# Assignment A

- **Give a math deduction to the general form of "discrete Laplacian operator".**

**Deduction:**

In the case of a 2-D function$f(x, y)$, the Laplace operator produces:

$$\nabla^2 f = \frac{\partial^2 f}{\partial^2 x} + \frac{\partial^2 f}{\partial^2 y}$$

While in discrete case, the second-order differentiation becomes second-order difference:

In 1-D case, the second-order difference is defined as:

$$\begin{aligned}\Delta f[x] &= \nabla(\nabla f[x]) \\ &= \nabla f[x] - \nabla f[x-1] \\ &= (f[x+1] - f[x]) - (f[x] - f[x-1]) \\ &= f[x+1] - 2f[x] + f[x-1]\end{aligned}$$

In 2-D case, Laplace operator is the sum of two second-order differences in both dimensions:

$$\begin{aligned}\Delta f[x, y] &= \Delta_x \left[f[x, y]\right] + \Delta_y \left[f[x, y]\right] \\ &= f[x+1, y] - 2f[x, y] + f[x-1, y] + f[x, y+1] - 2f[x, y] + f[x, y-1] \\ &= f[x+1, y] + f[x-1, y] + f[x, y+1] + f[x, y-1] - 4f[x, y]\end{aligned}$$

So the operation can be carried out in the form of matrix:

$$\begin{bmatrix} 0 & -1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- **Give a math deduction to the form of “Laplacian of Gaussian”.**
  - **Math deduction**
  - **Give an example of LoG operator (of size 5*5)**

### 1) Deduction

To suppress the noise, before using Laplacian operator for edge detection, first we should use a convolution with a Gaussian filter to smooth the image:

$$\Delta[G_\sigma(x,y) * f(x,y)]$$

Due to the fact that:

$$\frac{d}{dt}[h(t) * f(t)] = \frac{d}{dt}\int f(\tau)h(t-\tau)d\tau \quad = \int f(\tau)\frac{d}{dt}h(t-\tau)d\tau = f(t) * \frac{d}{dt}h(t)$$

We can obtain the Laplacian of Gaussian operator first and then convolve it with the input image:

$$\Delta[G_\sigma(x,y) * f(x,y)] = [\Delta G_\sigma(x,y)] * f(x,y) = LoG * f(x,y)$$

In which the Gaussian is:

$$G_\delta(x,y) = \frac{1}{\sqrt{2\pi\delta^2}} exp\left(-\frac{x^2+y^2}{2\delta^2}\right)$$

So LoG can be defined as:

$$LoG \triangleq \Delta G_\sigma(x,y) = \frac{\partial^2}{\partial^2 x}G_\sigma(x,y) + \frac{\partial^2}{\partial^2 y}G_\sigma(x,y)$$

$$= -\frac{1}{\pi\sigma^4}(1-\frac{x^2+y^2}{2\sigma^2})\exp(-\frac{x^2+y^2}{2\delta^2})$$

### 2) The example of LoG operator (of size 5*5)

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 2 & -16 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$