1. The role of geodesics

December 25, 2021
Jan Střeleček

1.1 Minimizing the energy variance

From Bukov et al. [2019]. About transports using *fast forward* Hamiltonian means the system is driven to the target state in some fixed amount of time. The transport is done on the ground state manifold $\mathcal{M}$.

Conjecture 1. For any fast forward Hamiltonian $\hat{H}(\lambda(t))$ driven along one dimentional path $\lambda(t) : \mathbb{R} \mapsto \mathbb{R}$ using time t as parametrization, the energy fluctuations δE^2, averaged along the path, are larger than the geodesic length l_λ

$$\int_0^T \sqrt{\delta E^2(t)}\mathrm{d}t =: l_t \geq l_\lambda \int_{\lambda_i}^{\lambda_f} \sqrt{g_{\lambda\lambda}}\mathrm{d}\lambda = \int_0^T \sqrt{g_{\lambda\lambda}}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\mathrm{d}t. \tag{1.1}$$

The length $l_{\boldsymbol{\lambda}}$ is defined in control space (with metric tensor $g_{\lambda\lambda}$) and is generally larger than the distance between wave functions, i.e. the absolute geodesic (defined with $G_{\mu\nu}$). From its definition, we can see that it corresponds to the metric tensor as we use it.

The energy variance is

$$\delta E^2 = \langle o(t)|\hat{H}(t)^2|o(t)\rangle - \langle o(t)|\hat{H}(t)|o(t)\rangle^2 = \langle \partial_t(t)|\partial_t o(t)\rangle\rangle_c = G_{tt} \tag{1.2}$$

and the Metric tensor in control space is defined as

$$g_{\lambda\lambda} \coloneqq \langle \partial_\lambda o(t)|\partial_\lambda o(t)\rangle\rangle_c \tag{1.3}$$

Proof.

$$\delta E^2 \equiv \langle o(t)|\hat{H}(t)^2|o(t)\rangle_c = \dot{\lambda}^2 G_{\lambda\lambda} + \mathcal{O}(\dot{\lambda}^4), \tag{1.4}$$

where $\mathcal{O}(\dot{\lambda}^4)$ needs to be positive for any real-valued Hamiltonian. This comes from the fact, that it has instantaneous time-reversal symmetry. □

The conjecture only applies to unit fidelity protocols and can be extended to an arbitrary dimensional path.

1.2 APT

From Rigolin et al. [2008].

The power series will be derived using a small parameter $v = 1/T$.

Starting again with

$$|\Psi(s)\rangle = \sum_{p=0}^{\infty} v^p |\Psi^{(p)}(s)\rangle, \tag{1.5}$$

for

$$|\Psi^{(p)}(s)\rangle = \sum_{n=0} e^{-\frac{i}{v}\omega_n(s)} e^{i\gamma_n(s)} b_n^{(p)}(s) \, |n(s)\rangle \, . \tag{1.6}$$

Here the

$$\omega_n(s) := \frac{1}{\hbar}\int_0^s E_n(s')\mathrm{d}s' \tag{1.7}$$

$$\gamma_n(s) := i\int_0^s \langle n(s')| \frac{\mathrm{d}}{\mathrm{d}s'} n(s')\rangle \, \mathrm{d}s' \equiv i\int_0^s M_{nn}(s')\mathrm{d}s' \tag{1.8}$$

are so-called dynamical resp. Berry (geometric) phase and $|n(s)\rangle$ are solution to

$$\hat{H}(s)\,|n(s)\rangle = E_n(s)\,|n(s)\rangle \, . \tag{1.9}$$

The problem again lies in determining $b_n^{(p)}(s)$, which will be done iteratively. Because it is dependent on its relative geometric and dynamical phases to other energy levels, lets write it as a series

$$b_n^{(p)}(s) = \sum_{m=0} e^{\frac{i}{v}\omega_{nm}(s)} e^{-i\gamma_{nm}(s)} b_{nm}^{(p)}(s), \tag{1.10}$$

where $\omega_{nm} := \omega_m - \omega_n$, $\gamma_{nm} := \gamma_m - \gamma_n$. The new coefficients $b_{mn}^{(p)}(s)$ now **depend only on the metric structure around point** s and therefore one may perform adiabatic expansion from any point on manifold.

Inserting all to original series 1.5, we get

$$|\Psi(s)\rangle = \sum_{n,m=0} \sum_{p=0}^{\infty} v^p e^{-\frac{i}{v}\omega_m(s)} e^{i\gamma_m(s)} b_{nm}^{(p)}(s) \, |n(s)\rangle \, . \tag{1.11}$$

Because the initial state is eigenstate, we get initial conditions $b_{nm}^{(0)}(s) = 0$. In addition, one can rewrite equation 1.11 to the iteratively solvable form

$$\frac{i}{\hbar}\Delta_{nm}(s) b_{nm}^{(p+1)}(s) + \dot{b}_{nm}^{(p)}(s) + W_{nm}(s) b_{nm}^{(p)}(s) + \sum_{k=0, k\neq n} M_{nk}(s) b_{km}^{(p)}(s) = 0, \tag{1.12}$$

for $W_{nm}(s) := M_{nn}(s) - M_{mm}(s)$, where M_{mn} is defined in Eq. 1.8. We can see that $b_{mn}^{(p)}$, as a solution to Eq. 1.12, only depends on difference between energy levels, eigenstates during the path and their directional derivatives. All of those are easily obtained, once the driving path is prescribed.

Bibliography

Marin Bukov, Dries Sels, and Anatoli Polkovnikov. Geometric Speed Limit of Accessible Many-Body State Preparation. *Physical Review X*, 9(1):011034, feb 2019. ISSN 2160-3308. doi: 10.1103/PhysRevX.9.011034. URL `https://doi.org/10.1103/PhysRevX.9.011034https://link.aps.org/doi/10.1103/PhysRevX.9.011034`.

Gustavo Rigolin, Gerardo Ortiz, and Victor Hugo Ponce. Beyond the quantum adiabatic approximation: Adiabatic perturbation theory. *Physical Review A - Atomic, Molecular, and Optical Physics*, 78(5):1–26, 2008. ISSN 10502947. doi: 10.1103/PhysRevA.78.052508.