1. The role of geodesics

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1.1 Minimizing the energy variance

From Bukov et al. [2019]. About transports using *fast forward* Hamiltonian means the system is driven to the target state in some fixed amount of time. The transport is done on the ground state manifold $\mathcal{M}$.

Conjecture 1. For any fast forward Hamiltonian $\hat{H}(\lambda(t))$ driven along one dimentional path $\lambda(t) : \mathbb{R} \mapsto \mathbb{R}$ using time t as parametrization, the energy fluctuations δE^2, averaged along the path, are larger than the geodesic length l_λ

$$\int_0^T \sqrt{\delta E^2(t)}\mathrm{d}t =: l_t \geq l_\lambda \int_{\lambda_i}^{\lambda_f} \sqrt{g_{\lambda\lambda}}\mathrm{d}\lambda = \int_0^T \sqrt{g_{\lambda\lambda}}\frac{\mathrm{d}\lambda}{\mathrm{d}t}\mathrm{d}t. \tag{1.1}$$

The length $l_{\boldsymbol{\lambda}}$ is defined in control space (with metric tensor $g_{\lambda\lambda}$) and is generally larger than the distance between wave functions, i.e. the absolute geodesic (defined with $G_{\mu\nu}$). From its definition, we can see that it corresponds to the metric tensor as we use it.

The energy variance is

$$\delta E^2 = \langle o(t)|\hat{H}(t)^2|o(t)\rangle - \langle o(t)|\hat{H}(t)|o(t)\rangle^2 = \langle \partial_t(t)|\partial_t o(t)\rangle)_c = G_{tt} \tag{1.2}$$

and the Metric tensor in control space is defined as

$$g_{\lambda\lambda} \coloneqq \langle \partial_\lambda o(t)|\partial_\lambda o(t)\rangle)_c \tag{1.3}$$

Proof.

$$\delta E^2 \equiv \langle o(t)|\hat{H}(t)^2|o(t)\rangle_c = \dot{\lambda}^2 G_{\lambda\lambda} + \mathcal{O}(\dot{\lambda}^4), \tag{1.4}$$

where $\mathcal{O}(\dot{\lambda}^4)$ needs to be positive for any real-valued Hamiltonian. This comes from the fact, that it has instantaneous time-reversal symmetry. □

The conjecture only applies to unit fidelity protocols ($F(t) = 1\ \forall t \in [0, T_f]$) and can be extended to an arbitrary dimensional path.

1.2 APT

From Rigolin et al. [2008]. One important thing to bare in mind is *locality of variables*. Let's call variable $V(s)$ *local* if it depends only on infinitesimal sirrounding of s. These variables will be in shades of blue and non-local variables in shades of red.

The power series will be derived using a small parameter $v = 1/T$. Starting again with

$$|\Psi(s)\rangle = \sum_{p=0}^{\infty} v^p |\Psi^{(p)}(s)\rangle, \tag{1.5}$$

for

$$|\Psi^{(p)}(s)\rangle = \sum_{n=0} e^{-\frac{i}{v}\omega_n(s)} e^{i\gamma_n(s)} b_n^{(p)}(s)|n(s)\rangle. \tag{1.6}$$

Here the

$$\omega_n(s) \coloneqq \frac{1}{\hbar}\int_0^s E_n(s')\mathrm{d}s' \tag{1.7}$$

$$\gamma_n(s) \coloneqq i\int_0^s \langle n(s')|\frac{\mathrm{d}}{\mathrm{d}s'}n(s')\rangle \mathrm{d}s' \equiv i\int_0^s M_{nn}(s')\mathrm{d}s' \tag{1.8}$$

are so-called dynamical resp. Berry (geometric) phase and $|n(s)\rangle$ are solution to

$$\hat{H}(s)\,|n(s)\rangle = E_n(s)\,|n(s)\rangle. \tag{1.9}$$

Variables $\omega_n(s)$ and $\gamma_n(s)$ are defined using integration over the whole protocol, therefore there are *non/local variables*. The problem now lies in determining $b_n^{(p)}(s)$, which is also nonlocal. Because it depends on its relative geometric and dynamical phase to other energy levels, lets write it as a series

$$b_n^{(p)}(s) = \sum_{m=0} e^{\frac{i}{v}\omega_{nm}(s)} e^{-i\gamma_{nm}(s)} b_{nm}^{(p)}(s), \tag{1.10}$$

where $\omega_{nm} \coloneqq \omega_m - \omega_n$, $\gamma_{nm} \coloneqq \gamma_m - \gamma_n$. The reason for locality of $b_{nm}^{(p)}(s)$ will be clear soon.

Inserting all to original series 1.5, we get

$$|\Psi(s)\rangle = \sum_{n,m=0}\sum_{p=0}^{\infty} v^p e^{-\frac{i}{v}\omega_m(s)} e^{i\gamma_m(s)} b_{nm}^{(p)}(s)|n(s)\rangle. \tag{1.11}$$

Because the initial state is eigenstate, we get initial conditions $b_{nm}^{(0)}(s) = 0$. In addition, one can rewrite equation 1.11 to the iteratively solvable form

$$\frac{i}{\hbar}\Delta_{nm}(s)b_{nm}^{(p+1)}(s) + \dot{b}_{nm}^{(p)}(s) + W_{nm}(s)b_{nm}^{(p)}(s) + \sum_{k=0,k\neq n} M_{nk}(s)b_{km}^{(p)}(s) = 0, \tag{1.12}$$

for $\Delta_n m(s) \coloneqq E_m - E_n$, $W_{nm}(s) \coloneqq M_{nn}(s) - M_{mm}(s)$, where M_{mn} is defined in Eq. 1.8. We can see that $b_{mn}^{(p)}$, as a solution to Eq. 1.12, **only depends on difference between energy levels, eigenstates during the path and their directional derivatives. Not on the path itself**. All of those are easily obtained, once the driving path is prescribed.

Bibliography

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