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# Bäcklund transformation and multiple soliton solutions for a cylindrical KdV equation to model electron-acoustic waves in the Saturnian magnetosphere

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## ABSTRACT

In this work, we investigate nonlinear electron-acoustic waves (EAWs) in the Saturn's magnetosphere, modeled as a plasma system with cold inertial electrons, inertia-less kappa-distributed hot electrons, and stationary ions. Using the reductive perturbation technique, the cylindrical Korteweg–de Vries (CKdV) equation for small-amplitude EAWs is derived. The Bäcklund transformation is employed to analyze the CKdV equation. This approach yields novel analytical multi-soliton solutions in terms of Airy functions, with a recursive scheme for N-soliton solutions. Parametric analysis using Cassini data shows that only rarefactive solitary waves are supported for the system of interest. The impact of related plasma parameters on the profile of the cylindrical electron-acoustic soliton is numerically examined. These results elucidate nonlinear electrostatic structure formation in the planetary magnetospheres and provide a framework for interpreting spaceborne data.



## I. INTRODUCTION

The propagation of nonlinear electrostatic waves in multi-component plasmas remains a subject of profound interest, central to understanding energy dissipation and particle acceleration mechanisms in both laboratory and space environments.[1,2] Among these, electron-acoustic waves (EAWs) constitute a fundamental high-frequency mode that requires a plasma with at least two distinct electron populations: a cold, inertial component and a hot, Boltzmann-like component providing the restoring force, alongside stationary ions forming a neutralizing background.[3,4] Since their theoretical prediction by Fried and Gould[1] and subsequent experimental verification,[5] EAWs have been instrumental in explaining a myriad of phenomena, notably the broadband electrostatic noise (BEN) observed in various regions of the terrestrial magnetosphere, such as the plasma sheet boundary layer, the cusp, and the auroral zones.[6–8] Satellite missions, such as FAST, GEOTAIL, and, more recently, Cassini, have provided compelling evidence of EAWs and their solitary structures, confirming their prevalence in space plasmas.[9–11]

A critical feature of space plasmas, as revealed by these observations, is the frequent deviation of particle velocity distributions from the conventional Maxwellian equilibrium. The presence of superthermal particles, characterized by enhanced high-energy tails, is more accurately described by a kappa distribution.[12,13] This distribution has been successfully employed to model data from Saturn's magnetosphere, where a mixture of cold and hot kappa-distributed electrons is a common feature.[14,15] The incorporation of kappa statistics significantly alters the linear and nonlinear dynamics of plasma waves, including EAWs, influencing their dispersion, stability, and the characteristics of resulting solitary structures.[16,17]

The nonlinear evolution of small-amplitude EAWs is often governed by the Korteweg–de Vries (KdV) equation, derived via the reductive perturbation technique (RPT).[18] While extensive studies have investigated planar EA solitary waves (EASWs) and their interactions,[19,20] the realistic geometry of many plasma systems, such as experimental devices (e.g., shock tubes) or natural phenomena in planetary magnetospheres, often exhibits cylindrical or spherical symmetry. This nonplanar geometry introduces a crucial geometric factor into the dynamics, rendering the standard KdV equation inadequate and requiring the use of its cylindrical or spherical counterparts.[21,22] The cylindrical KdV (CKdV) equation admits solutions that are structurally distinct from their planar analogs, often expressed in terms of Airy functions rather than hyperbolic functions using the Bäcklund transformation, adding a layer of complexity to their analysis.[23,24]

The investigation of multi-soliton solutions and their interactions is paramount for understanding complex wave phenomena. While powerful methods such as Hirota's direct method have been successfully used to study overtaking collisions of EASWs in planar geometry,[25] the analytical treatment of nonplanar multi-solitons remains a significant challenge. In this context, the Bäcklund transformation (BT) emerges as a powerful analytical tool. The BT provides a systematic procedure for generating a hierarchy of exact solutions, including multi-soliton solutions, from a known seed solution.[26,27] Its application to the cylindrical KdV equation has been demonstrated for ion-acoustic waves in a plasma system,[23] but its potential remains largely unexplored for the study of electron acoustic waves in nonplanar geometry.

In this paper, we bridge this gap by investigating the nonlinear propagation of electron acoustic waves in a cylindrical geometry with application to Saturn's magnetosphere. Our plasma model consists of cold inertial electrons, hot kappa-distributed electrons, and stationary ions. Using the RPT, we derive a CKdV equation. We then employ the Bäcklund transformation to analytically derive novel single-, two-, and three-soliton solutions expressed in terms of Airy functions. Furthermore, we establish a general recursive scheme for constructing N-soliton solutions. A comprehensive parametric analysis is conducted to elucidate the effects of the hot-to-cold electron density ratio and the spectral index $\kappa$ on the properties of these cylindrical electron acoustic solitons (CEASs), using realistic parameters from Cassini satellite observations.

This paper is structured as follows: Sec. II details the fluid model and the derivation of the CKdV equation. Section III presents the Bäcklund transformation and the derivation of the multi-soliton solutions. The results of our parametric analysis and a discussion of their implications for Saturn's magnetosphere are presented in Sec. IV. Finally, Sec. V provides a summary of our principal conclusions.

## II. PLASMA FLUID MODEL

We consider a homogeneous, unmagnetized, collisionless plasma embedded within the specific conditions of Saturn's magnetosphere. The plasma comprises two distinct electron populations (cold and hot) along with stationary ions that provide a neutralizing background. Contrary to the typical Cartesian geometry used for uniform plasmas, the particular environment of a planetary magnetosphere often requires a cylindrical coordinate system to accurately model wave propagation along magnetic field lines or within specific plasma structures. Consequently, we model the wave propagation along the radial direction in cylindrical coordinates, i.e., $\nabla = (\partial/\partial r, 0, 0)$, assuming azimuthal and axial symmetry for the studied waves.

While the inclusion of a finite temperature for the cold electron population is possible, we adopt the common simplification of neglecting it ($T_c \approx 0$) for this theoretical treatment. This assumption is well-justified under the condition that $T_c n_{h0} \ll T_h n_{c0}$, which is robustly satisfied in the regions of interest within Saturn's magnetosphere where the hot electron component is significantly hotter and denser.[14] The fundamental model equations, appropriately normalized, thus describe the propagation of EAWs, as given in the following.

The normalized continuity equation for the cold electrons in cylindrical geometry is

$$\frac{\partial n_c}{\partial t} + \frac{1}{r}\frac{\partial (r n_c u_c)}{\partial r} = 0, \tag{1}$$

where $n_c$ and $u_c$ are the cold electron number density and radial velocity, respectively. Furthermore, the normalized equation of motion and Poisson's equations are given as

$$\frac{\partial u_c}{\partial t} + u_c\frac{\partial u_c}{\partial r} = \frac{\partial \phi}{\partial r}, \tag{2}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) = \frac{1}{\alpha}n_c + n_h - \left(1 + \frac{1}{\alpha}\right). \tag{3}$$

Here, the cold electron number density, $n_c$, and its radial fluid velocity, $u_c$, are normalized by their equilibrium value, $n_{c0}$, and the effective electron acoustic speed, $C_e = \sqrt{k_B T_h/(\alpha m_e)}$, respectively. Here, $\alpha = n_{h0}/n_{c0}$ is the density ratio of hot to cold electrons, a critical parameter that defines the plasma environment. The electrostatic wave potential, $\phi$, is scaled by the thermal energy of the hot electron population, $k_B T_h/e$. The temporal and spatial scales are normalized by the cold electron plasma period, $\omega_{pc}^{-1} = (4\pi n_{c0} e^2/m_e)^{-1/2}$, and the hot electron Debye length, $\lambda_{Dh} = \sqrt{k_B T_h/(4\pi n_{h0} e^2)}$, respectively. This choice of scaling is physically intuitive as it naturally takes into account the inertial scale of the cold electrons and the screening scale provided by the hot electrons. The charge neutrality at equilibrium requires $n_{c0} + n_{h0} = n_{i0}$, where $n_{i0}$ denotes the equilibrium density of the stationary ion background.

The hot electrons, which are not trapped in the wave potential, are treated as obeying a kappa distribution function, accurately capturing the prevalent superthermal characteristics observed in Saturn's outer magnetosphere and around its moons.[14] The three-dimensional isotropic kappa distribution is given by[27]

$$f_\kappa(v) = \frac{n_{h0}}{(\pi\kappa\theta^2)^{3/2}}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-1/2)}\left(1 + \frac{v^2}{\kappa\theta^2} - \frac{2e\phi}{m_e\theta^2}\right)^{-(\kappa+1)}, \tag{4}$$

where the spectral index $\kappa > 3/2$ quantifies the degree of superthermality—a lower $\kappa$ signifies an energy spectrum with a more pronounced non-Maxwellian tail. Empirical data from Cassini typically indicate $2 \lesssim \kappa \lesssim 6$ for Saturn's magnetospheric plasma.[14] The quantity $\theta$ represents the effective thermal speed, related to the conventional thermal speed, $\upsilon_{th} = \sqrt{2k_B T_h/m_e}$, through the relation,

$$\theta^2 = \left(\frac{\kappa - 3/2}{\kappa}\right)\upsilon_{th}^2. \tag{5}$$

This relation ensures the second moment of the kappa distribution correctly defines the temperature $T_h$ for any $\kappa > 3/2$. In the limit $\kappa \to \infty$, the distribution[4] converges smoothly to the Maxwell–Boltzmann distribution and $\theta \to \upsilon_{th}$.[29]

By taking the velocity moment of Eq. (4), we obtain the number density for the hot electrons, which exhibits a power-law dependence on the electrostatic potential,

$$n_h = n_{h0}\left(1 - \frac{e\phi}{(\kappa - 3/2)k_B T_h}\right)^{-\kappa+1/2}. \tag{6}$$

In terms of the normalized units, this simplifies to the form

$$n_h = \left(1 - \frac{\phi}{\kappa - 3/2}\right)^{-\kappa+1/2}. \tag{7}$$

The functional form of the hot-electron density is pivotal in introducing the nonlinear characteristics associated with the suprathermal population, thereby introducing a strong nonlinearity into Poisson's equation that precludes a straightforward analytical solution. To render the system tractable, we consider the regime $|\phi| \ll 1$, which allows a series expansion of the hot-electron density. Expanding Eq. (7) in a Taylor series about $\phi = 0$ yields the expression,

$$n_h = 1 + a_1\phi + a_2\phi^2 + \mathcal{O}(\phi^3), \tag{8}$$

where the expansion coefficients $a_n$ are functions of the spectral index $\kappa$. The first two coefficients, which govern the linear and leading-order nonlinear response, are found to be

$$a_1 = \frac{\kappa - \frac{1}{2}}{\kappa - \frac{3}{2}}, \qquad a_2 = \frac{a_1}{2}\left(\frac{\kappa + \frac{1}{2}}{\kappa - \frac{3}{2}}\right). \tag{9}$$

It is straightforward to verify that for the physically permissible range of the spectral index (i.e., $\kappa > 3/2$), these coefficients are strictly positive and real-valued. This ensures the well-posedness of the subsequent perturbation analysis.

### A. Nonlinear CKdV equation for EAWs

To investigate the evolution of small-amplitude electron-acoustic solitary waves (EASWs), we employ the reductive perturbation technique (RPT).[18] This method systematically balances weak nonlinearity against dispersion by introducing appropriately stretched space and time coordinates. For the geometry under consideration, the coordinates accounting for the cylindrical case are defined as

$$\xi = \epsilon^{1/2}(r - \lambda t), \tag{10}$$

$$\tau = \epsilon^{3/2} t. \tag{11}$$

Within the perturbative framework, the quantity $\lambda$ represents the normalized linear phase velocity of the electron-acoustic wave, a fundamental plasma property determined self-consistently. The parameter $0 < \epsilon \ll 1$ is a small, dimensionless ordering parameter that formally characterizes the relative amplitude of the nonlinear perturbation.

Consistent with the reductive perturbation technique, the dependent field variables, namely, the cold electron density $n_c$, the fluid velocity $u_c$, and the electrostatic potential $\phi$, are also expanded in asymptotic power series in terms of this small parameter $\epsilon$,

$$n_c = 1 + \epsilon n_{c1} + \epsilon^2 n_{c2} + \cdots, \tag{12}$$

$$u_c = \epsilon u_{c1} + \epsilon^2 u_{c2} + \cdots, \tag{13}$$

$$\phi = \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots. \tag{14}$$

Here, the superscript denotes the order of the perturbation, and the equilibrium values (1 for the densities and 0 for the velocity and potential) have been explicitly separated, ensuring the perturbations describe deviations from the homogeneous, stationary background state.

We now proceed with the formal perturbative analysis by substituting the asymptotic expansions from Eqs. (12)–(14) into the governing system of equations, namely, the continuity equation,[1] the momentum Eq. (2), and Poisson's Eq. (3), supplemented by the series representation of the hot electron density given by Eq. (8). Isolating terms of order $\mathcal{O}(\epsilon)$ yields the following set of linear equations governing the first-order perturbations:

$$-\lambda\frac{\partial n_{c1}}{\partial\xi} + \frac{\partial u_{c1}}{\partial\xi} = 0, \tag{15}$$

$$-\lambda\frac{\partial u_{c1}}{\partial\xi} = \frac{\partial\phi_1}{\partial\xi}, \tag{16}$$

$$\frac{1}{\alpha}n_{c1} + a_1\phi_1 = 0. \tag{17}$$

This system describes the linear properties of the electron-acoustic wave. A non-trivial solution exists only if a specific relationship between the phase velocity $\lambda$ and the plasma parameters is satisfied, leading to the following dispersion relation:

$$\lambda = \sqrt{\frac{1}{\alpha a_1}}. \tag{18}$$

Proceeding to the next order in the perturbation hierarchy, we isolate terms of $\mathcal{O}(\epsilon^2)$. This yields a system of inhomogeneous equations that govern the second-order perturbations. The source terms for these equations are nonlinear products of the first-order solutions, which act to drive the evolution of the wave profile on the slower timescale $\tau$. The resulting second-order system is given by

$$-\lambda\frac{\partial n_{c2}}{\partial\xi} + \epsilon\frac{\partial u_{c2}}{\partial\xi} = -\lambda\frac{\partial n_{c1}}{\partial\tau} - \frac{\partial}{\partial\xi}(n_{c1}u_{c1}) - \frac{\epsilon\lambda(n_{c1}u_{c1})}{\tau}, \tag{19}$$

$$-\lambda\frac{\partial u_{c2}}{\partial\xi} - \frac{\partial\phi_2}{\partial\xi} = -\frac{\partial u_{c1}}{\partial\tau} - u_{c1}\frac{\partial u_{c1}}{\partial\xi}, \tag{20}$$

and

$$\frac{1}{\alpha}n_{c2} + a_1\phi_2 + a_2\phi_1^2 = \epsilon\frac{\partial^2\phi_1}{\partial\xi^2} + \frac{\epsilon\lambda}{\tau}\frac{\partial\phi_1}{\partial\xi}. \tag{21}$$

After careful elimination of second-order terms using the first-order solutions, we obtain

$$\frac{\partial\phi_1}{\partial\tau} + A\phi_1\frac{\partial\phi_1}{\partial\xi} + B\frac{\partial^3\phi_1}{\partial\xi^3} + \frac{\phi_1}{2\tau} = 0, \tag{22}$$

where the final term accounts for cylindrical geometry. The coefficients A and B represent, respectively, the coefficients of the nonlinearity and dispersion,

$$A = -\frac{1}{2}\left(\frac{3\alpha}{\lambda} + 2\lambda^3 a_2\right), \tag{23}$$

$$B = \frac{\lambda^3}{2}. \tag{24}$$

Applying the transformation,

$$\phi_1 = \left(\frac{6B^{1/3}}{A}\right)u, \quad \xi = B^{-1/3}x, \quad \tau = t.$$

The following CKdV equation is obtained:

$$u_t + 6uu_x + u_{xxx} + \frac{u}{2t} = 0. \tag{25}$$

Within the framework of one-dimensional nonlinear wave theories, the cylindrical Korteweg–de Vries (CKdV) equation represents a fundamental and paradigmatic model for the study of soliton dynamics in geometries exhibiting cylindrical symmetry. This equation was first derived and studied in the context of cylindrical ion-acoustic solitons by Maxon and Viecelli,[22] with subsequent experimental observations reported by Hershkowitz and Romesser.[30] A profound mathematical property of the CKdV equation, established by Nakamura,[31] is the existence of an infinite hierarchy of conservation laws, a hallmark of its integrable nature. This integrability is further underscored by the existence of a Bäcklund transformation for the equation, as demonstrated by Nakamura.[24] In Eq. (25), the subscripts $x$ and $t$ denote partial differentiation with respect to these variables.

## III. PAINLEVÉ ANALYSIS TO THE CKdV EQUATION

The Painlevé test is applied to evaluate the integrability of Eq. (25). Based on the findings in Refs. 32–34, we can determine whether Eq. (1) is integrable. Applying the WTC methodology, we identify a singular branch with three resonances: $(-1, 4, 6)$. The freedom of the singular manifold $\phi(x,t) = 0$ relates to the resonance at $-1$. In addition, the compatibility conditions[4,6] are satisfied, indicating that this equation is Painlevé integrable according to the WTC approach.

## IV. BÄCKLUND TRANSFORMATION FOR THE CKdV EQUATION

The integrable nature of the CKdV equation is explicitly demonstrated through the existence of a Bäcklund transformation. Following the methodology,[24] we introduce two distinct solutions, $w$ and $\widetilde{w}$, of the CKdV equation, which are related to the physical fields via the Miura-type transformations $u = -w_x$ and $\widetilde{u} = -\widetilde{w}_x$ or, equivalently, $w = -2(\log f)_x$ and $\widetilde{w} = -2(\log\widetilde{f})_x$. Here, the tilde notation denotes an alternative solution and does not represent a derivative.

In this framework, the Bäcklund transformation for the CKdV equation can be expressed as the following coupled system:

$$(\widetilde{W}_+)_x = \frac{1}{2}(-\widetilde{W}_-)^2 - \frac{x + x_1}{6t}, \tag{26}$$

$$(-\widetilde{W}_-)_t = (\widetilde{W}_-)_{xxx} - \frac{3}{2}\left[(\widetilde{W}_-)(\widetilde{W}_+)_x\right]_x + \frac{1}{4}\left[(\widetilde{W}_-)^3\right]_x + \left[\frac{x + x_1}{4t}(\widetilde{W}_-)\right]_x, \tag{27}$$

where we define the sum and difference variables as $\widetilde{W}_+ \equiv w + \widetilde{w}$ and $\widetilde{W}_- \equiv w - \widetilde{w}$, respectively.

The constant $x_1$, which arises naturally and can be chosen arbitrarily, carries physical significance as it parameterizes the initial position (or phase) of the soliton solution generated by the transformation.[24] It is a straightforward exercise to verify that the system of Eqs. (26) and (27) is mutually consistent, satisfying the integrability condition $(W_+)_{xt} = (-W_-)_{tx}$.

To generate a non-trivial soliton solution from a known seed solution, we begin with the vacuum state, defined by $\widetilde{w} = 0$, and, for simplicity, choose $x_1 = 0$. Substituting this into the Bäcklund system (26) and (27) yields the following reduced equations:

$$\frac{dw}{dx} = \frac{1}{2}w^2 - \frac{x}{6t}, \tag{28}$$

$$\frac{dw}{dt} = -\frac{d^3w}{dx^3} + \frac{3}{2}\left(\frac{dw}{dx}\right)^2 + \frac{3}{2}w\left(\frac{d^2w}{dx^2}\right) - \frac{3}{4}w^2\left(\frac{dw}{dx}\right) - \frac{x}{4t}\left(\frac{dw}{dx}\right) - \frac{w}{4t}. \tag{29}$$

We then introduce a similarity transformation,

$$\eta = \frac{x}{t^{1/3}}, \tag{30}$$

$$w = t^{-1/3}g(\eta). \tag{31}$$

In the new variables, $\eta$ and $g(\eta)$ can be written as follows:

$$\frac{dg}{d\eta} = \frac{1}{2}g^2 - \frac{\eta}{6}, \tag{32}$$

$$\frac{d^3g}{d\eta^3} - \frac{3}{2}\left(\frac{dg}{d\eta}\right)^2 - \frac{3}{2}g\frac{d^2g}{d\eta^2} = \eta\frac{dg}{d\eta} - \frac{3}{4}g^2\frac{dg}{d\eta} - \frac{g}{4}. \tag{33}$$

Now, Eq. (32) is similar to the "Riccati equation," where the general Riccati equation is given by

$$\frac{dg}{dg} = a(\eta)g^2(\eta) + b(\eta)g(\eta) + c(\eta).$$

Now, we will perform the appropriate substitution,

$$g(\eta) = -\frac{1}{a(\eta)T}\frac{dT}{d\eta}, \tag{34}$$

and from Eqs. (32)–(34), we can easily write the following equation:

$$\frac{d^2T}{d\eta^2} - \frac{1}{12}\eta T = 0. \tag{35}$$

This equation is the standard "Airy's equation." The solution for it reads as

$$T(\eta) = c_1\mathrm{Ai}\left(\frac{\eta}{2^{2/3}\cdot 3^{1/3}}\right) + c_2\mathrm{Bi}\left(\frac{\eta}{2^{2/3}\cdot 3^{1/3}}\right), \tag{36}$$

where $c_1$ and $c_2$ are arbitrary constants. For simplicity, choose $c_1 = c_2 = 1$. Now, the value of $g(\eta)$ can be easily found from Eq. (34), and by using Eqs. (30) and (31), the final solution $w$ of Eq. (28) comes out to be

$$w(x,t) = -t^{1/3}\frac{\frac{1}{2^{2/3}.3^{1/3}}\left[\mathrm{Ai}'\left(\frac{x/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}'\left(\frac{x/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}{\frac{1}{2}\left[\mathrm{Ai}\left(\frac{x/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}\left(\frac{x/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}. \tag{37}$$

This solution can be written in terms of $x_1$,

$$w_1(x,t) = -t^{1/3}\frac{\frac{1}{2^{2/3}.3^{1/3}}\left[\mathrm{Ai}'\left(\frac{(x+x_1)/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}'\left(\frac{(x+x_1)/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}{\frac{1}{2}\left[\mathrm{Ai}\left(\frac{(x+x_1)/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}\left(\frac{(x+x_1)/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}. \tag{38}$$

According to the transformation $u = w_{1x}$, the analytical solution of the CKdV equation reads as

$$u(x,t) = \frac{\left(\frac{2}{3}\right)^{1/3}\left(\frac{1}{t}\right)^{1/3}\left[\frac{\left(\frac{1}{t}\right)^{2/3}(x+x_1)\mathrm{Ai}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)+\left(\frac{1}{t}\right)^{2/3}(x+x_1)\mathrm{Bi}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)}{2^{4/3}3^{2/3}}\right]}{\mathrm{Ai}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right) + \mathrm{Bi}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)}$$

$$- \frac{\left(\frac{2}{3}\right)^{1/3}\left(\frac{1}{t}\right)^{1/3}\left[\mathrm{Ai}'\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right) + \mathrm{Bi}'\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)\right]\left[\frac{\left(\frac{1}{t}\right)^{1/3}\mathrm{Ai}'\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)+\left(\frac{1}{t}\right)^{1/3}\mathrm{Bi}'\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)}{2^{2/3}3^{1/3}}\right]}{\left[\mathrm{Ai}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right) + \mathrm{Bi}\left(\frac{(x+x_1)t^{-1/3}}{2^{2/3}3^{1/3}}\right)\right]^2}. \tag{39}$$

This gives the one soliton solution of the CKdV equation through the transformation $\phi_1 = \left(6B^{1/3}/A\right)u$ obtained in terms of Airy's function type 1 and Airy's function type 2.

## V. THEOREM OF PERMUTABILITY (TP)

The permutability theorem asserts that two successive Bäcklund transformations commute. In other words, if two Bäcklund transformations with distinct parameters, $x_1$ and $x_2$, are applied to a solution $w$, producing a new solution $w_{12}$, the order in which the transformations are performed does not affect the outcome. This means that the final solution remains the same regardless of whether the first or second transformation is applied first. Using the permutability theorem, higher-order solutions can be constructed algebraically. TP for two solitons solution is given as

$$w \xrightarrow{x_1} w_1 \xrightarrow{x_2} w_{12}, \quad w \xrightarrow{x_2} w_2 \xrightarrow{x_1} w_{21}.$$

From the figure, we conclude that TP requires $w_{12} = w_{21}$, so after applying TP, the two-soliton solution is given by

$$w_{12} = w + \frac{x_1 - x_2}{3t(w_1 - w_2)}. \tag{40}$$

Here, $w_1$ and $w_2$ are single-soliton solutions with distinct parameters $x_1$ and $x_2$, respectively.

Similar to $w_1$, the expression for $w_2$ can be written as

$$w_2(x,t) = -t^{1/3}\frac{\frac{1}{2^{2/3}.3^{1/3}}\left[\mathrm{Ai}'\left(\frac{(x+x_2)/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}'\left(\frac{(x+x_2)/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}{\frac{1}{2}\left[\mathrm{Ai}\left(\frac{(x+x_2)/t^{1/3}}{2^{2/3}.3^{1/3}}\right) + \mathrm{Bi}\left(\frac{(x+x_2)/t^{1/3}}{2^{2/3}.3^{1/3}}\right)\right]}. \tag{41}$$

This shows that $w_1$ and $w_2$ maintain symmetry in accordance with the requirement of TP. Therefore, the two-soliton solution is

$$u_{12} = w_{12x}.$$

By setting $w = 0$ in Eq. (40) and using the expression above, we obtain

$$u_{12}(x,t) = \frac{(x_1 - x_2)\left[-\frac{\frac{C(DA_1+DB_1)}{2^{4/3}3^{2/3}}}{(A_1+B_1)} + \frac{\frac{C(EA_2+EB_2)}{2^{4/3}3^{2/3}}}{(A_2+B_2)} + \frac{C(A_1'+B_1')\left(\frac{\left(\frac{1}{t}\right)^{1/3}A_1'+\left(\frac{1}{t}\right)^{1/3}B_1'}{2^{2/3}3^{1/3}}\right)}{(A_1+B_1)^2} - \frac{C(A_2'+B_2')\left(\frac{\left(\frac{1}{t}\right)^{1/3}A_2'+\left(\frac{1}{t}\right)^{1/3}B_2'}{2^{2/3}3^{1/3}}\right)}{(A_2+B_2)^2}\right.}{3\left[\left(-\frac{C(A_1'+B_1')}{A_1+B_1}\right) + \left(\frac{C(A_2'+B_2')}{A_2+B_2}\right)\right]^2}, \tag{42}$$

with

$$A_1 = Ai\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_1)}{2^{2/3}.3^{1/3}}\right),\quad B_1 = Bi\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_1)}{2^{2/3}.3^{1/3}}\right),$$

$$A_2 = Ai\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_2)}{2^{2/3}.3^{1/3}}\right),\quad B_2 = Bi\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_2)}{2^{2/3}.3^{1/3}}\right),$$

$$A_1' = Ai'\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_1)}{2^{2/3}.3^{1/3}}\right),\quad B_1' = Bi'\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_1)}{2^{2/3}.3^{1/3}}\right),$$

$$A_2' = Ai'\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_2)}{2^{2/3}.3^{1/3}}\right),\quad B_2' = Bi'\left(\frac{\left(\frac{1}{t}\right)^{1/3}(x+x_2)}{2^{2/3}.3^{1/3}}\right),$$

$$C = \left(\frac{2}{3}\right)^{1/3}\left(\frac{1}{t}\right)^{1/3},\quad D = \left(\frac{1}{t}\right)^{2/3}(x+x_1),\quad E = \left(\frac{1}{t}\right)^{2/3}(x+x_2).$$

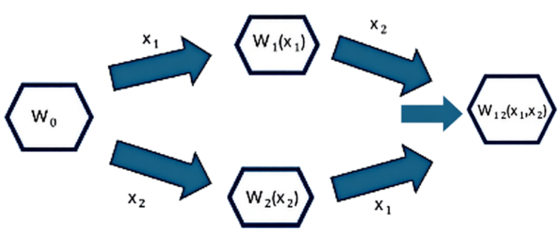


**FIG. 1.** Bianchi diagram for the two-soliton solution having seat solution $w_0 = 0$.

Figure 1 gives a schematic diagram for constructing the two soliton solutions. From this procedure, the general expression for the $n$th soliton solution is constructed, recursively,

$$w_{(n)} = w_{(n-2)} + \frac{x_{n-1} - x_n}{3t\left(w_{(n-1)} - w_{(n-1)'}\right)}, \tag{43}$$

with

$$w_n = w_{(k_1,k_2,\ldots,k_n)} = w_{12\ldots n}$$

and

$$w_n' = w_{(k_1,\ldots,k_{n-1},k_n+1)} = w_{12\ldots n-1,n+1},$$

where the seed solution is given as

$$w = w_{(0)} = 0.$$

So from this, the three-soliton solution can be written as

$$w_{123} = w_1 + \frac{x_2 - x_3}{3t(w_{12} - w_{13})}, \tag{44}$$

where $w_1$ is the first-soliton solution. The schematics for constructing the three-soliton solution are given in Fig. 2.

The recursive nature of this scheme underscores the power and generality of the Bäcklund transformation method. It is crucial to note that this formalism is not limited to the specific case of unmagnetized plasma in the cylindrical geometry presented here. The same methodological framework has been successfully applied to derive soliton solutions in diverse settings, including magnetized plasmas where the external field introduces anisotropic effects,[35–37] unmag-

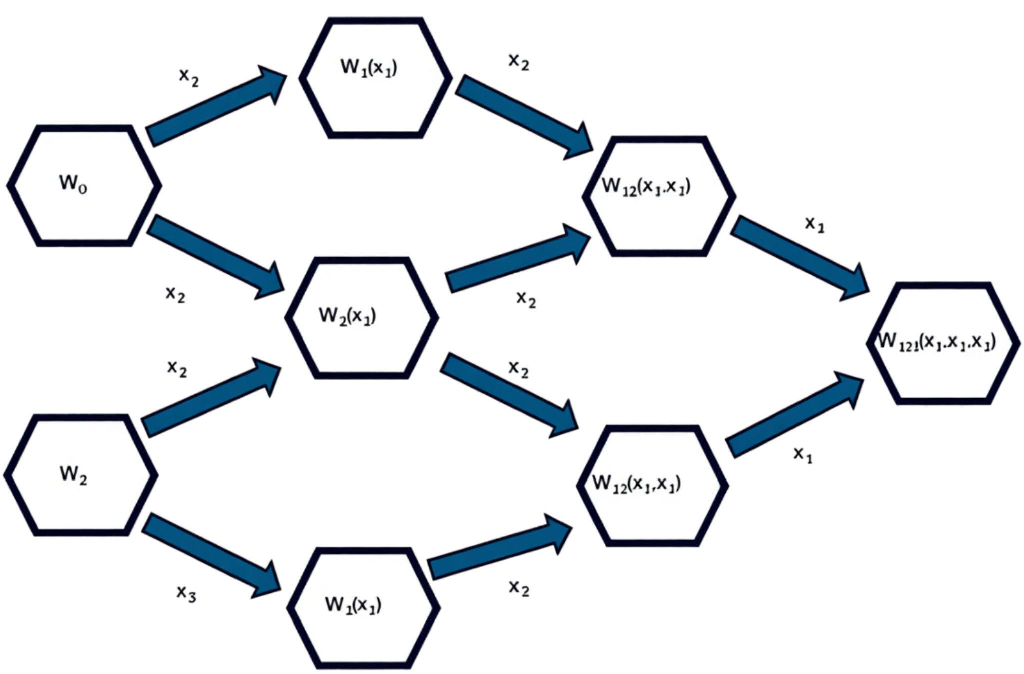


**FIG. 2.** Bianchi diagram for the three-soliton solution having seat solution $w_0 = 0$.

netized plasmas with various particle distributions in Cartesian geometry,[38–43] and other nonplanar geometries beyond the cylindrical case discussed here.[44–46] This establishes the Bäcklund transformation and related analytical techniques as versatile and unifying approaches for constructing nonlinear wave solutions across a wide range of plasma physics contexts.

## VI. RESULTS AND DISCUSSION

Here, we present the results of the nonlinear propagation of EAWs in a cylindrical geometry within plasma comprising cold inertial electrons and kappa-distributed hot electrons, a model directly related to the Cassini spacecraft observations of Saturn's magnetosphere. For this system, the CKdV equation admits rarefactive solitary wave structures, obtained via the Bäcklund transformation method. In contrast to the planar KdV equation, the analytical solution for the cylindrical case is expressed in terms of Airy functions of the first and second kind, Ai and Bi, as given by Eq. (37).

We present a detailed parametric analysis of this solution, focusing specifically on the influence of two key parameters: the superthermality index ($\kappa$) of the hot electrons and the density ratio ($\alpha = n_{h0}/n_{c0}$). The variation of $\kappa$ is found to critically modify the amplitude and width of the solitary waves, while $\alpha$ significantly alters the dispersion properties. This investigation provides crucial insights into the behavior of nonlinear EAWs under realistic conditions in space plasma. While this study details the single-soliton solution, the mathematical framework also permits the construction of multi-soliton solutions. A complete numerical exploration of these interactions, although computationally arduous, remains a compelling objective for future work.

### A. Influence of hot electron superthermality ($\kappa$)

The spectral index $\kappa$ is a fundamental parameter governing the non-Maxwellian character of the hot electron population. A lower $\kappa$ value signifies an energy spectrum characterized by a more pronounced suprathermal tail. Conversely, as $\kappa \to \infty$, the distribution converges to a Maxwellian case. Figure 3 illustrates the impact of varying $\kappa$ on the profile of the electron-acoustic solitary wave, specifically on the potential perturbation $\phi_1$.

The structures observed in Fig. 3 are rarefactive solitary waves, evidenced by the negative potential well ($\phi_1 < 0$). Physically, this corresponds to a localized region characterized by a *depression* in electron density—a density *rarefaction*—that propagates as a coherent entity. This is a canonical feature of electron-acoustic waves in plasmas where the cold electron component provides the inertia.

The results demonstrate a clear and consistent trend: as the hot electron distribution becomes more superthermal (i.e., as $\kappa$ decreases from 10 to 3), the amplitude of the rarefactive soliton decreases significantly. For instance, the depth of the potential well is lowest for $\kappa = 3$ and highest for the nearly case of $\kappa = 10$. This further deduces that the amplitude of EASW in a cylindrical geometry would become maximum for the Maxwellian case (i.e., $\kappa \to \infty$). This leads us to the conclusion that the Maxwellian distribution is hotter than the kappa distribution. The direct relationship between the amplitude of the soliton and $\kappa$ is a critical result linking microscopic particle statistics (the kappa distribution) to the macroscopic nonlinear wave phenomenology.

### B. Influence of the hot-to-cold electron density ratio ($\alpha$)

The density ratio $\alpha = n_{h0}/n_{c0}$ is a fundamental parameter controlling the composition of the plasma of two-electrons. Its variation profoundly affects the characteristics of the electron-acoustic solitary structures. Figure 4 presents the electrostatic potential profiles computed for different values of $\alpha$, with all other parameters, including the spectral index $\kappa$, held constant.

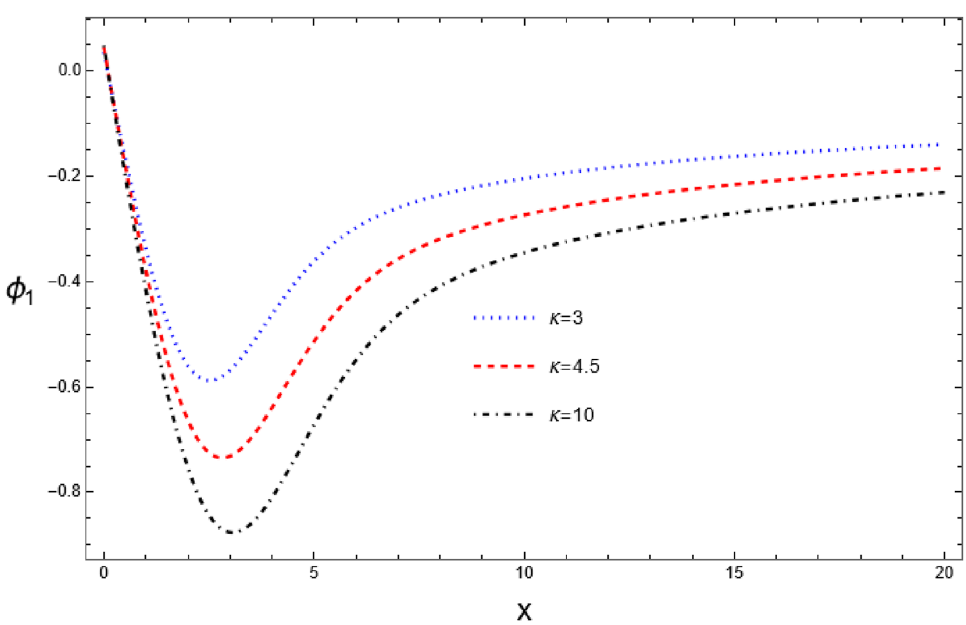


**FIG. 3.** Variational effect of kappa on the solitary wave structure.

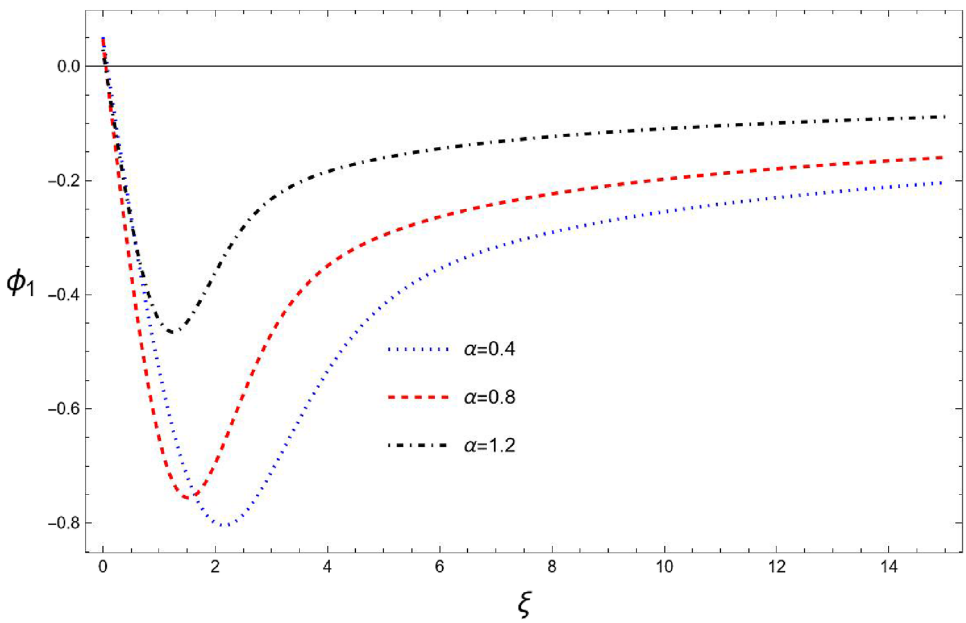


**FIG. 4.** Variational effect of alpha on solitary wave structures.

The solutions presented in Fig. 4 are, once again, rarefactive solitary waves, characterized by a negative potential perturbation ($\phi_1 < 0$). This confirms that the fundamental nature of the electron-acoustic mode in this cylindrical plasma geometry is to support density depressions, regardless of the specific value of $\alpha$. This can also be verified from the expression of the nonlinearity coefficient, which lacks competing terms and remains negative for all values of plasma parameters.

The central finding of this analysis is the dramatic influence of $\alpha$ on the amplitude of the soliton. The results demonstrate a clear and robust trend: as the density ratio $\alpha$ increases, the amplitude (depth) of the rarefactive soliton increases substantially. The primary reason for this behavior is that the dynamics of cold electrons drives electron-acoustic waves and decreasing the population of cold electrons (i.e., increasing $\alpha$) will reduce the amplitude of the nonlinear electron-acoustic solitary waves.

## VII. CONCLUSION

This study has presented a rigorous theoretical investigation of the nonlinear propagation of electron-acoustic waves within the unique cylindrical geometry of Saturn's magnetosphere. By constructing a plasma model comprising cold inertial electrons, kappa-distributed hot electrons, and stationary ions, we have derived the cylindrical Korteweg–de Vries (CKdV) equation governing the dynamics of small-amplitude electron acoustic solitary waves. The integrable nature of this equation has been explicitly leveraged through the application of the Bäcklund transformation, a powerful analytical tool that has enabled us to construct novel, exact analytical solutions.

Our primary achievements are threefold. First, we have successfully derived a hierarchy of multi-soliton solutions, including explicit expressions for single-, two-, and three-soliton profiles in terms of the Airy functions Ai and Bi. This represents a significant advancement beyond the standard sech$^2$-type solutions of the planar KdV equation, capturing the essential geometric dispersion inherent to cylindrical systems. The recursive scheme, established to generate $N$-soliton solutions, provides a robust mathematical framework for future studies of complex soliton interactions in nonplanar geometries.

Second, our parametric analysis, grounded in realistic data from the Cassini mission, has elucidated the fundamental role of two key plasma parameters: the hot-to-cold electron density ratio, $\alpha$, and the spectral index, $\kappa$, which characterizes the superthermality of the hot electron population. We have unequivocally demonstrated that the electron-acoustic mode in this environment exclusively supports rarefactive solitary waves. A central finding of this work is the identification of a strong direct relationship between the amplitude of the soliton and both the above mentioned parameters; amplitude increases with a more Maxwellian distribution (higher $\kappa$) and, more significantly, with a higher density ratio $\alpha$.

Finally, we have employed the permutability theorem to generate higher-order solutions, confirming the system's integrability and providing a method for constructing complex waveforms from simpler ones.

In summary, this work has provided a self-consistent, analytical model that bridges microscopic particle statistics (via the kappa distribution) with macroscopic nonlinear wave phenomena in a realistic space plasma geometry. The solutions and insights presented here offer a valuable theoretical framework for interpreting *in situ* satellite observations of electrostatic solitary waves in planetary magnetospheres, particularly those of Saturn, and contribute fundamentally to the understanding of nonlinear wave propagation in nonplanar plasmas.

### A. Future work

In light of the widespread adoption of fractional calculus, which effectively elucidates certain perplexing behaviors of nonlinear phenomena that traditional calculus fails to identify, we plan to examine the fractional CKdV equation in forthcoming research to enhance our understanding of the propagation dynamics of fractional cylindrical solitons or cnoidal waves in diverse plasma systems. This problem poses a significant challenge for many researchers because the geometric term renders it rigid when analyzed using conventional approaches. Nevertheless, using the pivotal method for analyzing fractional differential equations, the Tantawy technique,[47–53] we can overcome this challenge, effectively diagnose the problem, and produce high-order approximations that replicate diverse nonplanar structures encountered not only in plasma physics but also across numerous interdisciplinary domains.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

In the preparation of this manuscript, each author contributed equally to this work.

**Khizra Qaiser**: Formal analysis (equal); Investigation (equal); Methodology (equal); Resources (equal); Writing – original draft (equal). **Waqas Masood**: Formal analysis (equal); Resources (equal); Supervision (equal); Writing – review & editing (equal). **Rabia Jahangir**: Formal analysis (equal); Investigation (equal); Writing – original draft (equal). **Hanan Al-Ghamdi**: Investigation (equal); Supervision (equal); Writing – review & editing (equal). **Muhammad Shahnewaz Bhuyan**: Investigation (equal); Validation (equal); Writing – original draft (equal).

## DATA AVAILABILITY

The authors confirm that the data supporting the findings of this study are available within the article. In addition, we confirm that this investigation does not involve any clinical trial.

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