1. Two level system (Landau-Zener)

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Let's have Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix} \Omega(t) & \Delta(t) \\ \Delta(t) & -\Omega(t) \end{pmatrix} \tag{1.1}$$

and driving along the path parametrized by time $t \in [0, 1]$

$$d(t) := \begin{pmatrix} -s\cos(\omega(T_f)t) \\ 0 \\ s\sin(\omega(T_f)t) \end{pmatrix} \tag{1.2}$$

for speed regulating function $\omega(T_f) := \pi/T_f$. This means, the driving will always be along half-sphere, as in Fig. 1.1.

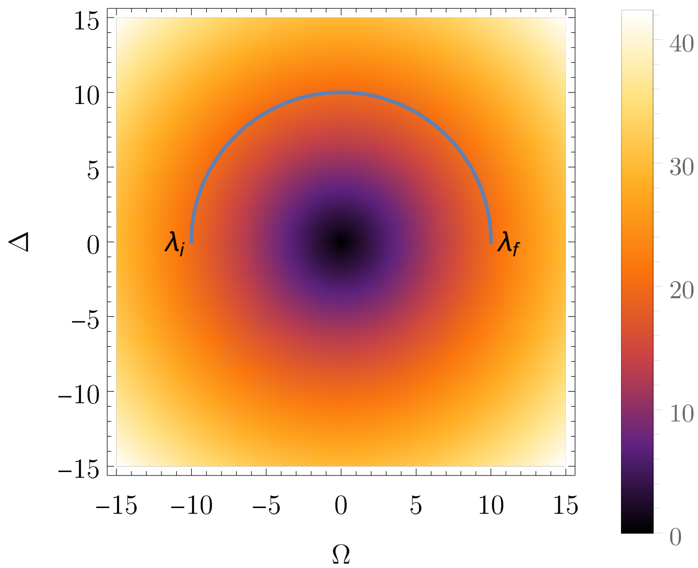

Figure 1.1: Driving along the geodesic. DensityPlot shows the difference between Hamiltonian eigenvalues.

Because the Hamiltonian can be rewritten using Pauli matrices

$$\mathcal{H}(t) = \Delta(t)\sigma_x + \Omega(t)\sigma_z \tag{1.3}$$

one can see that changing to moving frame of reference (let's omit the final time dependence $\omega = \omega(T_f)$ for a while and use different colors for different frames)

$$\color{purple}{\psi(t)} \color{black}{=: e^{-\frac{i\omega}{2}\hat{\sigma}_y t}} \color{blue}{\tilde{\psi}(t)} \tag{1.4}$$

reflects rotational symmetry of the problem. This change of reference frame in Schrödinger equation is in fact

$$
\begin{aligned}
\mathcal{H}(t)\psi(t) &= i\psi'(t) \\
\mathcal{H}(t)e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) &= ie^{-\frac{i\omega}{2}\hat{\sigma}_y t}\left(-\frac{i\omega\hat{\sigma}_y}{2}\right)\tilde{\psi}(t) + ie^{-\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}'(t) \\
\underbrace{\left(e^{\frac{i\omega}{2}\hat{\sigma}_y t}\mathcal{H}(t)e^{-\frac{i\omega}{2}\hat{\sigma}_y t} - \frac{\omega\hat{\sigma}_y}{2}\mathbb{1}\right)}_{\tilde{\mathcal{H}}(t)}\tilde{\psi}(t) &= i\tilde{\psi}'(t).
\end{aligned}
\tag{1.5}
$$

Therefore we can equivalently solve the Fidelity problem in this new coordinate system.

Hamiltonian in the moving frame is

$$
\tilde{\mathcal{H}} = \begin{pmatrix} -\omega(T_f)/2 & -s \\ -s & \omega(T_f)/2 \end{pmatrix}, \tag{1.6}
$$

which is time independent. The Schrödinger equation can now be easily solved using evolution operator

$$
\hat{U}(t) = e^{-i\tilde{\mathcal{H}}t} = \begin{pmatrix} \cos(qt) - \frac{i\omega}{2q}\sin(qt) & -\frac{is}{q}\sin(qt) \\ -\frac{is}{q}\sin(qt) & \cos(qt) + \frac{i\omega}{2q}\sin(qt) \end{pmatrix}, \tag{1.7}
$$

for $q = \frac{1}{2}\sqrt{4s^2 + \omega(T_f)^2}$.

In the original frame we get the evolution of the state $\psi(0)$

$$
\psi(t) = e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}(t)\tilde{\psi}(0) = \underbrace{e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}e^{\frac{i\omega}{2}\hat{\sigma}_y t}}_{\hat{U}(t)}\underbrace{e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(0)}_{\psi(0)}. \tag{1.8}
$$

The evolution matrix is then

$$
\hat{U}(t) = \begin{pmatrix} \cos(qt) + \frac{i\sin(tqt)(\omega\cos(t\omega) - 2s\sin(t\omega))}{2q} & \frac{i\sin(qt)(2s\cos(t\omega) + \omega\sin(t\omega))}{2q} \\ \frac{i\sin(qt)(2s\cos(t\omega) + \omega\sin(t\omega))}{2q} & \cos(qt) + \frac{i\sin(qt)(2s\sin(t\omega) - \omega\cos(t\omega))}{2q} \end{pmatrix} \tag{1.9}
$$

and the evolved wavefunction

$$
|\psi(t)\rangle = \begin{pmatrix} \cos(qt) + i\sin(qt)\left(\frac{1}{2}(\omega(T_f))\cos(t\omega(T_f)) - s\sin(t\omega(T_f))\right) \\ \frac{i\sin(qt)}{q}\left(s\cos(t\omega(T_f)) + \frac{1}{2}(\omega(T_f))\sin(t\omega(T_f))\right). \end{pmatrix} \tag{1.10}
$$

Fidelity during the transport is then

$$
F = |\langle 0(t)|\psi(t)\rangle|^2, \tag{1.11}
$$

where ground state is

$$
|0(t)\rangle = \mathcal{N}\begin{pmatrix} \sec(t\omega(T_f)) - \tan(t\omega(T_f)) \\ 1 \end{pmatrix}, \tag{1.12}
$$

for normalization constant $\mathcal{N}$.

Bibliography