left(\frac{A}{x-a}+\frac{B}{(x-a)^2}+\frac{C}{x-c}\right)dx$
$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)}$	$\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}$	$\int\left(\frac{A}{x-a}+\frac{Bx+C}{x^2+bx+c}\right)dx$

Example 10.26

Find $\int \frac{2x}{x^2-4x+3}dx$.

Solution

Given

$$\int \frac{2x}{x^2-4x+3}dx = \int \frac{2x}{(x-1)(x-3)}dx$$

Decompose the integrand into partial fractions as follows:

$$\frac{2x}{(x-1)(x-3)} = \frac{A}{x-1}+\frac{B}{x-3}$$

Solving for A and B gives $A=-1$ and $B=3$.

Thus, $\frac{2x}{(x-1)(x-3)} = \frac{-1}{x-1}+\frac{3}{x-3}$

Now, the integral becomes,

$$\int \frac{2x}{x^2-4x+3}dx = \int\left(\frac{-1}{x-1}+\frac{3}{x-3}\right)dx$$

$$= \int \frac{-1}{(x-1)}dx + \int \frac{3}{(x-3)}dx$$

$$= -\ln|x-1| + 3\ln|x-3| + c$$

$$= 3\ln|x-3| - \ln|x-1| + c$$

$$= \ln\left|(x-3)^3\right| - \ln|x-1| + c$$

$$= \ln\left|\frac{(x-3)^3}{(x-1)}\right| + c \cdot$$

Therefore,

$$\int \frac{2x}{x^2-4x+3}\,dx = \ln\left|\frac{(x-3)^3}{(x-1)}\right| + c\,.$$

Example 10.27

Determine $\int \frac{dx}{x^2-9}$.

Solution

Given $\int \frac{dx}{x^2-9} = \int \frac{dx}{(x-3)(x+3)}$.

Decompose the integrand into partial fractions as follows:

$$\frac{1}{x^2-9} = \frac{A}{x-3} + \frac{B}{x+3}$$

Solving for A and B gives, $A = \frac{1}{6}$ and $B = -\frac{1}{6}$.

Thus, $\frac{1}{x^2-9} = \frac{\frac{1}{6}}{(x-3)} + \frac{-\frac{1}{6}}{(x+3)}$.

$$= \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$$

Now, the integral becomes,

$$\int \frac{dx}{x^2-9} = \int\left(\frac{1}{6(x-3)} - \frac{1}{6(x+3)}\right)dx$$

$$= \int \frac{1}{6(x-3)}\,dx - \int \frac{1}{6(x+3)}\,dx$$

$$= \frac{1}{6}\ln|x-3| - \frac{1}{6}\ln|x+3| + c$$

$$= \ln\left|(x-3)^{\frac{1}{6}}\right| - \ln\left|(x+3)^{\frac{1}{6}}\right| + c$$

$$= \ln\left|\frac{(x-3)^{\frac{1}{6}}}{(x+3)^{\frac{1}{6}}}\right| + c$$

Therefore, $\int \frac{dx}{x^2-9} = \ln\left|\frac{(x-3)^{\frac{1}{6}}}{(x+3)^{\frac{1}{6}}}\right| + c.$

Example 10.28

Find $\int \frac{5x-2}{(x+3)^2}\,dx$.

Solution

Given $\int \frac{5x-2}{(x+3)^2}\,dx$.

Write the integrand in partial fractions. That is,

$$\frac{5x-2}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2}$$

Solving for A and B gives, $A = 5$ and $B = -17$. Substitute the values of A and B into the partial fractions.

$$\Rightarrow \frac{5x-2}{(x+3)^2} = \frac{5}{(x+3)} - \frac{17}{(x+3)^2}$$

Now, the integral becomes;

$$\int \frac{5x-2}{(x+3)^2}\,dx = \int\left(\frac{5}{(x+3)} - \frac{17}{(x+3)^2}\right)dx$$

$$= \int \frac{5}{(x+3)}\,dx - \int \frac{17}{(x+3)^2}\,dx$$

$$= 5\int \frac{dx}{(x+3)} - 17\int \frac{dx}{(x+3)^2}$$

$$= 5\ln|x+3| + \frac{17}{(x+3)} + c.$$

Therefore, $\displaystyle\int \frac{5x-2}{(x+3)^2}\,dx = 5\ln|x+3| + \frac{17}{(x+3)} + c.$

Example 10.29

Find $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx.$

Solution

Given $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx.$

Write the integrand into partial fractions.

That is, $\displaystyle\frac{-2x+4}{(x^2+1)(x-1)} = \frac{Ax+B}{(x^2+1)} + \frac{C}{(x-1)}$

Solving for A, B, and C gives $A=-1$, $B=-3$, and $C=1$.

Thus, $\displaystyle\frac{-2x+4}{(x^2+1)(x-1)} = \frac{-x-3}{(x^2+1)} + \frac{1}{(x-1)}.$

Now, the integral becomes;

$$\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx = \int \left(\frac{-x-3}{(x^2+1)} + \frac{1}{(x-1)} \right) dx$$

$$= \int \frac{-x-3}{(x^2+1)}\,dx + \int \frac{1}{(x-1)}\,dx$$

$$= -\int \frac{x}{x^2+1}\,dx - 3\int \frac{dx}{x^2+1} + \int \frac{dx}{x-1} \text{ but } \int \frac{dx}{x^2+1} = \tan^{-1} x$$

$$= -\frac{1}{2}\ln\left(x^2+1\right) - 3\tan^{-1} x + \ln|x-1| + c$$

Therefore, $\displaystyle\int \frac{-2x+4}{(x^2+1)(x-1)}\,dx = -\frac{1}{2}\ln\left(x^2+1\right) - 3\tan^{-1} x + \ln|x-1| + c.$

Example 10.30

Integrate $\dfrac{x^2+x-3}{(x+1)(x-2)(x-5)}$ with respect to x.

Solution

Given $\displaystyle\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx.$

Decompose the integrand into partial fractions. That is,

$$\frac{x^2+x-3}{(x+1)(x-2)(x-5)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-5)}$$

Solving for A, B, and C gives $A=-\frac{1}{6}$, $B=-\frac{1}{3}$, and $C=\frac{3}{2}$.

$$\Rightarrow \frac{x^2+x-3}{(x+1)(x-2)(x-5)} = \frac{-\frac{1}{6}}{(x+1)} + \frac{-\frac{1}{3}}{(x-2)} + \frac{\frac{3}{2}}{(x-5)}$$

Now, the integral becomes,

$$\begin{aligned}\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx &= \int\left(-\frac{1}{6(x+1)} - \frac{1}{3(x-2)} + \frac{3}{2(x-5)}\right)dx\\ &= -\frac{1}{6}\int\frac{dx}{(x+1)} - \frac{1}{3}\int\frac{dx}{(x-2)} + \frac{3}{2}\int\frac{dx}{(x-5)}\\ &= -\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{3}{2}\ln|x-5| + c\end{aligned}$$

Therefore, $\displaystyle\int \frac{x^2+x-3}{(x+1)(x-2)(x-5)}\,dx = -\frac{1}{6}\ln|x+1| - \frac{1}{3}\ln|x-2| + \frac{3}{2}\ln|x-5| + c.$

Exercise 10.6

Evaluate each of the following integrals:

1. $\displaystyle\int \frac{x-9}{(x+5)(x-2)}\,dx$
2. $\displaystyle\int \frac{1}{(x+5)^2(x-1)}\,dx$
3. $\displaystyle\int \frac{10}{(x^2+9)(x-1)}\,dx$
4. $\displaystyle\int \frac{1}{s^2(s-1)^2}\,ds$
5. $\displaystyle\int \frac{x^2-5x+16}{(2x+1)(x-2)^2}\,dx$
6. $\displaystyle\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx$

7. $\int \frac{1}{x^2-1}dx$
8. $\int \frac{4y^2-7y-12}{y(y+2)(y-3)}dy$
9. $\int \frac{x-1}{x^2+3x+2}dx$
10. $\int \frac{x^2+2x-1}{x^3-x}dx$
11. $\int \frac{dt}{(t+4)(t-1)}$
12. $\int \frac{x+3}{(x-1)^3}dx$
13. $\int \frac{x-1}{(x+3)^6}dx$
14. $\int \frac{x^4}{x^4-1}dx$
15. $\int \frac{5x-17}{x^2-6x+9}dx$
16. $\int \frac{2-t}{t^2+5t}dt$
17. $\int \frac{x^2-1}{x^2-16}dx$
18. $\int \frac{x^4+x^3+x^2+1}{x^2+x-2}dx$
19. $\int \frac{r^2+r-1}{r(r^2-1)}dr$
20. $\int \frac{x+14}{(x+5)(x+2)}dx$
21. $\int \frac{x^5+1}{x^3(x+2)}dx$

Integration of trigonometric functions

Integration of trigonometric functions depends on the type of the integrand.

(a) ***Integrals of the form*** $\int \sin ax \cos bx\, dx$, $\int \sin ax \sin bx\, dx$, ***or*** $\int \cos ax \cos bx\, dx$, where “a” and “b” are constants.

Integrals which are in this form are evaluated using the factor formula. The factor formula is used to transform the integrand which is in the form of the product of two trigonometric functions into a sum of two trigonometric functions. This makes it easier to integrate the resulting integrands. Table 10.3 shows the factor formulae and their corresponding integrals given that P and Q are angles containing the variable x.

Table 10.3: *Factor formulae and the corresponding integrals*

Factor formula	Integral
$\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	$\int\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\sin P+\sin Q)\,dx$
$\sin P-\sin Q=2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$	$\int\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\sin P-\sin Q)\,dx$
$\cos P+\cos Q=2\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$	$\int\cos\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)dx=\frac{1}{2}\int(\cos P+\cos Q)\,dx$
$\cos P-\cos Q=-2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$	$\int\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)dx=-\frac{1}{2}\int(\cos P-\cos Q)\,dx$

Example 10.31

Find $\int\sin 4x\cos 2x\,dx$.

Solution

Given $\int\sin 4x\cos 2x\,dx$.

The integrand is the product of sine and cosine functions. Thus, it can be expressed as the sum or difference of two sine functions. In this case, it will be expressed as a sum of two sine functions because the sine angle is greater than the cosine angle.

In this case, the factor formula

$$\sin P+\sin Q=2\sin\left(\frac{P+Q}{2}\right)\cos\left(\frac{P-Q}{2}\right)$$

can be used.

The values of P and Q are computed as follows.

Let $\frac{P+Q}{2}=4x$ and $\frac{P-Q}{2}=2x$.

Solving for P and Q gives $P=6x$ and $Q=2x$.

Thus, $2\sin 4x\cos 2x=\sin 6x+\sin 2x$

$$\Rightarrow \sin 4x\cos 2x=\frac{1}{2}(\sin 6x+\sin 2x)$$

Now,

$$\int\sin 4x\cos 2x\,dx=\int\frac{1}{2}(\sin 6x+\sin 2x)\,dx$$

$$=\frac{1}{2}\int\sin 6x dx+\frac{1}{2}\int\sin 2x dx$$

$$=\frac{1}{2}\left(-\frac{\cos 6x}{6}\right)+\frac{1}{2}\left(-\frac{\cos 2x}{2}\right)+c$$

$$=-\frac{1}{12}\cos 6x-\frac{1}{4}\cos 2x+c$$

Therefore,

$$\int\sin 4x\cos 2x dx=-\frac{1}{12}\cos 6x-\frac{1}{4}\cos 2x+c.$$

Example 10.32

Find $\int \cos 7x \sin 5x \, dx$.

Solution

Given $\int \cos 7x \sin 5x dx$.

In this case, the cosine angle is greater than the sine angle. Thus, the integrand can be expressed as the difference of two sine functions using the following factor formula:

$$\sin P - \sin Q = 2\cos\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Compute the values of P and Q as follows:

Let $\frac{P+Q}{2} = 7x$ and $\frac{P-Q}{2} = 5x$. Solving for P and Q gives $P = 12x$ and $Q = 2x$.

Thus, $2\cos 7x \sin 5x = \sin 12x - \sin 2x$

$$\Rightarrow \cos 7x \sin 5x = \frac{1}{2}(\sin 12x - \sin 2x)$$

Now, $\int \cos 7x \sin 5x \, dx = \frac{1}{2}\int (\sin 12x - \sin 2x)\, dx$

$$= \frac{1}{2}\int \sin 12x \, dx - \frac{1}{2}\int \sin 2x \, dx$$

$$= \frac{1}{2}\left(-\frac{\cos 12x}{12}\right) - \frac{1}{2}\left(-\frac{\cos 2x}{2}\right) + c$$

$$= \frac{1}{4}\left(\cos 2x - \frac{1}{6}\cos 12x\right) + c$$

Therefore, $\int \cos 7x \sin 5x \, dx = \frac{1}{4}\left(\cos 2x - \frac{1}{6}\cos 12x\right) + c$.

Example 10.33

Integrate $\sin 4x \sin x$ with respect to x.

Solution

Given $\int \sin 4x \sin x \, dx$.

The integrand is a product of sine functions. It can be expressed as the difference of two cosine functions using the following factor formula:

$$\cos P - \cos Q = -2\sin\left(\frac{P+Q}{2}\right)\sin\left(\frac{P-Q}{2}\right)$$

Compute the values of P and Q as follows:

Let $\frac{P+Q}{2} = 4x$ and $\frac{P-Q}{2} = x$. Solving for P and Q gives $P = 5x$ and $Q = 3x$

Thus, $-2\sin 4x \sin x = \cos 5x - \cos 3x$

$$\Rightarrow \sin 4x \sin x = -\frac{1}{2}(\cos 5x - \cos 3x)$$

$$\Rightarrow \int \sin 4x \sin x \, dx = -\frac{1}{2}\int (\cos 5x - \cos 3x)dx$$

$$= -\frac{1}{2}\int \cos 5x \, dx + \frac{1}{2}\int \cos 3x \, dx$$

$$= -\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + c$$

Therefore, $\int \sin 4x \sin x dx = -\frac{1}{10}\sin 5x + \frac{1}{6}\sin 3x + c.$

(b) *Integrals of the form* $\int \sin^n x \, dx$ *or* $\int \cos^n x \, dx$*, where* n *is an odd or even integer*

In this form of integrals, the exponent of trigonometric function determines the approach to be used. Integrals of even exponents of $\sin x$ or $\cos x$ are integrated with the help of the double angle formula of cosine which reduce the exponent of the given trigonometric function. If the integrand contains odd exponent of $\sin x$ or $\cos x$, the exponent is split, and the remaining part is expressed as an even exponent of $\sin x$ or $\cos x$. For instance, $\sin^3 x = \sin^2 x \sin x$.

Example 10.34

Find $\int \sin^2 x \, dx$.

Solution

Given $\int \sin^2 x\,dx$.

From $\cos 2x = 1-2\sin^2 x$, it implies that

$\sin^2 x = \frac{1}{2}(1-\cos 2x)$.

Now, the integral becomes,

$$\int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x\,dx$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

Therefore,

$$\int \sin^2 x\,dx = \frac{1}{2}x - \frac{1}{4}\sin 2x + c.$$

Example 10.35

Find $\int \sin^3 x\,dx$.

Solution

Given $\int \sin^3 x\,dx$, then

$$\int \sin^3 x\,dx = \int \sin^2 x \sin x\,dx$$

But $\sin^2 x = 1-\cos^2 x$.

$$\Rightarrow \int \sin^3 x\,dx = \int (1-\cos^2 x)\sin x\,dx$$

$$= \int \sin x\,dx - \int \sin x \cos^2 x\,dx$$

$$= -\cos x - \int \sin x \cos^2 x\,dx$$

$\int \sin x \cos^2 x\,dx$ can be integrated by substitution method as follows:

Let $u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$.

Thus, $dx = -\frac{1}{\sin x}du$.

Substitute these values into $\int \sin x \cos^2 x\,dx$ as follows,

$$\int u^2 \sin x\left(-\frac{1}{\sin x}du\right)$$

$$= \int -u^2\,du$$

$$= -\frac{1}{3}u^3 + c$$

Hence, $\int \cos^2 x \sin x\,dx = -\frac{1}{3}\cos^3 x + c$

Therefore,

$$\int \sin^3 x\,dx = -\cos x + \frac{1}{3}\cos^3 x + c.$$

Example 10.36

Find $\int \cos^4 x\,dx$.

Solution

Given $\int \cos^4 x\,dx$.

$$\Rightarrow \int \cos^4 x\,dx = \int (\cos^2 x)^2\,dx$$

But $\cos^2 x = \frac{1}{2}(1+\cos 2x)$.

$$\Rightarrow \int \cos^4 x\,dx = \int \left(\frac{1}{2}(1+\cos 2x)\right)^2 dx$$

$$= \frac{1}{4}\int (1+\cos 2x)^2\,dx$$

$$= \frac{1}{4}\int \left(1+2\cos 2x+\cos^2 2x\right)dx$$

$$=\frac{1}{4}\int dx+\frac{2}{4}\int\cos 2x dx+\frac{1}{4}\int\cos^2 2x\,dx$$

$$=\frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{4}\int\cos^2 2x\,dx$$

$\int\cos^2 2x\,dx$ requires further integration as follows:

From $\cos 2x=2\cos^2 x-1\Rightarrow\cos 4x=2\cos^2 2x-1$

$$\Rightarrow\cos^2 2x=\frac{1}{2}(\cos 4x+1)$$

Now, $\int\cos^2 2x\,dx=\int\frac{1}{2}(1+\cos 4x)\,dx$

$$=\frac{1}{2}\int dx+\frac{1}{2}\int\cos 4x\,dx$$

$$=\frac{1}{2}x+\frac{1}{8}\sin 4x+c$$

Hence, $\int\cos^4 x\,dx=\frac{1}{4}x+\frac{1}{4}\sin 2x+\frac{1}{4}\left(\frac{1}{2}x+\frac{1}{8}\sin 4x+c\right).$

Therefore, $\int\cos^4 x\,dx=\frac{3}{8}x+\frac{1}{4}\sin 2x+\frac{1}{32}\sin 4x+B$, where $B=\frac{1}{4}c$

Example 10.37

Find $\int\sin^2 x\cos^5 x\,dx.$

Solution

Given $\int\sin^2 x\cos^5 x\,dx.$

$$\Rightarrow\int\sin^2 x\cos^5 x\,dx=\int\sin^2 x\cos^4 x\cos x\,dx$$

$$=\int\sin^2 x\left(\cos^2 x\right)^2\cos x\,dx$$

$$=\int\sin^2 x\left(1-\sin^2 x\right)^2\cos x\,dx$$

$$=\int\sin^2 x\left(1-2\sin^2 x+\sin^4 x\right)\cos x\,dx$$

$$=\int\left(\sin^2 x-2\sin^4 x+\sin^6 x\right)\cos x\,dx$$

Let $u=\sin x\Rightarrow du=\cos x\,dx$

Thus, $\int\left(\sin^2 x - 2\sin^4 x + \sin^6 x\right)\cos x\,dx = \int (u^2 - 2u^4 + u^6)\,du$

$$= \frac{u^3}{3} - \frac{2u^5}{5} + \frac{u^7}{7} + c$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$$

Therefore, $\int \sin^2 x\cos^5 x\,dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$.

Note that,

1. if the power of sine is an odd positive integer then save one sine factor and convert the remaining factors to cosine. Then, expand and integrate.
2. If the power of cosine is an odd positive integer, save one cosine factor and convert the remaining factors to sine. Then, expand and integrate.
3. If the powers of both the sine and cosine terms are even positive integers, make repeated use of the identities, $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$.

(c) *Integrals of the form $\int \tan^n x\,dx$ or $\int \sec^n x\,dx$ where n is an odd or even integer*

In this form of integrals, the following approaches can be used:

1. If the power of the tangent is an odd positive integer, save a tangent factor and convert the remaining factors to secant. Expand and integrate the resulting function.
2. If the power of the secant is an even positive integer, save a secant-squared factor and convert the remaining factors to tangent. Expand and integrate the resulting function.
3. If there is no secant term and the power of the tangent term is an even positive integer, convert a tangent-squared factor to secant-squared factor. Expand and integrate the resulting function.
4. If the integral is of the form $\int \sec^n x\,dx$, where n is odd positive integer, then use integration by parts.

Example 10.38

Find $\int \tan^4\theta\,d\theta$

Solution

Given $\int \tan^4\theta\,d\theta$.

$$\Rightarrow \int \tan^4\theta\, d\theta = \int \tan^2\theta\left(\tan^2\theta\right) d\theta$$
$$= \int \tan^2\theta\left(\sec^2\theta - 1\right) d\theta$$
$$= \int \tan^2\theta \sec^2\theta\, d\theta - \int \tan^2\theta\, d\theta$$
$$= \int \tan^2\theta \sec^2\theta\, d\theta - \int \left(\sec^2\theta - 1\right) d\theta$$

Let $u = \tan\theta \Rightarrow du = \sec^2\theta\, d\theta$

$$\Rightarrow \int \tan^4\theta\, d\theta = \int u^2 du - \int \sec^2\theta\, d\theta + \int d\theta$$
$$= \frac{u^3}{3} - \tan\theta + \theta + c$$
$$= \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c$$

Therefore, $\int \tan^4\theta d\theta = \frac{1}{3}\tan^3\theta - \tan\theta + \theta + c.$

Example 10.39

Determine $\int \sec^4 6x \tan^3 6x\, dx$.

Solution

Given $\int \sec^4 6x \tan^3 6x\, dx$.

$$\Rightarrow \int \sec^4 6x \tan^3 6x\, dx = \int \sec^2 6x \tan^3 6x\left(\sec^2 6x\right) dx$$
$$= \int \left(1 + \tan^2 6x\right) \tan^3 6x \sec^2 6x\, dx$$
$$= \int \tan^3 6x \sec^2 6x\, dx + \int \tan^5 6x \sec^2 6x\, dx$$

Let $u = \tan 6x \Rightarrow \frac{1}{6} du = \sec^2 6x\, dx$

$$\Rightarrow \int \sec^4 6x \tan^3 6x\, dx = \int u^3 \cdot \frac{1}{6} du + \int u^5 \cdot \frac{1}{6} du$$
$$= \frac{1}{6}\left[\frac{u^4}{4} + \frac{u^6}{6}\right] + c$$

Therefore, $\int \sec^4 6x \tan^3 6x\, dx = \frac{1}{36}\tan^6 6x + \frac{1}{24}\tan^4 6x + c.$

Exercise 10.7

Evaluate each of the following integrals:

1. $\int \sin 4x \sin 3x \, dx$
2. $\int \sin 5x \cos 3x \, dx$
3. $\int \cos 6x \cos 4x \, dx$
4. $\int \cos 4x \sin 2x \, dx$
5. $\int \sin\theta \sin 2\theta \sin 3\theta \; d\theta$
6. $\int \sin 2\theta \sin 4\theta \, d\theta$
7. $\int \sin^5 2x \, dx$
8. $\int \sin^2 (2x-1) \, dx$
9. $\int \cos^4 (3x+1) \, dx$
10. $\int \sin^4 3x \, dx$
11. $\int \cos^3 2x \, dx$
12. $\int 2\sin^4 (x-1) \, dx$
13. $\int \cos^3 x \, dx$
14. $\int \sin^5 x \, dx$
15. $\int \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta \, d\theta$
16. $\int \sin^3 x \cos^2 x \, dx$
17. $\int \sin^4 x \cos^3 x \, dx$
18. $\int \sin^2 2x \cos^2 2x \, dx$
19. $\int \tan^5 x \, dx$
20. $\int \sin^5 x \cos^2 x \, dx$
21. $\int \tan^3 (3\theta) \sec^4 (3\theta) \, d\theta$

Trigonometric substitutions

Integrals of rational function whose denominators cannot be factorized and cannot be expressed into the form $x^2 - a^2$ require trigonometric substitutions. The common useful trigonometric substitutions includes tangent, sine, and cosine.

(a) ***Integrals of the form*** $\int \frac{1}{x^2+a^2} dx$**,**

where "a" is a constant

The suitable trigonometric substitution for integrals of this kind is the tangent function.

Let $x = a\tan\theta \Rightarrow \frac{dx}{d\theta} = a\sec^2\theta.$

Thus, $dx = a\sec^2\theta d\theta.$

Substituting these values into the integral as follows,

$$\int \frac{1}{x^2+a^2} dx = \int \frac{1}{(a\tan\theta)^2 + a^2} \times a\sec^2\theta d\theta$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2\tan^2\theta + a^2}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2(\tan^2\theta + 1)}$$

$$= \int \frac{a\sec^2\theta d\theta}{a^2\sec^2\theta}$$

$$= \frac{1}{a}\int d\theta$$

$$= \frac{1}{a}\theta + c$$

But $x = a\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{a}\right).$

Therefore,

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c.$$

Example 10.40

Find $\int \frac{1}{x^2 + 25} dx.$

Solution

Given $\int \frac{1}{x^2 + 25} dx = \int \frac{1}{x^2 + 5^2} dx.$

Let $x = 5\tan\theta \Rightarrow \frac{dx}{d\theta} = 5\sec^2\theta.$

Thus, $dx = 5\sec^2\theta d\theta$.

Substitute these values into the integral to get;

$$\int \frac{1}{(5\tan\theta)^2 + 5^2} \times 5\sec^2\theta\, d\theta$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2\tan^2\theta + 5^2}$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2(\tan^2\theta + 1)}$$

$$= \int \frac{5\sec^2\theta d\theta}{5^2\sec^2\theta}$$

$$= \frac{1}{5}\int d\theta$$

$$= \frac{1}{5}\theta + c.$$

But $x = 5\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{x}{5}\right)$.

Therefore,

$$\int \frac{1}{x^2 + 25} dx = \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + c.$$

Example 10.41

Find $\int \frac{1}{x^2 + 2} dx.$

Solution

Given $\int \frac{1}{x^2 + 2} dx = \int \frac{1}{x^2 + \left(\sqrt{2}\right)^2} dx.$

Let $x = \sqrt{2}\tan\theta \Rightarrow \frac{dx}{d\theta} = \sqrt{2}\sec^2\theta.$

Thus, $dx = \sqrt{2}\sec^2\theta d\theta.$

Substituting these values, the integral becomes,

$$\int \frac{1}{\left(\sqrt{2}\tan\theta\right)^2 + \left(\sqrt{2}\right)^2} \times \sqrt{2}\sec^2\theta d\theta$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\tan^2\theta + \left(\sqrt{2}\right)^2}$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\left(\tan^2\theta + 1\right)}$$

$$= \int \frac{\sqrt{2}\sec^2\theta d\theta}{\left(\sqrt{2}\right)^2\sec^2\theta}$$

$$= \frac{1}{\sqrt{2}}\int d\theta$$

$$= \frac{\sqrt{2}}{2}\theta + c$$

But $x = \sqrt{2}\tan\theta \Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{2}x}{2}\right)$.

Therefore,

$$\int \frac{1}{x^2 + 2} dx = \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}x}{2}\right) + c.$$

Example 10.42

Show that $\int \frac{1}{x^2+6x+16}dx = \frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)+c.$

Solution

Given $\int \frac{1}{x^2+6x+16}dx$.

By completing the square, the denominator $x^2+6x+16$ becomes,

$$\begin{aligned}x^2+6x+16 &= x^2+6x+3^2+16-3^2\\ &= x^2+6x+3^2+7\\ &= (x+3)^2+7\\ &= (x+3)^2+7\end{aligned}$$

Thus, $\int \frac{1}{x^2+6x+16}dx = \int \frac{1}{(x+3)^2+7}dx = \int \frac{1}{(x+3)^2+\left(\sqrt{7}\right)^2}dx$

Let $x+3=\sqrt{7}\tan\theta \Rightarrow \frac{dx}{d\theta}=\sqrt{7}\sec^2\theta$

Thus, $dx=\sqrt{7}\sec^2\theta d\theta.$

Substitute these values into the integral as follows,

$$\begin{aligned}\int \frac{1}{(x+3)^2+\left(\sqrt{7}\right)^2}dx &= \int \frac{1}{\left(\sqrt{7}\tan\theta\right)^2+\left(\sqrt{7}\right)^2}\times\sqrt{7}\sec^2\theta d\theta\\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2\tan^2\theta+\left(\sqrt{7}\right)^2}\\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2(\tan^2\theta+1)}\\ &= \int \frac{\sqrt{7}\sec^2\theta d\theta}{\left(\sqrt{7}\right)^2\sec^2\theta}\\ &= \frac{1}{\sqrt{7}}\int d\theta\end{aligned}$$

$$=\frac{\sqrt{7}}{7}\theta+c$$

But $x+3=\sqrt{7}\tan\theta \Rightarrow \theta=\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)$

Therefore, $\displaystyle\int\frac{1}{x^2+6x+16}dx=\frac{\sqrt{7}}{7}\tan^{-1}\left(\frac{\sqrt{7}}{7}(x+3)\right)+c.$

(b) ***Integrals of the form*** $\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}dx$, *where "**a**" is a constant*

In this case, sine or cosine trigonometric substitutions are suitable for this kind of integrals.

Let $x=a\sin\theta \Rightarrow \frac{dx}{d\theta}=a\cos\theta.$

Thus, $dx=a\cos\theta d\theta$.

Substituting these values, the integral as follows;

$$\int\frac{1}{\sqrt{a^2-x^2}}dx=\int\frac{1}{\sqrt{a^2-(a\sin\theta)^2}}\times a\cos\theta d\theta$$

$$=\int\frac{1}{\sqrt{a^2-a^2\sin^2\theta}}\times a\cos\theta d\theta$$

$$=\int\frac{1}{\sqrt{a^2\left(1-\sin^2\theta\right)}}\times a\cos\theta d\theta$$

$$=\int\frac{1}{\sqrt{a^2\cos^2\theta}}\times a\cos\theta d\theta$$

$$=\int\frac{a\cos\theta}{a\cos\theta}d\theta$$

$$=\int d\theta$$

$$=\theta+c$$

But $x=a\sin\theta \Rightarrow \theta=\sin^{-1}\left(\frac{x}{a}\right).$

Therefore, $\displaystyle\int\frac{1}{\sqrt{a^2-x^2}}dx=\sin^{-1}\left(\frac{x}{a}\right)+c$.

Example 10.43

Determine $\int \frac{1}{\sqrt{4 - x^2}} dx.$

Solution

Given $\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{2^2 - x^2}} dx.$

Let $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$.

Thus, $dx = 2\cos\theta d\theta$.

Substitute these values in the integral to get,

$$\int \frac{1}{\sqrt{2^2 - x^2}} dx = \int \frac{1}{\sqrt{2^2 - (2\sin\theta)^2}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{2^2 - 2^2\sin^2\theta}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{2^2\left(1 - \sin^2\theta\right)}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{2\sqrt{1 - \sin^2\theta}} \times 2\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{\cos^2\theta}} \times \cos\theta d\theta$$

$$= \int \frac{1}{\cos\theta} \times \cos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

But $x = 2\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{2}\right)$.

Therefore,

$$\int \frac{1}{\sqrt{4 - x^2}} dx = \sin^{-1}\left(\frac{x}{2}\right) + c.$$

Example 10.44

Find $\int \frac{1}{\sqrt{7 - 6x - x^2}} dx.$

Solution

Given $\int \frac{1}{\sqrt{7 - 6x - x^2}} dx$.

By completing the square, the quadratic equation $7 - 6x - x^2$ becomes,

$$\begin{aligned} 7 - 6x - x^2 &= 7 - (x^2 + 6x) \\ &= 7 + 9 - (x^2 + 6x + 9) \\ &= 16 - (x+3)^2 \end{aligned}$$

Hence, the integral as follows,

$$\int \frac{1}{\sqrt{7 - 6x - x^2}} dx = \int \frac{1}{\sqrt{16 - (x+3)^2}} dx$$

$$= \int \frac{1}{\sqrt{4^2 - (x+3)^2}} dx$$

Let $x + 3 = 4\sin\theta \Rightarrow \frac{dx}{d\theta} = 4\cos\theta$.

Thus, $dx = 4\cos\theta d\theta$.

Substitute into the integral as follows,

$$\int \frac{1}{\sqrt{4^2 - (x+3)^2}} dx$$

$$= \int \frac{1}{\sqrt{4^2 - \left(4\sin\theta\right)^2}} \times 4\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{4^2 - 4^2\sin^2\theta}} \times 4\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{4^2\left(1 - \sin^2\theta\right)}} \times 4\cos\theta d\theta$$

$$= \int \frac{1}{4\sqrt{1 - \sin^2\theta}} \times 4\cos\theta d\theta$$

$$= \int \frac{1}{\sqrt{\cos^2\theta}} \times \cos\theta d\theta$$

$$= \int \frac{1}{\cos\theta} \times \cos\theta d\theta$$

$$= \int d\theta$$

$$= \theta + c$$

But $(x+3) = 4\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x+3}{4}\right)$.

Therefore, $\int \frac{1}{\sqrt{7-6x-x^2}} dx = \sin^{-1}\left(\frac{x+3}{4}\right) + c.$

(c) ***Integrals of the form*** $\int \sqrt{a^2 - x^2}\, dx$, *where "**a**" is a constant*

In this case, sine or cosine trigonometric substitution is suitable for this kind of integrals.

Let $x = a\sin\theta \Rightarrow \frac{dx}{d\theta} = a\cos\theta$.

Thus, $dx = a\cos\theta d\theta$.

Substituting these values into the integral as follows,

$$\int \sqrt{a^2 - x^2}\, dx = \int \sqrt{a^2 - (a\sin\theta)^2} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2 - a^2\sin^2\theta} \times a\cos\theta d\theta$$

$$= \int \sqrt{a^2\left(1-\sin^2\theta\right)} \times a\cos\theta d\theta$$

$$= \int a\sqrt{1-\sin^2\theta} \times a\cos\theta d\theta$$

$$= \int a\sqrt{\cos^2\theta} \times a\cos\theta d\theta$$

$$= a^2 \int \cos\theta \times \cos\theta d\theta$$

$$= a^2 \int \frac{1}{2}(1+\cos 2\theta) d\theta$$

$$= \frac{1}{2}a^2\left[\int d\theta + \int \cos 2\theta d\theta\right]$$

$$= \frac{1}{2}a^2\theta + \frac{1}{4}a^2\sin 2\theta + c.$$

But $x = a\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$.

$$\Rightarrow \int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{4}a^2 \sin 2\left(\sin^{-1}\left(\frac{x}{a}\right)\right) + c$$

From the relation $\sin 2\theta = 2\sin\theta\cos\theta$, it implies that

$$\sin 2\left(\sin^{-1}\left(\frac{x}{a}\right)\right) = 2\sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right)$$

Thus, $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{2}{4}a^2 \sin\left(\sin^{-1}\left(\frac{x}{a}\right)\right)\cos\left(\sin^{-1}\left(\frac{x}{a}\right)\right) + c$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}a^2 \times \left(\frac{x}{a}\right)\sqrt{1 - \sin^2\left(\sin^{-1}\left(\frac{x}{a}\right)\right)} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}ax\sqrt{1 - \left(\frac{x}{a}\right)^2} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}ax\sqrt{\frac{a^2 - x^2}{a^2}} + c$$

$$= \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2} + c$$

Therefore, $\int \sqrt{a^2 - x^2}\,dx = \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}x\sqrt{a^2 - x^2} + c.$

Example 10.45

Find $\int \sqrt{36 - x^2}\,dx.$

Solution

Given $\int \sqrt{36 - x^2}\,dx = \int \sqrt{6^2 - x^2}\,dx$.

Let $x = 6\sin\theta \Rightarrow \frac{dx}{d\theta} = 6\cos\theta$.

Thus, $dx = 6\cos\theta\, d\theta$.

Substituting these values into the integral as follows,

$$\int \sqrt{6^2 - x^2}\,dx = \int \sqrt{6^2 - (6\sin\theta)^2} \times 6\cos\theta\, d\theta$$

$$= \int \sqrt{6^2 - 6^2\sin^2\theta} \times 6\cos\theta\, d\theta$$

$$= \int \sqrt{6^2(1 - \sin^2\theta)} \times 6\cos\theta\, d\theta$$

$$= \int 6\sqrt{1-\sin^2\theta} \times 6\cos\theta d\theta$$

$$= \int 6\sqrt{\cos^2\theta} \times 6\cos\theta d\theta$$

$$= 36\int \cos\theta \times \cos\theta d\theta$$

$$= 36\int \frac{1}{2}(1+\cos 2\theta)d\theta$$

$$= 18\left[\int d\theta + \int \cos 2\theta d\theta\right]$$

$$= 18\theta + 9\sin 2\theta + c.$$

But $x = 6\sin\theta \Rightarrow \theta = \sin^{-1}\left(\frac{x}{6}\right)$.

Thus, $\int \sqrt{36- x^2}\,dx = 18\sin^{-1}\left(\frac{x}{6}\right) + 9\sin 2\left(\sin^{-1}\left(\frac{x}{6}\right)\right) + c$

More simplification gives: $\int \sqrt{36 - x^2}\,dx = 18\sin^{-1}\left(\frac{x}{6}\right) + \frac{1}{2}x\sqrt{36-x^2} + c.$

Therefore, $\int \sqrt{36 - x^2}\,dx = 18\sin^{-1}\left(\frac{x}{6}\right) + \frac{1}{2}x\sqrt{36-x^2} + c.$

Example 10.46

Show that $\int \sqrt{3-2x- x^2}\,dx = 2\sin^{-1}\left(\frac{x+1}{3}\right) + \frac{1}{2}(x+1)\sqrt{3-2x- x^2} + c.$

Solution

Given $\int \sqrt{3-2x-x^2}\,dx = 2\sin^{-1}\left(\frac{x+1}{3}\right) + \frac{1}{2}(x+1)\sqrt{3-2x- x^2} + c$.

Consider the left-hand side:

By completing the square, $3-2x-x^2$ gives, $3-2x-x^2 = 2^2 - (x+1)^2$.

Now, the integral becomes,

$$\int \sqrt{3-2x-x^2}\,dx = \int \sqrt{2^2 - (x+1)^2}\,dx$$

Let $x+1 = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta$.

Thus, $dx = 2\cos\theta d\theta$.

Substituting these values into the integral as follows,

$$\int\sqrt{2^2-(x+1)^2}\,dx=\int\sqrt{2^2-(2\sin\theta)^2}\times 2\cos\theta d\theta$$

$$=\int\sqrt{2^2-2^2\sin^2\theta}\times 2\cos\theta d\theta$$

$$=\int\sqrt{2^2\left(1-\sin^2\theta\right)}\times 2\cos\theta d\theta$$

$$=\int 2\sqrt{(1-\sin^2\theta)}\times 2\cos\theta d\theta$$

$$=\int 2\sqrt{\cos^2\theta}\times 2\cos\theta d\theta$$

$$=4\int\cos\theta\times\cos\theta d\theta$$

$$=4\int\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$=2\left[\int d\theta+\int\cos 2\theta d\theta\right]$$

$$=2\theta+\sin 2\theta+c$$

But $x+1=2\sin\theta\Rightarrow\theta=\sin^{-1}\left(\frac{x+1}{2}\right)$

$$\Rightarrow\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\sin 2\left(\sin^{-1}\left(\frac{x+1}{2}\right)\right)+c$$

More simplification gives:

$$\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+c.$$

Therefore, $\int\sqrt{3-2x-x^2}\,dx=2\sin^{-1}\left(\frac{x+1}{2}\right)+\frac{1}{2}(x+1)\sqrt{3-2x-x^2}+c.$

Exercise 10.8

Evaluate each of the following integrals:

1. $\int\frac{1}{x^2+49}\,dx$
2. $\int\frac{1}{5x^2+81}\,dx$
3. $\int\frac{9}{3+x^2}\,dx$
4. $\int\frac{1}{z^2+2z+5}\,dz$
5. $\int\frac{dy}{y\sqrt{9+4y^2}}$
6. $\int\frac{dz}{z^2\sqrt{4+z^2}}$

7. $\int \frac{1}{1+x^2}dx$

8. $\int \frac{y+3}{y^2+9}dy$

9. $\int \frac{14}{49+16y^2}dy$

10. $\int \frac{1}{\sqrt{1-2x^2}}dx$

11. $\int \frac{12}{\sqrt{9-4x^2}}dx$

12. $\int \frac{1}{\sqrt{5-x^2}}dx$

13. $\int 2\sqrt{1-4x^2}\,dx$

14. $\int \sqrt{6-4x-x^2}\,dx$

15. $\int \frac{1}{x^2\sqrt{9-x^2}}dx$

16. $\int \sqrt{-2x-x^2}\,dx$

17. $\int \sqrt{5+4x-x^2}\,dx$

18. $\int \frac{t^3}{\sqrt{t^2+9}}dt$

19. $\int \frac{x^2}{\sqrt{4-x^2}}dx$

20. $\int x^2\sqrt{1-x^2}\,dx$

21. $\int \frac{1}{x^2+4x+8}dx$

22. $\int \frac{2}{x^2+10x+30}dx$

23. $\int \frac{1}{\sqrt{4-(x+2)^2}}dx$

24. $\int \frac{2x-7}{\sqrt{9-x^2}}dx$

25. $\int \frac{2}{13-4x+x^2}dx$

26. $\int \frac{s^3}{(4s^2+9)^{\frac{3}{2}}}ds$

27. $\int \left(\frac{7}{\sqrt{1+x^2}}+\frac{\sqrt{1-2x^2}}{4}\right)dx$

28. Show that

$$\int \frac{1}{a^2x^2+b^2}dx = \frac{1}{ab}\tan^{-1}\left(\frac{a}{b}x\right)+c,$$

where a, b, and c are constants.

(d) *Integrals of the form* $\int \frac{1}{a+b\sin^2 x + c\cos^2 x}dx$ *where* a, b, *and* c *are constants*

Integrals of this type are evaluated by applying the trigonometric substitution $t=\tan x$. Consider a right-angled triangle with $t=\tan x$ as shown in Figure 10.1.

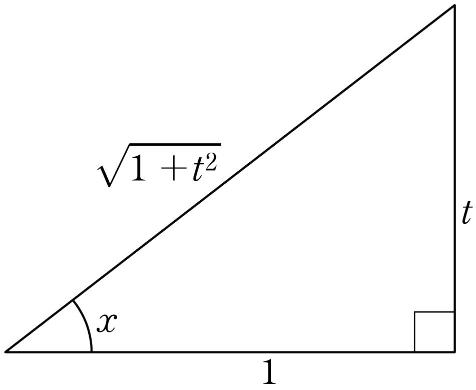


Figure 10.1: *A right-angled triangle*

From Figure 10.1, it can be observed that:

If $t = \tan x \Rightarrow \sin x = \dfrac{t}{\sqrt{1+t^2}}$ and $\cos x = \dfrac{1}{\sqrt{1+t^2}}$.

Differentiate $t = \tan x$ with respect to x, and re-write the integral in terms of t as follows:

$dt = \sec^2 x dx$

But $\sec^2 x = 1 + \tan^2 x$.

Now, $dt = (1 + \tan^2 x)dx$, $dt = (1+t^2)dx$

Thus, $dx = \dfrac{1}{(1+t^2)} dt$.

Substituting these values in the given integral and proceeding with integration leads to the required solution.

Example 10.47

Find $\displaystyle\int \frac{1}{1+\sin^2 x} dx.$

Solution

Given $\displaystyle\int \frac{1}{1+\sin^2 x} dx.$ In this case, the substitution of $t = \tan x$ can be used.

Let $t = \tan x \Rightarrow \sin x = \dfrac{t}{\sqrt{1+t^2}}$ and $dx = \dfrac{1}{(1+t^2)} dt$.

Thus, $\int \frac{1}{1+\sin^2 x}dx = \int \frac{1}{1+\left(\frac{t}{\sqrt{1+t^2}}\right)^2} \times \frac{1}{1+t^2}dt$

$$= \int \frac{1}{1+\frac{t^2}{1+t^2}} \times \frac{1}{1+t^2}dt$$

$$= \int \frac{1+t^2}{1+2t^2} \times \frac{1}{1+t^2}dt$$

$$= \int \frac{1}{1+2t^2}dt$$

$$= \int \frac{1}{1+\left(\sqrt{2}t\right)^2}dt$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}t\right)+c$$

Therefore, $\int \frac{dx}{1+\sin^2 x} = \frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\tan x\right)+c.$

Example 10.48

Show that $\int \frac{4}{\cos^2 x+9\sin^2 x}dx = \frac{4}{3}\tan^{-1}\left(3\tan x\right)+c.$

Solution

Given $\int \frac{4}{\cos^2 x+9\sin^2 x}dx$. In this case, the substitution of $t=\tan x$ can be used.

Let $t=\tan x \Rightarrow \sin x = \frac{t}{\sqrt{1+t^2}}$, $\cos x = \frac{1}{\sqrt{1+t^2}}$, and $dx = \frac{1}{1+t^2}dt$

Thus, $\int \frac{4}{\cos^2 x+9\sin^2 x}dx = \int \frac{4}{\frac{1}{1+t^2}+9\left(\frac{t^2}{1+t^2}\right)} \times \frac{1}{1+t^2}dt$

$$= \int \frac{4(1+t^2)}{1+9t^2} \cdot \frac{1}{1+t^2} dt$$

$$= 4\int \frac{dt}{1+9t^2}$$

Let $3t = \tan\theta \Rightarrow dt = \frac{1}{3}\sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{4}{\cos^2 x + 9\sin^2 x} dx = 4\int \frac{\frac{1}{3}\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \frac{4}{3}\int \frac{\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \frac{4}{3}\int d\theta = \frac{4}{3}\theta + c$$

But $\theta = \tan^{-1}(3t)$ and $t = \tan x$

$$\Rightarrow \int \frac{4}{\cos^2 x + 9\sin^2 x} dx = \frac{4}{3}\tan^{-1}(3\tan x) + c$$

Therefore, $\int \frac{4}{\cos^2 x + 9\sin^2 x} dx = \frac{4}{3}\tan^{-1}(3\tan x) + c$.

(e) *Integrals of the form* $\int \frac{1}{a + b\sin x + c\cos x} dx$***, where*** a***,*** b ***and*** c ***are constants***

Integrals of these type are also evaluated using the trigonometric substitution of $t = \tan\left(\frac{x}{2}\right)$ and the double angle formula of tangent of an angle, that is $\tan 2\text{A} = \frac{2\tan \text{A}}{1-\tan^2 \text{A}}$.

For instance, if $\text{A} = \frac{x}{2}$, then the double angle formula of tangent will be as follows;

$$\tan 2\left(\frac{x}{2}\right) = \tan x = \frac{2\tan\left(\frac{x}{2}\right)}{1 - \tan^2\left(\frac{x}{2}\right)}.$$

But $\tan\left(\frac{x}{2}\right) = t \Rightarrow \tan x = \frac{2t}{1-t^2}$.

The relation $\tan x = \frac{2t}{1-t^2}$ can be represented in a right-angled triangle ABC as shown in Figure 10.2.

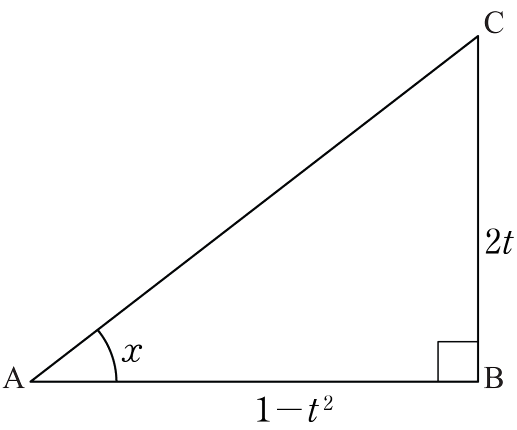


Figure 10.2: *Right-angled triangle*

In Figure 10.2, the hypotenuse is obtained by Pythagoras' theorem. That is,

$$\left(\overline{AC}\right)^2 = \left(\overline{AB}\right)^2 + \left(\overline{BC}\right)^2.$$

$$\left(1-t^2\right)^2 + \left(2t\right)^2$$

$$\Rightarrow \left(\overline{AC}\right)^2 = 1 - 2t^2 + t^4 + 4t^2$$

$$= 1 + 2t^2 + t^4$$

$$= (1+t^2)^2$$

Therefore, $\overline{AC} = 1 + t^2$.

Now, the sides of the triangle can be represented as shown in Figure 10.3.

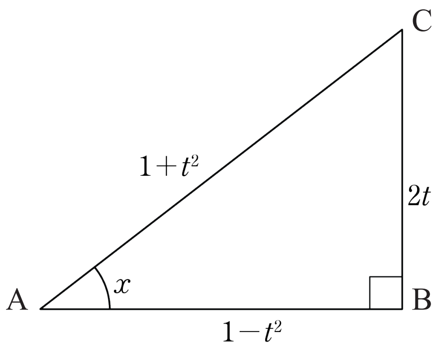


Figure 10.3: *Right-angled triangle*

In Figure 10.3, the ratios of sine and cosine formulae will be as follows;

$\sin x = \frac{2t}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$.

Also, if $t = \tan\frac{x}{2}$, then $\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$.

But $\sec^2\frac{x}{2} = 1 + \tan^2\frac{x}{2}$.

Now, $\frac{dt}{dx} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right)$.

Substituting $t = \tan\left(\frac{x}{2}\right)$ gives,

$$\frac{dt}{dx} = \frac{1}{2}\left(1+t^2\right) \Rightarrow dx = \frac{2}{1+t^2}dt.$$

Example 10.49

Show that

$$\int \sec x\,dx = \text{In}\left|\sec x + \tan x\right| + c.$$

Solution

Given $\int \sec x\,dx = \int \frac{1}{\cos x}dx.$

Substituting $\cos x = \frac{1-t^2}{1+t^2}$ and

$dx=\dfrac{2}{1+t^2}dt$, the integral becomes,

$$=\int\frac{1+t^2}{1-t^2}\times\frac{2}{1+t^2}dt$$

$$=\int\frac{2}{1-t^2}dt$$

$$=2\int\frac{1}{1-t^2}dt$$

But $\dfrac{1}{1-t^2}=\dfrac{1}{(1-t)(1+t)}$.

Decompose the fraction into partial fractions. That is,

$$\frac{1}{(1-t)(1+t)}=\frac{\text{A}}{(1-t)}+\frac{\text{B}}{(1+t)}$$

Solving for A and B gives

$\text{A}=\dfrac{1}{2}$ and $\text{B}=\dfrac{1}{2}$.

Thus, $$\frac{1}{(1-t)(1+t)}=\frac{\frac{1}{2}}{1-t}+\frac{\frac{1}{2}}{1+t}$$

$$=\frac{1}{2(1-t)}+\frac{1}{2(1+t)}$$

Now, the integral becomes;

$$\int\sec x dx=2\int\left(\frac{1}{2(1-t)}+\frac{1}{2(1+t)}\right)dt$$

$$=\int\frac{1}{1-t}dt+\int\frac{1}{1+t}dt$$

$$=-\ln|1-t|+\ln|1+t|+c$$

$$=\ln|1+t|-\ln|1-t|+c$$

$$=\ln\left|\frac{1+t}{1-t}\right|+c$$

But $t=\tan\dfrac{x}{2}$.

Hence, $\displaystyle\int\sec x\,dx=\ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right|+c$.

More simplification gives,

$$\int\sec x\,dx=\ln\left|\frac{1+\dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}{1-\dfrac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}\right|+c$$

$$=\ln\left|\frac{\dfrac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\frac{x}{2}}}{\dfrac{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)}}\right|+c$$

$$=\ln\left|\frac{\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)}\right|+c$$

Multiplying the numerator and denominator by $\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)$ gives,

$$=\ln\left|\frac{\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}{\left[\cos\left(\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)\right]\left[\cos\left(\frac{x}{2}\right)+\sin\left(\frac{x}{2}\right)\right]}\right|+c$$

$$=\ln\left|\frac{\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right|+c$$

$$=\ln\left|\frac{\cos^2\left(\frac{x}{2}\right)+\sin^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)-\sin^2\left(\frac{x}{2}\right)}\right|+c$$

$$=\ln\left|\frac{1+\sin x}{\cos x}\right|+c$$

$$=\ln\left|\frac{1}{\cos x}+\frac{\sin x}{\cos x}\right|+c$$

Therefore,

$$\int\sec x dx=\ln|\sec x+\tan x|+c.$$

Alternatively,

Given $\int \sec x\,dx$.

Mutliplying by $\sec x + \tan x$ both the numerator and denominator gives,

$$\int \sec x\,dx = \int \sec x\left(\frac{\sec x + \tan x}{\sec x + \tan x}\right)dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Let $u = \sec x + \tan x \Rightarrow du = (\sec x \tan x + \sec^2 x)dx$

$$\Rightarrow dx = \frac{du}{\sec^2 x + \sec x \tan x}$$

Thus, $\displaystyle\int \sec x\,dx = \int \frac{\sec^2 x + \sec x \tan x}{u} \times \frac{du}{\sec^2 x + \sec x \tan x}$

$$= \int \frac{1}{u}\,du$$

$$= \ln|u| + c$$

But $u = \sec x + \tan x$

$$\Rightarrow \int \sec x\,dx = \ln|\sec x + \tan x| + c.$$

Therefore, $\int \sec x\,dx = \ln|\sec x + \tan x| + c.$

Example 10.50

Verify that $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + c.$

Solution

Given $\displaystyle\int \operatorname{cosec} x\,dx = \int \frac{1}{\sin x}\,dx.$

Substituting $\sin x = \dfrac{2t}{1+t^2}$ and $dx = \dfrac{2}{1+t^2}dt$, the integral as follows,

$$\int \operatorname{cosec} x\,dx = \int \frac{1+t^2}{2t} \times \frac{2}{1+t^2}\,dt$$

$$= \int \frac{1}{t} dt$$

$$= \ln|t| + c$$

$$= \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$$

$$= \ln\left|\frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} \times \frac{\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{\sin^2\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)}\right| + c$$

$$= \ln\left|\frac{\frac{1-\cos x}{2}}{\frac{\sin x}{2}}\right| + c$$

$$= \ln\left|\frac{1-\cos x}{\sin x}\right| + c$$

$$= \ln|\text{cosec}x - \cot x| + c$$

Therefore,

$\int \cos ec\, x\, dx = \ln|\text{cosec}x - \cot x| + c$.

Example 10.51

Find $\int \frac{3\,dx}{15+9\cos x}$.

Solution

Given $\int \frac{3\,dx}{15+9\cos x}$.

Substituting $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, the integral becomes,

$$\int \frac{3\,dx}{15+9\cos x} = \int \frac{3}{15+9\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2\,dt}{1+t^2}$$

$$= 6\int \frac{1+t^2}{15(1+t^2)+9(1-t^2)} \times \frac{dt}{1+t^2}$$

$$= 6\int \frac{dt}{6t^2+24} = \int \frac{dt}{t^2+4}$$

Let $t = 2\tan\theta \Rightarrow dt = 2\sec^2\theta\, d\theta$

Thus, $\int \frac{3\,dx}{15+9\cos x} = \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta+4}$

$$= \frac{1}{2}\int \frac{\sec^2\theta}{\sec^2\theta} d\theta = \frac{1}{2}\int d\theta$$

$$= \frac{1}{2}\theta + c.$$

But $\theta = \tan^{-1}\left(\frac{t}{2}\right)$

$$\Rightarrow \int \frac{3\,dx}{15+9\cos x} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c$$

where $t = \tan\left(\frac{x}{2}\right)$

Therefore,

$$\int \frac{3\,dx}{15+9\cos x} = \frac{1}{2}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{2}\right) + c.$$

Exercise 10.9

Evaluate each of the following integrals:

1. $\int \frac{1}{1+\cos x} dx$
2. $\int \frac{1}{5+4\cos\theta} d\theta$

3. $\int \frac{1}{1+\sin 2x}\,dx$

4. $\int \frac{1}{1+\cos^2 x}\,dx$

5. $\int \frac{\sin x}{1-\cos^2 x}\,dx$

6. $\int \frac{\operatorname{cosec} x \cot x}{1+\operatorname{cosec}^2 x}\,dx$

7. $\int \frac{\sin 2x}{1+\cos^2 x}\,dx$

8. $\int \frac{1}{\cos 2x+3\sin^2 x}\,dx$

9. $\int \frac{1}{1-10\sin^2 x}\,dx$

10. $\int \frac{1}{1+\cos 2\theta}\,d\theta$

11. $\int \frac{\sec x \tan x}{\sec x+\sec^2 x}\,dx$

12. $\int \frac{\sec x}{\tan x}\,dx$

13. $\int \frac{1}{1+10\sin^2 x}\,dx$

14. $\int \frac{1}{\cos^2 2\theta-\sin^2 2\theta}\,d\theta$

15. $\int \frac{1}{5+4\cos\theta}\,d\theta$

16. $\int \frac{\cos x}{\sqrt{4-\sin^2 x}}\,dx$

17. $\int \frac{\cos 2x}{\cos x}\,dx$

18. $\int \frac{1}{3+5\cos\left(\frac{x}{2}\right)}\,dx$

19. $\int \frac{1}{4+5\cos x}\,dx$

20. $\int \operatorname{cosec}\frac{1}{2}x\,dx$

21. $\int \frac{1}{1+\cos y}\,dy$

22. $\int \frac{1}{1+2\sin^2 x}\,dx$

23. $\int \frac{dx}{2-\sin x}$

24. $\int \frac{dx}{2+\sin x}$

25. $\int \frac{1}{5+3\cos\left(\frac{\theta}{2}\right)}\,d\theta$

26. $\int \frac{\sin x}{\cos^2 x-3\cos x}\,dx$

27. $\int \frac{1}{2+2\sin\theta+\cos\theta}\,d\theta$

28. $\int \frac{dx}{2+\cos x}$

29. $\int \frac{\cos x}{\sin^2 x-3\sin x+2}\,dx$

30. $\int \frac{\sec^2 x}{\tan^2 x+2\tan x+2}\,dx$

31. $\int \frac{3+\cos\theta}{2-\cos\theta}\,d\theta$

32. $\int \frac{1}{2+2\cos x+\sin x}\,dx$

33. $\int \frac{\sqrt{1+\sin x}}{\sec x}\,dx$

Integration by splitting the numerator

If an integrand is a rational function with a quadratic denominator which cannot be expressed in simple partial fractions, then it can be expressed in terms of two fractions by splitting its numerator in the following form:

$$\text{Numerator} = \text{A}\frac{d}{dx}(\text{denominator}) + \text{B},$$

where A and B are constants which split the numerator and $\frac{d}{dx}(\text{denominator})$ is the derivative of the denominator with respect to the given variable.

Example 10.52

Determine $\int \frac{2+x}{4+x^2}\,dx$.

Solution

Given $\int \frac{2+x}{4+x^2}\,dx$.

Split the numerator of the integrand that is,

$$\text{Numerator} = \text{A}\frac{d}{dx}(\text{denominator}) + \text{B}$$

$$\Rightarrow 2+x = \text{A}\frac{d}{dx}(4+x^2) + \text{B}$$

$$\Rightarrow 2+x = 2\text{A}x + \text{B}$$

Comparing the coefficients of x and for constants of the two sides, gives

$\text{A} = \frac{1}{2}$ and $\text{B} = 2$.

Now, the integral becomes,

$$\int \frac{2+x}{4+x^2}\,dx = \int \frac{\frac{1}{2}(2x)+2}{4+x^2}\,dx$$

$$= \int \frac{x}{4+x^2}\,dx + \int \frac{2}{4+x^2}\,dx$$

$$= \frac{1}{2}\ln(4+x^2) + \tan^{-1}\left(\frac{x}{2}\right) + c$$

Therefore,

$$\int \frac{2+x}{4+x^2}\,dx = \tan^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\ln(4+x^2) + c.$$

Example 10.53

Find $\int \frac{5x+7}{x^2+4x+8}\,dx$.

Solution

Given $\int \frac{5x+7}{x^2+4x+8}\,dx$.

Split the numerator of the integrand. That is,

$$\text{Numerator} = \text{A}\frac{d}{dx}(\text{denominator}) + \text{B}$$

$$\Rightarrow 5x+7 = \text{A}\frac{d}{dx}(x^2+4x+8) + \text{B}$$

$$\Rightarrow 5x+7 = (2x+4)\text{A} + \text{B}$$

$$\Rightarrow 5x+7 = 2\text{A}x + 4\text{A} + \text{B}$$

Comparing the coefficients of x and for constants of the two sides, gives $A=\frac{5}{2}$ and $B=-3$.

Now, the integral becomes,

$$\int \frac{5x+7}{x^2+4x+8}dx = \int\left(\frac{\frac{5}{2}(2x+4)}{x^2+4x+8}-\frac{3}{x^2+4x+8}\right)dx$$

$$= \int \frac{\frac{5}{2}(2x+4)}{x^2+4x+8}dx - \int \frac{3}{x^2+4x+8}dx$$

$$= \frac{5}{2}\ln\left|x^2+4x+8\right| - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c$$

Therefore, $\displaystyle\int \frac{5x+7}{x^2+4x+8}dx = \frac{5}{2}\ln\left|x^2+4x+8\right| - \frac{3}{2}\tan^{-1}\left(\frac{x+2}{2}\right)+c.$

Integrals of the form $\displaystyle\int \frac{a\cos x \pm b\sin x}{p\cos x \pm q\sin x}dx.$

The integral in this form can be easily integrated by splitting the numerator of the integrand. In this case, the numerator is expressed into the following form:

$$\text{Numerator} = \mathrm{A}\frac{d}{dx}(\text{denominator})+\mathrm{B}(\text{denominator})$$

The constants A and B are determined by comparing both sides of the equation.

Example 10.54

Determine $\displaystyle\int \frac{2\cos x+3\sin x}{\cos x+\sin x}dx.$

Solution

Given $\displaystyle\int \frac{2\cos x+3\sin x}{\cos x+\sin x}dx.$

Split the numerator of the integrand as follows:

$$\text{Numerator} = \mathrm{A}\frac{d}{dx}(\text{denominator})+\mathrm{B}(\text{denominator}).$$

Thus, $2\cos x+3\sin x = \mathrm{A}\frac{d}{dx}(\cos x+\sin x)+\mathrm{B}(\cos x+\sin x)$

$$\Rightarrow 2\cos x + 3\sin x = (-\sin x + \cos x)A + B(\cos x + \sin x)$$

Comparing the coefficients of cosines and sines from both sides gives,

$2 = A + B$............................(i)

$3 = -A + B$.........................(ii)

Solving equations (i) and (ii) simultaneously gives,

$A = -\dfrac{1}{2}$ and $B = \dfrac{5}{2}$.

Now, the integral becomes,

$$\int \frac{2\cos\ x + 3\sin x}{\cos x + \sin x}\,dx = \int \frac{-\frac{1}{2}(-\sin x + \cos x)}{\cos x + \sin x}\,dx + \int \frac{\frac{5}{2}(\cos x + \sin x)}{\cos x + \sin x}\,dx$$

$$= -\frac{1}{2}\ln\left|\cos x + \sin x\right| + \frac{5}{2}x + c$$

Therefore, $\displaystyle\int \frac{2\cos\ x + 3\sin x}{\cos x + \sin x}\,dx = -\frac{1}{2}\ln\left|\cos x + \sin x\right| + \frac{5}{2}x + c.$

Example 10.55

Show that $\displaystyle\int \frac{2\cos x + 5\sin x}{3\cos x - \sin x}\,dx = -\frac{17}{10}\ln\left|3\cos x - \sin x\right| + \frac{1}{10}x + c.$

Solution

Given $\displaystyle\int \frac{2\cos x + 5\sin x}{3\cos x - \sin x}\,dx.$

Split the numerator of the integrand as follows:

$$2\cos x + 5\sin x = A\frac{d}{dx}(3\cos x - \sin x) + B(3\cos x - \sin x)$$

$$\Rightarrow 2\cos x + 5\sin x = A(-3\sin x - \cos x) + B(3\cos x - \sin x)$$

comparing the coefficients of cosines and sines on both sides gives,

$-A + 3B = 2$....................... (i)

$-3A - B = 5$....................... (ii)

Solving equations (i) and (ii) simultaneously gives, $A = -\dfrac{17}{10}$ and $B = \dfrac{1}{10}$.

Thus, the integral becomes,

$$\int \frac{2\cos x+5\sin x}{3\cos x-\sin x}dx=-\frac{17}{10}\int \frac{-3\sin x-\cos x}{3\cos x-\sin x}dx+\frac{1}{10}\int \frac{3\cos x-\sin x}{3\cos x-\sin x}dx$$

$$=-\frac{17}{10}\ln\left|3\cos x-\sin x\right|+\frac{1}{10}x+c$$

Therefore, $\displaystyle\int \frac{2\cos x+5\sin x}{3\cos x-\sin x}dx=-\frac{17}{10}\ln\left|3\cos x-\sin x\right|+\frac{1}{10}x+c.$

Exercise 10.10

Find each of the following:

1. $\displaystyle\int \frac{x+7}{x^2+5}dx$
2. $\displaystyle\int \frac{x+2}{2x^2+6x+5}dx$
3. $\displaystyle\int \frac{\sin\theta}{\cos\theta+\sin\theta}d\theta$
4. $\displaystyle\int \frac{2\sin\theta-\cos\theta}{\sin\theta+3\cos\theta}d\theta$
5. $\displaystyle\int \frac{2x+3}{x^2+2x+10}dx$
6. $\displaystyle\int \frac{y+4}{y^2+6y+9}dy$
7. $\displaystyle\int \frac{3u+8}{u^2+2u+5}du$
8. $\displaystyle\int \frac{2\cos\theta-3\sin\theta}{\cos\theta+4\sin\theta}d\theta$
9. $\displaystyle\int \frac{\cos\theta}{\cos\theta+\sin\theta}d\theta$
10. $\displaystyle\int \frac{3\cos x-2\sin x}{\cos x+\sin x}dx$
11. $\displaystyle\int \frac{1}{2+3\tan\theta}d\theta$
12. $\displaystyle\int \frac{\sin 2x-5\cos 2x}{7\cos 2x-2\sin 2x}dx$

Integrals of exponential functions of the form $\int a^x dx$

Integrals of this type, where x is a variable and a is a constant are easily evaluated by substitution method, with the application of natural logarithms.

Given $\int a^x dx$.

Let $u=a^x$.

Applying natural logarithm on both sides gives,

$\ln u=\ln a^x$

$\Rightarrow \ln u = x\ln a$

Differentiating with respect to x on both sides of the equation gives,

$$\frac{1}{u}\frac{du}{dx} = \ln a \Rightarrow dx = \frac{1}{u \ln a} du$$

Substituting these values, the integral becomes,

$$\int u \times \frac{1}{u \ln a} du$$

$$= \frac{1}{\ln a} \int du$$

$$= \frac{1}{\ln a} u + c$$

But $u = a^x$.

Therefore, $\int a^x dx = \frac{1}{\ln a} a^x + c.$

Example 10.56

Determine $\int 3^x dx.$

Solution

Given $\int 3^x dx$.

Let $u = 3^x$.

Apply natural logarithm on both sides of the equation to get,

$$\ln u = \ln 3^x$$

$$\Rightarrow \ln u = x \ln 3$$

Differentiating with respect to x on both sides of the equation gives,

$$\frac{1}{u}\frac{du}{dx} = \ln 3 \Rightarrow dx = \frac{1}{u \ln 3} du$$

Now, the integral becomes,

$$\int u \times \frac{1}{u \ln 3} du$$

$$= \frac{1}{\ln 3} \int du$$

$$= \frac{1}{\ln 3} u + c$$

But $u = 3^x$.

Therefore, $\int 3^x dx = \frac{3^x}{\ln 3} + c.$

Example 10.57

Find $\int 2^{4x+3} dx.$

Solution

Given $\int 2^{4x+3} dx.$

Let $u = 2^{4x+3}$.

Apply natural logarithm on both sides of the equation to get,

$$\ln u = \ln 2^{4x+3}$$

$$\ln u = (4x+3)\ln 2$$

Differentiate with respect to x on both sides of the equation to get,

$$\frac{1}{u}\frac{du}{dx} = 4\ln 2 \Rightarrow dx = \frac{1}{4u \ln 2} du$$

Substitute these values into the integral as follows,

$$\int 2^{4x+3} dx = \int u \times \frac{1}{4u \ln 2} du$$

$$= \frac{1}{4\ln 2} \int du$$

$$= \frac{1}{4\ln 2} u + c$$

But $u = 2^{4x+3}$.

Therefore, $\int 2^{4x+3} dx = \frac{2^{(4x+3)}}{4\ln 2} + c.$

Integrals of logarithmic functions of the form $\int \log ax\,dx$, where a is a constant

Integrals of logarithmic functions of the form $\int \log ax\,dx$ are determined by the method of integration by parts. This is achieved after converting the function from logarithm of base 10 to natural logarithm. The integration of $\int \log ax\,dx$ can be evaluated as follows.

Convert $\int \log ax\,dx$ to natural logarithmic form as follows:

$$\int \log ax\,dx = \int \frac{\ln ax}{\ln 10}dx$$

$$= \frac{1}{\ln 10}\int \ln ax\,dx$$

Let $u = ax \Rightarrow du = a\,dx$.

$$\Rightarrow dx = \frac{1}{a}du$$

Now, the integral becomes,

$$\frac{1}{\ln 10}\int (\ln u)\left(\frac{1}{a}du\right)$$

$$= \frac{1}{a\ln 10}\int \ln u\,du$$

But $\int \ln u\,du = u\ln|u| - u + c$

$$\Rightarrow \int \log ax dx = \frac{1}{a\ln 10}u(\ln|u| - 1) + c$$

But $u = ax$.

Thus,

$$\int \log ax\,dx = \frac{1}{a\ln 10}\times ax\left(\ln|ax| - 1\right) + c.$$

Therefore,

$$\int \log ax\,dx = \frac{1}{\ln 10}x\left(\ln|ax| - 1\right) + c.$$

Example 10.58

Find $\int \log 4x\,dx$.

Solution

Given $\int \log 4x\,dx$.

Convert the integral into natural logarithmic form as follows:

$$\int \log 4x\,dx = \frac{1}{\ln 10}\int \ln 4x\,dx$$

Let $u = 4x \Rightarrow du = 4dx$.

$$\Rightarrow dx = \frac{1}{4}du$$

Substitute these values into the integral as follows,

$$\int \log 4x\,dx = \frac{1}{\ln 10}\int \ln u\left(\frac{1}{4}du\right)$$

$$= \frac{1}{4\ln 10}\int \ln u\,du$$

$$= \frac{1}{4\ln 10}\left(u\ln|u| - u\right) + c$$

$$= \frac{1}{4\ln 10}u\left(\ln|u| - 1\right) + c$$

But $u = 4x$.

Thus,

$$\int \log 4x dx = \frac{1}{4\ln 10}\times 4x\left(\ln|4x| - 1\right) + c$$

Therefore,

$$\int \log 4x\,dx = \frac{x}{\ln 10}\left(\ln|4x| - 1\right) + c.$$

Example 10.59

Determine $\int \frac{\log_3 2x}{5x}dx$.

Solution

Given $\int \frac{\log_3 2x}{5x}dx = \int \frac{\log 2x}{5x \log 3}dx.$

Convert the integral into natural logarithmic form as follows:

$$\int \frac{\log_3 2x}{5x}dx = \int \frac{\frac{\ln 2x}{\ln 10}}{5x\frac{\ln 3}{\ln 10}}dx$$

$$= \frac{1}{5\ln 3}\int \frac{\ln 2x}{x}dx$$

Let $u = \ln 2x \Rightarrow du = \frac{1}{x}dx$.

Thus, $\int \frac{\log_3(2x)}{5x}dx = \frac{1}{5\ln 3}\int u du$

$$= \frac{1}{5\ln 3}\left(\frac{u^2}{2}\right) + c$$

$$= \frac{1}{10\ln 3}u^2 + c$$

But $u = \ln 2x$

Therefore,

$$\int \frac{\log_3(2x)}{5x}dx = \frac{1}{10\ln 3}(\ln 2x)^2 + c.$$

Exercise 10.11

Determine each of the following integrals:

1. $\int 7^x dx$
2. $\int \frac{\log x}{x}dx$
3. $\int 3(8^x)dx$
4. $\int \log(e^{-2x})dx$
5. $\int 3^{2x}dx$
6. $\int 6^{1-2y}dy$
7. $\int 4^{-x}dx$
8. $\int 9^{3-7x}dx$
9. $\int 10^{x-1}dx$
10. $\int \log 3x dx$
11. $\int \frac{\log_2 x}{x}dx$
12. $\int (2^x + \log 5x)dx$
13. $\int \log(2x+1)dx$
14. $\int (2x+1)\log(x^2+x-1)dx$
15. $\int \frac{\log 2x}{6}dx$
16. $\int \log \frac{3}{4}x\, dx$
17. $\int \left(\frac{1}{3}\right)^x dx$
18. $\int \frac{\log(\tan x)}{\cos^2 x}dx$
19. $\int \log_2 x\, dx$
20. $\int \frac{\log_5 3x}{2x}dx$

Definite integrals

When a function is integrated over a specified interval, then the integral is referred to as a definite integral.

Suppose that a function $f(x)$ is integrated over $a \le x \le b$ or $[a, b]$, then the integral is represented as $\int_a^b f(x)dx$. In this integral, a and b are called the boundaries or limits of the integral, where a is the lower limit and b is the upper limit.

Definite integrals of polynomial, trigonometric, exponential, logarithmic, and rational functions

A definite integral defines the area of the region bounded by a curve or curves, the lines $x = a$, $x = b$, and the x-axis. Figure 10.4 shows the bounded region of the function $y = f(x)$ on the interval $[a,b]$ on the xy-plane.

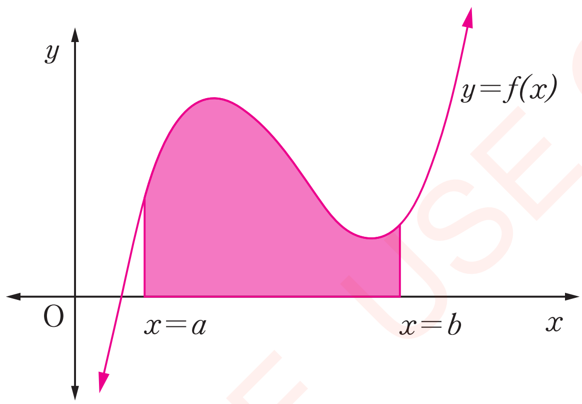


Figure 10.4: *Bounded region of the function* $y = f(x)$

The area of the region defined by the integral $\int_a^b f(x)dx$ over the interval $a \le x \le b$ is evaluated as follows:

$$\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a).$$

That is, the value of the integral is obtained by taking the difference between the values of the integral of the given function $f(x)$ for an upper limit b and a lower limit a of the variable x.

Example 10.60

Evaluate $\int_0^2 \left(x^3 + 2x^2 + x - 1\right) dx$.

Solution

Given $\int_0^2 \left(x^3+2x^2+x-1\right)dx.$

Integrate the function and substitute the given limits,

$$\int_0^2 \left(x^3+2x^2+x-1\right)dx = \left[\frac{x^4}{4}+\frac{2x^3}{3}+\frac{x^2}{2}-x\right]_0^2$$

$$=\left[\left(\frac{2^4}{4}+\frac{2(2)^3}{3}+\frac{(2)^2}{2}-2\right)-\left(\frac{(0)^4}{4}+\frac{2(0)^3}{3}+\frac{(0)^2}{2}-0\right)\right]$$

$$=\frac{28}{3}$$

Therefore, $\int_0^2 \left(x^3+2x^2+x-1\right)dx = \frac{28}{3}.$

Example 10.61

Show that $\displaystyle\int_1^2 \frac{x^3+4x^2+3x-2}{x^2+4x+3}dx = \frac{3}{2}+\ln\left(\frac{5}{6}\right).$

Solution

Given $\displaystyle\int_1^2 \frac{x^3+4x^2+3x-2}{x^2+4x+3}dx.$

The integrand is an improper fraction.

By division,

$$\frac{x^3+4x^2+3x-2}{x^2+4x+3} = x-\frac{2}{x^2+4x+3}$$

$$= x-\frac{2}{(x+1)(x+3)}$$

Let $\displaystyle\frac{2}{(x+1)(x+3)} = \frac{\text{A}}{x+1}+\frac{\text{B}}{x+3}$

Solving for A and B gives $\text{A}=1$ and $\text{B}=-1$.

$$\Rightarrow \frac{2}{(x+1)(x+3)} = \frac{1}{x+1}-\frac{1}{x+3}$$

Thus,

$$\int_1^2 \frac{x^3+4x^2+3x-2}{x^2+4x+3}dx = \int_1^2 x\,dx - \int_1^2 \frac{1}{x+1}dx + \int_1^2 \frac{1}{x+3}dx$$

$$=\left[\frac{1}{2}x^2-\ln|x+1|+\ln|x+3|\right]_1^2$$

$$=\left[\frac{1}{2}x^2+\ln\left|\frac{x+3}{x+1}\right|\right]_1^2$$

$$=\left[\frac{1}{2}(4)+\ln\left(\frac{5}{3}\right)\right]-\left[\frac{1}{2}(1)+\ln\left(\frac{4}{2}\right)\right]$$

$$=2+\ln\frac{5}{3}-\ln 2-\frac{1}{2}$$

$$=\frac{3}{2}+\ln\left(\frac{5}{3}\div 2\right)$$

$$=\frac{3}{2}+\ln\left(\frac{5}{6}\right)$$

Therefore,

$$\int_1^2\frac{x^3+4x^2+3x-2}{x^2+4x+3}dx=\frac{3}{2}+\ln\left(\frac{5}{6}\right).$$

Example 10.62

Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx.$

Solution

Given $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx.$

Let $u=\sin x\Rightarrow\dfrac{du}{dx}=\cos x$

$$\Rightarrow dx=\frac{du}{\cos x}$$

Change the limits of integration into the variable u as follows:

Since the lower limit in terms of x is $\dfrac{\pi}{4}$, then its limit in terms of u will be $u=\sin\dfrac{\pi}{4}=\dfrac{\sqrt{2}}{2}$.

Also, the upper limit in term of x is $\dfrac{\pi}{2}$, then its limit in terms of u will be $u=\sin\dfrac{\pi}{2}=1$.

Now, the integral becomes,

$$\int_{\frac{\sqrt{2}}{2}}^{1}\frac{\cos x}{u^2}\times\frac{1}{\cos x}du=\int_{\frac{\sqrt{2}}{2}}^{1}\frac{1}{u^2}du$$

$$=\int_{\frac{\sqrt{2}}{2}}^{1}u^{-2}du$$

$$=\left[-\frac{1}{u}\right]_{\frac{\sqrt{2}}{2}}^{1}$$

$$=\left[\left(-\frac{1}{1}\right)-\left(-\frac{1}{\frac{\sqrt{2}}{2}}\right)\right]$$

$$=\frac{2}{\sqrt{2}}-1=\sqrt{2}-1$$

Therefore, $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx=\sqrt{2}-1.$

Alternatively, the integral is evaluated as an indefinite integral and the limits are applied after integration where the constant term is not considered. That is;

Given $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\cos x}{\sin^2 x}dx.$

Let $u = \sin x \Rightarrow du = \cos x$

Thus, $\int \frac{\cos x}{\sin^2 x} dx = \int \frac{du}{u^2}$
$= -\frac{1}{u}.$

Substitute back $u = \sin x$.

Thus, $\int \frac{\cos x}{\sin^2 x} dx = -\frac{1}{\sin x}$

Now, introduce the limits:

$$\Rightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \left[-\frac{1}{\sin x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}.$$

$$= \left[-\frac{1}{\sin\frac{\pi}{2}} - \left(\frac{-1}{\sin\frac{\pi}{4}}\right)\right]$$

$$= \frac{2}{\sqrt{2}} - 1 = \sqrt{2} - 1$$

Therefore, $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin^2 x} dx = \sqrt{2} - 1.$

Example 10.63

Showthat $\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}} dx = 2\ln\left(\frac{4}{3}\right).$

Solution

Given $\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}} dx$

Let $u = \sqrt{e^x} \Rightarrow u^2 = e^x.$

$$\Rightarrow 2u\,du = e^x\,dx \Leftrightarrow dx = \frac{2u}{e^x} du$$

$$\Rightarrow dx = \frac{2u}{u^2} du \Rightarrow dx = \frac{2}{u} du$$

Change the limits of integration into the variable u as follows:

x	$\ln 4$	$\ln 9$
u	2	3

Now, the integral becomes,

$$\int_2^3 \frac{u}{1+u} \times \frac{2}{u} du = 2\int_2^3 \frac{1}{1+u} du$$
$$= 2\left[\ln|1+u|\right]_2^3$$
$$= 2[\ln 4 - \ln 3]$$
$$= 2\ln\left(\tfrac{4}{3}\right)$$

Therefore,

$$\int_{\ln 4}^{\ln 9} \frac{\sqrt{e^x}}{1+\sqrt{e^x}} dx = 2\ln\left(\frac{4}{3}\right).$$

Example 10.64

Evaluate $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx$ correct to five significant figures.

Solution

Given $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\,dx.$

Let $u = \ln 2x \Rightarrow du = \frac{1}{x}dx$ and $dv = x^4 dx \Rightarrow v = \frac{x^5}{5}$

Using the method of integration by parts, the integral becomes,

$$\int_{\frac{1}{2}}^{1} x^4 \ln 2x\, dx = \left[\left(\frac{x^5}{5}\right)\ln 2x\right]_{\frac{1}{2}}^{1} - \int_{\frac{1}{2}}^{1}\left(\frac{1}{x}\right)\left(\frac{x^5}{5}\right)dx$$

$$\int_{\frac{1}{2}}^{1} x^4 \ln 2x\, dx = \left[\left(\frac{x^5}{5}\right)\ln 2x\right]_{\frac{1}{2}}^{1} - \frac{1}{5}\left[\left(\frac{x^5}{5}\right)\right]_{\frac{1}{2}}^{1}$$

$$\int_{\frac{1}{2}}^{1} x^4 \ln 2x\, dx = \frac{1}{5}\left[1\times\ln 2 - \left(\frac{1}{2}\right)^5 \ln 1\right] - \frac{1}{25}\left[1 - \left(\frac{1}{2}\right)^5\right]$$

$$= \frac{1}{5}[1\times\ln 2] - \frac{1}{25}\left[1 - \left(\frac{1}{2}\right)^5\right]$$

$$= \frac{1}{5}\ln 2 - \frac{1}{25} + \frac{1}{800}$$

$$= \frac{1}{5}\ln 2 - \frac{31}{800}$$

$$= 0.099879436$$

Therefore, $\int_{\frac{1}{2}}^{1} x^4 \ln 2x\, dx = 0.099879.$

Exercise 10.12

Evaluate each of the following integrals:

1. $\int_{-1}^{3}(6-2x)dx$
2. $\int_{-4}^{0}\left(x^2+x+1\right)dx$
3. $\int_{0}^{4}(x^3-2x-3\sqrt{x})dx$
4. $\int_{2}^{6}(3t^2-2t+1)dt$
5. $\int_{0}^{\ln 2}(7e^{-x}+4e^{x})\,dx$
6. $\int_{1}^{2}\frac{y^3+1}{y^2}dy$

7. $\int_{2}^{4} \frac{dx}{x}$

8. $\int_{0}^{\frac{\pi}{4}} \frac{4\sin x}{3\tan x} dx$

9. $\int_{-1}^{1} \left(x^3 + 7x - 3\cos\frac{\pi}{2}x \right) dx$

10. $\int_{1}^{4} \left(\frac{x^4 - x^3 + \sqrt{x} - 1}{x^2} \right) dx$

11. $\int_{1}^{2} \frac{t^2 + 5t^6 - 4}{t^3} dt$

12. $\int_{1}^{2} \left(3r + \frac{1}{r^2} - \frac{1}{r^4} \right) dr$

13. $\int_{0}^{1} (6x-9)^8 dx$

14. $\int_{-1}^{0} (1-2x)^{\frac{3}{2}} dx$

15. $\int_{1}^{3} (3t-1)^{-3} dt$

16. $\int_{-3}^{-1} 3(1-x)^{-1} dx$

17. $\int_{0}^{3} e^{\frac{x}{3}} dx$

18. $\int_{0}^{\frac{1}{3}} \frac{dt}{4-3t}$

19. $\int_{0}^{\frac{\pi}{2}} 6\cos(\pi - 3x)\, dx$

20. $\int_{1}^{3} \frac{dx}{5-7x}$

21. $\int_{\frac{\pi}{2}}^{\pi} \tan\left(\theta + \frac{\pi}{4} \right) d\theta$

22. $\int_{1}^{4} \frac{dx}{2x-1}$

23. $\int_{-2}^{1} e^{3-4x} dx$

24. $\int_{0}^{\frac{\pi}{2}} 5\sin\left(\frac{1}{2}x - 1 \right) dx$

25. $\int_{\ln 2}^{\ln 4} e^x dx$

26. $\int_{0}^{1} e^{2x-2} dx$

27. $\int_{2}^{4} \frac{1}{4} \sqrt[4]{x^5}\, dx$

28. $\int_{0}^{1} x^3 \sqrt{1-x^2}\, dx$

29. $\int_{-2}^{4} \frac{2t+1}{t^2+t+3} dt$

30. $\int_{0}^{2} \frac{1}{t^2 - 6t + 9} dt$

31. $\int_{0}^{\frac{\pi}{2}} \cos\theta\, d\theta$

32. $\int_{0}^{\pi} 3\sin\theta\, d\theta$

33. $\int_{0}^{\frac{\pi}{2}} (\cos 2\theta + \sin 2\theta)\, d\theta$

34. $\int_{1}^{e} \frac{1+\ln x}{x} dx$

35. $\int_{0}^{2\sqrt{3}} \frac{x^3}{\sqrt{16-x^2}} dx$

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36. $\int_{-1}^{0} \frac{\ln(x+2)}{2x+4}\,dx$

37. $\int_{0}^{\frac{\pi}{4}} 4\cos^4\theta\, d\theta$

38. $\int_{0}^{9} \frac{\sqrt{x}}{1+\sqrt{x}}\,dx$

39. $\int_{1}^{\sqrt{10}} x\sqrt{x^2-1}\,dx$

40. $\int_{\sqrt{\ln\frac{\pi}{2}}}^{\sqrt{\ln\pi}} xe^{x^2}\cos(3e^{x^2})\,dx$

41. $\int_{0}^{1} (x+2)e^{x^2+4x}\,dx$

42. $\int_{\ln 2}^{\ln 8} \frac{1-e^t}{1+e^t}\,dt$

43. $\int_{e^2}^{e^5} \frac{dx}{x(\ln x)^2}$

44. $\int_{1}^{9} \frac{\ln x}{\sqrt{x}}\,dx$

45. $\int_{1}^{e} \frac{\ln x}{x^7}\,dx$

46. $\int_{1}^{2} (x+1)^2 \ln x\,dx$

47. $\int_{2}^{3} \frac{1}{x^2-1}\,dx$

48. $\int_{0}^{1} \frac{x-1}{x^2+3x+2}\,dx$

49. $\int_{0}^{\frac{2\pi}{3}} \cos^3 x\,dx$

50. $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x\,dx$

51. $\int_{0}^{\pi} \sin\frac{1}{2}\theta \sin\frac{3}{2}\theta\, d\theta$

52. $\int_{0}^{\frac{\pi}{2}} \sin^4 x\cos^3 x\,dx$

53. $\int_{0}^{\frac{\pi}{2}} \sin^3 x\cos^2 x\,dx$

54. $\int_{0}^{2\pi} \cos^2 3x\,dx$

55. $\int_{0}^{1} \frac{5}{x^2+2x+1}\,dx$

56. $\int_{0}^{1} \frac{1}{1+x^2}\,dx$

57. $\int_{0}^{1} \frac{1}{x^2+4x+8}\,dx$

58. $\int_{2}^{3} \frac{4}{13-4x+x^2}\,dx$

59. $\int_{0}^{\frac{\pi}{2}} \frac{\sin\theta}{\cos\theta+\sin\theta}\,d\theta$

60. $\int_{0}^{1} \frac{2x-7}{\sqrt{9-x^2}}\,dx$

Applications of integration

Definite integrals can be used to find the area under a curve and area between two curves, volume of solid of revolution about x-axis, about y-axis, and about any line, length of an arc and to determine area of a sector.

Area under a curve and area between two curves

Integration can be used to find the area under the curve and area enclosed between two curves.

Area under a curve and the *x*-axis

When calculating the area under a curve, the first step is to make a sketch of the curve. Remember that the area lying above the x-axis will have a positive value, whereas the area lying below the x-axis will have a negative value. In some cases, the required area at the same time may lie on above and below the x-axis.

Consider Figure 10.5 which shows the curve $y = f(x)$, enclosing the shaded area with the positive x-axis. If the area is denoted by A, then it is given by

$$A = \left| \int_a^b f(x)\,dx - \int_b^c f(x)dx \right|$$

where a, b, and c are values of x. These values are usually referred to as limits when evaluating the area under the curve and between two curves.

Note that, the area under the curve is taken as the magnitude of A. When calculating the area under a curve $y = f(x)$, follow the following steps:

1. Sketch the area and identify the area enclosed.
2. Determine the limits of integration if not given.
3. Set up the definite integral.
4. Integrate to get the area.

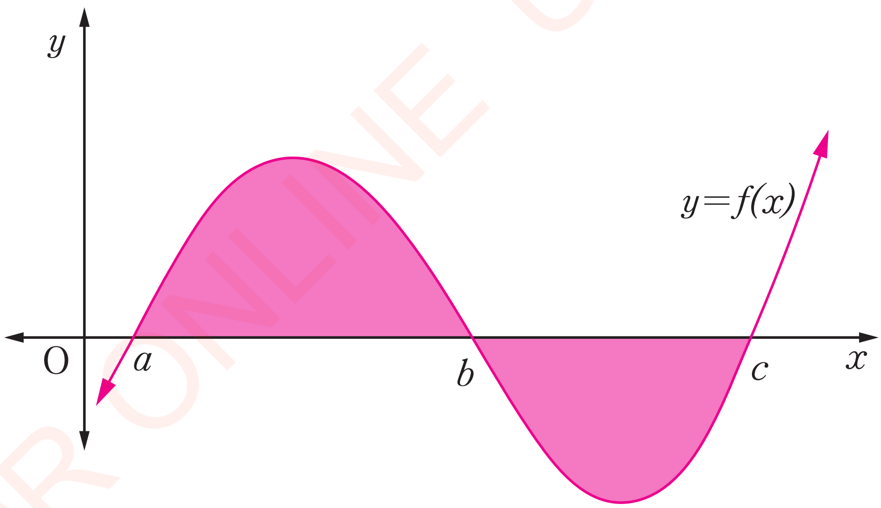


Figure 10.5: *Area under a curve*

Example 10.65

Determine the area between the curve $f(x) = -x^2 - 2x$ and the x–axis.

Solution

The curve intersect the x-axis where $f(x) = 0$.

That is, $f(x) = -x^2 - 2x$

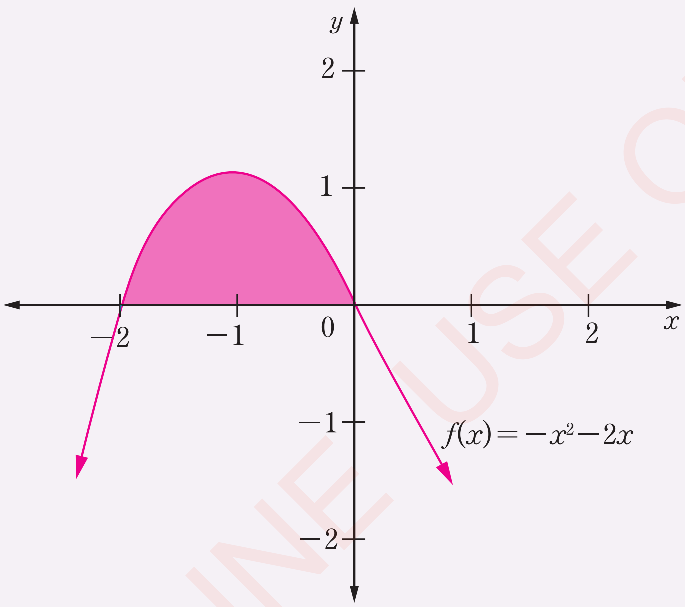
$\Rightarrow -x(x+2) = 0$

Either $-x = 0$ or $x + 2 = 0$

$\Rightarrow x = 0$ or $x = -2$

Hence, $f(x)$ intersects the x-axis at $x = 0$ and $x = -2$.

The coefficient of x^2 is negative. Thus, the curve for $f(x)$ opens downwards as shown in the following figure.



From the figure, the area between the curve and the x-axis is shaded and given by;

From $A = \int_a^b f(x)dx$

$= \int_{-2}^{0} \left(-x^2 - 2x\right)dx$

$= \left[\frac{-1}{3}x^3 - x^2\right]_{-2}^{0}$

$= \left[-\frac{1}{3}(0) - (0)\right] - \left[-\frac{1}{3}(-2)^3 - (-2)^2\right]$

$= -\left(\frac{8}{3} - 4\right)$

$$=-\left(-\frac{4}{3}\right)$$

$$=\frac{4}{3}\text{ square units.}$$

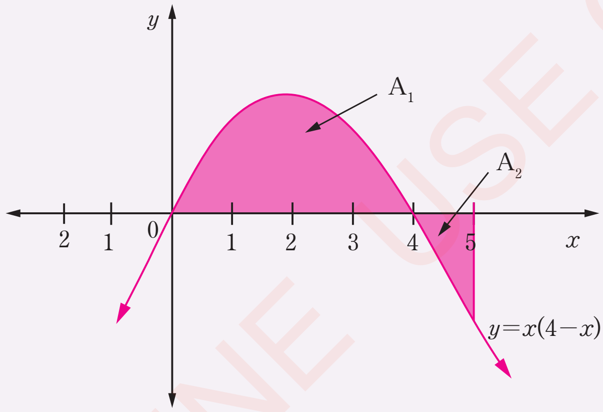
Therefore, the area is $\frac{4}{3}$ square units.

Example 10.66

Find the area between the curve $y = x(4-x)$ and the x–axis from $x = 0$ to $x = 5$.

Solution

The following is a sketch of the curve $y = x(4-x)$.



The sketch shows that the required area is divided in two parts. One part lies above the x-axis and hence has positive area. The other part lies below the x-axis and has negative area.

Using $A = \int_a^b y\,dx$, and calculating the two areas separately gives,

$$A_1 = \int_0^4 x(4-x)\,dx$$

$$=\left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$=\left(32-\frac{64}{3}\right)-0$$

$$=\frac{32}{3}$$

$$=10\frac{2}{3}\text{ square units.}$$

Thus, $A_1=\frac{32}{3}$ square units.

$$A_2=\int_4^5 x(4-x)\,dx$$

$$=\left[2x^2-\frac{x^3}{3}\right]_4^5$$

$$=\left(50-\frac{125}{3}\right)-\left(32-\frac{64}{3}\right)$$

$$=(50-32)+\left(\frac{64}{3}-\frac{125}{3}\right)$$

$$=18+\left(-\frac{61}{3}\right)$$

$$=-2\frac{1}{3}$$

Thus, $A_2=\left|\frac{-7}{3}\right|=\frac{7}{3}$ square units.

The total area under the curve between $x = 0$ and $x = 5$ is the sum of the numerical values of the two areas A_1 and A_2. That is,

$$A=\frac{32}{3}+\frac{7}{3}\text{ square units}$$

$$=13\text{ square units}$$

Therefore, the area is 13 square units.

Note that, it is possible to evaluate the area as $A=\int_0^5 x(4-x)\,dx$ but this would not give the correct answer for the required area.

Area under a curve and the y-axis

Suppose it is required to find the area between a curve $y = f(x)$ and the y-axis, from $y = a$ to $y = b$.

Consider a sketch of $y = f(x)$ as shown in Figure 10.6.

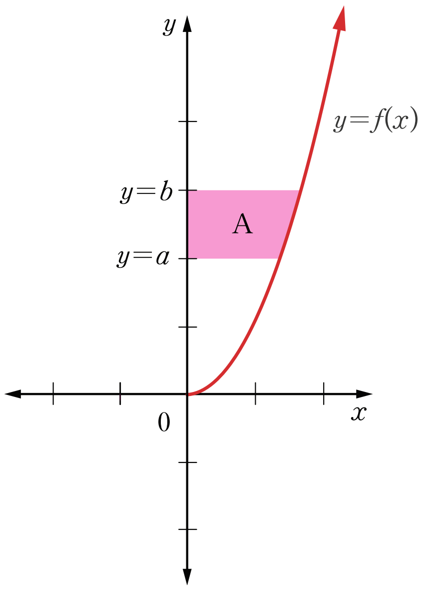


Figure 10.6: *Area between a curve $y = f(x)$ and the y-axis*

The enclosed area between the curve $y = f(x)$ and the y-axis, from $y = a$ to $y = b$ is given by:

$A = \int_a^b x\,dy$, where x is a function of y.

Example 10.67

A curve is defined by the parametric equations $x = at^2$ and $y = 2at$. Find the area bounded by the curve, the x-axis and the ordinates at $t = 1$ and $t = 2$.

Solution

Given that $A = \left|\int_a^b y\,dx\right|$, where a and b are the limits or bounding values of the variable. Replacing y by $2at$ gives $A = \left|\int_a^b 2at\,dx\right|$.

But, it is not possible to integrate a function of t with respect to x directly. Therefore, change the variable of the integral.

Given $x = at^2$,

$\Rightarrow \dfrac{dx}{dt} = 2at$, so $dx = 2at\,dt$

Now, the integral becomes,

$$A = \left|\int_1^2 (2at).(2at)\,dt\right|$$

$$= \left|\int_1^2 4a^2t^2\,dt\right|$$

$$= \left|4a^2\left[\frac{t^3}{3}\right]_1^2\right|$$

$$= 4a^2\left|\left(\frac{8}{3} - \frac{1}{3}\right)\right| = \frac{28a^2}{3}$$

Therefore, the area bounded by the curve is $\dfrac{28a^2}{3}$ square units.

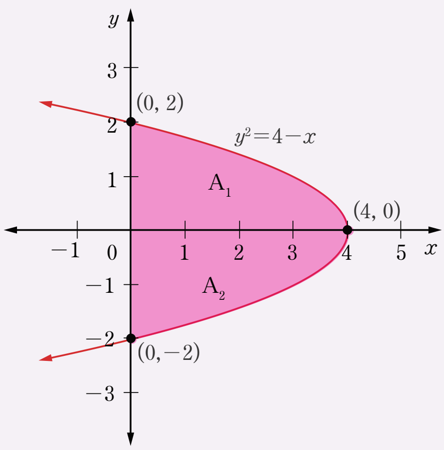
Example 10.68

Find the area enclosed between the curve $y^2 = 4 - x$ and the y-axis.

Solution

Sketch the curve of $y^2 = 4 - x$.

The curve crosses the y-axis at $(0, 2)$ and $(0, -2)$, and crosses the x– axis at $(4, 0)$ as shown in the following figure.



From the figure, the area A enclosed between the curve and the y-axis is given by:

$$A = \int_{y=-2}^{y=2} x\,dy$$

In this case, the required area is lying below the x-axis and above x axis. Thus, the required area A is divided into two areas,

$$A_1 = \int_{-2}^{0} x\,dy \text{ and } A_2 = \int_{0}^{2} x\,dy.$$

From $y^2 = 4 - x$, $x = 4 - y^2$.

Thus, $A_1 = \int_{-2}^{0} (4 - y^2)\,dy$

$$= \left[4y - \frac{y^3}{3}\right]_{-2}^{0}$$

$$= 0 - \left(-8 + \frac{8}{3}\right)$$

$$= -\left(-\frac{16}{3}\right) = \frac{16}{3}$$

$$A_2 = \int_{0}^{2} (4 - y^2)\,dy$$

$$= \left[4y - \frac{y^3}{3}\right]_{0}^{2}$$

$$= 4(2) - \frac{8}{3}$$

$$= \frac{16}{3} \text{ square units.}$$

Hence, $= \frac{16}{3} + \frac{16}{3} = \frac{32}{3}$ square units.

Therefore, the area enclosed is $\frac{32}{3}$ square units.

Area of a region between two curves

The area of a region enclosed between the curves is obtained by determining the integral of the difference between the two functions using common limits. The common limits are obtained from the points of intersection of the two curves. In Figure 10.7, the area enclosed between the two curves $f(x)$ and $g(x)$ is given by:

$$A = \left| \int_a^b [f(x) - g(x)]\,dx \right|$$

where a and b are the points at which f(x) and g(x) intersect. In this case, the function $f(x)$ is a higher function and $g(x)$ is a lower function.

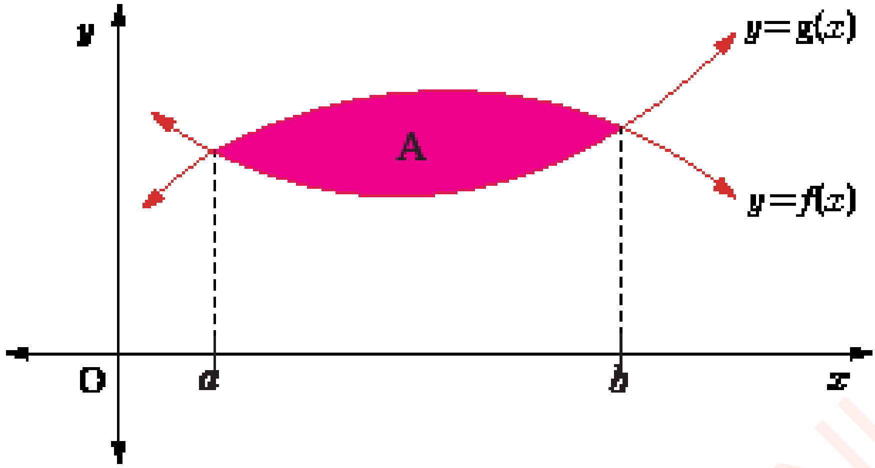


Figure 10.7: *Area enclosed between two curves*

Example 10.69

Determine the area enclosed between the curve $y = (x - 1)^2$ and the straight line $y = x + 1$.

Solution

The limits of the enclosed area are given by the x-values at the points of intersection. These can be obtained by solving the two equations simultaneously. That is,

$$(x-1)^2 = x+1$$

$$\Rightarrow x^2 - 2x + 1 = x + 1$$

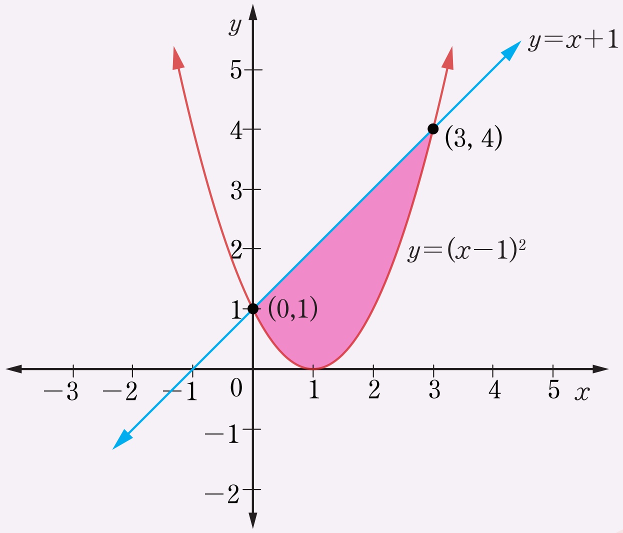
$$\Rightarrow x^2 - 3x = 0$$

$$\Rightarrow x(x-3) = 0$$

$$\Rightarrow x = 0 \text{ or } x - 3 = 0$$

Hence, the curve and the line meet at $x = 0$ and $x = 3$.

The area enclosed between the curve and the straight line at $x = 0$ and $x = 3$ is the shaded region shown in the following figure.



From the figure, the area A enclosed between the curve and the straight line is given by,

$$A = \int_0^3 \left((x+1)-(x-1)^2\right)dx$$

$$= \int_0^3 \left(x+1-x^2-1+2x\right)dx$$

$$= \int_0^3 \left(3x-x^2\right)\, dx$$

$$= \left[\frac{3}{2}x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \left[\frac{3}{2}(3)^2 - \frac{(3)^3}{3}\right] - [0-0]$$

$$= \left(\frac{3}{2}\right)(9)-\frac{27}{3}-0$$

$$= \frac{9}{2}$$

$= 4.5$ square units

Therefore, the area enclosed between the curve and the straight line is 4.5 square units.

Example 10.70

Find the area enclosed between the curves representing the functions

$f(x)=x^2+2x+2$ and $g(x)=-x^2+2x+10$.

Solution

The limits of the enclosed area are given by the x-values at the points of intersection. These can be obtained by solving the two equations simultaneously. That is,

$$x^2+2x+2=-x^2+2x+10$$

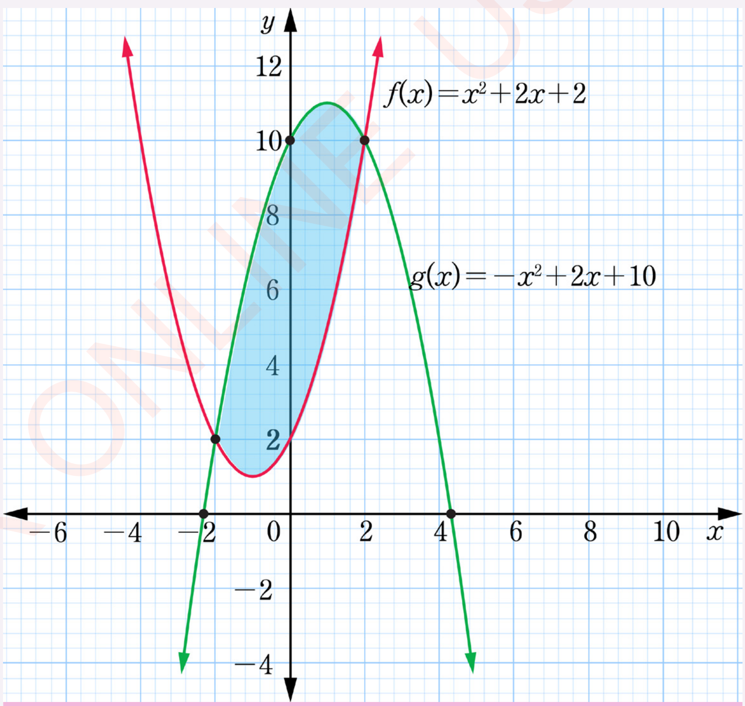
$$\Rightarrow 2x^2-8=0$$

$$\Rightarrow (x-2)(x+2)=0$$

$$\Rightarrow x=2 \text{ and } x=-2.$$

Hence, the two curves intersect at $x=2$ and $x=-2$.

The area enclosed between the two curves within $x=2$ and $x=-2$ is the shaded region as shown in the following figure.



From the figure, the area A enclosed between the two curves given by;

$$A = \int_{-2}^{2}\left[\left(-x^2+2x+10\right)-\left(x^2+2x+2\right)\right]dx$$

$$= \int_{-2}^{2}\left(-2x^2+8\right)dx = \left[-\tfrac{2x^3}{3}+8x\right]_{-2}^{2}$$

$$= \left(-\frac{16}{3}+16\right)-\left(\frac{16}{3}-16\right) = \frac{32}{3}+\frac{32}{3}$$

$$= \frac{64}{3} \text{ square units}$$

Therefore, the area of the region enclosed between the two curves is $\frac{64}{3}$ square units.

Length of an arc

The length of a curve can be obtained by taking the sum of the lengths of small chords $\overline{P_1P_2}$, $\overline{P_2P_3}$,... on the curve as shown in Figure 10.8.

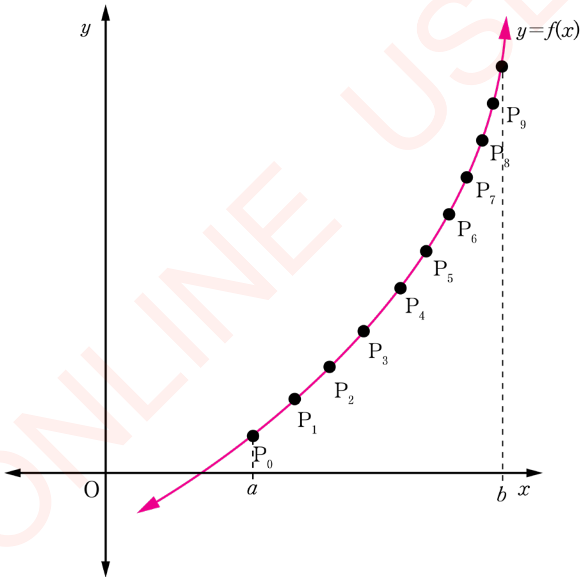


Figure 10.8: *Length of an arc of a function*

If δx and δy are the small increments in x and y, respectively, from P_{r-1} to P_r as shown in Figure 10.9, then by Pythagoras' theorem,

$\left(\overline{P_{r-1}P_r}\right)^2 = (\delta x)^2 + (\delta y)^2$. But $\overline{P_{r-1}P_r} = \delta s$.

Factorizing $(\delta x)^2$ to obtain,

$$(\delta s)^2 = \left[1+\left(\frac{\delta y}{\delta x}\right)^2\right](\delta x)^2 \qquad (10.1)$$

For an equation of the form $y = f(x)$, it is convenient to use the variable x. Equation (10.1) can be written as,

$$\delta s = \sqrt{\left(1 + \left(\frac{\delta y}{\delta x}\right)^2\right)(\delta x)^2}$$

Hence, $\delta s = \sqrt{\left(1+\left(\frac{\delta y}{\delta x}\right)^2\right)}\delta x.$

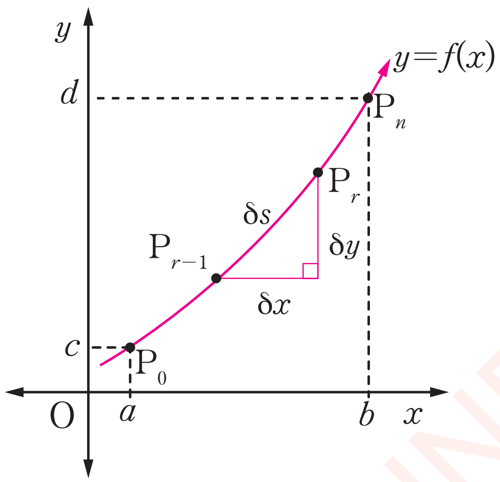


Figure 10.9: *Length of an arc*

The sum of the δs of all the chords gives the length of the arc . That is,

$$S = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx\,,$$

where a and b are values of x at the end points of the arc.

Therefore, the arc length to any point x on the curve is given by,

$$S = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx \text{ if } y = f(x),$$

$a \le x \le b$, and $S = \int_c^d \sqrt{\left(1 + \left(\frac{dx}{dy}\right)^2\right)}\, dy$

if $x = h(y)$, $c \le y \le d$.

Length of an arc with parametric equations

Suppose that the curve is in parametric form, $x = f(t)$ and $y = g(t)$. It is more convenient to find the length of an arc when the integral is evaluated with respect to t.

From equation (10.1) in the previous section, the variable t is introduced in the equation as follows:

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$= \left(\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right)\times(\delta t)^2$$

Thus, $\delta s = \sqrt{\left(\left(\frac{\delta x}{\delta t}\right)^2 + \left(\frac{\delta y}{\delta t}\right)^2\right)}\delta t.$

The sum of the δs of all the chords gives the length of the arc. That is,

$$S = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

where t_1, t_2 are the values of t corresponding to the arc ends.

Length of an arc in polar coordinates

If $r = f(\theta)$ describes a polar curve, the arc of this curve swept out as the angle θ varies from θ_1 to θ_2 can be derived as follows:

Suppose $r = f(\theta)$ is a given polar curve where the angle θ varies over the interval $\theta_1 < \theta < \theta_2$. By converting to cartesion coordinates, the following parametric form can be obtained:

$$x(\theta) = r\cos\theta = f(\theta)\cos\theta \quad (10.2)$$

$$y(\theta) = r\sin\theta = f(\theta)\sin\theta \quad (10.3)$$

Differentiating equations (10.2) and (10.3) with respect to θ gives;

$$x'(\theta) = f'(\theta)\cos\theta - f(\theta)\sin\theta \quad (10.4)$$

$$y'(\theta) = f'(\theta)\sin\theta + f(\theta)\cos\theta \quad (10.5)$$

Squaring and adding equations (10.4) and (10.5) gives,

$$\begin{aligned}
x'(\theta)^2 + y'(\theta)^2 &= \left[f'(\theta)\cos\theta - f(\theta)\sin\theta\right]^2 + \left[f'(\theta)\sin\theta + f(\theta)\cos\theta\right]^2 \\
&= f'(\theta)^2\cos^2\theta - 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\sin^2\theta + f'(\theta)^2\sin^2\theta + \\
&\quad 2f'(\theta)f(\theta)\sin\theta\cos\theta + f(\theta)^2\cos^2\theta \\
&= \left(f'(\theta)^2 + f(\theta)^2\right)\sin^2\theta + \left[f'(\theta)^2 + f(\theta)^2\right]\cos^2\theta \\
&= \left(f'(\theta)^2 + f(\theta)^2\right)\left(\sin^2\theta + \cos^2\theta\right) \\
&\Rightarrow x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2
\end{aligned}$$

Using the formula for parametric arc length, the arc length of this curve can be obtained as,

$$\begin{aligned}
\text{S} = \int_{\theta_1}^{\theta_2} \sqrt{x'(\theta)^2 + y'(\theta)^2}\, d\theta &= \int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{f'(\theta)^2 + r^2}\, d\theta \\
&= \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta
\end{aligned}$$

Therefore, the length of an arc is given as $S = \displaystyle\int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$.

Example10.71

Find the length of an arc in the first quadrant of the curve $y = 2x^{\frac{3}{2}}$, from $x = 0$ to $x = \frac{1}{3}$.

Solution

The length of the arc is given by,

$$S = \int_a^b \sqrt{\left(1 + \left(\frac{dy}{dx}\right)^2\right)}\, dx$$

Given $y = 2x^{\frac{3}{2}}$, then $\frac{dy}{dx} = 2 \times \frac{3}{2} x^{\frac{3}{2}-1}$

$$\frac{dy}{dx} = 3x^{\frac{1}{2}}$$

Thus, $S = \int_0^{\frac{1}{3}} \sqrt{1 + \left(3x^{\frac{1}{2}}\right)^2}\, dx$

$$= \int_0^{\frac{1}{3}} \sqrt{1+9x}\, dx$$

$$= \left[\frac{2(1+9x)^{\frac{3}{2}}}{27}\right]_0^{\frac{1}{3}}$$

$$= \frac{2}{27}\left[\left(1+9\left(\frac{1}{3}\right)\right)^{\frac{3}{2}} - (1+0)^{\frac{3}{2}}\right]$$

$$= \frac{2}{27}(8-1)$$

$$= \frac{14}{27}.$$

Therefore, the length of the arc is $\frac{14}{27}$ units.

Example 10.72

Find the length of an arc of the cardioid $r = 1 + \sin\theta$ from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$.

Solution

The length of the arc is given by

$$S = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Given $r = 1 + \sin\theta$ then $\frac{dr}{d\theta} = \cos\theta$

Thus,

$$S = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1+\sin\theta)^2 + \cos^2\theta}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 + 2\sin\theta + \sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{2(1+\sin\theta)}\, d\theta$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sqrt{1+\sin\theta}\cdot\sqrt{1-\sin\theta}}{\sqrt{1-\sin\theta}}\, d\theta$$

$$= \sqrt{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos\theta}{\sqrt{1-\sin\theta}}\, d\theta,$$

Let $u = \sqrt{1-\sin\theta} \Rightarrow u^2 = 1 - \sin\theta$

$$\Rightarrow 2u\,du = -\cos\theta\, d\theta$$

$$\Rightarrow -2u du = \cos\theta\, d\theta$$

By changing limits;

θ	$-\frac{\pi}{2}$	$\frac{\pi}{2}$
u	$\sqrt{2}$	0

$$\Rightarrow S=\sqrt{2}\int_{\sqrt{2}}^{0}\frac{-2u}{u}du$$

$$=-2\sqrt{2}\int_{\sqrt{2}}^{0}du=-2\sqrt{2}\left[u\right]_{\sqrt{2}}^{0}$$

$$=-2\sqrt{2}\left[0-\sqrt{2}\right]$$

$$=4$$

Therefore, the length of the arc is 4 units.

Example 10.73

Determine the length of the curve given by the parametric equations $x=3\sin(3t)$ and $y=3\cos(3t)$, $0\le t\le 2\pi$.

Solution

Given

$x=3\sin(3t)$ and $y=3\cos(3t)$,

$$\Rightarrow \frac{dx}{dt}=9\cos(3t),$$

and $\frac{dy}{dt}=-9\sin(3t)$,

The length of the curve is given by;

$$S=\int_{0}^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

$$=\int_{0}^{2\pi}\sqrt{(9\cos3t)^2+(-9\sin3t)^2}\,dt$$

$$=\int_{0}^{2\pi}\sqrt{9^2\left(\cos^2 3t+\sin^2 3t\right)}\,dt$$

$$=\int_{0}^{2\pi}9\,dt$$

$$=\left[9t\right]_{0}^{2\pi}$$

$$=\left[9\times 2\pi\right]-\left[9(0)\right]$$

$$=18\pi.$$

Therefore, the length of the curve is 18π units.

Example 10.74

Find the length of the arc of the curve $6xy=3+x^4$ between the points whose abscissae are 1 and 4.

Solution

Given $6xy=3+x^4 \Rightarrow y=\frac{1}{2x}+\frac{x^3}{6}$

$$\Rightarrow \frac{dy}{dx}=-\frac{1}{2x^2}+\frac{x^2}{2}=\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)$$

$$\Rightarrow 1+\left(\frac{dy}{dx}\right)^2=1+\left[\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)\right]^2$$

$$=1+\frac{1}{4}\left(x^4-2+\frac{1}{x^4}\right)$$

$$=\left[\frac{1}{2}\left(x^2+\frac{1}{x^2}\right)\right]^2$$

The length of the curve is given by;

$$S=\int_{a}^{b}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$=\int_{1}^{4}\sqrt{\left[\tfrac{1}{2}\left(x^2+\tfrac{1}{x^2}\right)\right]^2}\,dx$$

$$=\int_{1}^{4}\tfrac{1}{2}\left(x^2+\tfrac{1}{x^2}\right)dx$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-\frac{1}{x}\right]_{1}^{4}$$

$$=\frac{1}{2}\left[\left(\frac{64}{3}-\frac{1}{4}\right)-\left(\frac{1}{3}-1\right)\right]$$

$$=10\frac{7}{8} \text{ units}$$

Therefore, the length of the arc is $10\frac{7}{8}$ units.

Exercise 10.13

1. Find the area of the region enclosed by the parabolas $y=x^2$ and $y=2x-x^2$.

2. If $x=a\sin\theta$, $y=b\cos\theta$, find the area under the curves between $\theta=0$ and $\theta=\pi$.

3. If $x=-\sin\theta, y=1-\cos\theta$, find the area under the curve between $\theta=0$ and $\theta=\pi$.

4. Find the area bounded by the curves $y=3e^{2x}$ and $y=3e^{-x}$, and the ordinates at $x=1$ and $x=2$.

5. Determine the area of a region enclosed between the curves $x=\frac{1}{2}y^2-3$ and the straight line $y=x-1$.

6. The parametric equations of a curve are $x=2+2t$ and $y=2\sin\frac{\pi t}{10}$. Find the area under the curve between $t=0$ and $t=10$.

7. Determine the area of an arc of the cycloid, $x=\theta-\sin\theta$, $y=1-\cos\theta$ between $\theta=0$ and $\theta=2\pi$.

8. Find the area enclosed by the curves $y=\sin x$ and $y=\sin^2 x$, between $x=0$ and $x=\frac{\pi}{3}$.

9. Find the area of the region enclosed between the two curves, $y=x^3-6x^2+8x$ and $y=x^2-4x$.

10. Show that the area enclosed by the curves, $x=a(2t-\sin 2t), y=2a\sin^2 t$ and the x-axis between $t=0$ and $t=\pi$ is $3\pi a^2$ square units.

11. Find the length of the curve $y=16x^{\frac{3}{2}}$ in the interval $0\le x\le 1$.

12. Find the length of an arc of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 4$, between the lines $x = 4$ and $x = 8$.

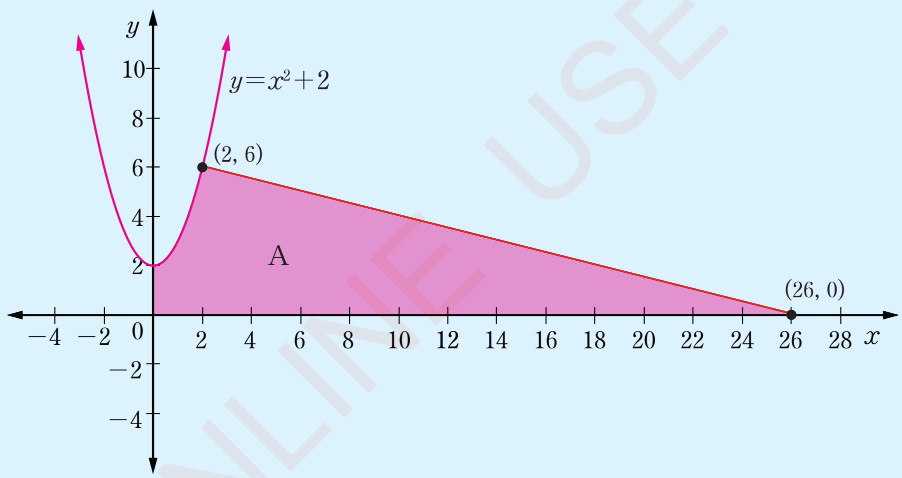
13. Find the length of an arc of the curve $y^2 = 8x^3$, between the lines $x = 1$ and $x = 3$.

14. Find the length of an arc of the curve $x^2 + y^2 = r^2$, between the points whose abscissae are $x = 0$ and $x = r$.

15. Determine the length of the curve with parametric equations $x = t\sin t,\ y = t\cos t;\ 0 \le t \le 2\pi$, correct to 4 decimal places.

16. Compute the length of the curve defined by $x = 2\cos^2\theta$ and $y = 2\cos\theta\sin\theta$, between 0 and π.

17. The region A is bounded by the curve $y = x^2 + 2$, the x and y-axes, and the line joining the points (2, 6) and (26, 0) as shown in the figure. Find the area A.



18. Find the length of the curve $r = a\sin^3\frac{\theta}{3}$ from $\theta = 0$ to $\theta = 3\pi$.

19. Prove that the length of the curve $r = ae^{k\theta}$ from $\theta = 0$ to $\theta = 2\pi$ is $\frac{a\sqrt{1+k^2}}{k}\left(e^{2\pi k} - 1\right)$.

20. Verify that the length of a curve $x = 2\cos^3\theta$ and $y = 2\sin^3\theta$ between the point corresponding to $\theta = 0$ to $\theta = \frac{\pi}{2}$ is 3 units.

Volumes of solid of revolution

Suppose the plane in Figure 10.10(a) is bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is rotated through a complete revolution about the x-axis, it will generate a solid symmetric about the x-axis.

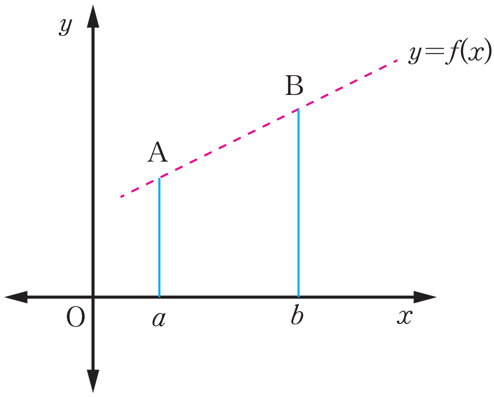


Figure 10.10 (a): *The curve and its boundaries*

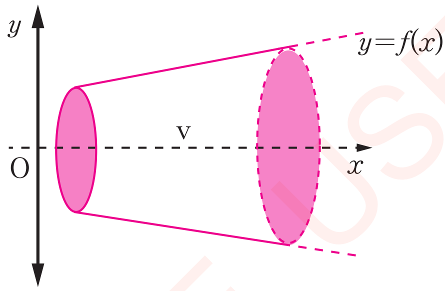


Figure 10.10 (b): *Solid rotated through a complete revolution about the x-axis*

Let V be the volume of the solid of revolution obtained after rotating a complete revolution about the x-axis. To find V in Figure 10.10(b), consider a thin strip of the original plane as in Figure 10.11(a).

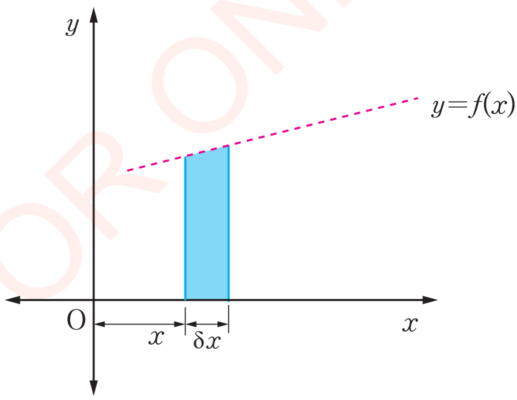


Figure 10.11(a): *Thin strip of the plain*

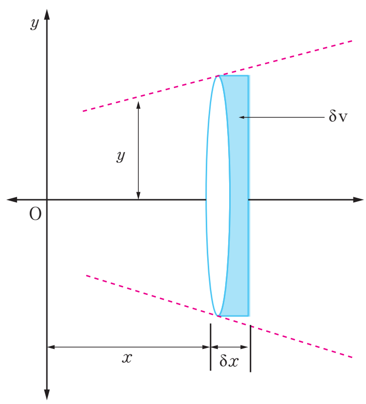


Figure 10.11(b): *Solid sliced into small cylinders*

The volume generated by the strip is approximately equal to the volume generated by the rectangle, that is $\delta v = y^2\delta x$. Since the solid generated is a flat cylinder, divide the plane into a number of such strips. Each strip will contribute its own flat disc with volume $\pi y^2 \delta x$ as shown in Figure 10.12.

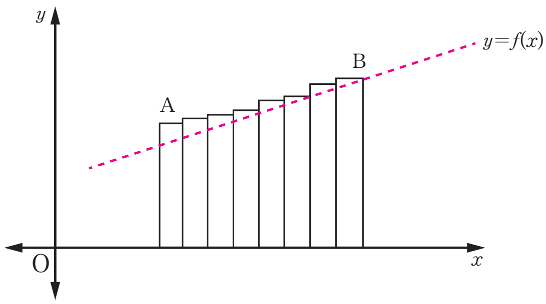


Figure 10.12(a): *Slices of a rotated solid*

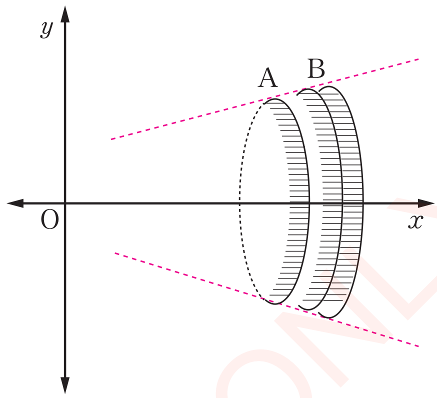


Figure 10.12(b): *A rotated solid*

The total volume, V of the slices of the solid generated is approximately given by;

$$V \simeq \sum_{x=a}^{x=b} \pi y^2 \delta x \qquad (10.6)$$

The error in approximation is due to the areas of the rectangles in Figure 10.12(a) causes the step formation in the solid as shown in Figure 10.12(b). The error disappears when $\delta x \to 0$. Therefore, the volume of revolution of the solid

$$V = \int_a^b \pi y^2 \, dx$$

The volume of the solid of revolution formed by rotating an area through one revolution about the y-axis can be found in a similar way. In this case, the volume is $V = \int_a^b \pi x^2 dy$ (10.7)

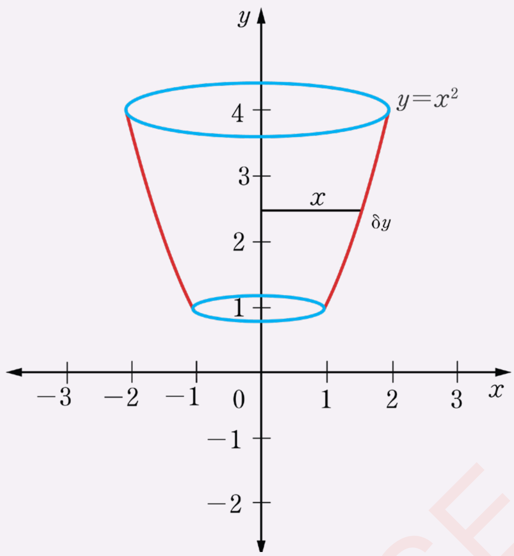
Note that, the integration in (10.7) will be with respect to y, and hence the limits of integration will be values of y.

Example 10.75

Find the volume of the solid generated by rotating about the y-axis the area in the first quadrant enclosed by $y = x^2$, $y = 1$, $y = 4$ and the y-axis.

Solution

The volume generated will be obtained by rotating the shaded area about the y-axis as shown in the following figure.



The element of volume = $\pi x^2\delta y = \pi y\delta y$

The required volume = $\int_1^4 \pi x^2 dy$ but $x^2 = y$

$$= \int_1^4 \pi y dy$$

$$= \left[\frac{\pi y^2}{2}\right]_1^4$$

$$= \frac{\pi}{2}\left(4^2 - 1^2\right)$$

$$= \frac{15\pi}{2} \text{cubic units.}$$

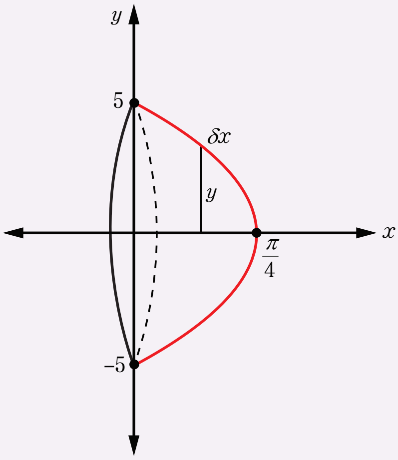
Therefore, the volume (V) = $\frac{15\pi}{2}$ cubic units.

Example 10.76

Find the volume generated when the plane bounded by $y = 5\cos 2x$, the x-axis and ordinates at $x = 0$ and $x = \frac{\pi}{4}$ rotates about the x-axis through a complete revolution.

Solution

Given the curve $y = 5\cos 2x$, then the volume generated is shown in the following figure.



The volume generated is given by;

$$V = \int_a^b \pi y^2 dx$$

$$= \int_0^{\frac{\pi}{4}} \pi y^2 dx = 25\pi \int_0^{\frac{\pi}{4}} \cos^2 2x dx$$

Express $\cos^2 2x$ in terms of double angle. That is,

$$\cos^2 2x = \frac{1}{2}(1 + \cos 4x)$$

Thus,

$$V = 25\pi \int_0^{\frac{\pi}{4}} \frac{1}{2}(1 + \cos 4x)\,dx$$

$$= \frac{25\pi}{2} \int_0^{\frac{\pi}{4}} (1 + \cos 4x)\,dx$$

$$= \frac{25\pi}{2}\left[x + \frac{\sin 4x}{4}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{25\pi}{2}\left(\left(\frac{\pi}{4} + 0\right) - (0 + 0)\right)$$

$$= \frac{25\pi^2}{8} \text{cubic units.}$$

Therefore, the volume is

$\frac{25\pi^2}{8}$ cubic units.

Example 10.77

The parametric equations of a curve are $x = 3t^2$ and $y = 3t - t^2$. Find the volume generated when the plane bounded by the curve, the x-axis and the ordinates corresponding to $t = 0$ and $t = 2$, rotates about the x-axis (give your answer correct to 3 significant figures).

Solution

The volume generated is given by;

$$V = \int_a^b \pi y^2 dx$$

But $x = 3t^2$ and $y = 3t - t^2$.

$$\Rightarrow \frac{dx}{dt} = 6t \Leftrightarrow dx = 6tdt$$

Thus,

$$V = \int_{t=0}^{t=2} \pi(3t - t^2)^2 6tdt$$

$$= \pi\int_0^2 \left(9t^2 - 6t^3 + t^4\right) 6tdt$$

$$= 6\pi\int_0^2 (9t^3 - 6t^4 + t^5)dt$$

$$= 6\pi\left[\frac{9t^4}{4} - \frac{6t^5}{5} + \frac{t^6}{6}\right]_0^2$$

$$= 6\pi\left[36 - \frac{192}{5} + \frac{64}{6}\right]$$

$$= \frac{248}{5}\pi = 155.8229956 \simeq 156$$

Therefore, the volume is approximately 156 cubic units.

In case the limits are given, it is possible to evaluate the volume generated when the plane bounded by the curve, $y=f(x)$, the x-axis and the ordinates $x=a$ and $x=b$ rotates completely about the y-axis by evaluating the definite integral

$$V=2\pi\int_a^b xy\,dx \qquad (10.9)$$

Example 10.78

Find the volume generated when the plane bounded by the curve, $y=x^2+5$, the x-axis and the ordinates $x=1$ and $x=3$ rotates about y-axis through a complete revolution.

Solution

Given the curve $y=x^2+5$.

From $V=2\pi\int_a^b xy\,dx$.

$$\Rightarrow V=2\pi\int_1^3 x\left(x^2+5\right)dx$$

$$=2\pi\int_1^3 (x^3+5x)dx$$

$$=2\pi\left[\frac{x^4}{4}+\frac{5x^2}{2}\right]_1^3$$

Thus,

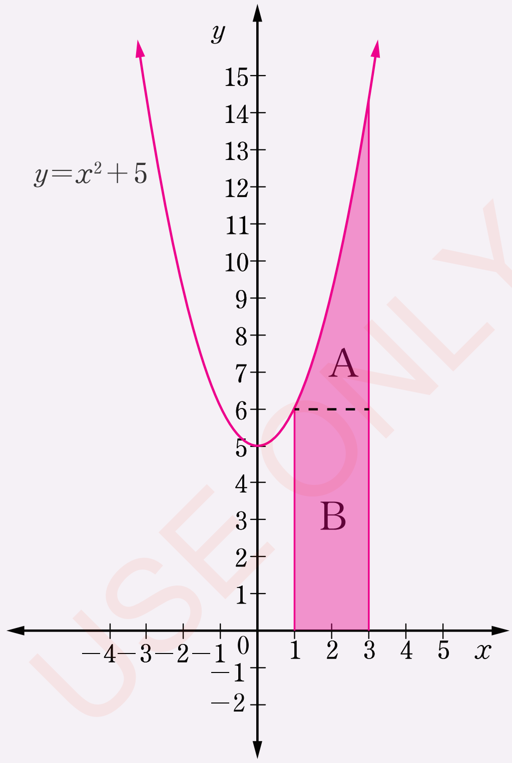
$$V=2\pi\left[\left(\frac{3^4}{4}+\frac{5(3^2)}{2}\right)-\left(\frac{1^4}{4}+\frac{5(1)^2}{2}\right)\right]$$

$$=2\pi\left[\frac{81}{4}+\frac{45}{2}-\frac{1}{4}-\frac{5}{2}\right]$$

$$=80\pi$$

Therefore, the volume generated is 80π cubic units.

Alternatively, the volume can also be obtained by rotating about the y-axis the shaded region as shown in the following figure.



Thus,

$$VA=\int_6^{14}\left[\pi(3)^2-\pi\left(\sqrt{y-5}\right)^2\right]dy$$

$$=\pi\int_6^{14}(9-y+5)\,dy$$

$$=\pi\int_6^{14}(14-y)\,dy$$

$$=\pi\left[14y-\frac{y^2}{2}\right]_6^{14}$$

$$=\pi[(196-98)-(84-18)]$$

$$=32\pi$$

$$VB=\int_0^6\left[\pi(3)^2-\pi(1)^2\right]dy$$

$$=\int_0^6(9\pi-\pi)\,dy$$

$$= 8\pi\int_0^6 dy = 8\pi\left[y\right]_0^6$$

$$= 48\pi$$

Thus, total volume = VA + VB = $32\pi + 48\pi$

$$= 80\pi$$

Therefore, the total volume is 80π cubic units.

Volume of solid of revolution about any line

(a) *Disk method*

In this case, the volume of a solid of revolution is given by,

$$V = \pi\int_a^b\left[f(x) - (\text{axis of rotation})\right]^2 dx \text{ or } V = \pi\int_a^b\left[f(y) - (\text{axis of rotation})\right]^2 dy$$

Example 10.79

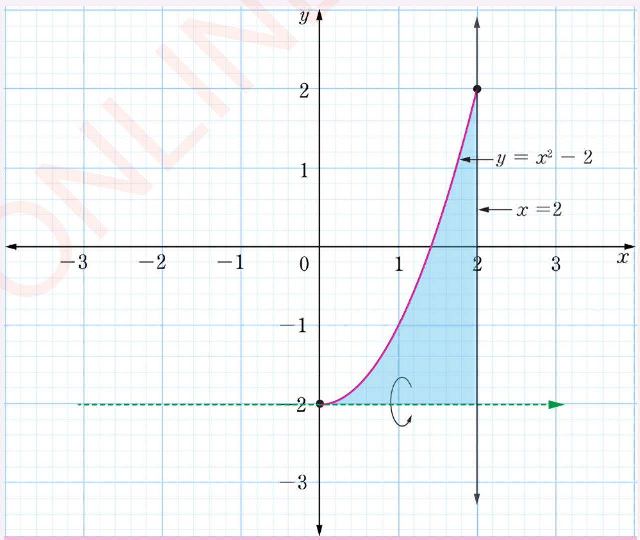
The area enclosed by the graphs $y = x^2 - 2$, $y = -2$, and $x = 2$ is rotated about the line $y = -2$. Find the volume of the resulting solid.

Solution

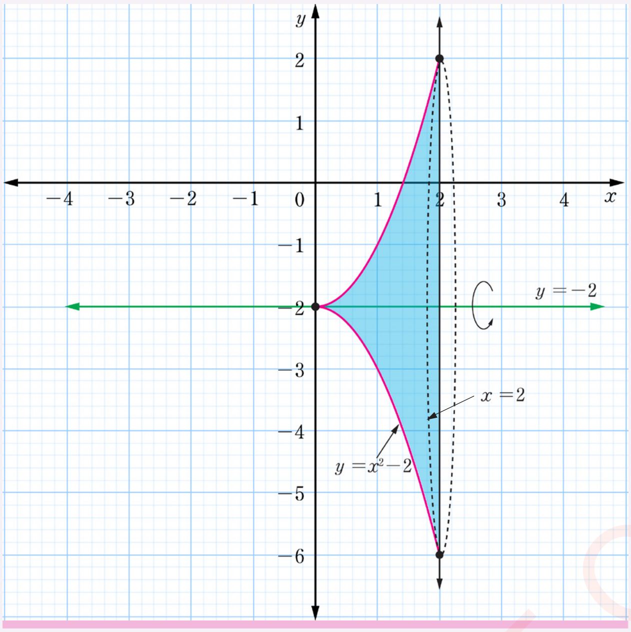
Given $y = x^2 - 2$, $y = -2$.

Point of intersection will be; $x^2 - 2 = -2 \Rightarrow x = 0$

Since the rotation is about the horizontal axis, then the cross-sectional area will be a function of x.



Before Rotation



After Rotation

Using disk method, $V = \pi\int_a^b [f(x) - \text{axis of rotation}]^2 dx.$

$$\Rightarrow V = \pi\int_0^2 (f(x) - \text{axis of rotation})^2 dx$$

$$= \pi\int_0^2 [(x^2 - 2) - (-2)]^2 dx$$

$$= \pi\int_0^2 (x^2)^2 dx$$

$$= \pi\int_0^2 x^4 dx$$

$$= \left[\frac{x^5}{5}\pi\right]_0^2$$

$$= \frac{32}{5}\pi \text{ cubic units}$$

Therefore, the volume is $\frac{32}{5}\pi$ cubic units.

(b) ***Washer method***

In this case the volume of solid of revolution is given by,

$$V = \pi\int_a^b \left[(\text{outer curve} - \text{axis of rotation})^2 - (\text{inner curve} - \text{axis of rotation})^2\right] dx$$

A similar formula hold when the region that lies between the two curves is revolved about the y-axis that is,

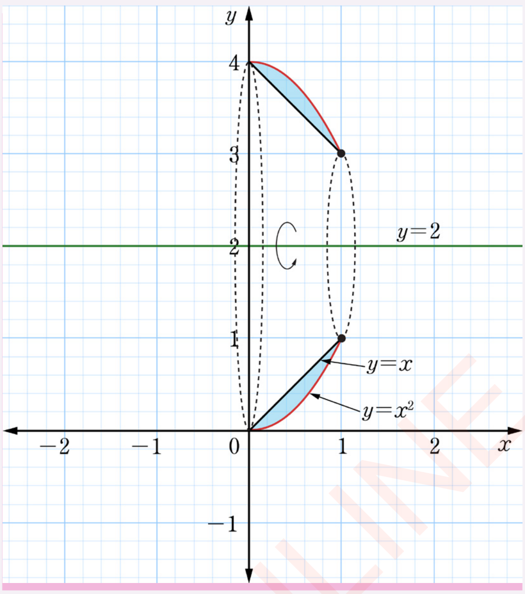
$$V = \pi\int_a^b \left[(\text{outer curve} - \text{axis of rotation})^2 - (\text{inner curve} - \text{axis of rotation})^2\right]dy$$

Example 10.80

The region enclosed by $y = x$ and $y = x^2$ is rotated about the line $y = 2$. Find the volume of the resulting solid.

Solution

The volume generated will be obtained by rotating the shaded region as shown in the following figure.



Given $y = x$, and $y = x^2$.

Points of intersection will be;

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

Outer radius = $x^2 - 2$

Inner radius = $x - 2$

$$\Rightarrow V = \pi\int_a^b \begin{bmatrix} (\text{outer curve-axis of rotation})^2 \\ -(\text{inner curve-axis of rotation})^2 \end{bmatrix} dx.$$

Thus, $V = \pi\int_0^1 \left[(x^2 - 2)^2 - (x - 2)^2\right]dx$

$$= \pi\int_0^1 \left[\left(x^4 - 4x^2 + 4\right) - \left(x^2 - 4x + 4\right)\right] dx$$

$$= \pi\int_0^1 \left(x^4 - 5x^2 + 4x\right) dx$$

$$= \pi\left[\frac{x^5}{5} - \frac{5x^3}{3} + 2x^2\right]_0^1$$

$$= \pi\left[\left(\frac{1^5}{5} - \frac{5(1)^3}{3} + 2(1)^2\right) - \left(\frac{0^5}{5} - \frac{5(0)^3}{3} + 2(0)^2\right)\right]$$

$$= \pi\left(\frac{1}{5} - \frac{5}{3} + 2\right) = \frac{8}{15}\pi \text{ cubic units}.$$

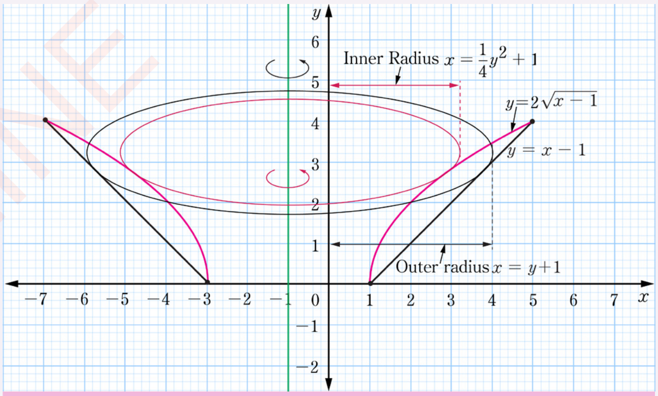
Therefore, the volume is $\frac{8}{15}\pi$ cubic units.

Example 10.81

Calculate the volume of the solid obtained by rotating the region bounded by the curve $y = 2\sqrt{x-1}$ and the straight line $y = x - 1$ about the line $x = -1$.

Solution

Since the rotation is made about a vertical axis, the cross-sectional area will be a function of y.



Given $y = 2\sqrt{x-1} \Rightarrow x = \frac{y^2}{4} + 1$

and $y = x - 1 \Rightarrow x = y + 1$

Points of intersection will be;

$$\frac{y^2}{4} + 1 = y + 1$$

$$\Rightarrow y^2 = 4y$$

$$\Rightarrow y^2 - 4y = 0$$

$$\Rightarrow y(y-4) = 0$$

$$\Rightarrow y = 0,\ y = 4.$$

The solid region obtained after rotation is shown in the figure.

$$\Rightarrow V = \pi\int_a^b \left[(\text{outer curve} - \text{axis of rotation})^2 - (\text{inner curve} - \text{axis of rotation})^2\right] dy$$

$$= \pi \int_0^4 \left[(y+1+1)^2 - \left(\frac{y^2}{4} + 1 + 1 \right)^2 \right] dy$$

$$= \pi \int_0^4 \left[(y+2)^2 - \left(\frac{y^2}{4} + 2 \right)^2 \right] dy$$

$$= \pi \int_0^4 \left(4y - \frac{y^4}{16} \right) dy$$

$$= \pi \left[2y^2 - \frac{y^5}{80} \right]_0^4$$

$$= \pi \left[\left(2(4)^2 - \frac{(4)^5}{80} \right) - \left(2(0)^2 - \frac{(0)^5}{80} \right) \right]$$

$$= \frac{96}{5}\pi \text{ cubic units}$$

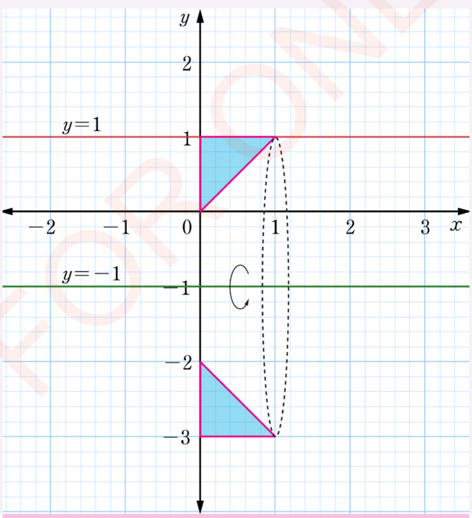
Therefore, the volume is $\frac{96}{5}\pi$ cubic units.

Example 10.82

Find the volume of the solid of revolution generated by revolving the region bounded by the lines $y = x$, $y = 1$, and $x = 0$ about the line $y = -1$.

Solution

The region generated is as shown in the following figure.



The limits of integration are $x = 0$ and $x = 1$. The outer radius $R(x) = 1 - (-1) = 2$ and the inner radius $r(x) = x - (-1) = x + 1$.

Thus,

$$V = \pi \int_0^1 (R^2 - r^2)\, dx$$

$$= \pi \int_0^1 (2^2 - (x+1)^2)\, dx$$

$$= \pi \int_0^1 (3 - 2x - x^2)\, dx$$

$$= \pi \left[3x - x^2 - \frac{x^3}{3} \right]_0^1$$

$$= \pi \left[\left(3 - 1^2 - \frac{1^3}{3} \right) - (0) \right]$$

$$= 2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$$

Therefore, the volume is $\frac{5\pi}{3}$ cubic units.

(c) *Shell method*

Consider the solid of revolution obtained by revolving about the y-axis the region R in the first quadrant between the x-axis and the curve $y = f(x)$ and lying between $x = a$ and $x = b$.

In this case, the volume of the solid of revolution is given by,

$$V = 2\pi\int_a^b (\text{shell radius})(\text{height of the shell})\,dx$$

$$V = 2\pi\int_a^b (x\text{-axis of rotation})(\text{top curve} - \text{bottom curve})\,dx$$

A similar formula holds when the region R is rotated in the first quadrant between the y-axis and the curve $x = f(y)$ lying between $y = a$ and $y = b$. That is,

$$V = 2\pi\int_a^b (\text{shell radius})\times(\text{height of the shell})\,dx$$

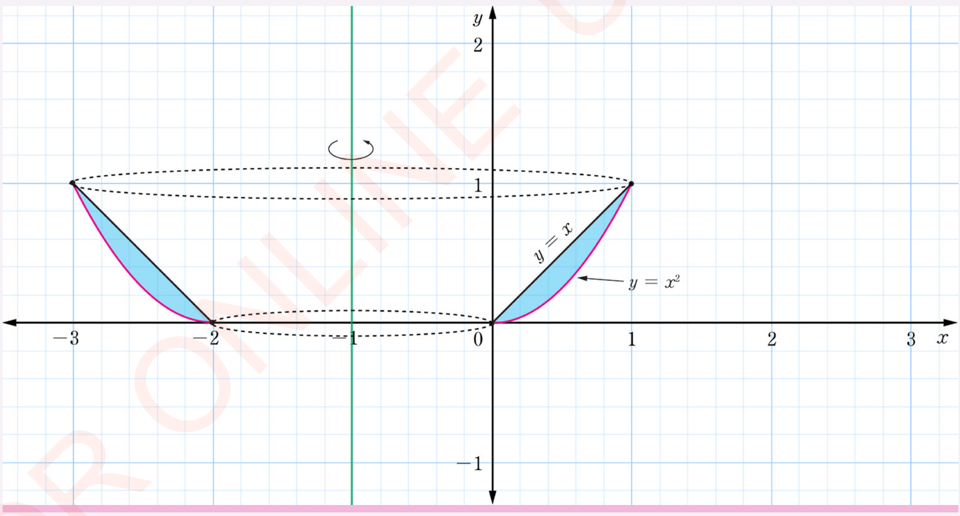
$$V = 2\pi\int_a^b (y-\text{axis of rotation})\times(\text{top curve} - \text{bottom curve})\,dy$$

Example 10.83

The region enclosed by $y = x$ and $y = x^2$ is rotated about the line $x = -1$. Find the volume of the resulting solid.

Solution

The volume generated is obtained by rotating the shaded region in the following figure.



Given $y = x$ and $y = x^2$.

Points of intersection of the curve and the line are obtained as follows.

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

Thus, $V = 2\pi\int_a^b (x\text{-axis of rotation})(\text{top curve} - \text{bottom curve})\,dx$

$$\Rightarrow V = 2\pi\int_0^1 (x-(-1))(x-x^2)\,dx$$
$$= 2\pi\int_0^1 (x+1)(x-x^2)\,dx$$
$$= 2\pi\int_0^1 (x^2-x^3+x-x^2)\,dx$$
$$= 2\pi\int_0^1 (x-x^3)\,dx$$
$$= 2\pi\left[\frac{x^2}{2}-\frac{x^4}{4}\right]_0^1$$

$$V = 2\pi\left[\left(\frac{1^2}{2}-\frac{1^4}{4}\right)-\left(\frac{0^2}{2}-\frac{0^4}{4}\right)\right]$$
$$= 2\pi\left(\frac{2-1}{4}\right)$$
$$= \frac{\pi}{2} \text{ cubic units .}$$

Therefore, the volume is $\frac{\pi}{2}$ cubic units.

Area of a sector

Consider a sector OPR in Figure 10.13 with the central angle θ and radius r. Suppose that a small sector OPQ with the central angle $\delta\theta$ and radius $r+\delta r$ is partitioned.

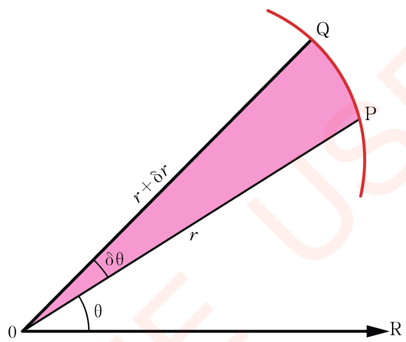


Figure 10.13: *A sector OPR*

In Figure 10.13, the radius vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ are r and $r+\delta r$, respectively. The angles between the radius vectors and the fixed line $\overline{OR}$ are θ and $\theta+\delta\theta$, respectively. If δ θ is small, the area of the sector OPQ is approximately equal to the area of the triangle.

Area of the sector $OPQ = \frac{1}{2}r(r+\delta r)\sin\delta\theta$

$$= \frac{1}{2}r^2\sin\delta\theta + \frac{1}{2}r\delta r\sin\delta\theta$$

By Maclaurin's series, $\sin\delta\theta = \delta\theta - \frac{(\delta\theta)^3}{3!} + \cdots$ and the product $\delta r\sin\delta\theta$ is small compared with $\delta\theta$.

Thus, the area of the sector OPQ up to terms in $\delta\theta$ is given by;

$$\text{Area of the sector OPQ} = \frac{1}{2}r^2\delta\theta$$

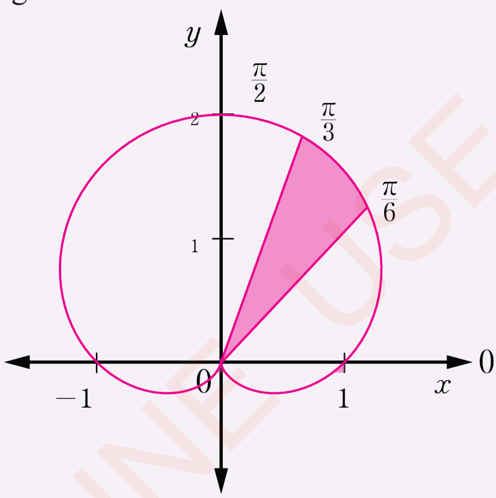
Here, it is assumed that the difference between the sector OPQ and the triangle OPQ is small compared to $\delta\theta$.

Summing all elements in the sector and proceeding to the limit, the area A of a sector becomes; $A=\int_{\alpha}^{\beta}\frac{1}{2}r^2 d\theta$

where α and β are the values of θ corresponding to the bounding radius vectors of the sector.

Example 10.84

Find the area of the cardioid $r=1+\sin\theta$ bounded between $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$, as in the following figure.



Solution

The formula for the area of a sector is given by;

$$\text{Area}=\frac{1}{2}\int_{a}^{b} r^2 d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+\sin\theta)^2 d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}(1+2\sin\theta+\sin^2\theta)d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(1+2\sin\theta+\frac{1-\cos 2\theta}{2}\right)d\theta$$

$$=\frac{1}{2}\left[\frac{3\theta}{2}-\frac{1}{4}\sin 2\theta-2\cos\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\frac{1}{2}\left[\frac{\pi}{2}-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-(1)\right]-\frac{1}{2}\left[\frac{\pi}{4}-\frac{1}{4}\left(\frac{\sqrt{3}}{2}\right)-2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$=\frac{1}{2}\left(\sqrt{3}+\frac{\pi}{4}-1\right)$$

Therefore, the area of the cardioid is $\frac{1}{2}\left(\sqrt{3}+\frac{\pi}{4}-1\right)$ square units.

Example 10.85

Find the area of the cardioid $r=3+4\cos 2\theta$ bounded between $\theta=\frac{\pi}{4}$ and $\theta=\frac{3\pi}{4}$.

Solution

The formula for the area A of a sector is given by,

$$A=\frac{1}{2}\int_a^b r^2\,d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(3+4\cos 2\theta)^2\,d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left(9+24\cos 2\theta+16\cos^2 2\theta\right)d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\left[9+24\cos 2\theta+16\left(\frac{1+\cos 4\theta}{2}\right)\right]d\theta$$

$$=\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}(17+24\cos 2\theta+8\cos 4\theta)\,d\theta$$

$$=\frac{1}{2}[17\theta+12\sin 2\theta+2\sin 4\theta]_{\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$=\frac{1}{2}\left[\left(\frac{51\pi}{4}-12+0\right)-\left(\frac{17\pi}{4}+12+0\right)\right]$$

$$=\frac{17\pi}{4}-12$$

Therefore, the area of the cardioid is $\left(\frac{17\pi}{4}-12\right)$ square units.

Exercise 10.14

1. The parametric equations of a curve are $x = 3t^2$ and $y = 3t - t^2$. Find the volume generated when the plane bounded by the curve, the x-axis and the ordinates corresponding to $t = 0$ and $t = 2$ rotates about the y-axis.
2. The part of the curve $y = x^3$ from $x = 1$ to $x = 2$ is rotated completely around the y-axis. Find the volume of the solid generated.
3. Find the volume generated when the plane figure bounded by $y = 5\cos 2x$, the x-axis and abscissae $x = 0$ and $x = \frac{\pi}{4}$ rotates about the x-axis through 2π radians.
4. Find the volume generated by revolving the area bounded by the parabola $y^2 = 8x$ and its latus rectum $x = 2$ in the first quadrant about the x-axis.
5. Find the volume generated when the area enclosed between the line $y = x + 2$, the x-axis, from $x = -2$ to $x = 2$ is rotated about the x-axis.
6. The region bounded by the y-axis, the line $y = 27$ and the curve $y = \frac{x^3}{8}$ is rotated completely about the y-axis. Find the volume generated.
7. Find the volume generated when the region bounded by the curve, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (an ellipse of length $2a$ and height $2b$) rotates through $180°$ about,
   (a) x-axis
   (b) y-axis
8. Find the volume of the solid obtained by rotating about the y-axis the region bounded by the curve $y = 2x^2 - x^3$ and the x-axis.
9. Find the volume generated by revolving the area bounded by the curve, $y = 2x^2$, from $y = 0$ and $x = 5$ about the line $y = 0$.
10. Find the volume generated by revolving the area cut off from the parabola $y = 4x - x^2$ by the x-axis about the line $y = 3$.
11. The region R is bounded by the curve $4y = x^2$ and the lines, $x = 4, y = 1$. Find the volume of the solid formed when R is rotated completely about the
    (a) x-axis
    (b) y-axis
    (c) line $y = 1$
    (d) line $x = 4$.
12. Find the volume generated when the area bounded by the x and y-axes, the line $x = 1$, and the curve $y = e^x$ is rotated through one revolution about the x-axis.

13. Find the volume of the solid formed by revolving the region bounded by the graph of $y=\sqrt{x}$ and $y=x^2$ about the x-axis.
14. Find the volume of the solid generated by rotating about the x-axis the region bounded by the curve $y=\frac{4}{x}$ and the line $y=5-x$.
15. Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.
16. Find the area of the cardioid $r=\cos\theta$ bounded between $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$.
17. Find the area of the curve $r=\sin 4\theta$, where $\frac{\pi}{2}\le\theta\le\pi$.
18. Determine the area of a sector bounded by $r=1-\cos\theta$, where $0\le\theta\le\frac{\pi}{3}$.

Chapter summary

1. The reverse process of differentiation is called integration.
2. An indefinite integral of a function $f(x)$ is a differentiable function $F(x)$ such that its derivative gives the original function $f(x)$.
3. Integration by substitution method is used in reversing the chain rule. It is sometimes called chain rule backward.
4. If $u(x)$ and $v(x)$ are any two differentiable functions, then $\int udv=uv-\int vdu$ is a formula for integrating by parts.
5. The area enclosed between two curves $f(x)$ and $g(x)$ is given by $A=\left|\int_a^b[f(x)-g(x)]dx\right|$.
6. Integrals of the form $\int\sin^n x\,dx$ or $\int\cos^n x\,dx$, where n is odd or even, the power of the trigonometric function determines the approach to be used.
7. The formulae for the length of an arc are $s=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$,

   $s=\int_a^b\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$, $s=\int_{t_1}^{t_2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$ and $s=\int_{\theta_1}^{\theta_2}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$
8. Integrals of the form $\int\frac{1}{a+b\cos^2x+c\sin^2x}dx$ are evaluated by applying the trigonometric substitution $t=\tan x$.

9. Integration by splitting the numerator, the numerator of the integrand is expressed in the form:

$$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B, \text{ for rational polynomial function.}$$

$$\text{Numerator} = A\frac{d}{dx}(\text{denominator}) + B(\text{denominator}), \text{ for rational}$$

trigonometric function.

10. The volume of the solid of revolution formed by rotating an area through one revolution about the x- axis is $v = \pi\int_a^b y^2 dx$ and y-axis is $v = \pi\int_a^b \pi x^2\, dy$.

11. The volume of a sector $A = \int_\alpha^\beta \frac{1}{2}r^2\, d\theta$ where α and β are the values of θ corresponding to the bounding radius vectors of the sector.

12. A function which is integrated over a specified interval, its integral is referred to as a definite integral and given by $\int_a^b f(x)dx$.

Revision exercise 10

1. Determine each of the following:

(a) $\int\sqrt{3x-1}\, dx$ (c) $\int\frac{e^x}{e^x+e^{-x}}dx$ (e) $\int x\sqrt{x-1}\, dx$

(b) $\int(x^2+3)x^2 dx$ (d) $\int x(x+4)^3\, dx$

2. Use the substitution $x = \sec\theta$ to evaluate $\int_{\sqrt{2}}^{2}\frac{dx}{x\sqrt{x^2-1}}$.

3. Use the substitution $u = \sin x$ to evaluate $\int_0^{\frac{\pi}{2}}\frac{\cos x}{3+\cos^2 x}dx$.

4. Find each of the following:

(a) $\int_1^4 (1-x)\sqrt{x}\, dx$ (c) $\int\theta\sqrt{2\theta-3}\, d\theta$

(b) $\int\frac{d\theta}{5+4\cos\theta}$ (d) $\int x^2 e^x\, dx$

5. By using the substitution $x^2 = \frac{1}{u}$, evaluate $\int_1^2 \frac{dx}{x^2\sqrt{(5x^2-1)}}$ to 4 decimal places.

6. Show that $\int_0^\pi \frac{2\cos\theta + 11\sin\theta}{3\cos\theta + 4\sin\theta}d\theta = 2\pi$.

7. Evaluate each of the following integrals:

(a) $\int_0^{\frac{\pi}{2}} \frac{d\theta}{5\sin\theta + 3}$

(b) $\int \frac{dx}{x^2\sqrt{-x^2+4}}$

(c) $\int \sin 4x \sin 3x \, dx$

(d) $\int e^x \sin x \, dx$

(e) $\int \frac{d\theta}{5+\sin\theta+\cos\theta}$

(f) $\int \frac{d\theta}{\theta^2 - 6\theta + 13}$

(g) $\int_{-1}^{1} \frac{d\theta}{1+\sin\theta\cos\theta}$

8. Show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \operatorname{cosec} 2x \, dx = \frac{1}{2}\ln 3$.

9. Evaluate each of the following integrals:

(a) $\int 2x\cos(x^2-5)\,dx$

(b) $\int (3x^2-4x)\sin(x^3-2x^2+1)\,dx$

(c) $\int \sqrt{1-\cos x}\,dx$

(d) $\int \sin^8 3z \cos^5 3z \, dz$

(e) $\int \cos^4 2t \, dt$

(f) $\int \sin 3x \sin(\cos 3x)\,dx$

(g) $\int \sec^6 3y \tan^2 3y \, dy$

(h) $\int \cos 3t \sin 8t \, dt$

(i) $\int \tan^5 x \, dx$

(j) $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$

(k) $\int (\cos x)e^{4+\sin x}\,dx$

(l) $\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \sin 8x \sin x \, dx$

(m) $\int_0^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)dw$

(n) $\int (\sin x + \cos x)(\sin x - \cos x)^5 \, dx$

10. Derive the reduction formula for $I_n = \int \sin^n x \, dx \; (n \geq 2)$ and use it to find I_6 and I_7.

11. Evaluate each of the following integrals:

(a) $\int \frac{dx}{(1+x^2)^2}$

(b) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

(c) $\int \frac{\sqrt{1+\ln x}}{x}dx$

(d) $\int \frac{x\,dx}{\sqrt{x-1}}$

(e) $\int \frac{x+1}{x^2+4x+8}dx$

12. The portion of the curve $y = x^2$ between $x = 0$ and $x = 2$ is rotated completely about the x-axis. Find the volume of the solid created.

13. Find the volume generated by rotating the straight line $y = x + 1$ from $x = 1$ to $x = 2$ completely around the x-axis.

14. Show that the area enclosed by the curves $x = 2(2t - \sin 2t)$, $y = 2a\sin^2 t$ and the x-axis, between $t = 0$ and $t = \pi$ is $3\pi a^2$ sq. units.

15. Evaluate each of the following:

(a) $\int_2^e \frac{1}{x^2 \ln x}\left(1 + \frac{1}{\ln x}\right)dx$

(b) $\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{dx}{\tan x - \sin x}$

(c) $\int_3^4 \frac{x^3 - 2x^2 - 4}{x^3 - 2x^2}dx$

(d) $\int_0^2 \frac{xdx}{\sqrt{4x^2 + 9}}$

16. Find the area swept out by the radius vector of the equiangular spiral $r = ae^{k\theta}$ as θ increases from $-\pi$ to π, where a and k are constants.

17. A curve has parametric equations $x = t^2 - 2t$, and $y = t + 1$. Find the area bounded by the curve, the x-axis and the ordinates at $t = -1$ and $t = 3$.

18. Show that the area of a sector in polar form is given by $A = \int_{\theta_1}^{\theta_2} \frac{1}{2}r^2 d\theta$, where r is the radius and θ is the measure of the central angle in radians.

19. Show that

$$\int_0^{\frac{1}{3}} \frac{dx}{3 - \sin x + \cos x} = \ln\frac{1}{4}\left(5 + \sqrt{3}\right)$$

20. Determine each of the following:

(a) $\int \frac{\sqrt{x^2 - 25}}{x}dx$, $x > 5$

(b) $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$

(c) $\int_1^{100} \frac{dx}{x(\ln x)(\ln x)}$

21. Determine each of the following:

(a) $\int \cos x \sin x\sqrt{1 - \sin^4 x}\,dx$

(b) $\int \frac{e^{\sqrt{x-2}}}{\sqrt{x-2}}dx$

(c) $\int \frac{dx}{3^{\sqrt{x}}\sqrt{x}}$

(d) $\int \frac{x + 1}{x^2 + 2x + 5}dx$

(e) $\int \frac{x^3 + x^2 + 2x + 1}{(x^2 + 1)(x^2 + 2)}dx$

(f) $\int \frac{x^3 - 4x - 10}{x^3 - x - 6}dx$

(g) $\int \frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)}dx$

(h) $\int \frac{x^2+x+1}{(x^2+1)^2}dx$

(i) $\int \frac{dx}{\sqrt{14-12x-2x^2}}$

(j) $\int \frac{\ln x}{x(1+\ln x)}dx$

(k) $\int \frac{2x^2}{(x^3+1)^2}dx$

(l) $\int \frac{dx}{\sqrt{x+1}-\sqrt{x-1}}$

22. Find the arc length of the curve $y=\ln(\cos x)$ over the interval $\left(0, \frac{\pi}{4}\right)$.

23. (a) Find the area of the region enclosed by the curve $y=\ln x$, the lines $x=1$ and $x=e$, and the x-axis.

(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

24. A particle moving along the x-axis has velocity function $v(t)=t^2e^{-t}$. How far does the particle travel from $t=0$ to $t=5$?

25. Show that the volume generated by rotating about the x – axis the area inscribed between the x-axis and the curve $cy=(x-a)(x-b)$ is $\frac{\pi(a-b)^5}{30c^2}$, where a and b are the limits and c is a constant.

26. The trend of sales of a certain bookshop is given by $s(t)=1500\sin\frac{\pi}{6}(t-7)+2000$, where t is the time in months and $t=0$ represents 1st January. Estimate the total sales over the four month period beginning 1st March.

27. Find the area of the cardioid $r=1+\sin 2\theta$ bounded by $\theta=\frac{\pi}{6}$ and $\theta=\frac{\pi}{3}$.

28. Find the length of the curve $24xy=x^4+48$ from $x=2$ to $x=4$.

29. Prove that the arc length of the curve $(y-1)^3=x^2$ over the interval $0\le x\le 8$ is 9.0734 units.

30. The line $y=4x$ meets the curve $y=\frac{9x^2}{25-x^2}$ at the origin and at points P and Q. Find:

(a) The coordinates of the points P and Q.

(b) The area enclosed by the curve and the line in the first quadrant.

Answers

Chapter One

Exercise 1.1

1. (a) 6.752×10^{-1} (e) 1.507×10^{-2}
   (b) 2.335×10^{0} (f) 2.897×10^{-1}
   (c) 1.079×10^{2} (g) 1.291×10^{33}
   (d) 6.731×10^{0} (h) 9.498×10^{-1}

2. (a) 0.1215 (f) 0.041279
   (b) 1.65×10^{-4} (g) 0.6819
   (c) 3.32515×10^{7} (h) 2.7
   (d) 3.3474 (i) –0.630587
   (e) 9.0450×10^{-5} (j) 0.57536

Exercise 1.2

1. (a) 0.785398163 (e) 9.427032344
   (b) 0.523598775 (f) 1.363213653
   (c) 6.283185307 (g) 0.002298017
   (d) 0.331612557 (h) 0.785485430

2. (a) 120° (e) 17.19°
   (b) 300° (f) 7.076°
   (c) 540° (g) 44.69°
   (d) 45° (h) 45.034°

3. (a) 1.320789186 rad
   (b) 1.592721078 rad
   (c) 56.94°
   (d) 88.81°
   (e) 1.815432093 rad
   (f) 1.871007926 rad
   (g) 292.01°
   (h) 7.55826 rad

Exercise 1.3

1. (a) 0.501233882 (f) 1.051462224
   (b) 0.707106781
   (c) 60° (g) $\frac{2}{3}$
   (d) 60° (h) 180°
   (e) 53.13° (i) 45°6′27″

2. (a) 0.7754 (f) –0.191255
   (b) 3.659 (g) 0.390016994
   (c) 0.96641 (h) 0.561901978
   (d) 0.219
   (e) 2.54067

3. (a) 4.0851°
   (b) 0.0713 rad

Exercise 1.4

1. (a) 0.2
   (b) $3\frac{11}{84}$
   (c) 1.639794667
   (d) 1.228398574
   (e) 7.019313087
   (f) 4.228814214

2. (a) 4.17×10^{-5}
   (b) 10.3282
   (c) 0.3886
   (d) 0.0021
   (e) 2.344
   (f) 1.95
   (g) 0.173384
   (h) 1.096
   (i) 6.878
   (j) 4.4
   (k) 2.23758
   (l) 3.4421

Exercise 1.5

1. 2.1
2. (a) 53.5, 374.5, 20287.75, 6, 36
   (b) $0.908\dot{3}$, 5.45, 4.9591, 0.0380, 1.4472×10^{-3}
   (c) 122.82, 12282, 1508600, 1.1170, 1.2476
   (d) 29.54, 7385, 218775, 1.5775, 2.4884
3. 30.5833 km/h, 5.7729 km/h^2
4. 4.6, 2
5. 27.1176, 13.0170
7. 6745, 455803, 67.45

Exercise 1.6

1. (a) 120 (b) 5038 (c) 0.125
2. (a) 945 (b) 350
3. (a) 0.80234 (b) 0.08426
   (c) 0.91574 (d) 0.44696
   (e) 0.93319

Exercise 1.7

1. $x=0$, $x=8$, $x=15$
2. $x=3.333333333$, $x=12$
3. $x=2$, $x=-1$, $x=1$
4. $x=1.414213562$
5. $x=-0.5\pm0.707106781i$
6. $x=-1$, $y=2$
7. $x=1$, $y=2$, $z=3$
8. $x=3$, $y=-5$
9. $x=1.16$, $y=2.04$
10. $x=1.783$, $y=1.39$, $z=4.52$
11. $x=3$ $y=-2$ $z=4$
12. $x=1$ $y=3$ $z=2$

Exercise 1.8

1. (a) $\det A=-2$, $A^{T}=\begin{pmatrix}1&1&1\\1&0&1\\0&-1&2\end{pmatrix}$

   (b) $A^{-1}=\begin{pmatrix}-0.5&1&0.5\\1.5&-1&-0.5\\-0.5&0&0.5\end{pmatrix}$

   (c) $A+A^{-1}=\begin{pmatrix}0.5&2&0.5\\2.5&-1&-1.5\\0.5&1&2.5\end{pmatrix}$

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(d) $A^2 = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & -2 \\ 4 & 3 & 3 \end{pmatrix}$

2. (a) -236

(b) $\begin{pmatrix} 21 & 24 & -5 \\ 4 & -4 & 40 \\ 10 & 20 & 15 \end{pmatrix}$

(d) $\begin{pmatrix} 213 & 62 & 241 \\ 80 & 496 & 288 \\ 161 & 268 & 355 \end{pmatrix}$

Exercise 1.9

1. 1.3734 3. 72 5. 0.7854
2. −6.873 4. 1.292707 6. 1.4833

Exercise 1.10

1. (a) $3\underline{i}+2\underline{j}+5\underline{k}$ (b) $-\underline{i}-4\underline{j}-3\underline{k}$
2. (a) -11 (b) $12\underline{i}+2\underline{j}+16\underline{k}$
   (c) 20.09975124
3. $-10\underline{i}+15\underline{j}-15\underline{k}$

Exercise 1.11

1. (a) $5+3i$ (b) $11+10i$
   (c) $-0.5+2.5i$ (d) $-32.08-22.38i$
2. (a) 3.16 (b) $-108.4°$ (c) $0.7-0.1i$

Exercise 1.12

1. (a) 7.845265319 (b) 471.8513119
2. (a) 2.234828456
   (b) −0.662496845
3. $\bar{x} = 39.69230769$, $\sum x^2 = 30370$,
   $\delta_x = 27.58032915$
4. (a) 1.047197551 (b) 404685.6m^2
   (c) 0.001 m^3 (d) 1 litre
5. (a) 0.34, 17.71, 124
   (b) 31.31, 1633.86, 11437
   (c) 17.98, 938.43, 6569

Exercise 1.13

1. (a) Excel sheet showing active cells P, Q, R , and S.
   (b) A3, F1, C6, D4
3. (a) −20.4 (c) 9.95548
   (b) 425.5 (d) 1
6. Mean = 49.6111
   Median = 53
   Mode = 73
   Variance = 718.12654
   Standard deviation = 26.79788

Revision exercise 1

1. (a)3166757.638
   (b) 7.87915876
   (c) 35305373.83
2. (a) – 3.17929
   (b) 1.024295

3. (a) 17.367
   (b) 30.83333333
4. (a) $x = 30, \quad y = 50$
   (b) $x = 60, \quad y = 90$
5. (a) $x = 1, \quad y = 3, \quad z = 3$
   (b) $x = 3, \ y = -2, \ z = 1$
6. (a) $x_1 = 0.833333, \ x_2 = 0.75$
   (b) $x_1 = 16, \ x_2 = -25$
   (c) $x_1 = 12, \ x_2 = -9, \ x_3 = 2$
   (d) $x_1 = \frac{3}{2}, \quad x_2 = -\frac{2}{5}, \quad x_3 = \frac{1}{3}$
7. (a) 62.1 (b) 26.3522
   (c) 12420 (d) 910170
10. (a) 53.46666 (b) 23.1845
    (c) 50943.06 (d) 802
11. (a) –0.094920908
    (b) 4330
    (c) 75750.50703
    (d) 1772776810
12. (a) 0.885756084
    (b) 1.000473295
    (c) $\overline{x} = 161, \ \delta_x = 19.5115$
13. (a) $AB = \begin{pmatrix} 106 & 90 & -46 \\ 11 & 9 & 27 \\ 50 & -72 & 14 \end{pmatrix}$

    (b) $A^{-1} + B^{-1} = \begin{pmatrix} 0.059 & 0.393 & 0.103 \\ 0.077 & -0.227 & 0.193 \\ 0.187 & 0.062 & -0.057 \end{pmatrix}$

    (c) $(AB)^T = \begin{pmatrix} 106 & 11 & 50 \\ 90 & 9 & -72 \\ -46 & 27 & 14 \end{pmatrix}$

14. (a) 52.5
    (b) 22.78
    (c) 655,000
15. (a) 157.2°C
    (b) 643.74 km
    (c) 576 km/h
16. (a) $-\frac{125}{2}$
    (b) $-\frac{3}{2}\underline{i} - 4\underline{j} + 11\underline{k}$
    (c) 11.80042372
    (d) $84\underline{i} - 59\underline{j} - 10\underline{k}$
17. (a) $-16 - 63\underline{i}$
    (b) $\frac{56}{25} - \frac{33}{25}\underline{i}$
    (c) 48.83646179
    (d) −30.51023745°
    (e) $-16 + 63\underline{i}$
18. (a) 0.4286
    (b) 0.09259
    (c) 2.546

Answers — Chapter Two

Exercise 2.1

1. (a) $A = \{x : x \text{ is a cube of a natural number}\}$
   (b) $B = \{x : x \text{ is a multiple of three}\}$
   (c) $C = \{x : x \text{ is an integer from } -4 \text{ to } 4 \text{ inclusive}\}$
   (d) $D = \{x : x \text{ is a domestic animal}\}$
2. (a) $A = \{x : x^2,\ x \in \mathbb{N}\}$ (b) $B = \{x : x^3,\ x \in \mathbb{N}\}$ (c) $C = \{x : x = -3 \text{ and } x = 3\}$.
   (d) $D = \{x : x \in \mathbb{N}\}$ (e) $E = \{x : x \text{ is a positive even number}\}$
   (f) $F = \{x : x \in \text{whole numbers}\}$
3. (a) $A = \{-1, 1\}$ (b) $B = \{3, 4, 5, 6, \ldots\}$ (c) $C = \{1, 3\}$

Exercise 2.2

1. (a) Infinite set – Number of all plants on the earth is uncountable
   (b) Infinite – Real numbers between 10 and 30 are uncountable
   (c) Infinite numbers between 10 and 20 inclusive are uncountable
2. T and R are equivalent
   T and R are equal
   H and T are unequal
   H and R are unequal
3. (a) True (b) False (c) True (d) False (e) False
4. (a) $\{\}, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a,b, d\}, \{b, c, d\}, \{a, c, d\}, \{a, b, c, d\}$.
   (b) $n(k) = 16$.
5. (a) $p(J) = \{\{\ \}, \{\text{Dog}\}, \{\text{Cat}\}, \{\text{Lion}\}, \{\text{Zebra}\}, \{\text{Dog, Cat}\}, \{\text{Dog, Lion}\}$
   $\{\text{Cat, zebra}\}, \{\text{Dog, zebra}\}, \{\text{Cat, Lion}\}, \{\text{Lion, zebra}\}, \{\text{Dog, Cat, Lion}\},$
   $\{\text{Dog, Cat, Zebra}\}, \{\text{Cat, Lion, Zebra}\}, \{\text{Dog, Lion, Zebra}\},$
   $\{\text{Dog, Cat, Lion, Zebra}\}\}$
   (b) $n[p(\text{J})] = 16$
6. $U = \{2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 17, 19\}$

Exercise 2.3

1. $A \cup B = \{a, b, c, d, e, f, g, h\}$

   $A \cap B = \{a, b, c, d, e\}$

2. $A \cup B = \{\text{Counting numbers}\}$

   $A \cap B = \{\text{Even numbers}\}$

3. $G \cup H = \{20, 25, 30, 45\}$

   $G \cap H = \{25\}$

4. $J \cup K = \{0, \Delta, 3\}$

   $J \cap K = \{\Delta\}$

5. $A \cup B = \{4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 44, 48, 54\}$

   $A \cap B = \{12, 24, 36\}$

6. $W \cup Z = \{14, 16, 18, 20\}$

   $W \cap Z = \{\ \}$ or $\varnothing$

7. $A \cup B = \{94, 110, 120, 131, 140, 265\}$

   $A \cap B = \{94, 110\}$

8. $A \cup B = \{2, 3, 5\}$

   $A \cap B = \{3\}$

9. $A \cup B = \{2, 3, 4, 6, 8, 9, 10, 12, 14, 15, 16, 18, 20, 21, 22, 24, 26\}$

   $A \cap B = \{6, 12, 18, 24\}$

10. $A \cup B \cup C = \{a, b, c, d, e\}$

    $A \cap B \cap C = \{\ \}$

11. $M \cap N = \{\ \}$

12. Disjoint sets

13. $M' = \{5, 6, 7, 8, 9, 10, 11, 12\}$

    $N' = \{2, 4, 5, 6, 8, 10, 12\}$

14. $A = \{\text{Orange, cabbage, pineapple}\}$

15. $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

16. (a) $A \cap B' = \{3, 4\}$

    (b) $B \cup A' = \{5, 6, 7, 8, 9, 10, 11, 12, 13\}$

    (c) $B' \cap A' = \{3, 4, 7, 8, 9, 10, 11, 12, 13\}$

    (d) $A' \cap B = \{7, 8\}$

17. (a) F (b) T (c) F (d) F

18. $A' = \{\text{magenta, red, blue}\}$

19. (a) $A \cap B = \{a, d, e\}$

    (b) $A \cup B = \{a, b, c, d, e, f\}$

    (c) $A' \cap B = \{\ \}$

    (d) $A \cap B' = \{b, c, f\}$

    (e) $A' \cap B' = \{\text{g, h, i, j}\}$

    (f) $A' \cup B' = \{\text{b, c, f, g, h, i, j}\}$

20. (a) $A \cup F = \{22, 24, 26, 28, 30\}$

    $A \cap F = \{24, 26\}$

    (b) $Y \cup W = \{2 < x < 23\}$

    $Y \cap W = \{7 \leq x \leq 18\}$

21. (a) $A - B = \{13\}$

    (b) $B - A = \{16, 17, 18\}$

22. (a) $M - K = \{32, 42\}$

    (b) $K - M = \{33, 35, 37\}$

    (c) $(M - K) \cup (K - M) = \{32, 33, 35, 37, 42\}$

23. (a) $P - R = \{8, 10, 12\}$

    (b) $R - P = \{14, 16, 18\}$

    (c) $Q - P = \{14, 16, 18, 20, 22\}$

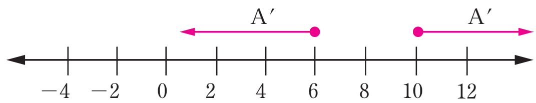
24. (a) $B - A = \{3, 5, 9\}$ (b) $(A - B)' = \{1, 3, 5, 9, 11, 17, 19\}$

25. $X - Y = \{2, 4, 6, 8, 10\}$

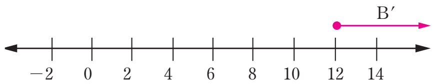
26. $x = 9$ and $y = 12$

Exercise 2.4

1. (a) A'

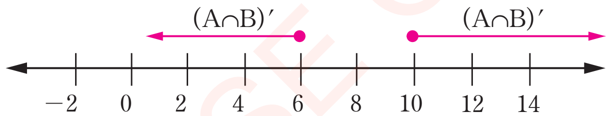


(b) B'

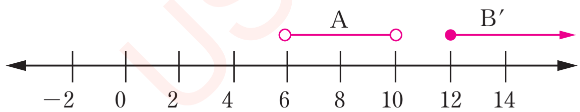


(c) $(A \cup B)'$

(d) $(A \cap B)'$

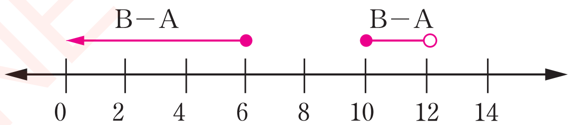


(e) $A - B$

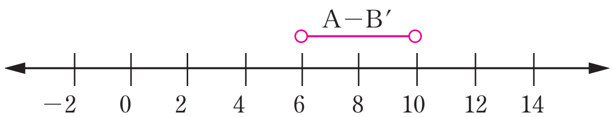


$A - B = A \cap B' = \emptyset$

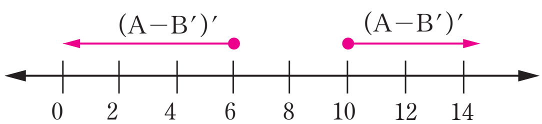
(f) $B - A$



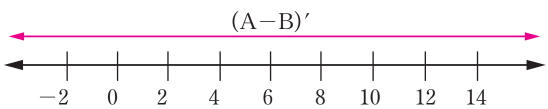
(g) $A - B'$



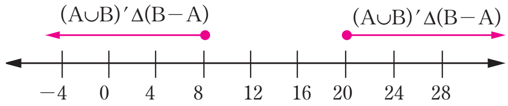
(h) $(A - B')'$



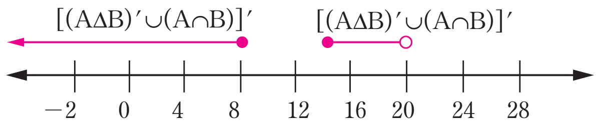
(i) $(A - B)'$



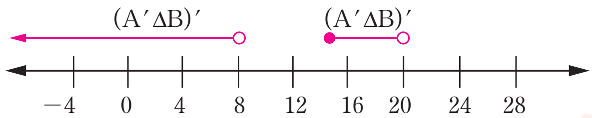
2. (g) $(A\cup B)'\Delta(B-A)$



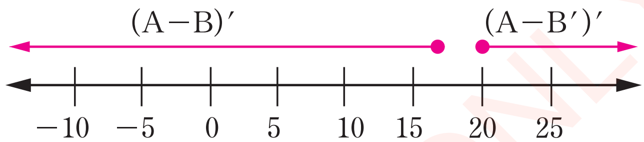
(h) $\left[(A\Delta B)'\cup(A\cap B)\right]'$



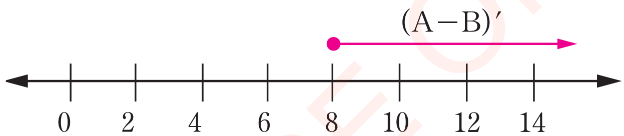
(j) $(A'\Delta B)'$



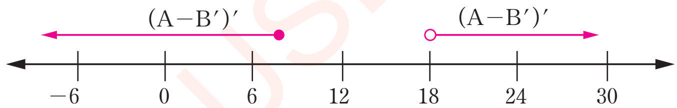
3. (a) $(A-B)'$



(b) $(B-A)'$



(c) $(A-B')'$

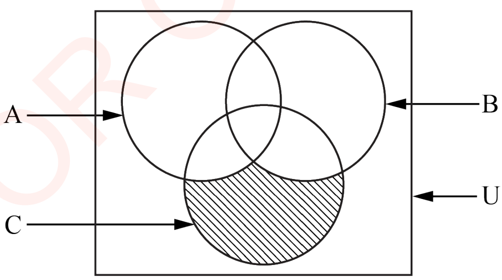


Exercise 2.5

3. (a) $A\cup(B\cap C)$ (b) $(A\cup B)'$ (c) A (d) $\varnothing$
(e) $A'\cup B$ (f) $\varnothing$ (g) $A\cup B'$
(h) $(P\cap M')\cap H'$ (i) B (j) $\varnothing$

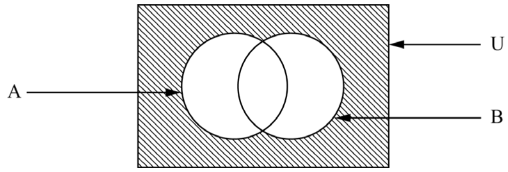
Exercise 2.6

1.



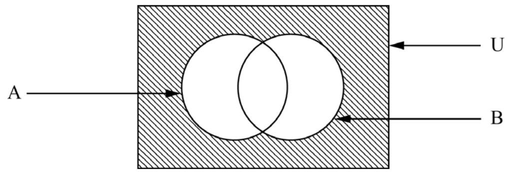
$(A\cup B)'\cap(A\cup C)$

2.



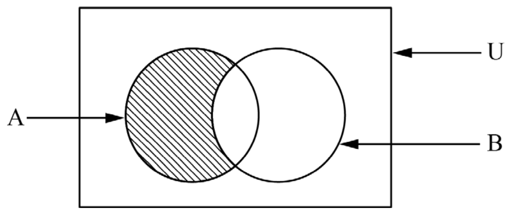
$(A \cup B)'$

3.



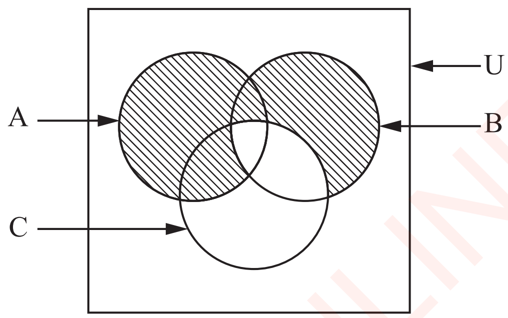
$A' \cap B'$

5.



$A - B$

6.



$(A-B) \cup (A-C) \cup (B-C)$

Exercise 2.7

1. (a) $n(A \cap B) = 3$ (c) $n(A \cup B) = 5$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 1$

2. (a) $n(A \cap B) = 4$ (c) $n(A \cup B) = 10$
   (b) $n(A \Delta B) = 6$ (d) $n(A - B) = 4$

3. (a) $n(A \cap B) = 0$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 3$ (d) $n(A - B) = 1$

4. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 3$
   (b) $n(A \Delta B) = 2$ (d) $n(A - B) = 0$

5. (a) $n(A \cap B) = 1$ (c) $n(A \cup B) = 2$
   (b) $n(A \Delta B) = 1$ (d) $n(A - B) = 1$

6. $n(A \cap B)' = 7$

7. (a) $n(A) = 150$ (b) $n(B) = 100$

8. All three diseases = 20 patient

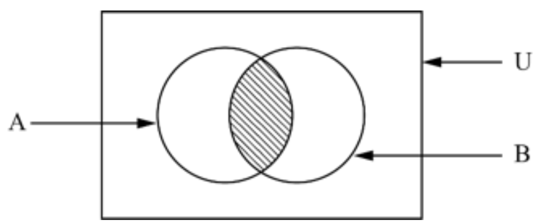
9. (a) 5 (b) 131 (c) 138 (d) 145

Revision exercise 2

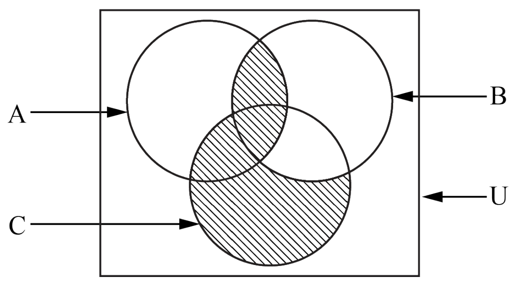
1. (a) $A \cup B \cup C = \{0, 2, 4, 6, 8, 12, 16, 18\}$
   (b) $A \cap B \cap C = \{0\}$
   (c) (i) $n(A \cup B \cup C) = 8$
   (ii) $n(A \cap B \cap C) = 1$

3. (a) 8 (b) 16 (c) 16

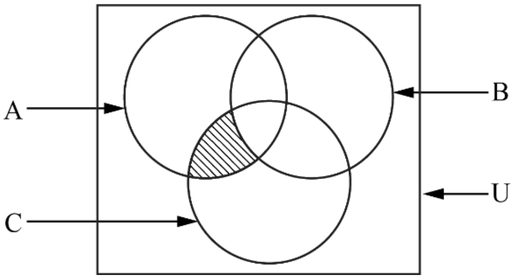
4. (i) $B - A'$



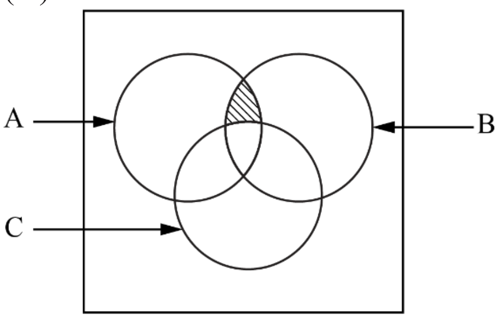
(ii) $(A \cap B) \cup (B' \cap C)$



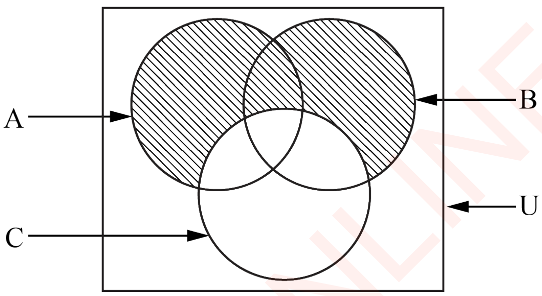
(iii) $A \cap B' \cap C$



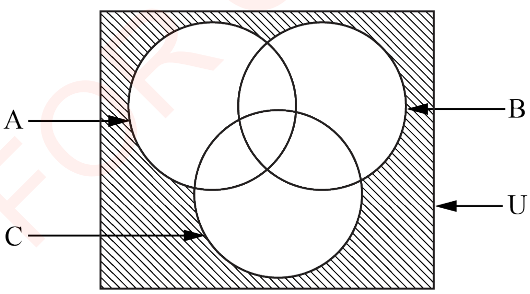
(iv) $A \cap B \cap C'$



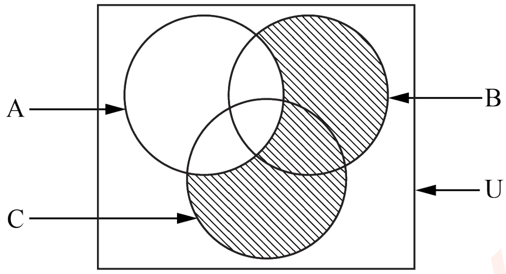
(v) $(A \cup B) \cap C'$



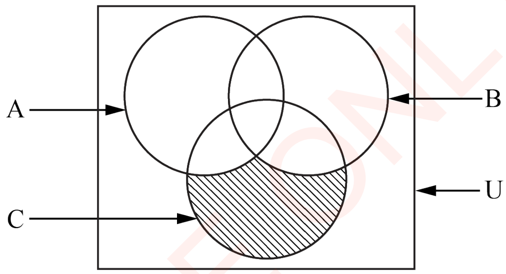
(vi) $A' \cap B' \cap C'$



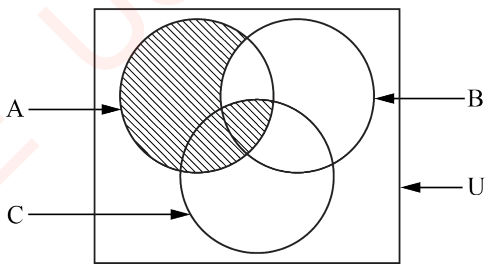
(vii) $A' \cap (B \cup C)$



(viii) $A' \cap B' \cap C$



(ix) $A \cap (B' \cup C)$



5. (a) $(B - C) = \{b, c\}$

(b) $(A - C) = \{a, b, c\}$

(c) $(A - B) = \{a\}$

(d) $(C - B) = \{e, f, g\}$

(e) $(B - A) = \{d\}$

(f) $(C - A) = \{d, e, f, g\}$

7. (a) False (b) True (c) True (d) True
   (e) False (f) False (g) False (h) False

8. (a) (i) $A=\{5, 10, 15, 20, 25, 30, 35, 40, 45\}$
   $B=\{4, 8, 12, 16, 20, 24, 28, 32, 36, 40, 44, 48\}$ $A \cap B=\{20, 40\}$

   $A \cup B=\{4, 5, 8, 10, 12, 15, 16, 20, 24, 25, 28, 30, 32, 35, 36, 40, 44, 45, 48\}$

   (ii) $A \cap B=\{$positive integers less than 50, which are both multiples of 4 and 5$\}$
   (iii) $n(A) = 9$, $n(B) = 12$, $n(A \cap B)=2$
   (b) $A=\{q,r,s\}$, $B=\{q,r,p\}$, $C=\{s,t,q\}$

10. (a) U (b) $\varnothing$ (c) A (d) B (e) $A' \cup B$
    (f) $A \cap B'$ (g) C (h) $X \cup Y'$ (i) $\varnothing$ (j) $A \cup B$

11. (a) $\varnothing$

14. (a) 240 (b) 504 (c) 304 (d)40
15. (a) 11 (b) 3 (c) 12 (d) 8
16. (a) 5 (b) 42 (c) 87
17. (a) 5% (b) 35% (c) 60%
18. (a) 83.8% (b) 16.2% (c) 95.6%.
19. 19. (a) 6 (b) 18 (c) 33
20. (a) Maximum 25, Minimum 15
    (b) Maximum 10, Minimum 0.
21. (a) 5 (b) 12 (c) 43 (d) 22
22. (a) B (b) $A' \cap B'$ (c) $A \cap B$ (d) B
25. (a) 15 (b) 10 (c) 45 (d) 40.

Answers

Chapter Three

Exercise 3.1

1. a, b, c, e.
2. (a) Simple statement
   (b) Compound statement
   (c) Simple statement
   (d) Compound statement
   (e) Compound statement
3. (a)

p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

(b)

p	q	r	s
T	T	T	T
T	T	T	F
T	T	F	T
T	T	F	F
T	F	T	T
T	F	T	F
T	F	F	T
T	F	F	F
F	T	T	T
F	T	T	F
F	T	F	T
F	T	F	F
F	F	T	T
F	F	T	F
F	F	F	T
F	F	F	F

4. (a) Pili is a woman.
   (b) It is not raining now.
   (c) Masai do not maintain their culture.
   (d) Summer does not come after spring.
   (e) Industries are not friendly to the environment.
   (f) Tomorrow is not Saturday.

Exercise 3.2

1. (a) False (b) False (c) True
2. (a) $p \wedge q$ (d) $\sim q \wedge \sim p$
   (b) $p \wedge \sim q$ (e) $p \vee (\sim p \wedge q)$
   (c) $\sim(\sim p \vee q)$
3. Let $a \equiv \sim p \leftrightarrow \sim q$ and $b \equiv \sim (p \to q)$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow \sim q$	$p \to q$	$\sim(p \to q)$	$a \leftrightarrow b$
T	T	F	F	T	T	F	F
T	F	F	T	F	F	T	F
F	T	T	F	F	T	F	T
F	F	T	T	T	T	F	F

4. (a) $p \to q$
   (b) $q \to p$
   (c) $\sim(\sim p \leftrightarrow \sim q)$
5. (a) You like Physics or Chemistry but not Biology
   (b) You like Physics and Chemistry or you do not like Physics and Biology
   (c) It is not true that you like Physics but not Biology.

6.

p	q	$p \wedge q$	$p \vee q$	$p \wedge q) \rightarrow (p \vee q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

7. (a) $\sim(p \wedge \sim q)$ (b) $\sim p \vee \sim q$

8.

p	q	r	$\sim p$	$q \rightarrow r$	$\sim p \leftrightarrow (q \rightarrow r)$
T	T	T	F	T	F
T	T	F	F	F	T
T	F	T	F	T	F
T	F	F	F	T	F
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

9. (a) False (b) False (c) True

10. (a) $(\sim P \wedge Q) \vee (P \wedge \sim Q)$

(b) $[(P \rightarrow Q) \wedge (Q \leftrightarrow R)] \rightarrow (\sim R \wedge \sim Q)$

11. (a) Halima works hard or she is poor.
(b) It is not true that Halima is poor and she works hard.
(c) If Halima is poor, then she works hard.
(d) Halima is poor and she works hard.
(e) If Halima is not poor, then she doesn't work hard.
(f) Halima is not poor or she doesn't work hard.
(g) Halima works hard if and only if she is poor.
(h) Halima does not work hard or she is poor and she works hard.

12. (a) Swimming in the poor has danger.
(b) It is not true that few people have been drawn in the swimming pool if and only if Swimming in the pool has no danger.
(c) If swimming in the pool has a danger, then few people have been drawn in the Swimming pool.

13. (a)

p	q	$p \wedge q$	$\sim(p \wedge q)$	$(p \wedge q) \wedge \sim (p \wedge q)$
T	T	T	F	F
T	F	F	T	F
F	T	F	T	F
F	F	F	T	F

(b)

p	q	$q \rightarrow p$	$p \rightarrow q$	$(q \rightarrow p) \leftrightarrow (p \rightarrow q)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

(c)

p	q	r	$p \vee q$	$(p \vee q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(d)

p	q	r	$\sim r$	$p \vee q$	$(p \vee q) \rightarrow \sim r$
T	T	T	F	T	F
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	T
F	T	T	F	T	F
F	T	F	T	T	T
F	F	T	F	F	T
F	F	F	T	F	T

(e)

p	q	r	t	$\sim p$	$\sim p \leftrightarrow q$	$(\sim p \leftrightarrow q) \wedge r$	$[(\sim p \leftrightarrow q)] \wedge r \rightarrow t$
T	T	T	T	F	F	F	T
T	T	T	F	F	F	F	T
T	T	F	T	F	F	F	T
T	T	F	F	F	F	F	T
T	F	T	T	F	T	T	T
T	F	T	F	F	T	T	F
T	F	F	T	F	T	F	T
T	F	F	F	F	T	F	T
F	T	T	T	T	T	T	T
F	T	T	F	T	T	T	F
F	T	F	T	T	T	F	T
F	T	F	F	T	T	F	T
F	F	T	T	T	F	F	T
F	F	T	F	T	F	F	T
F	F	F	T	T	F	F	T
F	F	F	F	T	F	F	T

Exercise 3.3

1.(a) *Inverse*:If tomorrow is not Saturday, then Paul will not go to the beach.
Converse: If Paul will go to the beach, then tomorrow is Saturday.
Contrapositive: If Paul will not go to the beach then tomorrow is not Saturday.

(b) *Inverse*: If it is not raining, then the shop is not closed.
Converse: If the shop is closed, then it is raining.
Contrapositive: If the shop is not closed, then it is not raining.

(c) *Inverse*: Industries are not environmentally friendly, if they are not in harmony with the surroundings.
Converse: If the industries are environmentally friendly, then they are in harmony with the surroundings.
Contrapositive: If the industries are not environmentally friendly, then they are not in harmony with the surroundings.

(d) *Inverse*: If ABC is not an equilateral triangle, then it is not a right-angled triangle.
Converse: If ABC is a right-angled triangle, then it is an equilateral triangle.
Contrapositive: If ABC is not a right-angled triangle, then it is not an equilateral triangle.

(e) *Inverse*: If $f(x)$ is not a rational function, then it has no asymptotes.
Converse: If $f(x)$ has asymptotes, then it is a rational function.
Contrapositive: If $f(x)$ has no asymptotes, then it is not a rational function.

2. (a)

p	q	$\sim p$	$q \to \sim p$	$(q \to \sim p) \to p$
T	T	F	F	T
T	F	F	T	T
F	T	T	T	F
F	F	T	T	F

(b)

p	q	$\sim p$	$q \rightarrow \sim p$	$\sim(q \rightarrow \sim p)$	$\sim(q \rightarrow \sim p) \rightarrow \sim p$
T	T	F	F	T	F
T	F	F	T	F	T
F	T	T	T	F	T
F	F	T	T	F	T

3. (a) If two vectors are not orthogonal then their dot product is not zero.
   (b) If the dot product of two vectors is zero then they are orthogonal.
   (c) If the dot product of two vectors is not zero then the two vectors are not orthogonal.
4. (a) If Halima wont win, then she has no courage.
   (b) One cannot be a sailor if one is not strong.
   (c) If a geometrical figure is not a rectangle, then it is not a square.
5. Converse $(q \wedge r) \rightarrow p$
   Inverse $\sim p \rightarrow \sim(q \wedge r)$
   Contrapositive $\sim(q \wedge r) \rightarrow \sim p$
6. (a) $q \rightarrow p$
   (b) $\sim(q \rightarrow \sim r) \rightarrow p$
   (c) $\sim(p \leftrightarrow q) \rightarrow \sim(\sim p \leftrightarrow \sim r)$

Exercise 3.4

1. Equivalent 2. Not equivalent
3. Equivalent 4. Equivalent
5. Equivalent 6. Equivalent
7. Equivalent 8. Equivalent
9. Equivalent 10. Not equivalent

Exercise 3.5

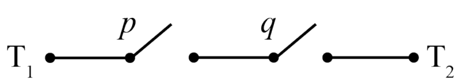
1. (a) Tautology (b) Tautology
   (c) Tautology (d) Not a tautology
   (e) Tautology
2. $\sim(pvq)$
4. (a) $p \vee \sim q$ (c) p
   (b) $\sim p$ (d) T
5. Tautology.

Exercise 3.6

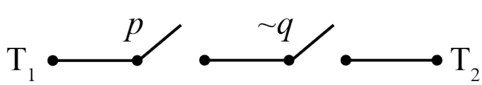
1. Not valid 6. Not valid
2. Not valid 7. Valid
3. Valid 8. Valid
4. Not valid 9. Valid
5. Not valid 10. Valid

Exercise 3.7

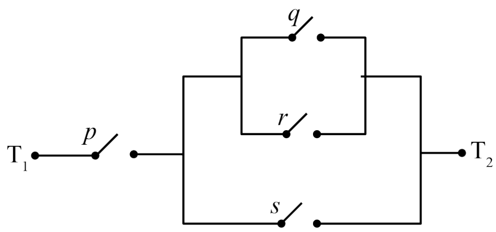
1. (a) $p \wedge q$



(b) $p \wedge \sim q$



(c) $p \wedge (q \vee r) \vee s$

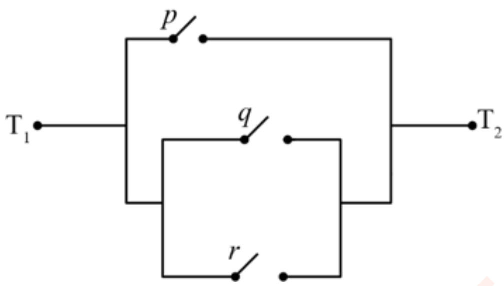


3 (a) $(p \wedge q) \vee \sim q \vee \sim r$

(b) $[\sim p \vee ((p \wedge q) \wedge \sim q)] \wedge [r \vee \sim r]$

(c) $(p \vee q \vee s) \wedge r$

5.



Exercise 3.8

1. $p \wedge \sim q$
2. $(p \wedge q) \vee (\sim p \wedge \sim q)$
3. $\sim p \wedge \sim q$
4. $(p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$
5. $p \wedge q$
6. $s_1 : (p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge r)$

$\equiv \sim q \vee r$

$s_2 : (p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \quad s_1 : q \wedge r$

Revision exercise 3

1. Not a mathematical statement.
2. (a) Converse: If a person is an adult, then a person is 20 years old.

Contrapositive: If a person is not an adult, then a person is not 20 years old.

Inverse: If a person is not 20 years old then a person is not an adult.

(b) *Converse:* If I have a test today then, today is Friday.

Contrapositive: If I have no test today then, today is not Friday.

Inverse: If today is not Friday, then I have no test.

(c) *Converse:* If you are attractive, then you bought our clothes.

Contrapositive: If you are not attractive then you did not buy our clothes.

Inverse: If you did not buy our clothes then you are not attractive.

3. (c)

q	r	$q \vee r$	$\sim(q \vee r)$	$\sim q$	$\sim q \wedge r$	$\sim(q \vee r) \leftrightarrow (\sim q \wedge r)$
T	T	T	F	F	F	T
T	F	T	F	F	F	T
F	T	T	F	T	T	F
F	F	F	T	T	F	F

4. (a) Not a tautology (b) Tautology (c) Tautology (d) Tautology
5. They have the same truth values (They are equivalent).
6. (a) Tautology (b) Tautology (c) Not a tautology (d) Tautology
(e) Not a tautology (f) Not a tautology (g) Tautology (h) Tautology
(i) Not a tautology (j) Tautology
7. (a), (b), (c), and (e)
9. (a) Not logically equivalent (b) Logically equivalent (c) Logically equivalent
(d) Logically equivalent
10. (d) and (f)
12. (a) Not a tautology (b) Not a tautology

13. (a) (i) $(p \wedge q) \vee r \equiv$ Lightness is clever and polite or she is humble.
(ii) $\sim p \vee q \equiv$ Lightness is not clever or she is humble.

(b) (i) $(p \vee q) \wedge p \rightarrow \sim q$ (ii) $(q \vee r) \wedge \sim p$ (iii) $(\sim q \vee \sim r) \rightarrow \sim p$

14. (a) Valid (b) Valid (c) Not valid (d) Valid (e) Valid
(f) Not valid (g) Not valid (h) Not valid (i) Not valid (j) Not valid

16. (a) $p \wedge (q \vee r) \wedge s$ (b) $p \wedge [(\sim q \wedge r) \vee (p \vee r)]$ (c) $p \vee q \vee r$

17. (a) $(p \wedge \sim q) \vee (\sim p \wedge q)$ (b) $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$

Answers

Chapter Four

Exercise 4.2

3. $B(6,1)$

4. $y=-2x$ and $y=-\frac{2}{5}x$

5. (c) 32.5 square units

6. $(1,-1)$

Exercise 4.3

1. $109.4°$ 2. $78.7°$

3. (a) $36.9°, 40.6°, 102.5°$

(b) $40.6°, 63.4°, 76°$

(c) $17.7°, 60.3°, 102°$

4. $x-3y+7=0$

5. $m=\frac{18}{5}$ and $n=\frac{22}{15}$, $\theta=53.6°$ or $126.4°$

6. $\theta=47.7°$ and $\theta=132.3°$

Exercise 4.4

1. (a) 1 unit (b) 2.55 units

(c) $\frac{3}{5}a$ units (d) $\frac{2h+4k}{5}$ units

(e) 0 units, the point lies on the line

2. $P=\frac{-9\pm\sqrt{2}}{3}$ units

4. $\overline{PR}=5.06$ units, $\overline{QS}=0.95$ units and $\overline{RS}=0.32$ units

Exercise 4.5

1. (a) $3x-y-2=0$, $3x+9y-12=0$

(b) $x=0$, $y=0$

(c) $\frac{3x+4y}{5}=\frac{2x-y-1}{\sqrt{5}}$, $\frac{3x+4y}{5}=\frac{-2x+y+1}{\sqrt{5}}$

2. $2x+2y+15=0$

4. $4x-7y+3=0$; $2x+3y=5$; $x-3y+2=0$; $3x+y+1=0$; $15x-10y+8=0$; and $49x+28-12=0$

Exercise 4.6

1. (a) $x=4$ (b) $y=5$

(c) $x^2+y^2=25$

(d) $x^2+y^2+4x+6y-51=0$

(e) $x^2+y^2-8x+12=0$

(f) $x^2-3y^2+32y-64=0$

(g) $3x^2+4y^2+6x-9=0$

(h) $x=\frac{3}{8}$

2. $8x-6y-7=0$ 3. $y^2=4ax$

4. $y^2-3x^2+34x=91$

5. $3x-11y=0$ and $11x+3y=0$

Exercise 4.7

1. (a) $\left(-5,\frac{32}{5}\right)$ and $(7, 40)$

(b) $(-1, 3)$, no external division

(c) $\left(-5,\frac{32}{5}\right)$ and $(7, 40)$

(d) $\left(\frac{19}{3},6\right)$ and $(8, 6)$

2. $C(20, 22)$ 3. 0.48 units

4. $P(-41, -83)$, $Q(22, 43)$

5. $m=3$, $n=2$ 6. $p=4$, $q=5$

6. $2:9$

Exercise 4.8

1. (a) $x^2+y^2-4y-12=0$
(b) $x^2+y^2+4x-4y-17=0$
(c) $x^2+y^2-4=0$
(d) $4x^2+4y^2+4x-40y-299=0$
(e) $8x^2+8y^2-32x+48y+103=0$
2. (a) Centre $(1,-2)$, radius $=\sqrt{45}$
(b) Centre $(0, 0)$, radius $=5$
(c) Centre $(0, 2)$, radius $=2$
(d) Centre $(-1, 0)$, radius $=2$
(e) Centre $\left(\frac{1}{2}, -\frac{3}{2}\right)$, radius $=1$
3. $x^2+y^2-8x-8y+16=0$
4. $(x+41)^2+(y+161)^2=144$
5. $4x^2+4y^2+40x-16y+67=0$

Exercise 4.9

1. (a) $x^2+y^2+7x-5y+16=0$
(b) $x^2+y^2-4x+4y-17=0$
(c) $x^2+y^2+x+2y-3=0$
(d) $x^2+y^2-x+4y-53=0$
(e) $x^2+y^2+4x-3y=0$
2. (a) $x^2+y^2-6x+4y-12=0$
(b) $x^2+y^2-x+4y-12=0$
(c) $x^2+y^2-14x+10y-95=0$
(d) $x^2+y^2-10x-8y+16=0$
(e) $x^2+y^2-13x+3y+2=0$
3. $x^2+y^2-5x-6y=0$

Exercise 4.10

1. (a) Tangent $6x-2y-5=0$;
Normal $x+3y=0$
(b) Tangent $3x-4y+25=0$;
Normal $4x+3y=0$.
(c) Tangent $2x-y-9=0$;
Normal $x+2y+3=0$
(d) Tangent $2x+4y-7=0$;
Normal $2x-y-2=0$
(e) Tangen $2x+3y-22=0$;
Normal $3x-2y-7=0$
(f) Tangent $2x+3y=0$;
Normal $3x-2y=0$.
2. $(3,-1)$
3. $k=40$, or $k=-10$
4. $x+2y-3=0$
7. $x+y-5=0$
8. $5x+2y-21=0$ and $5x+2y+28=0$
10. $4x-3y+25=0$ and $4x-3y-25=0$
11. $2x+y+4=0$ and $2x+y-6=0$
13. $x+2y-4=0$ and $x+2y+6=0$
14. 5 units
15. $x^2+y^2+6x+10y+9=0$.

Exercise 4.11

3. $y=-2x$
4. $(0, 0)$
5. $\left(\frac{27+8\sqrt{43}}{17}, \frac{23-2\sqrt{43}}{17}\right)$
and $\left(\frac{27-8\sqrt{43}}{17}, \frac{23+2\sqrt{43}}{17}\right)$
6. $x^2+y^2-4x+2y-2=0$; $(5,-5)$

Exercise 4.12

2. $x^2+y^2-10x-2y+9=0$

3. (a) $k=47$ (b) $k=34$

4. $x^2+y^2-4x-2y+3=0$

5. (a) $5x^2+5y^2+24x-36y=0$;
   (b) $2x-3y=0$.

Exercise 4.13

1. 11.58 units
2. 12 cm
3. 2 units
4. 6 units
5. 7 units
7. $x^2+y^2-10x-4y-2=0$

Revision exercise 4

1. $k=5$
2. M(3, 6), L(6, 3), N$\left(\frac{25}{4},1\right)$
3. 3 square units
5. $x^2+y^2+6x+6y+2=0$
6. $m=2$, $m=8$
7. $\left(\frac{8}{5},\frac{24}{5}\right)$, (16, 12)
8. (a) $\left(\frac{18}{5},\frac{14}{5}\right)$ (b) $(6,-2)$
9. $14x+112y=43$
10. $4x+3y-25=0$; (4, 3)
11. L(4, 4), M(7, 6)
12. 9.5 square units
13. 3 square units
14. $x+6y-34=0$, $x-y+1=0$,
    $A=5\sqrt{85}$ square units
15. 5.9 units
16. 53.13°
17. $2x-y-4=0$, $x+2y-7=0$
18. $x-3y+3=0$, $3x+y-11=0$
19. 47.7°
20. $(a, b)=(6, 0)$
21. $x^2+y^2-2x+2y-23=0$
23. $x^2+y^2-4x+6y+8=0$
24. C(–4, 8)
25. B(6, 4)
26. $x^2+y^2-2x-2y+1=0$
27. $3x-y-2=0$, $x+3y-4=0$
28. $18x+12y+69=0$
29. $16x^2+9y^2+24xy-156x-42y+249=0$
30. $x^2-4x+2y-3=0$
31. $x^2+y^2-10x-8y+16=0$
33. $3x^2+3y^2+40x-40y-28=0$
34. $x^2+y^2-4x-8y+15=0$
35. $a=2$, $b=0$
36. (1, 2)
37. $x^2+y^2-12x-10y+36=0$
38. centre (–2, 4), radius = 4,
    length of tangent = 5 units
39. $2x^2+2y^2-8x-12y+25=0$

40. $y = 2x$

42. $c = -32$

43. $x - y - 2 = 0,\ x + 7y + 10 = 0$

44. $12x + 5y + 119 = 0,\ 12y - 5x + 49 = 0$

45. $y = 2x,\ 22x + 19y = 0$

46. 6.08 units

47. 3.46 units

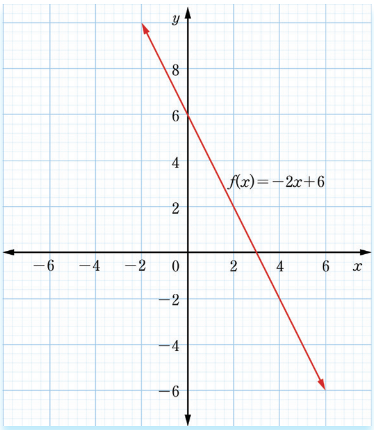
49. $4x - 3y - 18 = 0;$

Area $= 13.5$ square units

Answers

Exercise 5.1

1.



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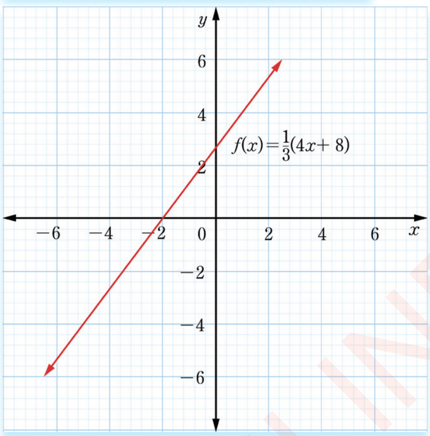

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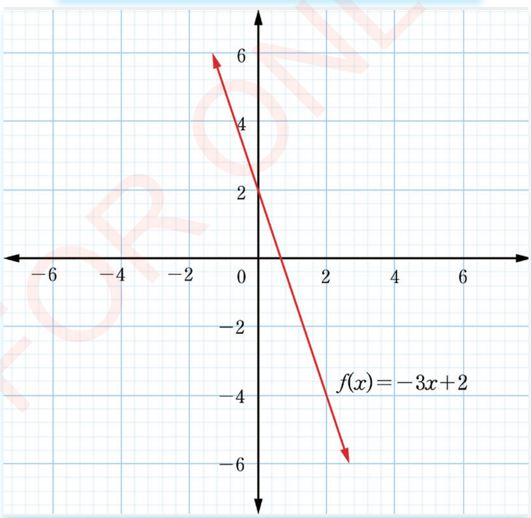

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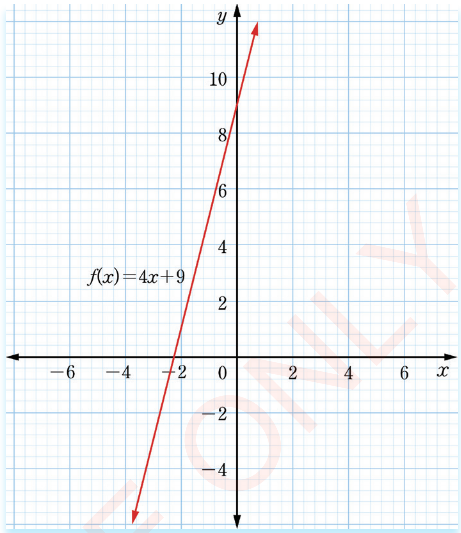

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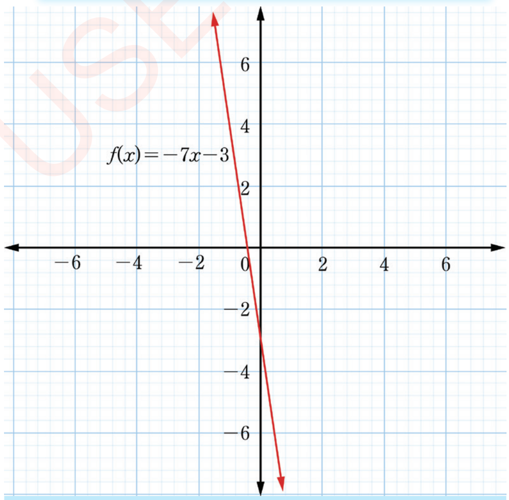

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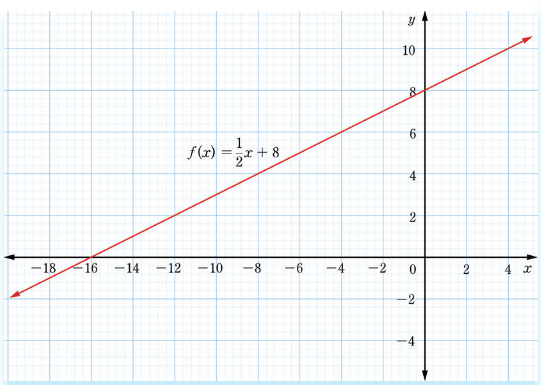

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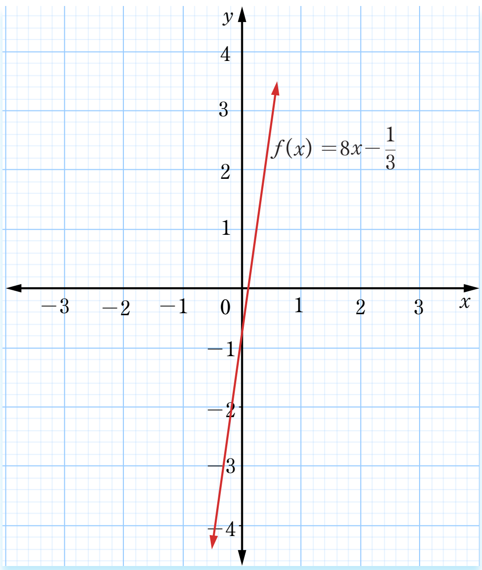

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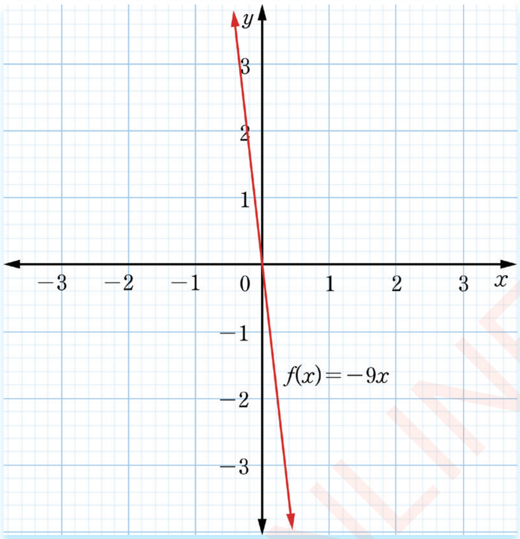

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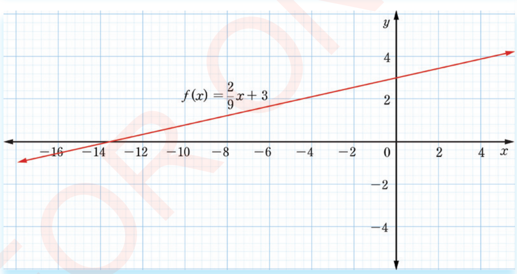
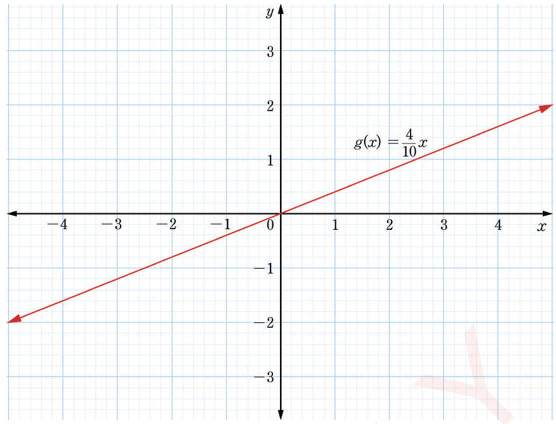
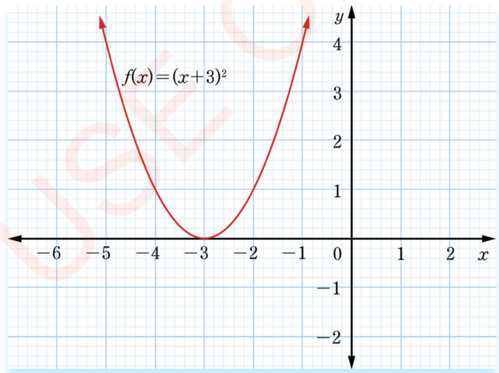


10.

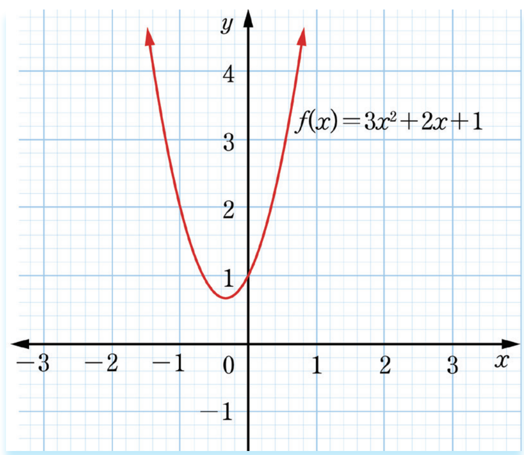


Exercise 5.2

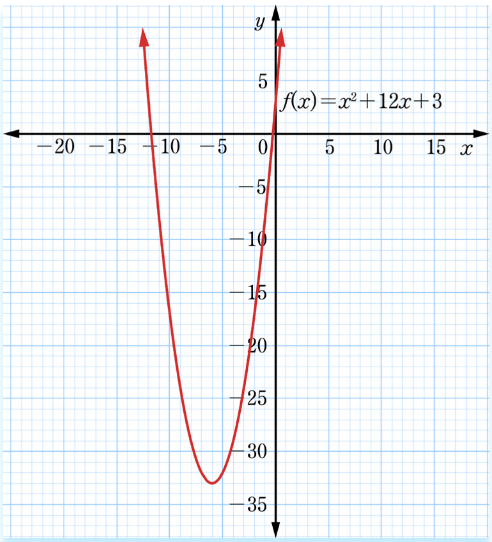
1.



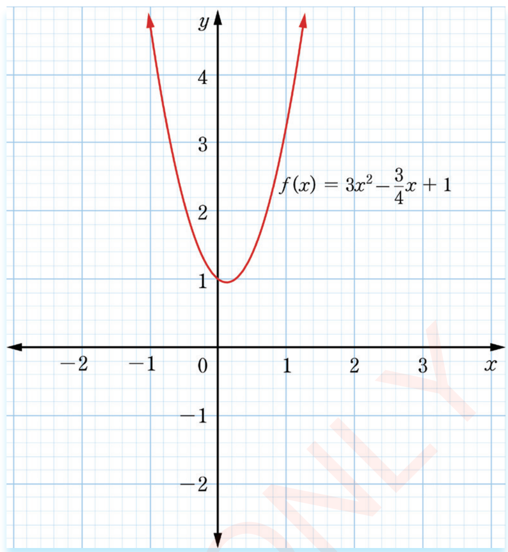
2.



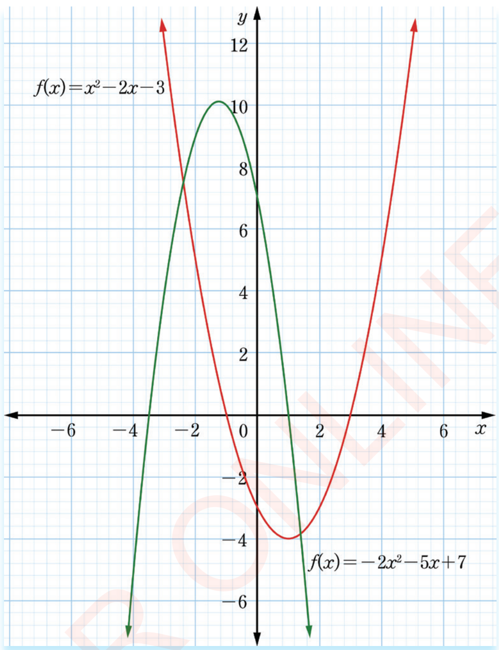
3.



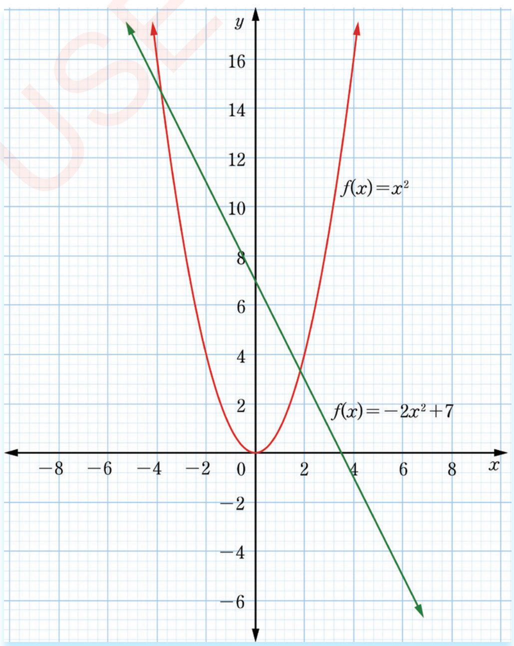
5.



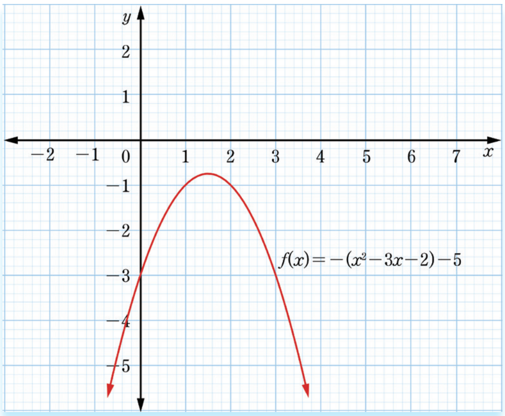
4.



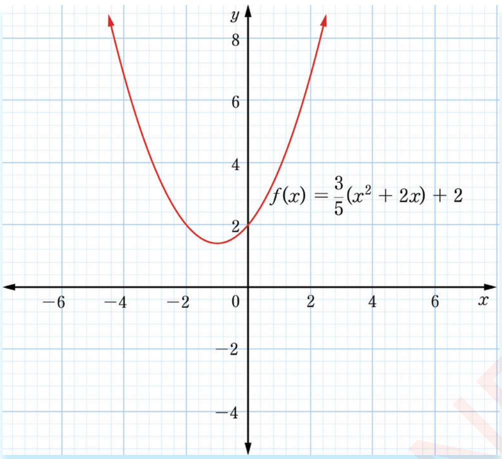
6.



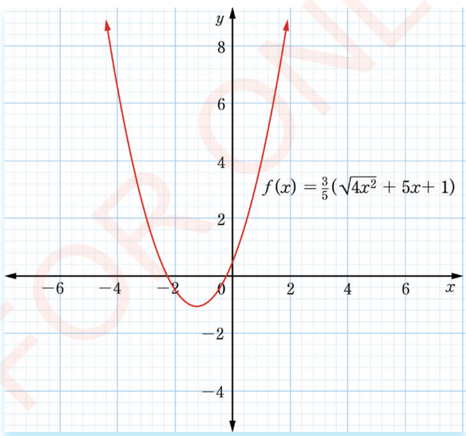
7.



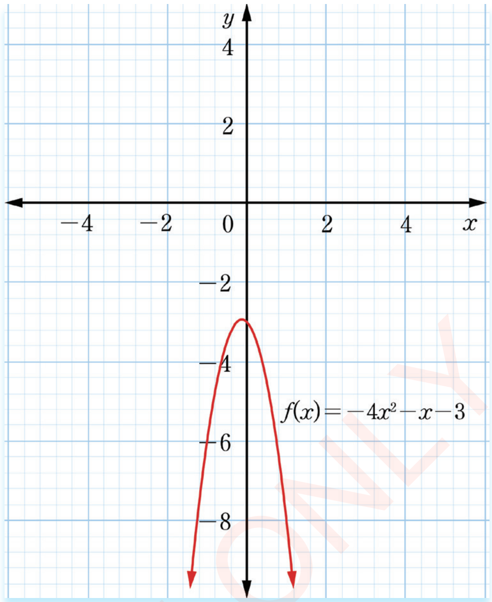
8.



9.

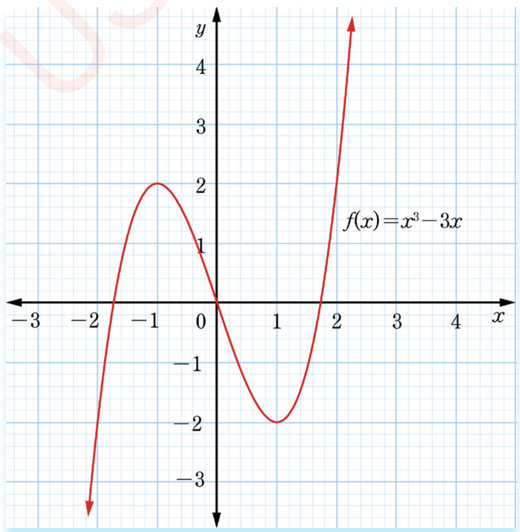


10.



Exercise 5.3

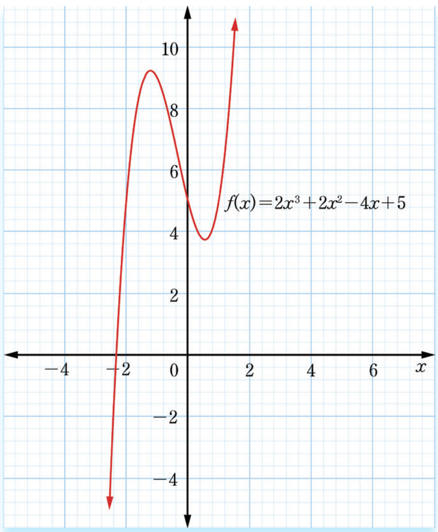
1.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

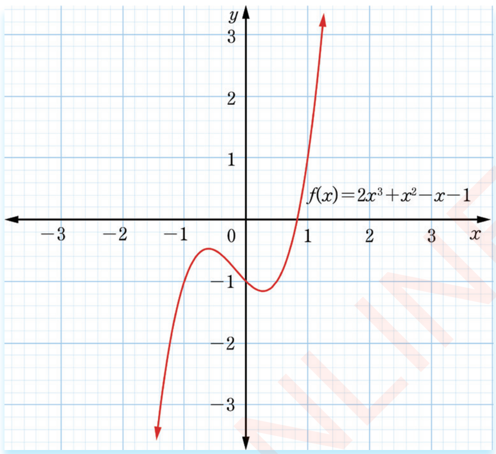
2.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

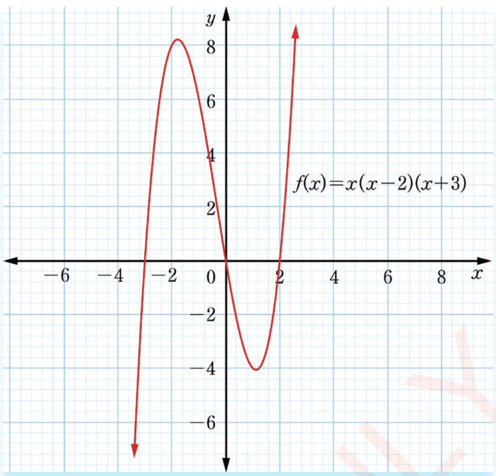
3.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

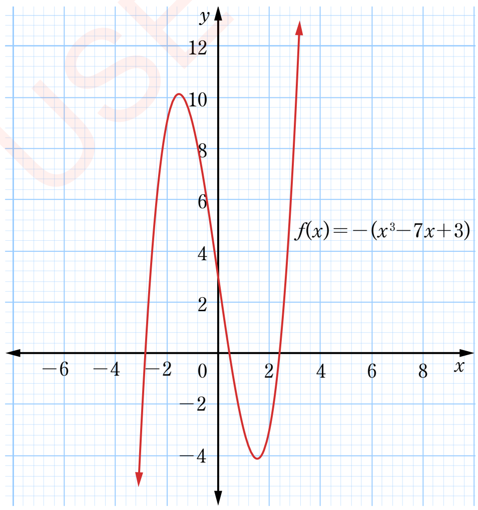
4.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

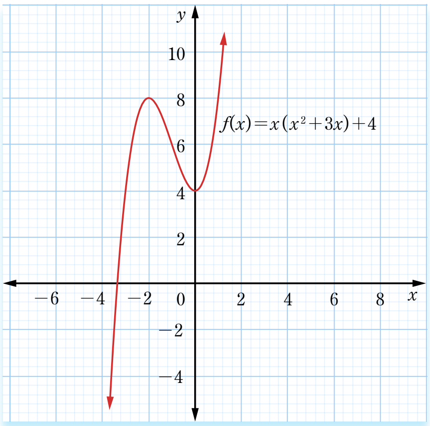
5.



Domain = $\{x : x \in \mathbb{R}\}$

Range = $\{y : y \in \mathbb{R}\}$

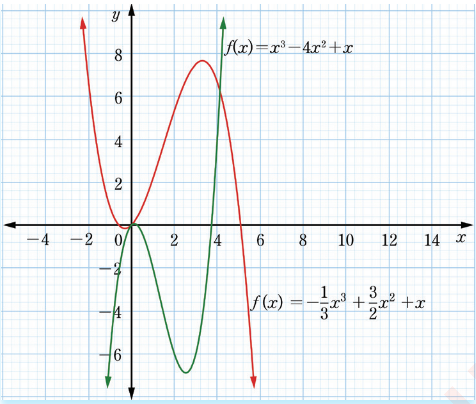
6.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

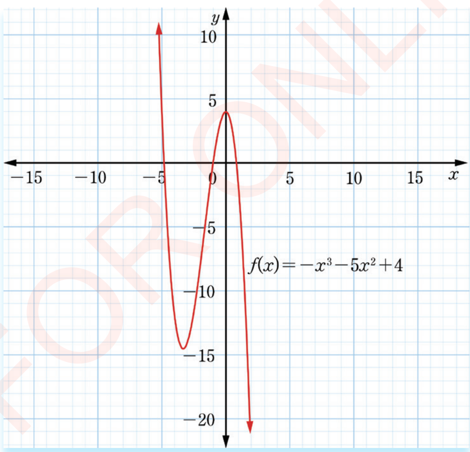
7.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

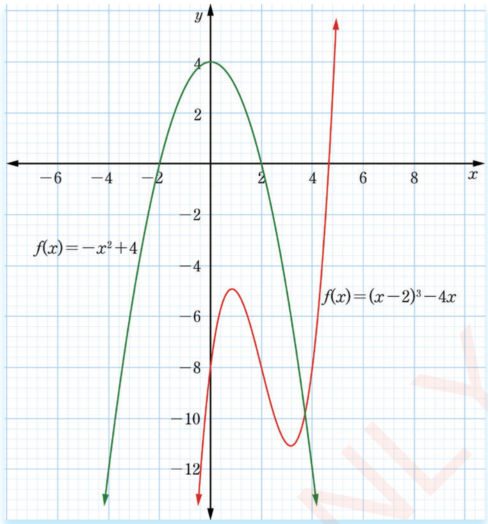
8.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

9.



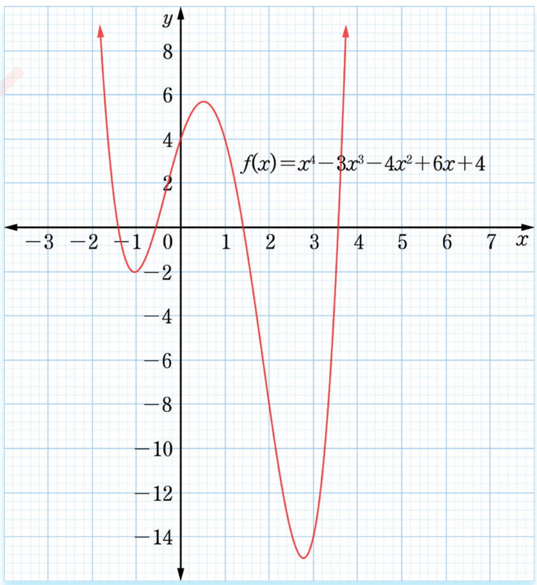
Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

Exercise 5.4

1. $x = -1, 0, 1, 2$

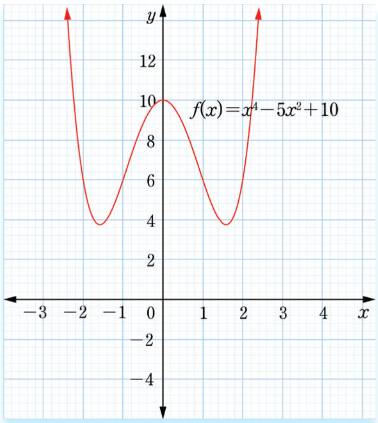
2.



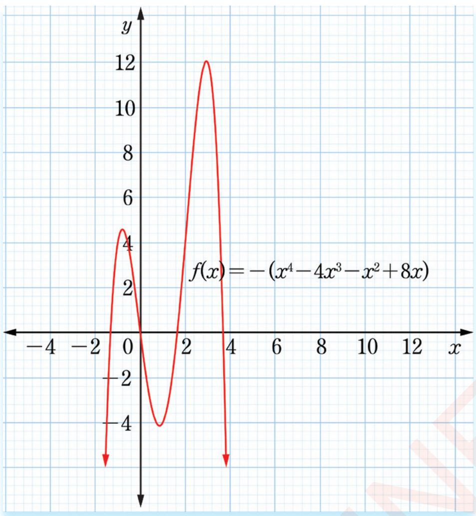
Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \geq -15\}$

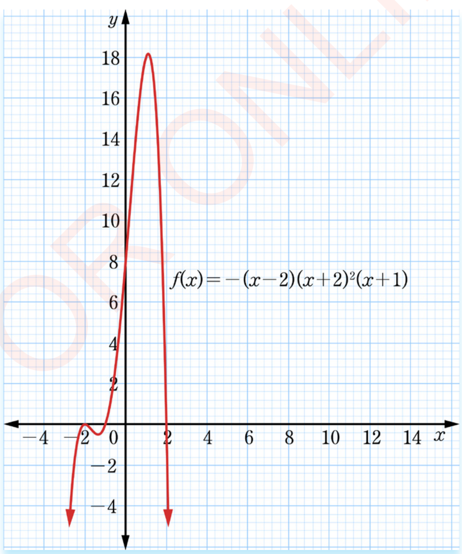
3. (a)



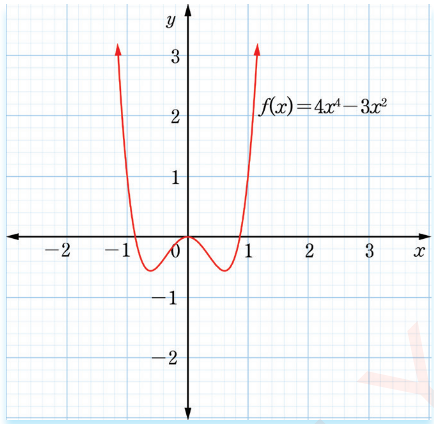
(b)



(c)

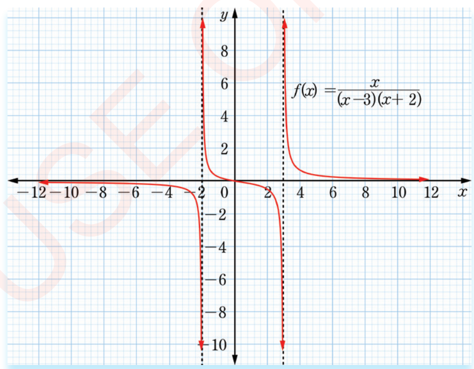


4.



Exercise 5.5

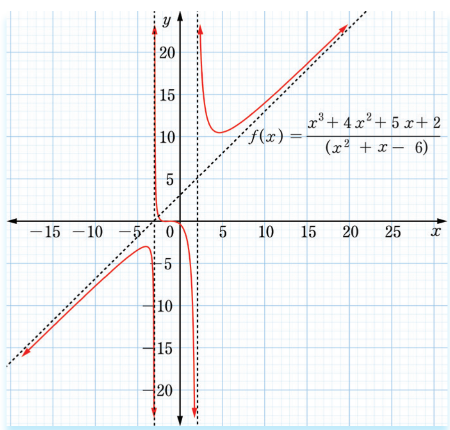
1.



Domain $= \{x : x \in \mathbb{R}, x \neq -3,\ x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

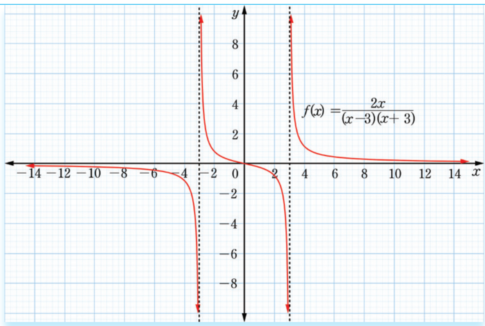
2.



Domain $= \{x : x \in \mathbb{R}, x \neq -3, x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

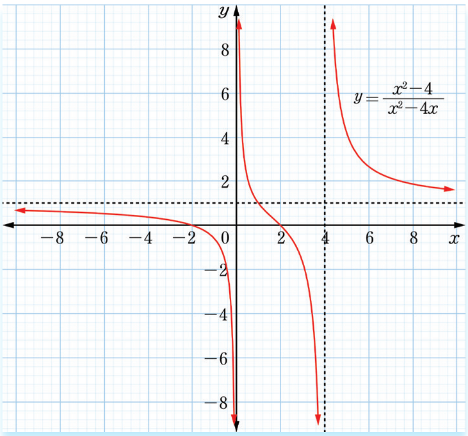
3.



Domain $=\{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $=\{y : y \in \mathbb{R}\}$

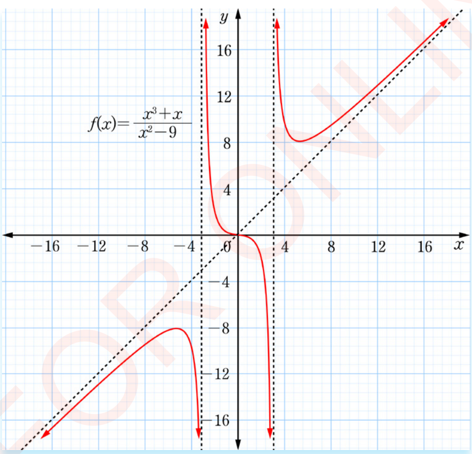
4.



Domain $=\{x : x \in \mathbb{R}, x \neq 0, x \neq 4\}$

Range $=\{y : y \in \mathbb{R}\}$

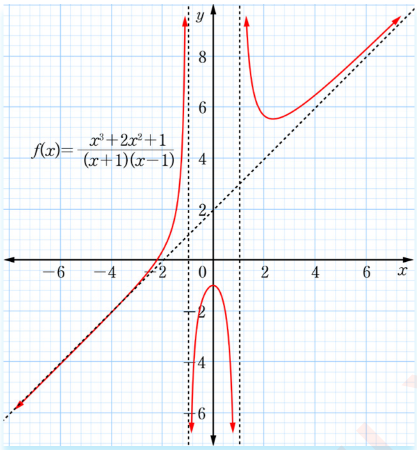
5.



Domain $=\{x : x \in \mathbb{R}, x \neq -3, x \neq 3\}$

Range $=\{y : y \in \mathbb{R}\}$

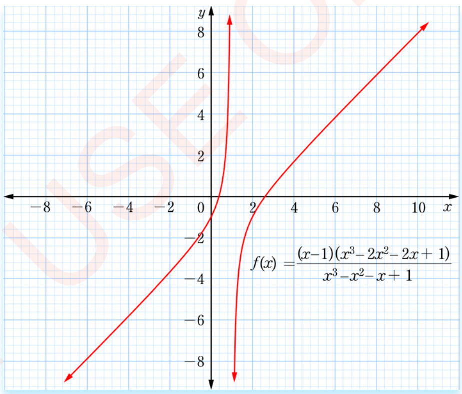
6.



Domain $=\{x : x \in \mathbb{R}, x \neq -1, x \neq 1\}$

Range $=\{y : y \in \mathbb{R}\}$

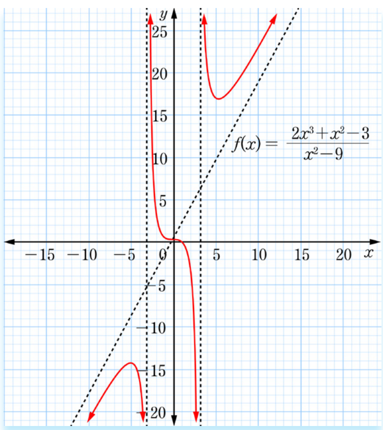
7.



Domain $=\{x : x \in \mathbb{R}, x \neq 1\}$

Range $=\{y : y \in \mathbb{R}\}$

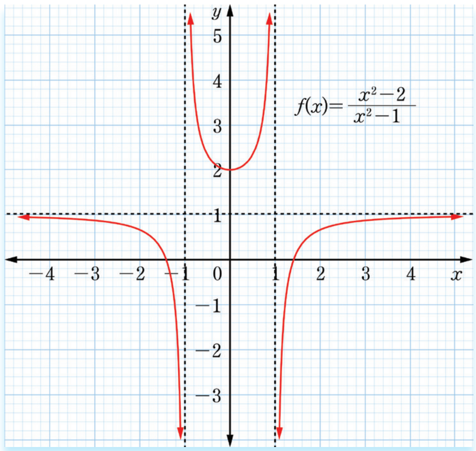
8.



Domain $=\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 3\}$

Range $=\{y : y \in \mathbb{R}\}$

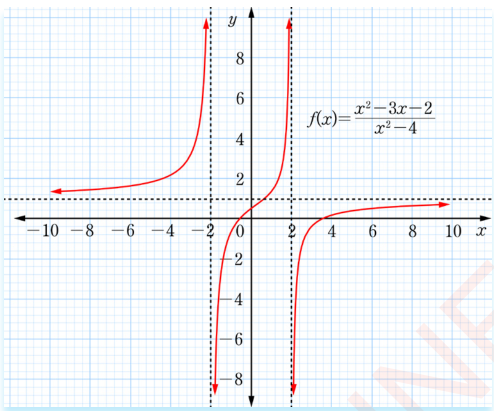
9.



Domain $= \{x : x \in \mathbb{R}, x \neq -1,\ x \neq 1\}$

Range $= \{y : y \in \mathbb{R}, y \neq 1, y \neq 2\}$

10.



Domain $= \{x : x \in \mathbb{R}, x \neq -2,\ x \neq 2\}$

Range $= \{y : y \in \mathbb{R}\}$

Exercise 5.6

1. $f(g(x)) = -3x + 17$, $g(f(x)) = -3x - 1$. Is not commutative.

2. $f\left(\frac{1}{2}\right) = \frac{7}{2}$

3. $(g \circ f)(x) = \{(-1, 1), (0, 5)\}$ Domain$=\{x : -1, 0\}$, Range$=\{y : 1, 5\}$

4. (a) $(f \circ g)(x) = x^2 + 6x + 7$

   (b) $(f \circ g)(x) = x^2 - 7$

   (c) $(f \circ g)(x) = e^{x^3} - 2$

   (d) $f \circ g = x$

6. -7

7. $x = \pm 5$

8. (a) 34 (b) 112

   (c) 584 (d) 130

   (e) 2706 (f) $x^2 - 2x + 3$

9. (a) $32x^2 - 312x + 685$

   (b) $512x^3 - 4576x + 10149$

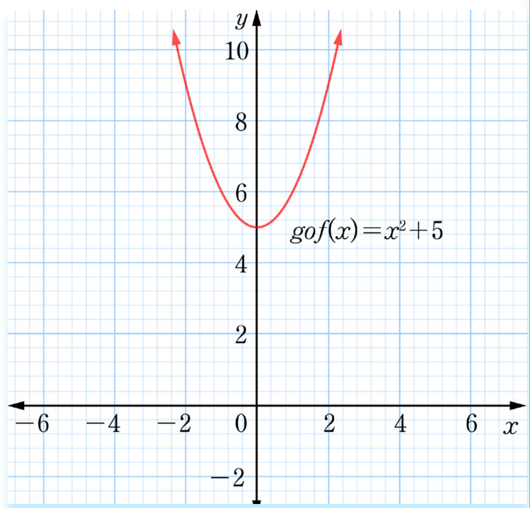
10. (a) $f \circ g = \{(7, 5), (-5, 8)\}$ and $g \circ f = \{(3, 10), (2, 7), (4, 3)\}$

11. $g \circ f = \{(-2, 1), (0, 3)\}$

12. $f \circ g = \{(2, 6), (4, 7)\}$

Exercise 5.7

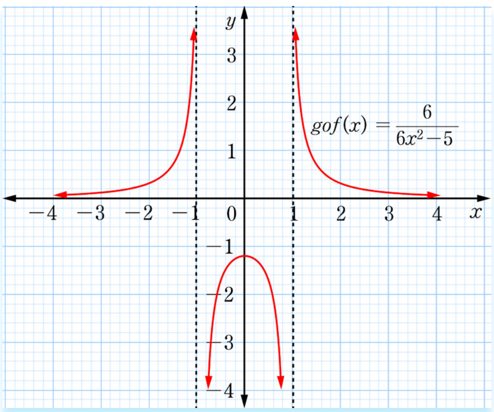
1.



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}, y \geq 5\}$

2. $g(x) = x - 3$ and $g(x) = -x + 1$
3. $(f \circ g)(x) = \{(3, 6)\}$
4.



5. $-1+\sqrt{3x^2+10x+15}$

   or $-1-\sqrt{3x^2+10x+15}$

6. $f(x) = \dfrac{x+1}{x^2-2x+5}$ 7. $k = \dfrac{5}{3}$

8. $x = \pm 4$ or $x = \pm 2$

9. x-intercept $(x, y) = (1, 0)$

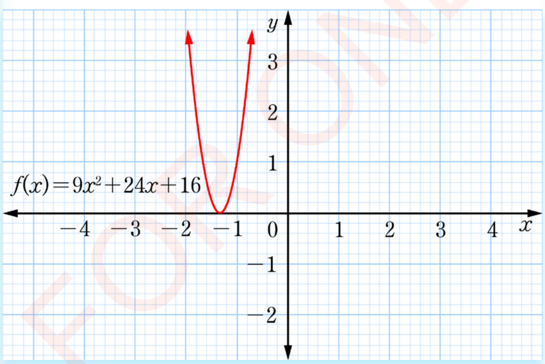
   y-intercept $(x, y) = \left(0, -\frac{1}{2}\right)$

   Vertical asymptote $x = -2$

   Horizontal asymptote $y = 1$

10. (a) $3x^2 + 4$

    (b) $9x^2 + 24x + 16$



11. (a) $\frac{1}{4}x^2 - x + 1$ (b) 7.5

12. (a) $(f \circ g)(x) = x + 9$ and $(g \circ f)(x) = x + 9$

    (b) $(f \circ g)(x) = x^2 + 2$ and $(g \circ f)(x) = x^2 + 2$

    (c) $(f \circ g)(x) = -9x + 19$

    and $(g \circ f)(x) = 3 - 9x$

Exercise 5.8

1. Domain $= \{x : x \in \mathbb{R}\}$

   Range $= \{y : y \in \mathbb{R},\ y > 0\}$

   (a) Domain $= \{x : x \in \mathbb{R}\}$, Range $= (-4, \infty)$

   (b) Domain $= \{x : x \in \mathbb{R}\}$, Range $= (-\infty, 0)$

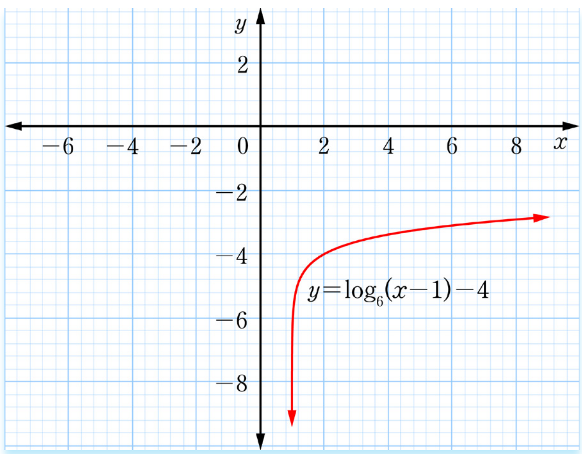
   (c) Domain $= \{x : x \in \mathbb{R}\}$, Range $= (-3, \infty)$

   (d) Domain $= \{x : x \in \mathbb{R}\}$, Range $= (-\infty, 1)$

   (e) Domain $= \{x : x \in \mathbb{R}\}$, Range $= (3, \infty)$

Exercise 5.9

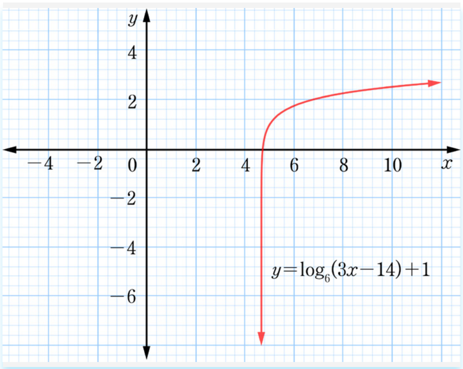
1. (a)



Domain $= \{x : x \in \mathbb{R}, x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

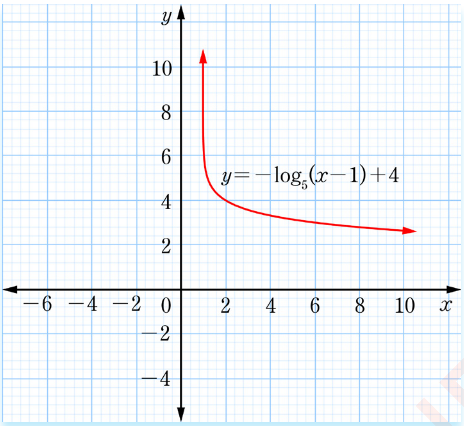
(b)



Domain $=\left\{x : x \in \mathbb{R},\ x > \frac{14}{3}\right\}$

Range $= \{y : y \in \mathbb{R}\}$

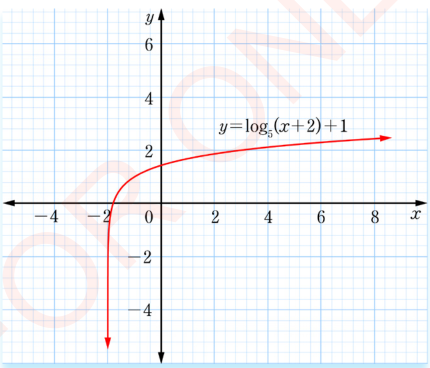
(c)



Domain $= \{x : x \in \mathbb{R},\ x > 1\}$

Range $= \{y : y \in \mathbb{R}\}$

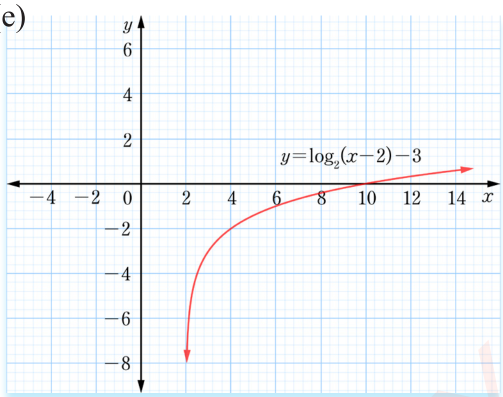
(d)



Domain $= \{x : x \in \mathbb{R},\ x > -2\}$

Range $= \{y : y \in \mathbb{R}\}$

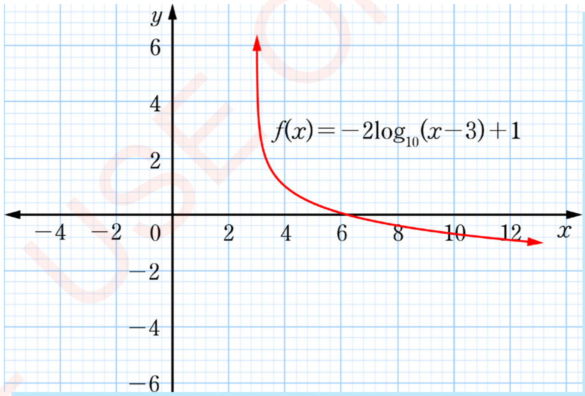
(e)



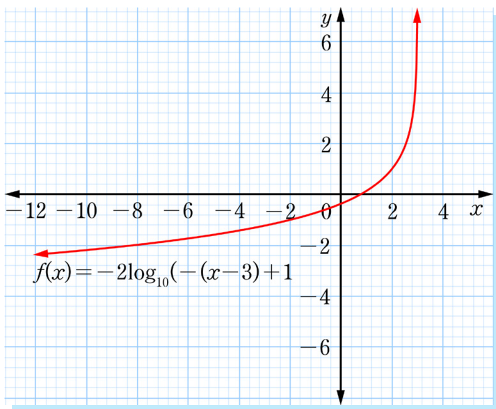
Domain $= \{x : x \in \mathbb{R},\ x > 2\}$

Range $= \{y : y \in \mathbb{R}\}$

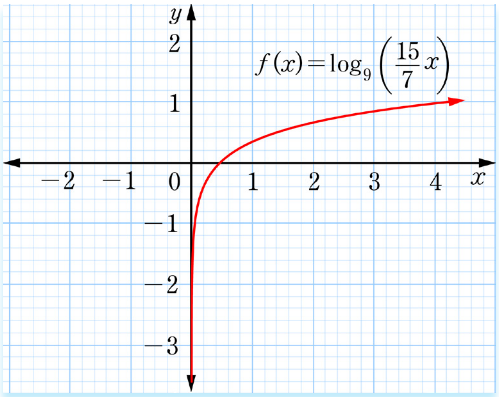
2.



3.

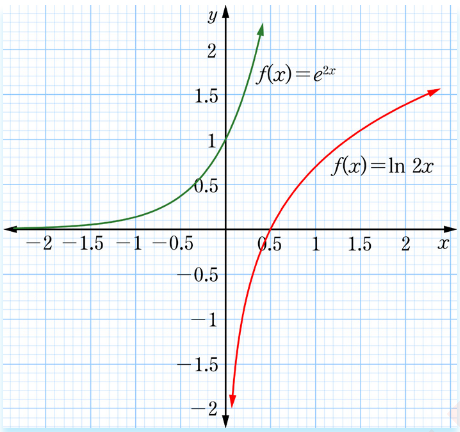


4.

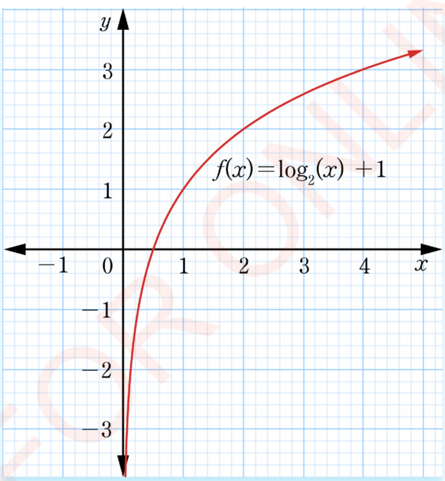


Domain $=\{x : x \geq 0\}$, Range $=\{y : y \in \mathbb{R}\}$

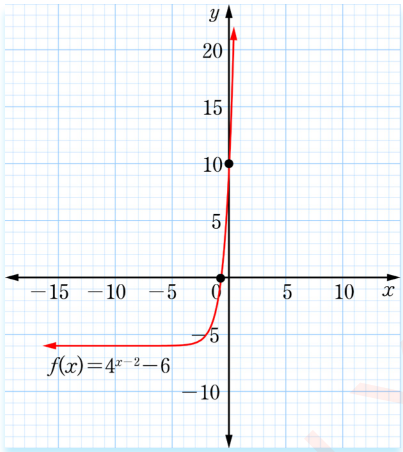
5.



6.



7.



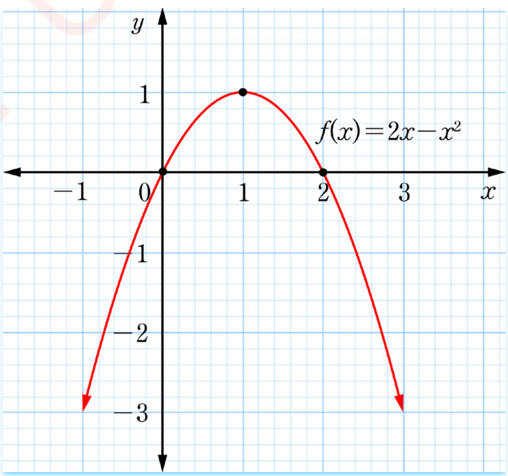
(a) Domain $=\{x : x \in \mathbb{R}\}$,

Range $=\{y : y > -6\}$

(b) x-intercept $= (-0.7, 0)$

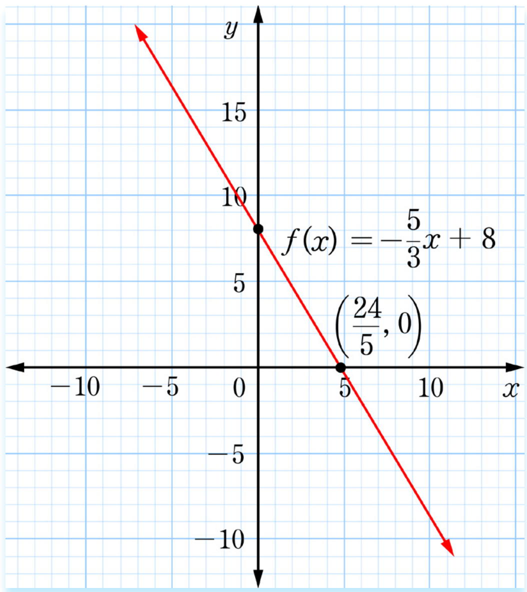
y-intercept $= (0, 10)$

Revision exercise 5

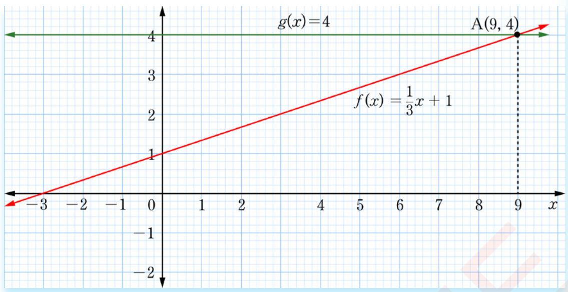


Domain $=\{x : x \in \mathbb{R}\}$, Range $=\{y : y \leq 1\}$.

2. (a)

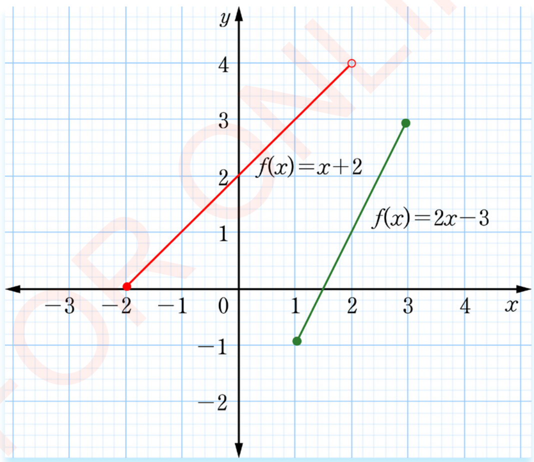


(b)

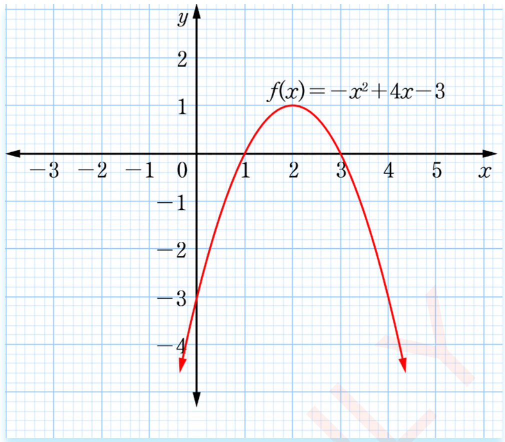


The point is (9, 4).

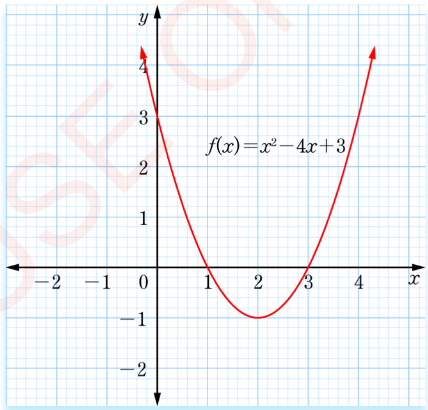
(c)



3. (a)

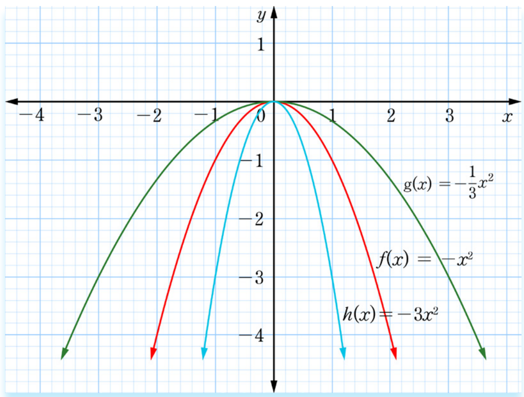


(b)



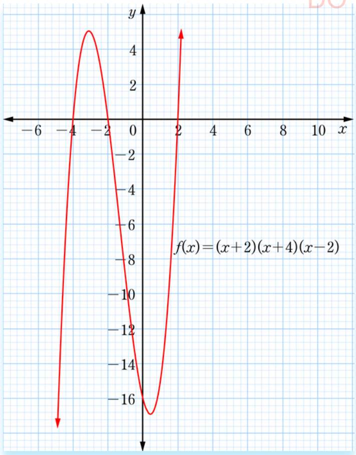
Domain$=\{x : x \in \mathbb{R}\}$, range $=\{y : y \geq -1\}$

(c)



4. Domain$=\{x : x = 1 \text{ and } x = 2\}$

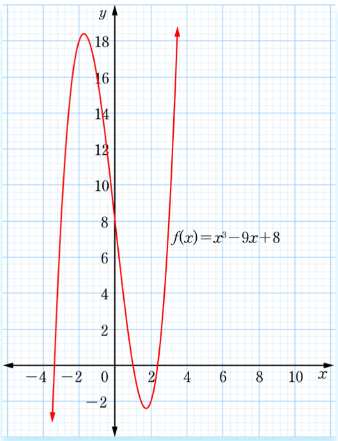
5. (a)



Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

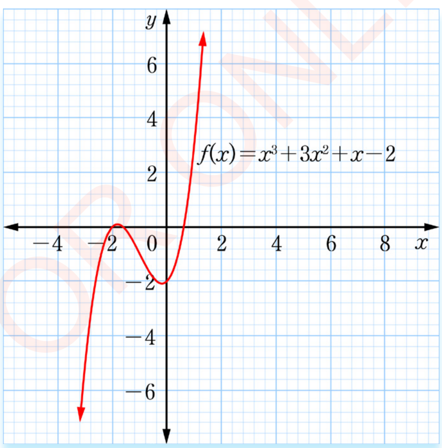
(b)



Domain $= \{x : x \in \mathbb{R}, -4 \leq x \leq 4\}$

Range $= \{y : y \in \mathbb{R}, -20 \leq y \leq 36\}$

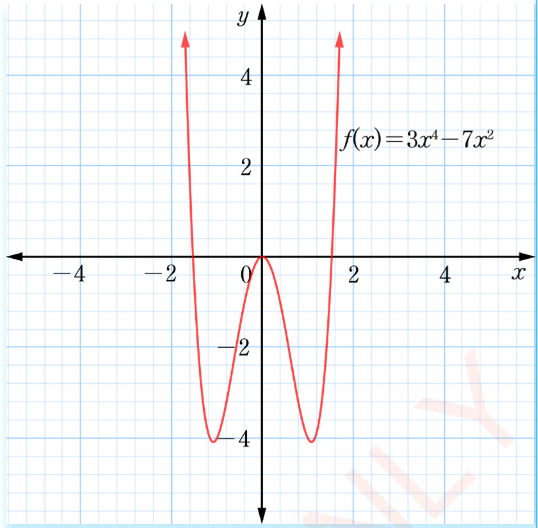
(c)



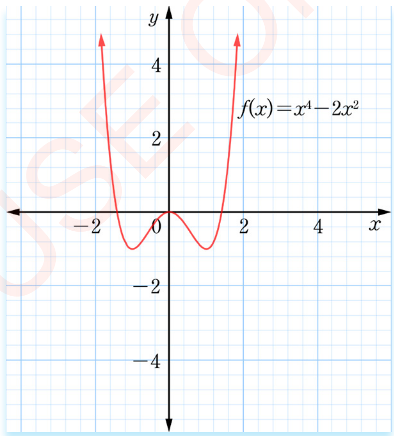
Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y : y \in \mathbb{R}\}$

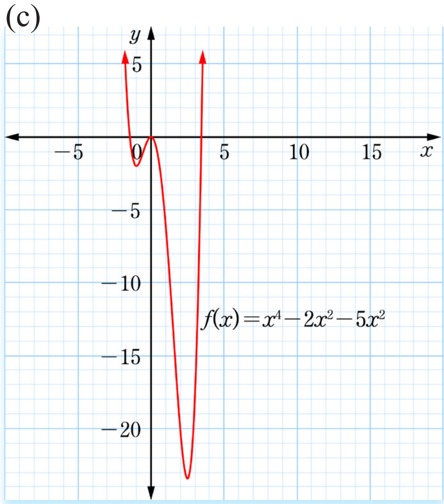
6. (a)



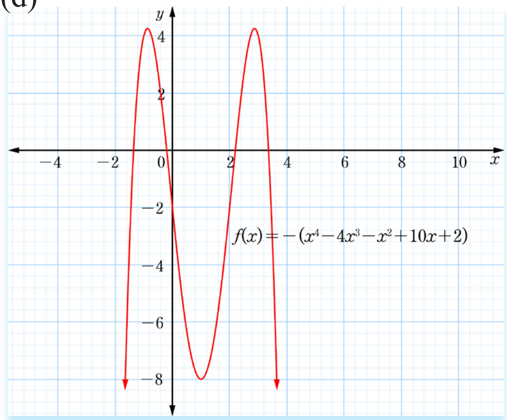
(b)



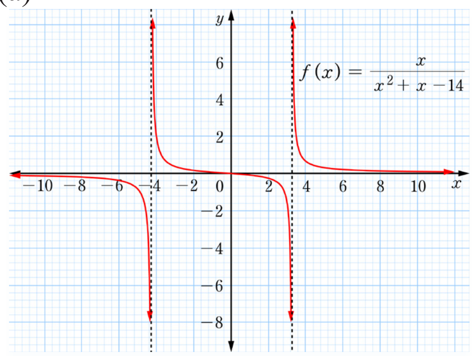
(c)



(d)

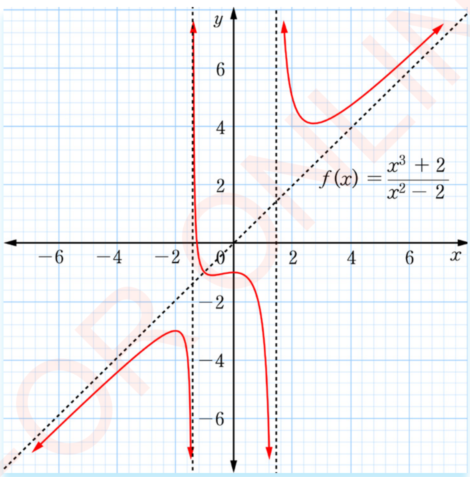


7. (a)



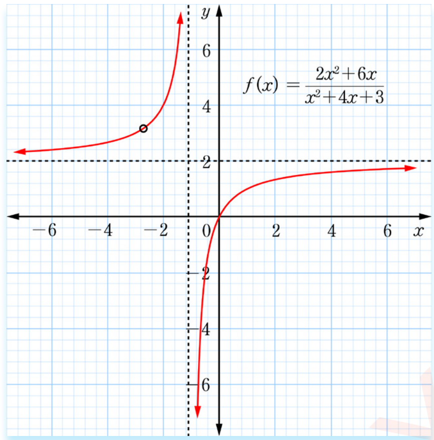
Domain = $\{x : x \in \mathbb{R},\ x \neq -4.27,\ x \neq 3.27\}$

(b)



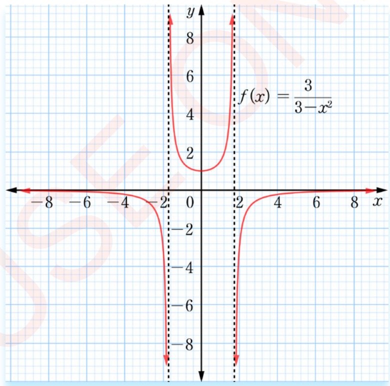
Domain $= \left\{x : x \in \mathbb{R}, x \neq \pm\sqrt{2}\right\}$

(c)



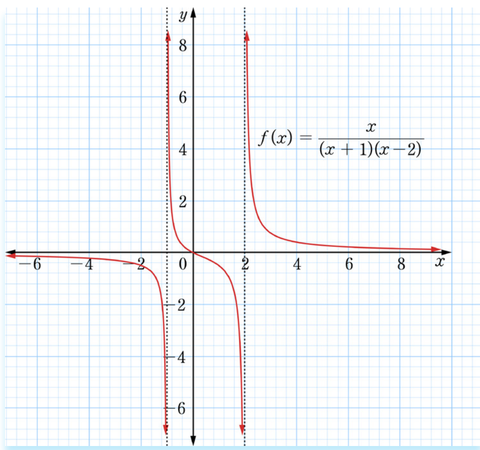
Domain=$\{x : x \in \mathbb{R}, x \neq -3,\ x \neq -1\}$

(d)



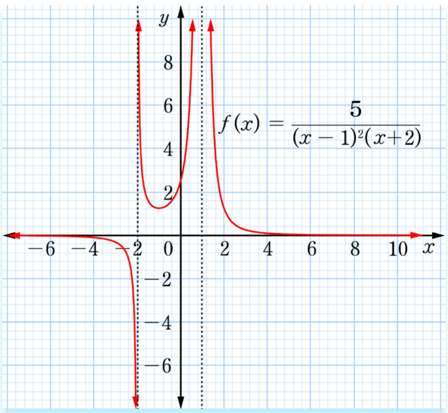
Domain $= \left\{x : x \in \mathbb{R}, x \neq \pm\sqrt{3}\right\}$

(e)



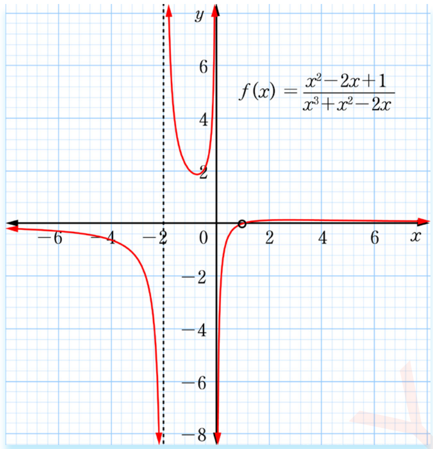
Domain = $\{x : x \in \mathbb{R},\ x \neq -1,\ x \neq 2\}$

(f)



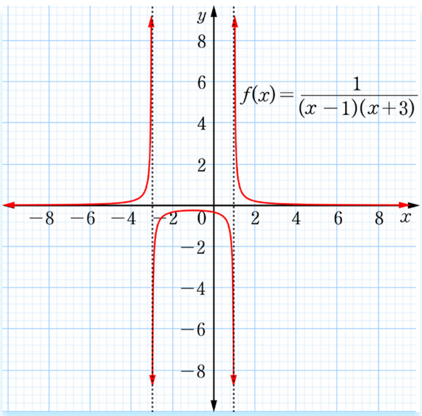
Domain $= \{x : x \in \mathbb{R},\ x \neq -2, x \neq 1\}$

(i)



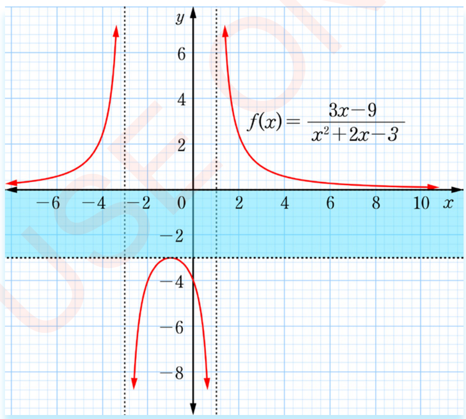
Domain$=\{x : x \in \mathbb{R}, x \neq -2,\ x \neq 0, x \neq 1\}$

(g)



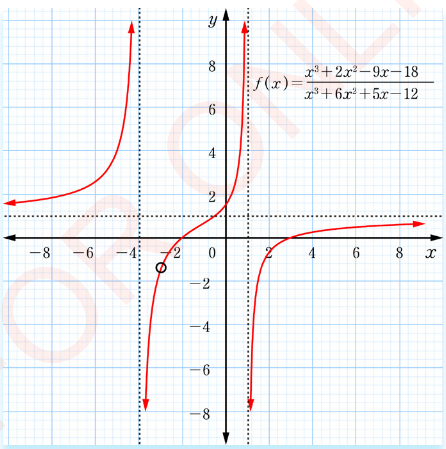
Domain$=\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 1\}$

(j)



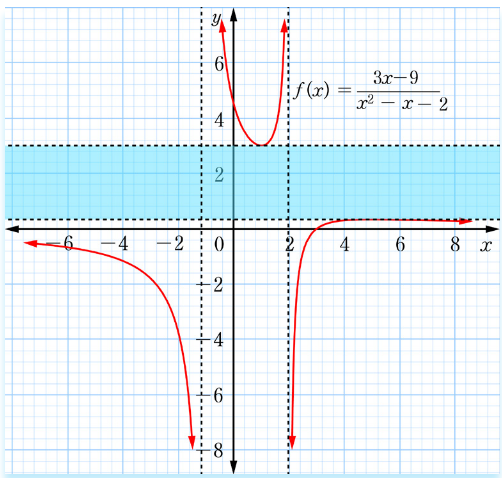
Domain$=\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 1\}$

(h)



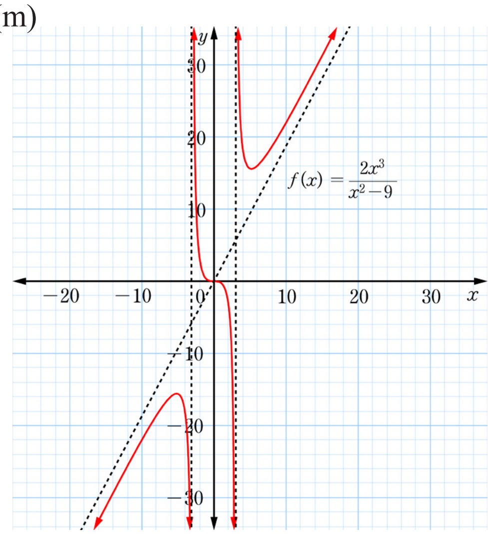
Domain$=\{x : x \in \mathbb{R}, x \neq -4, -3, x \neq 1\}$

(k)



Domain$=\{x : x \in \mathbb{R}, x \neq -1,\ x \neq 2\}$

(m)



Domain $=\{x : x \in \mathbb{R}, x \neq -3,\ x \neq 3\}$

8. (a) $f \circ g(x) = \{(2,-2),\ (3,\ 2)\}$

   (b) $g \circ f(x) = \{(3,\ 1),\ (4,\ 6)\}$.

9. (a) $f \circ g(x) = \{(7,12), (-1,19), (9,15)\}$

   (b) $g \circ f(x) = \{(1,3), (2,4)\}$

10. (a) $f \circ g(x) = x^2 - 6x + 10$

    (b) $g \circ f(x) = x^2 + 2x - 2$

    (c) $g \circ f \circ h(x) = \cos^2 x + 2\cos x - 2$

11. $f(x) = \dfrac{-5 \pm \sqrt{144x^2 - 23}}{4}$

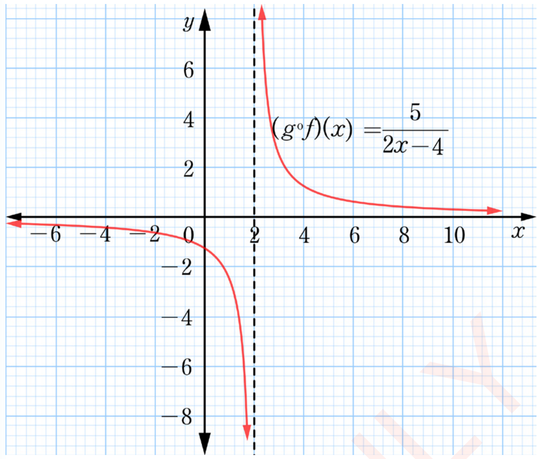
13. (a) $(f \circ g)(x) = \dfrac{-2x+14}{x-2}$

    (b) $(g \circ f)(x) = \dfrac{5}{2x-4}$

    (c) $(f \circ g)(3) = 8$

    (d) $(g \circ f)(x) = \dfrac{5}{4}$

(e)

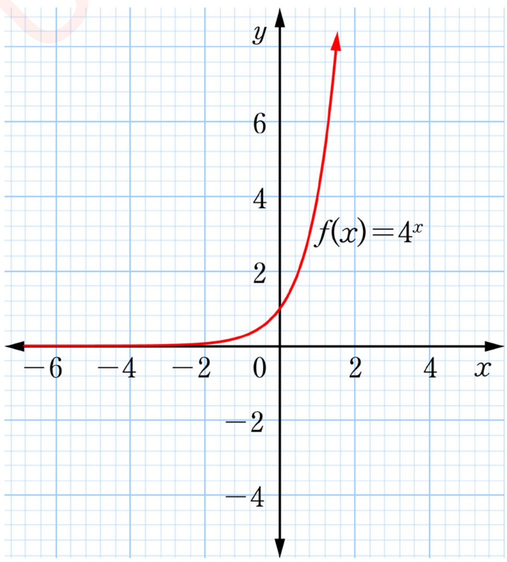


14. $x = \pm 2.91$

15. (a) $\dfrac{3-3x}{x^2+4x+4}$ (b) $\dfrac{1}{4}$

16. (a) 2 (b) 4

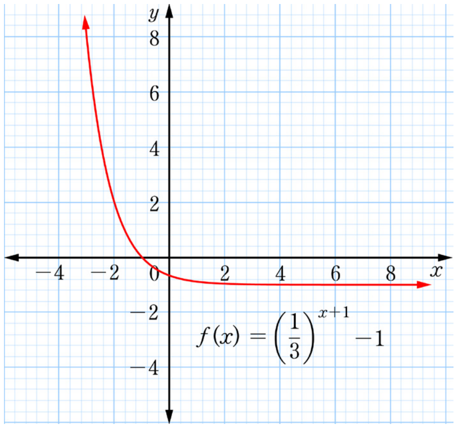
17. (b)



Domain $=\{x : x \in \mathbb{R}\}$

Range $=\{y : y > 0\}$

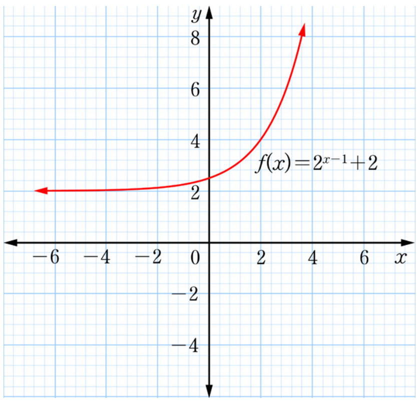
(c)



Domain $= \{x : x \in \mathbb{R}\}$

Range$=\{y : y > -1\}$

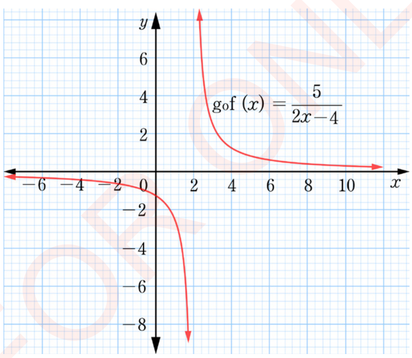
(d)



Domain$=\{x : x \in \mathbb{R}\}$

Range$=\{y : y > 2\}$

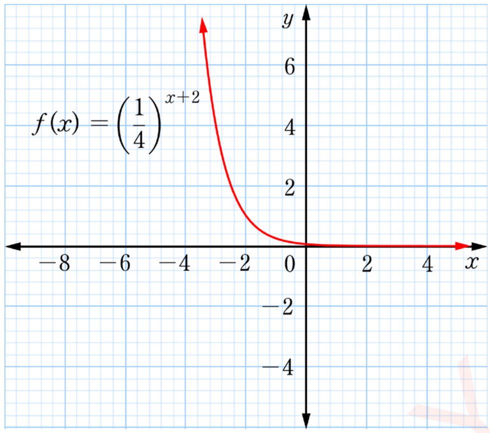
(e)



Domain$=\{x : x \in \mathbb{R}\}$

Range $= \{y : y > 0\}$

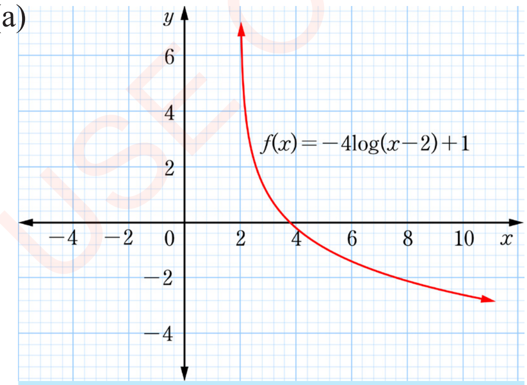
(f)



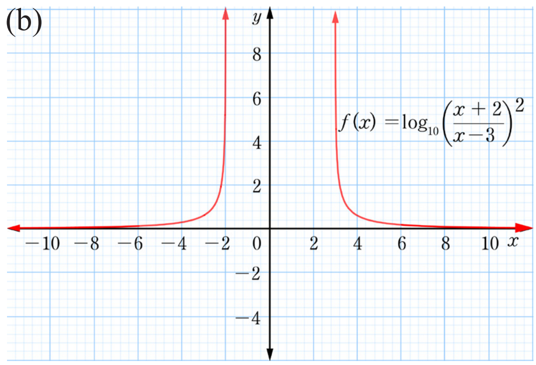
Domain$=\{x : x \in \mathbb{R}\}$

Range $=\{y : y \geq 0\}$

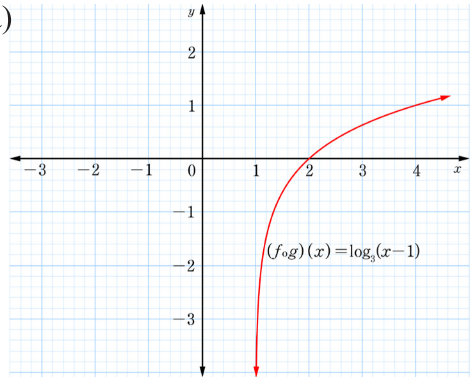
18. (a)



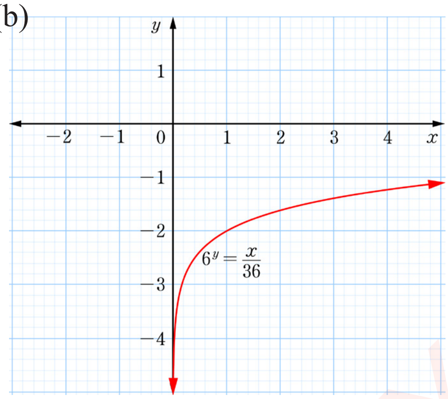
(b)



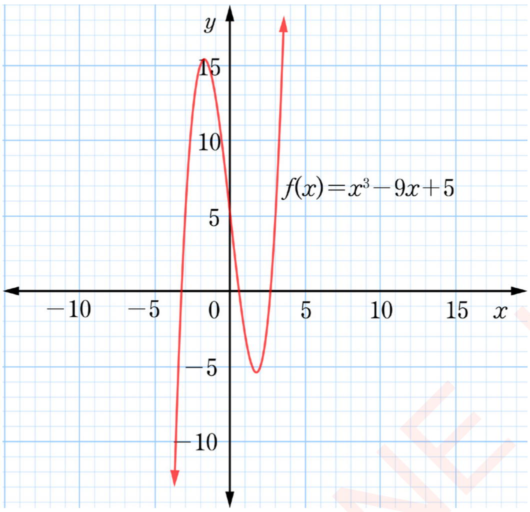
19. (a)



(b)



20.



Answers

Chapter Six

Exercise 6.1

1. (a) $n=3$ (b) $x=-2$ (c) $u=1$ (d) $x=3$

2. $\frac{p}{q}=\frac{9}{16}$

3. $x=2,\ y=0$

4. 36

5. $x=\left(2\pm\sqrt{2}\right)^{-5}$

6. $x=3$

7. $x=1$

8. $x=2$

9. $x=-\frac{1}{2}$

10. $y=-1$ and $y=3$

Exercise 6.2

1. (a) $m=3$ (b) $y=3$ (c) $x=0.9061$
   (d) $z=3$ and $z=27$ (e) $x=0.8830$ (f) $x=\frac{2}{3}$

3. $(m,\ n)=(2,\ 8)$ or $(m,\ n)=(8,\ 2)$

4. (a) $x=50$ (b) $x=11.6927$

5. $(f,\ g)=\left(65536,\ \frac{1}{4}\right)$

6. (a) 0.7177 (b) 5.1127 (c) 0.6881

7. $x=4$ and $y=\frac{1}{2}$

8. $p=3$ or $p=27$

10. $z=-4$ or $z=3$

Exercise 6.3

1. (a) $\sum_{k=1}^{5}(3k+5)$ (b) $\sum_{k=1}^{14}2k$ (c) $\sum_{k=1}^{7}32\left(\frac{1}{2}\right)^{k-1}$ (d) $\sum_{k=1}^{11}k^2$
   (e) $\sum_{k=1}^{6}\frac{1}{2^k}$ (f) $\sum_{k=1}^{n}(-1)^k k^2$

2. (a) $4+5+6+7+8+9$ (b) $29+70+145$
   (c) $-2-10-24-44-70-102-140$ (d) $-3-7-15-31-63-127-255$
   (e) $\frac{1}{4}+\frac{1}{2}+1+2+4+8+16+32+64+128$

3. (a) 156 (b) $\frac{63}{2}$ (c) $\frac{42}{5}$ (d) 380 (e) 1

4. (a) $2n(n+1)$ (b) $\frac{1}{3}n(n+1)(n+5)$ (c) $\frac{1}{12}n(n+1)\left(3n^2+23n+46\right)$

5. (a) $\sum_{k=1}^{4} k(2k-1)$ (b) $\sum_{k=1}^{5}(-x)^{k-1}$ (c) $\sum_{k=1}^{n} ny^{k}$ (d) $\sum_{k=1}^{8}(3k-7)$

   (e) $\sum_{k=1}^{4}\frac{1}{2k+1}$ (f) $\sum_{k=1}^{4}(3k-2)(3k+1)$

6. (a) 4, 8, 16 (b) 2, 6, 12 (c) $2, \frac{1}{2}, \frac{4}{15}$ (d) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}$

   (e) 1, −4, 9 (f) 18, 300, 1,134

Exercise 6.5

1. $\frac{2}{1-q}, \frac{-1}{1-q}$ 2. (a) 6 (b) $-\frac{117}{8}$ 3. (a) $x^2-5x-3=0$

   (b) $3x^2-10x-4=0$

5. $\sqrt{\frac{140}{17}}$ 6. $9x^2+55x+6=0$

7. (a) $4x^2+40x+51=0$ (b) $4x^2-37x+9=0$ (c) $x^2-5x-6=0$

   (d) $3x^2+37x+12=0$ (e) $2x^2-7x=0$ (f) $2x^2+25x+72=0$

10. $4x^3-13x^2+48x-64=0$

11. (a) $x^3-3x^2-2x-32=0$ (b) $x^3+4x^2-x-11=0$

12. (a) $-\frac{1}{12}$ (b) 23 (c) 64 (d) $-\frac{1}{12}$ (e) $-\frac{5}{12}$

Exercise 6.6

1. (a) $-43x^5+86x^4-56x^3+12x^2-24x+20$

   (b) $7x^5-58x^4+8x^3-12x^2-24x+120$

2. $a=\frac{41}{7}, b=-\frac{114}{7}$, and $c=-\frac{648}{7}$ 3. 890

4. $e=9, f=-2, g=-11$ 5. (a) $x^4-4x^3-2x^2+12x+9$

   (b) $-3x^4+7x^3-6x^2+3x-1$

6. (a) $m=10, n=-7, p=4$

7. (a) quotient x^2+2x+1, remainder 3
   (b) quotient $x^3-8x+10$, remainder -11
   (c) quotient x^5-2x^4+2, remainder -5
   (d) quotient $x-13$ or remainder 64
   (e) quotient $x^3+4x^2+12x+50$, remainder 206.
   (f) quotient $2x^3-x^2+x-1$, remainder 2.
8. (a) quotient $2x^3+2x+8$ Remainder $-x+6$.
   (b) quotient x^2-x-1, remainder $2x+2$.
   (c) quotient $2x^3-x^2+x-1$, remainder 2.
   (d) quotient $x^3+ax^2+a^2x+a^3$, remainder 0.
9. (a) 2 (b) 7 (c) 319 (d) 2549 (e) 131
10. $t=30$, no factors

Exercise 6.7

1. $\left\{x\in\mathbb{R}:-\frac{1}{3}<x<7\right\}$
2. $\left\{x\in\mathbb{R}:\frac{5}{3}<x<11, x\neq 4\right\}$
3. $\left\{x\in\mathbb{R}:x>\frac{3}{2}\text{ or } x<-\frac{7}{2}\right\}$
4. $\left\{x\in\mathbb{R}:x<-\frac{1}{2}\text{ or } x>\frac{3}{2}\right\}$
5. $\left\{x\in\mathbb{R}:x\leq\frac{1}{4}\text{ or } x\geq\frac{1}{2}\right\}$
6. $\{x\in\mathbb{R}:x>3\text{ or } x<-1\}$
7. $\{x\in\mathbb{R}:-1<x<1\}$
8. $\{x\in\mathbb{R}:-1<x<0\text{ and } x>1\}$
9. $\{x\in\mathbb{R}:x<-1,\ 0<x<1\}$
10. $\{x\in\mathbb{R}:x\leq 3\text{ or } x\geq 5\}$
11. $\{x\in\mathbb{R}:0\leq x\leq 5\}$
12. $\{x\in\mathbb{R}:x\geq -1\text{ and } x\leq -5\}$
13. $\left\{x\in\mathbb{R}:x<-\frac{3}{2}\text{ or } x>1\right\}$
14. $\{x\in\mathbb{R}:1<x<3\}$
15. $\{x\in\mathbb{R}:-1<x<0\text{ and } 0<x<1\}$
16. $\left\{x\in\mathbb{R}:\frac{2}{3}<x<1\text{ or } \frac{3}{2}<x<4\right.$
17. $\left\{x\in\mathbb{R}:-1<x<\frac{1}{3}\text{ or } 1<x<24\right.$
18. $\left\{x\in\mathbb{R}:-\frac{3}{7}<x<1\right.$
19. $\left\{x\in\mathbb{R}:-\frac{1}{4}<x<\frac{5}{4}\right.$

Exercise 6.8

2. $\begin{pmatrix} 25 & 37 & 53 \\ 119 & 31 & 55 \\ 68 & -9 & 0 \end{pmatrix}$

4. (a) $\begin{pmatrix} -5890 \\ -190 \\ 9310 \end{pmatrix}$

   (b) Not possible

   (c) Not possible

5. $c = -50$, $d = 4$, $e = 9$.

6. $\begin{pmatrix} 1220 \\ 620 \\ 10 \end{pmatrix}$

8. (a) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$

   (b) $\begin{pmatrix} -11 & -15 & -10 \\ 13 & 15 & 8 \\ -31 & -36 & -23 \end{pmatrix}$

9. $\begin{pmatrix} 2580 \\ 2170 \\ 2292 \end{pmatrix}$ where, 2580, 2170, and 2292 are the total points from the departments of geography, chemistry, and biology, respectively.

10. (a) $\begin{array}{r c} & \begin{array}{ccc} D_1 & D_2 & D_3 \end{array} \\ 1^{st} \text{ student} & \\ 2^{nd} \text{ student} & \begin{pmatrix} 8 & 12 & 16 \\ 5 & 9 & 10 \\ 15 & 18 & 12 \end{pmatrix} \\ 3^{rd} \text{ student} & \end{array}$

   (b) The total sum of money spent by the 1st, 2nd and 3rd students are Tshs 18,000, Tshs 12,000 and Tshs 22,500 respectively.

Exercise 6.9

1. (a) −532 (b) 380

2. (a) $\begin{pmatrix} -16 & 8 & -34 \\ -14 & -46 & 63 \\ -24 & 12 & 2 \end{pmatrix}$

   (b) $\begin{pmatrix} 7 & -9 & -10 \\ 1 & -12 & -5 \\ -10 & -4 & 0 \end{pmatrix}$

3. (a) $\begin{pmatrix} 3 & -9 & -5 \\ -4 & 1 & 3 \\ -5 & 4 & 1 \end{pmatrix}$

   (b) $\begin{pmatrix} 15 & -3 & -21 \\ -6 & -18 & 2 \\ -30 & 6 & 10 \end{pmatrix}$

4. (a) −212 (b) 25

9. (a) $m = 4$ (b) $r = -10$

10. (a) $\begin{pmatrix} 17 & 2 & -6 \\ -4 & 4 & 6 \\ -2 & 6 & 9 \end{pmatrix}$ (b) 144

Exercise 6.10

1. (a) $R^{-1} = \frac{1}{100}\begin{pmatrix} 2 & 10 & 26 \\ 16 & -20 & 8 \\ 22 & 10 & -14 \end{pmatrix}$ (b) $x = 1,\ y = -2,\ z = 3$

2. (a) $(x,\ y,\ z) = (3,\ 1,\ 4)$ (b) $(x,\ y,\ z) = (3,\ 2,\ 1)$
   (c) $(x,\ y,\ z) = (-4.714,\ -2.071,\ -5.86)$ (d) $(x,\ y,\ z) = (-1,\ 2,\ 3)$
   (e) $(x,\ y,\ z) = (8,\ -11,\ 3)$.

3. (a) $\begin{pmatrix} 8 & 8 & 8 \\ -7 & 11 & -1 \\ -5 & 1 & 13 \end{pmatrix}$ (b) $x = 1,\ \ y = 3,\ z = -2$

4. $\frac{1}{15}\begin{pmatrix} -4 & 11 & 0 \\ -7 & 8 & 0 \\ 27 & -3 & 15 \end{pmatrix}$ 5. $(a,\ b,\ c) = (2,\ 5,\ 1)$

6. $(I_1,\ I_2,\ I_3) = (1.3,\ -4.5,\ -3.7)$ 7. $(k,\ m,\ n) = (-5,\ 10,\ 2)$

8. $q = 40{,}000,\ r = 75,\ s = 30$

9. 5 units of simple, 8 units of medium, 8 units of complex

10. $(x,\ y,\ z) = (2,\ 3,\ 1)$ units of complex

Exercise 6.11

1. (a) $32x^5 + 240x^4z + 720x^3z^2 + 1080x^2z^3 + 810xz^4 + 243z^5$
   (b) $a^7 - 7a^6b + 21a^5b^2 - 35a^4b^3 + 35a^3b^4 - 21a^2b^5 + 7ab^6 - b^7$
   (c) $32x^5 + 160x^3 + 320x + \frac{320}{x} + \frac{160}{x^3} + \frac{32}{x^5}$

2. (a) $140\sqrt{2}$ (b) 24476 (c) $40\sqrt{2}$ (d) 98

3. $1 + 10x + 55x^2 + 210x^3 + \cdots$; 1.106

4. (a) 1.0743 (b) 1279.2 (c) 7272.2

5. $m = -9$ and $n = 46$

7. $32 + 80y + 80y^2 + 40y^3 + 10y^4 + y^5$; 32.08008

8. $1024 + 1280y + 720y^2 + 240y^3$; 1159

9. $4-28c+85c^2-146c^3+155c^4+\cdots$

10. $81a^4-54a^3b+\dfrac{27}{2}a^2b^2-\dfrac{3}{2}ab^3+\dfrac{1}{16}b^4$; 757335.0625

Exercise 6.12

3. $1+2x+\dfrac{5}{2}x^2+\cdots; -\dfrac{1}{3}<x<\dfrac{1}{3}$

4. $-\dfrac{2}{3}<y<\dfrac{2}{3}$

5. $1-a-\dfrac{7}{2}a^2+\cdots; -\dfrac{1}{3}<a<\dfrac{1}{3}$

6. $1-4y-24y^2-224y^3$; 2.499

8. $1+\dfrac{1}{3}t-\dfrac{1}{9}t^2+\cdots$; 2.080

10. $-\dfrac{1}{4}-\dfrac{1}{2}n-\dfrac{3}{16}n^2-\cdots, -1<n<1$

Exercise 6.13

2. $n=55$ 3. 27 4. $r=7$

5. (a) ${}^9C_r(-1)^r\dfrac{2^{18-3r}}{5^{9-2r}}y^{9-2r}$ (b) ${}^5C_r(2y)^{5-r}\left(\dfrac{-1}{y}\right)^r$ (c) $(-1)^r\,{}^{12}C_r y^{24-3r}$

(d) $(-1)^r\,{}^6C_r t^{12-2r}s^r$

8. $p=3, q=5, n=6$

9. (a) ${}^{15}C_{10}\left(\dfrac{1}{6}\right)^5$ (b) 45 (c) $\dfrac{17}{54}$ (d) -3432

10. (a) 14 (b) -16

Exercise 6.14

1. $\dfrac{4}{x+2}-\dfrac{2}{x+1}$

2. $\dfrac{2}{x-3}-\dfrac{4}{x-1}$

3. $\dfrac{5}{12(x+1)}+\dfrac{34}{3(x-2)}-\dfrac{43}{4(x-3)}$

4. $\dfrac{3}{13(x+1)}+\dfrac{5x-17}{13(x^2+3)}$

5. $1-\dfrac{14}{x}-\dfrac{12x-6}{x^2-x+1}$

6. $\dfrac{8x+13}{3(x^2+2)}+\dfrac{1}{3(x+1)}$

7. $y-5+\dfrac{2}{y+1}+\dfrac{1}{y+2}-\dfrac{3}{y+3}$

8. $-\dfrac{3}{2(x-1)}+\dfrac{1}{(x-1)^2}+\dfrac{2}{x-2}-\dfrac{3}{2(x-3)}$

9. $\dfrac{16}{3(x+1)}+\dfrac{-13+11}{3(x^2-x+1)}$

10. $\dfrac{36}{5(t^2+2)}+\dfrac{8}{5\sqrt{3}\left(t+\sqrt{3}\right)}-\dfrac{8}{5\sqrt{3}\left(t-\sqrt{3}\right)}$

Exercise 6.15

1. $\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2};\ s_\infty=\frac{1}{4}$

3. $2-\frac{1}{n+1}-\frac{2}{n+2};\ s_\infty=2$

4. $\frac{1}{2n-1}-\frac{1}{2n+1};\ \frac{1}{2}-\frac{1}{2n+1}$

5. $\frac{1}{2}+\frac{n+3}{(2n+1)(2n+3)}$

6. $\frac{1}{n}-\frac{1}{n+2}$

9. $\frac{3}{8}-\frac{2n+1}{(2n+2)(2n+6)}$

10. (a) $\frac{1}{6}-\frac{n+2}{(n+3)(n+4)}$ (b) $\frac{11}{18}-\frac{1}{3}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}\right)$ (c) $\frac{1}{9}-\frac{1}{9(n+1)}$

Revision exercise 6

4. Factors: $x+1, x-1$, and x^2+4, $\frac{1}{x+1}-\frac{2}{x-1}+\frac{3x+2}{x^2+4}$

5. $a=\frac{5}{2}, b=\frac{9}{2}$

6. $\frac{3}{1+x}+\frac{4-3x}{1+x^2};\ a=7, b=-6, c=-6, d=0, e=7, f=-6$

7. $(0,\ 0), (9, 27)$ 8. $\frac{2}{(x-2)^6}+\frac{5}{(x-2)^7}+\frac{4}{(x-2)^8}$ 12. 4

13. (a) $x=80,\ y=2$ (b) $x=75$

14. $\begin{pmatrix} 11 & -4 & 1 \\ -25 & 9 & -2 \\ 15 & -5 & 1 \end{pmatrix}$ 15. $x=30,\ y=20,\ z=-60$ 16. $x=\frac{1}{2},\ y=1,$ and $z=\frac{3}{2}$

17. (a) $1-3x+6x^2-10x^3+\cdots$ (b) $1-\frac{2}{3}x^2-\frac{1}{9}x^4-\frac{4}{81}x^6+\cdots$

18. $\frac{21875}{128}x^4$ 20. $\frac{3}{3x-1}-\frac{1}{x+1}+\frac{1}{(x+1)^2}$, Coefficient $=n(-1)^n-3^{n+1}$

21. (a) 1.00499 (b) 0.9933

22. $n=11$; 55, 165, and 330.

24. (a) 0 (b) $(y-x)(z-x)(z-y)$

30. $x=1,\ y=-4,\ z=-4$

31. (a) $\begin{pmatrix}19 & 31\end{pmatrix}$ (b) $\begin{pmatrix}-6 & -4 & 13\\ -1 & -2 & 3\\ 3 & -6 & -10\end{pmatrix}$

33. $\frac{2}{9}+\frac{11}{27}x+\frac{20}{27}x^2;\quad -\frac{3}{2}<x<\frac{3}{2}$

36. (a) $k=-6,\ k=6$ (b) $k<-6$ or $k>6$

37. (a) $x<-5$ or $x>\frac{1}{5}$ (b) $2<x<3$ or $x>8$

38. $n\left(n^2+6n+11\right)$; 296616

39. 3 carton of concentrated, 1 carton of diluted and 2 cartones of drt product.

Answers

Chapter Seven

Exercise 7.1

1. $x = 8.1\text{cm},\ h = 12.8676\text{ cm}$
2. $\overline{VW} = 18.9148$ units,
   $\overline{UV} = 37.8297$ units,
   $\overline{UW} = 42.2949$ units
3. $\overline{BD} = 13.4$ units
4. $\overline{JK} = 10$ cm, $\overline{KL} = 10$ cm,
   $\overline{JL} = 14.14$ cm
5. $\operatorname{cosec} D = \frac{169}{119}$, $\sec D = \frac{169}{120}$
6. (a) $1\frac{1}{12}$ (b) $2\frac{2}{5}$
7. $\widehat{M} = 45°$, $\widehat{N} = 45°$
8. $5\sqrt{5}$ cm, 53.13° and 126.87°
9. $\overline{PR} = 13.97$ cm
10. 407 metres
11. $h = 46.86$ metres

Exercise 7.2

1. (a) $\cos y$ (b) $\cos\theta$
   (c) 1 (d) $\sec\theta$
4. (a) $x^2 + y^2 = 1$
   (b) $(x-2)^2 + (y+1)^2 = 1$
   (c) $xy = 3$
5. (a) $\theta = 60°, 120°, 240°, 300°$
   (b) $\theta = 70.5°, 289.5°$
   (c) $\theta = 26.6°, 45°, 225°, 206.6°$

Exercise 7.3

1. (a) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$
   (b) $-\frac{36}{325}, \frac{323}{325}, -\frac{36}{323}$
2. (a) $\frac{204}{325}, -\frac{253}{325}, -\frac{204}{253}$
   (b) $\frac{56}{65}, \frac{33}{65}, \frac{56}{33}$
6. (a) $157\frac{1}{2}°$, $337\frac{1}{2}°$
   (b) 49.1°, 229.1°
   (c) 56.5°, 236.5°
8. (a) $\frac{\sqrt{3}}{2}$ (c) $\frac{\sqrt{6}+\sqrt{2}+6}{8}$
   (b) $\sqrt{3}$

Exercise 7.4

3. (a) 2 (c) $1 - \sin x$
   (b) $\cot\theta$ (d) $\cot x$
5. (a) $x + 2y^2 = 0$
   (b) $x = 2y^2 + 4y + 1$
   (c) $y = \frac{4x(1-x^2)}{1-6x^2+x^4}$
   (d) $\frac{(x-4)^2}{9} + \frac{(y-7)^2}{81} = 1$
6. (a) 22.5°, 112.5°, 202.5°, 292.5°
   (b) 45°, 121°, 225°, 301°
   (c) 14.47°, 165.53°
   (d) 0°, 48.18°, 180°, 311.82°, 360°

8. (a) $\frac{336}{527}, -\frac{336}{625}, -\frac{625}{527}$

(b) $-\frac{28560}{239}, -\frac{28560}{28561}, \frac{28561}{239}$

11. (a) $-\frac{\sqrt{3}}{3}$

(b) $\frac{\sqrt{3}}{2}$

(c) $-\frac{1}{2}$

(d) $\frac{\sqrt{8-2\sqrt{6}-2\sqrt{2}}}{4}$

Exercise 7.5

1. (a) 0º, 112º, 360º

(b) 53.1°, 323.1°

(c) 48.4°, 205.34°

(d) 119.56°, 346.7°

(e) 114.2°, 335.7°

(f) 188.4°, 319.72°

(g) 101.34°, 355.4°

(h) 36.9°, 241.92°

(i) 80.72°, 234.04°

(j) 51.32°, 308.68°

2. (a) $-30°$, 90°

(b) 76.71°, $-150.45°$

(c) $-4.9°$, $-129.88°$

(d) $-20.52°$, 28.11°, 159.48°

(e) $-24.3°$, 114.29°

(f) 0°, 45°, 180°

(g) $-8.3°, -155.43°$

(h) $-180°, -106.85°, -90°, -16.85°, 0°, 73.16°, 90°, 163.16°, 180°$

(i) $-40.72°, 72.6°$

(j) 51.33°, 128.67°

3. (a) Maximum $\sqrt{2}$, minimum $-\sqrt{2}$
Maximum occurs at $45°$
Minimum occurs at $225°$

(b) Maximum $\sqrt{2-\sqrt{3}}$, minimum $-\sqrt{2-\sqrt{3}}$ Maximum occurs at $255°$ Minimum occurs at $75°$

(c) Maximum 5, minimum -5
Maximum occurs at $143.13°$
Minimum occurs at $306.87°$

(d) Max. $\sqrt{\frac{201}{4}+7\sqrt{3}}$, min. $-\sqrt{\frac{201}{4}+7\sqrt{3}}$ Maximum occurs at $83.42°$ Minimum occurs at $263.42°$

(e) Maximum $\sqrt{10}$, minimum $-\sqrt{10}$ Maximum occurs at $71.57°$ Minimum occurs at $251.57°$

(f) Maximum 5, minimum -5
Maximum occurs at $36.87°$
Minimum occurs at $216.87°$

(g) Maximum 17, minimum -17
Maximum occurs at $118.07°$
Minimum occurs at $293.96°$

(h) Maximum $\sqrt{34}$, minimum $-\sqrt{34}$,
Maximum occurs at $30.96°$
Minimum occurs at $210.96°$

(i) Maximum $\sqrt{37}$, minimum $-\sqrt{37}$
Maximum occurs at $99.46°$
Minimum occurs at $279.46°$

(j) Maximum $\frac{1}{2}$, minimum $-\frac{1}{2}$
Maximum occurs at $45°$ Minimum occurs at $135°$.

4. Maximum $\sqrt{13}$, minimum $-\sqrt{13}$
Maximum occurs at $33.69°$
Minimum occurs at $-146.31°$

5. $\cos\theta + 2\sin\theta \equiv \sqrt{5}\sin(\theta+\alpha)$
Maximum $\sqrt{5}$, minimum $-\sqrt{5}$
Maximum occurs at $63.43°$
Minimum occurs at $-116.565°$

Exercise 7.6

1. $\frac{\pi}{2}n + (-1)^n \frac{\pi}{12}$
2. $\frac{\pi}{5}n + \frac{\pi}{20}$
3. $2\pi n + \frac{\pi}{4}$, $2\pi n + \frac{7}{4}\pi$
4. $2\pi n - 0.19$, $2\pi n + \pi - 0.89$
5. $\pi n + 1.6512$
6. $\pi n + 1.0616$, $\pi n - 1.22678$
7. $2\pi n + \frac{7}{6}\pi$, $2\pi n + \frac{11}{6}\pi$, $2\pi n + \frac{1}{2}\pi$
8. $2\pi n + 1.7$, $2\pi n + (2\pi - 0.72)$
9. $\pi n + 1.365$, $\pi n + 0.2058$
10. $2\pi n + \frac{11}{6}\pi$.

Exercise 7.7

1. (a) $2\sin 45°\cos 5°$
(b) $2\cos 45°\sin 25°$
(c) $2\cos 40°\cos 15°$
(d) $2\sin 55°\sin 20°$

2. (a) $-\frac{1}{2}(\cos 95° - \cos 15°)$
(b) $\frac{1}{2}(\sin 165° - \sin 55°)$
(c) $\frac{1}{2}(\sin 70° + \sin 10°)$
(d) $\frac{1}{2}(\cos 85° + \cos 15°)$.

3. (a) $\theta = xn,\ \theta = \frac{\pi - 2\pi n}{2}$,
$\theta = \frac{\pi + 4\pi n}{4},\ \theta = \frac{3\pi + 4\pi n}{4}$,
$\theta = \frac{\pi(3n+1)}{3},\ \theta = \frac{\pi(3n+2)}{3}$.

(b) $x = \pi n,\ n\pi - \frac{\pi}{2},\ 2\pi n \pm \frac{2\pi}{3}$

(c) $x = \frac{1}{4}\pi n$, *or* $\frac{2}{3}\pi n \pm \frac{2}{9}\pi$

(d) $\theta = \frac{n\pi}{3}$ or $\frac{\pi n}{2} - \frac{\pi}{8}$

(e) $x = \frac{n\pi}{4}$ or $2\pi n \pm \frac{\pi}{3}$

(f) $x = \pi n + (-1)^n \frac{\pi}{4}$

6. $\frac{a}{b}$, $\frac{2a - b + a^2 b}{1 - a^2 + 2ab}$

7. $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and
$\cos 2\theta = 2\cos^2\theta - 1$

8. (a) $60°, 90°, 120°, 180°$

9. (b) $50°$

 (c) $20°, 90°, 100°, 140°$

 (d) $0°, 45°, 90°, 120°, 135°, 180°$

10. $0, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}, 2\pi$

11. (a) $\tan\frac{5\lambda}{2}$ (b) $18°, 90°$

12. (a) $\tan\frac{17A}{2}$ (b) $\cot\frac{2}{3}\pi$

Exercise 7.8

1. (a) $\frac{\pi}{6}$ (b) $\frac{4\pi}{3}$ (c) $\frac{\pi}{2}$

 (d) $-\frac{3\pi}{4}$ (e) $\frac{2\pi}{9}$ (f) $\frac{3\pi}{2}$

 (g) $\frac{7\pi}{4}$ (h) $\frac{9\pi}{4}$

2. (a) $60°$ (b) $300°$
 (c) $210°$ (d) $-120°$
 (e) $540°$ (f) $252°$
 (g) $382.5°$ (h) $360°$

3. (a) $\frac{\pi}{6}, \frac{11\pi}{6}$ (b) $\frac{\pi}{4}, \frac{5\pi}{4}$

 (c) $\frac{\pi}{4}, \frac{3\pi}{4}$ (d) $\frac{7\pi}{6}, \frac{11\pi}{6}$

 (e) $\frac{3\pi}{4}, \frac{5\pi}{4}$ (f) $\frac{\pi}{3}, \frac{4\pi}{3}$

4. (a) $28.07°$ (b) $98.55°$
 (c) $-135.22°$ (d) $-48.70°$
 (e) $251.53°$ (f) $-291.06°$

5. (a) 1.14 (b) 0.559 (c) 1.48

6. $\frac{7\pi}{12}, \frac{\pi}{4}$

7. (a) $\frac{7\pi}{8}$ (b) $\frac{5\pi}{9}$ (c) $\frac{11\pi}{36}$

8. (a) 0.71π (b) $\frac{2\pi}{3}$ (c) $\frac{4\pi}{5}$

9. $360°, 135°$

10. (a) $-64.3°$ (b) $270°$
 (c) $-114.6°$ (d) $267.4°$

Exercise 7.9

1. (a) $1+5\theta$ (b) 4

 (c) $\frac{1}{1+2\theta}$ (d) $\frac{4\theta}{1-2\theta^2}$

 (e) $\frac{1}{4}\sqrt{2}\left(\theta^2+2\theta-2\right)$

 (f) $-\frac{1}{2}\theta$

2. 5

3. 0.294

6. $\frac{16\theta^2}{1+\theta}$

7. $\frac{2}{3}\left[\left(1-2\theta^2\right)\sin 3\alpha + 2\theta\cos 3\alpha\right]$

9. $\frac{18\theta^2}{1+2\theta+6\theta^2}$

10. (a) $1+2\theta-\theta^2$
 (b) $\tan\alpha + \theta\sec^2\alpha$

Exercise 7.10

1. Domain $= \left\{x \in \mathbb{R} : x \neq (2n+1)\dfrac{\pi}{2}\right\}$

   Range $= \{y \in \mathbb{R} : \text{except } -1 < y < 1\}$

2. Domain $= \left\{x \in \mathbb{R} : x \neq \dfrac{\pi}{4} + \dfrac{\pi n}{2}\right\}$ Range $= \{y \in \mathbb{R}\}$

3. Domain $= \{x \in \mathbb{R} : x \neq \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$ Range $= \{y : y \in \mathbb{R}\}$

4. Domain $= \{x \in \mathbb{R} : x \neq 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots\}$

   Range $= \{y \in \mathbb{R} : y \in (-\infty, -1] \cup [1, \infty)\}$

5. Domain $= \left\{x \in \mathbb{R} : x \neq \pm\dfrac{\pi}{4}, \pm\dfrac{3\pi}{4}, \pm\dfrac{5\pi}{4}, \ldots\right\}$

   Range $= \{y \in \mathbb{R} : \text{except } -1 < y < 1\}$

6. Domain $= \{x \in \mathbb{R} : -\pi \leq x \leq \pi\}$ Range $= \{y \in \mathbb{R} : -2 \leq y \leq 2\}$

7. Domain $= \left\{x \in \mathbb{R} : x \neq \dfrac{\pi}{4} + \dfrac{\pi n}{2},\ n \in \mathbb{Z}\right\}$ Range $= \{y : y \in \mathbb{R}\}$

8. Domain $= \left\{x \in \mathbb{R} : x \neq (2n+1)\dfrac{\pi}{2}\right\}$ Range $= \left\{y \in \mathbb{R} : \text{except } -\dfrac{1}{2} < y < \dfrac{1}{2}\right\}$

9. Domain $= \{x \in \mathbb{R} : 0 \leq x \leq \pi\}$ Range $= \{y \in \mathbb{R} : -6 \leq y \leq 6\}$

10. Domain $= \left\{-2\pi \leq x \leq 2\pi,\ x \neq \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}\right\}$ Range $= \{y : y \in \mathbb{R}\}$

Exercise 7.11

1. (a) $\dfrac{\sqrt{2}-\sqrt{6}}{4}$ (b) $\dfrac{\pi}{4}$ (c) $\sqrt{3}$ (d) 1

2. (a) $\dfrac{4}{3}$ (b) $-\dfrac{13}{85}$ (c) $\dfrac{87}{425}$ (d) $\dfrac{56}{65}$

4. (a) $\sqrt{1-x^2}$ (b) $\dfrac{x}{\sqrt{1+x^2}}$

5. (a) $x = 0.7862$ (b) $x = \pm 1,\ x = \dfrac{1}{2}$ (c) $x = 0.2808$ (d) $x = \sqrt{6-4\sqrt{2}}$

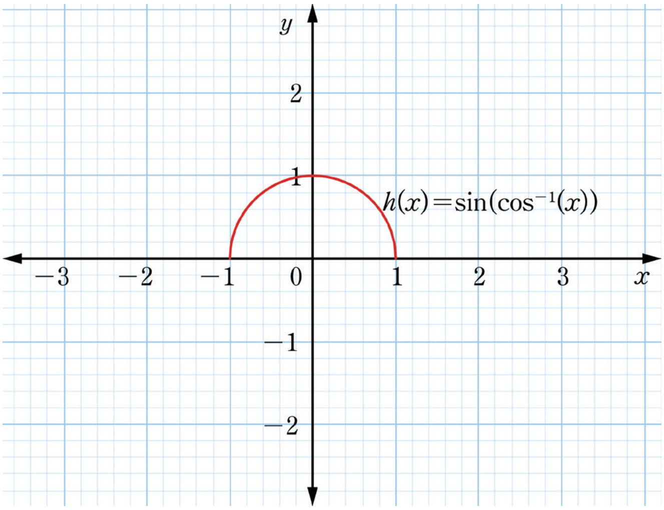
   (e) $\dfrac{1}{3}$ (f) $\dfrac{1}{6}$ (g) $\pm\sqrt{2}$ (h) 0

   (i) $\pm\sqrt{2}$ (j) $\pm\dfrac{3}{4}$

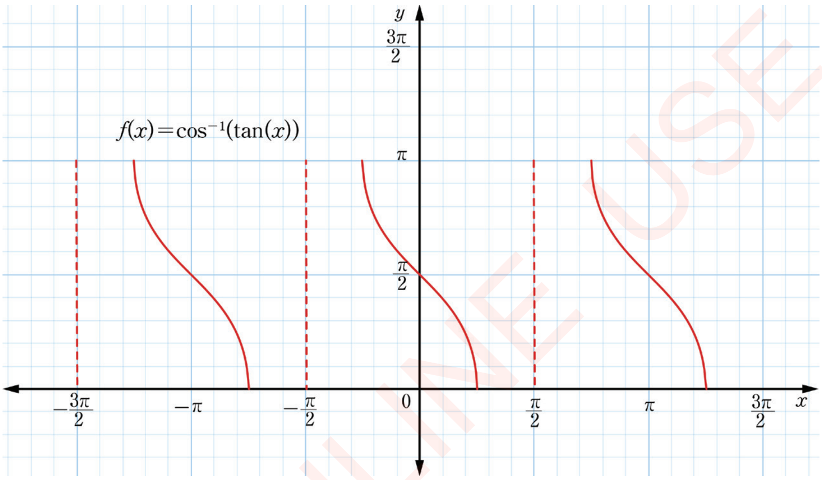
10. 0

Exercise 7.12

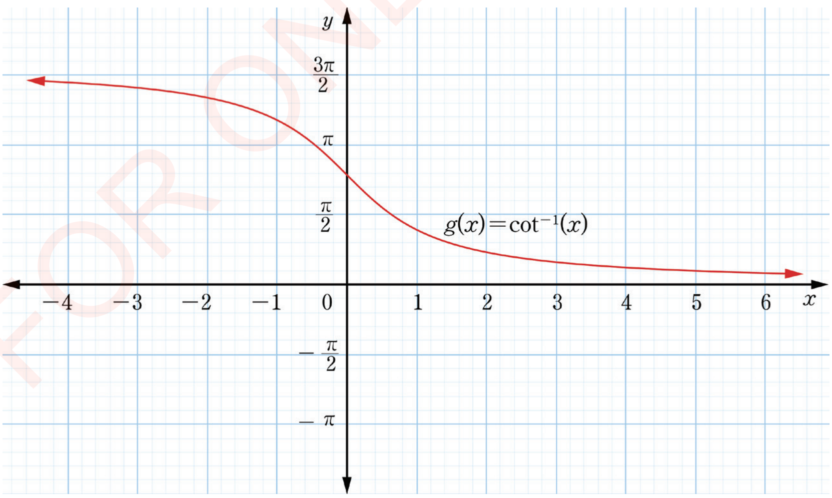
1. (a)



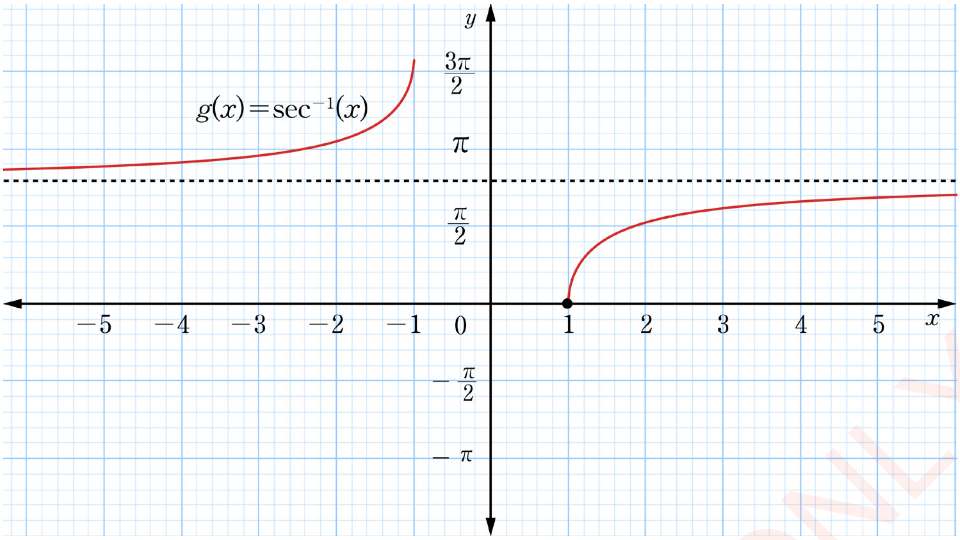
(b)



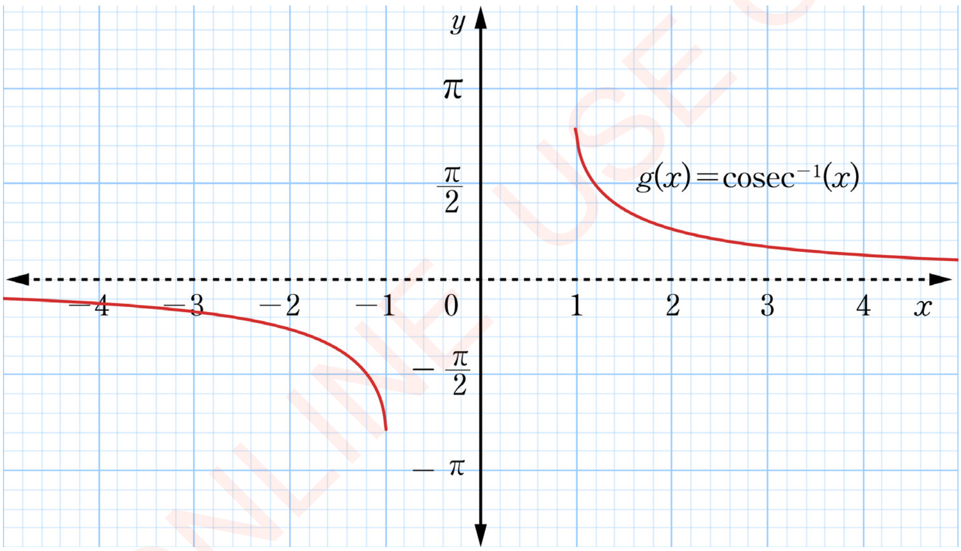
(c)



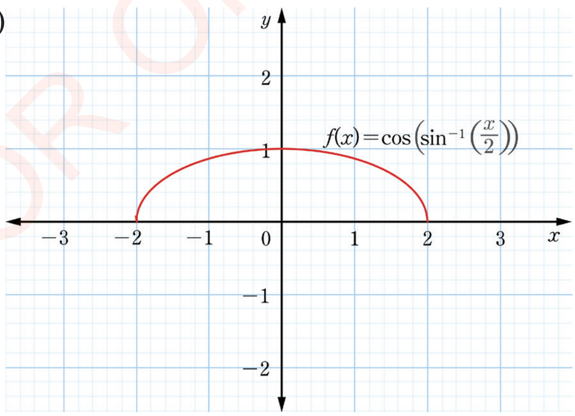
(d)


y
$\frac{3\pi}{2}$
$g(x)=\sec^{-1}(x)$
π
$\frac{\pi}{2}$
-5 -4 -3 -2 -1 0 1 2 3 4 5 x
$-\frac{\pi}{2}$
$-\pi$

(e)


y
π
$\frac{\pi}{2}$
$g(x)=\text{cosec}^{-1}(x)$
-4 -3 -2 -1 0 1 2 3 4 x
$-\frac{\pi}{2}$
$-\pi$

(f)


y
2
$f(x)=\cos\left(\sin^{-1}\left(\frac{x}{2}\right)\right)$
1
-3 -2 -1 0 1 2 3 x
-1
-2

2. (a) Domain $= \{x \in \mathbb{R} : -1 \le x \le 1\}$

Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} \le y \le \frac{\pi}{2}\right\}$

(b) Domain $= \{x : x \in \mathbb{R}\}$

Range $= \{y \in \mathbb{R} : 0 < y < \pi\}$

(c) Domain $= \{x : x \in \mathbb{R}\}$

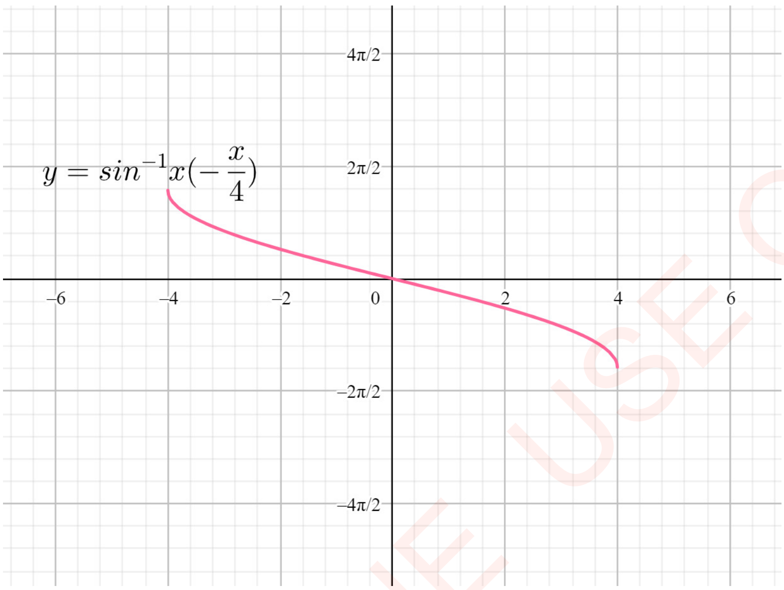
Range $= \left\{y \in \mathbb{R} : -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}$

(d) Domain $= \{x \in \mathbb{R} : -1 \le x \le 1\}$

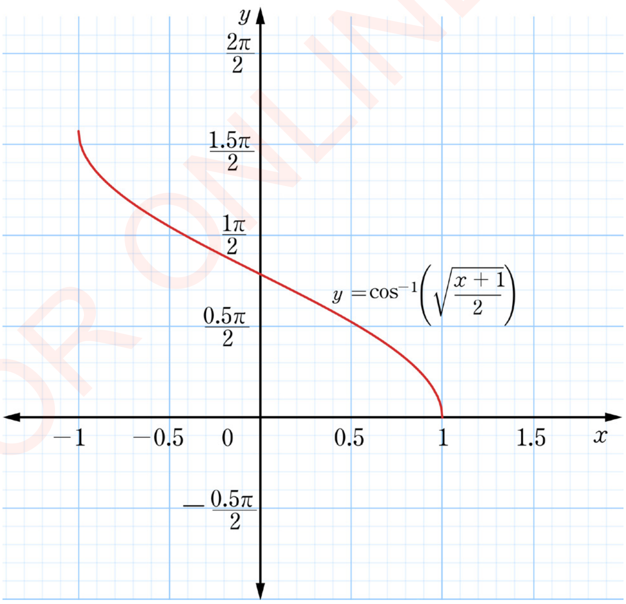
Range $= \{y \in \mathbb{R} : 0 \le y \le \pi\}$

3. (a) $0 < \cot^{-1} x < \pi$

4.



5.



Revision exercise 7

1. (a) $-\left(\dfrac{15\sqrt{135}+168}{315-8\sqrt{135}}\right)$ (b) $-\left(\dfrac{315-8\sqrt{135}}{408}\right)$ (c) $\dfrac{15\sqrt{135}-168}{408}$

2. (a) $\left(\dfrac{x-9}{4}\right)^2+\left(\dfrac{y-7}{15}\right)^2=1$ (b) $y=\dfrac{4x^2-46x+94}{x^2-10x+16}$

   (c) $x=2\left(1-y^2\right)$ (d) $yx^3+12x^2-256=0$

3. (a) $2\tan 2\theta$ (b) $-\dfrac{3\sin 3\theta}{4}$ (c) $-\tan x$ (d) $\dfrac{\cos 3y}{\cos 8y}$

5. (a) $460°$ (b) $316.55°$ (c) $429.7°$ (d) $-150°$

6. (a) $\dfrac{73}{72}\pi$ (b) $\dfrac{19}{30}\pi$

7. (a) -3π (b) $-\dfrac{35}{6}\pi$ (c) $\dfrac{9641}{14400}\pi$

8. (a) $9\theta+2$ (b) $\dfrac{4}{3}$ (c) $-\dfrac{9}{8}$ (d) $6\theta^2$

10. (a) $-144.1°, -58.5°, 35.9°, 121.5°$ (b) $139.26°, -107.38°$

11. (a) $2\pi n,\ 2\pi n+\dfrac{\pi}{2},\ \dfrac{\pi(1-2n)}{5}$ (b) $\theta=2\pi n$, or $2\pi n+\dfrac{1}{2}\pi$

    (c) $\theta=\dfrac{2}{5}\pi n+\dfrac{1}{10}\pi$, or $\theta=-\dfrac{2}{3}\pi n+\dfrac{1}{6}\pi$

    (d) $\theta=360°n+69.2°$ or $\theta=360°n-32.3°$ (e) $\dfrac{4\pi n\pm\pi}{a+b},\ \dfrac{4\pi n\pm\pi}{a-b}$

    (f) $\theta=360°n-19°47'$ or $\theta=360°n-47°35'$

    (g) $\theta=\dfrac{\pi n}{3}$ or $\theta=\pi n\pm\dfrac{\pi}{3}$ (h) $\theta=180°n\pm60°$ or $\theta=180°n\pm24.09°$

12. (a) $(x, y)=(15°, 75°); (75°, 15°); (195°, 255)$
    $(15°, 255); (195°, 75°)$.

    (b) $(x, y)=(45°, 165°)$ or $(x, y)=(165°, 45°)$

13. $\dfrac{52}{39}$

14. $27.2°, 152.8°, 207.2°, 332.8°$

15. $26.6°, 206.6°$

16. (a) 103.1°, 330° (b) 36.9°, 270° (c) 0°, 112.6°
(d) 28.1°, 208.1°, 159.5°, 339.5°
(e) 76.7°, 209.6° (f) 90°, 330°

17. (a) 13, 67.4° (b) 5, 53.13° (c) 10, 53.13° (d) $\sqrt{2}$, 45°

18. (a) $\sqrt{2}$, 45°; $-\sqrt{2}$, 225° (b) 5, 53.1°; −5, 233.1°
(c) 17, 298.1°; −17, 118.1° (d) $\sqrt{37}$, 170.5°; $-\sqrt{37}$, 350.5°

19. (a) Maximum = 5; Minimum = −5 (b)
$x = 53.13°$; and $-126.9°$

20. (a) $-\frac{1}{2}\sin\alpha$ (b) $\frac{2}{3}(1+\theta)$ (c) $\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$

22. 22.5°, 112.5°

24. $t = \frac{1}{2}$ or $t = \frac{1}{p}$

25. (a) $2\cos\frac{23}{2}\theta\sin\frac{9}{2}\theta$ (b) $2\sin 11\theta\cos 2\theta$ (c) $-2\sin\frac{11}{10}\theta\sin\frac{43}{10}\theta$
(d) $2\cos\left(\frac{7}{2}\theta\right)\cos\theta$

28. $x = \frac{1}{2}$, $x = \frac{1}{3}$

35. −360°, −90°, 0°, 360°

36. 0°, 10.32°, 60°, 70.32°, 120°, 130.32°, 180°,
190.32°, 240°, 250.32°, 300°, 310.32, 360°

37. $\tan 2x = \frac{2t}{1-t^2}$, $x = \pi n, \frac{2\pi}{3}+\pi n, \frac{\pi}{3}+\pi n$

39. (a) $\pm\sqrt{\frac{4+\sqrt{15}}{4}}$ (b) $\pm\sqrt{\frac{4-\sqrt{15}}{8}}$ (c) $\pm\sqrt{\frac{4+\sqrt{15}}{4-\sqrt{15}}}$

Chapter Eight

Exercise 8.1

1. Let: x be the number of units paint of quality P_1
   y be the number of units paint of quality P_2
   Max $z = 2,000x + 2,500y$
   Subject to: $x + 2y \leq 6$
   $3x + 2y \leq 10$
   $x \geq 0,\ y \geq 0$

2. Let: x be the number of tables
   y be the number of chairs
   Max $z = 3,250x + 3,165y$
   Subject to: $4x + 3y \leq 140$
   $2x + y \leq 100$
   $x \geq 0,\ y \geq 0$

3. Let: x be the number of circuit of type C_1
   y be the number of circuit of type C_2
   Max $z = 4,450x + 3,570y$
   Subject to: $2x + y \leq 20$
   $x + y \leq 12$
   $2x + 3y \leq 15$
   $x \geq 0,\ y \geq 0$

4. Let: x be the number of chairs of type A
   y be the number of chairs of type B
   Max $z = 2,330x + 1,890y$
   Subject to: $2x + 3y \leq 30$
   $x + y \leq 12$
   $x \geq 0,\ y \geq 0$

5. Let: x be the number of packages of type A
   y be the number of packages of type B
   Max $z = 5,000x + 7,000y$
   Subject to: $2x + y \leq 4000$
   $x + 2y \leq 4000$
   $5x + 6y \leq 13500$
   $x \geq 0,\ y \geq 0$

6. Let: x be the number of kg of food type I
   y be the number of kg of food type II
   Min $z = 2500x + 2850y$
   Subject to: $x + 2y \geq 10$
   $x + y \geq 6$
   $3x + y \geq 8$
   $x \geq 0,\ y \geq 0$

7. Let: x be the number of mangoes
   y be the number of oranges
   Min $w = 400x + 150y$
   Subject to: $\frac{1}{2}x + y \geq 5$
   $x + y \geq 10$
   $x + y \geq 5$
   $x \geq 0,\ y \geq 0$

8. Let: x be number of kilogram of ingredient type 1
   y be number of kilogram of ingredient type 2
   Min $z = 8,000x + 7,500y$
   Subject to: $2x + 3y \geq 6$
   $3x + 4y \geq 8$
   $x \geq 0,\ y \geq 0$

9. Let: x be the number of units of product type A

y be the number of units product type B

Max $z = 10,000x + 13,000y$

Subject to: $x + 2y \leq 45$

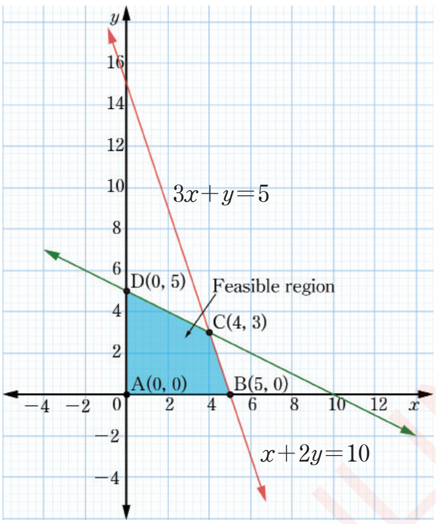
$2x + 5y \leq 80$

$4x + y \leq 60$

$x \geq 0,\ y \geq 0$

Exercise 8.2

1. (a)



Corner points

$z = 2x + 3y$

A (0, 0)

$z = 2(0) + 3(0) = 0$

B (5, 0)

$z = 2(5) + 3(0) = 10$

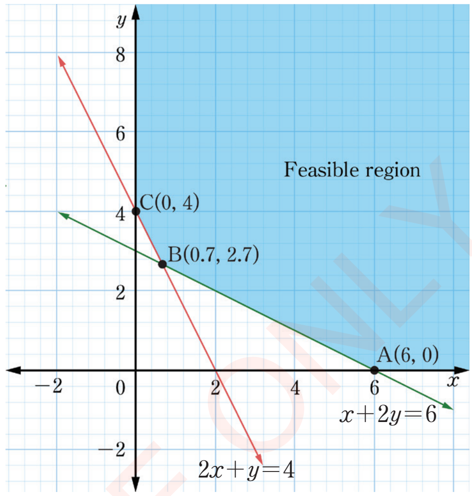
C (4, 3)

$z = 2(4) + 3(3) = 17$

D (0, 5)

(d) $z = 2(0) + 3(5) = 15$

Therefore, the maximum value is at (4, 3) which is 17.



Corner points

$z = x + 2y$

A (6, 0)

$z = (6) + 2(0) = 6$

$B\left(\frac{2}{3},\ \frac{8}{3}\right)$

$z = \frac{2}{3} + 2\left(\frac{8}{3}\right) = 6$

C (0, 4)

$z = 0 + 2(4) = 8$

Therefore, the minimum value is 6.

2. The maximum value is at (28, 0) which is 306,600.
   In order to maximize profit, 28 standard tents and 0 expedition tents should be manufactured.
3. The maximum value is at (6, 6) which is 167,700.
4. The maximum value is at $\left(\frac{64}{35},\ \frac{76}{7}\right)$ which is Tshs 860,571.43.
5. The minimum cost Tshs 352,500 which is at point (150, 150)

6. The maximum value is at $(2,\ 3)$ which is 8,720.
   The carpenter should make 2 tables and 3 chairs to get a profit of Tshs 8,720.
7. The maximum value is at $B(0,\ 80)$ which is 790,400. Should sell 0 mixture type I and 80 units of mixture type II.
8. The maximum profit is at (150, 400) which is 11,521,000.
9. The maximum contribution of each product is at (80, 88) which is 22,480,000.

Exercise 8.3

1. Min $f = 2500x + 2500y + 705,000$
   Subject to : $x \leq 90$
   $y \leq 70$
   $x + y \geq 60$
   $x + y \leq 120$
   $x \geq 0,\ y \geq 0$

2. Min $w = x - y + 500$
   Subject to : $x \leq 70$
   $y \leq 40$
   $x + y \leq 90$
   $x + y \geq 30$
   $x \geq 0,\ y \geq 0$

3. Min $w = -(2500x + 2000y) + 575,000$
   Subject to : $x \leq 30$
   $y \leq 25$
   $x + y \leq 40$
   $x + y \geq 20$
   $x \geq 0,\ y \geq 0$

4. Min $w = 25,000x + 35,000y + 2,100,000$
   Subject to : $x \leq 30$
   $y \leq 25$
   $x + y \geq 10$
   $x + y \leq 50$
   $x \geq 0,\ y \geq 0$

5. Min $f = 500x + 1,407,500$
   Subject to : $x \leq 40$
   $y \leq 55$
   $x + y \leq 80$
   $x + y \geq 30$
   $x \geq 0,\ y \geq 0$

6. Min $f = -(3x + y) + 82$
   Subject to : $x \leq 8$
   $y \leq 5$
   $x + y \geq 4$
   $x + y \leq 6$
   $x \geq 0,\ y \geq 0$

7. Min $w = 4x + 2y + 11,000$
   Subject to : $x \leq 2000$
   $y \leq 1500$
   $x + y \geq 2000$
   $x + y \leq 3000$
   $x \geq 0,\ y \geq 0$

8. Min $w = 1000x - 7000y + 190,000$
   Subject to : $x \leq 5$
   $y \leq 5$
   $x + y \geq 4$
   $x + y \leq 8$
   $x \geq 0,\ y \geq 0$

9. Min $w = 400x + 300y + 380,000$

Subject to : $x \leq 500$

$y \leq 600$

$x + y \geq 500$

$x + y \leq 900$

$x \geq 0,\ y \geq 0$

Exercise 8.4

1.

From / To	D_1	D_2
C_1	80	0
C_2	40	10

Minimum cost is 2,500

2.

From / To	Misungwi	Ngudu
Mabuki	20	0
Misasi	5	10

Minimum cost is 53, 000

3.

From / To	H_1	H_2
P	0	500
Q	500	100
R	400	0

Minimum cost is Tshs 5,300

4.

From / To	P	Q
A	0	5
B	5	0
C	3	1

Minimum cost is Tshs 1,550

5.

From / To	A	B
Malaga	0	15,000
Falulu	20,000	0
Tina	10,000	5,000

Minimum cost is Tshs 3,200,000

6.

From / To	S_1	S_2
P_1	5,000	30,000
P_2	20,000	0
P_3	25,000	0

Minimum cost is Tshs 3,200,000

7.

From / To	W_1	W_2
C_1	600,000	20,000
C_2	1,500,000	0
C_3	1,000,000	0

Minimum cost is Tshs 5,100,000

8.

From / To	F_1	F_2
S_1	0	20
S_2	5	10
S_3	20	0

Minimum cost is Tshs 9,500 when

$x = 5, y = 0$

9.

From / To	K_1	K_2
Lundo	35,000	0
Hamisi	45,000	15,000

Minimum cost is Tshs 1,015,000,000

Revision exercise 8

1. (a) Max $z = 4,000x + 6,000y$
   Subject to: $x + 2y \leq 30$
   $$3x + 2y \leq 80$$
   $$x \geq 0,\ y \geq 0$$
   (b) Product mix should be 30 products from Alpha and 0 products from Beta
   (c) 10 Beta products
2. The maximum value is at (0, 34) which is Tshs s 1,020,000.
3. The minimum value is at (100, 100) which is Tshs s 6,000,000.
4. The maximum value is at (5, 8) which is Tshs s 210,000. Hence,5 machines of type A and 8 machines of type B should be produced.
5. The maximum profit is at $(2.4, 7.2) \approx (3, 7)$ which is Tshs s 5,400,000.
6. The maximum value is at (2, 4) which is 8 cups. Thus, the students should take 2 cups of coffee and 4 cups of tea.
7. Greatest possible areas he can sow is 5 hectares.
8. The minimum value is at (18, 0). Hence, 8 tables and 0 chairs.
9. (a) 11ways
   (b) Product of 7 toys of each type give the greates balance of Tshs 8,500.
   (c) 18 toys
   (d) The maximum value is at (10, 8). Thus, 5 cheetah toys and 8 cat toys.
10. Maximum loss of calories is at $\left(1, \frac{1}{2}\right)$ which is 9. 5 calories. This implies that, 1 hour for aerobics and 30 minutes for flexibility exercises.
11. (a) 70 grams of A and 40 grams of B.
    (b) Tshs 5,200
12. The minimum cost is at (0, 5) which is Tshs s 500. This implies, 0 feed from A and 5 feed from B.
13. The minimum cost is Tshs 44,000. The transportation is as follows;
    500 litres from N to D
    3,000 litres from N to E
    3,500 litres from N to F
    4,000 litres from Q to D
    0 litres from Q to E
    0 litres from Q to F
14. The minimum cost Tshs 64,000. The crates should be supplied as follows;
    0 crates from G_1 to R_1
    200 crates from G_1 to R_2
    700 crates from G_1 to R_3
    600 crates from G_2 to R_1
    300 crates from G_2 to R_2
    0 crates from G_2 to R_3
15. The maximum number of cakes that can be made is 30.

16. 100kg of F_1 and 80kg of F_2.
17. (a) 10 units of M and 4 units of N
    (b) The maximum Profit is Tshs 7,600.

18.

To \ From	A	B
P_1	70	0
P_2	90	30

Minimum Transportation cost is Tshs 40,600

19.

	A	B	C
Deport 1	50	150	0
Deport 2	100	0	200

20. (a) The objective function is Min $z = (40x + 20y) + 45,200$
    (b) Inequalities associated to the transportation problems are

$x + y \leq 160$

$x + y \geq 50$

$x \leq 70$

$y \leq 120$

$x \geq 0,\ y \geq 0$

(c) The problem is not balanced since total supply from sources is not equal to the total demands.

Chapter Nine

Exercise 9.1

1. (a) $f'(x)=5$

(b) $f'(t)=3t^2-2t$

(c) $f'(x)=-3$

(d) $f'(x)=4x-3$

(e) $f'(v)=2$

(f) $f'(x)=(3-x)^{-2}$

(g) $f'(t)=5kt^4$

(h) $f'(x)=(2x)^{-\frac{1}{2}}$

2. (a) $f'(x)=5;\ f'(1)=5$

(b) $f'(x)=3x^2-2x;\ f'(2)=8$

(c) $f'(x)=1;\ f'(2)=1$

(d) $f'(x)=2x;\ f'(3)=6$

(e) $f'(x)=-2x^{-3};\ f'(4)=-\frac{1}{32}$

3. $f'(x)=5-4x;\ f'(3)=-7;\ f'(-1)=9$

4. (a) $y=1080;\ \frac{dy}{dx}=540$

(b) $y=2;\ \frac{dy}{dx}=9$

(c) $y=1;\ \frac{dy}{dx}=-1$

Exercise 9.2

1. (a) $f'(1)=-2$

(b) $f'(0)=0$

(c) $f'(-1)=20$

(d) $f'(-2)=-37$

2. (a) $f'(x)=\frac{1}{2}ax-3b$

(b) $f'(x)=\frac{9}{25}x^{\frac{4}{5}}-\frac{4}{25}x^{-\frac{6}{5}}$

(c) $f'(x)=\frac{3}{2}\sqrt{x}-\frac{2}{\sqrt{x}}$

3. (a) $f'(x)=4x-12$

(b) $g'(x)=\frac{3}{4}(2x-3)$

(c) $h'(x)=\frac{2}{3}(x-1)$

4. $a=2$ 5. $\left(-2, \frac{1}{2}\right)$

6. $x=\pm\frac{2\sqrt{3}}{3}$

7. $t=0,\ t=-1;\ x'(2)=36$

8. $h'(1)=9$

9. $4x-8x^{-3};\ 15\frac{7}{8}$

10. $\frac{dy}{dx}=200(x+2)^{-3};\ 200$

Exercise 9.3

1. (a) $f'(x)=4x^3+6x^2-2$

   (b) $f'(s)=24s^3+60s^2+64s-32$

   (c) $f'(y)=-96y^3+36y^2+96y-27-3y^{-4}-2y^{-2}$

   (d) $f'(t)=70t^4-32t^3+12t^2+30t-22$

2. $x=\dfrac{-6\pm\sqrt{46}}{5}$

3. $\dfrac{dy}{dx}=\dfrac{17\sqrt{3}}{6}$

4. (a) $f'(x)=\dfrac{5x^2-23}{2\sqrt{x-3}}$

   (b) $g'(x)=2x^5\left[6(x+2)^5+5x(x+2)^4\right]$

   (c) $h'(x)=\dfrac{-3-2x^2}{x^4\left(x^2+1\right)^{\frac{1}{2}}}$

   (d) $k'(x)=-9x^8+8x^7+6x^5-5x^4-6x^2$

5. $g'(x)=2(4x-1)(x+4)^{-3}$

7. (a) 1 (b) 7, −7

8. $f'(r)=2r\left(1+r^2\right)\left(1-3r^2\right)$

9. $\dfrac{dz}{dy}=6y^2-10y-4$

10. $f'(u)=2(u+1)(3u+1)(u-1)^3;\ f'(-5)=-24192$

Exercise 9.4

1. (a) $-\dfrac{16}{(3x-2)^2}$ (b) $-\dfrac{4(x+1)}{(x-1)^3}$ (c) $-\dfrac{2\left(x^2+x+2\right)}{\left(x^2-2\right)^2}$

2. (a) $(1+x)^{-\frac{1}{2}}(3+x)^{-\frac{3}{2}};\ f'(2)=0.052$

   (b) $\dfrac{3}{2(x+3)^2}\sqrt{\dfrac{x+3}{x}};\ f'(2)=0.095$

   (c) $\dfrac{1}{2}x\left(10x^2+33x-12\right)(x+3)^{-\frac{3}{2}}$; 8.408

3. (a) $-\frac{1}{t^4}$ (b) $1-\frac{3}{q^2}$ (c) $\frac{2}{3}$

4. $\frac{9}{4}$

5. $x=0$ and $x=-2$

9. $\frac{-4s^2+20s-22}{(s-3)^2(s-4)^2}, -\frac{208}{9}$

Exercise 9.5

1. (a) $f'(x)=2x^3\left(x^4-2\right)^{-\frac{1}{2}}$ (b) $f'(x)=-3x^2\left(x^3+1\right)^{-2}$

2. (a) $\frac{dy}{dx}=32x\left(x^2+1\right)^{15}$

   (b) $\frac{dy}{dz}=-6\left(3z^2+8z-3\right)\left(z^3+4z^2-3z-3\right)^{-7}$

   (c) $\frac{dy}{dx}=18x\left(3x^2-4\right)^2$

   (d) $\frac{dy}{dx}=x\left(x^2+5\right)^{-\frac{1}{2}}$

3. (a) $f'(t)=(t-3)\left(t^2-6t+7\right)^{-\frac{1}{2}}$

   (b) $f'(t)=-21(3-t)^{20}$

   (c) $g'(z)=\frac{3}{2}z^2\left(3-z^3\right)^{-\frac{3}{2}}$

   (d) $z'=\frac{2}{5}\left(x+\frac{1}{x}\right)^{-\frac{3}{5}}\left(1-\frac{1}{x^2}\right)$

   (e) $y'=2.9\left(t^3-\sqrt{t}\right)^{1.9}\left(3t^2-\frac{1}{2\sqrt{t}}\right)$

5. (a) $f'(t)=-\frac{8t^3+9t^2-5}{3\sqrt[3]{\left(2t^4+3t^3-5t+6\right)^4}}$ (b) $f'(-1)=0.06189$

6. $r'=\frac{r^4+1}{r^2\sqrt{r^4-1}}$

7. $\theta'=-10(4-2\theta)^4$

8. $x'=\frac{1}{3u^{\frac{3}{2}}\left(1-\frac{2}{\sqrt{u}}\right)^{\frac{2}{3}}}$

9. $\frac{15}{16}$

Exercise 9.6

1. (a) $3y^2\frac{dy}{dx}$ (b) $1+\frac{dy}{dx}$

3. $\frac{dy}{dx}=\frac{5}{2y(x+3)^2}$

4. $\frac{dy}{dx}=\frac{12(3x+6)^3}{15y^2}$ 7. $\frac{\sqrt{2}}{4}$

8. $\frac{2}{3}$

9. $y'=y^{-1}$ or $y'=(2x+1)^{-\frac{1}{2}}$

10. $z'=\pm\frac{4}{3}$

Exercise 9.7

1. (a) $\frac{dy}{dx}=-\frac{4x}{y(1+x^2)^2}$

   (b) $\frac{dy}{dx}=\frac{1-2y}{2x+2y-1}$

   (c) $\frac{dy}{dx}=\frac{3x^2+y^2}{y(3y-2x)}$

   (d) $\frac{dy}{dx}=\frac{6x^2y^4-8x^3y^5-4x^2y^3-y}{12x^4y^4-12x^3y^3+4x^3y^2+x}$

2. $\frac{dy}{dx}=\frac{1-2xy^3}{3x^2y^2}$

3. $\frac{dy}{dx}=\frac{13}{9}$

5. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2+2y-2xy}$

6. $\frac{dz}{dx}=\frac{6z-3-2x}{2z-6x-2}$

9. $\frac{dy}{dx}=\frac{y^2-2x-2xy}{x^2+2y-2xy}$

10. $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$

13. (a) $\frac{dy}{dx}=-\sqrt[3]{\frac{y}{x}}$

   (b) $\frac{dy}{dx}=\frac{4x\sqrt{xy}-y}{2\sqrt{xy}+x}$

   (c) $\frac{dy}{dx}=\frac{y-4x^3-4xy^2}{4x^2y+4y^3-x}$

Exercise 9.8

1. (a) $2\cos 2x$

   (b) $2x\sin x+x^2\cos x$

   (c) $-\frac{2}{x^2}\cos\left(\frac{2}{x}\right)$

   (d) $-12\sin 2x$

   (e) $2x\sin\left(1-x^2\right)$

   (f) $-\frac{1}{x^4}(x\sin x+3\cos x)$

   (g) $\cos x+\cos 2x$

   (h) $\frac{\sqrt{x+2}}{2(x+2)}\sec^2\sqrt{x+2}$

   (i) $4\sec^2 4x$

(j) $12x\cos\left(3x^2-1\right)\sin\left(3x^2-1\right)$

(k) $\dfrac{1}{1+\cos x}$

(l) $\dfrac{-\sin x+\cos x+1}{(1+\cos x)^2}\sin\left(\dfrac{1-\sin x}{1+\cos x}\right)$

2. (a) $2\pi-2$ (b) 4

5. $\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$

7. $\dfrac{\theta\cos\sqrt{\left(\theta^2-1\right)}}{\sqrt{\left(\theta^2-1\right)}}$

10. (a) $-\dfrac{y^2}{2xy-2\sin 2y}$

(b) $\dfrac{\sin\left(x+y^2\right)-3x^2}{4y^3-2y\sin\left(x+y^2\right)}$

(c) $\sec^2 x$

(d) $\dfrac{2\sec^2 x}{(1-\tan x)^2}$

Exercise 9.9

2. $-\dfrac{1}{1+x^2}$

4. $\dfrac{x^2}{\left(a^2-x^2\right)\sqrt{a^2-x^2}}$

5. $\dfrac{1-\left(1+x^2\right)y^3\cos x}{\left(1+x^2\right)\left(1+3y^2\sin x\right)}$

6. $-\dfrac{1}{1+x^2}$

9. $x\left(\dfrac{x}{\sqrt{1-x^2}}+2\sin^{-1}x\right)$

10. (a) $\dfrac{5}{\sqrt{1-25x^2}}$

(b) $-\dfrac{\sqrt{2}}{\sqrt{1-2x^2}}$

Exercise 9.10

1. (a) $4x(5x-16)(x-8)^2$

(b) $2\sec^2 2x+10\tan 5x\sec^2 5x$

(c) $-\dfrac{\left(x^2+6x+4\right)}{\left(x^2-4\right)^2}$

(d) $-\left(3x^2+2\right)\sin\left(x^3+2x+1\right)$

(e) $7\cos(7x+4)$

2. (a) $2\cos 2x\cos 4x-4\sin 2x\sin 4x$

(b) $\dfrac{2}{\sqrt{1-4x^2}}-\dfrac{2}{\sqrt{2-x^2}}$

(c) $3\sec(3x+6)\tan(3x+6)$

(d) $-\dfrac{(7x+29)}{15}(2x+1)^{-\frac{4}{5}}(x-8)^{-\frac{5}{3}}$

(e) $2\operatorname{cosec}2x\cot 2x\operatorname{cosec}^2(\operatorname{cosec}2x)$

Exercise 9.11

1. (a) $y'=\dfrac{5}{5x-4}$ (b) $y'=\ln x$

(c) $y'=2\sin(\ln x)$

2. $y' = \dfrac{2}{(x+3)(x+5)}$

3. $y' = \dfrac{2x}{x^2+3} + \dfrac{3x^2}{x^3+2}$

4. (a) $f'(x) = \dfrac{1}{\sqrt{1+x^2}}$

   (b) $f'(x) = \dfrac{4}{x} - \dfrac{6}{3x-5}$

   (c) $f'(x) = (\sin x)^x [\ln(\sin x) + x\cot x]$

   (d) $f'(x) = \dfrac{6x}{2+x^2} - \dfrac{12x^2}{1-x^3}$

5. (a) $y' = (\sin x)^{\tan x}\left(1+\sec^2 x \ln \sec x\right)$

   (b) $y' = \dfrac{2x}{x^2+3} - 7\operatorname{cosec} 2x$

   (c) $y' = \dfrac{2x^2-4}{x\left(x^2-4\right)}$

   (d) $y' = x^x(1+\ln x)$

6. $f'(x) = \dfrac{2\operatorname{cosec} 2x}{\ln \tan x}$

7. $x' = -\dfrac{2(\sin\theta - \cos\theta)}{(\sin\theta+\cos\theta)(1-\sin 2\theta)}$

11. $y' = \dfrac{3}{4\ln 2}$

12. $\dfrac{dy}{dx} = \dfrac{1}{2}$

Exercise 9.12

1. (a) $f'(x) = 2xe^{x^2}$

   (b) $g'(x) = e^x\left(1-12x-6x^2\right)$

   (c) $h'(x) = e^x\left(\dfrac{1}{x} + \ln x\right)$

   (d) $f'(x) = \dfrac{xe^{\sqrt{x^2-1}}}{\sqrt{x^2-1}}$

2. $y' = \left(\cos x - \sin^2 x\right)e^{\cos x}$

5. $f'(x) = 2xe^x + x^2e^x$

6. (a) $y' = 4\sin(8x)e^{\sin^2 4x}$

   (b) $y' = (2\ln 3)3^{2x}$

   (c) $y' = (1+x\ln 3)3^x$

   (d) $x' = \sec^2\theta e^{\tan\theta}$

   (e) $x' = 3t^2 + 6te^{-3t^2} + \sec t\tan t$

   (f) $v' = (2u+2)e^{u^2+2u-8}$

10. $y' = \dfrac{x^3\ln 2x}{e^x \sin x}\left[\dfrac{3}{x} + \dfrac{1}{x\ln 2x} - 1 - \cot x\right]$

Exercise 9.13

2. $\frac{dy}{dx}=\frac{x\cos x-2\sin x}{x^3}$;

$$\frac{d^2y}{dx^2}=\frac{\left(6-x^2\right)\sin x-4x\cos x}{x^4}$$

3. (a) $\frac{dy}{dx}=\frac{(2t+1)^2}{t^2}$

(b) $\frac{d^2y}{dx^2}=\frac{2(2t+1)^3}{3t^3}$

(c) $\frac{d^2y}{dx^2}=\frac{9}{4}$

4. (a) $\frac{dy}{dx}=-\tan t$

(b) $\frac{d^2y}{dx^2}=-\frac{1}{2}e^{-t}\sec^3 t$

(c) $-\frac{1}{2}$

6. (a) -1 (b) 48

7. $\frac{dy}{dx}=\frac{1}{5}$

9. $\frac{2\ln(x)-3}{x^3}$; $\frac{2\ln(2)-3}{8}$

10. 1

Exercise 9.14

1. $\Delta y=0.18$

2. (a) 3.01 (c) 2.84

(b) 127.26

3. $0.03x^3\text{cm}^3$

4. $80\pi\ \text{m}^3$; $16\pi\ \text{m}^2$

5. $40\pi\ \text{cm}^3$

6. 0.015

7. $1.8\pi\ \text{cm}^2$

8. 1.25%

10. $\frac{1}{2}y\%$

Exercise 9.15

1. $32\pi\ \text{cm}^2/\text{s}$

2. 157.5 m

3. $\frac{1}{8\pi}\text{cm/s}$

4. $384\pi\ \text{cm}^3/\text{min}$

5. $192\text{cm}^2/\text{s}$

6. 1.5 cm/s

7. $0.04\text{cm}^2/\text{s}$

8. 4.096cm, 0.3576cm / min

9. $3\text{cm}^2/\text{s}$

10. $9.3\pi\text{m}^2/\text{s}$

11. $v=15$ m/s, $a=12$ m/s^2

12. (a) $v(t)=3t^2-24t+36$, $a(t)=6t-24$

(c) $t=4$

(b) when $t=2$ and $t=6$

(d) $2<t<6$

Exercise 9.16

1. $(-1, 3)$ and $\left(\frac{1}{2}, -3.75\right)$ 2. $(0, 0)$ and $(4, -32)$ 3. $(5, -210)$ and $(-1, 6)$

4. (a) $(1, -9)$ (b) $\left(\frac{3}{2}, \frac{9}{4}\right)$ (c) $(1, 2)$ and $(-1, -2)$

5. (a) $\left(\frac{1}{3}, \frac{76}{27}\right)$ and $(-1, 4)$ (c) $(2, -8)$

(b) $(-2, 0)$ and $\left(\frac{2}{3}, \frac{256}{27}\right)$ (d) $(3, 4)$

6. $k = 2$; Turning point $(1, -3)$ and $\left(\frac{1}{3}, -\frac{77}{27}\right)$

7. $(0, -9)$ 10. $y = x^3 - 3x + 5$

Exercise 9.17

1. $48\,\text{m}^2$ 2. $x = \frac{3}{2}$; $v = \frac{243}{4}\,\text{m}^3$ 3. 20 4. Maximum point $\left(\frac{2}{3}, 5\sqrt{13}\right)$

5. Maximum point $\left(\frac{7}{3}, \frac{4}{27}\right)$, Minimum point $(3, 0)$

6. (a) Maximum point $\left(\frac{4}{3}, \frac{4}{27}\right)$ min point $(2, 0)$; x-intercept $= 1$ and 2, y-intercept -4

(b) Inflexion point $(0, 0)$, Minimum $(3, -27)$; x intercept $= 0$ and 4, y intercept $= 0$

(c) Maximum point $(2, 44)$, min point $(-3, -81)$; x-intercept $= 3.6$ and -5.1, y-intercept $= 0$

(d) Maximum point $\left(\frac{1}{3}, \frac{1}{81}\right)$, Min point $(0, 0)$; x-intercept $= 0$, and $\frac{5}{12}$, y-intercept $= 0$

(e) Maximum point $(0, -4)$; x-intercept $= 2, -2$; y-intercept $= -4$

(f) Maximum point $\left(-\frac{1}{6}, -\frac{49}{54}\right)$; Min point $\left(\frac{1}{2}, -\frac{3}{2}\right)$.

7. $\frac{2x^2 + 24}{x}$ cm, $4\sqrt{12}$ cm 8. $10.98\,\text{cm}^2$ 9. $614.2\,\text{cm}^2$ 10. $1250\,\text{m}^2$

Exercise 9.18

2. $\frac{4}{5}+\frac{3}{5}x-\frac{2}{5}x^2-\frac{1}{10}x^3+\cdots$

3. (a) $4x-8x^2+\frac{64}{3}x^3-64x^4+\cdots$

(b) $1+\frac{1}{2}\left(x-\frac{\pi}{2}\right)^2+\frac{5}{24}\left(x-\frac{\pi}{2}\right)^4+\cdots$

(c) $\sqrt{2}-\sqrt{2}\left(x+\frac{\pi}{4}\right)+\frac{3}{\sqrt{2}}\left(x+\frac{\pi}{4}\right)^2-\frac{11}{3\sqrt{2}}\left(x+\frac{\pi}{4}\right)^3+\frac{19}{4\sqrt{2}}\left(x+\frac{\pi}{4}\right)^4+\cdots$

(d) $\frac{1}{2}-\frac{\sqrt{3}}{2}x-\frac{1}{4}x^2+\frac{1}{4\sqrt{3}}x^3+\frac{1}{48}x^4+\cdots$

5. (a) $1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots$

(d) $1+2x-2x^2+4x^3-10x^4+\cdots$

(b) $x-\frac{1}{3!}x^3+\cdots$

(c) $1+(\ln a)x+\frac{(\ln a)^2}{2}x^2+\frac{(\ln a)^3}{6}x^3+\frac{(\ln a)^4}{24}x^4+\cdots$

6. $1-4\theta^2+\frac{16}{3}\theta^4-\frac{128}{45}\theta^6+\cdots$

7. $1+\frac{3}{2}x+\frac{9}{8}x^2+\frac{9}{16}x^3+\cdots$

9. $1-\frac{1}{2}h^2+\frac{1}{24}h^4+\cdots$

Exercise 9.19

1. $\frac{\partial p}{\partial V}=-k\frac{T}{V^2};\ \frac{\partial p}{\partial T}=k\frac{1}{V}$

2. $\frac{\partial z}{\partial x}=2xy^3,\ \frac{\partial z}{\partial y}=3x^2y^2$

3. $\frac{\partial z}{\partial x}=\frac{1}{y^2}+\frac{2y}{x^3}$ and $\frac{\partial z}{\partial y}=-\frac{2x}{y^3}-\frac{1}{x^2}$

4. (a) $y(\cos xy-xy\sin xy)$ (b) $\frac{2y}{(x+y)^2}$ (c) $6(3x+y)$

5. (a) $\frac{\partial f}{\partial x}=3yx^2+2y^2x,\ \frac{\partial f}{\partial y}=x^3+2x^2y$ (b) $\frac{\partial f}{\partial x}=2x+3y,\ \frac{\partial f}{\partial y}=3x$

(c) $\dfrac{\partial f}{\partial x}=4x^3y^3+16xy+5,\ \dfrac{\partial f}{\partial y}=3x^4y^2+8x^2+4y^3$

(d) $\dfrac{\partial f}{\partial x}=\dfrac{2x}{y}-\dfrac{y^2}{x^2},\ \dfrac{\partial f}{\partial y}=-\dfrac{x^2}{y^2}+\dfrac{2y}{x}$

6. (a) $\dfrac{\partial^2 z}{\partial x^2}=-x^2\sin(x-2y)+4x\cos(x-2y)+2\sin(x-2y)$

(b) $\dfrac{\partial^2 z}{\partial y^2}=-4x^2\sin(x-2y)$

(c) $\dfrac{\partial^2 z}{\partial x\partial y}=-4x\cos(x-2y)+2x^2\sin(x-2y)$

(d) $\dfrac{\partial^2 z}{\partial y\partial x}=-4x\cos(x-2y)+2x^2\sin(x-2y)$

7. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{x}{\left(y^2-x^2\right)^{\frac{3}{2}}},\ \dfrac{\partial^2 z}{\partial y^2}=\dfrac{2x^3y-4xy^3}{2\left(y^4-y^2x^2\right)^{\frac{3}{2}}}$

8. $\dfrac{\partial^2 z}{\partial x^2}=-0.71$

10. (a) $\dfrac{\partial t}{\partial l}=\dfrac{\pi}{\sqrt{gl}}$ (b) $\dfrac{\partial t}{\partial g}=-\dfrac{\pi}{g}\sqrt{\dfrac{l}{g}}$

9. $\dfrac{\partial^2 z}{\partial y^2}=9$

Revision exercise 9

1. (a) $\dfrac{\cos x}{x}-\dfrac{\sin x}{x^2}$ (b) $-2\sin 2x$ (c) $x\cos x+\sin x$ (d) $-5\sin(5x+2)$

3. $5\cos x+4\sin x$

4. $4\sin x\left(1-2\tan x\right)-2\sec^2 x\left(5-4\cos x\right)$

5. (a) $\dfrac{-2y^3-6xy}{6xy^2+3x^2}$ (c) $-\left(\dfrac{2x^2\ln x+x^2+y^2}{2xy\ln x}\right)$

(b) $\dfrac{y\left(x^2+y^2+2\right)+2x^3\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}{x\left(x^2+y^2+2\right)-2x^2y\sec^2\left[\ln\left(x^2+y^2+2\right)\right]}$

6. (a) $e^x + xe^x$ (b) $\dfrac{1-x}{e^x}$ (c) $e^x\left(x^2-3\right)+2xe^x$

(d) $\dfrac{1-3\ln x}{x^4}$ (e) $-5e^{1-x}$

7. (a) $2xe^{4x}+4x^2e^{4x}$ (b) $\dfrac{e^x-1}{2\sqrt{e^x-x}}$

(c) $\dfrac{2(x+2)^2}{x}+4(x+2)\ln 2x$ (d) $\dfrac{6x}{\left(1+x^2\right)^2}$

9. (a) $(2+x)^2 e^{-x}=4-x^2+\dfrac{1}{3}x^3+\cdots$

(b) $\log_e y=(y-1)-\dfrac{1}{2}(y-1)^2+\dfrac{1}{3}(y-1)^3-\dfrac{1}{4}(y-1)^4+\dfrac{1}{5}(y-1)^5+\cdots$

13. (a) $\sin(x+h)=\sin x+\dfrac{h\cos x}{1!}-\dfrac{h^2\sin x}{2!}-\dfrac{h^3\cos x}{3!}+\dfrac{h^4\sin x}{4!}+\cdots$

(b) $e^x=1+x+\dfrac{1}{2!}x^2+\dfrac{1}{3!}x^3+\dfrac{1}{4!}x^4+\cdots$

(c) $\sin\left(\dfrac{\pi}{6}+h\right)=\dfrac{1}{2}+\dfrac{\sqrt{3}}{2}h-\dfrac{1}{4}h^2-\dfrac{\sqrt{3}}{12}h^3+\dfrac{1}{48}h^4+\cdots$

(d) $\tan\left(\dfrac{\pi}{3}+h\right)=\sqrt{3}+4h+4\sqrt{3}h^2+\dfrac{40}{3}h^3+\dfrac{44}{3}h^4+\cdots$

(e) $\tan\left(\dfrac{\pi}{6}+x\right)=\dfrac{\sqrt{3}}{3}+\dfrac{4}{3}x+\dfrac{4}{3\sqrt{3}}x^2+\dfrac{8}{9}x^3+\dfrac{4}{3\sqrt{3}}x^4+\cdots$

14. $e^{\sin x}=1+x+\dfrac{1}{2}x^2-\dfrac{1}{8}x^4+\cdots$

15. $\dfrac{\cos x}{\sqrt{1-x}}=1+\dfrac{1}{2}x-\dfrac{1}{8}x^2+\dfrac{1}{16}x^3+\dfrac{49}{384}x^4+\cdots$

17. Turning point $=(0,-1)$

18. (a) Gradient $=-3$ (b) $(2,0)$ and $\left(-\dfrac{1}{2},\,0\right)$ (c) 5 and -5

19. $\frac{1}{14}\text{cm/s}$

20. 16 cm

21. $\frac{7}{100\pi}\text{cm/s}$

24. (a) $\frac{-\sin x}{2+\cos x}+\frac{\cos x}{3-\sin x}$ (b) $-\frac{1}{x^2+1}$

(c) $2\sec 2x$ (d) $\frac{-4-x^3}{2x^3\sqrt{x^3+1}}$

25. (b) $x+\frac{1}{3}x^3+\cdots$

26. (a) $1-\frac{9}{2}x^2+\frac{27}{8}x^4$ (b) $2x-\frac{4}{3}x^3+\frac{4}{15}x^5$

27. $1+2x+2x^2+\frac{4}{3}x^3$

28. (a) $\frac{\partial z}{\partial x}=2\cos(2x+3y)$; $\frac{\partial z}{\partial y}=3\cos(2x+3y)$

(b) $\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{(x^2+y^2)^3}}$; $\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{(x^2+y^2)^3}}$

(c) $\frac{\partial z}{\partial x}=3x^2+y^3$; $\frac{\partial z}{\partial y}=2y+3xy^2$

29. (a) $\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{25-x^2-y^2}}$; $\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{25-x^2-y^2}}$

(b) $\frac{\partial z}{\partial x}=(2x+y)e^{x^2+xy}$; $\frac{\partial z}{\partial y}=xe^{x^2+xy}$

(c) $\frac{\partial z}{\partial x}=3x^2+2y$; $\frac{\partial z}{\partial y}=2x+4y^3$

(d) $\frac{\partial z}{\partial x}=\frac{3x^2}{y^2}+\frac{y^3}{x^2}$; $\frac{\partial z}{\partial y}=-\frac{2x^3}{y^3}-\frac{3y^2}{x}$

(e) $\frac{\partial z}{\partial x}=4x-6y$; $\frac{\partial z}{\partial y}=-6x+2y$

(f) $\frac{\partial z}{\partial x}=y$; $\frac{\partial z}{\partial y}=x$

30. 1.85 cm

31. Length $l_1 = \dfrac{100\pi}{4+\pi}$ cm and $l_2 = \dfrac{400}{4+\pi}$ cm

32. $\dfrac{30}{\pi}$ m / min

33. -86 m/sec

34. Velocity $= (-4t+10)$ m/sec, acceleration $= -4$ m/s^2

35. (a) $\dfrac{\partial z}{\partial x} = 16$

(b) $\dfrac{\partial z}{\partial x} = 5$

36. (a) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{2(x-y)}{(x+y)^3}$

(b) $\dfrac{\partial^2 z}{\partial y \partial x} = -\dfrac{2(y-x)}{(x+y)^3}$

39. 0.16%

40. $\dfrac{d^2 y}{dx^2} = -\dfrac{2}{3}$

Answers

Chapter Ten

Exercise 10.1

1. (a) $x^6 + c$ (b) $2x^{\frac{1}{2}} + c$

(c) $3x^{-7} + c$ (d) $x^{\frac{1}{2}} + c$

(e) $x^{\frac{2}{3}} + c$ (f) $\frac{3}{2}x^{-2} + c$

2. (a) $\frac{1}{3}x^3 + c$ (b) $5e^x + c$

(c) $\frac{1}{4}\ln|x| + c$ (d) $2\sin\theta + c$

(e) $-\frac{1}{2}\cos\theta + c$

(f) $4\ln|\sec\theta| + c$

3. (a) $-\frac{3}{x} + c$ (b) $2x^{\frac{1}{2}} + c$

(c) $\frac{12}{5}t^{\frac{5}{4}} + c$ (d) $-2e^x + c$

(e) 10cosθ+c (f) $\frac{5}{4}x^{\frac{8}{5}} + c$

Exercise 10.2

1. $\frac{5}{4}x^4 + \frac{8}{3}x^3 - \frac{3}{2}x^2 + 5x + c$
2. $-\frac{3}{2}x^4 + 3x^3 + 2x^2 - 3x + c$
3. $\frac{2}{5}x^{\frac{5}{2}} + x^2 + 3x + c$
4. $\frac{48}{7}t^{\frac{7}{4}} - \frac{18}{7}t^{\frac{7}{2}} + c$
5. $\frac{1}{12}x^4 + \frac{5}{3}x^3 - \frac{7}{2}x^2 + 2x + c$
6. $-\frac{1}{2}t^{-2} - \frac{2}{5}t^5 + 7\ln|t| + t + c$
7. $2x^{\frac{1}{2}} - 2x^{\frac{3}{2}} + \frac{6}{5}x^{\frac{5}{2}} - \frac{2}{7}x^{\frac{7}{2}} + c$
8. $-\frac{3}{x} + \frac{\sin x}{2} + c$
9. $\frac{2}{3}x^{\frac{3}{2}} + \frac{2}{3}x^{\frac{1}{2}} + c$
10. $8\ln|x| + \frac{5}{x} - \frac{3}{x^2} + c$
11. $\ln|t| + \frac{5}{4}t^4 + \frac{2}{t^2} + c$
12. $\ln|x| - \frac{\cos x}{5} - \frac{\sin x}{3} + c$
13. $6x - x^2 + c$
14. $\frac{1}{3}x^3 + 2x^2 + x + c$
15. $\frac{1}{5}x^5 - x^2 - 2x^{\frac{3}{2}} + c$
16. $t^4 - t^3 - t^2 + t + c$
17. $7e^x + 4e^x + c$
18. $\frac{y^2}{2} - \frac{2}{y} + c$
19. 2sinθ – 2cosθ + c
20. $\frac{4}{3}\sin x + c$
21. $x^4 + \frac{7}{3}x^3 - \frac{3}{2}\sin x + c$
22. $\frac{1}{3}x^3 - \frac{1}{2}x^2 - \frac{3}{2}x^{-\frac{2}{3}} + \frac{1}{x} + c$

23. $\ln|t|+\frac{5}{4}t^4+\frac{2}{t^2}+c$

24. $-\frac{1}{r}+\frac{1}{3r^3}+\frac{3r^2}{2}+c$

Exercise 10.3

1. $\frac{1}{54}(6x-9)^9+c$
2. $-\frac{1}{5}(1-2x)^{\frac{5}{2}}+c$
3. $-\frac{1}{6}(3t-1)^{-2}+c$
4. $-3\ln|1-x|+c$
5. $12e^{\frac{x}{3}}+c$
6. $-\frac{1}{3}\ln|4-3t|+c$
7. $-2\sin(1-3x)+c$
8. $-\frac{1}{7}\ln|5-7x|+c$
9. $\frac{1}{2}\ln|\sec(2\theta+1)|+c$
10. $\frac{\ln|2x-1|}{2}+c$
11. $-\frac{e^{3-4x}}{14}+c$
12. $-10\cos\left(\frac{x}{2}-1\right)+c$
13. $\frac{e^{2x+2}}{2}+c$
14. $\frac{1}{a}\sin(ax\pm b)+c$
15. $\frac{1}{a}\ln|\sec(ax\pm b)|+c$

Exercise 10.4

1. $\frac{1}{2}(x^3+x^2-3)^2+c$
2. $-\frac{1}{2}(1+e^{-t})^2+c$
3. $\frac{4}{9}(3t^3-1)^2+c$
4. $\ln|1+e^x|+c$
5. $\ln|\ln|x||+\ln|x|+c$
6. $-\frac{1}{2}\sin^2\theta+c$
7. $-\ln|\cos x-\sin x|+c$
8. $\frac{x^3}{2}+\frac{6}{5}x^{\frac{5}{2}}+c$
9. $\ln|\cos\theta+\sin\theta|+c$
10. $\frac{1}{2}(1+\ln x)^2+c$
11. $\ln|3+4x+9x^2-6x^3|+c$
12. $\frac{1}{4}[\ln(x+2)]^2+c$
13. $\frac{2}{3}\ln\left|1+x^{\frac{3}{2}}\right|+c$
14. $x-2\sqrt{x}+2\ln|\sqrt{x}+1|-3+c$
15. $\frac{1}{3}(x^2-1)^{\frac{3}{2}}+c$

16. $\frac{1}{2}e^{x^2+4x}+c$

17. $\frac{1}{6}\sin(3e^{x^2})+c$

18. $t-2\ln\left|1+e^t\right|+c$

19. $-\frac{1}{x\ln x}+c$

20. $-\frac{1}{\ln x}+c$

21. $e^{\tan^{-1}x}+c$

22. $\ln\left|\tan(x+1)\right|+c$

Exercise 10.5

1. $\frac{1}{2}e^{2x}\left(x^2-x+\frac{1}{2}\right)+c$

2. $x\tan^{-1}(3x)-\frac{1}{6}\ln(1+9x^2)+c$

3. $x\cos^{-1}(0.5x)-\sqrt{4-x^2}+c$

4. $-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x+c$

5. $\frac{1}{2}x(\sin x)^2+\frac{1}{4}\cos x\sin x-\frac{1}{4}x+c$

6. $x^2\sin x+2x\cos x-2\sin x+c$

7. $\frac{1}{4}x^4\ln|x|-\frac{1}{16}x^4+c$

8. $\frac{1}{2}\left(x^2-2x\right)\ln|2x|+x-\frac{x^2}{4}+c$

9. $x\ln|3x|-x+c$

10. $\frac{1}{5}e^{2x}(2\cos x+\sin x)+c$

11. $-\frac{1}{5}e^{-x}\left(2\cos(2x)+\sin(2x)\right)+c$

12. $\frac{2}{5}e^{\frac{1}{2}x}\left(2\sin x+\cos x\right)+c$

13. $\frac{1}{10}e^{4x}\left(2\sin 2x-\cos 2x\right)+c$

14. $\frac{1}{13}e^{2x}\left(3\sin 3x+2\cos 3x\right)+c$

15. $\theta\tan\theta+\ln\left|\cos\theta\right|+c$

16. $-e^{-3x}\left(\frac{1}{3}x^2+\frac{2}{9}x+\frac{2}{27}\right)+c$

17. $x\sin^{-1}(2x)+\frac{1}{2}\sqrt{1-4x^2}+c$

18. $\frac{1}{3}x^3\ln|x|-\frac{1}{9}x^3+c$

19. $2\sqrt{x}\ln|x|-4\sqrt{x}+c$

20. $-\frac{1}{6x^6}\ln|x|-\frac{1}{36x^6}+c$

21. $\frac{1}{3}(1+x)^3\ln|3x|-\frac{1}{3}\ln|x|-\frac{1}{9}x^3-\frac{1}{2}x^2-x+c$

23. $I_n=x^ne^x-nI_{n-1}$,

$I_4=x^4e^x-4x^3e^x+12x^2e^x-24xe^x+24e^x+c$

27. $\frac{1}{216}\left(8x^3-1\right)\left(1+x^3\right)^8+c$

Exercise 10.6

1. $\ln\left|\frac{(x+5)^2}{x-2}\right|+c$

2. $\frac{1}{36}\ln|x-1|+\frac{1}{6(x+5)}-\frac{1}{36}\ln|x+5|+c$

3. $\ln\left|\frac{x-1}{\left(9+x^2\right)^{\frac{1}{2}}}\right|-\frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)+c$

4. $\ln\left|\frac{s^2}{(s-1)^2}\right|-\frac{1}{s-1}-\frac{1}{s}+c$

5. $\ln\left|\frac{(2x+1)^{\frac{3}{2}}}{x-2}\right|+\frac{2}{2-x}+c$

6. $\ln\left|x^2(x+2)^3\right|+\frac{1}{x}+c$

7. $\ln\left|\frac{x-1}{x+1}\right|^{\frac{1}{2}}+c$

8. $\ln\left|(y-3)^{\frac{1}{5}}y^2(y+2)^{\frac{9}{5}}\right|+c$

9. $\ln\left|\frac{(x+2)^3}{(x+1)^2}\right|+c$

10. $\ln\left|\frac{x(x-1)}{x+1}\right|+c$

11. $\ln\left|\left(\frac{t-1}{t+4}\right)^{\frac{1}{5}}\right|+c$

12. $-\frac{(x+1)}{(x-1)^2}+c$

13. $-\frac{5x-1}{20(x+3)^5}+c$

14. $x-\frac{1}{2}\tan^{-1}(x)+\ln\left|\left(\frac{x-1}{x+1}\right)^{\frac{1}{4}}\right|+c$

15. $5\ln|x-3|+\frac{2}{x-3}+c$

16. $\ln\left|\frac{t^{\frac{2}{5}}}{(t+5)^{\frac{7}{5}}}\right|+c$

17. $\ln\left|\left(\frac{x-4}{x+4}\right)^{\frac{15}{8}}\right|+x+c$

18. $3x+\frac{1}{3}x^3+\frac{4}{3}\ln|x-1|-\frac{13}{3}\ln|x+2|+c$

19. $\ln\left|r\left(\frac{r-1}{r+1}\right)^{\frac{1}{2}}\right|+c$

20. $\ln\left|\frac{(x+2)^4}{(x+5)^3}\right|+c$

21. $\frac{x^2}{2}+\frac{1}{4x}-\frac{1}{4x^2}-2x+\frac{1}{8}\ln x+\frac{31}{8}\ln(x+2)+c$

Exercise 10.7

1. $\frac{1}{2}\left(\sin x-\frac{1}{7}\sin 7x\right)+c$

2. $\frac{1}{16}(-4\cos 2x-\cos 8x)+c$

3. $\frac{1}{20}(5\sin 2x+\sin 10x)+c$

4. $\frac{1}{12}(3\cos 2x-\cos 6x)+c$

5. $\frac{1}{24}\cos 6\theta-\frac{1}{16}\cos 4\theta-\frac{1}{8}\cos 2\theta+c$

6. $\frac{1}{12}(3\sin 2\theta-\sin 6\theta)+c$

7. $\frac{1}{30}\cos 2x\left(5\cos 4x-3\cos^4 2x-10\right)+c$

8. $\frac{1}{8}(4x+\sin(2-4x))+c$

9. $\frac{1}{96}\sin(12x+4)+\frac{1}{12}\sin(6x+2)+\frac{3}{8}x+c$

10. $\frac{1}{96}(\sin 12x-8\sin 6x+36x)+c$

11. $\frac{1}{2}\left(\sin 2x-\frac{1}{3}\sin^3 2x\right)+c$

12. $\frac{1}{8}\left(6x+3\sin(2-2x)+4\cos(x-1)(\sin^3(1-x)\right)+c$

13. $\sin x-\frac{1}{3}\sin^3 x+c$

14. $\cos x\left(\frac{2}{3}\cos^2 x-\frac{1}{5}\cos^4 x-1\right)+c$

15. $\frac{1}{2}\sin\theta-\frac{1}{4}\sin 2\theta+c$

16. $\cos^3 x\left(\frac{1}{5}\cos^2 x-\frac{1}{3}\right)+c$

17. $\frac{1}{5}\sin^5 x-\frac{1}{7}\sin^7 x+c$

18. $\frac{1}{64}(8x-\sin 8x)+c$

19. $\frac{1}{4}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c$

20. $-\frac{1}{3}\cos^3 x+\frac{2}{5}\cos^5 x-\frac{1}{7}\cos^7 x+c$

21. $\frac{1}{12}\tan^4(3\theta)+\frac{1}{18}\tan^6(3\theta)+c$

Exercise 10.8

1. $\frac{1}{7}\tan^4 x-\frac{1}{2}\tan^2 x+\ln|\sec x|+c\tan^{-1}\left(\frac{x}{7}\right)+c$

2. $\frac{\sqrt{5}}{45}\tan^{-1}\left(\frac{\sqrt{5}x}{9}\right)+c$

3. $3\sqrt{3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c$

4. $\frac{1}{2}\tan^{-1}\left(\frac{z+1}{2}\right)+c$

5. $\frac{1}{3}\ln\left|\frac{\sqrt{9+4y^2}-3}{y}\right|+c$

6. $-\frac{\sqrt{4+z^2}}{4z}+c$

7. $\tan^{-1} x+c$

8. $\frac{1}{2}\ln\left(y^2+9\right)+\tan^{-1}\frac{y}{3}+c$

9. $\frac{1}{2}\tan^{-1}\left(\frac{4y}{7}\right)+c$

10. $\frac{\sqrt{2}}{2}\sin^{-1}\sqrt{2}x+c$

11. $6\sin^{-1}\left(\frac{2}{3}x\right)+c$

12. $\sin^{-1}\left(\frac{\sqrt{5}}{5}x\right)+c$

13. $x\sqrt{1-4x^2}+\frac{1}{2}\sin^{-1}(2x)+c$

14. $5\left(\sin^{-1}\left(\frac{10}{\sqrt{10}}(x+2)\right)\right)+$
$\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{\sqrt{10}}{10}(x+2)\right)\right)+c$

15. $-\frac{\sqrt{9-x^2}}{9x}+c$

16. $\frac{1}{2}\left(\sin^{-1}(x+1)+\frac{1}{2}\sin\left(2\sin^{-1}(x+1)\right)\right)+c$

17. $\frac{9}{2}\left[\sin^{-1}\frac{1}{3}(x-2)+\frac{1}{9}(x-2)\sqrt{5+4x-x^2}\right]+c$

18. $\frac{1}{3}(t^2+9)^{\frac{3}{2}}-9\sqrt{t^2+9}+c$

19. $2\sin^{-1}(\frac{1}{2}x)-\sin(2\sin^{-1}(\frac{1}{2}x))+c$

20. $\frac{1}{8}(\sin^{-1}x-\frac{1}{4}\sin(4\sin^{-1}x))+c$

21. $\frac{1}{2}\tan^{-1}(\frac{x+2}{2})+c$

22. $\frac{2}{5}\sqrt{5}\tan^{-1}\left(\frac{\sqrt{5}}{5}(x+5)\right)+c$

23. $\sin^{-1}(\frac{1}{2}x+1)+c$

24. $-2\sqrt{9-x^2}-7\sin^{-1}(\frac{1}{3}x)+c$

25. $\frac{2}{3}\tan^{-1}(\frac{1}{3}(x-2))+c$

26. $\frac{2s^2+9}{8(4s^2+9)^{\frac{1}{2}}}+c$

27. $7\ln\left|\sqrt{x^2+1}+x\right|+\frac{\sqrt{2}}{16}(\sin^{-1}(\sqrt{2}x))+$
$\frac{1}{2}\sin(2\sin^{-1}(\sqrt{2}x)))+c$

Exercise 10.9

1. $\tan\frac{1}{2}x+c$

2. $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{1}{2}\theta\right)\right)+c$

3. $\frac{-1}{1+\tan x}+c$

4. $\frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\sqrt{2}}{2}\tan x\right)+c$

5. $\ln\left|\tan\left(\frac{x}{2}\right)\right|+c$

6. $\tan^{-1}(\sin x)+c$

7. $-\ln\left|1+\cos^2 x\right|+c$

8. $\frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan x\right)+c$

9. $\frac{1}{6}\ln\left|\frac{3\tan x+1}{3\tan x-1}\right|+c$

10. $\frac{1}{2}\tan\theta+c$

11. $\ln\left|\sec^2\left(\frac{x}{2}\right)\right|+c$

12. $\ln\left|\tan(\frac{x}{2})\right|+c$

13. $-\frac{1}{\sqrt{11}}\tan^{-1}\left(\frac{\cot x}{\sqrt{11}}\right)+c$

14. $\frac{1}{4}\ln\left|\tan(4x)+\sec(4x)\right|+c$

15. $\frac{2}{3}\tan^{-1}(\frac{1}{3}\tan(\frac{x}{2}))+c$

16. $\sin^{-1}(\frac{1}{2}\sin x)+c$

17. $-\ln|\tan x+\sec x|+2\sin x+c$

18. $\frac{1}{2}\ln\left|\frac{\tan\frac{x}{4}+2}{\tan\frac{x}{4}-2}\right|+c$

19. $\frac{1}{3}\left(\ln\left|\frac{\tan\frac{x}{2}}{3}+1\right|-\ln\left|\frac{\tan\frac{x}{2}}{3}-1\right|\right)+c$

20. $2\ln|\tan(\frac{x}{4})|+c$

21. $\tan(\frac{y}{2})+c$

22. $\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)+c$

23. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}\right)-1\right)+c$

24. $\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{1}{\sqrt{3}}\left(2\tan\frac{x}{2}+1\right)\right]+c$

25. $\tan^{-1}\left(\frac{1}{2}\tan\left(\frac{\theta}{4}\right)\right)+c$

26. $\frac{1}{3}\ln\left|\frac{\cos x}{\cos x-3}\right|+c$

27. $\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)+1}{\tan\left(\frac{\theta}{2}\right)+3}\right|+c$

28. $\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\left(\frac{x}{3}\right)\right)+c$

29. $\ln\left|\frac{2\sin x-4}{2\sin x-2}\right|+c$

30. $\tan^{-1}(\tan x+1)+c$

31. $-\theta+\frac{10\sqrt{3}}{3}\tan^{-1}(\sqrt{3}\tan(\frac{\theta}{2}))+c$

32. $\ln|\tan(\frac{x}{2})+2|+c$

33. $\sin x\sqrt{1+\sin x}-\frac{1}{3}(1+\sin x)^{\frac{3}{2}}+\sqrt{1+\sin x}+c$

Exercise 10.10

1. $\frac{1}{2}\ln|x^2+5|+\frac{7}{\sqrt{5}}\tan^{-1}(\frac{x}{\sqrt{5}})+c$

2. $\frac{1}{4}\ln|4x^2+12x+10|+\frac{1}{2}\tan^{-1}(2x+3)+c$

3. $\frac{1}{2}\theta-\frac{1}{2}\ln|\cos\theta+\sin\theta|+c$

4. $-\frac{7}{10}\ln|\sin\theta+3\cos\theta|-\frac{1}{10}\theta+c$

5. $\ln|x^2+2x+10|+\frac{1}{3}\tan^{-1}(\frac{1}{3}(x+1))+c$

6. $\ln|y+3|-\frac{1}{y+3}+c$

7. $\frac{3}{2}\ln|u^2+2u+5|+\frac{5}{2}\tan^{-1}(\frac{1}{2}(u+1))+c$

8. $\frac{11}{17}\ln|\cos\theta+4\sin\theta|-\frac{10}{17}\theta+c$

9. $\frac{1}{2}\theta+\frac{1}{2}\ln|\cos\theta+\sin\theta|+c$

10. $\frac{1}{2}x+\frac{5}{2}\ln|\cos x+\sin x|+c$

11. $\frac{3}{13}\ln|2\cos\theta+5\sin\theta|+\frac{2}{13}\theta+c$

12. $\frac{3}{106}\ln|7\cos 2x-2\sin 2x|-\frac{37}{53}x+c$

Exercise 10.11

1. $\frac{7^x}{\ln 7}+c$

2. $\frac{(\ln x)^2}{2\ln 10}+c$

3. $\frac{8^x}{\ln 2}+c$

4. $-\frac{x^2}{\ln 10}+c$

5. $\frac{3^{2x}}{2\ln 3}+c$

6. $-\frac{6^{1-2y}}{2\ln 6}+c$

7. $\frac{-1}{4^x\ln 4}+c$

8. $\frac{-9^{3-7x}}{7\ln 9}+c$

9. $\frac{10^{x-1}}{\ln 10}+c$

10. $\frac{x}{\ln 10}(\ln|3x|-1)+c$

11. $\frac{1}{2\ln 2}(\ln x)^2+c$

12. $\frac{2^x}{\ln 2}+\frac{x}{\ln 10}(\ln|5x|-1)+c$

13. $\frac{(2x+1)\ln|2x+1|}{2\ln 10}-\frac{x}{\ln 10}+c$

14. $\frac{(x^2+x-1)\left[\ln|x^2+x-1|-1\right]}{\ln 10}+c$

15. $\frac{x}{6\ln 10}(\ln|2x|-1)+c$

16. $\frac{x}{\ln 10}(\ln|\tfrac{3}{4}x|-1)+c$

17. $-\frac{(\frac{1}{3})^x}{\ln 3}+c$

18. $\frac{\tan x}{\ln 10}(\ln|\tan x|-1)+c$

19. $\frac{x}{\ln 2}(\ln|x|-1)+c$

20. $\frac{(\ln|3x|)^2}{4\ln 5}+c$

Exercise 10.12

1. 16
2. $\frac{52}{3}$
3. 32
4. 180
5. $\frac{15}{2}$
6. 2
7. $\ln 2$
8. $\frac{2\sqrt{2}}{3}$
9. $-\frac{12}{\pi}$
10. $\frac{55}{4}$
11. $\ln 2+\frac{69}{4}$
12. $\frac{113}{24}$
13. 7174089
14. $\frac{9\sqrt{3}-1}{5}$
15. $\frac{15}{128}$
16. $3\ln 2$
17. $3(e-1)$
18. $\frac{1}{3}(2\ln 2-\ln 3)$
19. 2
20. $\frac{-3}{7}\ln 2$
21. 0
22. $\frac{1}{2}\ln 7$
23. $\frac{e^{12}-1}{4e}$
24. $10\left(\cos(1)-\cos\left(\frac{\pi-4}{4}\right)\right)$
25. 2
26. $\frac{1}{2}(1-e^{-2})$
27. $\frac{16\sqrt{2}-4\sqrt[4]{2}}{9}$
28. $\frac{2}{15}$
29. $\ln(\frac{23}{5})$
30. $\frac{2}{3}$
31. 1
32. 6
33. 1
34. $\frac{3}{2}$
35. $\frac{40}{3}$
36. $\frac{(\ln 2)^2}{4}$
37. $\frac{3\pi}{8}+1$
38. $3+4\ln 2$
39. 9
40. $\frac{1}{6}$
41. $\frac{e^5-1}{2}$
42. $2\ln 2-2\ln 3$
43. $\frac{3}{10}$
44. $12\ln 3-8$
45. $\frac{1}{36}-\frac{7}{36e^6}$
46. $\frac{26}{3}\ln 2-\frac{59}{18}$
47. $-\frac{\ln 2-\ln 3}{2}$
48. $3\ln 3-5\ln 2$

49. $\dfrac{3\sqrt{3}}{8}$

50. $\dfrac{203}{480}$

51. 0

52. $\dfrac{2}{35}$

53. $\dfrac{2}{15}$

54. π

55. $\dfrac{5}{2}$

56. $\dfrac{\pi}{4}$

57. $\dfrac{1}{2}(\tan^{-1}(\tfrac{3}{2})-\tfrac{\pi}{4})$

58. $\dfrac{4}{3}\tan^{-1}(\tfrac{1}{3})$

59. $\dfrac{\pi}{4}$

60. $6-4\sqrt{2}-7\sin^{-1}(\tfrac{1}{3})$

Exercise 10.13

1. $\dfrac{1}{3}$ square units

2. $\dfrac{1}{2}ab\pi$ square units

3. $\dfrac{\pi}{2}$ square units

4. 70.116 square units

5. 18 square units

6. $\dfrac{80}{\pi}$ square units

7. π square units

8. $\left(\dfrac{1}{2}-\dfrac{4\pi-3\sqrt{3}}{24}\right)$ square units

9. $\dfrac{71}{6}$ square units

11. 16.04 units

12. $\dfrac{3}{64}$ units

13. 12.0397 units

14. $\dfrac{\pi}{2}$ units

15. 21.2563 units

16. 2π units

17. $78\tfrac{2}{3}$ square units

18. $\dfrac{3\pi a}{2}$ square units

Exercise 10.14

1. $\dfrac{96}{5}\pi$ cubic units

2. $\dfrac{93}{5}\pi$ cubic units

3. $\dfrac{25}{8}\pi^2$ cubic units

4. 16π cubic units

5. $\dfrac{64}{3}\pi$ cubic units

6. $\dfrac{80}{\pi}$ square units

7. (a) $\dfrac{4\pi ab^2}{3}$ cubic units (b) $\dfrac{4\pi a^2 b}{3}$ cubic units

8. $2\pi\left(3\sqrt{3}-\dfrac{4}{3}\pi\right)$

9. 2500π cubic units

10. $\dfrac{16}{15}\pi$ cubic units

11. (a) $\frac{52\pi}{5}$ cubic units (b) 18π cubic units

(c) $\frac{76\pi}{15}$ cubic units (d) $\frac{10\pi}{3}$ cubic units

12. $\frac{\pi}{2}\left(e^2-1\right)$ cubic units 13. $\frac{3\pi}{10}$ cubic units 14. 9π cubic units

15. $\left(2-\frac{\pi}{4}\right)$ square units 16. $\frac{\pi}{24}$ square units 17. $\frac{\pi}{8}$ square units

18. $\frac{4\pi-7\sqrt{3}}{16}$ square units

Revision exercise 10

1. (a) $\frac{2}{9}(3x-1)^{\frac{3}{2}}+c$ (b) $\frac{x^5}{5}+x^3+c$ (c) $\frac{1}{2}\ln\left|e^{2x}+1\right|+c$

(d) $\frac{1}{5}x^5+3x^4+16x^3+32x^2+c$ (e) $\frac{2}{5}(x-1)^{\frac{5}{2}}+\frac{2}{3}(x-1)^{\frac{3}{2}}+c$

2. $\frac{\pi}{12}$ 3. $\frac{\ln 3}{4}$

4. (a) $-\frac{116}{15}$ (b) $\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\tan\left(\frac{\theta}{2}\right)\right)+c$

(c) $\frac{1}{5}(\theta+1)(2\theta-3)^{\frac{3}{2}}+c$ (d) $x^2e^x-2xe^x+2e^x+c$

5. 0.1794

7. (a) $\frac{1}{4}\ln 3$ (b) $-\frac{\sqrt{-x^2+4}}{4x}+c$

(c) $\frac{1}{2}\left(\sin x-\frac{1}{7}\sin 7x\right)+c$ (d) $-\frac{e^x\cos x}{2}+\frac{e^x\sin x}{2}+c$

(e) $\frac{2}{\sqrt{23}}\tan^{-1}\left(\frac{4}{\sqrt{23}}\tan\left(\frac{\theta}{2}\right)+\frac{1}{\sqrt{23}}\right)+c$ (f) $\frac{1}{2}\tan^{-1}\left(\frac{\theta-3}{2}\right)+c$

(g) 2.3752

9. (a) $\sin(x^2-5)+c$ (b) $-\cos(x^3-2x^2+1)+c$ (c) $-2\sqrt{2}\cos\left(\frac{x}{2}\right)+c$

(d) $\frac{1}{3}\left(\frac{\sin^9(3z)}{9}-\frac{2\sin^{11}(3z)}{11}+\frac{\sin^{13}(3z)}{13}\right)+c$

(e) $\frac{1}{2}\left(\cos^3 2t\sin 2t+\frac{3}{8}\left(2t-\frac{1}{4}\sin 8t\right)\right)+c$

(f) $\frac{1}{3}\cos(\cos 3x)+c$

(g) $\frac{1}{3}\left(\frac{\tan^3 3y}{3}+\frac{2\tan^5 3y}{5}+\frac{\tan^7 3y}{7}\right)+c$

(h) $\frac{1}{2}\left(-\frac{1}{11}\cos 11t-\frac{1}{5}\cos 5t\right)+c$ (i) $\ln|\sec x|-\sec^2 x+\frac{1}{4}\sec^4 x+c$

(j) $\frac{3}{2}\left(-\frac{1}{5}\cos^5\left(\frac{2x}{3}\right)+\frac{1}{7}\cos^7\left(\frac{2x}{3}\right)\right)+c$ (k) $\frac{-32-5\sqrt{2}}{252\sqrt{2}}$

(l) $e^{4+\sin x}+c$ (m) 0 (n) $\frac{1}{6}(\sin x-\cos x)^6+c$

10. $I_6=-\frac{1}{6}\cos x\sin^5 x+\frac{5}{6}\left[-\frac{1}{4}\sin^3 x\cos x+\frac{3}{8}\left(x-\frac{1}{2}\sin 2x\right)\right]+c$ and

$I_7=\frac{1}{7}\cos^7 x-\frac{3}{5}\cos^5 x+\cos^3 x-\cos x+c$

11. (a) $\frac{1}{2}\left(\tan^{-1}x+\frac{x}{1+x^2}\right)+c$ (b) $\frac{2}{3}(x+1)^{\frac{3}{2}}+\frac{2}{3}x^{\frac{3}{2}}+c$ (c) $\frac{2}{3}(1+\ln x)^{\frac{3}{2}}+c$

(d) $2\left(\frac{1}{3}(x-1)^{\frac{3}{2}}+\sqrt{x-1}\right)+c$ (e) $\frac{1}{2}\ln|x^2+4x+8|-\frac{1}{2}\tan^{-1}\left(\frac{1}{2}(x+2)\right)+c$

12. $\frac{32}{5}\pi$ cubic units 13. $\frac{19}{3}\pi$ cubic units

15. (a) $\frac{1}{2\ln 2}-\frac{1}{e}$ (b) $-\frac{1}{2}\left(\frac{1}{2}\ln 3-1\right)$

(c) $\frac{7}{6}+\ln 2-\ln 3$ (d) $\frac{1}{2}$

16. $\frac{a^2}{4k}\left[e^{2\pi k}-e^{-2\pi k}\right]$ square units 17. $\frac{32}{3}$ square units

20. (a) $-5\sec^{-1}\frac{1}{5}x+\sqrt{x^2-25}+c$ (b) $\frac{4\sqrt{3}\pi-\pi+2\ln 2-4}{6}$ (c) $\frac{-1}{2\ln 10}$

21. (a) $\frac{1}{4}\left(\sin^{-1}\left(\sin^2 x\right)\right)+\frac{1}{2}\sin\left(2\sin^{-1}\left(\sin^2 x\right)\right)+c$ (b) $2e^{\sqrt{x-2}}+c$

(c) $-\frac{2}{3^{\sqrt{x}}\ln 3}+c$ (d) $\frac{1}{2}\ln\left|x^2+2x+5\right|+c$

(e) $\frac{1}{2}\ln\left|x^2+1\right|+\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$

(f) $x-\frac{10}{11}\ln|x-2|+\frac{5}{11}\ln\left|\frac{3}{2}+\frac{1}{2}x^2+x\right|-\frac{3}{11\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}(x+1)+c$

(g) $\ln|x-1|+\frac{1}{x-1}-\frac{1}{2}\ln\left|x^2+1\right|+\tan^{-1}x+c$

(h) $\tan^{-1}x-\frac{1}{2(x^2+1)}+c$ (i) $\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x+3}{4}\right)+c$

(j) $1+\ln|x|-\ln\left|1+\ln|x|\right|+c$ (k) $-\frac{2}{3(x^3+1)}+c$

(l) $\frac{1}{3}(x+1)^{\frac{3}{2}}+\frac{1}{3}(x-1)^{\frac{3}{2}}+c$

22. $\ln(1+\sqrt{2})$ units

23. (a) 1 square unit (b) $\frac{\pi(e^2+1)}{2}$ cubic units

24. $\left(-37e^{-5}+2\right)$ units

26. The total sales for 4 months is 3702.80.

27. $\frac{3\pi}{8}-1$ cubic units.

28. $\frac{17}{6}$ units

30. (a) P(4, 16), Q(-6.25, -25) (b) $\left(68-45\ln 3\right)$ square units.

Glossary

Absolute value inequalities	an expression with absolute function as well as inequality signs.
Adjoint of a matrix	a transpose of a matrix containing co-factors of elements of the matrix.
Angle bisector	a line that splits an angle into two equal angles.
Anti-derivative	opposite of the derivative of a function.
Argument	a series of connected propositions that form a definite statement.
Asymptote	a line that the distance between the curve and the line approaches zero as one or both of the x or y coordinates tends to infinity.
Biconditional statement	a statement formed by a combination of a conditional statement.
Binomial theorem	an algebraic method of expanding the binomial expression to any power without lengthy multiplication.
Cartesian coordinate system	a system used to determine a point uniquely in xy-plane through two numbers.
Cofactor of an element	the matrix obtained when the column and row of a chosen element in a matrix are removed.
Column matrix	a matrix whose elements are arranged in an order of a single column.
Common logarithms	logarithms to base 10, usually written without showing the base.
Complement of a set	a set of elements in the universal that are not in the set under consideration.
Composite function	a function formed by substituting one function into another function.
Compound angle	an algebraic sum of two or more angles which are added or subtracted through trigonometric functions.
Compound statement	a declarative sentence formed by more than one declarative sentence.
Conditional statement	a statement formed by two statements in which the second statement is a logical consequence of the first statement.
Conjunction statement	a compound statement formed by joining two statements with the connector “and”.

Constraints	inequalities or equations which connect the decision variables under certain restrictions or limitations.
Contrapositive of a conditional statement	a statement formed by interchanging the premise and the conclusion of the inverse statement.
Converse of a conditional statement	a statement formed by exchanging the premises and conclusion of the original conditional statement.
Definite integral	the integral function which has limit of integration.
Derivative of a function	the rate of change of a function with respect to a given variable.
Determinant	a scalar value that is a function of the entries of a square matrix.
Disjunction statement	a compound statement formed by joining two statements with the connector "or".
Double angle formulae	are formulas in trigonometry that deals with the double angles of trigonometric functions.
Equivalent logical expressions	expressions which have the same truth values in the truth table.
Feasible region	the region of the graph which contains all the points which satisfy all the constraints of the system.
Feasible solution	a solution that satisfies all constraints.
Finite set	a set with a fixed number of elements.
Hole	a removable discontinuity that exist on the graph of rational function at any input value that causes both the numerator and denominator of the function to be equal to zero
Horizontal asymptote	a horizontal line parallel to the axis of the independent variable.
Identity matrix	a square matrix in which all elements of the principal diagonal are 1's and all other elements are zeros.
Indefinite integral	is the integral function which has no limit of integration.
Indices	is a number which shows how many times it has been multiplied by itself.
Infinite set	a set containing uncountable number of elements.
Integration by parts	a process that finds the integral of the products of functions in terms of the integral of the product of their derivative and anti-derivative.
Intersection of sets	a set which contains all the elements that are common to all involved sets.

Inverse of a conditional statement	a conditional statement formed by negating both the premise and conclusion.
Inverse of a matrix	a matrix that, when multiplied by the given matrix gives the multiplicative identity.
Limits	limits describe how a function behaves near a point, instead of at that point.
Locus	a set of points whose location is determined by one or more specified conditions.
Logarithms	the exponent or power to which a base must be raised to yield a given number.
Mathematical induction	a mathematical technique which is used to prove that if the statement is true for the n^{th} iteration, then it is also true for $(n+1)^{th}$ iteration.
Matrix	a set of numbers arranged in a rectangular array, having m rows and n columns and enclosed by the square or ordinary brackets.
Minor of determinant	the determinant of a sub-matrix formed by deleting the rows and columns in a given matrix.
Natural logarithms	logarithms to the base e (where $e =$ 2.718281828459).
Normal to a circle	a straight line drawn at 90° to the tangent line at the point where the tangent touches the circle.
Objective function	a linear function whose value is to be either minimized or maximized subject to the constraints defined over the set of feasible solutions.
Oblique asymptote	occurs when the degree of the numerator is greater than that of the denominator.
Optimal points	a point where the objective function attains its maximum or minimum value.
Optimal value	the value from the optimal solution that maximizes or minimizes the objective function of the linear programming problems.
Orthogonal circles	circles intersecting in such a way that the tangents at the points of intersection are perpendicular.
Proposition	a declarative sentence which can be either true or false but not both.
Radian	the measure of an angle subtended at the centre of a circle by an arc whose length is equal to the radius of that circle.
Remainder theorem	a theorem used to calculate the remainder of the division of any polynomial by another polynomial without applying the long division method.

Row matrix	a matrix whose elements are arranged in a single row.
Scalar matrix	a diagonal matrix with equal-valued elements along the diagonal.
Sequence	a set of numbers, or algebraic expressions, which can be obtained from a proceeding one by a definite rule.
Series	an expression showing the sum of terms in a sequence by linking the terms of the sequence with sign of addition.
Simple statement	a declarative sentence that is either true or false.
Spreadsheet	an interactive computer application program for organization, analysis, and storage of data.
Square matrix	a matrix having the same number of rows and columns.
Tangent to a circle	a straight line which touches the circle at only one point.
Trigonometric identities	equations that are true for every value of variables occurring on both sides of an equation.
Trigonometric ratios	are values of all trigonometric functions based on the value of the ratio of sides in a right-angled triangle.
Trigonometric substitution	the substitution of trigonometric functions for other expressions.
Truth table	a table that shows the validity of a compound statement depending on the truth values of its simple statements.
Union sets	a set containing all the elements available in the individual sets under consideration.
Universal set	a set containing all elements or members of all related sets without any repetition of elements.
Valid argument	an argument that has all true premises and a true conclusion.
Venn diagram	a diagram that uses circles or ovals to show relationship between sets and their elements.

Bibliography

A. Page. (1951). *Trigonometry*. University of London Press.

Arora, S.R. (2001). *Bussiness Mathematics with Applications*. Rajendra Ravindra Printers (PVt) Ltd.

Backhouse, J.K. (2011). *Pure Mathematics (International Edition – East Africa)*. Pearson Longman.

Backhouse, J.K & Houldsworth, S.P.T. (1984). *Pure Mathematics Book 1*. Pearson Education Ltd.

Backhouse, J.K, Houldsworth S.P.T. & B.E.D. Cooper. (1985). *Pure Mathematics Book 2*. Longman Group Ltd.

Bostock, L.S. & Chandler, S. (1981). *Mathematics*; *The Core Course for A – Level.* Stanley Throne (Publisher) Ltd.

Bostock & Chandler, S. (1984). *Core Mathematics for A – Level*. Stanley Throne (Publisher) Ltd.

Brian & Mark Gaulter. (2001). *Further Pure Mathematics*. Oxford University Press.

Celia, C.W, Nice A.T.F & Elliot (1980). *Advanced Mathematics 1*. Macmillan Education Ltd.

Chand's, S, Dass, H. K & Verma, R. (2015). *Mathematics for class XI (6th Revised Edition)*. Vikas Publishing House Pvt. Ltd & Published by S. Chand and Sons Company Pvt. Ltd.

Dass, H.K. (2002). *Engineering Mathematics.* S. Chand and Sons Company Ltd.

John, B. (2010). *Higher Engineering Mathematics*. Elsevier's Publisher.

Mannall. G & Kenwood. M. (2000). *Pure Mathematics 1*. Heinemann Educational Publishers (Oxford) Ltd.

Sadler, A.J & Thorning, D.W.S. (1987). *Understanding Pure Mathematics*. Oxford University Press.

Shayo L.K.S & Masenge, R.W.P. *Advance Level Mathematics (Volume 1)*. Dar es Salaam University Press.

Sancheti, D.C & Kapoor. V.K (1993). *Bussines Mathematics (11th Edition)*. Sultan Chand & Sons Educational Publisher New Delhi.

Stroud, K. A (2001). *Engineering Mathematics (5th Edition)* Antony Rowe Ltd.

Tranter, C.J. & Lamb, C. G. (1961). *Advanced Level Mathematics (Pure and Applied) 4th Edition*. Hodder and Stoughton Educational Ltd.

Tranter, C.J & O.B.E. D.Sc. (1953). *Advanced Level Pure Mathematics*. The English Universities Press Ltd.

Webber, R.P. (1945). *Collage Algebra and trigonometry*. Brooks/Cole Publishing Company; Monterey, California.

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