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So who gains from a small tick size?

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Abstract

We investigate the relation between price discreteness and the number of dealers in a dealer market. We present a model featuring a finite number of dealers competing in prices for supplying liquidity to a forthcoming market order. We find that a decrease in tick size benefits dealers and tends to hurt investors when the number of dealers for a stock is small. In contrast, a decrease in tick size hurts dealers and benefits investors when the number of dealers is large. This result yields several new empirical implications relating a change in tick size to entry and exit of dealers, order aggressiveness and transaction rates.

JEL classification: G10; D43

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1. Introduction

Prices in financial markets are discrete. Until recently, US stock markets traded stocks in multiples of $\$\frac{1}{16}$, and previously $\$\frac{1}{8}$. As of April 2001, all US stock markets use a decimal system, and most prices are quoted in multiples of one penny. The choice of the minimum price variation (hereafter called the "tick size") in any trading mechanism is at the hands of

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the mechanism designer.[1] It is commonly believed that this decision can alter the division of gains from trade between different market participants. Moreover, decreasing the tick size is considered welfare improving, in the sense that total gains from trade in the market increase. This is so, because a positive tick size is considered to be a "friction," preventing trades from taking place even when they are beneficial.

Furthermore, changing the tick size has been used in financial markets as a policy tool to transfer rents from one type of market participant to another. For example, it is commonly believed that a large tick size benefits professional traders, e.g. dealers who quote prices. Thus, decreasing the tick size in markets such as NASDAQ was aimed to deprive dealers from some of their profits, and transfer these profits to investors. The reason for this belief is the fact that the tick size forms a lower bound on the bid–ask spread. If competition among dealers is intense, this lower bound becomes effective, and decreasing the tick size lowers the bid–ask spread. On the other hand, it is not entirely clear what the impact of a decrease in tick size would be when competition among dealers is lax.

Following the rush of exchanges to lower the tick size, many recent empirical papers test the implications of such a move on various liquidity measures. A survey of this literature can be found in Bacidore (1997), Harris (1994, 1997), Goldstein and Kavajecz (2000) and others. The typical result is that a decline in tick size decreases the average bid–ask spread, but it also decreases the depth at the inside quotes. Thus, the overall liquidity effect of a decrease in tick size is not clear. Jones and Lipson (2001) study the change in tick size in NYSE from eighths to sixteenths. They show that execution costs for institutional traders actually increase following the decrease in tick size. They conclude that "smaller tick sizes may actually reduce market liquidity" and that "spreads are not a sufficient statistic for market quality." Christie et al. (2002) study the cross-sectional effects of changing the tick size. They show that the bid–ask spread of high volume stocks declines following a decrease in tick size, whereas the impact of a decrease in tick size on the spread of lower volume stocks is ambiguous.

Do investors always gain from a lower tick size? How are the gains and losses of investors and dealers following a tick size reduction affected by the number of dealers in the market? What are the cross-sectional differences in the effects of a tick size reduction? In particular, does a tick size reduction affect the number of dealers, order aggressiveness, and the transaction rate differently in different markets? Should market designers take into account the number of dealers when they decide on the tick size for a specific stock?

In this paper we provide a model to answer these questions. We do this by investigating the welfare consequences of imposing a positive tick size on a dealer market, and relating these consequences to the number of dealers for a stock. Following the conclusion of Jones and Lipson (2001), we focus on the total welfare of market participants, and not specifically on the bid–ask spread. Our model features a finite number of dealers for a specific stock in the spirit of the fragmented market of Biais (1993). The dealers compete in prices for supplying liquidity to a forthcoming market order by posting simultaneous quotes. The market order may or may not be submitted, depending on the reservation value of a repre-

[1] Currency markets are an exception. In these markets there is no "designer." Yet, maybe by convention, dealers choose their quotes on a grid (see Lyons, 1995, for instance).

sentative investor. Each dealer has his own privately-known valuation of the stock, which could be driven by idiosyncratic inventory positions, differences of opinions, or liquidity shocks. Dealers' quotes are restricted to a finite grid of prices, even though their private valuations may take any value in a finite interval.

We model this interaction as a game of incomplete information (Harsanyi, 1967–1968), and study the welfare properties of non-decreasing, symmetric, pure strategy Nash equilibria. We view this setting as one out of many encounters between dealers and investors. Thus, when deciding on the "correct" tick size, the market designer would like to maximize ex ante welfare, represented by the expected gains from trade calculated before any market participant knows his reservation value. We use this ex ante approach to assess the welfare properties of different discretization policies.

Our main result is that the welfare effects of a change in tick size are affected by the number of dealers in the market.[2] Specifically, when the number of dealers is large, they prefer a coarse grid of prices (large tick size), whereas when the number of dealers is small, they prefer a fine grid of prices (small tick size). As for investors, if the number of dealers is large, they prefer a small tick size. If, however, the number of dealers is small, there is ambiguity in their preference, with some tendency toward larger ticks.

The relation between price discreteness and the number of dealers may be drawn from basic intuition regarding competition in oligopolistic markets. In order to articulate this intuition, we refer to the two extreme cases: the monopolistic dealer case, and the perfectly competitive case. Then, we study the gradual move from the former case to the latter.

Consider first the case of a monopolistic dealer. This dealer would like to maximize his ex ante profits, by submitting the optimal quote, given the distribution of reservation values of the upcoming market order. In a market with a continuous grid of prices the dealer is able to submit such an optimal quote. However, a positive tick size restricts his ability to do so, and his ex ante expected profit is reduced. Thus, a positive tick size serves as a *restraining device*, preventing the monopolistic dealer from fully exploiting his market power. By increasing the tick size, investors enjoy the extra welfare extracted from the monopolistic dealer. However, a large tick size might decrease the probability of transaction. These two opposing forces make the attitude of investors to the tick size in this case ambiguous. This attitude depends on the distribution of reservation values.

Consider now the perfectly competitive case—an infinite number of dealers. In this case, competition forces the dealers to post "honest quotes." This means that their quotes are as close as possible to their reservation values, given that quotes must lie on a discrete grid. No dealer shades his quote by more than one tick. As a result, in the perfectly competitive case, the tick size is the only source of profits to dealers, and may be viewed as a *commitment device* that enables them to earn positive profits despite competition. Thus, in the perfectly competitive case, decreasing the tick size decreases the welfare of dealers. The welfare loss of dealers is transferred to the public of investors who enjoy an increase in welfare following a decrease in tick size. Total market welfare increases as well, reflecting the fact that a decrease in tick size increases the ex ante probability of a transaction.

[2] We initially derive the model using a fixed and exogenous number of dealers. Later on, we allow the number of dealers to change endogenously in order to study entry and exit of dealers following a change in tick size.

Intermediate levels of competition feature gradual movement from the monopolistic case to the perfectly competitive case. Thus, when the number of dealers is small, they prefer small ticks, since a large tick size prevents them from exploiting their full market power. By contrast, when the number of dealers is large, dealers prefer large ticks. Furthermore, when the number of dealers is large, investors prefer fine price grids, while their preference for tick size in the case of a small number of dealers is ambiguous; a large tick size in this case restrains dealers and transfers gains to the investors, but it might also decrease the probability of a transaction and hurt investors.

Our results regarding a small number of dealers stand in contrast to the conventional wisdom among market practitioners that led to the tick size reduction, namely that decreasing the tick size is *always* beneficial for investors. Our results suggest that this view is incorrect, and that the number of dealers for a specific security should be taken into account when deciding on the tick size. In particular, if the number of dealers is small enough, setting a larger tick size might hurt dealers and improve the welfare of investors.

Our model yields several empirical predictions regarding the cross-sectional effects of a change in tick size. Our main findings are:

- A decrease in tick size would cause a decrease in the number of dealers in markets with a high extant number of dealers, while it will increase the number of dealers in markets with a small number of dealers.
- A decrease in tick size tends to increase the probability of transactions, and hence the transaction rate in markets with many dealers, while it has an ambiguous effect on the probability of transaction in markets with a small number of dealers.
- A decrease in tick size increases order aggressiveness, and hence decreases the bid–ask spread in markets with many dealers, while it has an ambiguous effect on order aggressiveness in markets with a small number of dealers.

While the last prediction is consistent with the findings of Christie et al. (2002), we are not aware of empirical evidence related to the former two.

The paper is organized as follows: In Section 2 we survey related theoretical literature. Section 3 presents the model. In Section 4 we prove existence of non-decreasing equilibria, and define our welfare measures. Section 5 analyzes the welfare implications of discrete prices in the case of a monopolistic dealer. Section 6 discusses the welfare implications of discrete prices for higher numbers of dealers. In Section 7 we study the empirical implications of our model, and the implications for market design. Section 8 concludes. Technical proofs are in Appendix A.

2. Related theoretical literature

Our paper highlights the relation between the number of dealers and the effect of a change in tick size. We are not, however, the first to demonstrate that a zero tick size might not be optimal. Seppi (1997) analyzes the impact of a change in tick size in a market where a specialist with market power competes against a limit order book. Liquidity providers in his model are perfectly competitive. He shows that while large and small traders may disagree on the optimal tick size, they would all prefer a tick size strictly larger than zero. The reason is that a small tick size benefits the specialist by allowing him to undercut the book with a lower cost. However, in Seppi's model, liquidity providers always profit from a large tick size. In our model, we do not assume perfect competition. Moreover, when the number of dealers is small, a decrease in tick size actually benefits liquidity suppliers.

Harris (1994), Cordella and Foucault (1999) and Foucault et al. (2004) provide arguments in favor of a non zero tick size in a dynamic setting. Harris (1994) suggests that a small tick size disrupts time priority, since the cost of undercutting the current quote by opportunistic "quote matchers" decreases with the tick size. Cordella and Foucault (1999) analyze dynamic price competition between two dealers. They show that a larger tick size expedites the convergence of prices to the competitive level. For this reason, a larger tick size may result in lower trading costs for investors. Foucault et al. (2004) show that a zero tick size impairs market resiliency, and hence if the population of traders tends to be impatient, a decrease in tick size might increase the expected spread. In contrast to these papers, our model is static and the arguments in favor of a positive tick size do not hinge on time priority, the arrival rate of traders, or their waiting costs.

Glosten (1995) models competition for quantities in a discrete setting. In his model, a decrease in tick size has two effects: it lowers the profitability of individual trades for dealers, but it also increases trade volume. Using numerical calculations he is able to demonstrate cases where decreasing the tick size improves the total profitability of providing quotes (the latter effect overrides the former one). By contrast, in our model dealers compete in prices. Furthermore, when the number of dealers is small, decreasing the tick size increases the profitability of individual trades for dealers, since they can better exploit their market power.

In general, none of the above papers relates the number of liquidity providers to the effect of tick size reduction; we do. This relation enables us to provide new empirical implications regarding the effect of a change in tick size on the entry and exit of dealers, the probability of transactions and order aggressiveness. These are absent from the extant literature.

Other authors have also considered different aspects of discrete prices in financial markets. Bernhardt and Hughson (1996) consider a model with two dealers arriving sequentially to compete for a market order that can arrive before or after the submission of quotes. In their model, discreteness always limits competition and increases the profits of dealers. Chordia and Subrahmanyam (1995) show that payment for order flow is a natural consequence of a positive tick size. Kandel and Marx (1997) show that a positive tick size yields multiplicity of equilibria in a Bertrand competition of dealers, and provide a rationale for odd-eighths avoidance (see Christie and Schultz, 1994). Anshuman and Kalay (1998) look for the tick size that maximizes the profits of a competitive market maker. They also show that discreteness per se can cause interesting phenomena such as location-dependent commissions and market breakdowns. None of these papers captures the relation between discreteness and the number of dealers that we have in this paper.

Finally, it is worth mentioning that the strategic relation between the dealers and the investor in our model makes it similar to a mechanism known as the "buyer's bid double auction." This mechanism was introduced by Chatterjee and Samuelson (1983), and its welfare attributes under different levels of competition have been explored by Wilson

(1985), Williams (1987, 1991), Satterthwaite and Williams (1989a, 1989b), Rustichini et al. (1994), and Zacharias and Williams (2001). Unfortunately, we were unable to use these results, because they rely on a first-order approach, which is not valid in the face of discrete prices.

3. Model

We model the "bid side" of a dealer market for a specific stock. In this bid side, dealers submit "buy" quotes anticipating a "sell" market order from investors.[3] Our market consists of $m+1$ agents ($m \geqslant 1$). The first m agents are designated as dealers. Each dealer is willing to buy one unit of stock. The last agent is a representative investor, this agent is endowed with one unit of stock that she is willing to sell. The fundamental value of the stock is assumed to be common knowledge among all market participants. Still each dealer has his own private reservation value of the stock. The diversity of reservation values might reflect differences in opinions, different inventory levels or liquidity shocks.

The distribution of reservation values for each dealer is given by the cumulative distribution function F. The representative investor's reservation value is distributed according to a cumulative distribution function denoted by G. The following assumptions are used throughout the paper:

A. 1. The support of F and G is $[0, 1]$.

A. 2. F and G are twice continuously differentiable, and have strictly positive density functions denoted by f and g respectively.

A. 3. Reservation values are statistically independent across dealers, and are each dealer's private information.

A. 4. The investor's reservation value is her own private information, and is statistically independent from the reservation values of the m dealers.

The choice of a $[0, 1]$ support for F and G is for convenience only. Any other closed interval would work just as well.[4] Also, the assumptions on independence and privacy of valuations are used to ease the derivation of the model. In Section 8 we discuss the robustness of our results to changes in these assumptions. In essence, our main results hold when the reservation values are non-negatively correlated (affiliated) and when there is a common component in valuations.

Reservation values may take any value in $[0, 1]$, however quotes (and prices) are restricted to a finite grid. The market designer chooses an integer $n \geqslant 1$, such that quotes

[3] Modeling the "offer side" of the market, in which dealers are sellers and the investors are buyers, is symmetric.

[4] Actually, the supports of valuations for the buyers and the seller need not be identical in order to get our results. All we need is that the interior of these supports have a non-empty intersection.

must be in the set: $P_n \equiv \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\}$. The minimum price variation or tick size is $\frac{1}{n}$.

The trading mechanism is as follows: first the m dealers approach the market at once. Each one of them submits a quote in the form: "I am willing to buy one unit of stock at any price not exceeding p" where $p \in P_n$. Quotes are submitted contemporaneously, thus dealers cannot see the quotes of other dealers. In the second stage, the investor approaches the market and views all the quotes submitted by dealers. She may either submit a "market order" or quit. A market order means that the investor sells one unit of stock to the dealer who submitted the best (highest) quote. If the investor quits then no transaction is made. In case of a tie between a few dealers the winner is selected using a fair lottery.[5] This trading mechanism simulates the situation in a fragmented market such as NASDAQ, where a potential investor "shops around" by making phone calls to several dealers, and chooses the best quote that exceeds her reservation value (if such a quote is found).

We assume that agents' payoffs are linear in money transfers. In particular, suppose that a transaction occurs between an investor with reservation value $x \in [0, 1]$ and a winning dealer with reservation value $y \in [0, 1]$ at a price $p \in P_n$. The investor's profit is the price she gets minus her reservation value: $p - x$, and the winning dealer's profit is his reservation value minus the price he pays: $y - p$. All other $m - 1$ dealers earn zero. In case of no transaction, all the $m + 1$ agents earn zero. All the parameters of the model, the rules of the trading mechanism, as well as the distributions F and G are assumed to be common knowledge among all agents.

This setting may be viewed as a game of incomplete information by identifying the agents' reservation values as their types. A strategy for a dealer in this game is a mapping $B : [0, 1] \to P_n$ that assigns a quote $B(y)$ to any possible reservation value y. A strategy for the investor is a mapping from $\{[0, 1] \times (P_n)^m\}$ to the set {"submit a market order," "quit"}. Thus, a strategy for the investor is a mapping that assigns any combination of the investor's reservation values and m dealers' quotes with a decision of submitting a market order or quitting.

It is readily seen that submitting a market order if and only if the highest dealer's quote is at least as high as the investor's reservation value is a (weakly) dominant strategy for the investor. To see this, denote by $b \in P_n$ the maximal dealer's quote, and denote the investor's reservation value by x. If $b < x$, submitting a market order will provide the investor with a negative payoff of $b - x$. However, by quitting the investor can assure herself a payoff of zero. If $b \geqslant x$, submitting a market order will provide the investor with a non-negative payoff, which is (weakly) preferred to quitting. From now on we will assume that the representative investor always employs her dominant strategy. Our focus will lie on the quoting strategies of the m dealers given this obvious investor's strategy.

Next, we obtain the dealer's objective function. Let $B(\cdot)$ denote a dealer's strategy, and let $y \in [0, 1]$, and $b \in P_n$. We denote by $\pi_{n,m}(b, y, B)$ the expected payoff to a dealer given that:

[5] Notice that since prices are discrete, ties are very likely when m is large relative to n.

(i) the dealer's reservation value is y and he submits a quote of b;
(ii) the investor uses her dominant strategy; and
(iii) all other $m-1$ dealers use the strategy $B(\cdot)$.

$\pi_{n,m}(b, y, B)$ is given by $y-b$ (the payoff in case of winning the transaction) times the probability that a transaction occurs, and that our specific dealer is the one who wins it. Formally:

$$\pi_{n,m}(b, y, B) = (y-b)G(b)\Pr\left\{\begin{array}{l}\text{The dealer wins against the other } m-1\\ \text{dealers, given that they all use the}\\ \text{strategy } B(\cdot).\end{array}\right\} \tag{1}$$

In a symmetric Nash equilibrium, each dealer's strategy maximizes his expected payoff, given that the other dealers use the same strategy.[6] Therefore, a symmetric, pure strategy Nash equilibrium in this game is given by a dealer's strategy $B:[0,1] \to P_n$, such that for all $y \in [0,1]$:

$$B(y) \in \arg\max_{b \in P_n} \pi_{n,m}(b, y, B). \tag{2}$$

Ex post efficiency requires that if the investor values the stock less than one of the dealers, then she should sell it to the dealer with the highest valuation. Unfortunately, due to Myerson and Satterthwaite (1983), no bilateral trade mechanism that allows for individual rationality and incentive compatibility is ex post efficient. Our mechanism is no different. Indeed, a dealer will never submit a buy-quote higher than his reservation value. However, typically dealers will submit buy-quotes lower than their reservation values because of strategic behavior. This behavior might prevent trade in situations where gains from trade exist.

While this inefficiency effect is also encountered in the continuous pricing literature, price discreteness worsens the situation. The reason is that there are only finitely many prices but a continuum of types. Even if bidders were not strategic, they would be forced to shade their bids to the closest price on the finite grid. For instance, if the highest dealer has a valuation of \$5.2 and the tick size is \$0.25, he will submit a quote no larger than \$5. If the investor's valuation is \$5.1, there will not be trade even though efficiency requires it. Moreover, typically, the highest quote will be posted by multiple dealers with different types. Since the winner is selected by chance, there is no special reason why the dealer with the highest reservation value will be selected.

We conclude that inefficiency in our market stems from three sources:

(i) strategic behavior of dealers (we call this the "strategic effect");
(ii) the discreteness of the grid (we call this the "discreteness effect"); and
(iii) clustering of dealers with different types on the same quote price (we call this the "clustering effect").

In order to achieve maximum efficiency, all these three effects must be minimized.

[6] For simplicity we confine our attention in this paper to symmetric equilibria only.

Reducing the strategic effect may be obtained by causing the dealers to submit quotes as close as possible to their reservation values. In order to formalize this, we define the "honest strategy," $H_n : [0, 1] \to P_n$, by:

$$H_n(y) \equiv \begin{cases} \frac{k}{n}, & \frac{k}{n} \leqslant y < \frac{k+1}{n} \text{ for } k = 0, \dots, n-1, \\ 1, & y = 1. \end{cases}$$

A dealer who uses the "honest strategy" H_n, shades his value as little as possible, given that he may only submit quotes in P_n. There is no way that the market designer can force dealers to use the "honest strategy," since whenever $n > 2$ it is not an equilibrium.[7]

Diminishing the discreteness effect and the clustering effect may be obtained by diminishing the tick size (increasing n). The tick size is definitely a decision variable of the market designer, and he can choose it as he likes. However, a priori the cross-impact of decreasing the tick size on the strategic effect is not clear, because decreasing the tick size changes the set of feasible prices for dealers and hence affects their bidding strategies.

4. Equilibrium and welfare measures

Following an approach introduced by Athey (2001), we focus in this paper on non-decreasing equilibria, i.e. equilibrium bidding strategies $B(y)$ that are non-decreasing in the dealer's reservation value y.

Since P_n contains only a finite number of actions, any non-decreasing strategy is just a step function with a finite number of steps. Therefore, we may identify any non-decreasing strategy with its jump points, i.e. the points on the unit interval at which the dealer jumps from one quote in P_n to another one. This is done using the following definition: We say that a vector $\sigma^B \equiv (\sigma_0^B, \sigma_1^B, \dots, \sigma_{n+1}^B) \in \mathbb{R}^{n+2}$ represents a dealer's strategy B, if for all $h \in \{0, \dots, n+1\}$, $\sigma_h^B = \inf\{y \in [0, 1]\colon\ B(y) \geqslant \frac{h}{n}\}$ whenever there is some $h' \geqslant h$ such that $B(y) = \frac{h'}{n}$ on an open interval of $[0, 1]$, and $\sigma_h^B = 1$ otherwise.

The space of all non-decreasing strategies' representations for a dealer is: $\Sigma \equiv \{\sigma \in [0, 1]^{n+2}\colon\ 0 = \sigma_0 \leqslant \sigma_1 \leqslant \cdots \leqslant \sigma_n \leqslant \sigma_{n+1} = 1\}$. This is a compact subset of $[0, 1]^{n+2}$.

To illustrate this definition, consider the case $n = 6$, and let $B^*(\cdot)$ be the non-decreasing strategy defined by:

$$B^*(y) = \begin{cases} 0, & 0 \leqslant y < \frac{1}{3}, \\ \frac{2}{6}, & \frac{1}{3} \leqslant y < \frac{3}{4}, \\ \frac{4}{6}, & \frac{3}{4} \leqslant y < 1, \\ 1, & y = 1. \end{cases} \tag{3}$$

Then $\sigma^{B^*} = (0, \frac{1}{3}, \frac{1}{3}, \frac{3}{4}, \frac{3}{4}, 1, 1, 1) \in \mathbb{R}^8$.

[7] Suppose $n > 2$ and $m \geqslant 1$, and let $k \in \{2, \dots, n-1\}$. Suppose that dealer 1 has reservation value $y \in (\frac{k}{n}, \frac{k+1}{n})$, and the other $m - 1$ dealers use H_n. As y gets closer to $\frac{k}{n}$, the expected profit to dealer 1 from submitting $\frac{k}{n}$ tends to zero. However, if he deviates and submits $\frac{k-1}{n}$, he gets a positive expected profit. This shows that H_n cannot be an equilibrium.

The following notation is useful: Given a non-decreasing strategy $B:[0,1]\to P_n$, that is represented by a vector $\sigma^B\in\mathbb{R}^{n+2}$, we denote by $\Delta_n(B)$ the subset of P_n of quote prices that are submitted with positive probability according to $B(\cdot)$. To illustrate this notation observe that $\Delta_n(B^*)=\{0,\frac{2}{6},\frac{4}{6}\}$ where B^* is the strategy defined by (3).

In order to further investigate the nature of equilibria in multi-dealer environments, we shall rephrase the objective function of a specific dealer given by Eq. (1) in a more explicit form. Let $B(\cdot)$ be any non-decreasing strategy for dealers, and let $y\in[0,1]$ and $b=\frac{k}{n}\in P_n$. Recall that $\pi_{n,m}(b,y,B)$ denotes the expected profit to a specific dealer given that his reservation value is y, he bids b, and all the other $m-1$ dealers use the strategy $B(\cdot)$.

If $b\in\Delta_n(B)$ then (1) may be written as follows:

$$\pi_{n,m}(b,y,B)=\left(y-\frac{k}{n}\right)G\left(\frac{k}{n}\right)\left(F\left(\sigma^B_{k+1}\right)\right)^{m-1}\sum_{r=0}^{m-1}\binom{m-1}{r}\left(\frac{F(\sigma^B_k)}{F(\sigma^B_{k+1})}\right)^r \times\left(\frac{F(\sigma^B_{k+1})-F(\sigma^B_k)}{F(\sigma^B_{k+1})}\right)^{m-r-1}\cdot\frac{1}{m-r}. \tag{4}$$

The term $(y-\frac{k}{n})G(\frac{k}{n})$ is the profit of the dealer in case of winning times the probability that the investor chooses to submit a market order. To see the logic behind the other terms, notice that in order for a dealer who bids $b=\frac{k}{n}$ to win the stock over the other $m-1$ dealers, there must not be any dealer with reservation value equal to or higher than σ^B_{k+1}. The probability of this event is $(F(\sigma^B_{k+1}))^{m-1}$. Conditional on this event, the $m-1$ other dealers may be divided such that r of them ($r\in\{0,1,\ldots,m-1\}$) have reservation values in $[0,\sigma^B_k)$ and $m-r-1$ of them have reservation values in $[\sigma^B_k,\sigma^B_{k+1})$. The r former dealers will surely lose to our dealer since their bid is less than $\frac{k}{n}$, while the $m-r-1$ latter dealers will tie with our specific dealer. Hence, he has a probability of $\frac{1}{m-r}$ of winning against them. Going over all r from 0 to $m-1$, and taking into account all the possible divisions of the $m-1$ dealers to two groups of r and $m-r-1$, yields (4).

After some simplifications and using the binomial formula, Eq. (4) reduces to

$$\pi_{n,m}\left(\frac{k}{n},y,B\right)=\frac{1}{m}\left(y-\frac{k}{n}\right)G\left(\frac{k}{n}\right)\frac{(F(\sigma^B_{k+1}))^m-(F(\sigma^B_k))^m}{F(\sigma^B_{k+1})-F(\sigma^B_k)}. \tag{5}$$

If $b=\frac{k}{n}\notin\Delta_n(B)$, then $\frac{k}{n}$ is submitted with zero probability according to $B(\cdot)$. In this case there is probability zero that submitting a quote of $\frac{k}{n}$ will result in a tie. Hence, in order to win the stock, the reservation values of the other $m-1$ dealers must be lower than σ^B_k. Therefore, in this case:

$$\pi_{n,m}(b,y,B)=\left(y-\frac{k}{n}\right)G\left(\frac{k}{n}\right)F\left(\sigma^B_k\right)^{m-1}. \tag{6}$$

Having established the expressions for the dealers' objective function we can now show that a non-decreasing, symmetric equilibrium always exists in our model.[8]

Proposition 1. *Consider a market with a tick size of $\frac{1}{n}$ and m dealers $(n, m \geqslant 1)$, and suppose the investor uses her dominant strategy. There exists a non-decreasing, symmetric, pure strategy Nash equilibrium.*

In the following sections we investigate the welfare properties of this kind of a non-decreasing equilibrium. We view this model as one encounter out of many between dealers and investors. Therefore, we gauge the ex ante gains from trade in our mechanism, i.e. the expected gains from trade calculated before any market participant knows his reservation value. This is done as follows.

Consider a market with a tick size of $\frac{1}{n}$ and m dealers, and let $B_{n,m}(\cdot)$ be a non-decreasing, pure strategy, symmetric Nash equilibrium, represented by $\sigma^{B_{n,m}} \in \mathbb{R}^{n+2}$. Let $F_{(m)}(y)$ denote the distribution of the highest among m draws of dealers' reservation values.[9] We have: $F_{(m)}(y) = (F(y))^m$. The investor's ex ante expected profit given this equilibrium is:

$$\Gamma_{n,m}^{\text{investor}}(B_{n,m}) \equiv \int_{y=0}^{1} \int_{x=0}^{B_{n,m}(y)} \big(B_{n,m}(y) - x\big)\, \mathrm{d}G(x)\, \mathrm{d}F_{(m)}(y).$$

If we denote: $\gamma(t) \equiv \int_0^t G(x)\,\mathrm{d}x$, then by applying integration by parts, the ex ante profits of the investor may be written as follows:

$$\begin{aligned}\Gamma_{n,m}^{\text{investor}}(B_{n,m}) &= \int_{y=0}^{1} \gamma\big(B_{n,m}(y)\big)\, \mathrm{d}F_{(m)}(y) \\ &= \sum_{k=1}^{n-1} \gamma\left(\frac{k}{n}\right)\Big[\big(F\big(\sigma_{k+1}^{B_{n,m}}\big)\big)^m - \big(F\big(\sigma_{k}^{B_{n,m}}\big)\big)^m\Big]. \qquad (7)\end{aligned}$$

The ex ante profits to the group of m dealers given the equilibrium $B_{n,m}$ is:

$$\Gamma_{n,m}^{\text{dealers}}(B_{n,m}) \equiv m \cdot \int_{y=0}^{1} \pi_{n,m}\big(B_{n,m}(y), y, B_{n,m}\big)\, \mathrm{d}F(y).$$

[8] In general, existence of equilibria in games of incomplete information is not obvious. Williams (1991) proved existence of equilibrium in a similar model with continuous prices. His proof uses a differential equations technique which is not applicable in our discrete setting.

[9] Also known as the mth order statistic.

Using Eq. (5) this can be written as follows:[10]

$$\Gamma_{n,m}^{\text{dealers}}(B_{n,m}) = \sum_{k=1}^{n-1} G\left(\frac{k}{n}\right) \frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})} \times \int_{\sigma_k^{B_{n,m}}}^{\sigma_{k+1}^{B_{n,m}}} \left(y - \frac{k}{n}\right) f(y)\,\mathrm{d}y. \tag{8}$$

The total gains from trade in the market are defined by $\Gamma_{n,m}^{\text{total}}(B_{n,m}) \equiv \Gamma_{n,m}^{\text{investor}}(B_{n,m}) + \Gamma_{n,m}^{\text{dealers}}(B_{n,m})$.

In general, although we know that an equilibrium exists, it is hard to calculate equilibria for arbitrary values of m and n, and distribution functions G and F. We can easily do so, however, when $m = 1$ (a monopolistic dealer). Moreover, in this case the equilibrium is unique. For $m \geqslant 2$ there are potentially multiple equilibria; however, we show that all possible equilibria converge to the "honest strategy" as m tends to infinity. In the following sections we investigate these two extreme cases (monopoly vs perfect competition). We also demonstrate that intermediate numbers of dealers feature a gradual move from the former case to the latter.

5. Discrete prices with a monopolistic dealer

Consider the case of a monopolistic dealer ($m = 1$). We would like to gauge the effect of a change in tick size on the welfare measures in this market. Suppose that the tick size is $\frac{1}{n}$. Since the investor uses her dominant strategy, the dealer's equilibrium strategy $B_{n,1}(y)$ solves:

$$B_{n,1}(y) \in \arg\max_{b \in P_n} \pi_{n,1}(b, y, B_{n,1}) \equiv (y - b)G(b). \tag{9}$$

In order to enhance the intuition, it is useful to consider first the case of a zero tick size ($n = \infty$). If prices are continuous then the optimal quoting strategy of the dealer is found by writing the first-order condition of (9) with respect to b.[11] This yields an implicit equation for $B_{\infty,1}(\cdot)$:

$$B_{\infty,1}(y) = y - \frac{G(B_{\infty,1}(y))}{g(B_{\infty,1}(y))}. \tag{10}$$

We see that the dealer posts a quote equal to his valuation less a positive term representing strategic shading. For instance, if G is uniform then we obtain: $B_{\infty,1}(y) = \frac{1}{2}y$. This

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[10] Notice that for the sake of calculating $\Gamma_{n,m}^{\text{buyers}}(B_{n,m})$ we can dispense with Eq. (6) because the probability of the event: $\{y\colon\ B_{n,m}(y) \notin \Delta_n(B_{n,m})\}$ is zero by definition.

[11] An in depth study of the case of continuous prices is found in Satterthwaite and Williams (1989a), and Kadan (2004). In particular they provide conditions assuring that the first-order condition is actually sufficient for an equilibrium.

demonstrates the "strategic effect" which is present when prices are discrete and when they are not. When we study the impact of discrete prices we can no longer use first-order conditions. Still, it is easy to see that the equilibrium in this case is determined uniquely.[12] We now study this equilibrium.

Proposition 2. *Consider a market with a monopolist dealer* $(m = 1)$, *and a tick size of* $\frac{1}{n}$. *The unique equilibrium* $B_{n,1}(\cdot)$ *is represented by the vector* $\sigma^{B_{n,1}} \in \mathbb{R}^{n+2}$ *where*: $\sigma_0^{B_{n,1}} = 0$, $\sigma_n^{B_{n,1}} = \sigma_{n+1}^{B_{n,1}} = 1$ *and for all* $k = 1, \ldots, n-1$:

$$\sigma_k^{B_{n,1}} = \min\left(\frac{k}{n} + \frac{1}{n} \cdot \frac{G(\frac{k-1}{n})}{G(\frac{k}{n}) - G(\frac{k-1}{n})}, 1\right). \tag{11}$$

The proof of this proposition is immediate. It follows from the fact that for $y = \sigma_k^{B_{n,1}}$, the dealer is indifferent between submitting a quote of $\frac{k}{n}$ and a quote of $\frac{k-1}{n}$. Therefore: $(\sigma_k^{B_{n,1}} - \frac{k}{n})G(\frac{k}{n}) = (\sigma_k^{B_{n,1}} - \frac{k-1}{n})G(\frac{k-1}{n})$. Solving for $\sigma_k^{B_{n,1}}$ yields (11).

Notice that the equilibrium does not depend on F at all. This is so because the dealer maximizes $\pi_{n,1}(\cdot)$ which does not depend on F due to lack of competition. It can also be checked that the equilibrium without a tick size given in (10) is just the limit of the equilibrium in Proposition 2 when the tick size tends to zero.

Let $\eta \geqslant 1$ be an integer and let $\tilde{n} = n\eta$. Observe that $P_{\tilde{n}}$ is a refinement of P_n in the sense that any price that is feasible when the tick size is $\frac{1}{n}$ is also feasible when the tick size is $\frac{1}{\tilde{n}}$. In this section we restrict our attention to this kind of grid refinements, which are fairly prevalent in reality (e.g. a change from a tick size of $\$\frac{1}{8}$ to $\$\frac{1}{16}$). Other kinds of changes in tick size would introduce some pathologies to the analysis.

Our first observation regarding welfare is that refining the grid of prices will *always* improve the position of the monopolistic dealer. This is so because by refining the grid of potential prices we enrich the set of actions he is allowed to take, and since there is no competitive pressure, he uses this richness and his "first mover advantage" to increase his payoff. This observation is formalized in the next proposition.

Proposition 3. *Consider a market with a monopolistic dealer, and let* $\eta > 1$ *be an integer. Suppose* $B_{n,1}(\cdot)$ *and* $B_{\tilde{n},1}(\cdot)$ *are the equilibrium strategies when the tick size is* $\frac{1}{n}$ *and* $\frac{1}{\tilde{n}} \equiv \frac{1}{\eta n}$ *respectively. The expected profits to the dealer given a tick size of* $\frac{1}{\tilde{n}}$ *are larger than when the tick size is* $\frac{1}{n}$. *Namely*: $\Gamma_{n,1}^{\text{dealer}}(B_{n,1}) \leqslant \Gamma_{\tilde{n},1}^{\text{dealer}}(B_{\tilde{n},1})$.

[12] More precisely, the equilibrium is unique up to changes of zero probability. To see this, let $b_1, b_2 \in P_n$, $y_0 \in [0,1]$ and suppose that $(y_0 - b_1)G(b_1) = (y_0 - b_2)G(b_2)$. The support of G is $[0,1]$, therefore: $G(b_1) \neq G(b_2)$ and hence y_0 is the only point in $[0,1]$ for which that equality holds. Since P_n is finite, we obtain that equality holds for at most finitely many points in $[0,1]$. Since F is differentiable, the probability assigned by F to any finite set of points is zero.

Proof. From Eq. (8), the dealer's expected profits are given by:

$$\begin{aligned}
\Gamma_{n,1}^{\text{dealer}}(B_{n,1}) &= \int_{y=0}^{1} \pi_{n,1}\big(B_{n,1}(y), y, B_{n,1}\big) f(y)\,\mathrm{d}y \\
&= \int_{y=0}^{1} \big(y - B_{n,1}(y)\big) G\big(B_{n,1}(y)\big) f(y)\,\mathrm{d}y, \\
\Gamma_{\tilde{n},1}^{\text{dealer}}(B_{\tilde{n},1}) &= \int_{y=0}^{1} \pi_{\tilde{n},1}\big(B_{\tilde{n},1}(y), y, B_{\tilde{n},1}\big) f(y)\,\mathrm{d}y \\
&= \int_{y=0}^{1} \big(y - B_{\tilde{n},1}(y)\big) G\big(B_{\tilde{n},1}(y)\big) f(y)\,\mathrm{d}y.
\end{aligned}$$

Since $P_{\tilde{n}}$ is finer than P_n, $(y - B_{n,1}(y))G(B_{n,1}(y)) \leqslant (y - B_{\tilde{n},1}(y))G(B_{\tilde{n},1}(y))$ for all y. Therefore: $\Gamma_{\tilde{n},1}^{\text{dealer}}(B_{\tilde{n},1}) \geqslant \Gamma_{n,1}^{\text{dealer}}(B_{n,1})$. $\square$

Although this result is almost trivial, it forms the basis for the main theme of this paper. It shows that in the absence of competition, dealers prefer fine grids, since they enable them to make better use of their "first mover advantage." It is, however, necessary to show that this result carries on to markets with a higher number of dealers. We will deal with this issue in Section 6.

In contrast to this unambiguous result, the effect of a decrease in tick size on the investor and on total welfare is ambiguous. To capture this point, notice that the investor's welfare may increase following two kinds of events:

(i) a transfer of welfare from the dealer to the investor, and
(ii) an increase in the probability of a transaction.

If the probability of a transaction were constant, any decrease in the welfare of the dealer would be immediately translated to an increase in the welfare of the investor (a zero sum game). Thus, following Proposition 3, any decrease in tick size would mean a lower welfare for the investor. However, the probability of transaction varies following a change in tick size; it might increase or decrease. Thus, a decrease in tick size in the monopolistic dealer case has an ambiguous effect on the investor and on total welfare. Notice, however, that a decrease in tick size in this case always has a negative effect on the investor because it benefits the dealer. Only in cases where the increase in the probability of trade compensates this negative effect will the investor be better off with a smaller tick size. This implies that in the monopolistic dealer case, investors have a preference for larger ticks. The extent of this preference depends on the distribution of reservation values.

We shall now provide two examples to demonstrate that in this monopolistic setting:

(i) decreasing the tick size might have no effect at all on total welfare, and
(ii) decreasing the tick size might even impair total welfare.

One can also devise examples where decreasing the tick size is welfare improving. Thus, the impact of refining the grid of prices on market welfare is counterintuitively ambiguous.

Example 1. A large tick size might be highly efficient.

Assume that F and G are uniformly distributed over [0, 1], and let n be an even integer. By Proposition 2 it follows that the unique equilibrium $B_{n,1}(\cdot)$ is given by:

$$B_{n,1}(y) = \begin{cases} 0, & 0 \leqslant y < \frac{1}{n}, \\ \frac{k}{n}, & \frac{2k-1}{n} \leqslant y < \frac{2k+1}{n} \text{ for } k = 1, \dots, \frac{n}{2} - 1, \\ \frac{1}{2}, & \frac{n-1}{n} \leqslant y \leqslant 1. \end{cases}$$

Given this equilibrium, we can calculate the welfare measures as follows:[13]

$$\begin{aligned} \Gamma_{n,1}^{\text{dealer}}(B_{n,1}) &= \sum_{k=1}^{n/2-1} \int_{y=\frac{2k-1}{n}}^{\frac{2k+1}{n}} \int_{x=0}^{\frac{k}{n}} \left(y - \frac{k}{n}\right) \mathrm{d}x\, \mathrm{d}y + \int_{y=\frac{n-1}{n}}^{1} \int_{x=0}^{\frac{1}{2}} \left(y - \frac{1}{2}\right) \mathrm{d}x\, \mathrm{d}y \\ &= \frac{1}{12} - \frac{1}{12n^2}, \\ \Gamma_{n,1}^{\text{investor}}(B_{n,1}) &= \sum_{k=1}^{n/2-1} \int_{y=\frac{2k-1}{n}}^{\frac{2k+1}{n}} \int_{x=0}^{\frac{k}{n}} \left(\frac{k}{n} - x\right) \mathrm{d}x\, \mathrm{d}y + \int_{y=\frac{n-1}{n}}^{1} \int_{x=0}^{\frac{1}{2}} \left(\frac{1}{2} - x\right) \mathrm{d}x\, \mathrm{d}y \\ &= \frac{1}{24} + \frac{1}{12n^2}, \\ \Gamma_{n,1}^{\text{total}}(B_{n,1}) &= \Gamma_{n,1}^{\text{dealer}}(B_{n,1}) + \Gamma_{n,1}^{\text{investor}}(B_{n,1}) = \frac{1}{8}. \end{aligned}$$

These calculations suggest that a decrease in tick size increases the gains of the dealer as predicted by Proposition 3, decreases the gains of the investor, and does not affect total gains at all. Indeed, total gains from trade are equal to $\frac{1}{8}$ regardless of the tick size. This result is striking since a very coarse grid (say $n = 2$) yields the same ex ante total gains from trade as any finer grid, which are identical to the ex ante total gains from trade that result in the continuous pricing case ($n = \infty$). The reason for this result is the fact that the probability of trade in this monopolistic, uniform case is equal to $\frac{1}{4}$ regardless of the tick size. Since there is no clustering in a monopoly, a constant probability of trade implies constant total gains from trade.

Example 2. Decreasing the tick size might impair total welfare.

[13] In the following calculations we use the identity: $1^2 + 2^2 + \cdots + r^2 = \frac{r(r+1)(2r+1)}{6}$, for any positive integer r.

Let us keep the assumption that G is uniform and suppose that f is given by[14]:

$$f(y)=\begin{cases}0, & 0\leqslant y<\frac{1}{2},\\ 4, & \frac{1}{2}\leqslant y<\frac{3}{4},\\ 0, & \frac{3}{4}\leqslant y\leqslant 1.\end{cases}$$

Consider the equilibrium strategies in the case $n=2$ and $n=4$, denoted by $B_{2,1}$ and $B_{4,1}$ respectively. Since equilibrium behavior does not depend on F, these strategies are the same as in Example 1. However, F does influence our welfare measures. Calculations show for the case $n=2$ that: $\Gamma_{2,1}^{\text{dealer}}(B_{2,1})=\frac{1}{16}$, $\Gamma_{2,1}^{\text{investor}}(B_{2,1})=\frac{1}{8}$ and $\Gamma_{2,1}^{\text{total}}(B_{2,1})=\frac{3}{16}$. For the case $n=4$ we obtain: $\Gamma_{4,1}^{\text{dealer}}(B_{4,1})=\frac{3}{32}$, $\Gamma_{4,1}^{\text{investor}}(B_{4,1})=\frac{1}{32}$ and $\Gamma_{4,1}^{\text{total}}(B_{2,1})=\frac{1}{8}$. Thus, by moving from a tick size of $\frac{1}{2}$ to a tick size of $\frac{1}{4}$, total ex ante welfare decreases from $\frac{3}{16}$ to $\frac{1}{8}$ (a decrease of 33% in total welfare). In this example, grid refinement decreases the probability of a transaction.[15] This causes a decline in total welfare.

To summarize: in the case of a monopolistic dealer, a decrease in tick size improves the position of the dealer, and has an ambiguous impact on the investor with some preference for coarser grids. In what follows we claim that these welfare patterns hold when m is larger than 1 but is sufficiently small.

6. Discrete prices in competitive markets

Our model features two "levers" that can be pulled: the tick size $\frac{1}{n}$, and the number of dealers m. We first study the effect of an increased number of dealers on bidding strategies given a fixed tick size. Eventually, the number of dealers m depends on the tick size and is determined endogenously. We study this effect in Section 7.

In the next proposition we show that as the number of dealers increases, dealers are forced to submit higher quotes. Actually, when m tends to infinity (perfect competition), any sequence of equilibrium strategies tends to the "honest strategy" H_n. Formally:

Proposition 4. *Assume a tick size of* $\frac{1}{n}$. *Let* $\{B_{n,m}\}_{m=1}^{\infty}$ *be any sequence of non-decreasing equilibrium strategies in markets with* m *dealers, then* $\lim_{m\to\infty} B_{n,m}=H_n$ *almost everywhere on* $[0,1]$.

Thus, although H_n is not an equilibrium, any equilibrium in markets with a high number of dealers would be very close to H_n. Figure 1 demonstrates the convergence process. It presents an equilibrium strategy in the case $n=4$. The shaded areas on the horizontal axis designate types for which the strategy specifies a dishonest quote. According to Proposition 4, the sum of the probabilities of these areas tends to zero as m tends to ∞.

[14] This f violates the assumption that the density function is strictly positive and continuous. However, this can by easily amended by increasing the mass over the zeroed intervals slightly, and decreasing the mass over the interval $[\frac{1}{2},\frac{3}{4})$ accordingly, in a way that will assure continuity as well. Continuity of the efficiency measures implies that our results will keep holding when we use the amended density.

[15] We elaborate more on the probability of transaction in Section 7.2.

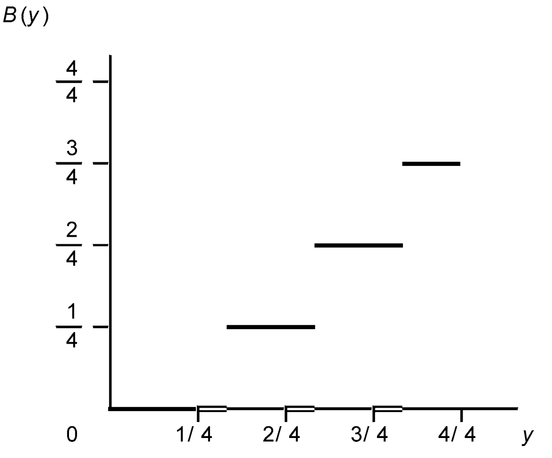

Fig. 1. Equilibrium with a limited level of competition.

In Section 5 we saw that decreasing the tick size in the monopolistic case may cause some odd results. Next we investigate the other limiting case, of decreasing the tick size in a perfectly competitive market. Then we examine the extent to which these results may be applied to markets with just a limited number of dealers. We demonstrate that if the number of dealers is small, the results of the monopolistic case may apply.

Let us fix a tick size of $\frac{1}{n}$, and consider any sequence of equilibrium strategies $\{B_{n,m}\}$ in markets with m dealers. We denote:

$$\Gamma_{n,\infty}^{\text{investor}} \equiv \lim_{m\to\infty} \Gamma_{n,m}^{\text{investor}}(B_{n,m}),$$
$$\Gamma_{n,\infty}^{\text{dealers}} \equiv \lim_{m\to\infty} \Gamma_{n,m}^{\text{dealers}}(B_{n,m}), \quad \text{and} \quad \Gamma_{n,\infty}^{\text{total}} = \Gamma_{n,\infty}^{\text{investor}} + \Gamma_{n,\infty}^{\text{dealers}}.[16]$$

These measures gauge the expected gains from trade to each group and to the market as a whole, when the market becomes perfectly competitive. In the following proposition, we use these measures to study the division of surplus in perfectly competitive markets. Before stating it, the following notation is needed. Denote: $\mu_G \equiv \int_0^1 x\,\mathrm{d}G(x)$. This is the ex ante expected value of the stock to the investor.

Proposition 5. *Consider a market with a tick size of* $\frac{1}{n}$. *Let* $\{B_{n,m}\}$ *be any sequence of non-decreasing equilibrium strategies in markets with* m *dealers. The following holds*[17]:

(1) $$\Gamma_{n,\infty}^{\text{investor}} = 1 - \mu_G - \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

(2) $$\Gamma_{n,\infty}^{\text{dealers}} = \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

[16] In what follows we show that these limits do exist, and thus justify this notation.

[17] Given two functions $f(n)$ and $g(n)$, we say that $f(n)$ is $O(g(n))$ if there exists a constant κ such that for all large enough n, $|f(n)| \leqslant \kappa g(n)$. Thus, a function is $O(\frac{1}{n^2})$ if it converges to zero at least as quickly as $\frac{1}{n^2}$.

(3) $\Gamma^{\text{total}}_{n,\infty} = 1 - \mu_G - \frac{1}{2n} + O\left(\frac{1}{n^2}\right).$

The intuition behind this result is as follows. As the number of dealers increases, they are forced to submit honest quotes. Thus, the winning dealer gets the stock and pays his reservation value minus an amount that is between zero and $\frac{1}{n}$. On average, the winning dealer gets a “discount” of approximately $\frac{1}{2n}$. This discount is the dealers’ last resort, since it is the only source of profits for them in a competitive market. The profits of the dealers are, thus, approximately proportionate to the tick size. It follows that a positive tick size serves as a *commitment device* for dealers, and enables them to shade their bids even in the face of severe competition.

As for the investor, since the number of dealers tends to infinity, the highest quote is likely to be $1 - \frac{1}{n}$. Therefore, the expected gains to the investor are approximately this quote minus the expected reservation value of the investor: $1 - \frac{1}{n} - \mu_G$. Finally, since the dealers gain one half of tick size while the investor loses a whole tick size, decreasing the tick size improves total welfare. This improvement reflects the higher probability of execution following a decrease in tick size in a highly competitive market.

While this result seems plausible, there is a problem in applying it to less then perfectly competitive markets. The reason is that the welfare measures converge gradually to the perfectly competitive outcome. When the number of dealers is small, the welfare measures tend to resemble the monopolistic case rather than the perfectly competitive case. Thus, decreasing the tick size helps the dealers and has an ambiguous effect on the investor and on total welfare. Only for a sufficiently large number of dealers, a decrease in tick size has the same implications as in the perfectly competitive case. The next example illustrates this point.

Example 3. Two dealers with uniformly distributed reservation values.

Suppose $m = 2$, and F and G are uniform. If $n = 2$ the unique equilibrium is given by:

$$B_{2,2}(y) = \begin{cases} 0, & 0 \leqslant y < \frac{1}{2}, \\ \frac{1}{2}, & \frac{1}{2} \leqslant y \leqslant 1. \end{cases}$$

The welfare measures in this case are: $\Gamma^{\text{investor}}_{2,2} = 0.09375$ and $\Gamma^{\text{dealers}}_{2,2} = 0.09375$, thus the investor and the two dealers split the potential gains from trade equally.

If $n = 4$, calculation shows that the unique equilibrium is:

$$B_{4,2}(y) = \begin{cases} 0, & 0 \leqslant y < \frac{1}{4}, \\ \frac{1}{4}, & \frac{1}{4} \leqslant y < 0.5897, \\ \frac{2}{4}, & 0.5897 \leqslant y \leqslant 1. \end{cases}$$

The welfare measures in this case are: $\Gamma^{\text{investor}}_{4,2} = 0.0904$ and $\Gamma^{\text{dealers}}_{4,2} = 0.1083$.

We see that the decline in tick size decreases the profit of the investor by 3.6%, and increases the profits of the dealers by 15.5%. Thus, despite the competition between the two dealers, decreasing the tick size is beneficial for them, and hurts the investor.

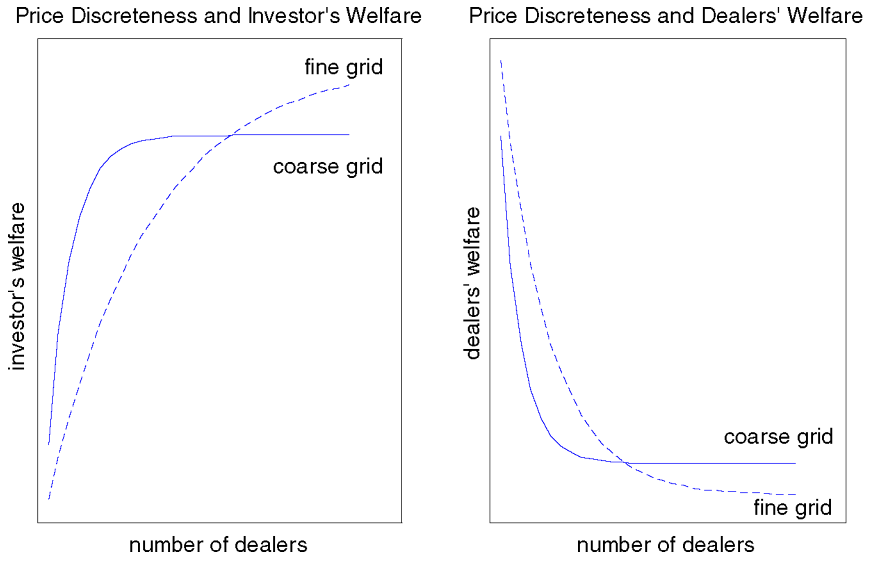

Fig. 2. Welfare implications of competition among dealers.

Figure 2 illustrates the gradual move of welfare measures from the monopolistic case to the fully competitive case. It shows that for a small number of dealers, decreasing the tick size helps the dealers and tends to hurt the investor.[18] By contrast, when the number of dealers is high, decreasing the tick size hurts the dealers and improves the welfare of the investor.

7. Empirical implications and market design

In this section we study the implications of our model on the cross-sectional effects of a decrease in tick size. We analyze the effect of a decrease in tick size on the number of dealers, on the probability of transactions, and on the aggressiveness of the quotes. Finally, as an application to market design, we look for an "optimal tick size."[19]

7.1. Tick size and the number of dealers

So far we have assumed that the number of dealers is given exogenously. However, the tick size is a determinant of dealers' profits. It follows that the number of dealers itself is affected by the tick size.

In order to study entry and exit of dealers, assume that each dealer incurs a transaction cost $C > 0$ for each submission of quote. This cost reflects variable as well as fixed costs

[18] As described in Section 4, decreasing the tick size has an ambiguous effect on the investor when competition is lax. The most likely scenario, however, is that a decrease in tick size would hurt the investor. Figure 2 demonstrates this most likely scenario.

[19] I am grateful to an anonymous referee for motivating much of the discussion in this section.

associated with supplying liquidity. The equilibrium number of dealers is determined such that there is no incentive for dealers to enter or exit the market. Thus, the equilibrium number of dealers is the largest positive integer m^* such that the per-dealer profits exceed C:

$$\frac{1}{m^*}\Gamma^{\text{dealers}}_{n,m^*}(B_{n,m^*}) \geqslant C.$$

We shall assume that C is small enough such that a monopolistic dealer finds it worthwhile to participate in the market, namely: $\Gamma^{\text{dealers}}_{n,1}(B_{n,1}) \geqslant C$. As the number of dealers, m, tends to infinity, the profit to the dealers as a group is on the order of $\frac{1}{2n}$ (Proposition 5). However, the profit per dealer is on the order of $\frac{1}{2nm}$, and hence tends to 0. It follows that when the number of dealers becomes too large, providing liquidity becomes unprofitable, regardless of the tick size. We conclude that there exists a number of dealers $m^* \geqslant 1$, that makes liquidity provision just marginally profitable, in the sense that any additional dealer would make the provision of liquidity unprofitable. This m^* is the equilibrium number of dealers. Thus, when n is sufficiently large, we find that the equilibrium number of dealers is given approximately by: $m^* \approx \frac{1}{2nC}$. If $\frac{1}{2nC}$ is relatively large, the current number of dealers will be large. Then, a decrease in tick size (increase in n) will cause some dealers to exit the market.

By contrast, if the current equilibrium number of dealers is small, a decrease in tick size will cause exactly the opposite effect. Recall that when the number of dealers is small, a decrease in tick size increases the profit to the dealers. For this reason, a decrease in tick size in this kind of a market will attract more dealers and m^* will increase.

To demonstrate the last argument, consider the following example.

Example 4. A decrease in tick size induces a market to attract more dealers when the current number of dealers is small.

Assume that F and G are uniformly distributed over $[0, 1]$, and suppose that the entry cost is $C = 0.05$. Suppose first that $n = 2$. From Example 1 we know that if $m = 1$ then $\Gamma^{\text{dealers}}_{2,1}(B_{2,1}) = 0.0625 > C$. If $m = 2$ we get from Example 3 that the per dealer expected profits are $\frac{1}{2}\Gamma^{\text{dealers}}_{2,2} = 0.5 \cdot 0.09375 = 0.046875 < C$. It follows that the equilibrium number of dealers in this case is $m^* = 1$ (a monopoly). Suppose now that the tick size is reduced to $\$\frac{1}{4}$. From Example 1 we have that in a monopoly: $\Gamma^{\text{dealers}}_{4,1}(B_{4,1}) = 0.078125 > C$. While, if the number of dealers is 2, we have from Example 3 that the per dealer profit is: $\frac{1}{2}\Gamma^{\text{dealers}}_{4,2} = 0.5 \cdot 0.1083 = 0.5415 > C$. This implies that when the tick size has decreased to $\$\frac{1}{4}$, the equilibrium number of dealers has increased to $m^* = 2$.

In summary, a decrease in tick size is expected to cause a decrease in the number of dealers in stocks that currently have many dealers. By contrast, stocks that have a small number of dealers are expected to attract more dealers following a decrease in tick size. Thus, our model predicts that following a decrease in tick size, some dealers would migrate from highly liquid stocks, where the competition among dealers is intense, to less liquid stocks.

7.2. *Tick size and transaction probability*

One of the implications of changes in tick size is on the probability that a transaction will take place. Intuitively, the tick size seems to be a friction that prohibits transactions from taking place. Hence, it appears that a decrease in tick size would increase the probability of transactions. Again, this intuition is only correct in markets with a large number of dealers. To see this, let us fix m and n and let $B_{n,m}$ be an equilibrium strategy. The probability of a transaction is given by:

$$\begin{aligned} P(B_{n,m}) &\equiv \int\limits_{y=0}^{1} G\big(B_{n,m}(y)\big)\,\mathrm{d}F_{(m)}(y) \\ &= \sum_{k=1}^{n-1} G\left(\frac{k}{n}\right)\Big[\big(F\big(\sigma_{k+1}^{B_{n,m}}\big)\big)^m - \big(F\big(\sigma_{k}^{B_{n,m}}\big)\big)^m\Big]. \end{aligned} \tag{12}$$

When the number of dealers is very large, the winning dealer typically posts a quote of $1-\frac{1}{n}$, and hence the probability of a transaction is $G(1-\frac{1}{n})$. For this reason, when the number of dealers tends to infinity, the probability of a transaction is higher when the tick size is lower. In fact, as the number of dealers increases, the probability of *no transaction*, $1-P(B_{n,m})$, is on the order of $\frac{1}{n}$, the tick size itself. Formally:

Proposition 6. *Assume a tick size of $\frac{1}{n}$. Let $\{B_{n,m}\}_{m=1}^{\infty}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers, then* $\lim_{m\to\infty}(1-P(B_{n,m})) = O(\frac{1}{n})$.

We conclude that when the number of dealers in a specific market is large, decreasing the tick size increases the likelihood of transactions.

When the number of dealers is small, the situation is less straightforward, and a decrease in tick size might increase or decrease the probability of a transaction. For instance, using (12) we find that in Example 2 the probability of a transaction is $\frac{1}{2}$ when the tick size is $\$\frac{1}{2}$, while it declines to $\frac{1}{4}$ when the tick size is $\$\frac{1}{4}$. In Example 1, the probability of a transaction is $\frac{1}{4}$ regardless of the tick size. Other examples can be found such that the probability of a transaction will increase following a decrease in tick size even in a monopoly. Thus, the impact of a decrease in tick size on the probability of a transaction when the number of dealers is small is ambiguous.

Since a higher probability of a transaction contributes to a higher rate of transactions, our findings suggest that a decrease in tick size will tend to increase the transaction rate in markets with a high number of dealers, while it will have an ambiguous effect on the transaction rate in markets with a small number of dealers. We are not aware of empirical evidence related to this prediction.

7.3. *Tick size and order aggressiveness*

A dealer acts aggressively if he shades his bid in a minor way. Thus, buying dealers act aggressively if their quotes are high, and selling dealers are aggressive if their quotes are low. Taken together, an aggressive strategy will induce a small bid–ask spread in the security market. It is hard to measure the bid–ask spread in our model directly, but order aggressiveness can still be studied.

For a given equilibrium strategy $B_{n,m}$ we measure the expected aggressiveness of the equilibrium as the average quote of the highest bidder. We denote this expected aggressiveness by $\Omega(B_{n,m})$. Intuitively, when $\Omega(B_{n,m})$ is high, buying dealers shade their quotes downward only mildly. In a symmetric way, selling dealers would shade their quotes upward only mildly, and the bid–ask spread would be small. Formally:

$$\Omega(B_{n,m}) \equiv \int_0^1 B_{n,m}(y)\,\mathrm{d}F_{(m)}(y) = \sum_{k=1}^{n-1} \frac{k}{n}\Big[\big(F\big(\sigma_{k+1}^{B_{n,m}}\big)\big)^m - \big(F\big(\sigma_k^{B_{n,m}}\big)\big)^m\Big]. \tag{13}$$

When the number of dealers is high, each dealer tends to bid "almost honestly." Thus, competition between dealers induces them to submit aggressive quotes. The tick size in this case limits the extent of aggressiveness of dealers. For this reason, when the number of dealers is large, decreasing the tick size tends to increase the aggressiveness of dealers. To see this point formally, we study the effect of the tick size when the number of dealers tends to infinity.

Proposition 7. *Assume a tick size of $\frac{1}{n}$. Let $\{B_{n,m}\}_{m=1}^{\infty}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers, then* $\lim_{m\to\infty} \Omega(B_{n,m}) = 1 - \frac{1}{n}$.

When the number of dealers is small, this result does not hold. In particular, reducing the tick size might make dealers more or less aggressive. To demonstrate this, consider again Example 2. When the tick size is $\$\frac{1}{2}$ the expected aggressiveness is $\frac{1}{2}$, while when the tick size is $\$\frac{1}{4}$ the expected aggressiveness is just $\frac{1}{4}$.

In summary, when the number of dealers is large, a decrease in tick size is expected to decrease the bid–ask spread because it increases the aggressiveness of quotes. When the number of dealers is small, a decrease in tick size will have an ambiguous effect on dealers' aggressiveness. This prediction is consistent with the findings of Christie et al. (2002).

7.4. Optimal tick size

It is natural to ask whether our model yields a definite "optimal" tick size, namely a tick size that maximizes the total welfare in the economy (or perhaps the welfare of the investors). The model is successful in yielding such a definite answer in one case only: if the number of dealers in a specific stock is very large, a very small tick size will maximize the total welfare as well as the welfare of investors (Proposition 5). For markets with just a small number of dealers, our model does not yield a definite optimal tick size. We have shown that when the number of dealers in a specific stock is small, a sufficiently small tick size might transfer welfare gains from the investors to the dealers, whereas too large a tick size might adversely affect the probability of a transaction.

A normative implication of these observations is that as the number of dealers increases (for exogenous reasons), the tick size should be reduced. This is because the larger is the number of dealers, the closer is the market to the perfectly competitive outcome. Conversely, if the number of dealers decreases, market designers should generally increase the tick size.[20]

For a fixed number of dealers, a decrease in tick size might have a detrimental effect on investors' welfare. In order to further explore this point, we study the rate of convergence of markets to the competitive outcome. Namely, how far is the market from the perfectly competitive case given a finite number of dealers? We show below that this rate decreases as the grid of prices becomes finer. Thus, when we decrease the tick size, a higher number of dealers is needed to ensure behavior that is similar to the perfectly competitive case. Figure 3 demonstrates this finding. This figure presents the investor's welfare given three levels of price discreteness, and increasing numbers of dealers. The solid thick line represents the welfare of the investor given a coarse grid, the solid thin line represents the investor's welfare given a finer grid, while the dashed line represents the investor's welfare given a very fine grid. A finer grid implies a higher investor's welfare when we approach an infinite number of dealers. However, it is apparent that convergence to this "perfect competition" welfare is slower the finer is the grid. Now, suppose the current grid is coarse (solid thick line), and the market designer contemplates moving to a fine grid (solid thin

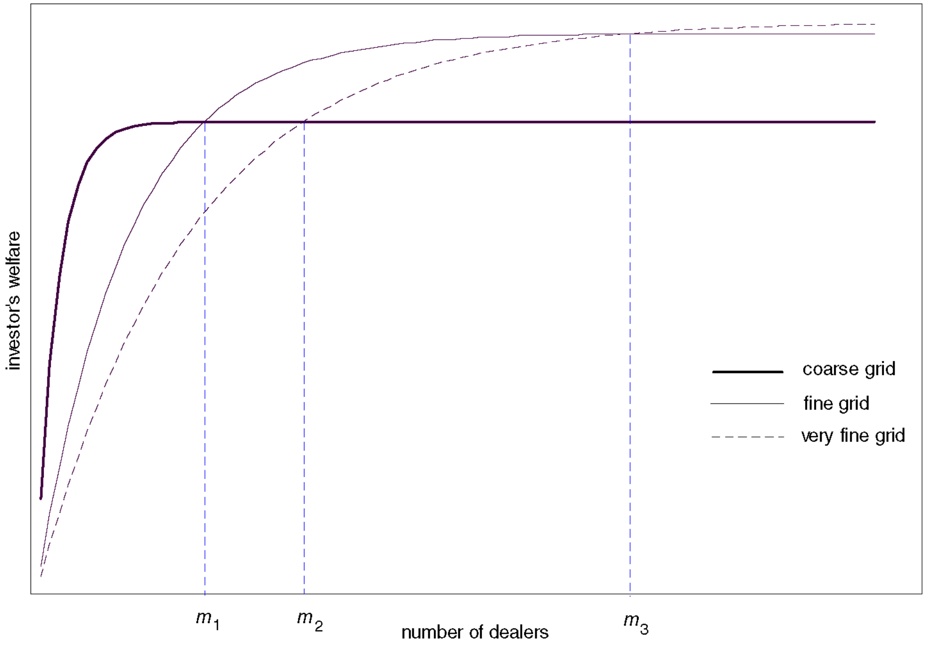

Fig. 3. The impact of the rate of convergence on investor's welfare.

[20] A caveat is in place. We are not able to derive a monotone relation between an "optimal tick size" and the number of dealers. Rather, our normative recommendation is based on the analysis of two extreme cases—the monopoly and the perfectly competitive case, and how close the actual number of dealers is to these two extreme cases.

line) or to a very fine grid (dashed line). If the number of dealers is smaller than m_1, decreasing the tick size to any one of the two alternatives would hurt the investor. If the number of dealers is between m_1 and m_3, moving to the fine grid dominates moving to the very fine grid in terms of investor's welfare. Only when the number of dealers is larger than m_3, moving to the very fine grid would be better for the investor. If the number of dealers is below m_2, moving from the coarse grid to the very fine grid actually impairs investor's welfare. Thus, a sharp decline in tick size might be harmful to investors, if the number of dealers is not sufficiently large.

The next proposition formalizes this idea by studying the rate of convergence to the competitive outcome.

Proposition 8. *Consider a market with a tick size of $\frac{1}{n}$, and let $\{B_{n,m}\}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers. The following holds*:

(1) $$\Gamma^{\text{investor}}_{n,\infty} - \Gamma^{\text{investor}}_{n,m}(B_{n,m}) = O\left(\left(F\left(1-\frac{1}{n}\right)\right)^m\right),$$

(2) $$\Gamma^{\text{dealers}}_{n,m}(B_{n,m}) - \Gamma^{\text{dealers}}_{n,\infty} = O\left(\left(F\left(1-\frac{1}{n}\right)\right)^m\right),$$

(3) $$\Gamma^{\text{total}}_{n,m}(B_{n,m}) - \Gamma^{\text{total}}_{n,\infty} = O\left(\left(F\left(1-\frac{1}{n}\right)\right)^m\right).$$

The intuition behind this result is as follows. As the number of dealers increases, it becomes increasingly likely that at least one of the dealers has a reservation value that is higher than $1-\frac{1}{n}$. By Proposition 4, $\sigma^{B_{n,m}}_{n-1} \to 1-\frac{1}{n}$, therefore it becomes highly likely that the winning quote will be $1-\frac{1}{n}$. Given m dealers, the probability that this is not the case is approximately $(F(1-\frac{1}{n}))^m$. Thus, the difference between the welfare measures in the perfectly competitive case and those measures given a finite number of dealers is approximately the probability that a quote smaller than $1-\frac{1}{n}$ will win the stock. This probability increases when the tick size becomes smaller.

For $n_2 > n_1$, the difference between $O((F(1-\frac{1}{n_1}))^m)$ and $O((F(1-\frac{1}{n_2}))^m)$ is large when m is small, and when the difference between n_1 and n_2 is large. Therefore, we would expect a detrimental effect on investors' welfare following a decrease in tick size in markets where: (i) the number of dealers is small; and (ii) the decrease in tick size is sharp.

8. Discussion and conclusion

We have investigated the relation between price discreteness and the number of dealers in a dealer market. Our analysis suggests that the welfare effects of a change in tick size are affected by the number of dealers in the market. When the number of dealers is small, dealers prefer fine price grids, while when the number of dealers is large they prefer coarse price grids (large tick size). When the number of dealers is small, investors' attitude toward the tick size is ambiguous with some preference toward larger ticks. When the number of dealers is large, investors prefer fine price grids. This observation yields several empirical implications. The number of dealers will decline following a decrease in tick size in markets with a large number of dealers. This number will increase in markets with a small number of dealers. The probability of transaction and hence the transaction rate will increase following a decrease in tick size in markets with many dealers. In markets with a small number of dealers, this effect is ambiguous. Finally, a decrease in tick size makes dealers more aggressive, implying a decline in bid–ask spreads in markets with a large number of dealers. This effect is ambiguous when the number of dealers is small.

In our view, these results should be useful to market designers. When they decide on the tick size for a specific stock, they should take into account the number of dealers for this stock. A very fine grid of prices for not highly competitive stocks enables the dealers to better exploit their market power, and extract a larger portion of the potential gains from trade.

While our model refers to dealer markets only, we claim that our main results and intuition carry over to other market mechanisms in which liquidity suppliers compete using limit orders for anticipated market orders (such as a limit order book). The role of dealers in our model should be played by liquidity suppliers (limit order submitters), and the role of investors should be played by liquidity demanders (market order submitters). The important feature needed in order to facilitate our analysis is that liquidity suppliers act before the liquidity demanders by submitting limit orders. This "first mover advantage" given to liquidity suppliers is sufficient to create the interaction between price discreteness and competition.

The intuition of our model also carries over to other selling mechanisms that are not directly related to financial markets, such as internet auctions. For example, e-Bay—the largest auctions site on the Web, induces automatic bid increments of approximately 2% of the prevailing highest bid.[21] The dealers in our model are equivalent to buyers in an auction and the investor is identified with the seller. The tick size is then set either by the seller or by the auctions web site. Our results indicate that in auctions that attract only a small number of buyers, sellers should mandate a positive and non negligible tick size. By contrast, highly popular auctions that attract many buyers should feature no tick size at all or a very small tick size.

In order to simplify the model, we assumed that dealers' valuations are statistically independent and that valuations are private. We claim that our main results are robust to these assumption. To see this, consider first the monopolistic dealer case. In this case, it is obvious that a high tick size restrains the monopoly, regardless of the structure of valuations. Thus, our insight about this case is intact, regardless of the specific assumption about dealers' valuations.

Consider now the perfectly competitive case. Reny and Perry (2003) provide a convergence result parallel to our Proposition 4 in a setting that allows for common components in valuation and correlation in valuations. They show that convergence to truth telling applies also in this case. Moreover, our results regarding the perfectly competitive market stand in line with the extant literature about discrete prices, that assumes zero profit up-front.

[21] See http://pages.ebay.com/help/buy/bid-increments.html. The second largest auctions website: Yahoo Auctions uses similar bid increments. See http://help.yahoo.com/help/us/auct/abid/abid-05.html.

Given that dealers have opposite attitudes toward the tick size in the monopolistic and in the perfectly competitive case, it must be that their attitude changes starting from some level of competition that depends on the current tick size. Thus, markets with a small number of dealers should resemble the monopolistic case while markets with a large number of dealers should resemble the perfectly competitive case. By imposing restrictions on the structure of valuations we created a relatively simple and tractable model. The discussion above suggests that the intuition underlying our main results is robust to relaxing some of our assumptions.

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Appendix A

Proof of Proposition 1. The following definition based on Milgrom and Shannon (1994) is needed first.

Definition 1. The function $h:\mathbb{R}^2 \to \mathbb{R}$ satisfies the Milgrom–Shannon single crossing property (SCP) in (x, θ) if, for all $x_H > x_L$ and all $\theta_H > \theta_L$, $h(x_H, \theta_L) \geqslant (>)\ h(x_L, \theta_L)$ implies $h(x_H, \theta_H) \geqslant (>)\ h(x_L, \theta_H)$.

Consider now our game of incomplete information with m dealers and a tick size of $\frac{1}{n}$. Let $(B_1, \ldots, B_m)$ be an m-tuple of strategies. We denote by $\pi_i(b_i, y_i, B_{-i})$ the expected profit to a dealer $i \in \{1, \ldots, m\}$ whose value is $y_i \in [0, 1]$, when he submits a quote of $b_i \in P_n$ and the $m-1$ other dealers use the strategies $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_m$.

Proposition (*Adapted from* Athey, 2001). *Suppose that given any m-tuple of non-decreasing strategies $B_i:[0, 1] \to P_n$, $\pi_i(b_i, y_i, B_{-i})$ satisfies the Milgrom–Shannon single crossing property in (b_i, y_i) for all $i = 1, \ldots, m$. Then there exists a non-decreasing pure strategy Nash equilibrium, i.e. an m-tuple $(B_1^*, \ldots, B_m^*)$ such that for all $y_i \in [0, 1]$, $B_i^*(y_i) \in \arg\max_{b \in P_n} \pi_i(b_i, y_i, B_{-i}^*)$, for all $i = 1, \ldots, m$.*

In order to accomplish our existence result we must show that in symmetric games Athey's result can be adapted to prove existence of symmetric equilibria, i.e. equilibria that satisfy $B_1 = B_1 = \cdots = B_m \equiv B$. Furthermore, we must show that our objective function satisfies the Milgrom–Shannon single crossing property given that all dealers use non-decreasing strategies. The former claim is standard, and we omit the proof.[22] For the latter claim, in the following lemma we show that the single crossing property is satisfied in this model no matter what strategies are used by other dealers.

Lemma 1. *For any given dealers' strategies* $(B_1, \ldots, B_m)$, $\pi_i(b_i, y_i, B_{-i})$ *satisfies the Milgrom–Shannon single crossing property for all* $i = 1, \ldots, m$.

Proof. Let $b_H, b_L \in P_n$, $y_H, y_L \in [0, 1]$ and suppose $b_H > b_L$, $y_H > y_L$. Let $i \in \{1, \ldots, m\}$ and assume:

$$\pi_i\left(b_H, y_L, B^{-i}\right) \geqslant \pi_i\left(b_L, y_L, B^{-i}\right). \tag{A.1}$$

Suppose on the contrary that

$$\pi_i\left(b_H, y_H, B^{-i}\right) < \pi_i\left(b_L, y_H, B^{-i}\right). \tag{A.2}$$

We denote:

$$\Phi_i(b, B_{-i}) = \Pr\left\{\begin{array}{l}\text{A transaction occurs, and dealer } i \text{ wins the stock} \\ \text{given that he bids } b \text{ and the other dealers use} \\ \text{the strategies } B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_m.\end{array}\right\}$$

Notice that $\Phi_i(b, B_{-i})$ is non-decreasing in b no matter what kind of strategies are used by the other $m - 1$ dealers.

Equations (A.1) and (A.2) may be rephrased as follows:

$$(y_L - b_H)\Phi_i(b_H, B_{-i}) \geqslant (y_L - b_L)\Phi_i(b_L, B_{-i}),$$
$$(y_H - b_H)\Phi_i(b_H, B_{-i}) < (y_H - b_L)\Phi_i(b_L, B_{-i}).$$

But these two inequalities together imply that: $\Phi_i(b_H, B_{-i}) < \Phi_i(b_L, B_{-i})$—a contradiction. A similar argument is used to show that the strict inequalities part of the single crossing property definition is satisfied. □

Proof of Proposition 4. The following four lemmas are required for the proof.

Lemma 2. *Let* $\{a_m\}, \{b_m\}, \{c_m\}$ *and* $\{d_m\}$ *be any convergent sequences of real numbers such that for all* $m = 1, 2, 3, \ldots$: $1 \geqslant a_m > b_m \geqslant c_m > d_m \geqslant 0$, *and denote the limits of these sequences by* a, b, c, *and* d *respectively. Then the following holds*:

(1) *If* $1 > a > b$ *then*

$$\frac{(a_m)^m - (b_m)^m}{a_m - b_m} \to 0.$$

[22] Proving that a symmetric equilibrium exists in this case can be done using a method originally introduced by John Nash in his dissertation. A formal proof for our case is available upon request.

(2) *If* $b \geqslant c > d$ *then*

$$\frac{(c_m)^m - (d_m)^m}{c_m - d_m} \Big/ \left(m \cdot (b_m)^{m-1}\right) = O\left(\frac{1}{m}\right).$$

(3) *If* $a \geqslant b$ *then the sequence*

$$m \cdot (b_m)^{m-1} \Big/ \frac{(a_m)^m - (b_m)^m}{a_m - b_m}$$

is bounded from above by 1.

(4) *If* $a \geqslant b \geqslant c > d$ *then*

$$\frac{(c_m)^m - (d_m)^m}{c_m - d_m} \Big/ \frac{(a_m)^m - (b_m)^m}{a_m - b_m} = O\left(\frac{1}{m}\right).$$

Proof. (1) Let $\varepsilon > 0$ satisfy $a < 1-\varepsilon$. For m large enough we have: $a_m < 1-\varepsilon$, and therefore: $0 \leqslant (a_m)^m < (1-\varepsilon)^m$. Hence by the sandwich rule $(a_m)^m \to 0$. The same argument shows that $(b_m)^m \to 0$, and since $a > b$ we are done.

(2)

$$\frac{(c_m)^m - (d_m)^m}{c_m - d_m} \Big/ \left(m \cdot (b_m)^{m-1}\right) = \frac{1}{m} \cdot \frac{b_m}{c_m - d_m} \cdot \left(\left(\frac{c_m}{b_m}\right)^m - \left(\frac{d_m}{b_m}\right)^m\right).$$

Now, $(\frac{c_m}{b_m})^m$ is bounded from above by 1, and $(\frac{d_m}{b_m})^m$ tends to zero by the argument of part (1) of this lemma. This establishes the required result.

(3) For any $m = 1, 2, \ldots,$ the following holds:

$$\begin{aligned} m \cdot (b_m)^{m-1} \Big/ \frac{(a_m)^m - (b_m)^m}{a_m - b_m} &= m \cdot (b_m)^{m-1} \Big/ \sum_{i=1}^{m} (a_m)^{m-i} (b_m)^{i-1} \\ &= m \Big/ \sum_{i=1}^{m} \left(\frac{a_m}{b_m}\right)^{m-i}. \end{aligned}$$

Since $a_m > b_m$, each term of the sum in the denominator is greater than or equal to 1; hence the denominator is greater than m. Thus, for each m this expression is bounded from above by 1 as required.

(4) For any $m = 1, 2, 3, \ldots,$ write:

$$\begin{aligned} &\frac{(c_m)^m - (d_m)^m}{c_m - d_m} \Big/ \frac{(a_m)^m - (b_m)^m}{a_m - b_m} \\ &\quad = \left(\frac{(c_m)^m - (d_m)^m}{c_m - d_m} \Big/ \left(m \cdot (b_m)^{m-1}\right)\right) \cdot \left(m \cdot (b_m)^{m-1} \Big/ \frac{(a_m)^m - (b_m)^m}{a_m - b_m}\right). \end{aligned}$$

The result follows now from an application of parts (2) and (3) of this lemma. □

Lemma 3. *Consider a market with a tick size of* $\frac{1}{n}$. *Suppose there exists a sequence of non-decreasing equilibrium strategies* $\{B_{n,m}\}_{m=1}^{\infty}$ *in markets with* m *dealers that converges to a non-decreasing strategy* B_n *almost everywhere on* $[0,1]$. *Then* $\Delta_n(B_{n,m}) = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$ *for all* m *large enough.*

Proof. Suppose on the contrary that there exists an $h \in \{0, \dots, n-1\}$ and a subsequence $\{m_t\}_{t=1}^{\infty}$ such that $\frac{h}{n} \notin \Delta_n(B_{n,m_t})$ for all $t = 1, 2, \dots$. Clearly[23] $h > 0$. For $k \in \{0, \dots, h-1\}$ define: $Q_k \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n})\colon B_n(y) = \frac{k}{n}\}$. There exists a $k \in \{0, \dots, h-1\}$ such that $\Pr(Q_k) > 0$. It follows that there exists a number t_0 such that for $t \geqslant t_0$ there exists a set of types $Q_{k,t} \subset (\frac{h}{n}, \frac{h+1}{n})$ such that $\Pr(Q_{k,t}) > 0$, and $B_{n,m_t}(y) = \frac{k}{n}$ for all $y \in Q_{k,t}$. Since B_n and $\{B_{n,m_t}\}_{t=t_0}^{\infty}$ are non-decreasing, we have that Q_k and $\{Q_{k,m_t}\}_{t=t_0}^{\infty}$ are all in the form of an interval contained in $(\frac{h}{n}, \frac{h+1}{n})$. Therefore, by choosing t_0 large enough we have that $S_k \equiv \bigcap_{t=t_0}^{\infty} Q_{k,t}$ is of positive probability.[24] Since B_{n,m_t} is an equilibrium, it follows from Eqs. (5) and (6) that for all $t \geqslant t_0$ and $y \in S_k$:

$$\frac{1}{m_t} G\left(\frac{k}{n}\right)\left(y - \frac{k}{n}\right) \frac{(F(\sigma_{k+1}^{B_{n,m_t}}))^{m_t} - (F(\sigma_k^{B_{n,m_t}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m_t}}) - F(\sigma_k^{B_{n,m_t}})}$$
$$\geqslant G\left(\frac{h}{n}\right)\left(y - \frac{h}{n}\right)\left(F\left(\sigma_h^{B_{n,m_t}}\right)\right)^{m_t - 1}.$$

Rearranging we obtain:

$$\frac{G(\frac{k}{n})(y - \frac{k}{n})}{G(\frac{h}{n})(y - \frac{h}{n})} \geqslant m_t\left(F\left(\sigma_h^{B_{n,m_t}}\right)\right)^{m_t - 1} \Big/ \frac{F((\sigma_{k+1}^{B_{n,m_t}}))^{m_t} - (F(\sigma_k^{B_{n,m_t}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m_t}}) - F(\sigma_k^{B_{n,m_t}})}$$
$$\text{for all } t \geqslant t_0.$$

The l.h.s. does not depend on t, however by Lemma 2 part (2), the r.h.s. tends to infinity as t increases. Thus, this inequality cannot hold for large enough t—a contradiction. □

Lemma 4. *Consider a market with a tick size of $\frac{1}{n}$. Suppose there exists a sequence of non-decreasing equilibrium strategies $\{B_{n,m}\}_{m=1}^{\infty}$ in markets with m dealers, that converges to a non-decreasing strategy B_n almost everywhere on* $[0, 1]$. *Then* $\Delta_n(B_n) = \{0, \frac{1}{n}, \dots, \frac{n-1}{n}\}$.

Proof. Suppose on the contrary that there exists an $h \in \{0, 1, \dots, n-1\}$, such that $\frac{h}{n} \notin \Delta_n(B_n)$. By Lemma 3 we may assume that $\frac{h}{n} \in \Delta_n(B_{n,m})$ for all $m = 1, 2, 3, \dots$. This implies that $\sigma_{h+1}^{B_{n,m}} > \sigma_h^{B_{n,m}}$ for all m, however $\sigma_{h+1}^{B_n} \equiv \lim_{m\to\infty} \sigma_{h+1}^{B_{n,m}} = \lim_{m\to\infty} \sigma_h^{B_{n,m}} \equiv \sigma_h^{B_n}$. We conclude that there exists a $k \in \{0, \dots, h-1\}$ such that the set: $Q_k \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n})\colon B_n(y) = \frac{k}{n}\}$ is of positive probability. It follows that there exists a number m_0 such that for $m \geqslant m_0$ the sets $Q_{k,m} \equiv \{y \in (\frac{h}{n}, \frac{h+1}{n})\colon B_{n,m}(y) = \frac{k}{n}\}$ are of positive probability. Furthermore, by choosing m_0 large enough, the set $T_k \equiv \bigcap_{m=m_0}^{\infty} Q_{k,m}$ is of positive probability.

[23] Any equilibrium strategy assumes 0 with positive probability.

[24] Denote $\xi_k = \inf Q_k$, and $\zeta_k = \sup Q_k$. Since $\Pr(Q_k) > 0$, we have: $\xi_k < \zeta_k$. Also, notice that $\lim_{t\to\infty} Q_{k,t} = Q_k$ (set convergence). Now, by choosing t_0 large enough and a sufficiently small $\varepsilon > 0$, we have: $Q_{k,t} \supset (\xi_k + \varepsilon, \zeta_k - \varepsilon)$ for all $t \geqslant t_0$. Therefore: $\bigcap_{t=t_0}^{\infty} Q_{k,t}$ is of positive probability.

All the $B_{n,m}$ are equilibrium strategies, and by Lemma 3 both $\frac{k}{n}$ and $\frac{h}{n}$ belong to $\Delta_n(B_{n,m})$. It follows that for all $y \in T_k$ and $m \geqslant m_0$:

$$\begin{aligned}&\frac{1}{m}G\left(\frac{k}{n}\right)\left(y-\frac{k}{n}\right)\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}\\&\quad\geqslant\frac{1}{m}G\left(\frac{h}{n}\right)\left(y-\frac{h}{n}\right)\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m-(F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}})-F(\sigma_h^{B_{n,m}})},\end{aligned}$$

or equivalently:

$$\frac{G(\frac{k}{n})(y-\frac{k}{n})}{G(\frac{h}{n})(y-\frac{h}{n})}\geqslant\frac{(F(\sigma_{h+1}^{B_{n,m}}))^m-(F(\sigma_h^{Bn,m}))^m}{F(\sigma_{h+1}^{B_{n,m}})-F(\sigma_h^{B_{n,m}})}\bigg/\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{Bn,m}))^m}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}.$$

The l.h.s. of this inequality does not depend on m while the r.h.s. tends to infinity by Lemma 2 part (4). This yields a contradiction. □

Lemma 5. *Consider a market with a tick size of $\frac{1}{n}$. Suppose there exists a sequence of non-decreasing equilibrium strategies $\{B_{n,m}\}_{m=1}^{\infty}$ in markets of m dealers, that converges to a non-decreasing strategy B_n almost everywhere on* $[0,1]$. *Then $B_n = H_n$ almost everywhere on* $[0,1]$.

Proof. In order to show that $B_n = H_n$ almost everywhere, we will show that $\sigma_k^{B_n} \equiv \lim_{m\to\infty}\sigma_k^{B_{n,m}} = \frac{k}{n}$ for all $k \in \{0,1,\dots,n-1\}$. First, notice that since each $B_{n,m}$ is an equilibrium we have $\sigma_k^{B_{n,m}} \geqslant \frac{k}{n}$ for all $k = 1,\dots,n-1$ and $m = 1,2,\dots$. By moving to the limit we obtain: $\sigma_k^{B_n} \geqslant \frac{k}{n}$ for all $k = 1,\dots,n-1$. Therefore, we only have to show that a strict inequality is impossible.

Let $k \in \{0,1,\dots,n-1\}$, and suppose on the contrary that $\sigma_k^{B_n} > \frac{k}{n}$. It follows that there exists a number m_0 such that for all $m \geqslant m_0$, $\sigma_k^{B_{n,m}} > \frac{k}{n}$. From Lemma 4 it follows that $\frac{k}{n} \in \Delta_n(B_n)$. This implies that there exists an $h \in \{k+1,\dots,n\}$, such the set $Q \equiv \{y \in (\frac{h}{n},\frac{h+1}{n})\colon B_n(y) = \frac{k}{n}\}$ is of positive probability. This in turn implies that if we choose m_0 large enough the sets $Q_{k,m} \equiv \{y \in (\frac{h}{n},\frac{h+1}{n})\colon B_{n,m}(y) = \frac{k}{n}\}$ are of positive probability for all $m \geqslant m_0$, and furthermore the set $T_k \equiv \bigcap_{m=m_0}^{\infty} Q_{k,m}$ is of positive probability. Applying Lemma 4 again, we conclude that $\frac{h}{n} \in \Delta_n(B_n)$, hence by choosing m_0 large enough we may assume that it is attained with positive probability also by $B_{n,m}$ for all $m \geqslant m_0$.

Since $B_{n,m}$ is an equilibrium, and both $\frac{k}{n}$ and $\frac{h}{n}$ appear with positive probability, we may write for all $y \in T_k$ and $m \geqslant m_0$:

$$\begin{aligned}&\frac{1}{m}G\left(\frac{k}{n}\right)\left(y-\frac{k}{n}\right)\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}\\&\quad\geqslant\frac{1}{m}G\left(\frac{h}{n}\right)\left(y-\frac{h}{n}\right)\frac{(F(\sigma_{k+1}^{B_{n,m}}))^m-(F(\sigma_k^{B_{n,m}}))^{m_t}}{F(\sigma_{k+1}^{B_{n,m}})-F(\sigma_k^{B_{n,m}})}.\end{aligned}$$

But Lemma 2 part (5) implies that this cannot hold for large enough m—a contradiction. □

We turn now to the proof of the proposition. Let $\{B_{n,m}\}$ be any sequence of non-decreasing equilibrium strategies in markets with m dealers. Each $B_{n,m}$ is represented by $\sigma^{B_{n,m}} \in \Sigma \subset \mathbb{R}^{n+2}$. By Lemma 5, all convergent subsequences of $\sigma^{B_{n,m}}$ converge to σ^{H_n}. Since Σ is compact, this implies that $\sigma^{B_{n,m}}$ itself converges the σ^{H_n}. But this in turn implies that $B_{n,m}$ converges to H_n almost everywhere. □

Proof of Proposition 5. (1) From Eq. (7) we have:

$$\Gamma_{n,m}^{\text{investor}}(B_{n,m}) = \sum_{k=1}^{n-1} \gamma\left(\frac{k}{n}\right)\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m - \left(F\left(\sigma_k^{B_{n,m}}\right)\right)^m\right]. \tag{A.3}$$

By Proposition 4, for all $k = 1, \ldots, n-1$, $\sigma_k^{B_{n,m}} \to \frac{k}{n} < 1$. Hence, by an argument similar to Lemma 2 part (1), $(F(\sigma_k^{B_{n,m}}))^m \to 0$. It follows that as m tends to infinity we may neglect all the terms of the summation except $k = n-1$, since they all tend to zero. On the other hand, $F(\sigma_n^{B_{n,m}}) = 1$ for all m. Thus,

$$\Gamma_{n,\infty}^{\text{investor}} = \lim_{m\to\infty} \gamma\left(\frac{n-1}{n}\right)\left[1 - \left(F\left(\sigma_{n-1}^{B_{n,m}}\right)\right)^m\right] = \gamma\left(1 - \frac{1}{n}\right). \tag{A.4}$$

By using the Taylor expansion for $\gamma(\cdot)$ and applying integration by parts we obtain:

$$\gamma\left(1 - \frac{1}{n}\right) = \gamma(1) - \frac{1}{n} + O\left(\frac{1}{n^2}\right) = 1 - \mu_G - \frac{1}{n} + O\left(\frac{1}{n^2}\right),$$

as required.

(2) From Eq. (8) we have:

$$\Gamma_{n,m}^{\text{dealers}}(B_{n,m}) = \sum_{k=1}^{n-1} G\left(\frac{k}{n}\right) \frac{(F(\sigma_{k+1}^{B_{n,m}}))^m - (F(\sigma_k^{B_{n,m}}))^m}{F(\sigma_{k+1}^{B_{n,m}}) - F(\sigma_k^{B_{n,m}})} \int\limits_{y=\sigma_k^{B_{n,m}}}^{\sigma_{k+1}^{B_{n,m}}} \left(y - \frac{k}{n}\right) f(y)\,\mathrm{d}y.$$

By the same argument used in part (1) of this proposition, all the terms in the summation except $k = n-1$ should be neglected as m tends to infinity.

It follows that:

$$\Gamma_{n,\infty}^{\text{dealers}} = \lim_{m\to\infty} G\left(\frac{n-1}{n}\right) \frac{1 - (F(\sigma_{n-1}^{B_{n,m}}))^m}{1 - F(\sigma_{n-1}^{B_{n,m}})} \int\limits_{y=\sigma_{n-1}^{B_{n,m}}}^{1} \left(y - \frac{n-1}{n}\right) f(y)\,\mathrm{d}y.$$

Denote: $\varphi(t) \equiv \int_0^t F(y)\,\mathrm{d}y$. By Proposition 4, $\sigma_{n-1}^{B_{n,m}} \to \frac{n-1}{n}$. Hence, from the continuity of F and by applying integration by parts we obtain:

$$\Gamma_{n,\infty}^{\text{dealers}} = \frac{G(1-\frac{1}{n})}{1 - F(1-\frac{1}{n})}\left[\frac{1}{n} - \varphi(1) + \varphi\left(1 - \frac{1}{n}\right)\right].$$

Using the Taylor expansion yields:

$$\varphi\left(1-\frac{1}{n}\right)=\varphi(1)-\frac{1}{n}+\frac{f(1)}{2n^2}+O\left(\frac{1}{n^3}\right),$$
$$F\left(1-\frac{1}{n}\right)=1-\frac{f(1)}{n}+O\left(\frac{1}{n^2}\right),$$
$$G\left(1-\frac{1}{n}\right)=1-O\left(\frac{1}{n}\right).$$

Therefore,

$$\Gamma^{\text{dealers}}_{n,\infty}(B_m)=\left(1-O\left(\frac{1}{n}\right)\right)\frac{\frac{f(1)}{2n^2}+O(\frac{1}{n^3})}{\frac{f(1)}{n}-O(\frac{1}{n^2})}=\frac{1}{2n}+O\left(\frac{1}{n^2}\right).$$

(3) Follows from adding up the results of parts (1) and (2). □

Proof of Proposition 6. The probability of transaction is given by:

$$P(B_{n,m})=\sum_{k=1}^{n-1}G\left(\frac{k}{n}\right)\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m-\left(F\left(\sigma_k^{B_{n,m}}\right)\right)^m\right].$$

By Proposition 4, for all $k=1,\dots,n-1$, $\sigma_k^{B_{n,m}}\to\frac{k}{n}<1$. Hence, by an argument similar to Lemma 2 part (1), $(F(\sigma_k^{B_{n,m}}))^m\to 0$. It follows that as m tends to infinity we may neglect all the terms of the summation except $k=n-1$, since they all tend to zero. On the other hand, $F(\sigma_n^{B_{n,m}})=1$ for all m. Thus:

$$\lim_{m\to\infty}P(B_{n,m})=G\left(1-\frac{1}{n}\right)\left[1-\left(F\left(\sigma_{n-1}^{B_{n,m}}\right)\right)^m\right]=G\left(1-\frac{1}{n}\right).$$

Using Taylor expansion we obtain:

$$G\left(1-\frac{1}{n}\right)=G(1)-\frac{1}{n}g(1)+O\left(\frac{1}{n^2}\right)=1-\frac{1}{n}g(1)+O\left(\frac{1}{n^2}\right).$$

It follows that the probability of no transaction is:

$$1-G\left(1-\frac{1}{n}\right)=\frac{1}{n}g(1)+O\left(\frac{1}{n^2}\right),$$

which is $O(\frac{1}{n})$ as required. □

Proof of Proposition 7. Let $B_{n,m}$ be an equilibrium. The expected aggressiveness is given by:

$$\Omega(B_{n,m})=\sum_{k=1}^{n-1}\frac{k}{n}\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m-\left(F\left(\sigma_k^{B_{n,m}}\right)\right)^m\right].$$

By Proposition 4, for all $k=1,\dots,n-1$, $\sigma_k^{B_{n,m}}\to\frac{k}{n}<1$. Hence, by an argument similar to Lemma 2 part (1), $(F(\sigma_k^{B_{n,m}}))^m\to 0$. It follows that as m tends to infinity we may neglect all the terms of the summation except $k = n-1$, since they all tend to zero. On the other hand, $F(\sigma_n^{B_{n,m}}) = 1$ for all m. Thus: $\lim_{m\to\infty} \Omega(B_{n,m}) = 1 - \frac{1}{n}$. □

Proof of Proposition 8. The following lemma is needed first.

Lemma 6. *Assume a market with a tick size of $\frac{1}{n}$, and consider a sequence of non-decreasing equilibrium strategies $\{B_{n,m}\}$ in markets with m dealers. The probability of submitting a dishonest quote by a dealer is $O(\frac{1}{m})$. Furthermore: $\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot O(\frac{1}{m})$, $h \in \{0, 1, \ldots, n-1\}$.*

Proof. We prove the second part of the lemma first.

By Proposition 4, there exists a number $M(n)$ such that for all $m \geqslant M(n)$ we have $\Delta_n(B_{n,m}) = \{0, \frac{1}{n}, \ldots, \frac{n-1}{n}\}$. Let $m \geqslant M(n)$. For any $h \in \{1, \ldots, n-1\}$, $k \in \{0, \ldots, h-1\}$ and $y = \sigma_h^{B_{n,m}}$ we have:

$$G\left(\frac{h-1}{n}\right)\left(\sigma_h^{B_{n,m}} - \frac{h-1}{n}\right)\frac{(F(\sigma_h^{B_{n,m}}))^m - (F(\sigma_{h-1}^{B_{n,m}}))^m}{F(\sigma_h^{B_{n,m}}) - F(\sigma_{h-1}^{B_{n,m}})}$$
$$= G\left(\frac{h}{n}\right)\left(\sigma_h^{B_{n,m}} - \frac{h}{n}\right)\frac{(F(\sigma_{h+1}^{B_{m,m}}))^m - (F(\sigma_h^{B_{n,m}}))^m}{F(\sigma_{h+1}^{B_{n,m}}) - F(\sigma_h^{B_{n,m}})}.$$

Solving for $\sigma_h^{B_{n,m}}$ yields:

$$\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot 1 \Big/ \left(\frac{G(\frac{h}{n})}{G(\frac{h-1}{n})} \cdot \frac{F(\sigma_h^{B_{n,m}}) - F(\sigma_{h-1}^{B_{n,m}})}{F(\sigma_{h+1}^{B_{n,m}}) - F(\sigma_h^{B_{n,m}})} \times \frac{(F(\sigma_{h+1}^{B_{n,m}}))^m - (F(\sigma_h^{B_{n,m}}))^m}{(F(\sigma_h^{B_{n,m}}))^m - (F(\sigma_{h-1}^{B_{n,m}}))^m} - 1\right).$$

By part (4) of Lemma 2 we obtain that: $\sigma_h^{B_{n,m}} = \frac{h}{n} + \frac{1}{n} \cdot O(\frac{1}{m})$ for all $h \in \{0, \ldots, n-1\}$ as required.

As for the first part of the proposition: the probability of getting a dishonest quote by a dealer is: $\sum_{h=0}^{n-1}(\sigma_h^{B_m} - \frac{h}{n})$. The previous result immediately implies that this probability is $O(\frac{1}{m})$. □

We turn now to the proof of the proposition.

(1) Let $m \geqslant 1$. We assume below that $n \geqslant 3$.[25] From Eqs. (A.3) and (A.4) we have:

$$\Gamma_{n,\infty}^{\text{investor}} - \Gamma_{n,m}^{\text{investor}}$$
$$= \gamma\left(1 - \frac{1}{n}\right) - \sum_{k=1}^{n-1} \gamma\left(\frac{k}{n}\right)\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m - \left(F\left(\sigma_k^{B_{n,m}}\right)\right)^m\right]$$

[25] The proof is the same for $n < 3$, only with minor modifications.

$$= \left[\gamma\left(1-\frac{1}{n}\right)-\gamma\left(1-\frac{2}{n}\right)\right]\left(F\left(\sigma_{n-1}^{B_{n,m}}\right)\right)^m+\gamma\left(1-\frac{2}{n}\right)\left(F\left(\sigma_{n-2}^{B_{n,m}}\right)\right)^m - \sum_{k=1}^{n-3}\gamma\left(\frac{k}{n}\right)\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m-\left(F\left(\sigma_{k}^{B_{n,m}}\right)\right)^m\right].$$

Therefore:

$$\frac{\Gamma_{n,\infty}^{\text{investor}}-\Gamma_{n,m}^{\text{investor}}}{(F(1-\frac{1}{n}))^m} = \left[\gamma\left(1-\frac{1}{n}\right)-\gamma\left(1-\frac{2}{n}\right)\right]\left(\frac{F(\sigma_{n-1}^{B_{n,m}})}{F(1-\frac{1}{n})}\right)^m+\gamma\left(1-\frac{2}{n}\right)\left(\frac{F(\sigma_{n-2}^{B_{n,m}})}{F(1-\frac{1}{n})}\right)^m - \sum_{k=1}^{n-3}\gamma\left(\frac{k}{n}\right)\left[\left(F\left(\sigma_{k+1}^{B_{n,m}}\right)\right)^m-\left(F\left(\sigma_{k}^{B_{n,m}}\right)\right)^m\right]\Big/\left(F\left(1-\frac{1}{n}\right)\right)^m.$$

By Proposition 4, as m tends to infinity $F(\sigma_k^{B_{n,m}}) \to F(\frac{k}{n})$ for all $k=1,\ldots,n-1$. This implies that the second and the third terms tend to zero. Let us denote the first term by A_1. By Lemma 6 we have:

$$A_1 = \left[\gamma\left(1-\frac{1}{n}\right)-\gamma\left(1-\frac{2}{n}\right)\right]\left(\frac{F(1-\frac{1}{n}+\frac{1}{n}O(\frac{1}{m}))}{F(1-\frac{1}{n})}\right)^m.$$

As m tends to infinity, the term in parentheses tends to $\exp(\frac{1}{n(1-1/n)})$, which is of course finite. Thus, we have shown that $(\Gamma_{n,\infty}^{\text{investor}}-\Gamma_{n,m}^{\text{investor}})/(F(1-\frac{1}{n}))^m$ tends to a finite limit as m tends to infinity, as required.

Similar arguments are used to prove part (2). Part (3) follows from parts (1) and (2). □

References

Anshuman, V., Kalay, A., 1998. Market making rents under discrete prices. Rev. Finan. Stud. 11, 81–109.

Athey, S., 2001. Single crossing properties and the existence of pure strategy equilibria in games of incomplete information. Econometrica 69, 861–889.

Bacidore, J.M., 1997. The impact of decimalization on market quality: an empirical investigation of the Toronto Stock Exchange. J. Finan. Intermediation 6, 92–120.

Bernhardt, D., Hughson, E., 1996. Discrete pricing and the design of dealership markets. J. Econ. Theory 71, 148–182.

Biais, B., 1993. Price formation and equilibrium liquidity in fragmented and centralized markets. J. Finance 48, 157–185.

Chatterjee, K., Samuelson, W., 1983. Bargaining under incomplete information. Operations Res. 31, 835–851.

Christie, W.G., Harris, J.H., Kandel, E., 2002. The impact of lower tick size on quoting behavior and trading cost on Nasdaq: double experiment. Working paper. Hebrew University.

Christie, W.G., Schultz, P.H., 1994. Why do NASDAQ market makers avoid odd-eighth quotes? J. Finance 49, 1813–1840.

Cordella, T., Foucault, T., 1999. Minimum price variation, time priority and quote dynamics. J. Finan. Intermediation 8, 141–173.

Chordia, T., Subrahmanyam, A., 1995. Market making, the tick size, and payment-for-order flow: theory and evidence. J. Bus. 68, 543–575.

Foucault T., Kadan O., Kandel E., 2004. Limit order book as a market for liquidity. Rev. Finan. Stud. In press.
Glosten, L.R., 1995. Competition and the set of allowable prices. Working paper. Columbia University.
Goldstein, M., Kavajecz, K.A., 2000. Eighths, sixteenths, and market depth: Changes in tick size and liquidity provision on the NYSE. J. Finan. Econ. 56, 125–149.
Harris, L.E., 1994. Minimum price variations, discrete bid–ask spreads, and quotation sizes. Rev. Finan. Stud. 7, 149–178.
Harris, L.E., 1997. Decimalization: a review of the arguments and evidence. Working paper. University of Southern California.
Harsanyi, J., 1967–1968. Games with incomplete information played by Bayesian players, Parts I, II, and III. Manage. Sci. 14, 159–182. 320–334, 486–502.
Jones, C.M., Lipson, M., 2001. Sixteenths: Direct evidence on institutional execution costs. J. Finan. Econ. 58, 253–278.
Kadan, O., 2004. Equilibrium in the two-player k-double auction with affiliated private values. Working paper. Washington University in St Louis.
Kandel, E., Marx, L., 1997. Nasdaq market structure and spread patterns. J. Finan. Econ. 45, 61–89.
Lyons, R.K., 1995. Tests of microstructural hypotheses in the foreign exchange market. J. Finan. Econ. 39, 321–351.
Milgrom, P., Shannon, C., 1994. Monotone comparative statics. Econometrica 62, 157–180.
Myerson, R., Satterthwaite, M., 1983. Efficient mechanisms for bilateral trading. J. Econ. Theory 29, 265–281.
Reny, P., Perry, M., 2003. Toward a strategic foundation for rational expectations equilibrium. Working paper. University of Chicago.
Rustichini, A., Satterthwaite, M., Williams, S.R., 1994. Convergence to efficiency in a simple market with incomplete information. Econometrica 62, 1041–1063.
Satterthwaite, M., Williams, S.R., 1989a. Bilateral trade with the sealed bid k-double auction: existence and efficiency. J. Econ. Theory 48, 107–133.
Satterthwaite, M., Williams, S.R., 1989b. The rate of convergence to efficiency in the buyer's bid double auction when the market becomes large. Rev. Econ. Stud. 56, 477–498.
Seppi, D.J., 1997. Liquidity provision with limit orders and a strategic specialist. Rev. Finan. Stud. 10, 103–150.
Williams, S.R., 1987. Efficient performance in two-agent bargaining. J. Econ. Theory 41, 154–172.
Williams, S.R., 1991. Existence and convergence of equilibria in the buyer's bid double auction. Rev. Econ. Stud. 58, 351–374.
Wilson, R., 1985. Incentive efficiency of double auctions. Econometrica 53, 1101–1115.
Zacharias, E., Williams, S.R., 2001. Ex post efficiency in the buyer's bid double auction when demand can be arbitrarily larger than supply. J. Econ. Theory 97, 175–190.