1. Two level system (Landau-Zener)

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Let's have Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix} \Omega(t) & \Delta(t) \\ \Delta(t) & -\Omega(t) \end{pmatrix} \tag{1.1}$$

and a driving along the path parametrized by time $t \in [0, 1]$

$$d(t) \coloneqq \begin{pmatrix} -s\cos(\omega(T_f)t) \\ 0 \\ s\sin(\omega(T_f)t) \end{pmatrix} \tag{1.2}$$

for speed regulating function $\omega(T_f) \coloneqq \pi/T_f$. This means, the driving will always be along half-sphere, as in Fig. 1.1.

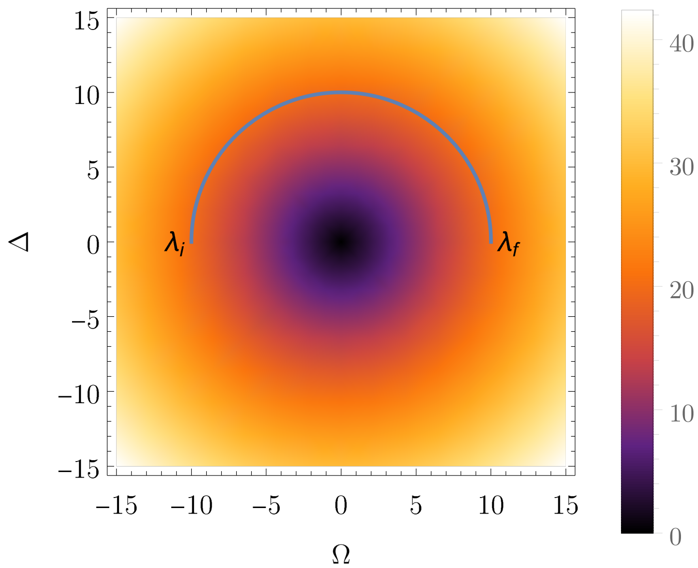

Figure 1.1: Driving along the geodesic. λ_i and λ_f are initial resp. final parameters. DensityPlot shows the difference between Hamiltonian eigenvalues.

1.1 Derivation of the fidelity

Because the Hamiltonian can be rewritten using Pauli matrices

$$\mathcal{H}(t) = \begin{pmatrix} -s\cos(t\omega) & s\sin(t\omega) \\ s\sin(t\omega) & s\cos(t\omega) \end{pmatrix} = \Delta(t)\sigma_x + \Omega(t)\sigma_z = d(t).\hat{\sigma} \tag{1.3}$$

one can see that changing from the original frame to moving frame of reference (let's omit the final time dependence $\omega = \omega(T_f)$ for a while)

$$\psi(t) =: e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) \tag{1.4}$$

reflects rotational symmetry of the problem. This change of reference frame transforms Schrödinger equation

$$\begin{aligned} \mathcal{H}(t)\psi(t) &= i\psi'(t) \\ \mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) &= ie^{\frac{i\omega}{2}\hat{\sigma}_y t}\left(\frac{i\omega\hat{\sigma}_y}{2}\right)\tilde{\psi}(t) + ie^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}'(t) \\ \underbrace{\left(e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t} + \frac{\omega}{2}\hat{\sigma}_y\right)}_{\tilde{\mathcal{H}}(t)}\tilde{\psi}(t) &= i\tilde{\psi}'(t). \end{aligned} \tag{1.5}$$

Therefore we can equivalently solve the Fidelity problem in this new coordinate system.

Hamiltonian in the moving frame is

$$\tilde{\mathcal{H}} = \begin{pmatrix} -s & -i\omega(T_f)/2 \\ i\omega(T_f)/2 & s \end{pmatrix}, \tag{1.6}$$

which is time independent. The Schrödinger equation can now be easily solved using evolution operator

$$\begin{aligned} \hat{U}(t) &= e^{-i\tilde{\mathcal{H}}t} \\ &= \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & -\frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & \cos\left(\frac{t}{2}q(T_f)\right) - \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix}, \end{aligned} \tag{1.7}$$

for $q(T_f) = \sqrt{4s^2 + \omega(T_f)^2}$.

In the original frame we get the evolution of the state $\psi(0)$

$$\psi(t) = e^{\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}(t)\tilde{\psi}(0) = \underbrace{e^{\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}e^{-\frac{i\omega}{2}\hat{\sigma}_y t}}_{\hat{U}(t)}\underbrace{e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(0)}_{\psi(0)}. \tag{1.8}$$

Then the evolved wavefunction is

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\cos(t\omega(T_f))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{(\omega(T_f)-2is\sin(t\omega(T_f)))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix} \tag{1.9}$$

and the ground state

$$|0(t)\rangle = \mathcal{N}\begin{pmatrix} -\cot(\frac{t}{2}\omega(T_f)) \\ 1 \end{pmatrix}, \tag{1.10}$$

for a normalization constant $\mathcal{N} := |\langle 0(t)|0(t)\rangle|^{-1}$. Fidelity during the transport is then[1]

$$F = |\langle 0(t)|\psi(t)\rangle|^2, \tag{1.11}$$

[1] If we would calculate the Fidelity in comoving frame, we would get exactly one. This is the essence of counterdiabatic driving.

Explicit formula for fidelity in time t and geodesic driving with final time T_f is then

$$F(t,T_f) = \frac{\pi^2\left(\cos\left(t\sqrt{\frac{\pi^2}{T_f^2}+4s^2}\right)+1\right)+8s^2T_f^2}{2\sin^4\left(\frac{\pi t}{2T_f}\right)\left(4s^2T_f^2+\pi^2\right)\left(\left|\cot\left(\frac{\pi t}{2T_f}\right)\right|^2+1\right)^2}. \tag{1.12}$$

1.2 Analysis of the fidelity formula

Fidelity for some fixed final time is just oscilating curve close to 1. For $T_f = 10$ it can be seen on Fig. 1.2

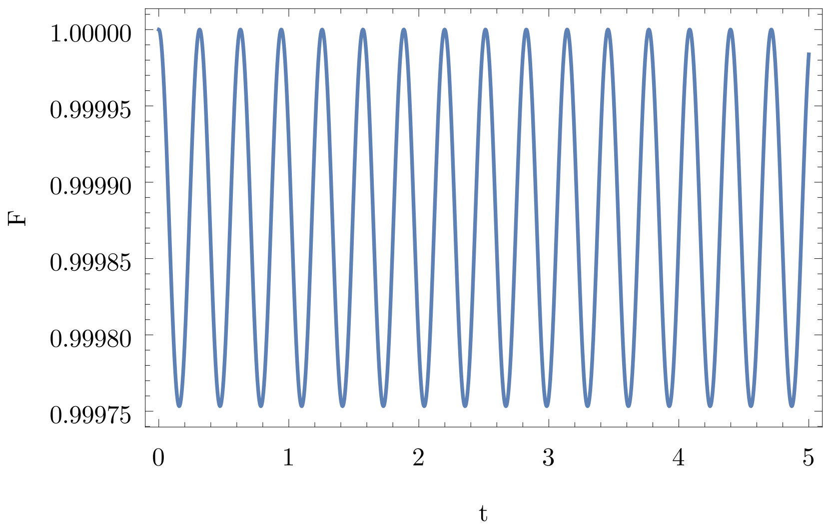

Figure 1.2: Fidelity in time for $T_f = 10$

The *final fidelity* (at $t = T_f$) dependence on final time T_f can be seen on Fig. 1.3 and 1.4.

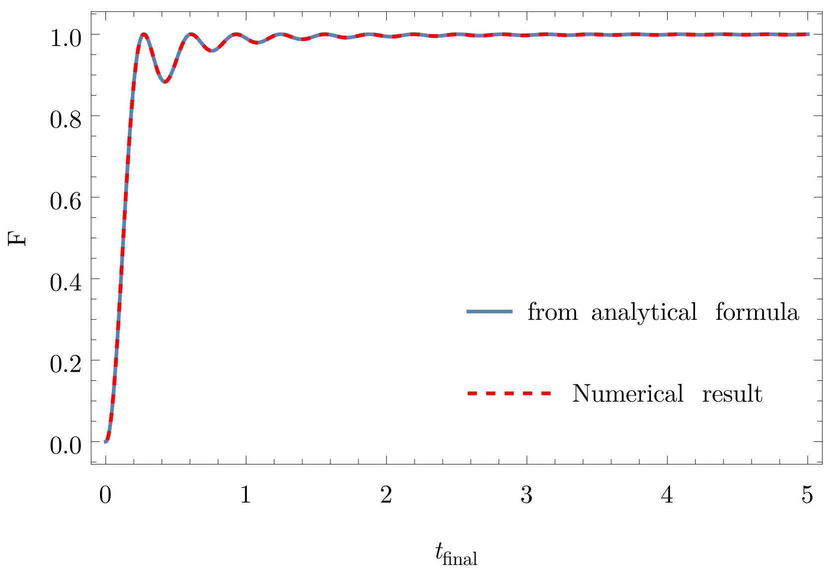

Figure 1.3: Final fidelity dependence on final time T_f.

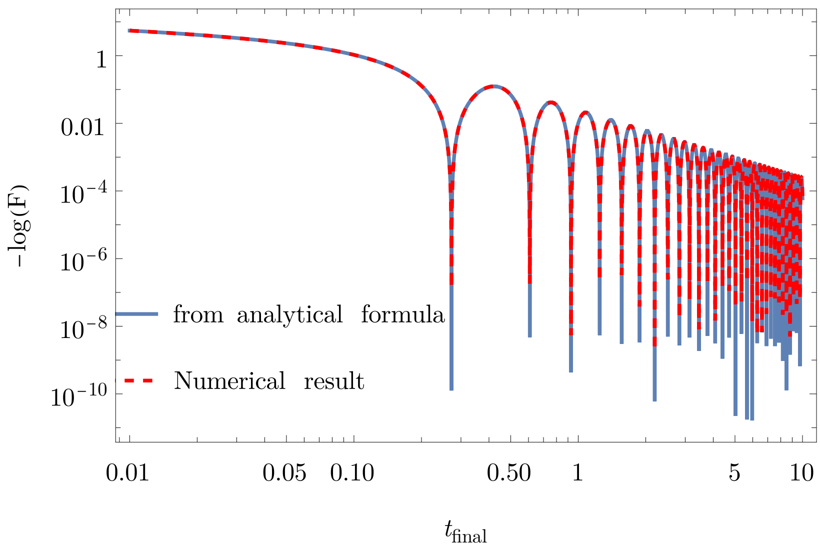

Figure 1.4: Fidelity dependence on final time in log-log scale. Spikes should go to zero, which does not hold due to numerical precision of the plotting algorithm.

From formula for fidelity 1.12 goes $F = 0$ is equivalent to

$$a \tag{1.13}$$

This proves that showing infidelity dependence on final time in log-log scale has spikes going to 0, see Fig. 1.4.

Fidelity as a function of time and final time can be seen on Figures 1.5, 1.8.

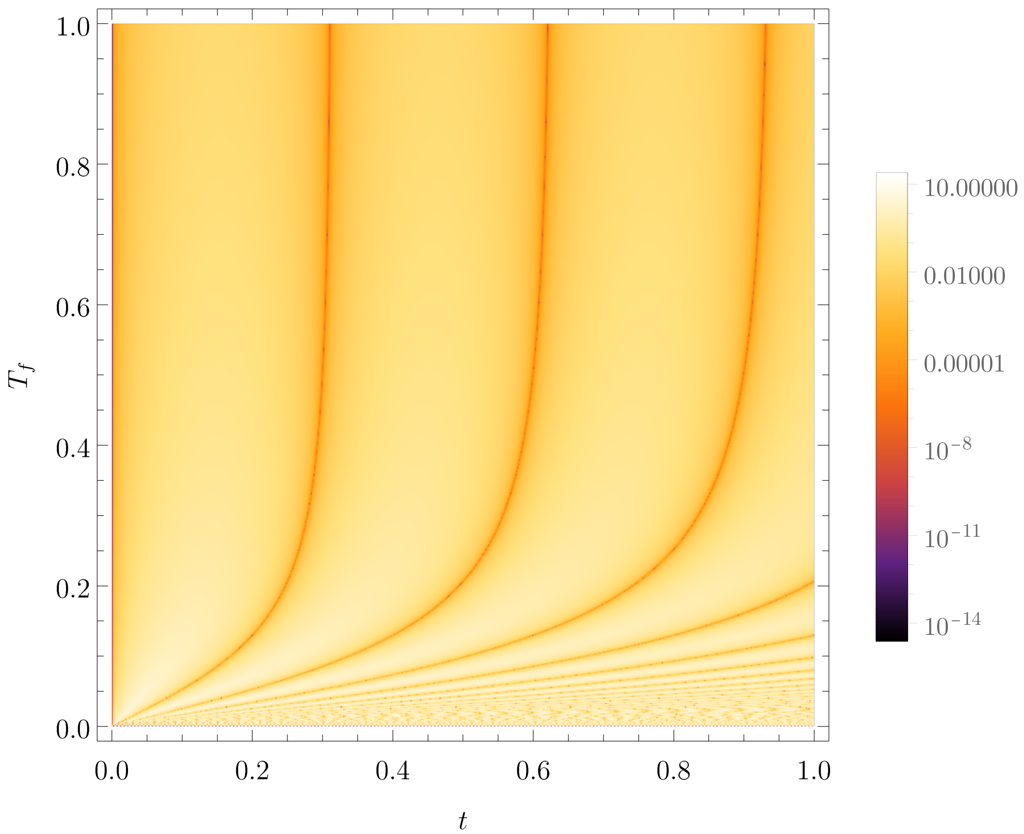

Figure 1.5: Fidelity dependence on time and final time.

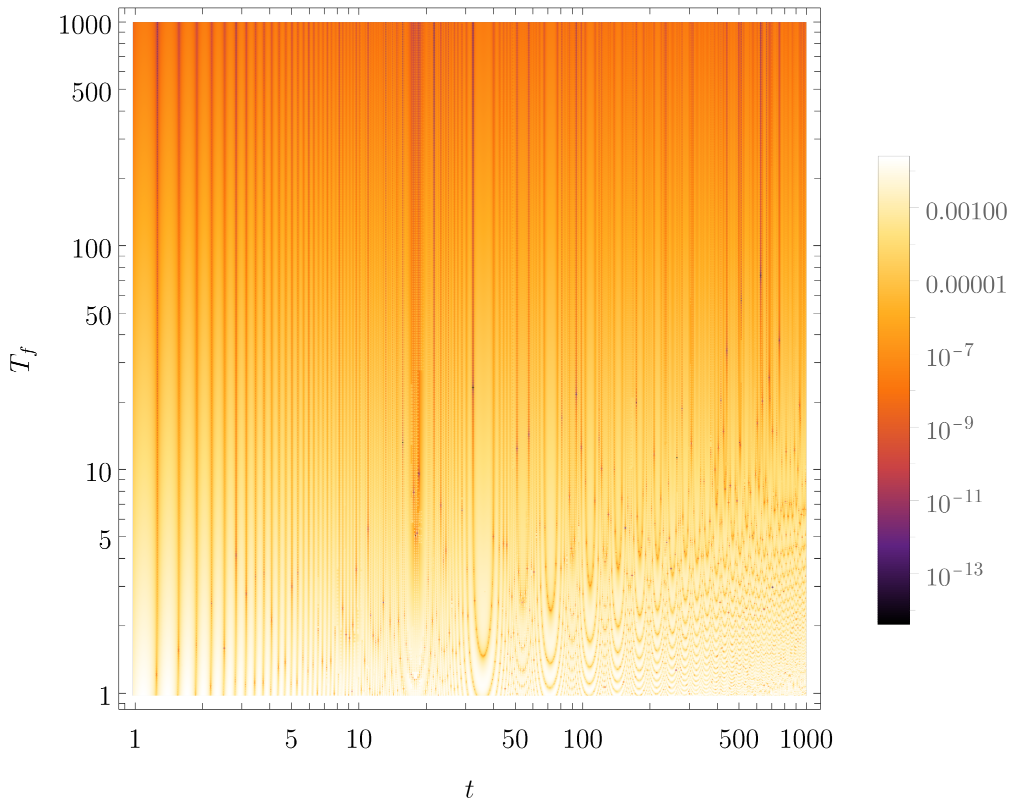

Figure 1.6: Fidelity dependence on time and final time in log log scale.

1.2.1 Energy variance

$$\begin{aligned}\delta E^2 =&\frac{s^2}{2q^2}\Bigg[\Big[16s^4 + 2s^2\left(\left(\omega^2 - 8s^2\right)\cos(2t\omega) - 8\omega^2\cos^2(t\omega)\cos\left(t\sqrt{q}\right)\right)\\ &+ 14s^2\omega^2 + \omega^4\Big] - \omega^2\left(\left(2s^2 + \omega^2\right)\cos(2t\omega) - 2s^2\right)\cos\left(2tq\right)\\ &+ 8s^2\omega q\sin(2t\omega)\sin\left(tq\right) + \omega^3 q\sin(2t\omega)\sin\left(2tq\right)\Bigg].\end{aligned} \tag{1.14}$$

It's valie as a function $\delta E^2(t, T_f)$ can be seen on Fig. 1.7.

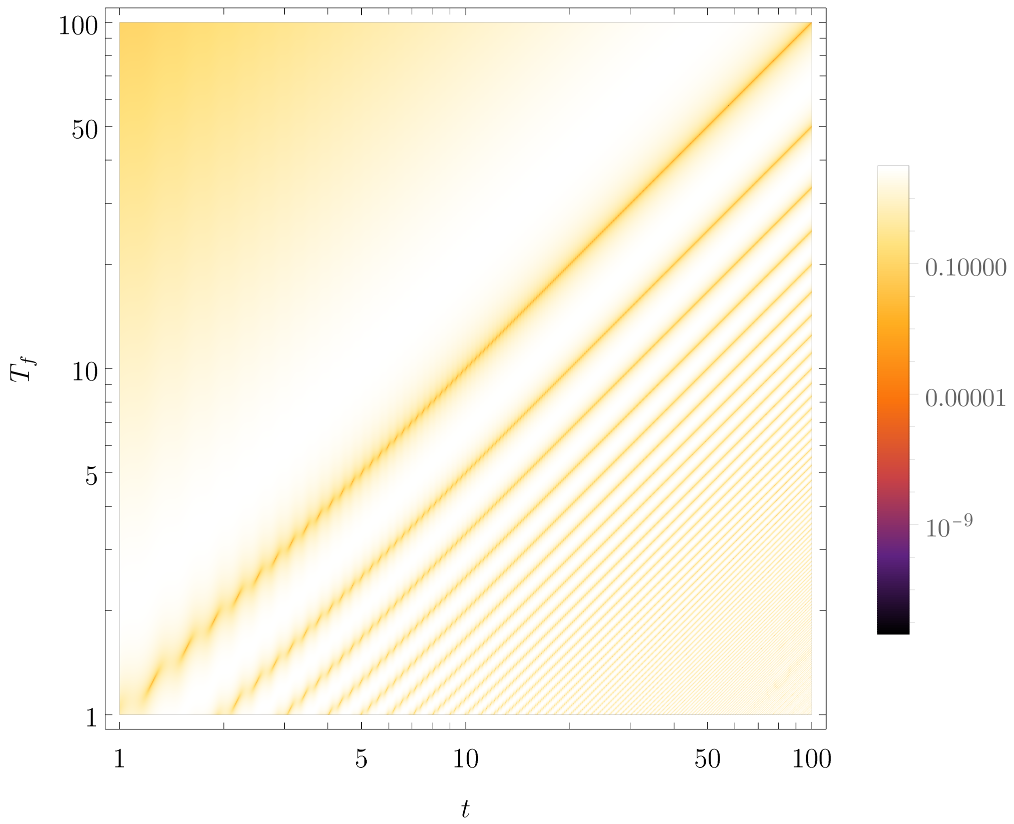

Figure 1.7: Energy variance for geodesical driving protocol.

1.3 Linear driving

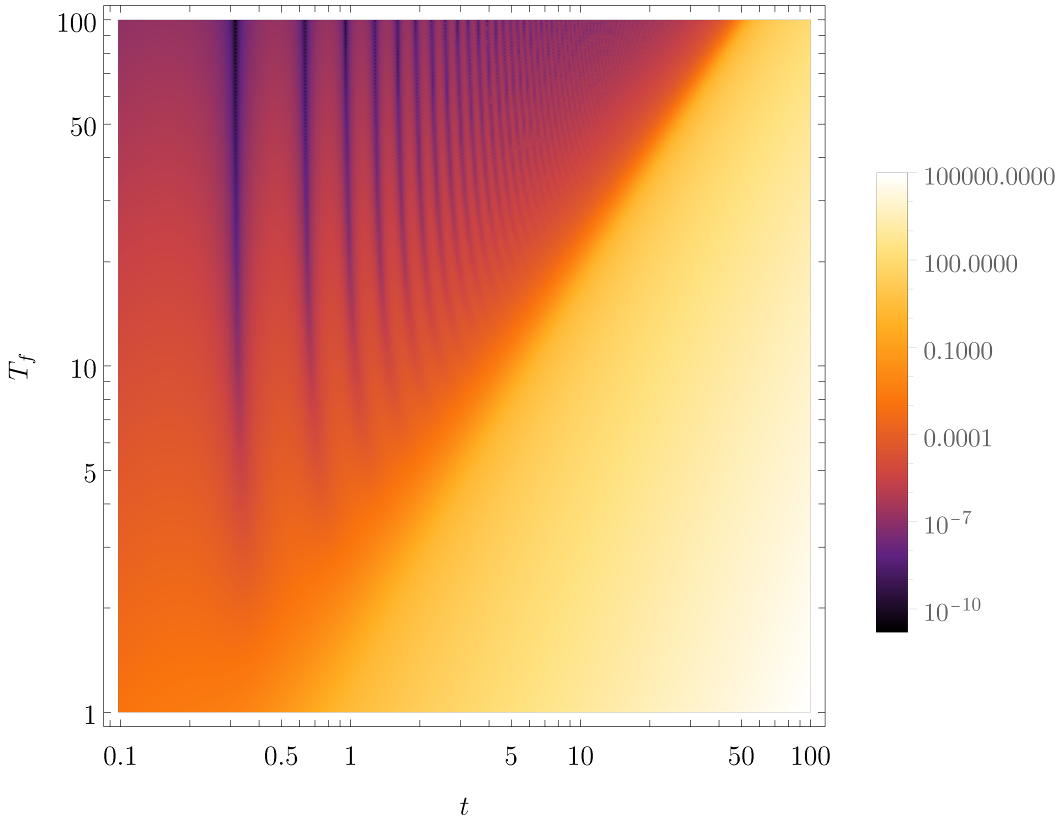

Figure 1.8: Energy variance for $(\Lambda; \Omega) = (0; 0.2)$ during the linear driving.