Market Making with Discrete Prices

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Exchange-mandated discrete pricing restrictions create a wedge between the underlying equilibrium price and the observed price. This wedge permits a competitive market maker to realize economic profits that could help recoup fixed costs. The optimal tick size that maximizes the expected profits of the market maker can be equal to $1/8 for reasonable parameter values. The optimal tick size is decreasing in the degree of adverse selection. Discreteness per se can cause time-varying bid-ask spreads, asymmetric commissions, and market breakdowns. Discreteness, which imposes additional transaction costs, reduces the value of private information. Liquidity traders can benefit under certain conditions.

Will Wall Street surrender its 'pieces of eight'?
Sure, share trading in 12.5-cent increments is antiquated—but so lucrative.

—Business Week, November 22, 1993

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Prices on organized exchanges in the U.S. are restricted to multiples of \$1/8th.[1] This phenomenon of discreteness has attracted the attention of financial economists [e.g., Niederhoffer (1966), Gottlieb and Kalay (1985), Harris (1991), and Hausman, Lo, and Mackinlay (1992)]. Such a restriction on the distribution of prices creates a wedge between the underlying "equilibrium price" and the observed price, where the equilibrium price is the market value of the asset conditional on all publicly available information in an otherwise identical economy with continuous prices. This difference forces investors to trade at off-equilibrium prices. Buyers can be forced to pay above equilibrium prices just as sellers can be forced to sell below equilibrium prices. One therefore wonders why an exchange would impose a discrete pricing restriction.

Clearly, all observed prices in our economy are discrete. Houses are traded in thousands of dollars, whereas cars are more likely to be traded in hundred dollar intervals. Here discreteness is an optimal outcome observed in markets, perhaps due to an attempt to reduce negotiation costs.[2] If discrete prices are optimal because they reduce negotiation costs, it would make sense to let investors endogenously decide the tick size. Why then do exchanges mandate a prespecified tick size?

This article suggests that an exchange can find it optimal to restrict the distribution of prices in order to maximize the profits of member firms. Consider the case of a competitive market maker. He will offer liquidity services as long as the gains from the bid-ask spread cover his costs. With continuous prices, all economic rents from the spread are eliminated in equilibrium by competing market makers. In the presence of discrete prices, however, the minimum bid-ask spread is constrained to be a tick. The tick size poses a barrier to competing forces, thereby creating positive expected profits.[3]

We present a market microstructure model that computes the economic profits arising from discreteness. We estimate the annual discreteness-related profits for a typical NYSE specialist firm dealing in a portfolio of moderately active stocks. It can vary between \$704,412.50 and \$14,703,010 per year for a range of parameter values. These num-

[1] See Chordia and Subrahmanyam (1995), and Ahn, Cao, and Choe (1996) for details on tick size rules.

[2] Harris (1991) suggests that a minimum tick size can lead to reduced negotiation costs. Brown, Laux, and Schachter (1991) argue that traders endogenously choose a tick size to control bargaining costs.

[3] Grossman and Miller (1988) suggest that the minimum tick size ensures profits on quick-turnaround transactions.

bers suggest that annual discreteness-related profits can be quite significant. It is possible that such profits could be used by market makers to recoup fixed costs. On the other hand, an exchange-mandated restriction on transaction prices could serve as a credible mechanism for enforcing a cartel-like agreement. Such rents can exist in equilibrium if potentially competing exchanges face significant start-up costs in opening and operating an exchange of equivalent depth.

If start-up costs are significant, incumbent market makers can choose a tick size that maximizes economic rents. A very large tick size is not in the best interests of the market maker, because the demand of liquidity traders is elastic with respect to transaction costs. On the other hand, a very low tick size would result in limited profits. Numerical simulations of the model show that the optimal tick size can be equal to \$1/8 for a set of reasonable parameter values. We also find that the optimal tick size is increasing in the amount of natural liquidity and decreasing in the degree of adverse selection. In a market with more natural liquidity the market maker's expected profits are less sensitive to the elasticity of liquidity traders' demand. Therefore the market maker can afford to choose a larger tick size when the amount of natural liquidity is high. Conversely, when the degree of adverse selection is severe, the optimal tick size is lower because liquidity traders are more sensitive to transaction costs.

Market making with discrete prices results in additional interesting phenomena. We show that discrete prices per se can cause (i) location-dependent commissions, that is, commissions that depend on the location of the underlying equilibrium price, (ii) asymmetric commissions on the ask and bid sides of the market, (iii) time-varying bid-ask spreads, and (iv) market breakdowns. None of these phenomena would occur in an otherwise identical economy with continuous prices.

This article also points out that discrete prices need not always result in additional transaction costs for liquidity traders. Informed traders would take into consideration the additional costs of discrete prices and, in equilibrium, invest less in acquiring information as compared to an otherwise identical economy with continuous prices. Thus discrete prices reduce the value of private information when information acquisition is endogenously determined. We show that, on both the ask and bid sides of the market, locations of the underlying equilibrium price exist for which the discrete-case equilibrium commissions are less than in the continuous case. For certain parameter values, depending on the location of the underlying equilibrium price, liquidity traders may even face a lower bid-ask spread under discrete prices. Thus some liquidity traders would be better off under discrete prices as compared to continuous prices.

For concreteness, we begin by adapting the trading mechanism presented in Admati and Pfleiderer (1989). In this setting, the market maker is risk neutral and therefore passive in the sense that he offsets his expected losses to informed traders with the expected profits from liquidity traders. Under continuous prices, the competitive market maker charges identical commissions on both the ask and bid sides of the market. This commission is a break-even commission, which satisfies the zero expected profit condition imposed on a competitive market maker.

Consider now the case when prices are restricted to multiples of a minimum tick size. The market-maker's requirement of nonnegative profits implies that the competitive market maker chooses the ask (bid) price to be the smallest (largest) feasible price that results in a commission that is at least as much as the break-even commission. Thus the ask (bid) commissions are greater under discrete prices.

We show that commissions in the discrete price economy depend on the location of the underlying equilibrium price. The discrete case equilibrium commission can be thought of as being made of two parts: (i) a break-even commission, which ensures that the market maker does not make negative expected profits and (ii) discreteness-related execution costs, which can be as low as zero or as high as the tick size. It is the latter component that varies with the location of the underlying equilibrium price. This is purely an artifact of discrete prices. The market-maker's pricing rule can be characterized by a rounding mechanism under which the ask price jumps when the underlying equilibrium price crosses an indifference point. This rounding mechanism differs from earlier models [e.g., Gottlieb and Kalay (1985)] in that the indifference point is not necessarily located at the midpoint of the tick or on the tick.

Interestingly, the impact of discreteness is not symmetric on the ask and bid sides of the market. Because the underlying equilibrium price is, in general, asymmetrically located with respect to the discrete ticks, discreteness-induced rounding is not the same on the ask and bid sides of the market. Thus the ask commission is not identical to the bid commission even when private information is drawn from a symmetric distribution. Moreover, the market-maker's pricing rule is asymmetric in the sense that the indifference points (describing the rounding mechanisms) are not identical on the ask and bid sides of the market. This result is consistent with the findings of Hausman, Lo, and Mackinlay (1992), who find that a simple rounding mechanism (where the indifference point is at the midpoint of the tick) does not satisfactorily describe empirical data.

Because the ask and bid commissions vary with the location of the underlying equilibrium price, so does the bid-ask spread. As information is made public at the end of each period, the location of the underlying equilibrium price changes. Therefore the bid-ask spread may change from period to period, even when the proportion of informed traders remains fixed. This phenomenon does not arise in an otherwise identical, continuous price economy, where the bid-ask spread changes only when the degree of adverse selection changes. Thus discrete prices per se can cause time-varying bid-ask spreads.

Our model also shows that discrete prices per se can cause market breakdowns. If the tick size is very large, feasible commissions may become too large to sustain the interest of liquidity traders and a market breakdown may occur depending on the location of the underlying equilibrium price. Such market breakdowns would not occur in an otherwise identical economy with continuous prices. More interestingly, a market breakdown occurring on the ask side need not be accompanied by a similar market breakdown on the bid side, because the commissions on both sides of the market are not identical. The occurrence of market breakdowns is more likely for low price stocks because the tick size is large relative to the set of feasible commissions.

Section 1 discusses the Admati and Pfleiderer (1989) model. Sections 2 and 3 discuss the benchmark continuous price equilibrium and the discrete price equilibrium, respectively. Section 4 discusses other implications of discrete prices. Section 5 derives the equilibrium under endogenous information acquisition. Section 6 concludes.

1. The Model

The following trading mechanism has been adapted from the Admati and Pfleiderer (1989) model (henceforth, AP model). For clarity, we retain the notation of their model. The market's valuation of the risky asset at time t is given by $P_t = P_{t-1} + \delta_t$, where P_{t-1} is the expected value of the asset conditional on all the publicly available information at the beginning of period t and δ_t is the unanticipated information made public at the end of period t. δ_t is drawn from a normal distribution with a mean of zero and a variance of σ^2. In the first period, the underlying price changes as shown in Equation (1), where the time subscript on δ has been suppressed.

$$P_1 = P_0 + \delta. \tag{1}$$

There are risk-neutral market makers who post ask and bid prices before trading commences. They do not face any inventory constraints and are willing to entertain all incoming orders at these prices. A batch of orders is cleared at the posted ask or bid price. Let the equilibrium ask (bid) price be $A(B)$. Traders therefore pay an ask commission, $a = A - P_0$, and a bid commission, $b = P_0 - B$.

Informed traders observe a signal $\delta + \varepsilon$, where ε is independent of δ and an $N(0, \psi^2)$ random variable. This information-acquiring technology is characterized by the precision of the signal, $(1/\psi^2)$. Initially we assume that the technology is exogenously fixed and can be accessed by paying a flat fee (normalized to zero, without loss of generality). Informed traders are identical in that their preferences can be represented by negative exponential utility functions [with risk aversion coefficient (H)]. Upon observing the signal, each informed trader buys or sells shares if and only if the expected utility of trading is greater than that of not trading. For convenience, we normalize the expected utility of not trading to -1, as in Subrahmanyam (1991). In equilibrium, each informed trader places an order x^* that maximizes his expected utility.

Liquidity traders face exogenous shocks in capital requirements which are not explicitly modeled. As in the AP model, liquidity traders are endowed with a "reservation commission" $(\tilde{s})$ that describes the opportunity cost of using the next best source of capital. We assume that the reservation commissions of liquidity traders are independent and identically distributed uniform random variables over the bounded support $(0, s)$. We assume that, out of the total trading population, a fraction λ are liquidity traders and the remaining $1 - \lambda$ are informed traders (the size of the trading population is normalized to 1, without loss of generality).

2. Continuous Price Equilibrium

This section discusses the continuous price equilibrium, which serves as a benchmark to compare with the discrete price equilibrium. Proposition 1 summarizes the strategic behavior of informed traders and the market maker. Since δ is symmetric, analysis of the ask side is analogous to that of the bid side. Shown below is the analysis for the ask side.

Proposition 1. *For a given technology of acquiring information, the informed trader's optimal buy order (x^*) and the market-maker's per trade expected profits on the ask side of the market (π^{MM}) are given by*

$$x^* = max\{0, (1/H\psi^2)[(\delta + \varepsilon) - aw]\} \text{ and} \tag{2}$$

$$\begin{aligned}\pi^{MM}(a) = & -[(1-\lambda)\sigma^2/(H\psi^2)] \\ & \times[(1+u(a)^2)(1-\Phi(u(a))) - u(a)\phi(u(a))] \\ & +(\lambda/s)(s-a)a,\end{aligned} \tag{3}$$

where $w = (\sigma^2 + \psi^2)/\sigma^2$, $u(a) = a\sqrt{w}/\sigma$, *and* $\phi(\cdot)$ *and* $\Phi(\cdot)$ *are the probability density function and the cumulative density function, respectively, of the standard normal distribution.*

(All proofs, unless otherwise stated, are shown in the Appendix.)

Using the notation, π^{MM}, π^{IT}, and π^{LT} to denote the per trade expected profits of the market maker, the expected profits of informed traders, and the expected losses of liquidity traders, respectively, it follows that $\pi^{MM} = -\pi^{IT} + \pi^{LT}$. As evident from Equation (2), the market maker infers that each informed trader is going to buy x^* (> 0) shares if he observes a signal $\delta + \varepsilon > aw$. The market maker is unaware of the actual signal value and, being risk neutral, computes expected losses over all values of the signal such that x^* is greater than zero. In contrast, the market-maker's expected profits from liquidity trades depend on the probability that the liquidity trader's "reservation commission" is greater than the ask commission. Thus $\pi^{MM} = -(1-\lambda)\Pr\{\delta+\varepsilon > aw\}E[x^*(\delta - a) \mid \delta+\varepsilon > aw] + \lambda\Pr\{\tilde{s} > a\}a$, which simplifies to Equation (3).

It can be shown that the expected profit function of the market maker is concave in the ask commission (see Appendix). Figure 1 depicts the expected profits of the market maker and informed traders as well as the expected losses of liquidity traders for the parameter set ($\lambda = 0.96$, $s = 0.18$, $\sigma^2 = 0.15$, $\psi^2 = 0.30$, and $H = 1$). The market maker makes zero expected profits when $a = \$0.067$ or when $a = \$0.158$. In general, the market-maker's expected profit function has, at most, two roots as shown in Figure 1.[4] Let the smaller root be a^* and the larger one be $a^{*\prime}$. The market maker can charge any commission in the interval (a^*, $a^{*\prime}$) and make nonnegative expected profits. If he sets the commissions outside this interval, he makes negative expected profits because the adverse selection problem is too severe relative to the supply of liquidity.

In equilibrium, the ask commission charged by the market maker has to satisfy two necessary and sufficient conditions of a Bertrand-type game: (i) the expected profits of the market maker should be nonnegative, and (ii) any lower commission should yield strictly negative expected profits. The two roots of the market-maker's expected profit function (a^* and $a^{*\prime}$) yield zero expected profits for the market maker. However, $a^{*\prime}$ cannot be sustained in equilibrium because a competing market maker can post a lower ask commission and still

[4] It is possible that informed traders' expected profit curve is tangential to the expected losses of liquidity traders. In this case there is exactly one root and the market maker can never make positive expected profits. It is also possible that the curves do not intersect at all and there are no roots to the market-maker's expected profit function. Intuitively this would happen when the adverse selection problem is so acute that the market is closed.

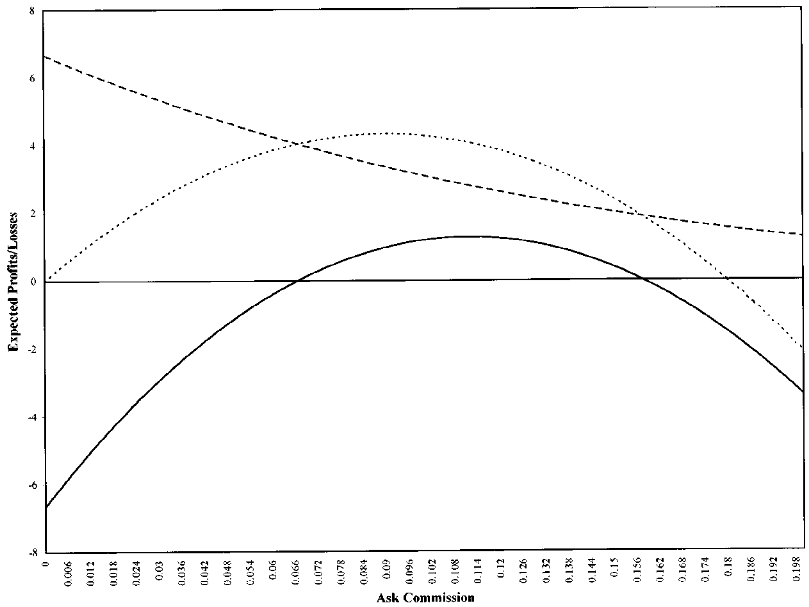

Figure 1
Expected profits and losses of market participants
This figure shows the expected profits of the market maker (——), the expected profits of informed traders (— — —), and the expected losses of liquidity traders (- - - - - -) for the parameter set ($\lambda = 0.96$, $s = 0.18$, $\sigma^2 = 0.15$, $\psi^2 = 0.30$, and $H = 1$). The roots of the market-maker's expected profit function are $a^* = \$0.067$ and $a^{*\prime} = \$0.158$. The market maker can charge any commission in the interval $(a^*, a^{*\prime})$ and make nonnegative expected profits.

make nonnegative expected profits. Therefore the equilibrium ask commission is equal to a^* and the equilibrium ask price is equal to $P_0 + a^*$.

3. Discrete Price Equilibrium

3.1 Market-maker's pricing policy

Table 1 demonstrates the market-maker's pricing rule. For this table, we assume a set of parameters that result in $a^* = \$0.03$. Consider the first row where $P_0 = \$10.05$. Under continuous prices, the ask price is equal to $\$10.05 + \$0.03 = \$10.08$ and the bid price is equal to $\$10.05 - \$0.03 = \$10.02$. However, under the discrete pricing restriction (tick size = \$0.125), these prices are not feasible. Thus the (discrete case) ask price would be equal to \$10.125 because it is the smallest feasible ask price that results in nonnegative expected profits for the market maker. A similar argument implies that the (discrete case) bid price is equal to \$10.00. The ask commission is equal to $\$10.125 - \$10.05 =$

Table 1
The market-maker's pricing rule under discrete prices

		Continuous case		Discrete case					
t	P_{t-1}	Ask price	Bid price	Ask price	Bid price	Ask commission	Bid commission	Bid-ask spread	δ_t
1	10.05	10.08	10.02	10.125	10.00	0.075	0.05	0.125	0.04
2	10.09	10.12	10.06	10.125	10.00	0.035	0.09	0.125	0.01
3	10.10	10.13	10.07	10.25	10.00	0.15	0.10	0.25	0.05
4	10.15	10.18	10.12	10.25	10.00	0.10	0.15	0.25	0.01
5	10.16	10.19	10.13	10.25	10.125	0.09	0.035	0.125	—

All figures in dollars.
This table shows the prices, commissions, and bid-ask spread under continuous prices and under discrete prices. We assume $P_0 = \$10.05$ and $a^* = \$0.03$. Under continuous prices, the bid-ask spread is always equal to \$0.06. Under discrete prices, the ask/bid commissions and the bid-ask spread depend on the location of the underlying equilibrium price at the beginning of the period (P_{t-1}). The location of the underlying equilibrium price changes when the private information (δ_t) is made public at the end of each period.

\$0.075 and the bid commission is equal to $\$10.05 - \$10.00 = \$0.05$ under discrete prices.

At the end of period 1, information $\delta = \$0.04$ is made public and $P_1 = \$10.05 + \$0.04 = \$10.09$ (see second row of table). In this case, the ask commission is equal to \$0.035 and the bid commission is equal to \$0.09 under discrete prices. These numbers differ from the ask and bid commission in the period 1. Thus we see that the discrete-case ask (bid) commissions depend on the location of the underlying equilibrium price (P_t).

Under discrete prices, the ask price is equal to \$10.125 when $P_1 = \$10.09$ (second row of Table 1), whereas the ask price increases to \$10.25 when $P_2 = \$10.10$ (third row). The underlying equilibrium price changes by only \$0.01, but the ask price jumps by \$0.125. The location of the underlying equilibrium price where the ask price first exhibits this jump is \$10.095. We refer to this location as the indifference point. Our result differs from the implicit assumption in Gottlieb and Kalay (1985) that observed prices are rounded upward (downward) if the underlying equilibrium price is above (below) the midpoint of ticks. Our model shows that the indifference points that describe the market-maker's rounding mechanism can lie anywhere on the real line. Thus the implication of our model is consistent with the empirical findings of Hausman, Lo, and Mackinlay (1992), who perform an ordered probit analysis to examine the impact of fundamental economic variables on transaction price changes. They take into account discreteness as well as irregular trading times and find that the ask and bid side indifference points are not symmetrically located, just as predicted by our model. Proposition 2 describes the market-maker's pricing rule, where d is the tick size.

Proposition 2. *If $d \leq (a^{*\prime} - a^*)$, the market-maker's pricing rule is given by*[5]

$$A = P_0 + a^* + [d - mod(P_0 + a^*, d)], \tag{4}$$

where A is the ask price, P_0 is the expected value conditional on all public information at the beginning of the period, and $mod(x, y)$ is the remainder left after dividing x by y. Within each discrete tick defined by the interval $[nd, (n+1)d]$, $n \geq 0$, the ask price exhibits a discontinuous jump if P_0 crosses an indifference point (f^n), where $mod(f^n, d) = (n+1)d - mod(a^*, d)$.

Under discrete prices, the competitive market maker rounds the continuous price $P_0 + a^*$ to the immediately higher feasible discrete price. Therefore, $A = P_0 + a^* + [d - \text{mod}(P_0 + a^*, d)]$. The ask commission $(A - P_0)$ is made of two components: (i) the breakeven commission (a^*) and (ii) the discreteness-related execution cost $[d - \text{mod}(P_0 + a^*, d)]$, which lies in the interval $(0, d)$. Consequently the ask commission lies in the interval $(a^*, a^* + d)$. Any commission less than a^* would result in negative expected profits—thus a^* is a lower bound on the equilibrium commission. On the other hand, the equilibrium commission cannot be greater than $a^* + d$. To see this, suppose the contrary, that is, $A - P_0 > a^* + d$. Then it follows that $A - d > P_0 + a^*$. Then $A - d$ is a "better" ask price than A because it is greater than the continuous-case ask price $(P_0 + a^*)$, but still less than A. Thus A cannot be posted as an equilibrium price. We reach a contradiction and thus conclude that the ask commission can vary only in the interval $(a^*, a^* + d)$. The additional transaction costs due to discreteness can be as low as 0 and as high as the tick size.

3.2 Discreteness-related profits

To compute the expected profits of the market maker under discrete prices, we first need to establish the distribution of the equilibrium commissions. As shown in Proposition 2, the equilibrium commission in the current period is equal to $[a^* + d - \text{mod}(P_0 + a^*, d)]$. The equilibrium commission in the next period would be equal to $[a^* + d - \text{mod}(P_1 + a^*, d)]$, where $P_1 = P_0 + \delta$. It is uncertain because δ is random (normally distributed). It follows that the distribution of the next period's equilibrium commission depends on the distribution of $\text{mod}(\delta, d)$. This distribution is a member of the wrapped normal

[5] This proposition only holds when the market maker cannot use mixed strategies in his pricing decisions. For instance, if the market maker could fragment orders and execute them at multiple prices, the discrete pricing restriction would not result in additional transaction costs. However, this strategy is costly to implement and there is no evidence indicating that it is widespread phenomenon on the stock markets.

distribution family, which arises when the normal density function is wrapped around a circle of unit radius. The wrapped normal distribution is unimodal and centered within the interval $(0, 2\pi)$. As discussed in the adjoining footnote, it follows that the distribution of the equilibrium commission is unimodal and centered on the support $(a^*, a^* + d)$.[6]

An interesting property of the wrapped normal distribution is that it converges to the uniform distribution when the variance of underlying normal distribution increases. Thus the distribution of the equilibrium commission in a distant period t, given the current price P_0, is asymptotically uniform because the underlying variance [≡ variance of $(\sum_t \delta_t)$] increases in proportion to time (t). This property is consistent with other models of discreteness [e.g., Ball, Torous, and Tschoegl (1985) and Gottlieb and Kalay (1985)]. It follows that the per trade expected profit of the market maker in any future period is given by $E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d)] = \int_{a^*}^{a^*+d} \pi^{MM}(a) \frac{1}{d} da$, where $\pi^{MM}(a^*) = 0$, as defined in Equation (3).

Our model can be simulated under reasonable parameter values to estimate the annual expected profits of market makers on the NYSE. Assuming a tick size (d) equal to \$0.125 and 250 trading days per year, we can compute the annual discreteness-related profits as the product of (250 · average daily trading volume · specialist participation rate) and $E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d)]$. To form an estimate of the annual figures we use data presented in Sofianos (1995), who examines specialist gross trading revenues for a sample of 200 common stocks (with a tick size of 1/8 of a dollar and a lot size of 100 shares) trading on the NYSE during April 1993.

In his sample of 200 NYSE stocks, Sofianos (1995) finds that, on average, stocks have a daily trading volume of 158,697 and a specialist participation rate of 17%. The most active decile of stocks has an average daily trading volume of 1,023,001 shares and an average specialist participation rate of 15.2%. On the other hand, the least active decile of stocks has an average daily trading volume of 4070 shares and an average specialist participation rate of 46.9%. Our estimates of discreteness-related profits are based on these three categories.

Table 2 shows the expected discreteness-related profits for a given set of parameters for a stock similar to (i) an average stock, (ii) a

[6] Patel and Read (1982) show that the wrapped normal distribution is unimodal, symmetric, and has a mean value of π (p. 38, Section 2.7.2). Further, this distribution converges to the uniform as the variance of the underlying normal distribution increases. Note, $\text{mod}(\delta, d) = (d/2\pi)\text{mod}(2\pi\delta/d, 2\pi)$, a result which follows after some algebra. The distribution of $\text{mod}(2\pi\delta/d, 2\pi)$, where $2\pi\delta/d \equiv N(0, 4\pi^2\sigma^2/d^2)$, is wrapped normal. Therefore the expected value of $\text{mod}(\delta, d)$ is equal to $(d/2\pi) \cdot \pi = d/2$. Further, for large σ^2, $\text{mod}(\delta, d)$ converges to $U(0, d)$. It follows that the equilibrium commission is $U(a^*, a^* + d)$.

Table 2
Estimates of the annual discreteness-related profits of specialists on the NYSE

						Annual discreteness-related profits		
λ	s	σ^2	ψ^2	H	a^*	Average stock	Most active stock	Least active stock
0.35	0.30	0.25	1.5	2	0.147	\$44,550	\$256,400	\$3,147
0.50	0.30	0.20	0.75	2.75	0.136	\$69,430	\$399,500	\$4,904
0.65	0.45	0.25	1	1	0.133	\$22,290	\$1,283,000	\$1,575
0.85	0.36	0.35	0.50	1	0.138	\$125,300	\$720,800	\$8,849
0.85	0.40	0.50	1	1	0.08	\$219,200	\$1,261,000	\$15,480
0.90	0.20	0.40	0.85	1	0.074	\$13,750	\$79,100	\$971
0.90	0.40	0.30	1	1	0.047	\$287,000	\$1,652,000	\$20,270

The per trade expected profits of the market maker depend on the following parameters: $\lambda \equiv$ proportion of liquidity traders, $s \equiv$ upper bound of liquidity traders' reservation commissions, $\sigma^2 \equiv$ variance of private information (δ), $\psi^2 \equiv$ variance of noise in informed trader's signal, and $H \equiv$ coefficient of risk aversion. a^* is the resulting break-even commission corresponding to the given parameter values. To compute the annual expected profits of the market maker, we assume that the tick size (d) is equal to \$0.125, and the trading volume and specialist participation rates are as described in Sofianos (1995): (i) the average trading volume is 158,697 and the average specialist participation rate is 17%; (ii) stocks in the most active decile have an average trading volume of 1,023,001 shares and the average specialist participation rate is 15.2%; and (iii) stocks in the least active decile of stocks have an average daily trading volume of 4070 shares and the average specialist participation rate is 46.9%. We also assume that there are 250 trading days in a year.

stock in the most active decile, and (iii) a stock in the least active decile, for given sets of parameter values. For instance, in the third row, the parameter set $(\lambda, s, \sigma^2, \psi^2, H)$ is given by the values (0.65, 0.45, 0.25, 1, 1). For these parameter values, $a^* = \$0.133$. Given a tick size of \$0.125, the per trade expected profit of the market maker, $E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d)]$, is completely determined by the parameter values. Assuming 250 trading days per year, we can compute the annual discreteness-related profits for the average stock (which has a daily trading volume of 158,697 and a specialist participation rate of 17%). The annual profits for the average stock are equal to \$22,290. The corresponding numbers for a stock in the most active decile and the least active decile are equal to \$1,283,000 and \$1575, respectively.

The annual profits can vary significantly depending on the parameter values. For the average stock, the annual profits range from \$13,750 to \$287,000 depending on the set of parameter values shown in Table 2. For the most active decile of stocks the annual profits range from \$79,100 to \$1,652,000. For the least active decile of stocks, the annual profits range from \$971 to \$20,270. Our computations can be compared to the empirical estimates provided in Sofianos (1995). For the average stock, his empirical estimate of gross trading revenues per stock per day is \$669. This translates into an annual gross trading revenue of \$28,432.50, after factoring in the specialist participation rate of 17% and assuming 250 trading days per year. For a stock in the most active decile, Sofianos' (1995) estimate of annual gross trading revenues is \$161,006. Finally, for the least active decile of stocks, Sofianos (1995) finds gross trading revenues to be statistically insignificantly different from zero. These empirical estimates are consistent with our estimates of annual discreteness-related profits shown in Table 2.

In general, the expected profits of the market maker increase with the fraction of liquidity traders (λ) and the upper bound of reservation commissions (s) of liquidity traders. Conversely, parameters that adversely affect informed trading (e.g., noise in signal) also increase the expected profits of the market maker. This point is well illustrated in the last two rows of the table. In both cases $\lambda = 0.90$. But in the last row, the remaining parameters suggest more inherent liquidity. For instance, in the last row $s = \$0.40$ as opposed to \$0.20 in the fifth row, which suggests that liquidity traders are less sensitive to commissions. There is also a lower amount of private information ($\sigma^2 = 0.3$ as compared to 0.4) and informed traders face more noise in their signals ($\psi^2 = 1$ as compared to 0.85). As expected, the expected profits of the market maker are higher in the last row. Interestingly, the expected profits in the fifth row, where $\lambda = 0.90$, are less than in the first row, where $\lambda = 0.35$, because the combined effect of the remaining parameters in the fifth row implies less inherent liquidity.

Let $\lambda = 1$ (i.e., no informed trading at all) and retain all the other parameter values as shown in the last row. We find that the expected profits of the market maker for a stock in the most active decile increase from \$1,652,000 to \$1,880,000. In this case, the only parameter that matters is s (because σ^2, ψ^2, and H are related to informed trading, which is absent when $\lambda = 1$). If we also assume that liquidity traders are inelastic (technically, $s \to \infty$), the per trade expected profit of the market maker is equal to $d/2$. Thus the annual discreteness-related profits for a stock in the most active decile would be equal to 250 (trading days) · 1,023,001 (average daily trading volume for a stock in the most active decile) · 0.152 (specialist participation rate) · \$0.0625 ($\equiv d/2$) = \$2,429,627.40. This number is an upper bound on annual discreteness-related profits because the model has only one parameter (λ), which has been set at its maximum value of 1. Thus annual discreteness-related profits for even the most active decile of stocks cannot exceed \$2.5 million. On the other hand, the upper bounds for the average stock and a stock in the least active decile are equal to \$424,018.55 and \$29,825, respectively.

Coughenour and Deli (1996) report that, on average, specialist firms on the NYSE make the market in 51.23 stocks. Thus the annual revenues for a portfolio of 51.23 stocks that are similar to the average stock (i.e., daily trading volume of 158,697 and specialist participation rate of 17%) range between \$704,412.50 and \$14,703,010 per year for the parameter sets shown in Table 2.[7] These numbers indicate that specialist firms enjoy significant discreteness-related profits.

3.3 Are discreteness-related profits economic rents?

Suppose the total trading volume (going through an exchange) per year is equal to T. The expected profits of the exchange over the entire year would be given by $TE[\pi^{MM}(a) \mid a \in U(a^*, a^*+d)]$. Further, note that $\pi^{MM}(a^*) = 0$ because a^* is the break-even commission. Let the fixed cost incurred by an exchange be equal to \$$F$ and the appropriate discount rate for all future profits be equal to r. Assuming for simplicity that the annual profits are realized at the end of each year, the NPV of creating an exchange is given by NPV $= -F + TE[\pi^{MM}(a) \mid a \in U(a^*, a^*+d)]/r$. Setting the NPV equal to zero yields the break-even tick size that would yield nonnegative NPV for a single exchange. This tick size is necessary to create a viable exchange.

Under simultaneous entry, a larger number of exchanges could exist, but a greater break-even tick size is required for each exchange. To see this, note that competing exchanges would have to share the annual order flow (T trades) with each other. Assuming simultaneous entry by N exchanges, which split the order flow equally, the NPV for each exchange would be equal to $-F + (T/N)E[\pi^{MM}(a) \mid a \in U(a^*, a^*+d)]/r$. It is easy to see that the break-even tick size required for nonnegative profits would have to be greater than that suggested for a single exchange, because each exchange has to rely on lower order flow to recoup the fixed cost (\$$F$). The break-even tick size, $d_{min}(N)$, is determined by solving the following equation:

$$-F + (T/N)E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d)]/r = 0. \qquad (5)$$

Once in place, competing exchanges would treat fixed costs as sunk costs. By undercutting the existing tick size, an exchange would be able to attract the entire order flow. Other exchanges would match the new tick size in order to survive. The order flow would still be equally shared among all exchanges, but Bertrand competition would ensure that the tick size ultimately went to zero. Thus exchanges would not be able to recover their fixed costs. Of course, all this can be anticipated and there would be no incentive to take up market-

[7] Coughenour and Deli (1996) also report that the smallest (largest) specialist firm handled 14 (249) stocks. The corresponding range of annual discreteness profits is \$192,500 to \$4,018,000 (\$3,423,750 to \$71,463,000).

making services in the first place. This deadlock forms part of the well-known Bertrand paradox.

Such a market collapse could be avoided if all exchanges mandated a prespecified tick size. Given a tick size (d), free entry would occur until the next entrant faces a negative-NPV project. In fact, if the tick size is a shade below $d_{min}(N+1)$, it would be sufficient to deter the $(N+1)$ entrant, and thus tick sizes in the range $d_{min}(N) \leq d < d_{min}(N+1)$ would support exactly N exchanges. Thus exchange-mandated tick sizes could be used as a mechanism to resolve a market collapse, which is inevitable under Bertrand competition if capacity is unlimited, products are homogeneous, and collusion is ruled out.[8] Of course, this mechanism is an exogenously imposed restriction to resolve the problem of recovery of fixed costs.

The annual discreteness-related profits of the average specialist firm dealing in a portfolio of moderately active stocks can vary between \$704,412.50 and \$14,703,010 per year, as shown in the previous section. These yearly profits may be just sufficient to recoup fixed costs and other capital requirements in setting up a specialist firm. If that were the case, the tick size would not be a source of economic rents. On the other hand, if these yearly profits exceed fixed costs, market makers would be enjoying economic rents due to the tick size restriction. Whether economic rents exist or not is an empirical issue that depends on the availability of accurate estimates of fixed costs.

Interestingly, even if discreteness-related profits are greater than fixed costs, it may not be possible for competing exchanges to attract order flow from incumbent exchanges by merely offering a finer tick size. It is likely that traders would be unwilling to switch over to a competing exchange until a critical trading volume has been attained. Market makers on the new exchange might be forced to charge a higher break-even commission (a^*) because of the lack of trading volume. The savings in discreteness-related execution costs would be more than offset by the lack of inherent depth in the new exchange.

In general, any attempt to lower the tick size would require a competing exchange to provide equivalent market depth (as the incumbent exchange) in order to attract order flow. To the extent that potentially competing exchanges may face, in addition to fixed costs, significant start-up costs in attaining equivalent market depth, incumbent exchanges can afford to choose a tick size that provides economic rents beyond fixed costs. As compared to the monopoly power due to capacity constraints, product differentiation, or collusive strategies,

[8] The assumptions related to capacity, product homogeneity, and collusion may be relaxed. Tirole (1990) discusses how the equilibrium price can be greater than the marginal cost, resulting in economic rents.

this explanation of economic rents implies that monopoly power can arise due to asymmetric costs faced by the entrant and the incumbent.

The existence of off-floor market-making services sheds some light on this phenomenon. Recall that we had earlier assumed that the proportion of informed to uninformed traders stays the same when order flow is split among competing exchanges. If, however, a competing exchange can attract a greater proportion of uninformed traders, it would be able to post a lower tick than an incumbent exchange. Off-floor market-making firms like Bernard L. Madoff Investment Securities attract order flow from brokers by making "payment-for-order flow." As Chordia and Subrahmanyam (1995) have shown payment-for-order flow can be used to undo the tick size restriction on organized exchanges. Interestingly, Madoff can afford to "lower" the effective tick size because he specializes in small-size order flow, which is more likely to represent uninformed trades. This specialization on trade size is an effective means of circumventing high start-up costs associated with opening a competing exchange of equivalent depth. By restricting the likelihood of informed trading, Madoff is able to pass off the savings in break-even commission in the form of lower tick sizes. Of course, this mode of competition can never be a serious threat to an incumbent exchange because it excludes large-size orders.

3.4 The optimal tick size

If potentially competing exchanges face significant start-up costs, incumbent exchanges could choose the tick size (d) that maximizes the expected profit per trade, that is, $E[\pi^{MM}(a) \mid a \in U(a^*, a^*+d)]$. Thus, by agreeing to trade in discrete prices, market makers are acting as if they have formed a cartel to extract rents.

Proposition 3. *There exists an optimal tick size (d^*) that maximizes the per trade expected profits of the competitive market maker. Numerical simulations of the model show that the optimal tick size can be **\$** 1/8. The optimal tick size is increasing in parameters that affect natural liquidity and decreasing in the degree of adverse selection. In particular, it is increasing in the fraction of liquidity traders (λ) and decreasing in the amount of private information (σ^2).*

It might seem that increasing the tick can arbitrarily increase the profits captured by the market maker. However, liquidity traders are sensitive to execution costs and gradually drop out of the market as the tick size increases. The trade-off between the economic profit per trade and the total number of liquidity trades determines the optimal tick size.

Table 3
An optimal tick size of $1/8

λ	s	σ^2	ψ^2	H	a^*	d^*	Annual discreteness-related profits
0.75	.25	.20	.65	3	.069	0.125	$131,800
.68	.28	.24	.76	2.5	.095	0.125	$110,000
.54	.30	.16	.80	2	.119	0.125	$115,900
.82	.24	.22	.50	3.5	.057	0.125	$138,200
.31	.40	.30	1	2.7	.205	0.125	$47,100

This table shows that the optimal tick size can be equal to $1/8 for a wide range of parameter values. The parameters are as follows: $\lambda \equiv$ proportion of liquidity traders, $s \equiv$ upper bound of liquidity traders' reservation commissions, $\sigma^2 \equiv$ variance of private information (δ), $\psi^2 \equiv$ variance of noise in informed trader's signal, and $H \equiv$ coefficient of risk aversion. Also, $a^* \equiv$ continuous-case equilibrium commission and $d^* \equiv$ optimal tick size. To compute the annual expected profits of the market maker, we assume that the daily trading volume is 158,697 and specialist participation rate is equal to 17%, which corresponds to an average stock in the sample used by Sofianos (1995). As in Table 2, we assume that there are 250 trading days in a year.

We employ numerical simulations of the model to examine whether the optimal tick size can be $1/8, as observed in the stock exchanges in the United States. Table 3 shows five different sets of parameter values for which the optimal tick size is equal to $0.125. The first row in Table 3 shows the case where the parameter set is given by $\lambda = 0.75$, $s = 0.25$, $\sigma^2 = 0.20$, $\psi^2 = 0.65$, and $H = 3$. In this case, $a^* = \$0.069$. The optimal tick size (d^*) turns out to be equal to $0.125. Another case, as shown in the last row, is described by the set of parameters ($\lambda = 0.31$, $s = \$0.40$, $\sigma^2 = 0.30$, $\psi^2 = 1$, and $H = 2.7$). This case also results in an optimal tick size of $0.125. Our simulations indicate that the optimal tick size can be equal to $1/8 for a wide range of reasonable parameter values. The last column in Table 3 shows the maximized annual discreteness-related profits of the market maker assuming that the stock is similar to an average stock in the Sofianos (1995) sample.

The stock exchanges in the United States use a uniform tick size of $1/8 for all stocks. Since stocks differ in their characteristics, a uniform tick size may not maximize economic profits accruing to market makers in *all* stocks. However, as shown in Table 3, the same optimal tick size of $1/8 exists for a wide range of parameter values. It can be seen in Table 3 that, given the various parameter values, the continuous-case equilibrium commission (a^*) varies between $0.057 and $0.205. In each of these cases, the optimal tick size is $1/8. Thus the optimal tick size of $1/8th applies to assets exhibiting differing degrees of adverse selection and it may be reasonable for an exchange to set a uniform tick size for a variety of stocks. Alternatively, a tick size of \$1/8 may be viewed as being optimal for a set of parameter values applicable to a typical stock.

Our model implies that if the exchange-mandated tick size is less than the optimal tick size, market makers would have an incentive to enforce the optimal tick size using alternative mechanisms. This incentive might explain the much-publicized controversy over the lack of odd-eighth quotes on the NASDAQ [Christie and Schultz (1994)]. Dealers could choose to avoid odd-eighth quotes in order to increase the effective tick size on the exchange. However, our theory cannot explain why odd-eighth quotes as opposed to even-eighth quotes are avoided.

Numerical simulations of the model reveal that the optimal tick size increases with the fraction of liquidity traders (λ), the upper bound of reservation commissions (s), the noise in the informed trader's signal (ψ^2), the informed trader's risk-aversion coefficient (H), and decreases with the amount of private information (σ^2).[9] In short, the optimal tick size increases as natural liquidity increases or as the degree of adverse selection diminishes. To see the intuition behind this result, note that liquidity traders begin to drop off the market as the tick size increases. When there is sufficient natural liquidity (or low adverse selection) in the market the negative externality due to a larger tick size is less significant. Therefore the market maker can afford to use a larger tick size. Conversely, when the adverse selection problem is acute, the optimal tick size is lower.

Our model implies that the optimal tick size should be lower when there is less natural liquidity in a stock. It is well known that low-price stocks have less inherent liquidity. Thus we should expect to see lower tick sizes for low-price stocks. Indeed, NYSE Rule 62 and AMEX Rule 127 explicitly reduce the tick size for lower-price stocks. Further evidence is found in Japan, Hong Kong, and other capital markets around the world, where the tick size is explicitly calibrated on the stock price level.

4. Other Implications of Discrete Prices

4.1 Asymmetric buyer and seller market

Table 1 shows the ask and bid commissions as the underlying equilibrium price varies. In the first row where $P_0 = \$10.05$, the ask commission is equal to \$0.075 and the bid commission is equal to \$0.05.

[9] For reasons of brevity, tables and figures showing the results of the numerical simulation have been omitted.

This asymmetry, which does not occur under continuous prices, holds in all the other cases discussed in the table. Unlike the model in Leach and Madhavan (1993), who show that the market maker may strategically post asymmetric commissions to gather information, our model shows that discrete prices per se can cause asymmetric commissions.

In addition, the indifference points on the ask and bid side of the market are not identically located. As discussed in Proposition 2, the indifference points on the ask side of the market lie at the locations $[(n+1)d - \text{mod}(a^*, d)]$, where $n \geq 0$. On the bid side, however, the indifference points are located at $[nd + \text{mod}(a^*, d)]$, where $n \geq 0$. Table 1 shows that at $t = 2$, the underlying equilibrium price is \$10.09 and the discrete-case ask price is \$10.125. When the underlying equilibrium price changes to \$10.10 at $t = 3$, the discrete-case ask price jumps to \$10.25. The location of the underlying equilibrium price where the jump first occurs is \$10.095. This location is referred to as the indifference point. In contrast, the discrete-case bid price jumps from \$10.00 (at $t = 4$) to \$10.125 (at $t = 5$) and the indifference point lies at \$10.155, which is not the same as the indifference point on the ask side of the market. This result is a distinct feature of our model. It differs from the assumption in Gottlieb and Kalay (1985) that the indifference points on the ask and bid side of the market are coincident and lie at the midpoint of a tick. In our model, the indifference points would coincide when $(n+1)d - \text{mod}(a^*, d) = nd + \text{mod}(a^*, d)$, that is, $\text{mod}(a^*, d) = d/2$. This would happen when a^* is a multiple of half the tick size.

4.2 Time-varying bid-ask spread

Under continuous prices, the bid-ask spread is equal to $2a^*$ and changes only when the degree of adverse selection changes. However, under discrete prices, the bid-ask spread changes even when there is no change in adverse selection. Table 1 shows that the bid-ask spread depends on the location of the underlying equilibrium price. It is equal to \$0.125 (at $t = 1$, 2, and 5) and changes to \$0.25 (at $t = 3$ and 4). Discrete prices per se can cause the bid-ask spread to vary with time. As shown in the adjoining footnote, depending on the location of P_0, the bid-ask spread can take either one of two levels: $[2a^* + 2d - \text{mod}(2a^*, d)]$ or $[2a^* + d - \text{mod}(2a^*, d)]$.[10]

[10] The ask commission is equal to $a^* + d - \text{mod}(P_0 + a^*, d)$. The bid commission can be shown to be equal to $a^* + \text{mod}(P_0 - a^*, d)$, after noting that the discrete-case bid price is rounded downward from the continuous-case bid price of $P_0 - a^*$. Therefore the bid-ask spread is equal to $2a^* + d - [\text{mod}(P_0 + a^*, d) - \text{mod}(P_0 - a^*, d)]$. It can be shown, after some algebra, that this expression reduces to $[2a^* + d - \text{mod}(2a^*, d)]$ or $[2a^* + 2d - \text{mod}(2a^*, d)]$, depending on whether $\text{mod}(P_0 + a^*, d) \geq \text{mod}(P_0 - a^*, d)$ or not.

Table 4
Market breakdown

$(\lambda, s, \sigma^2, \psi^2, H)$	a^*	$a^{*\prime}$	Ask side: Open $\forall P_0$:	Ask side: Closed $\forall P_0$:	Bid side: Open $\forall P_0$:	Bid side: Closed $\forall P_0$:
(0.60,0.20, 0.15,0.80,3)	0.09	0.186	[0,0.035] [0.064,0.125)	(0.035,0.064)	[0,0.061] [0.09,0.125)	(0.061,0.09)
(0.50,0.25, 0.15,1,1)	0.154	0.235	[0.015,0.096] (0.096,0.125)	[0,0.015)	[0.029,0.11] (0.11,0.125)	[0,0.029)
(0.80,0.14 0.05,0.40,3	0.049	0.138	[0,0.076] [0.112,0.125)	(0.076,0.112)	[0,0.013] [0.049,0.125)	(0.013,0.049)

This table shows the occurrence of market breakdowns when the tick size is equal to \$0.125. The parameters are as follows: $\lambda \equiv$ proportion of liquidity traders, $s \equiv$ upper bound of liquidity traders' reservation commissions, $\sigma^2 \equiv$ variance of private information (δ), $\psi^2 \equiv$ variance of noise in informed trader's signal, and $H \equiv$ coefficient of risk aversion. Also, $a^* \equiv$ smaller root of the solution to the market maker's zero expected profit condition, $a^{*\prime} \equiv$ larger root of the solution to the market-maker's zero expected profit condition, and $P_0 \equiv$ the location of the underlying equilibrium price. The last four columns show the locations of P_0 within a typical tick where the market is open or closed.

4.3 Market breakdown

So far we have assumed that $d \leq (a^{*\prime} - a^*)$. Table 4 shows the results of numerical simulations of the model when $d > (a^{*\prime} - a^*)$. For the parameter values shown in the first row of Table 4, $a^* = 0.09$ and $a^{*\prime} = 0.186$. Here, $d = 0.125 > (a^{*\prime} - a^*) = 0.096$. Suppose $P_0 = \$10.04$. The market maker cannot post \$10.125 as the ask price because it would result in an ask commission of \$0.085, which is less than the break-even commission of \$0.09. The next feasible ask price is \$10.25, which would result in an ask commission of \$0.21. However, $a^{*\prime}$, which is the highest commission at which the market maker gets nonnegative expected profits, is equal to \$0.186, and therefore \$10.25 cannot be the ask price. The market maker cannot post any other ask price because it would result in negative expected profits. In fact, the ask side of the market is closed whenever the underlying equilibrium price lies in the interval (0.035, 0.064). A similar analysis reveals that the bid side of the market is closed on the interval (0.061, 0.09). In this case, the interval for market breakdown on the ask side of the market is distinct from that on the bid side. It so happens in this case that for any location of P_0, the market is open either on the ask or bid side of the market. Nontrading would *not* be observed, although the market breaks down on one side of the market for many locations of P_0.

In other cases, as shown in the second row of the table, the intervals for market breakdown on the ask and bid sides of the market may overlap. In this case, $a^* = 0.154$, $a^{*\prime} = 0.235$, and $a^{*\prime} - a^* = 0.081$, which is less than the tick size. When the location of P_0 lies in the interval (0, 0.015) or in the interval (0.11, 0.125), both the ask and bid sides of the market would be closed. Here nontrading would be observed. The third row of the table illustrates another case. As in the first case, market breakdown on the ask and bid sides of the market occur over distinct sets of the location of P_0. The ask side of the market is closed on the interval (0.076, 0.012) and the bid side of the market is closed on the interval (0.013, 0.049). In this case, nontrading would not be observed.

The three examples in Table 4 show that market breakdowns occur when the a^* is equal to 0.09, 0.154, and 0.049. Thus stocks with differing levels of a^* can experience market breakdowns. The necessary condition for a market breakdown is $d > (a^{*\prime} - a^*)$, which is more likely to be satisfied for lower-price stocks. Note that $(a^{*\prime} - a^*)$ is small when s (the upper bound of liquidity traders' reservation commissions) is small. This is more likely to occur in low-price stocks because traders are more sensitive to small differences in transaction costs. Casual empirical evidence suggests that such stocks exhibit frequent nontrading and severe order imbalances.

5. Endogenous Information Acquisition

We endogenize the technology of information acquisition by allowing informed traders to choose the precision of their private signals. Informed traders observe $(\delta + \varepsilon)$, which is a noisy version of the true signal (δ) at the beginning of each period. For ease of exposition, let $p = \sigma^2$ (variance of δ) and let $v = \psi^2$ (variance of ε). Further, assume that informed traders face a cost schedule $C(v) > 0$, where, $C'(v) < 0$, $C''(v) > 0$.

First, consider the benchmark model with continuous prices. Informed traders choose the signal noise (v) that maximizes their unconditional expected utility of trading (i.e., over all possible signal values). Let $S(a, v)$ denote the unconditional expected utility of trading on the ask side of the market. The equilibrium is defined by the following two conditions: (i) the informed trader chooses v^* such that he maximizes $S(a^*, v)$, as defined in the equation below, and (ii) the market maker chooses an ask commission (a^*) such that he makes zero expected profits on the ask side of the market, as defined in Equation (3). We show that (see the Appendix)

$$\begin{aligned} S(a, v) \;=\; & -\exp^{HC(v)} \left\{ \sqrt{\frac{v}{p+v}} \exp^{-\frac{1}{2}\frac{a^2}{p}} \left[1 - \Phi\left(\frac{a\sqrt{v}}{p} \right) \right] \right. \\ & \left. + \Phi\left(\frac{a\sqrt{p+v}}{p} \right) - \Phi(0) \right\}. \end{aligned} \qquad (6)$$

The discrete price equilibrium is also derived in the same manner. Let a^{**} denote the equilibrium break-even commission on the ask (bid) side of the market under discrete prices. At this commission, the market maker makes zero expected profits under discrete prices. Informed traders are aware that the realized ask commissions would vary between $(a^{**}, a^{**}+d)$. Assuming that ask commissions are uniformly distributed over $(a^{**}, a^{**}+d)$, the unconditional expected utility of informed traders is given by $S^d(a^{**}, v) \equiv \int_{a^{**}}^{a^{**}+d} S(a, v)\frac{1}{d}da$. Informed traders choose v^{**} to maximize this expression. The equilibrium conditions are stated in Equations (7) and (8):

$$v^{**}\colon \max_v S^d(a^{**}, v) \equiv \int_{a^{**}}^{a^{**}+d} S(a, v)\tfrac{1}{d}da \tag{7}$$

$$\begin{aligned} a^{**}\colon & -[(1-\lambda)/(H\psi^2)][(1+u(a)^2)(1-\Phi(u(a))) - u(a)\phi(u(a))] \\ & + (\lambda/s)(s-a)a = 0. \end{aligned} \tag{8}$$

To summarize, the equilibrium commission under continuous prices is equal to a^* and informed traders choose v^* to maximize their unconditional expected utility. Further, the equilibrium commission is independent of the location of P_0. In contrast, the discrete-case equilibrium commissions vary over the interval $(a^{**}, a^{**}+d)$ and informed traders choose v^{**} to maximize their unconditional expected utility.

Lemma 1. $a^{**} < a^* < a^{**} + d/2$.

Proof. Suppose $a^{**} > a^*$. This implies that the discrete-case commissions for all possible locations of P_0 are strictly greater than under continuous prices. Since the unconditional expected utility, $S(a, v)$, is strictly decreasing in the commission charged, the informed trader would invest less under discrete prices and acquire a more noisy signal. In other words, $v^{**} > v^*$. But this inequality implies that $a^{**} < a^*$, because the market maker can break even at a lower commission if $v^{**} > v^*$. Thus our conjecture that $a^{**} > a^*$ leads us to a contradiction. It follows that, in equilibrium, the only consistent conjecture is that $a^{**} < a^*$ and $v^{**} > v^*$.

Because $S(a, v)$ is a convex function of the ask commission (a),[11] it follows from Jensen's inequality that the certainty equivalent commission under discrete prices (a^{ce}) is less than $E[a \mid a \in U(a^{**}, a^{**} + d)] = a^{**} + d/2$, where $a^{ce} \in (a^{**}, a^{**} + d)$. Therefore, to show that $a^* < a^{**} + d/2$, it would be sufficient to show that $a^* < a^{ce}$. Suppose the contrary, that is, $a^* > a^{ce}$. Then informed traders face more

[11] We rely on the result that indirect utility functions are convex in prices [see O'Hara (1995), p. 224, footnote 4]. Thus risk-averse informed traders prefer uncertainty in commissions (or the resulting prices).

commissions under continuous prices. They would invest less under continuous prices and choose $v^* > v^{**}$. This, in turn, implies $a^* < a^{**}$, which has already been shown to be impossible in equilibrium. Thus $a^* < a^{ce}$ and $a^{**} < a^* < a^{**} + d/2$ follows.

Proposition 4. *With endogenous information acquisition and discrete prices (i) informed traders invest less in information relative to the continuous case, (ii) locations of P_0 always exist for which either the ask or the bid commission is less than under continuous prices, and (iii) under certain parameter values, for some locations of P_0, the bid-ask spread is lower under discrete prices than under continuous prices.*

Items (i) and (ii) follow from the discussion in Lemma 1. Since $v^{**} > v^*$ informed traders would invest less under discrete prices. Further, because the discrete-case ask commission varies over $(a^{**}, a^{**} + d)$, where $a^{**} < a^*$, there would be at least some locations of P_0 with a lower ask commission under discrete prices. The same would hold on the bid side. We establish item (iii) in the Appendix and derive the condition under which the discrete-case bid-ask spread can be lower for certain locations of P_0. Thus some liquidity traders could be better off under discrete prices.

6. Conclusions

This article analyzes the economic effects of exchange-mandated discrete pricing restrictions. We find that (i) the discreteness-related profits can help recoup fixed costs; (ii) if competing exchanges face significant start-up costs, market makers can realize economic rents; (iii) a tick size of \$1/8 can be optimally chosen by profit-maximizing market makers; (iv) the optimal tick size is decreasing in the degree of adverse selection; (v) the market-maker's pricing rule can be described by asymmetric rounding mechanisms on the ask and bid sides of the market; (vi) discrete prices per se can cause time-varying bid-ask spreads, asymmetric commissions, and market breakdowns; and (vii) some liquidity traders can be better off with discrete prices under endogenous information acquisition.

It would be interesting to analyze the impact of discrete prices in a setting where inventory carrying costs also affect the bid-ask spread. In principle, the discrete pricing restriction should result in economic profits for the market maker. However, the market maker may find it optimal to give a price concession because of reasons related to inventory. The formal modeling of this trading mechanism could provide additional insights on asset price evolution.

Appendix

Proof of Proposition 1

Using the formula for the moment-generating function of a normal random variable, the expected utility of the risk-averse informed trader conditional on observing a signal $\delta+\varepsilon$ is computed as $EU[x(\delta-a) \mid \delta+\varepsilon] = E[-\exp\{-Hx(\delta-a)\} \mid \delta+\varepsilon] = -\exp(Hxa)\exp\{-HxE[\delta \mid \delta+\varepsilon] + (1/2)H^2x^2 var[\delta \mid \delta+\varepsilon]\}$. For the conditional normal distribution, $E[\delta \mid \delta+\varepsilon] = (1/w)(\delta+\varepsilon)$, $var[\delta \mid \delta+\varepsilon] = (1/w)\psi^2$, where $w \equiv (\sigma^2+\psi^2)/\sigma^2$. Then the conditional expected utility of the informed trader is given by $-\exp\{Hx[a-(1/w)(\delta+\varepsilon)+\frac{1}{2}Hx(1/w)\psi^2]\}$. Maximizing this expression with respect to x yields $x^* = (1/H\psi^2)[(\delta+\varepsilon)-aw]$. Equation (2) follows upon recognizing that when $\delta+\varepsilon \le aw$, the informed trader does not enter any trades on the ask side of the market, that is, $x^* = 0$.

Using the solution for x^*, the informed trader's expected profits are given by $\pi^{IT}(a) = (1-\lambda)E[x^*(\delta-a)] = (1-\lambda)\Pr[\delta+\varepsilon > aw]E[(1/H\psi^2)(\delta+\varepsilon-aw)(\delta-a) \mid \delta+\varepsilon > aw] = (1-\lambda)(1/H\psi^2)\Pr[\delta+\varepsilon > aw]E[E[(\delta+\varepsilon-aw)(\delta-a) \mid \delta+\varepsilon] \mid \delta+\varepsilon > aw] = (1-\lambda)(1/H\psi^2)\Pr[\delta+\varepsilon > aw]E[(\delta+\varepsilon-aw)E[(\delta-a) \mid \delta+\varepsilon] \mid \delta+\varepsilon > aw]$, which can be expressed as

$$\pi^{IT}(a) = (1-\lambda)(1/H\psi^2)(1/w)\Pr[y>0]E[y^2 \mid y>0], \text{ where } y = \delta+\varepsilon-aw. \tag{9}$$

Then because $\delta \equiv N(0,\sigma^2)$ and $\varepsilon \equiv N(0,\psi^2)$, y is $N[-aw, (\sigma^2+\psi^2)]$ or $N[-aw, \sigma^2 w]$. Note that $E[y^2 \mid y>0] = \text{var}[y \mid y>0] + \{E[y \mid y>0]\}^2$. Using the results in the adjoining footnote, we get

$$E[y^2 \mid y>0] = a^2w^2 + \sigma^2 w - a\sigma w^{3/2}\left[\frac{\phi\left(\frac{a\sqrt{w}}{\sigma}\right)}{1-\Phi\left(\frac{a\sqrt{w}}{\sigma}\right)}\right].^{12} \tag{10}$$

Using this result and noting that $\Pr[y>0] = [1-\Phi(\frac{a\sqrt{w}}{\sigma})]$, we can simplify the expression in Equation (9) to get $\pi^{IT}(a) = [\frac{(1-\lambda)T\sigma^2}{H\psi^2}]\{[1-$

[12] If $Y \equiv N(\xi,\sigma^2)$, then we have the following three identities [see Johnson and Kotz (1970), p. 83]:

$$E[Y \mid A \le Y \le B] = \xi + \sigma\left[\frac{\phi\left(\frac{A-\xi}{\sigma}\right)-\phi\left(\frac{B-\xi}{\sigma}\right)}{\Phi\left(\frac{B-\xi}{\sigma}\right)-\Phi\left(\frac{A-\xi}{\sigma}\right)}\right],$$

$$\text{var}[Y \mid A \le Y \le B] = \sigma^2\left[1+\frac{\left(\frac{A-\xi}{\sigma}\right)\phi\left(\frac{A-\xi}{\sigma}\right)-\left(\frac{B-\xi}{\sigma}\right)\phi\left(\frac{B-\xi}{\sigma}\right)}{\Phi\left(\frac{B-\xi}{\sigma}\right)-\Phi\left(\frac{A-\xi}{\sigma}\right)} - \left\{\frac{\phi\left(\frac{A-\xi}{\sigma}\right)-\phi\left(\frac{B-\xi}{\sigma}\right)}{\Phi\left(\frac{B-\xi}{\sigma}\right)-\Phi\left(\frac{A-\xi}{\sigma}\right)}\right\}^2\right],$$

$\Phi(u(a))][1 + u(a)^2] - u(a)\phi(u(a))\}$, where $u(a) = a\sqrt{w}/\sigma$. This term represents the expected profits of the informed traders. The expected losses of liquidity traders $[\pi^{LT}(a)]$ are equal to $\lambda \Pr(\tilde{s} > a)a = \lambda(s - a)a/s$, given that $\tilde{s} \equiv U(0, s)$. The market maker's expected profits is given by $\pi^{MM}(a) = -\pi^{IT}(a) + \pi^{LT}(a)$ and the result in Equation (3) follows.

Convexity of informed traders' expected profit function

Note that $\frac{\partial \pi(a)}{\partial a} = \frac{\partial \pi^{IT}(u(a))}{\partial u}\frac{\partial u}{\partial a} = \frac{\partial \pi^{IT}(u(a))}{\partial u}\frac{\sqrt{w}}{\sigma}$. Differentiating the expression for $\pi^{IT}(a)$ given above and using the following identities, $\Phi'(u) = \phi(u)$ and $\phi'(u) = -u\phi(u)$, we get

$$\frac{\partial \pi^{IT}(a)}{\partial a} = \left[\frac{2(1-\lambda)T\sigma\sqrt{w}}{H\psi^2}\right]\{[1 - \Phi(u)]u - \phi(u)\}. \tag{11}$$

Using the standard result for the normal distribution stated in the adjoining footnote, it follows that $[1 - \Phi(u)]u \leq \phi(u)\forall u$.[13] Thus $\frac{\partial \pi^{IT}(a)}{\partial} < 0$. Differentiating Equation (11) with respect to a, we have

$$\frac{\partial^2 \pi^{IT}(a)}{\partial a^2} = \left[\frac{2(1-\lambda)T\sigma\sqrt{w}}{H\psi^2}\right][1 - \Phi(u(a))]. \tag{12}$$

The right-hand side of Equation (12) is greater than zero and it follows that $\pi^{IT}(a)$ is convex. Also note that the expected loss function of liquidity traders $[\pi^{LT}(a) = \lambda(s - a)a/s]$ is a quadratic function of the ask commission (a). Thus it is concave in the ask commission. It follows that $\pi^{MM}(a)$ is concave in the ask commission because $\pi^{IT}(a)$ is convex and $\pi^{LT}(a)$ is concave in the ask commission.

Proof of Proposition 2

Rounding (upward) the continuous-case ask price ($P_0 + a^*$) yields the discrete-case ask price $A = P_0 + a^* + [d - \mathrm{mod}(P_0 + a^*, d)]$. At the indif-

and

$$E[Y^2 \mid A \leq Y \leq B] = \xi^2 + \sigma^2\left[1 + \frac{\left(\frac{A-\xi}{\sigma}\right)\phi\left(\frac{A-\xi}{\sigma}\right) - \left(\frac{B-\xi}{\sigma}\right)\phi\left(\frac{B-\xi}{\sigma}\right)}{\Phi\left(\frac{B-\xi}{\sigma}\right) - \Phi\left(\frac{A-\xi}{\sigma}\right)}\right] + 2\sigma\xi\left[\frac{\phi\left(\frac{A-\xi}{\sigma}\right) - \phi\left(\frac{B-\xi}{\sigma}\right)}{\Phi\left(\frac{B-\xi}{\sigma}\right) - \Phi\left(\frac{A-\xi}{\sigma}\right)}\right].$$

[13] This inequality is a fairly standard result in statistics. See *The Handbook of Normal Distribution* by Patel and Read (1982), Section 3.7.1, p. 64, for a result on the Mill's ratio: $R(u) = [1 - \Phi(u)]/\phi(u) < 1/u \ \forall \ u > 0$.

ference locations (f^n) the ask commission is equal to a^* under both discrete and continuous prices. Thus f^n represents the value of P_0 such that $[d - \text{mod}(P_0 + a^*, d)] = 0$, that is, $\text{mod}(f^n + a^*, d) = d$. Note, $\text{mod}(f^n + a^*, d) = \text{mod}(f^n, d) + \text{mod}(a^*, d) - d$, if $\text{mod}(f^n, d) + \text{mod}(a^*, d) > d$. Otherwise it is equal to $\text{mod}(f^n, d) + \text{mod}(a^*, d)$. Therefore the condition $\text{mod}(f^n + a^*, d) = d$ implies that $\text{mod}(f^n, d)$ is either equal to $[2d - \text{mod}(a^*, d)]$ or equal to $[d - \text{mod}(a^*, d)]$. Since the former is impossible, $\text{mod}(f^n, d) = d - \text{mod}(a^*, d)$.

Proof of Proposition 3

Market makers choose d^* to maximize $\int_{a^*}^{a^*+d} \pi^{MM}(a)\frac{1}{d}da$. The first-order condition is given by

$$d^*: -\int_{a^*}^{a^*+d} \pi^{MM}(a)\tfrac{1}{d}da + \pi^{MM}(a^* + d) = 0. \tag{13}$$

Note that Equation (13) has been derived after using the identity $\pi^{MM}(a^*) = 0$ and applying Leibnitz's rule. Differentiation of the first-order condition yields $\frac{\partial}{\partial a}(\pi^{MM}(a^* + d^*))$. To show that this term is less than zero (second-order condition), we proceed as follows. First, apply the mean value theorem for the function $\pi^{MM}(a)$ over the interval $[a^* + d^*/2, a^* + d^*]$ to get

$$\begin{gathered}[\pi^{MM}(a^* + d^*) - \pi^{MM}(a^* + d^*/2)] = (d^*/2)\tfrac{\partial \pi^{MM}(c)}{\partial a}, \\ \text{where, } c \in [a^* + d^*/2, a^* + d^*].\end{gathered} \tag{14}$$

Because $\pi^{MM}(a)$ is concave in a (as shown earlier), we can apply Jensen's inequality to get the following inequality: $E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d^*)] < \pi^{MM}(a^* + d^*/2)$. Further, since Equation (13) implies that $\pi^{MM}(a^* + d^*) = E[\pi^{MM}(a) \mid a \in U(a^*, a^* + d^*)]$, we can conclude that the left-hand side of Equation (14) is negative. It follows from Equation (14) that $\frac{\partial \pi^{MM}(c)}{\partial a} < 0$, $c \in (a^*, a^* + d^*)$. Concavity of $\pi^{MM}(a)$ implies that $\frac{\partial \pi^{MM}(a)}{\partial a}$ is monotonically decreasing. Therefore $\frac{\partial(\pi^{MM}(a^*+d^*))}{\partial a} < 0$, which satisfies the second-order condition.

Derivation of Equation (6)

Following the proof of Proposition 1, it can be shown that the optimal number of shares traded by the informed trader (x^*) is the same as in Equation (2) when a nonzero fixed cost $C(v)$ is incurred in acquiring the signal. Recall that $w = (\sigma^2 + \psi^2)/\sigma^2 = (p + v)/p$ and $x^* = \max\{0, (1/Hv)[(\delta + \varepsilon) - aw]\}$. The unconditional expected utility of the informed trader is given by $S(a, v) = EU[x^*(\delta - a) - C(v)] = \Pr\{\delta + \varepsilon > aw\}EU[(1/Hv)[\delta + \varepsilon - aw](\delta - a) - C(v) \mid \delta + \varepsilon >$

$aw] + \Pr\{\delta + \varepsilon \leq aw\} EU[0(\delta - a) - C(v) \mid \delta + \varepsilon \leq aw]$. Therefore,

$$\begin{aligned} S(a, v) &= \Pr\{\delta + \varepsilon > aw\} \\ &\quad \times EU[(1/Hv)[\delta + \varepsilon - aw](\delta - a) - C(v) \mid \delta + \varepsilon > aw] \\ &\quad - \exp(HC(v))\{\Phi[(a/p)(p + v)^{1/2}] - \Phi[0]\}. \qquad (15) \end{aligned}$$

The first term of Equation (15) can be expanded as $EU[(1/Hv)[\delta + \varepsilon - aw](\delta - a) - C(v) \mid \delta + \varepsilon > aw] = -E\{\exp[-(1/v)(\delta + \varepsilon - aw)(\delta - a) + HC(v) \mid \delta + \varepsilon > aw]\} = -\exp(HC(v))E\{E\{\exp[-(1/v)(\delta + \varepsilon - aw)(\delta - a) \mid \delta + \varepsilon]\} \mid \delta + \varepsilon > aw\}$. Let $t \equiv -(1/v)(\delta + \varepsilon - aw)$. Given $\delta + \varepsilon$, t is a known constant. The expression simplifies to $-\exp(HC(v))E\{E\{\exp[t(\delta - a) \mid \delta + \varepsilon]\} \mid \delta + \varepsilon > aw\} = -\exp(HC(v))E\{\exp(-ta)E\{\exp[t\delta \mid \delta + \varepsilon]\} \mid \delta + \varepsilon > aw\} = -\exp(HC(v))E\{\exp(-ta)\exp\{tE[\delta \mid \delta + \varepsilon] + (\frac{1}{2})t^2 \operatorname{var}[\delta \mid \delta + \varepsilon]\} \mid \delta + \varepsilon > aw\} = -\exp(HC(v))E\{\exp\{t(1/w)(\delta + \varepsilon - aw) + (\frac{1}{2})t^2(1/w)v\} \mid \delta + \varepsilon > aw\}$, after using results on the conditional normal distribution. Resubstituting $t = -(1/v)(\delta + \varepsilon - aw)$, this expression simplifies to

$$-\exp(HC(v))E\left\{\exp\left\{-\left(\frac{1}{2}\right)[p/(v(p + v))]y^2\right\} \mid y > 0\right\},$$

where $y = \delta + \varepsilon - aw$. After taking expectation, this expression further simplifies to $-\frac{\exp^{HC(v)}}{\left[1-\Phi\left(\frac{a\sqrt{p+v}}{p}\right)\right]} \exp^{-\frac{1}{2}\frac{a^2}{p}} \frac{\sqrt{v}}{p+v}[1-\Phi(a\frac{\sqrt{v}}{p})]$. Using this result and noting that $\Pr\{\delta+\varepsilon > aw\} = \Pr\{y > 0\} = [1-\Phi(\frac{a\sqrt{p+v}}{p})]$, Equation (15) can be reduced to Equation (6).

Proof of Proposition 4

Lemma 1 states that $a^* < a^{**} + d/2$, that is, $2a^{**} + d > 2a^*$. It follows that $2a^{**} + d + [d - \operatorname{mod}(2a^{**}, d)] > 2a^*$, because the expression in the square brackets is strictly positive. This inequality states that the (higher) bid-ask spread under discrete prices is always greater than the bid-ask spread under continuous prices (please refer to footnote 10). We turn to the lower level of the bid-ask spread: $[2a^{**} + d - \operatorname{mod}(2a^{**}, d)]$. This is a rounded-up version of $2a^{**}$ (to the nearest, higher tick). For this to be less than $2a^*$, a discrete tick has to lie in the interval $(2a^{**}, 2a^*)$, that is, the interval (a^{**}, a^*) should contain a multiple of the half-tick size. The likelihood of the condition being satisfied increases when a^* is incrementaly greater than a multiple of $d/2$. Thus discrete prices can lead to a lower bid-ask spread for some locations of P_0.

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