1. From the projective Hilbert space to state manifolds

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1.1 Projective Hilbert space

Consider the Hilbert space $\mathcal{H}$ to be a space of *bare states* and $\mathcal{S}$ to be the space of *normalized bare states*. Physical observables are related to the *space of rays*, defined as $\mathcal{PH} \coloneqq \mathcal{H}/U(1)$, for the factorization by elements of $U(1)$. This group consists of unitary transformations $e^{i\phi}$ for $\phi \in \mathbb{R}$, defining gauge symmetry between quantum states. $\mathcal{PH}$ is then considered to be the *space of pure states.* For the sake of generality, let's not normalize our vectors yet, which would lead to the *space of unnormalized pure physical states.*

It can be shown, that $\mathcal{PH}$ is of a Kähler structure, meaning it has two non-degenerate sesquilinear[1] 2-forms embedded along with operator complex unit

$$(J, G, \Omega),$$

such that

$$J^2 = \mathbb{1} \tag{1.1}$$

and any bracket of $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{PH}$ can be decomposed into real and imaginary part[Ashtekar and Schilling]

$$\langle\psi_1|\psi_2\rangle = \frac{1}{2}G(\psi_1, \psi_2) - \frac{i}{2}\Omega(\psi_1, \psi_2). \tag{1.2}$$

From braket sesquilinearity goes that G is symmetric and Ω antisymmetric form, thus they can be uniquely written into one 2-form called *Fubini-Study metric* with property

$$G = \mathrm{Re}Q; \qquad \Omega = \mathrm{Im}Q. \tag{1.3}$$

Because $\langle\psi_1|\psi_2\rangle \in [0, 1]$ we say, that the metric is measuring the geodesic distance on the Bloch sphere. Here if we define

$$|\langle\psi_1|\psi_2\rangle| = \cos^2\frac{\theta}{2}, \tag{1.4}$$

we get $\mathrm{d}\theta = 2\mathrm{d}s = 2\sqrt{|g_{\mu\nu}\mathrm{d}\boldsymbol{\Lambda}^\mu\mathrm{d}\boldsymbol{\Lambda}^\nu|}$, see Cheng.

To write the metric in a standard form, we need to realize how our space looks like. For finite $n+1$-dimensional Hilbert space, one dimension is lost in the gauge transformation, leaving us with n-dimensional $\mathcal{PH}$. Another dimension is lost due to normalization, which is usually done by mapping to an n-dimensional complex sphere

$$CP^n = \Big\{\mathbb{Z} = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{C}^{n+1}/\{0\}\Big\} \Big/ \{\mathbb{Z} \sim c\mathbb{Z} \text{ for } c \in \mathbb{C}\}.$$

[1] We are in physics, so complex conjugated is the first input of the 2-form.

Natural property of such complex spaces is splitting of its tangent space to holonomous and anholonomous part[2]

$$T^{1,0}\mathcal{M} = \mathrm{Span}\left\{\frac{\partial}{\partial Z_i}\right\}; \qquad T^{0,1}\mathcal{M} = \mathrm{Span}\left\{\frac{\partial}{\partial Z_{\bar{i}}}\right\}.$$

Distance on $\mathbb{C}^{n+1}$ is standardly defined using Hermitean metric[3]

$$\mathrm{d}s^2 = \mathrm{d}\overline{\mathbb{Z}} \otimes \mathrm{d}\mathbb{Z}. \tag{1.5}$$

For *normalized states* in quantum mechanics is $\mathrm{d}Z = |\psi + \delta\psi\rangle - |\psi\rangle$, which plugged into Eq. 1.5 yields (remember that complex conjugation changes the sign of variation)

$$\mathrm{d}s^2 = (\langle\psi - \delta\psi| - \langle\psi|)(|\psi + \delta\psi\rangle - |\psi\rangle) = 1 - \langle\delta\psi|\psi\rangle\,\langle\psi|\delta\psi\rangle + o(ket\delta\psi^2) \tag{1.6}$$

This can also be achieved by defining the distance between two infinitezimally separated states is $\mathrm{d}s = \langle\psi|\psi + \delta\psi\rangle$. The distance can also be rewritten as

$$\mathrm{d}s^2 = 1 - |\langle\delta\psi|\psi\rangle|^2\,. \tag{1.7}$$

1.2 Restriction to eigenstate manifolds

In quantum mechanics, one usually examine some system defined with Hamiltonian $\hat{H}(\mathbf{\Lambda})$, for some parameter $\mathbf{\Lambda} \in \mathbb{R}^n$. This gives us the eigenbasis of $\hat{H}$. Metric tensor of $\mathcal{PH}$ will not depend on this choise, but the Hamiltonian itself creates some natural sections on this manifold.

One of them might be the section along Schrödinger evolution of some state $|\psi_i\rangle$, meaning

$$S_{|\psi_i\rangle} := \left\{|\psi(t)\rangle \text{ following } i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|k\rangle \text{ on time interval } [t_i, t_f]\right\}.$$

Another more interesting sections are *eigenstate manifolds*, defined by setting only one non-zero coefficient Z_k in eigenbasis $|\psi\rangle = \sum_{k=0}^{n} Z_k |k\rangle$. From normalization goes automatically $Z_k = 1$. The distance is then

$$\mathrm{d}s^2 = 1 - \langle\delta k|\,k\rangle\langle k\,|\delta k\rangle = 1 - \langle\delta k|\left(\mathbb{1} - \sum_{j\neq k}|j\rangle\,\langle j|\right)|\delta k\rangle = \sum_{j\neq k}\langle\delta k|\,j\rangle\langle j\,|\delta k\rangle\,. \tag{1.8}$$

Using the Schrödinger equation $\hat{H}|k\rangle = E_k|k\rangle$, distributivity of derivative and projection to some state $\langle j|$, we get

$$\begin{aligned}
\hat{H}|k\rangle &= E_k|k\rangle \\
(\delta\hat{H})|k\rangle + \hat{H}|\delta k\rangle &= (\delta E_k)|k\rangle + E_k|\delta k\rangle \\
\langle j|\left(\delta\hat{H} - \delta E_k\right)|k\rangle &= \langle j|\left(E_k - \hat{H}\right)|\delta k\rangle = \langle j|\left(E_k - E_j\right)|\delta k\rangle\,.
\end{aligned} \tag{1.9}$$

[2] $T^{p,q}\mathcal{M}$ means $p+q$- cotravariant space (the space of vectors) on $\mathcal{M}$. The line over letter means complex conjugation.

[3] which is by definition sesquilinear, as one would expect in quantum mechanics later on

We can set[4] $\delta E_k = 0$, leading for $j \neq k$ to

$$\frac{\langle j|\delta\hat{H}\,|k\rangle}{(E_k - E_j)^2} = \langle j\,|\delta k\rangle\,. \tag{1.10}$$

Plugging to Equation 1.8 and considering $\hat{H} = \hat{H}(\mathbf{\Lambda})$, we get metric on a ground state manifold

$$\mathrm{d}s^2 = \mathrm{Re}\sum_{j\neq k}\frac{\langle 0|\,\partial_\mu\hat{H}\,|j\rangle\langle j|\partial_\nu\hat{H}\,|0\rangle}{(E_k - E_j)^2}\mathrm{d}\mathbf{\Lambda}^\mu\mathrm{d}\mathbf{\Lambda}^\nu \tag{1.11}$$

Definition of the k-state manifold is then

$$g^{(k)}_{\mu\nu} = \mathrm{Re}\sum_{j\neq k}\frac{\langle k|\frac{\partial\hat{H}(\mathbf{\Lambda})}{\partial\lambda^\mu}|j\rangle\,\langle j|\frac{\partial\hat{H}(\mathbf{\Lambda})}{\partial\lambda^\nu}|k\rangle}{(E_k - E_j)^2}. \tag{1.12}$$

The Fubini-Study metric on the eigenstate manifold is sometimes called *Geometric tensor*. For the Lipkin-Meshkov Glick model, those can be seen on Fig. 1.2.

If we compare first eigenstate manifold $\mathcal{M}_1$ with difference in infidelity transport done along ground state geodesic and along streight line, we can see they have one similarity. The higher curvature of $\mathcal{M}_1$ degrades geodesic even faster, making the streight line more advantageous. This is caused by the wavefunction running out from $\mathcal{M}_0$ to $\mathcal{M}_1$

[4] Can it be done only for E_0? It does not make sence generally, because $E = E(\mathbf{\Lambda})$, even $E_0 = E_0(\mathbf{\Lambda})$

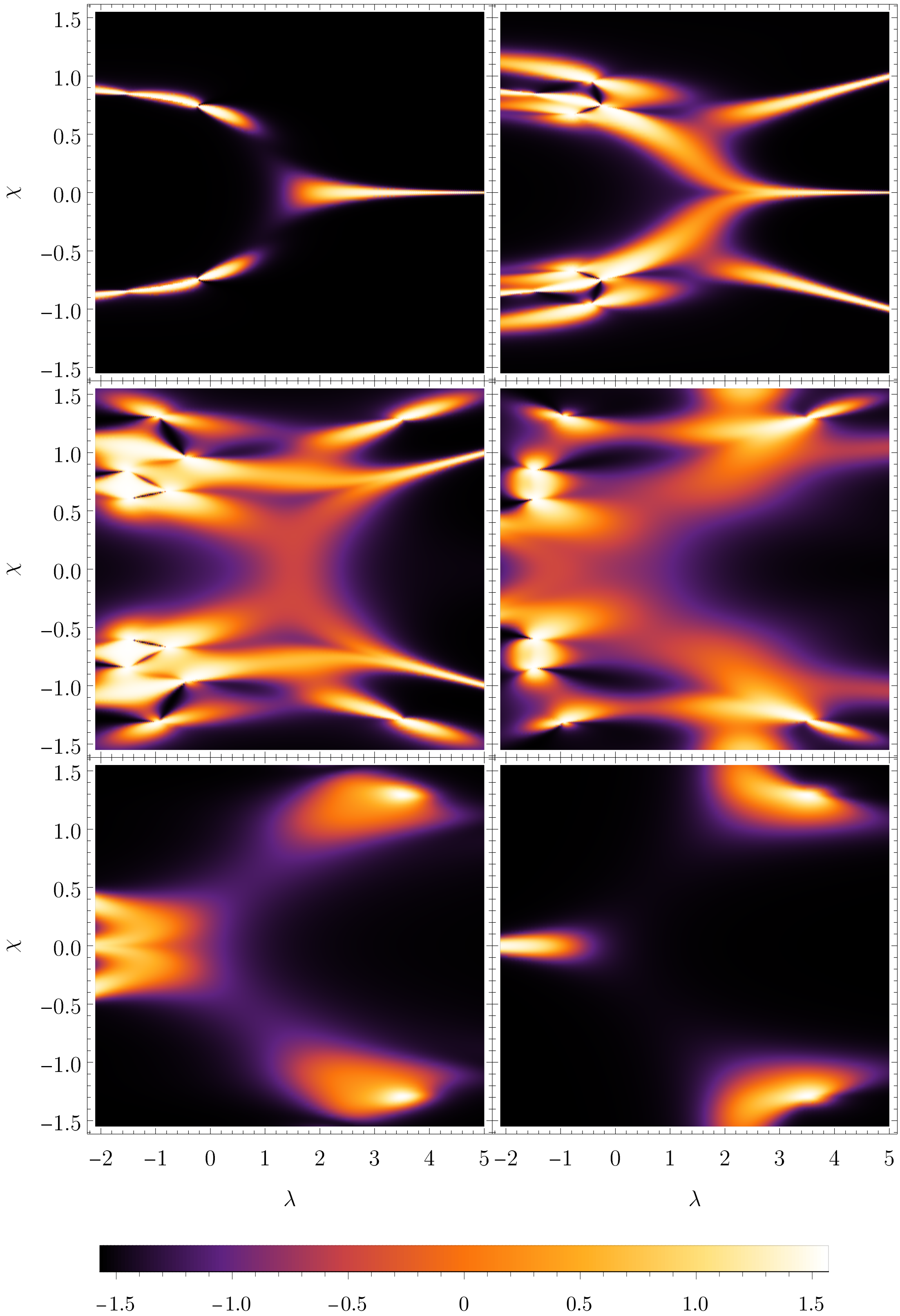

Figure 1.1: Arctangens of the metric tensor for higher state manifolds. By rows: M_0, M_1; M_2, M_3; M_4, M_5.

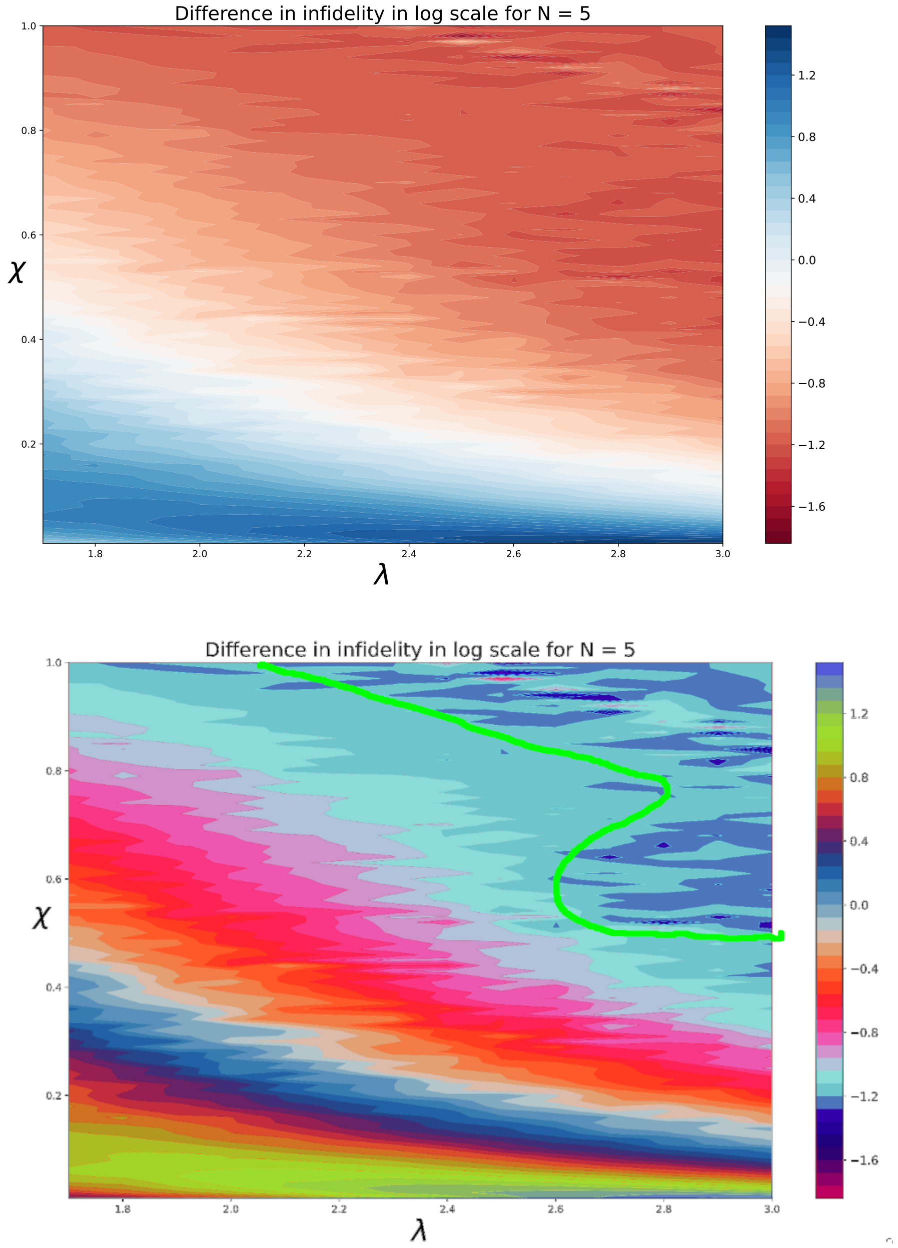

Figure 1.2: Infidelity difference between line and geodesic, above picture is original from Felipe, bottom one is edited in GIMP to see the difference more clearly.

2. Two level system

Having a vector $|\psi\rangle = (Z_0, Z_1)$, we can search for a geodesic on ground state manifold $\mathcal{M}_0$ by plugging metric tensor $g^{(0)}_{\mu\nu}$ from Eq. 1.12 into geodesic equation. This surely minimizes the distance on $\mathcal{M}_0$, but what meaning does it have on protocols inside the whole projective space $\mathcal{PH}$?

Let's once again consider Hamiltonian eigenbasis $|\psi\rangle = Z_0\,|0\rangle + Z_1\,|1\rangle$, only here it depends on parameter $\mathbf{\Lambda}$. Using normalization we get

$$|\psi(\mathbf{\Lambda})\rangle = \cos\theta(\mathbf{\Lambda})\,|0(\mathbf{\Lambda})\rangle + e^{i\phi}\sin\theta\,|1(\mathbf{\Lambda})\rangle \tag{2.1}$$

and it's variation (omitting the dependence on $\mathbf{\Lambda}$ in every element)

$$\delta\,|\psi\rangle = \delta Z_0\,|0\rangle + Z_0\,|\delta 0\rangle - \delta Z_0\,|1\rangle + (1 - Z_0)\,|\delta 1\rangle\,. \tag{2.2}$$

The projections $\langle j|\delta k\rangle$ are known from Eq. 1.10 and

$$\delta Z_0(\mathbf{\Lambda}) = Z_0(\mathbf{\Lambda} + \delta\mathbf{\Lambda}) - Z_0(\mathbf{\Lambda}) = \frac{\mathrm{d}Z_0}{\mathrm{d}\mathbf{\Lambda}^\mu}\mathrm{d}\mathbf{\Lambda}^\mu, \tag{2.3}$$

where the last fraction is close to zero for drivings with low excitation rates. Distance of some transport in $\mathcal{PH}$ with free parameters $\mathbf{\Lambda}, Z_0$ is then according to Eq. 1.7 (again imagine the $\mathbf{\Lambda}$ dependence in every element)

$$\begin{aligned}
\mathrm{d}s^2(\Lambda) =& 1 - |\,\langle\delta\psi|\psi\rangle\,|^2 = 1 - \Big|\underbrace{\langle\delta 0|\overline{Z}_0 Z_0|0\rangle}_{\propto\langle\delta 0|0\rangle=0} + \langle\delta 0|\overline{Z}_0(1-Z_0)|1\rangle \\
&+ \langle\delta 1|(1-\overline{Z}_0)Z_0|0\rangle + \underbrace{\langle\delta 1|(1-\overline{Z}_0)(1-Z_0)|1\rangle}_{\propto\langle\delta 1|1\rangle=0}\Big|^2 \\
=& 1 - \Big|\overline{Z}_0(1-Z_0)\frac{\langle 0|\delta H|1\rangle}{(E_1-E_0)^2} + \underbrace{(1-\overline{Z}_0)Z_0\frac{\langle 1|\delta H|0\rangle}{(E_1-E_0)^2}}_{\text{c.c. of the first part}}\Big|^2 \\
=& 1 - 2\underbrace{\Big|\overline{Z_0(\Lambda)}(1-Z_0(\Lambda))\Big|^2}_{Z_0\text{-term}}\underbrace{\frac{\big|\,\langle 1(\Lambda)|\delta H(\Lambda)|0\rangle\,\big|^2}{(E_1(\Lambda)-E_0(\Lambda))^2}}_{\mathbf{\Lambda}\text{-term}}\,.
\end{aligned} \tag{2.4}$$

The Z_0-term can be minimized by setting $Z_0 = 0.5$ and the $\mathbf{\Lambda}$-term is smallest for a ground state manifold geodesic. This does not mean, that some combination of them is not better, than fulfilling those two conditions simultaneously. Plus the initial and final conditions need to have $Z_0 = 0.5$.

3. The meaning of geodesics

3.1 Transport using quenches

Unifying the ground states $|o(\mathbf{\Lambda})\rangle$ over all points $\mathbf{\Lambda} \in \mathbb{R}^n$ in the parameter space, we get the ground state manifold. Here the fidelity f and distance s are defined

$$\mathrm{d}s^2 \equiv 1 - f^2 \equiv 1 - |\langle o(\mathbf{\Lambda} + \delta\mathbf{\Lambda})|o(\mathbf{\Lambda})\rangle|^2 . \tag{3.1}$$

The final fidelity of transport on $\mathcal{M}$ is then

$$F = \iint g_{\mu\nu}\mathrm{d}\lambda^\mu \mathrm{d}\lambda^\nu = \int_{t_i}^{t_f} \underbrace{\int_{t_i}^{\tau} g_{\mu\nu}\frac{\mathrm{d}\lambda^\mu}{\mathrm{d}t}\frac{\mathrm{d}\lambda^\nu}{\mathrm{d}t}\mathrm{d}t}_{\mathcal{L}(\lambda^\mu,\dot{\lambda}^\mu,\tau)}\,\mathrm{d}\tau. \tag{3.2}$$

Using Euler-Lagrange equations for time-independent $g_{\mu\nu} = g_{\mu\nu}(\lambda^\mu)$, leads to

$$\int_{t_i}^{\tau}\left[g_{\mu\nu,\kappa}\dot{\lambda}^\mu\dot{\lambda}^\nu - \frac{\mathrm{d}}{\mathrm{d}t}\left[g_{\mu\nu}\left(\delta^\mu_\kappa\dot{\lambda}^\nu + \dot{\lambda}^\mu\delta^\nu_\kappa\right)\right]\right]\mathrm{d}t = 0, \tag{3.3}$$

which needs to be zero for integration over any subset (t_i, τ). This can bne achieved for any path only it the integrand itself is zero, which happens if the geodesic equation holds.

The fidelity f measures transition probability between two neighboring eigenstates of two different Hamiltonians. Those two states belong to the same Fibre space $\mathcal{PH}(\mathbf{\Lambda}) \times \mathbb{R}^n$ from which the coefficients $(\mathbb{Z}, \mathbf{\Lambda})$ are taken. Because all $\mathcal{PH}(\mathbf{\Lambda})$ are canonically isomorphic, there is no problem in parallel transport from one space to another, which is needed for braket[1].

The distance minimization runs into some interpretation problems. On one hand, minimalization of the distance is equivalent to maximalization of the sum of infinitezimal fidelities along the path (we say we *maximize the fidelity along the path*). On the other hand we are using only ground states in every step of the transport, therefore defining the fidelity to be one. There are actually two ways out of this confusion. *Perturbed adiabatic driving* and *Transport using quenches.*

Perturbed adiabatic driving

It the first case, we imagine at every point of transport, that the fidelity is small enough, such that in eigenbasis and some small parameters $\delta_i \in \mathbb{C}$ we get

$$|o(\mathbf{\Lambda}_i)\rangle \equiv \begin{pmatrix} Z_0(\mathbf{\Lambda}_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{\text{transport } \mathrm{d}s} |o(\mathbf{\Lambda}_i + \delta\mathbf{\Lambda})\rangle \equiv \begin{pmatrix} Z_0(\mathbf{\Lambda}_i + \delta\mathbf{\Lambda}) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \delta_1(\mathbf{\Lambda}_i + \delta\mathbf{\Lambda}) \\ \vdots \\ \delta_n(\mathbf{\Lambda}_i + \delta\mathbf{\Lambda}) \end{pmatrix}}_{\mathbf{\Delta}(\mathbf{\Lambda}_i + \delta\mathbf{\Lambda})},$$

[1]This procedure is done without thinking in the back part of our brains such, that we don't even think about it. That's how trivial it is.

where the last term is neglected using

$$\langle \Delta(\mathbf{\Lambda}) | o(\mathbf{\Lambda} + \delta\mathbf{\Lambda}) \rangle \approx 0.$$

This might have interesting implication for slow transports, or small distance transports. For example when some slow thermalization is considered during a transport.

Transport using quenches

If we imagine $\delta\mathbf{\Lambda}$ to be finite (not infinitely small, as the notation suggests), the **transport** means **doing a sequence of quenches and measuring the system after every quench**.

Some notion of the space of our Hamiltonian can be seen by quenching from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$, as can be seen in Figure 3.1.

In Figure 3.2 are marked equidistant points, meaning $\int_a^b \mathrm{d}s = \text{const.}$ between every two neighboring points on curve. This means that if the system is measured periodically, the quenches jump smaller distances when closer to a singularity.

Decreasing time step Δt has no effect on the relative fidelity of quenches during the evolution but has an effect on their magnitude. As one would expect from *quantum Zeno effect*, when $\Delta t \to 0$, the transport becomes adiabatic, and the fidelity at any time will become 1. This can be observed in Figure 3.3, such that the shape of the point-like paths looks similar in the columns, and their magnitude decreases.

The quantum Zeno effect for this case can be shown dirrectly by splitting the distance s to N equal pieces. The fidelity for N splits will then be

$$f(N) = (1 - \Delta s)^N = \left(1 - \left(\frac{s}{N}\right)^2\right)^N \overset{N \to \infty}{\longrightarrow} 1, \tag{3.4}$$

meaning the consequent measurements will colapse the system to its instantenous eigenstate and the adiabatic condition for transport holds.

Such measurements can be achieved by fast thermalization of the system. If the finite speed thermalization with $N = T/\tau$ for the mean time between two measurements τ, we get

$$\begin{aligned} \log f(N) &= N \log\left(1 - \left(\frac{s}{N}\right)^2\right) = -\frac{s^2}{N} + o\left(\frac{s^4}{n^3}\right) \\ f(N) &= \exp\left(-s^2\frac{\tau}{T} - \frac{s^4}{2N^3}\dots\right) = \exp\left(-s^2\frac{\tau}{T}\right)\left(1 + o\left(\frac{s^4}{N^3}\right)\right) \end{aligned} \tag{3.5}$$

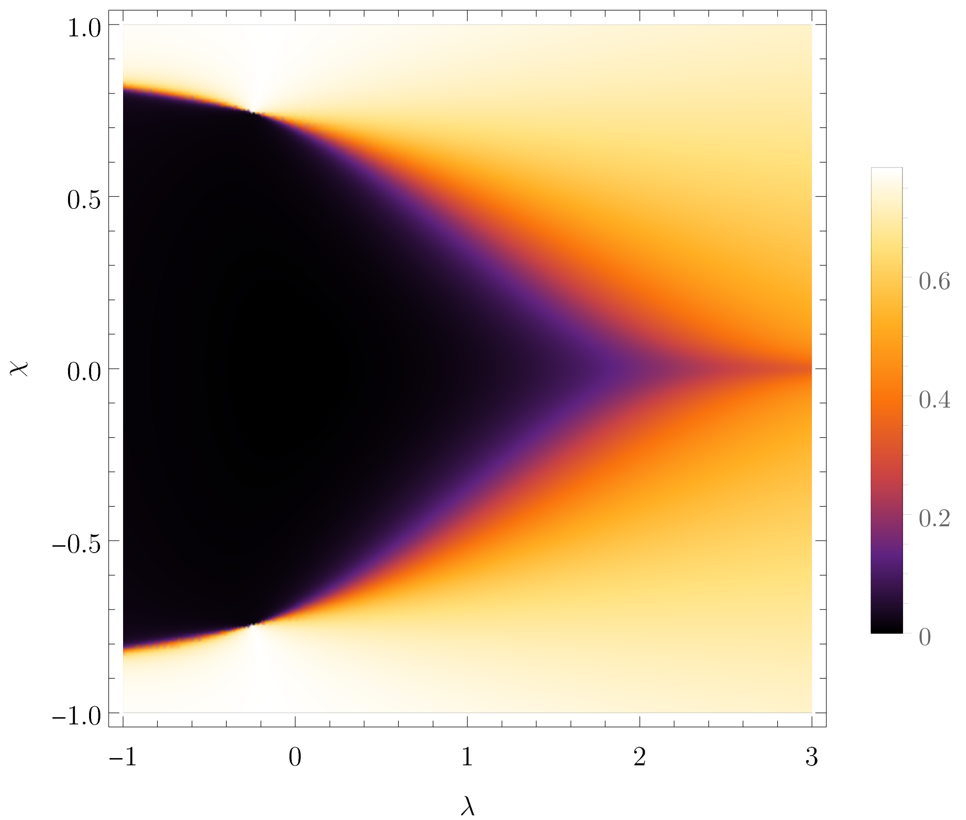

Figure 3.1: Arctangens of the fidelity of quenches from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$.

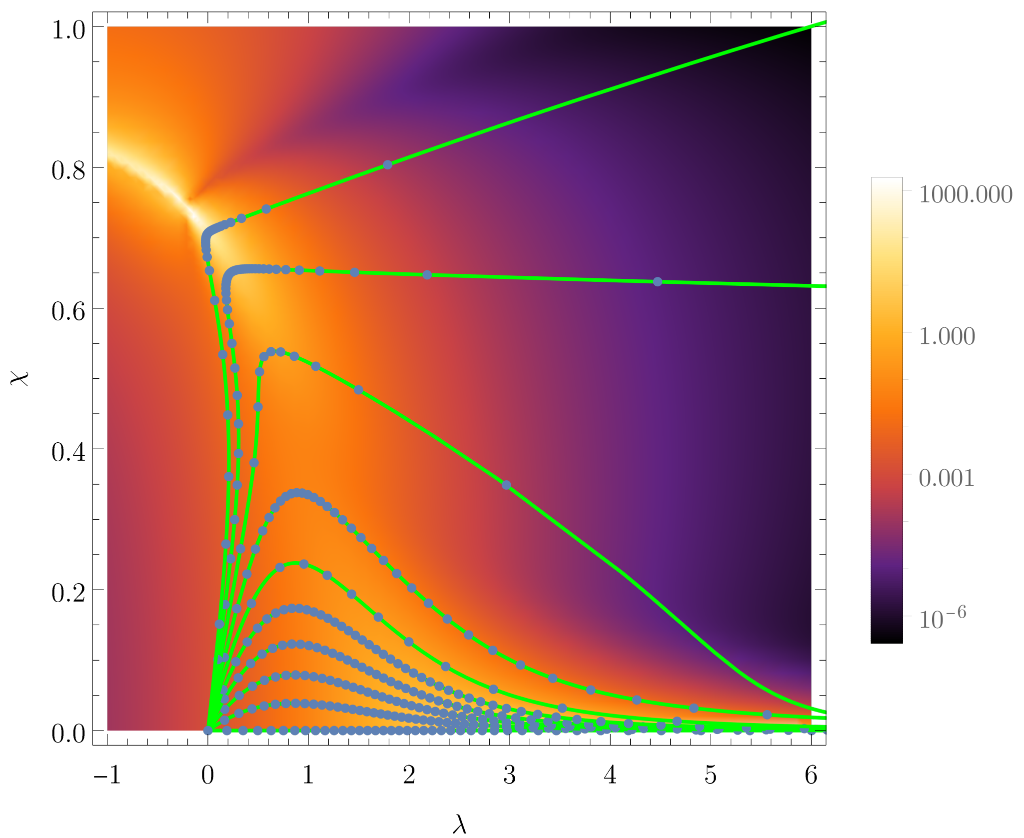

Figure 3.2: Equidistant points on geodesics of the ground state manifold.

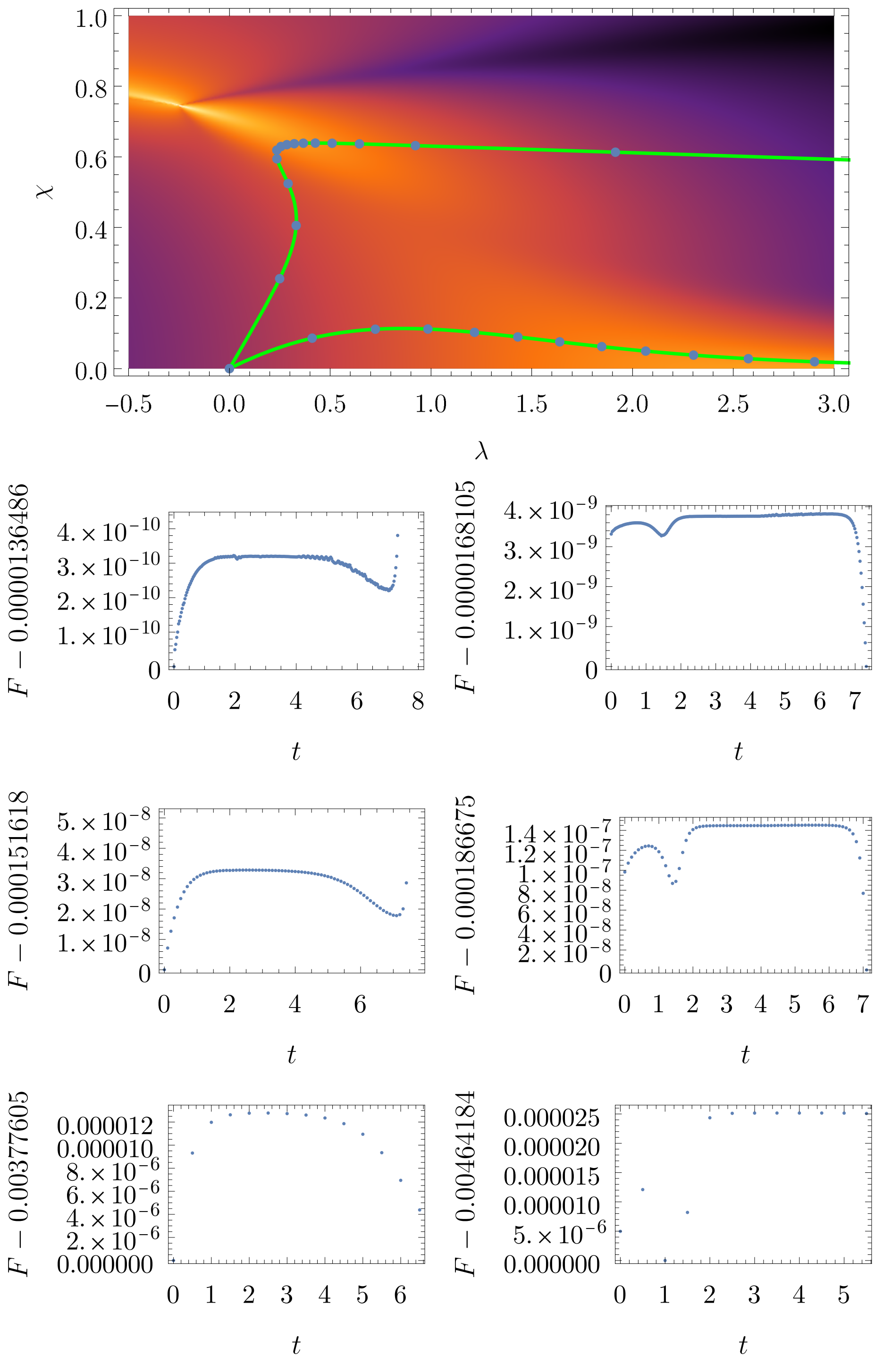

Figure 3.3: Fidelity for sequential quenches along geodesics (see green lines on top). Left (right) column corresponds to lower (upper) geodesic. Time steps from top are $\Delta t \in \{0.03, 0.1, 0.5\}$. Time difference between points in the plot on top is $\Delta t = 0.5$.

Bibliography

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