1. From the projective Hilbert space to state manifolds

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1.1 Projective Hilbert space

Consider the Hilbert space $\mathcal{H}$ to be a space of *bare states* and $\mathcal{S}$ to be the space of *normalized bare states*. Physical observables are related to the *space of rays*, defined as $\mathcal{PH} \coloneqq \mathcal{H}/U(1)$, for the factorization by elements of $U(1)$. This group consists of unitary transformations $e^{i\phi}$ for $\phi \in \mathbb{R}$, defining gauge symmetry between quantum states. $\mathcal{PH}$ is then considered to be the *space of pure states*. For the sake of generality, let's not normalize our vectors yet, which would lead to the *space of pure physical states*.

It can be shown, that $\mathcal{PH}$ is of Kähler structure, meaning it has two non-degenerate sesquilinear[1] 2-forms embedded along with operator complex unit

$$(J, G, \Omega)$$

such that

$$J^2 = \mathbb{1} \tag{1.1}$$

and any bracket of $\psi_1, \psi_2 \in \mathcal{PH}$ can be decomposed into real and imaginary part[Ashtekar and Schilling]

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{2} G(\psi_1, \psi_2) - \frac{i}{2} \Omega(\psi_1, \psi_2). \tag{1.2}$$

From braket sesquilinearity goes that G is symmetric and Ω antisymmetric form, thus they can be uniquely written into one 2-form called *Fubini-Study metric* with property

$$G = \mathrm{Re}Q; \qquad \Omega = \mathrm{Im}Q. \tag{1.3}$$

To write this metric in a standart form, we need to realize how our space looks like. For finite $n+1$-dimensional Hilbert space, one dimension is lost in the gauge transformation, leaving us with n-dimensional $\mathcal{PH}$. Here the orthonormal basis is some mapping to a n-dimensional complex space

$$CP^n = \left\{ \mathbb{Z} = (Z_0, Z_1, \ldots, Z_n) \in \mathbb{C}^{n+1}/\{0\} \right\} \Big/ \{ \mathbb{Z} \sim c\mathbb{Z} \text{ for } c \in \mathbb{C} \}.$$

The general property of this space is, that it naturally generates tangent space with splitting[2]

$$T^{1,0}\mathcal{M} = \mathrm{Span}\left\{ \frac{\partial}{\partial Z_i} \right\}; \qquad T^{0,1}\mathcal{M} = \mathrm{Span}\left\{ \frac{\partial}{\partial Z_{\bar{i}}} \right\}.$$

[1] We are in physics, so complex conjugated is the first input of the 2-form.

[2] $T^{p,q}\mathcal{M}$ means $p+q$- cotravariant space (the space of vectors) on $\mathcal{M}$. The line over letter means complex conjugation.

Distance on $\mathbb{C}^{n+1}$ is standardly defined using Hermitean metric

$$\mathrm{d}s^2 = \mathrm{d}\overline{\mathbb{Z}} \otimes \mathrm{d}\mathbb{Z}. \tag{1.4}$$

In quantum mechanics, it is handy to rewrite this formula using some basis

$$|\psi\rangle = \sum_{k=0}^{n} Z_k |k\rangle \tag{1.5}$$

$$\mathrm{d}s^2 = \frac{\langle\delta\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\delta\psi|\psi\rangle\langle\psi|\delta\psi\rangle}{\langle\psi|\psi\rangle^2}. \tag{1.6}$$

For normalization $\sum_{i=0}^{n} Z_i = 1$, one gets a distance on *space of pure physical states*

$$\mathrm{d}s^2 = 1 - |\langle\delta\psi|\psi\rangle|^2. \tag{1.7}$$

It's good to point out that the variation is taken in the $\mathbb{C}^{n+1}$ space. For example $\delta_i |\psi\rangle = (Z_0, \dots, \delta Z_i, \dots, Z_n)$.

1.2 Restriction to state manifolds

In quantum mechanics, one usually examine some system defined with Hamiltonian $\hat{H}(\mathbf{\Lambda})$, for some parameter $\mathbf{\Lambda} \in \mathbb{R}^n$. This implies natural selection for basis in Eq. 1.5 as the eigenbasis of $\hat{H}$. Metric tensor of $\mathcal{PH}$ will not depend on this choise, but the Hamiltonian itself creates some natural sections on this manifold.

One of them might be the section along Schrödinger evolution of some state $|\psi_i\rangle$, meaning

$$S_{|\psi_i\rangle} := \left\{ |\psi(t)\rangle \text{ following } i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|k\rangle \text{ on time interval } [t_i, t_f] \right\}.$$

Another more interesting sections are *state manifolds*, defined by setting only one non-zero coefficient Z_k in basis defined by Eq. 1.5. From normalization goes automatically $Z_k = 1$. The distance is then

$$\mathrm{d}s^2 = 1 - \langle\delta k|k\rangle\langle k|\delta k\rangle = 1 - \langle\delta k|\left(\mathbb{1} - \sum_{j\neq k}|j\rangle\langle j|\right)|\delta k\rangle = \sum_{j\neq k}\langle\delta k|j\rangle\langle j|\delta k\rangle. \tag{1.8}$$

Using the Schrödinger equation $\hat{H}|k\rangle = E_k|k\rangle$, distributivity of derivative and projection to some state $\langle j|$, we get

$$\begin{aligned} \hat{H}|k\rangle &= E_k|k\rangle \\ (\delta\hat{H})|k\rangle + \hat{H}|\delta k\rangle &= (\delta E_k)|k\rangle + E_k|\delta k\rangle \\ \langle j|\left(\delta\hat{H} - \delta E_k\right)|k\rangle &= \langle j|\left(E_k - \hat{H}\right)|\delta k\rangle. \end{aligned} \tag{1.9}$$

We can set[3] $\delta E_k = 0$, leading for $j \neq k$ to

$$\frac{\langle j|\delta\hat{H}|k\rangle}{(E_k - E_j)^2} = \langle j|\delta k\rangle. \tag{1.10}$$

[3] Can it be done only for E_0? It does not make sanse generally, because $E = E(\mathbf{\Lambda})$, even $E_0 = E_0(\mathbf{\Lambda})$

Plugging to Equation 1.8 and considering $\hat{H} = \hat{H}(\mathbf{\Lambda})$, we get metric on ground state manifold

$$\mathrm{d}s^2 = \sum_{j\neq k} \frac{\langle k|\, \partial_\mu \hat{H} |j\rangle\langle j| \partial_\nu \hat{H}\, |k\rangle}{(E_k - E_j)^2} \mathrm{d}\mathbf{\Lambda}^\mu \mathrm{d}\mathbf{\Lambda}^\nu \tag{1.11}$$

Definition of the k-state manifold is then

$$g^{(k)}_{\mu\nu} = \mathrm{Re} \sum_{j\neq k} \frac{\langle k| \frac{\partial \hat{H}(\mathbf{\Lambda})}{\partial \lambda^\mu} |j\rangle \, \langle j| \frac{\partial \hat{H}(\mathbf{\Lambda})}{\partial \lambda^\nu} |k\rangle}{(E_k - E_j)^2}. \tag{1.12}$$

1.3 Meaning of geodesics

1.3.1 Two level system

Having a vector $|\psi\rangle = (Z_0, Z_1)$, we can search for a geodesic on ground state manifold $\mathcal{M}_0$ by plugging metric tensor $g^(0)_{\mu\nu}$ from Eq. 3.1 into geodesic equation. This surely minimizes the distance on $\mathcal{M}_0$, but what meaning does it have on protocols inside the whole projective space $\mathcal{PH}$?

Let's once again consider Hamiltonian eigenbasis $|\psi\rangle = Z_0 |0\rangle + Z_1 |1\rangle$, only here it depends on parameter $\mathbf{\Lambda}$. Using normalization we get

$$|\psi(\mathbf{\Lambda})\rangle = Z_0(\mathbf{\Lambda}) |0(\mathbf{\Lambda})\rangle + (1 - Z_0(\mathbf{\Lambda})) |1(\mathbf{\Lambda})\rangle \tag{1.13}$$

and it's variation (omitting the dependence on $\mathbf{\Lambda}$ in every element)

$$\delta |\psi\rangle = \delta Z_0 |0\rangle + Z_0 |\delta 0\rangle - \delta Z_0 |1\rangle + (1 - Z_0) |\delta 0\rangle \,. \tag{1.14}$$

The projections $\langle j|\delta k\rangle$ are known from Eq. 1.10 and

$$\delta Z_0(\mathbf{\Lambda}) = Z_0(\mathbf{\Lambda} + \delta\mathbf{\Lambda}) - Z_0(\mathbf{\Lambda}) = \frac{\mathrm{d}Z_0}{\mathrm{d}\mathbf{\Lambda}^\mu} \mathrm{d}\mathbf{\Lambda}^\mu, \tag{1.15}$$

where the last fraction is close to zero for drivings with low excitation rates. Distance of some transport in $\mathcal{PH}$ with free parameters $\mathbf{\Lambda}, Z_0$ is then according to Eq. 1.7 (again imagine the $\mathbf{\Lambda}$ dependence in every element)

$$\begin{aligned}
\mathrm{d}s^2(\Lambda) =& 1 - |\langle \delta\psi|\psi\rangle|^2 = 1 - \Big| \underbrace{\langle \delta 0|\overline{Z}_0 Z_0|0\rangle}_{\propto \langle \delta 0|0\rangle = 0} + \langle \delta 0|\overline{Z}_0 (1 - Z_0)|1\rangle \\
& + \langle \delta 1|(1 - \overline{Z}_0) Z_0|0\rangle + \underbrace{\langle \delta 1|(1 - \overline{Z}_0)(1 - Z_0)|1\rangle}_{\propto \langle \delta 1|1\rangle = 0} \Big|^2 \\
=& 1 - \left| \overline{Z}_0 (1 - Z_0) \frac{\langle 0|\delta H|1\rangle}{(E_1 - E_0)^2} + \underbrace{(1 - \overline{Z}_0) Z_0 \frac{\langle 1|\delta H|0\rangle}{(E_1 - E_0)^2}}_{\text{c.c. of the first part}} \right|^2 \\
=& 1 - 2 \underbrace{\left| \overline{Z_0(\Lambda)} (1 - Z_0(\Lambda)) \right|^2}_{Z_0\text{-term}} \underbrace{\frac{\left| \langle 1(\Lambda)|\delta H(\Lambda)|0\rangle \right|^2}{(E_1(\Lambda) - E_0(\Lambda))^2}}_{\mathbf{\Lambda}\text{-term}} .
\end{aligned} \tag{1.16}$$

The Z_0-term can be minimized by setting $Z_0 = 0.5$ and the $\mathbf{\Lambda}$-term is smallest for a ground state manifold geodesic. This does not mean, that some combination of them is not better, than fulfilling those two conditions simultaneously. Plus the initial and final conditions need to have $Z_0 = 0.5$.

1.4 Higher state transition metric operator

When one assumes a function in a ground state at every point of transport, situation using quenches is reconstructed. During some general transport, in which a superposition of states is allowed, one also needs to minimize the flow of probability from higher states to even higher states and maximize the flow from higher states to lower ones. Let us try to construct a new metric tensor, which will count in those *higher-state transition* effects.

Imagine a driving

$$\begin{aligned} \gamma : \mathbb{R} &\to \mathbb{R}^n \\ t &\mapsto \mathbf{\Lambda} \equiv (\lambda_1, \dots \lambda_n) \end{aligned}$$

This function translates the driving parameter t to coordinates for Hamiltonian $\hat{H}(\mathbf{\Lambda})$. Assume the states

$$|\Psi(t)\rangle = \sum_m a_m(t) \, |m(t)\rangle \tag{1.17}$$

for eigenstates $|m(t)\rangle$ of the Hamiltonian $\hat{H}$. Now we can form a linear combination of effects on this wavefunction, where every excitation probability needs to be minimized and every deexcitation maximized. **Because $\partial_\mu \hat{H}$ is a hermitian operator, the excitation and deexcitation will occur with the same probability. Plus if we assume the Fidelity to be** $F > 0.5$, we can neglect the deexcitation probabilities and instead minimize only the excitations, which can be done by introducing *Higher state transition metric operator*, let's call it just the *metric operator*[4]

$$G^{\psi(t)}_{\mu\nu} = \mathrm{Re} \sum_{j=0}^{n} \sum_{m=j+1}^{n} a^*_m(t) a_m(t) \frac{\langle m(t)| \frac{\partial \hat{H}(\gamma(t))}{\partial \lambda^\mu} |j\rangle \langle j| \frac{\partial \hat{H}(\gamma(t))}{\partial \lambda^\nu} |m(t)\rangle}{(E_\psi(t) - E_j)^2} \tag{1.18}$$

for the coefficient $E_\Psi(t) = \sum_{m>j} a_m(t) E_m$, which is a complex number has no meaning of energy. If $a_0 = 1$ and $a_m = 0$ for $m \in \{1, \dots, n\}$, we get the metric tensor $g^{(0)}_{\mu\nu}$ from Eq. 3.1. The problem arises when those coefficients are nonzero, which makes the problem nonlinear in a sense

$$\begin{tikzcd} a_m \arrow[r] & \psi(t) \arrow[r] & G^{\psi(t)}_{\mu\nu} \arrow[ll, bend right] \end{tikzcd}$$

To see how much Higher state manifolds influence the driving, we can compare some driving results with individual elements of the metric operator, see Figure 3.2. One such result which might support the hypothesis postulated in Eq. 3.1 is that geodesics will be much worse in the areas where the space is more curved in M_1. This also holds for M_2 etc., but the effect from M_1 will be strongest. For comparison see the metric tensor for higher state manifolds in Figure 3.2.

[4] It's not a tensor, because it does not transform like a tensor. It depends on a path.

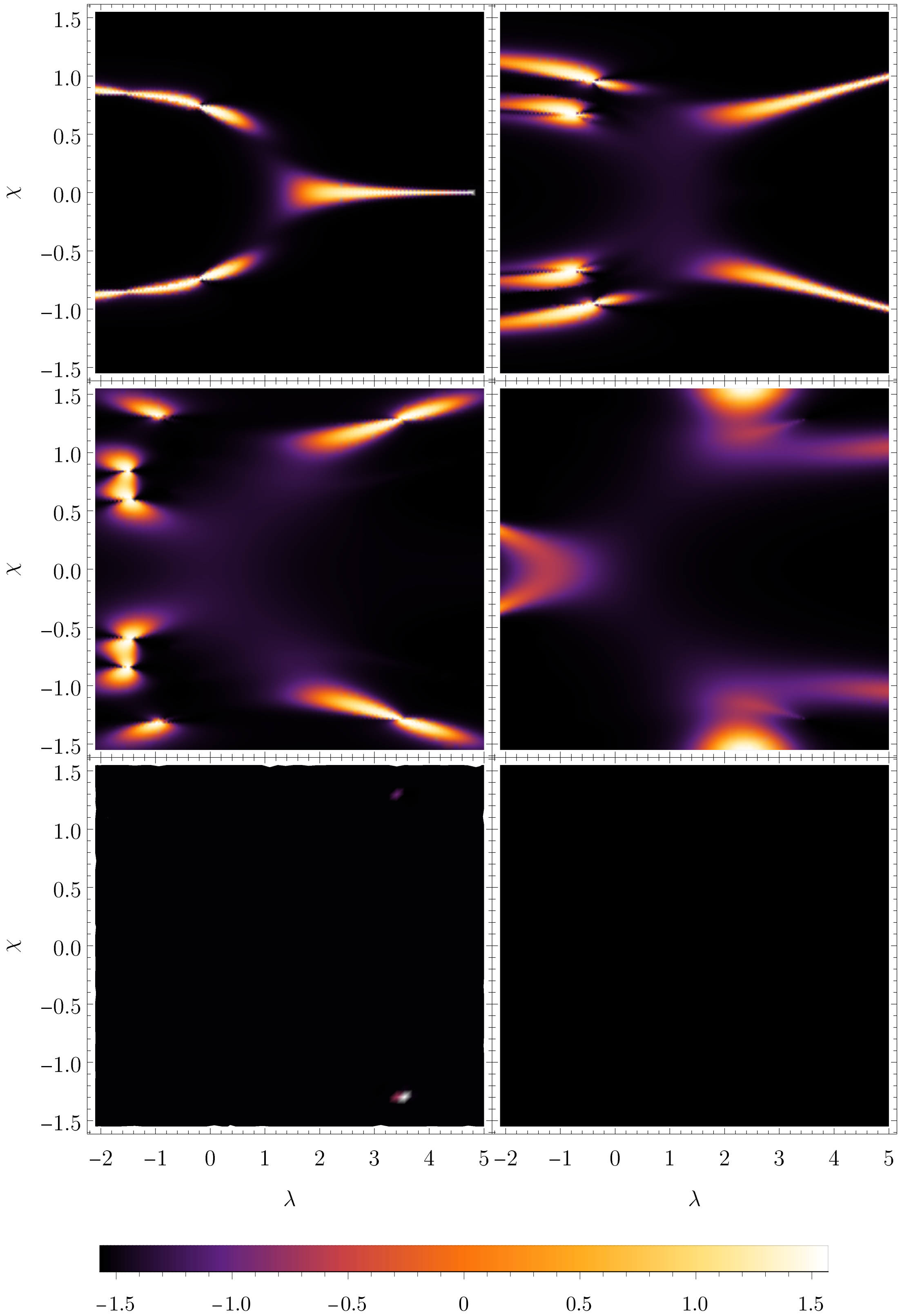

Figure 1.1: Arctangens of the elements of higher state manifolds. By rows using j from the sum in eq. 3.3: $j = 0$, $j = 1$; $j = 2$, $j = 3$; $j = 4$, $j = 5$.

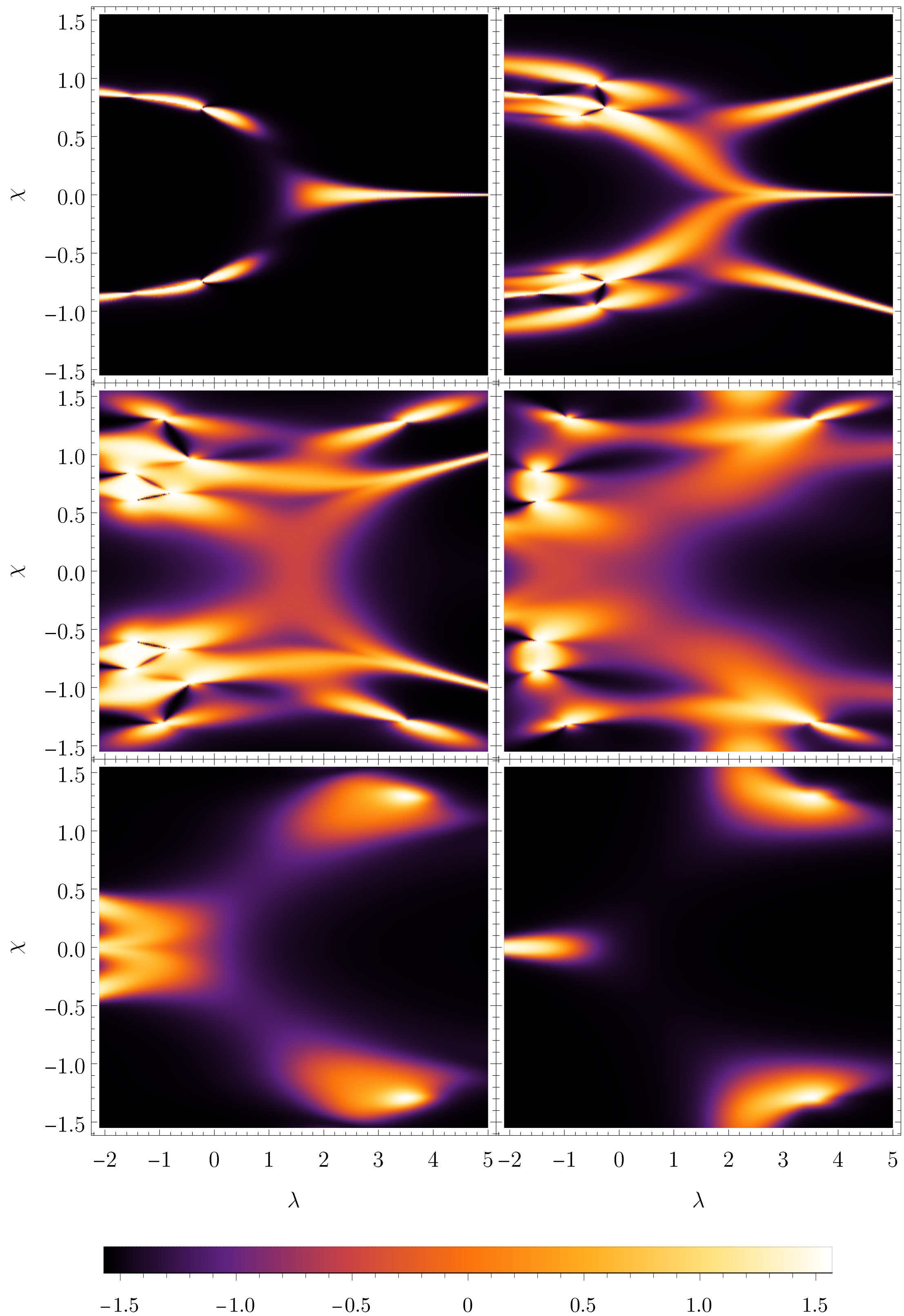

Figure 1.2: Arctangens of the metric tensor for higher state manifolds. By rows: M_0, M_1; M_2, M_3; M_4, M_5.

2. On the meaning of geodesics

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2.1 Transport using quenches

Unifying the ground states $|o(\mathbf{\Lambda})\rangle$ over all points $\mathbf{\Lambda} \in \mathbb{R}^n$ in the parameter space, we get the ground state manifold. Here the fidelity f and distance s are defined

$$\mathrm{d}s^2 \equiv 1 - f^2 \equiv 1 - \left|\langle o(\mathbf{\Lambda} + \delta\mathbf{\Lambda})|o(\mathbf{\Lambda})\rangle\right|^2 . \tag{2.1}$$

The final fidelity of transport on $\mathcal{M}$ is then

$$F = \iint g_{\mu\nu}\mathrm{d}\lambda^\mu \mathrm{d}\lambda^\nu = \int_{t_i}^{t_f} \underbrace{\int_{t_i}^{\tau} g_{\mu\nu}\frac{\mathrm{d}\lambda^\mu}{\mathrm{d}t}\frac{\mathrm{d}\lambda^\nu}{\mathrm{d}t}\mathrm{d}t}_{\mathcal{L}(\lambda^\mu,\dot{\lambda}^\mu,\tau)}\,\mathrm{d}\tau. \tag{2.2}$$

Using Euler-Lagrange equations for time-independent $g_{\mu\nu} = g_{\mu\nu}(\lambda^\mu)$, leads to

$$\int_{t_i}^{\tau}\left[g_{\mu\nu,\kappa}\dot{\lambda}^\mu\dot{\lambda}^\nu - \frac{\mathrm{d}}{\mathrm{d}t}\left[g_{\mu\nu}\left(\delta^\mu_\kappa\dot{\lambda}^\nu + \dot{\lambda}^\mu\delta^\nu_\kappa\right)\right]\right]\mathrm{d}t = 0, \tag{2.3}$$

which needs to be zero for integration over any subset (t_i, τ) leads to a zero condition for the integrand itself, which leads to the geodesic equation.

This means that the geodesic should *maximize the fidelity of the transport* between two points. One needs to stop for a while and realize what is the physical meaning of this transport. What the system at any point in the parameter space sees is how far away are all points in his surrounding. In the case of a geodesic, the direction of the smallest distance is then chosen, and the procedure repeats. The problem is that moving in the direction $\delta\mathbf{\Lambda}$ means physically changing the eigenstate to the new one, meaning one needs to project it to the newly diagonalized system at the point $\mathbf{\Lambda} + \delta\mathbf{\Lambda}$ leading to a transition amplitude

$$A_k = \langle o(\mathbf{\Lambda} + \delta\mathbf{\Lambda})|i(\mathbf{\Lambda})\rangle .$$

Next, we ignore higher states and continue only with the ground state. In the end, we calculate what fraction of states we can get to the final ground state, which is the probability of this event.

If we imagine $\delta\mathbf{\Lambda}$ to be finite (not infinitely small, as the notation suggests), the **transport** means **doing a sequence of quenches and measuring the system after every quench**.

Some notion of the space of our Hamiltonian can be seen by quenching from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$, as can be seen in Figure 2.1.

In Figure 2.2 are marked equidistant points, meaning $\int_a^b \mathrm{d}s$ = some const between every two neighboring points on curve. This means that if the system is measured periodically, the quenches jump smaller distances when closer to a singularity.

Decreasing time step Δt has no effect on the relative fidelity of quenches during the evolution but has an effect on their magnitude. As one would expect, when $\Delta t \to 0$, the transport becomes adiabatic, and the fidelity at any time will become zero. This can be observed in Figure 2.3, such that the shape of the point-like paths looks similar in the columns, and their magnitude decreases.

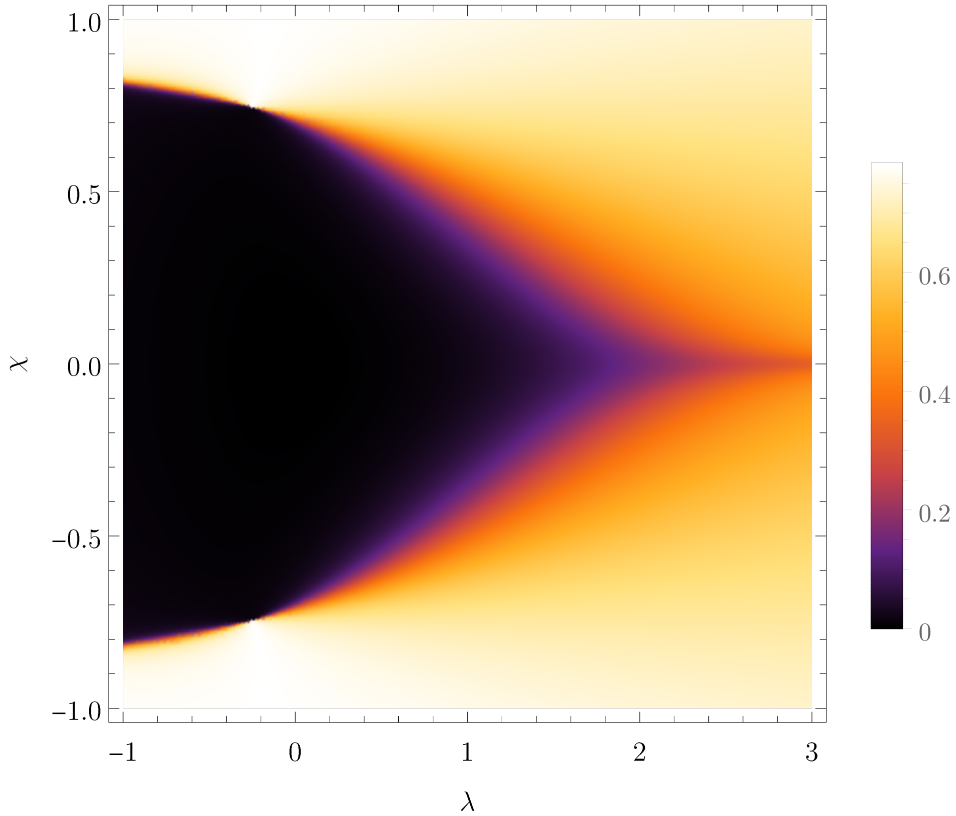

Figure 2.1: Arctangens of the fidelity of quenches from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$.

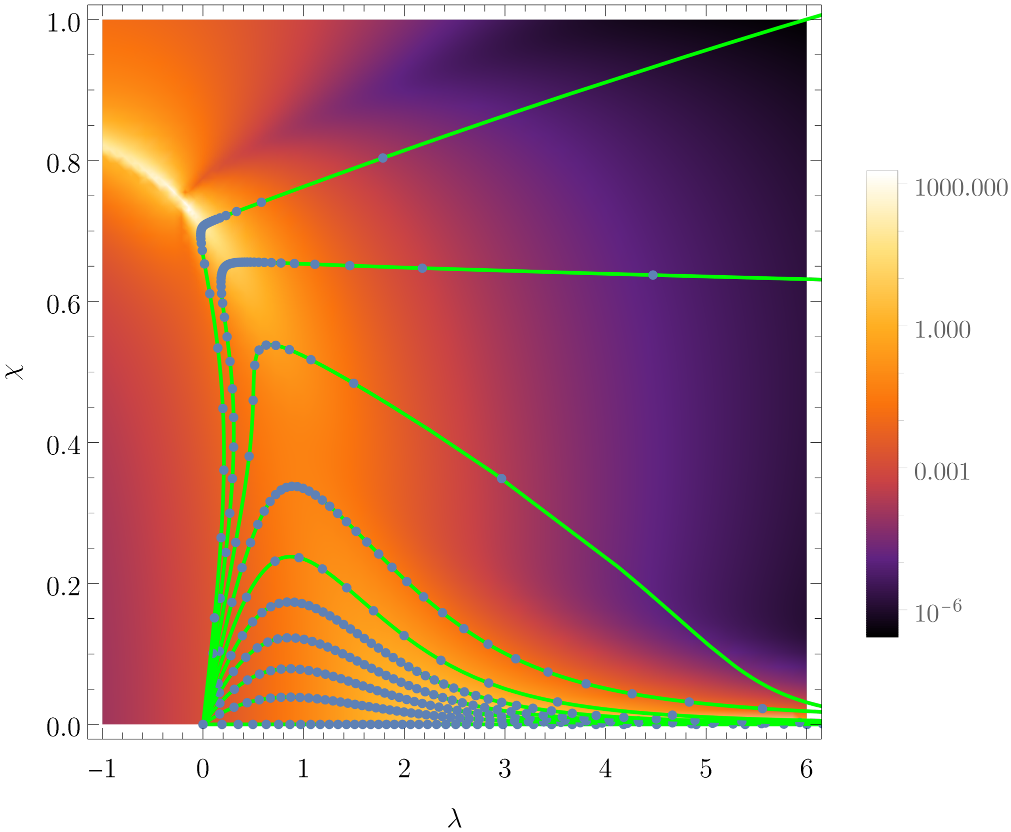

Figure 2.2: Equidistant points on geodesics of the ground state manifold.

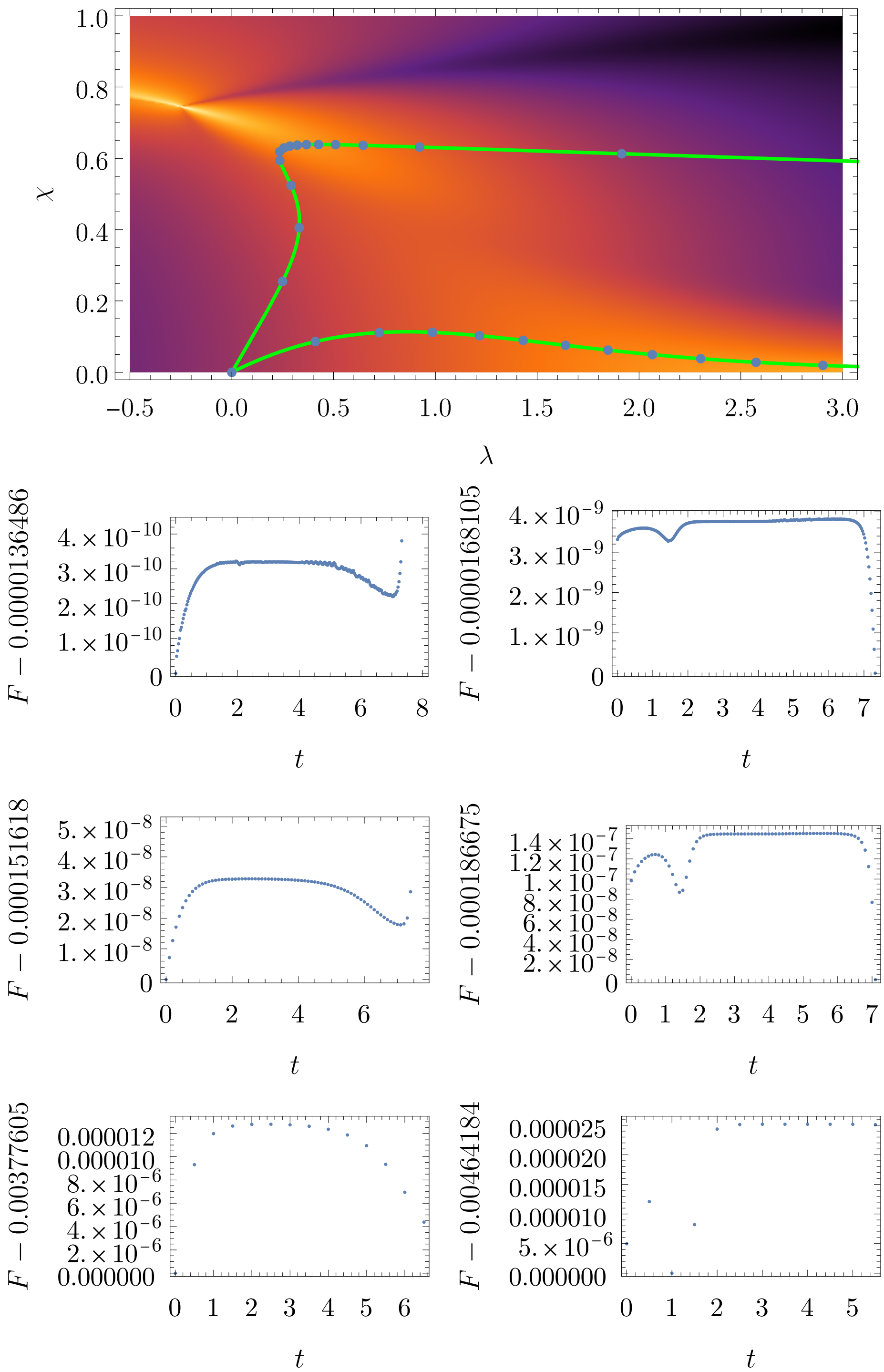

Figure 2.3: Fidelity for sequential quenches along geodesics (see green lines on top). Left (right) column corresponds to lower (upper) geodesic. Time steps from top are $\Delta t \in \{0.03, 0.1, 0.5\}$. Time difference between points in the plot on top is $\Delta t = 0.5$.

3. Metric tensor of the higher state manifold

Definition of the k-state manifold is

$$g_{\mu\nu}^{(k)} = \mathrm{Re} \sum_{j \neq k} \frac{\langle k| \frac{\partial \hat{H}(\mathbf{\Lambda})}{\partial \lambda^\mu} |j\rangle \langle j| \frac{\partial \hat{H}(\mathbf{\Lambda})}{\partial \lambda^\nu} |k\rangle}{(E_k - E_j)^2}. \tag{3.1}$$

One has to see it as a measure between the quantum states. The states are further away from each other when they are "more perpendicular" because the transition probability between the states is lower. The only difference here is that one state needs to be evolved using the derivative of Hamiltonian. From this goes that *minimizing the distance on path* means *going through the least excitation probability path.*

When one assumes a pure ground state at every point of transport, the situation using quenches, see section 2.1, is reconstructed. During some general transport, in which a superposition of states is allowed, one also needs to minimize the flow of probability from higher states to even higher states and maximize the flow from higher states to lower ones. Let us try to construct a new metric tensor, which will count in those *higher-state transition* effects.

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$$G_{\mu\nu}^{\psi(t)} = \mathrm{Re} \sum_{j=0}^{n} \sum_{m=j+1}^{n} a_m^*(t) a_m(t) \frac{\langle m(t)| \frac{\partial \hat{H}(\gamma(t))}{\partial \lambda^\mu} |j\rangle \langle j| \frac{\partial \hat{H}(\gamma(t))}{\partial \lambda^\nu} |m(t)\rangle}{(E_\psi(t) - E_j)^2} \tag{3.3}$$

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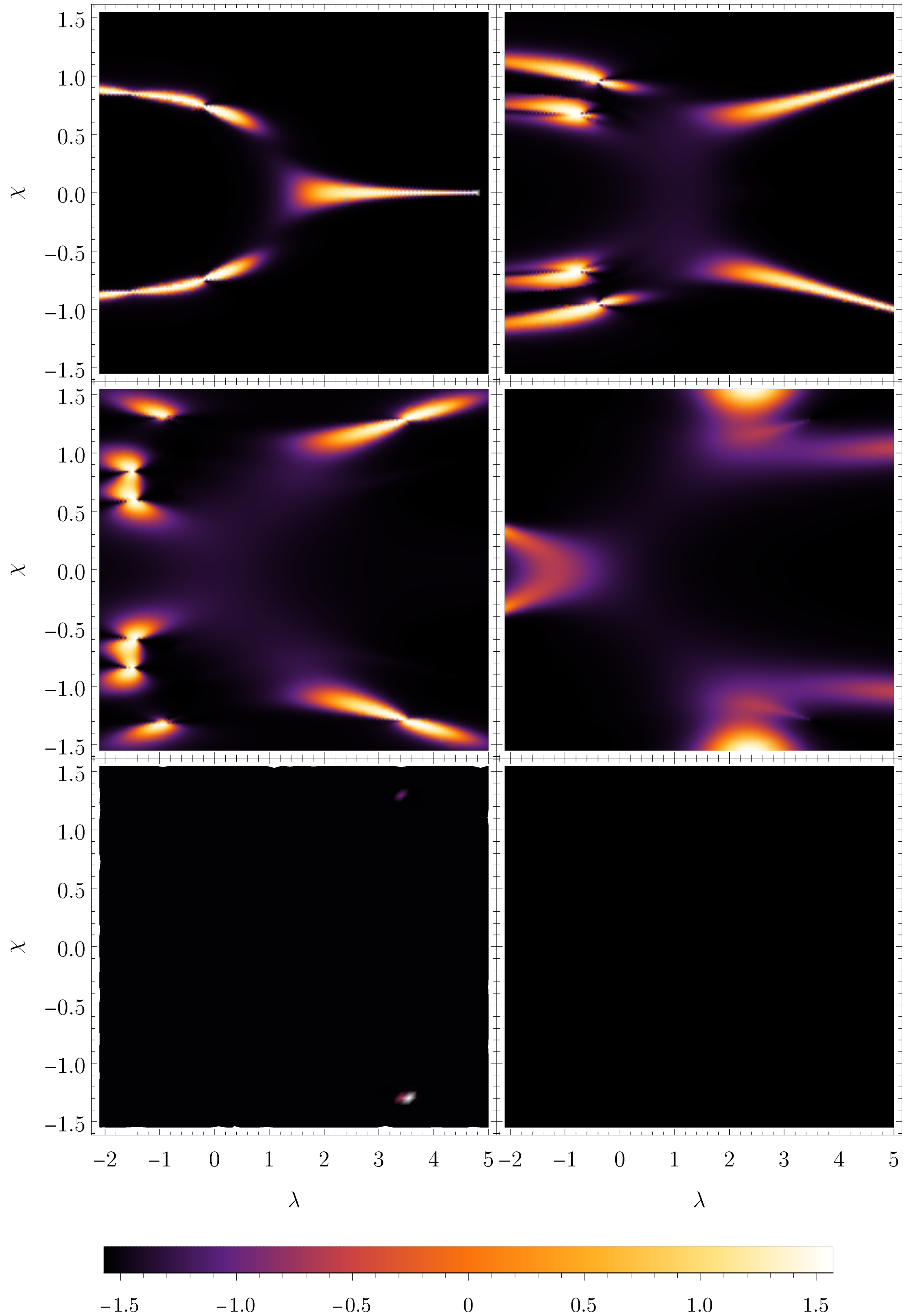

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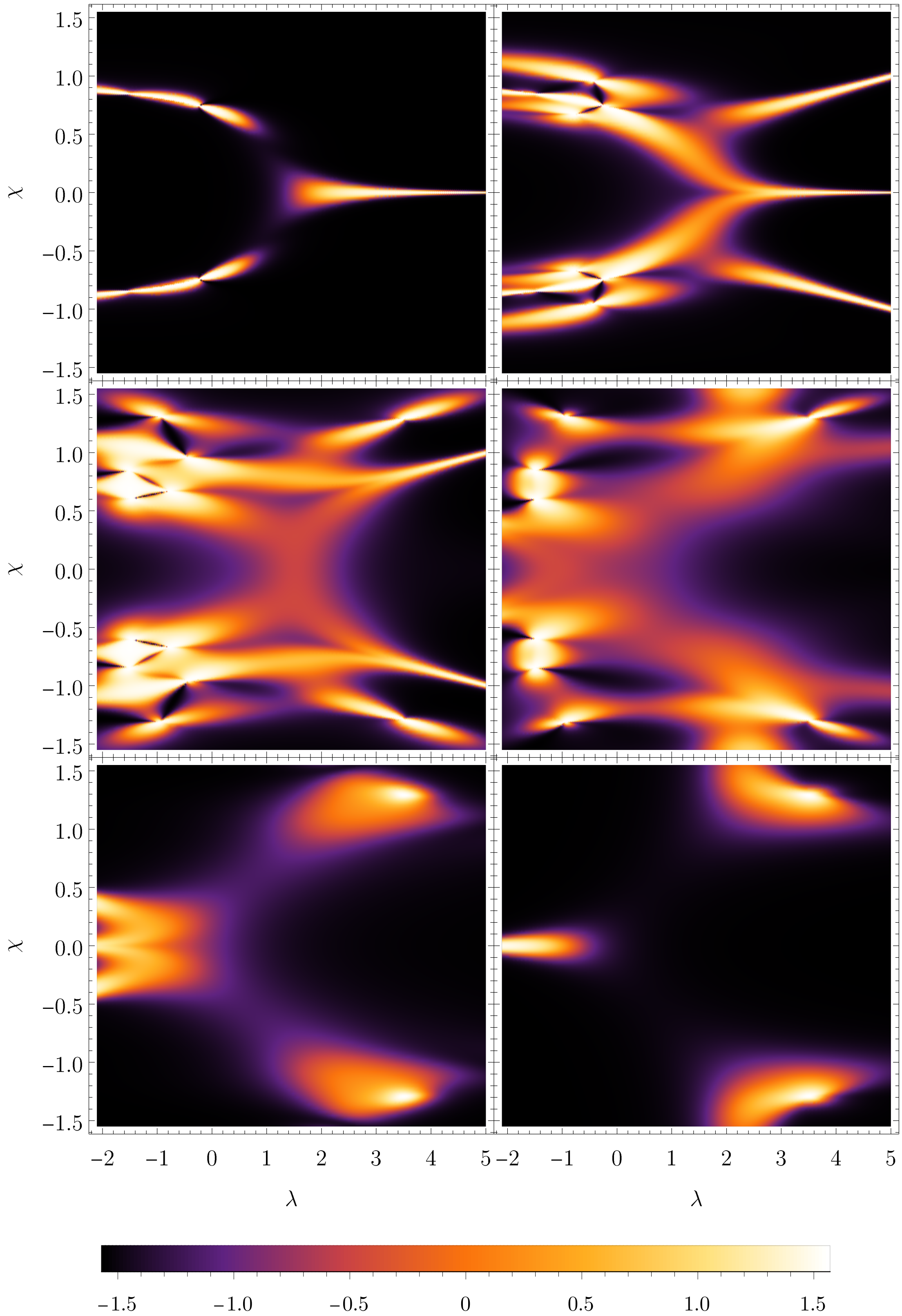

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Bibliography

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