

1. Two level system (Landau-Zener)

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Let's have Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix} \Omega(t) & \Delta(t) \\ \Delta(t) & -\Omega(t) \end{pmatrix} \quad (1.1)$$

and a driving along the path parametrized by time $t \in [0, 1]$

$$d(t) := \begin{pmatrix} -s \cos(\omega(T_f)t) \\ 0 \\ s \sin(\omega(T_f)t) \end{pmatrix} \quad (1.2)$$

for speed regulating function $\omega(T_f) := \pi/T_f$. This means, the driving will always be along half-sphere, as in Fig. 1.1.

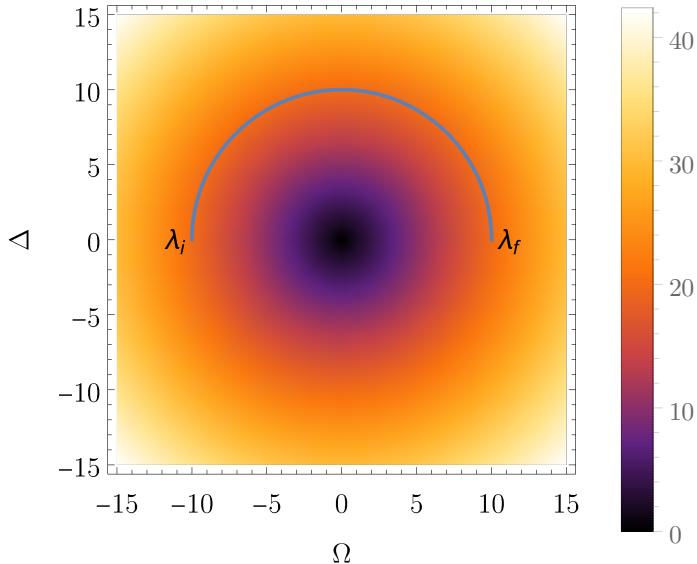


Figure 1.1: Driving along the geodesic. λ_i and λ_f are initial resp. final parameters. DensityPlot shows the difference between Hamiltonian eigenvalues.

1.1 Derivation of the fidelity

Because the Hamiltonian can be rewritten using Pauli matrices

$$\mathcal{H}(t) = \begin{pmatrix} -s \cos(t\omega) & s \sin(t\omega) \\ s \sin(t\omega) & s \cos(t\omega) \end{pmatrix} = \Delta(t)\sigma_x + \Omega(t)\sigma_z = d(t).\hat{\sigma} \quad (1.3)$$

one can see that changing from the original frame to moving frame of reference (let's omit the final time dependence $\omega = \omega(T_f)$ for a while)

$$\psi(t) = e^{\frac{i\omega}{2}\hat{\sigma}_y t} \tilde{\psi}(t) \quad (1.4)$$

reflects rotational symmetry of the problem. This change of reference frame transforms Schrödinger equation

$$\begin{aligned} \mathcal{H}(t)\psi(t) &= i\psi'(t) \\ \mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) &= ie^{\frac{i\omega}{2}\hat{\sigma}_y t} \left(\frac{i\omega\hat{\sigma}_y}{2} \right) \tilde{\psi}(t) + ie^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}'(t) \\ \underbrace{\left(e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t} + \frac{\omega}{2}\hat{\sigma}_y \right)}_{\tilde{\mathcal{H}}(t)} \tilde{\psi}(t) &= i\tilde{\psi}'(t). \end{aligned} \quad (1.5)$$

Therefore we can equivalently solve the Fidelity problem in this new coordinate system.

Hamiltonian in the moving frame is

$$\tilde{\mathcal{H}} = \begin{pmatrix} -s & -i\omega(T_f)/2 \\ i\omega(T_f)/2 & s \end{pmatrix}, \quad (1.6)$$

which is time independent. The Schrödinger equation can now be easily solved using evolution operator

$$\begin{aligned} \hat{U}(t) &= e^{-i\tilde{\mathcal{H}}t} \\ &= \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & -\frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & \cos\left(\frac{t}{2}q(T_f)\right) - \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix}, \end{aligned} \quad (1.7)$$

for $q(T_f) = \sqrt{4s^2 + \omega(T_f)^2}$.

In the original frame we get the evolution of the state $\psi(0)$

$$\psi(t) = e^{\frac{i\omega}{2}\hat{\sigma}_y t} \hat{U}(t) \tilde{\psi}(0) = \underbrace{e^{\frac{i\omega}{2}\hat{\sigma}_y t} \hat{U}}_{\hat{U}(t)} \underbrace{e^{-\frac{i\omega}{2}\hat{\sigma}_y t}}_{\psi(0)} \tilde{\psi}(0). \quad (1.8)$$

Then the evolved wavefunction is

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\cos(t\omega(T_f))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{(\omega(T_f) - 2is\sin(t\omega(T_f)))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix} \quad (1.9)$$

and the ground state

$$|0(t)\rangle = \mathcal{N} \begin{pmatrix} -\cot\left(\frac{t}{2}\omega(T_f)\right) \\ 1 \end{pmatrix}, \quad (1.10)$$

for a normalization constant $\mathcal{N} := |\langle 0(t)|0(t)\rangle|^{-1}$. Fidelity during the transport is then¹

$$F = |\langle 0(t)|\psi(t)\rangle|^2, \quad (1.11)$$

¹If we would calculate the Fidelity in comoving frame, we would get exactly one. This is the essence of counterdiabatic driving.

Explicit formula for fidelity in time t and geodesic driving with final time T_f is then

$$F(t, T_f) = \frac{\pi^2 \left(\cos \left(t \sqrt{\frac{\pi^2}{T_f^2} + 4s^2} \right) + 1 \right) + 8s^2 T_f^2}{2 \sin^4 \left(\frac{\pi t}{2T_f} \right) \left(4s^2 T_f^2 + \pi^2 \right) \left(\left| \cot \left(\frac{\pi t}{2T_f} \right) \right|^2 + 1 \right)^2}. \quad (1.12)$$

The domain can be extended to $t \in [0, T_f]$, $T_f \in [0, \infty]$ because

$$\lim_{t \rightarrow 0} F = 1, \quad \lim_{T_f \rightarrow 0} F = 0.$$

1.2 Analysis of the fidelity formula

Fidelity for some fixed final time is just oscillating curve close to 1. For $T_f = 10$ it can be seen on Fig. 1.2. The *final fidelity* (at $t = T_f$) dependence on final time T_f can be seen on Fig. 1.3 and 1.4.

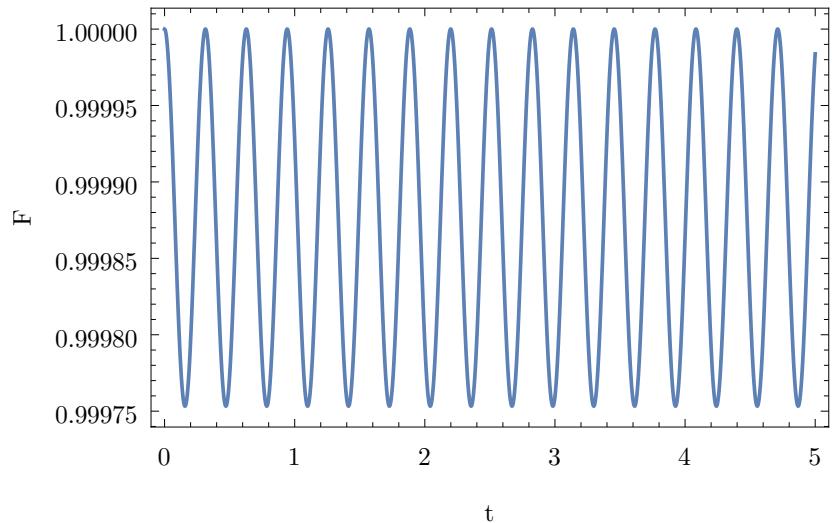


Figure 1.2: Fidelity in time for $T_f = 10$

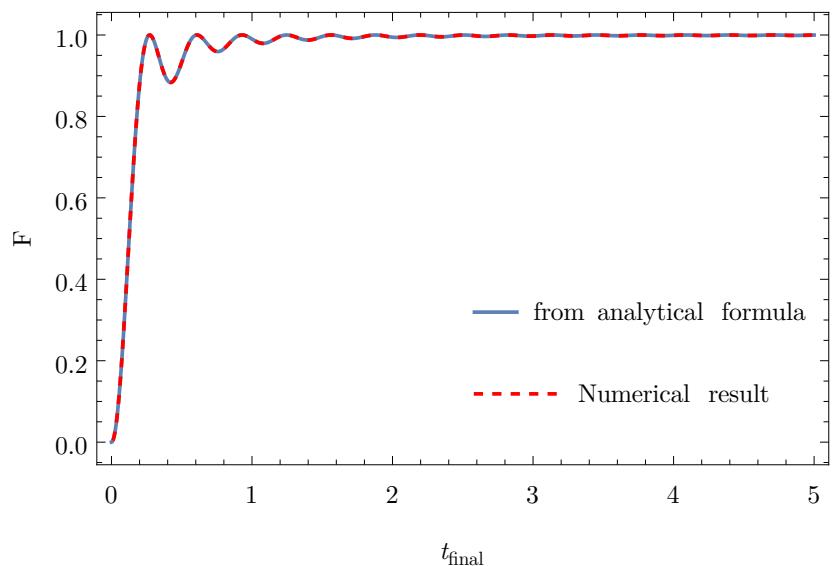


Figure 1.3: Final fidelity dependence on final time T_f .

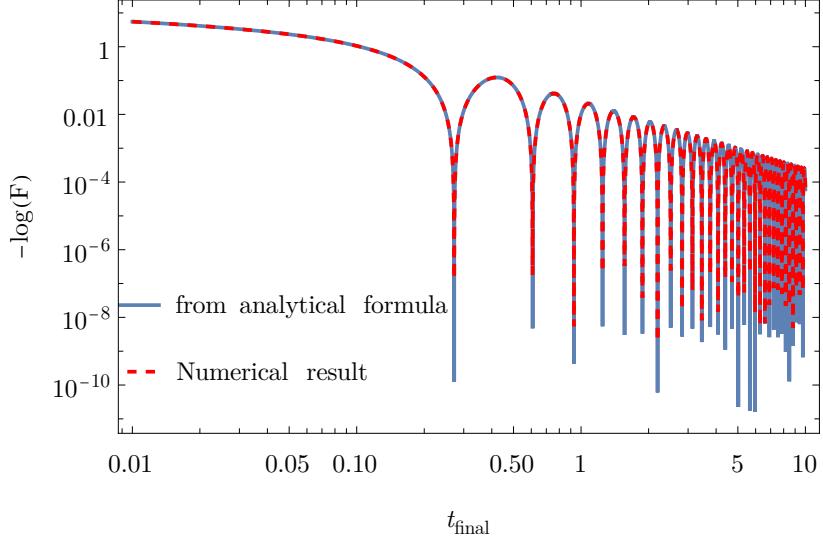


Figure 1.4: Fidelity dependence on final time in log-log scale. We can observe difference in numerical evaluation of different firmulas in the height of spikes.

From the fidelity Eq. 1.12 goes that $F = 1$ is equivalent to

$$\cos\left(\sqrt{T_s^2 + \pi^2}\right) = 1, \quad (1.13)$$

for $T_s := 2sT_f$. The solution to this equation is

$$T = \sqrt{(2\pi k)^2 - \pi^2} \text{ for } k \in \mathbb{N}. \quad (1.14)$$

The numerical check can be seen on Fig. 1.5. Because $F = 1$ has solutions 1.14, the its dependence on final time in logarithmic scale has spikes going to 0, see Fig. 1.4, and their density is proportional to T_f^2 .

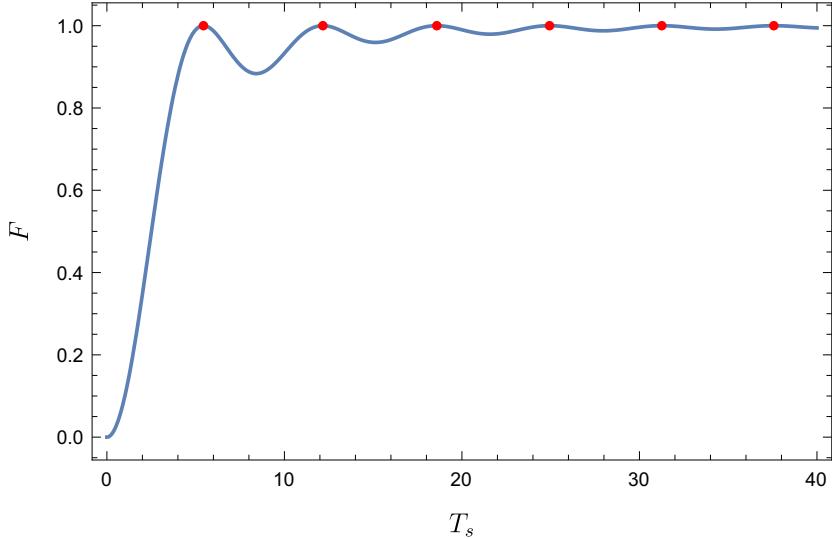


Figure 1.5: Rescaled final fidelity $T_s := 2sT_f$ dependence on final time. Red points are for $F = 1$.

Fidelity as a function of time and final time can be seen on Figures 1.6, 1.9.

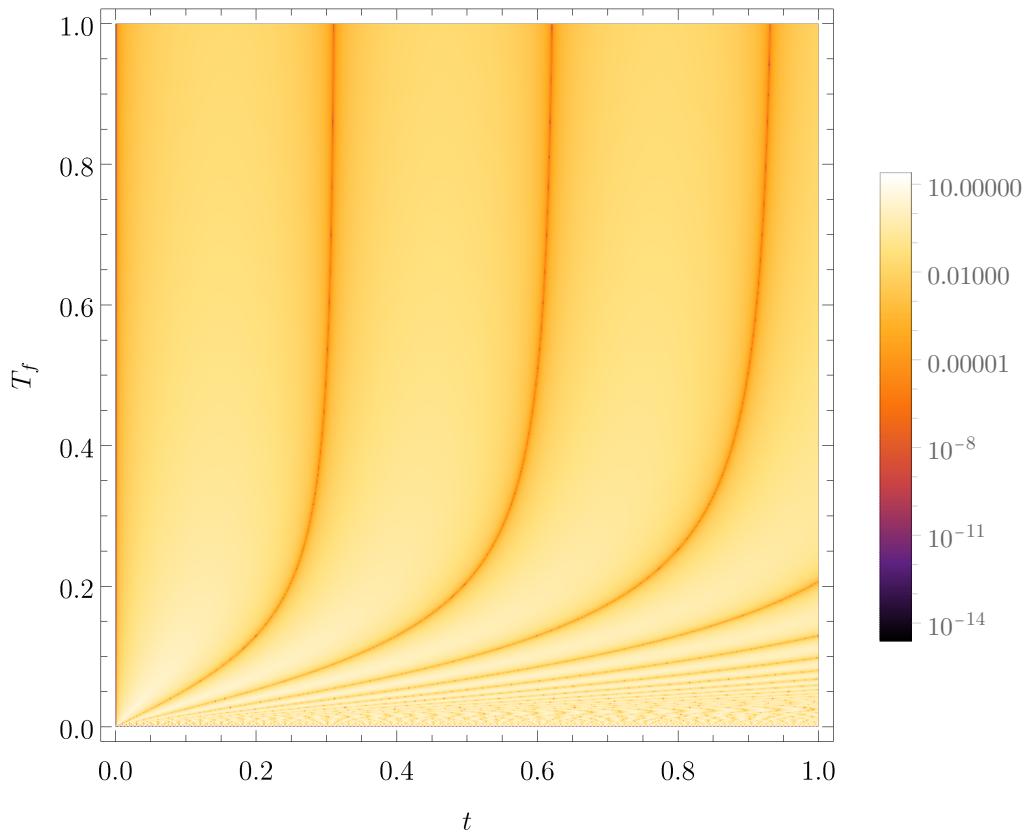


Figure 1.6: Fidelity dependence on time and final time.

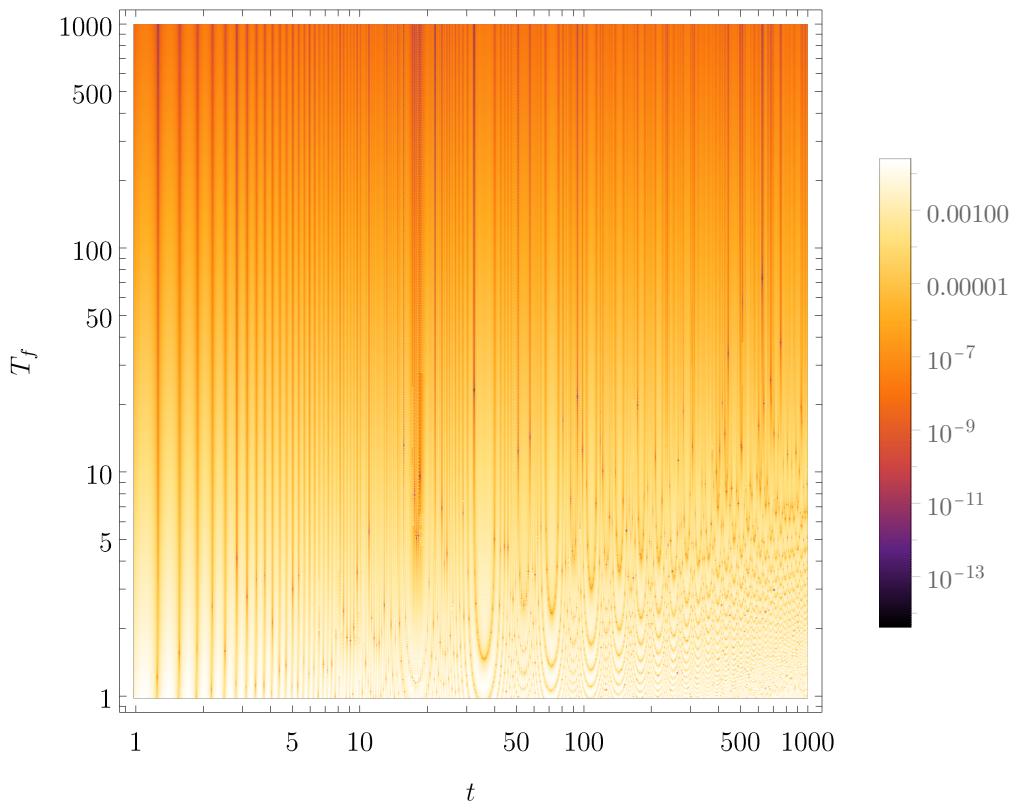


Figure 1.7: Fidelity dependence on time and final time in log log scale.

1.2.1 Energy variance

$$\delta E^2(t) := \langle \psi(t) | \hat{H}^2 | \psi(t) \rangle - \langle \psi(t) | \hat{H} | \psi(t) \rangle^2 \quad (1.15)$$

as a function of time t and final time T_f is for geodesical driving

$$\begin{aligned} \delta E^2 = & \frac{s^2}{2q^2} \left[\left[16s^4 + 2s^2 \left((\omega^2 - 8s^2) \cos(2t\omega) - 8\omega^2 \cos^2(t\omega) \cos(t\sqrt{q}) \right) \right. \right. \\ & + 14s^2\omega^2 + \omega^4 \Big] - \omega^2 \left((2s^2 + \omega^2) \cos(2t\omega) - 2s^2 \right) \cos(2tq) \\ & \left. \left. + 8s^2\omega q \sin(2t\omega) \sin(tq) + \omega^3 q \sin(2t\omega) \sin(2tq) \right], \right] \end{aligned} \quad (1.16)$$

see the definition of q under Eq. 1.7. It's value as can be seen on Fig. 1.8.

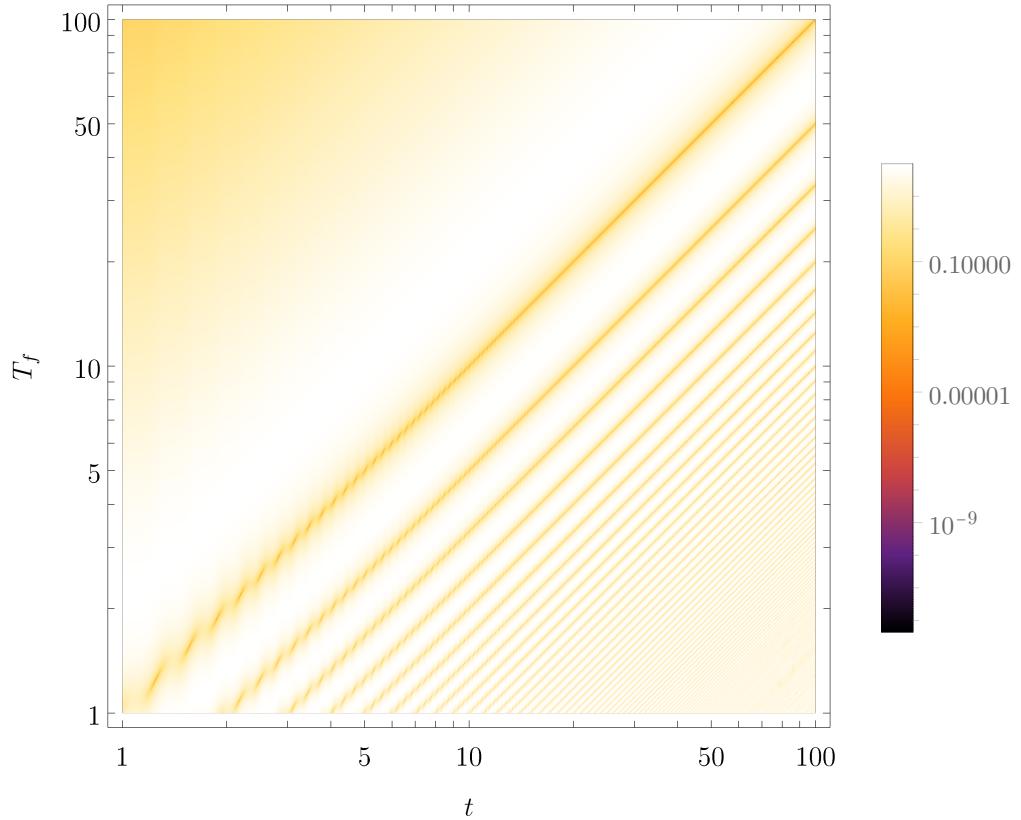


Figure 1.8: Energy variance for geodesical driving protocol.

1.3 Linear driving

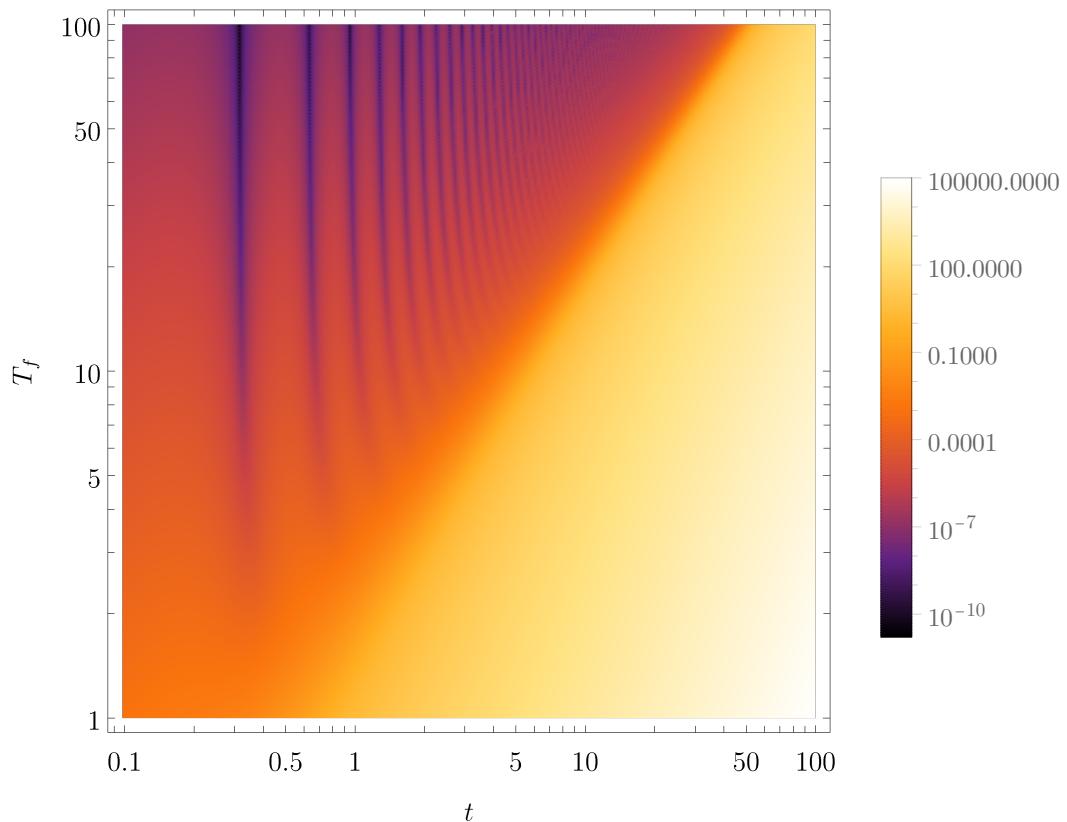


Figure 1.9: Energy variance for $(\Lambda; \Omega) = (0; 0.2)$ during the linear driving.