1. Two level system (Landau-Zener)

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1.1 Energy variance for two level system

For two level system, the variance

$$\delta E^2(t) := \langle\psi(t)|\hat{H}^2|\psi(t)\rangle - \langle\psi(t)|\hat{H}|\psi(t)\rangle^2 \tag{1.1}$$

can be rewritten inserting identity $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1|$ around Hamiltonian. Omitting the time dependence of every element we get

$$\begin{aligned}\delta E^2 &= \langle\psi|\mathbb{1}\hat{H}^2\mathbb{1}|\psi\rangle - \langle\psi|\mathbb{1}\hat{H}\mathbb{1}|\psi\rangle^2 \\ &= \langle\psi|0\rangle\langle 0|\hat{H}^2|0\rangle\langle 0|\psi\rangle + \langle\psi|1\rangle\langle 1|\hat{H}^2|1\rangle\langle 1|\psi\rangle \\ &+\langle\psi|0\rangle\langle 0|\hat{H}^2|1\rangle\langle 1|\psi\rangle + \langle\psi|1\rangle\langle 1|\hat{H}^2|0\rangle\langle 0|\psi\rangle \\ &-\Big(\langle\psi|0\rangle\langle 0|\hat{H}|0\rangle\langle 0|\psi\rangle + \langle\psi|1\rangle\langle 1|\hat{H}|1\rangle\langle 1|\psi\rangle \\ &+\langle\psi|0\rangle\underbrace{\langle 0|\hat{H}|1\rangle}_{\propto\langle 0|1\rangle=0}\langle 1|\psi\rangle + \langle\psi|1\rangle\underbrace{\langle 1|\hat{H}|0\rangle}_{\propto\langle 0|1\rangle=0}\langle 0|\psi\rangle\Big)^2.\end{aligned} \tag{1.2}$$

Using Fidelity definition $F(t) = |\langle 0(t)|\psi(t)\rangle|^2$ and Schrödinger equation $\hat{H}|k\rangle = E_k|k\rangle$ we have

$$\delta E^2 = FE_0^2 + (1-F)E_1^2 - (FE_0 + (1-F)E_1)^2 = F(1-F)(E_0-E_1)^2. \tag{1.3}$$

For three level system we have $\mathbb{1} = |0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2|$ and

$$\delta E^2 = \sum_{k=1}^{3} E_k^2 F_k(1-F_k) - 4\prod_{k=1}^{3} E_k F_k - 2F_0F_1E_0E_1 - 2F_0F_2E_0E_2 - 2F_1F_2E_1E_2, \tag{1.4}$$

for $F_k := \langle k|\psi\rangle$, which has no practical simplification.

1.2 Geodesical driving

Let's have Hamiltonian

$$\mathcal{H}(t) = \begin{pmatrix}\Omega(t) & \Delta(t)\\ \Delta(t) & -\Omega(t)\end{pmatrix} \tag{1.5}$$

and a driving along the path parametrized by time $t \in [0,1]$

$$d(t) := \begin{pmatrix}-s\cos(\omega(T_f)t)\\ 0\\ s\sin(\omega(T_f)t)\end{pmatrix} \tag{1.6}$$

for speed regulating function $\omega(T_f) := \pi/T_f$. This means, the driving will always be along half-sphere, as in Fig. 1.1.

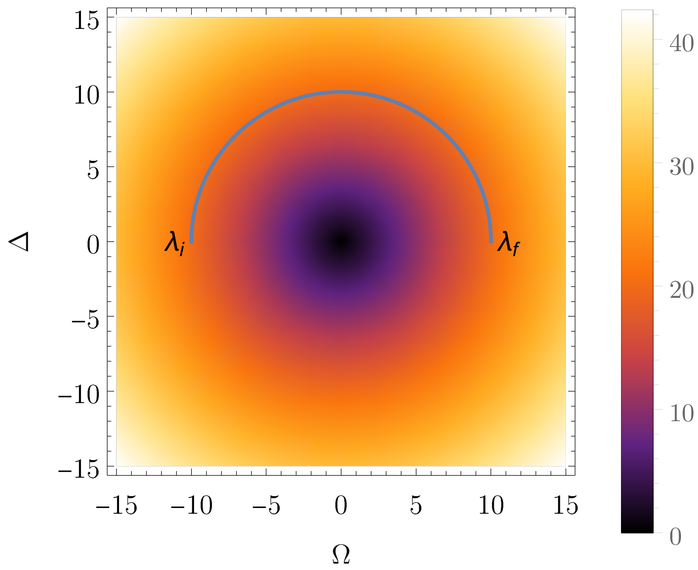

Figure 1.1: Driving along the geodesic. λ_i and λ_f are initial resp. final parameters. DensityPlot shows the difference between Hamiltonian eigenvalues.

1.2.1 Derivation of the fidelity

Because the Hamiltonian can be rewritten using Pauli matrices

$$\mathcal{H}(t) = \begin{pmatrix} -s\cos(t\omega) & s\sin(t\omega) \\ s\sin(t\omega) & s\cos(t\omega) \end{pmatrix} = \Delta(t)\sigma_x + \Omega(t)\sigma_z = d(t).\hat{\sigma} \tag{1.7}$$

one can see that changing from the original frame to moving frame of reference (let's omit the final time dependence $\omega = \omega(T_f)$ for a while)

$$\psi(t) =: e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) \tag{1.8}$$

reflects rotational symmetry of the problem. This change of reference frame transforms Schrödinger equation

$$\begin{aligned} \mathcal{H}(t)\psi(t) &= i\psi'(t) \\ \mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(t) &= ie^{\frac{i\omega}{2}\hat{\sigma}_y t}\left(\frac{i\omega\hat{\sigma}_y}{2}\right)\tilde{\psi}(t) + ie^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}'(t) \\ \underbrace{\left(e^{-\frac{i\omega}{2}\hat{\sigma}_y t}\mathcal{H}(t)e^{\frac{i\omega}{2}\hat{\sigma}_y t} + \frac{\omega}{2}\hat{\sigma}_y\right)}_{\tilde{\mathcal{H}}(t)}\tilde{\psi}(t) &= i\tilde{\psi}'(t). \end{aligned} \tag{1.9}$$

Therefore we can equivalently solve the Fidelity problem in this new coordinate system.

Hamiltonian in the moving frame is

$$\tilde{\mathcal{H}} = \begin{pmatrix} -s & -i\omega(T_f)/2 \\ i\omega(T_f)/2 & s \end{pmatrix}, \tag{1.10}$$

which is time independent. The Schrödinger equation can now be easily solved using evolution operator

$$\begin{aligned}\hat{U}(t) =& e^{-i\tilde{\mathcal{H}}t}\\ =& \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & -\frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{\omega(T_f)\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} & \cos\left(\frac{t}{2}q(T_f)\right) - \frac{2is\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix},\end{aligned} \tag{1.11}$$

for $q(T_f) = \sqrt{4s^2 + \omega(T_f)^2}$.

In the original frame we get the evolution of the state $\psi(0)$

$$\psi(t) = e^{\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}(t)\tilde{\psi}(0) = \underbrace{e^{\frac{i\omega}{2}\hat{\sigma}_y t}\hat{U}e^{-\frac{i\omega}{2}\hat{\sigma}_y t}}_{\hat{U}(t)}\underbrace{e^{\frac{i\omega}{2}\hat{\sigma}_y t}\tilde{\psi}(0)}_{\psi(0)}. \tag{1.12}$$

Then the evolved wavefunction is

$$|\psi(t)\rangle = \begin{pmatrix} \cos\left(\frac{t}{2}q(T_f)\right) + \frac{2is\cos(t\omega(T_f))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \\ \frac{(\omega(T_f)-2is\sin(t\omega(T_f)))\sin\left(\frac{t}{2}q(T_f)\right)}{q(T_f)} \end{pmatrix} \tag{1.13}$$

and the ground state

$$|0(t)\rangle = \mathcal{N}\begin{pmatrix} -\cot(\frac{t}{2}\omega(T_f)) \\ 1 \end{pmatrix}, \tag{1.14}$$

for a normalization constant $\mathcal{N} := |\langle 0(t)|0(t)\rangle|^{-1}$. Fidelity during the transport is then[1]

$$F = |\langle 0(t)|\psi(t)\rangle|^2, \tag{1.15}$$

Explicit formula for fidelity in time t and geodesic driving with final time T_f is then

$$F(t, T_f) = \frac{\pi^2\left(\cos\left(t\sqrt{\frac{\pi^2}{T_f^2} + 4s^2}\right) + 1\right) + 8s^2T_f^2}{2\sin^4\left(\frac{\pi t}{2T_f}\right)\left(4s^2T_f^2 + \pi^2\right)\left(\left|\cot\left(\frac{\pi t}{2T_f}\right)\right|^2 + 1\right)^2}. \tag{1.16}$$

The domain can be extended to $t \in [0, T_f]$, $T_f \in [0, \infty]$ because

$$\lim_{t\to 0} F = 1\,, \quad \lim_{T_f\to 0} F = 0.$$

1.2.2 Analysis of the fidelity formula

Fidelity for some fixed final time is just oscilating curve close to 1. For $T_f = 10$ it can be seen on Fig. 1.2. The *final fidelity* (at $t = T_f$) dependence on final time T_f can be seen on Fig. 1.3 and 1.4.

[1] If we would calculate the Fidelity in comoving frame, we would get exactly one. This is the essence of counterdiabatic driving.

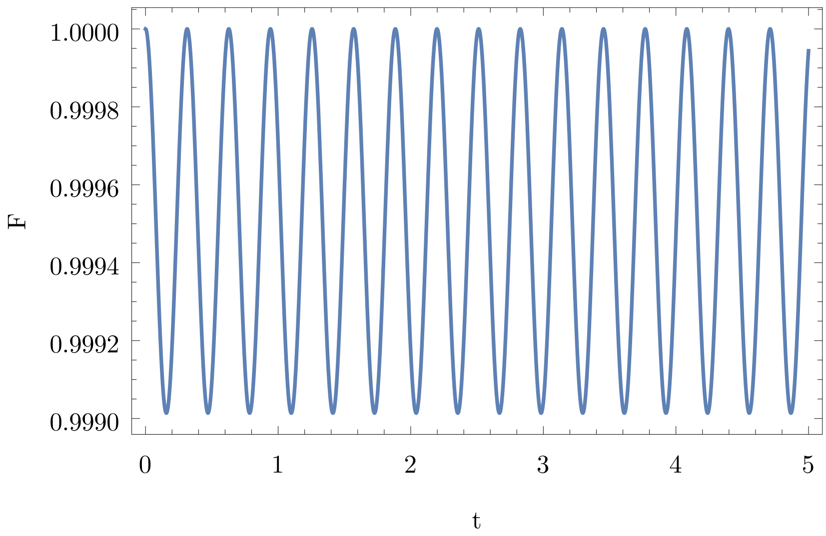

Figure 1.2: Fidelity in time for $T_f = 10$

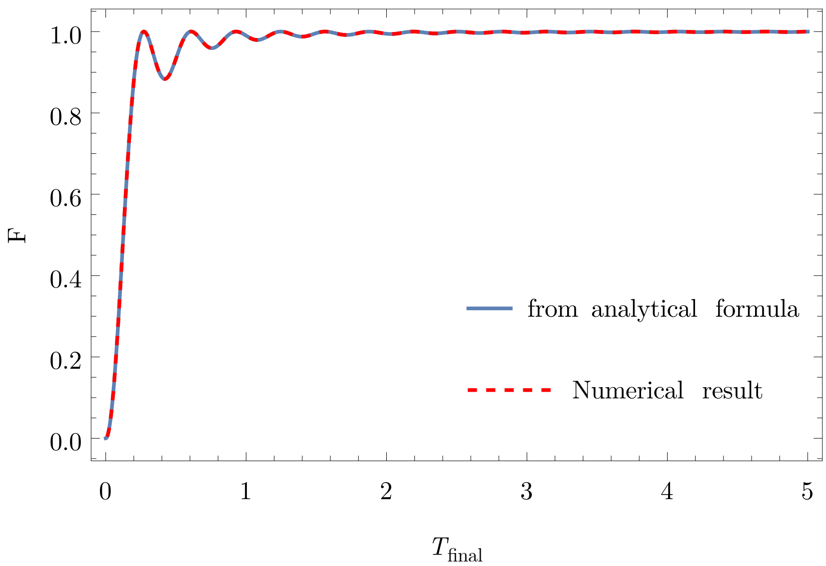

Figure 1.3: Final fidelity dependence on final time T_f.

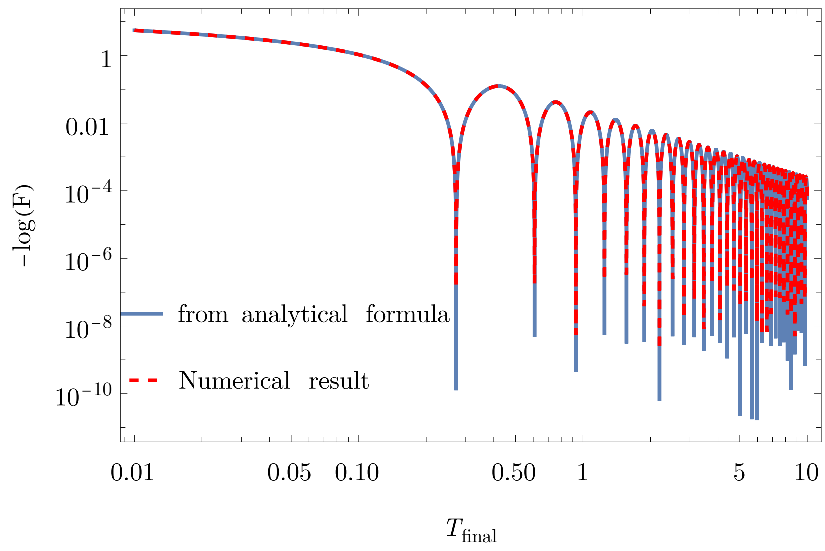

Figure 1.4: Fidelity dependence on final time in log-log scale. We can observe difference in numerical evaluation of different firmulas in the height of spikes.

From the fidelity Eq. 1.16 goes that $F = 1$ is equivalent to

$$\cos\left(\sqrt{T_s^2 + \pi^2}\right) = 1, \tag{1.17}$$

for $T_s := 2sT_f$. The solution to this equation is

$$T_s = \sqrt{(2\pi k)^2 - \pi^2} \text{ for } k \in \mathbb{N}. \tag{1.18}$$

This can be checked numerically, see Fig. 1.5. Because $F = 1$ has solutions 1.18, its dependence on final time in logarithmic scale has spikes going to 0, see Fig. 1.4, and their density is linear in T_f.

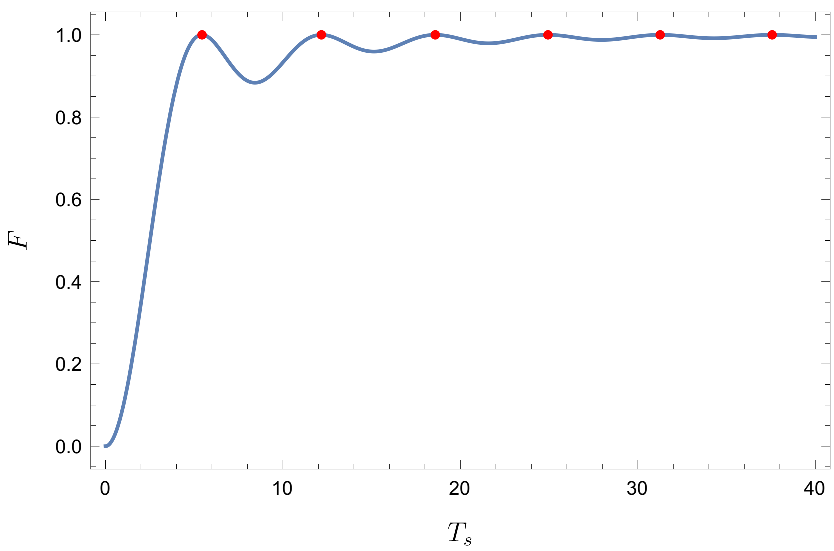

Figure 1.5: Rescaled final fidelity $T_s := 2sT_f$ dependence on final time. Red points are for $F = 1$.

Fidelity as a function of time and final time can be seen on Figures 1.6, 1.10.

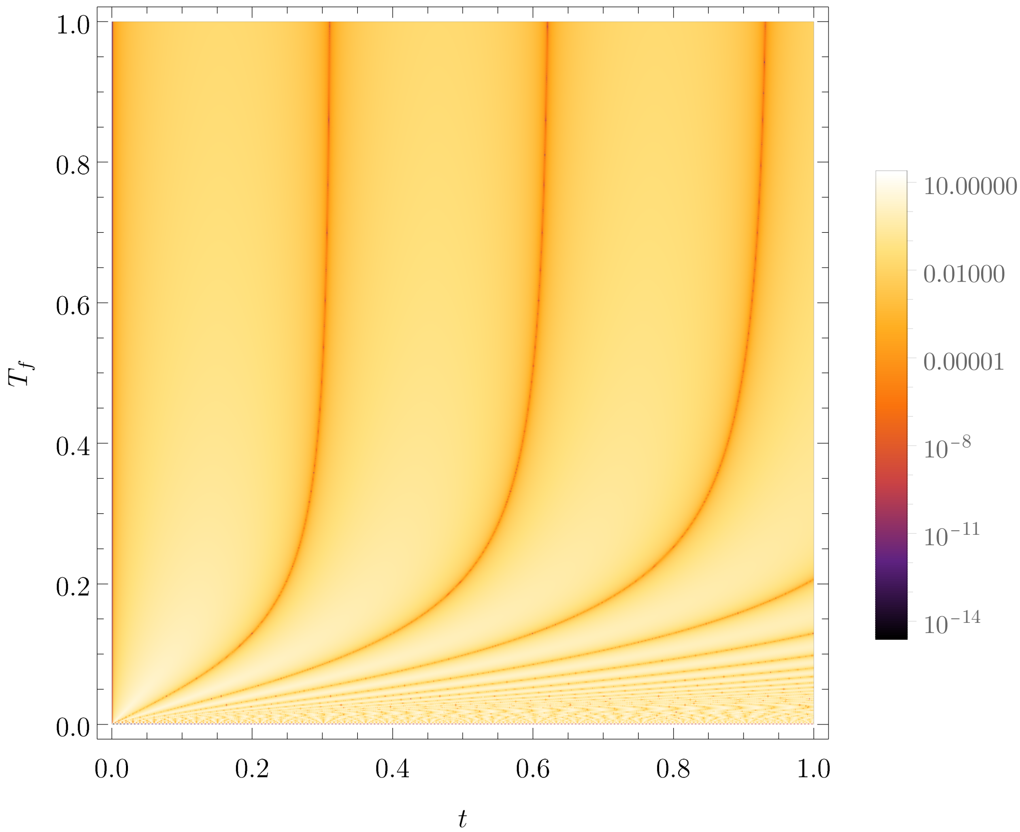

Figure 1.6: Fidelity dependence on time and final time.

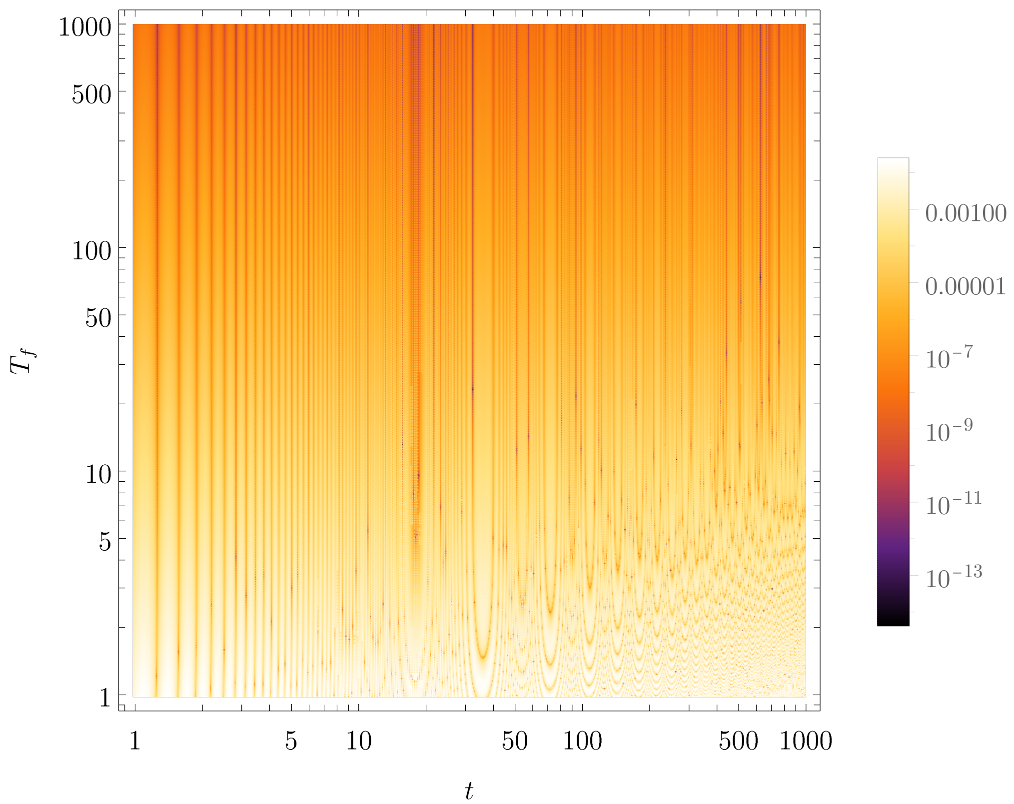

Figure 1.7: Fidelity dependence on time and final time in log log scale.

1.2.3 Energy variance

Evaluating the fidelity for geodesical driving gives a function of time t and final time T_f

$$\begin{aligned}\delta E^2 =& \frac{s^2}{2q^2}\Bigg[\bigg[16s^4 + 2s^2\left(\left(\omega^2 - 8s^2\right)\cos(2t\omega) - 8\omega^2\cos^2(t\omega)\cos\left(t\sqrt{q}\right)\right)\\ &+ 14s^2\omega^2 + \omega^4\bigg] - \omega^2\left(\left(2s^2+\omega^2\right)\cos(2t\omega) - 2s^2\right)\cos\left(2tq\right)\\ &+ 8s^2\omega q\sin(2t\omega)\sin\left(tq\right) + \omega^3 q\sin(2t\omega)\sin\left(2tq\right)\Bigg],\end{aligned} \tag{1.19}$$

see the definition of q under Eq. 1.11. It's value as can be seen on Fig. 1.12.

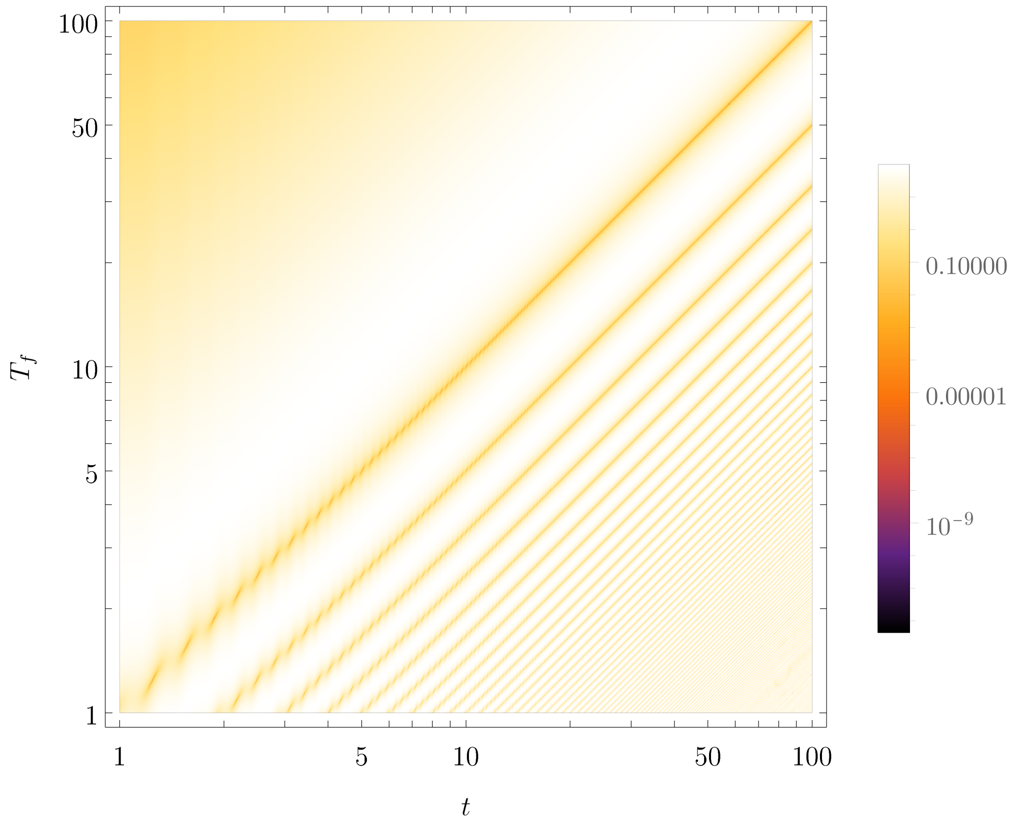

Figure 1.8: Energy variance for geodesical driving protocol.

1.3 Linear driving

Singularities are still present, at least according to function Root in Mathematica :). Their density might be $\propto \Omega^{-2}$ and is some funnction function of T_f.

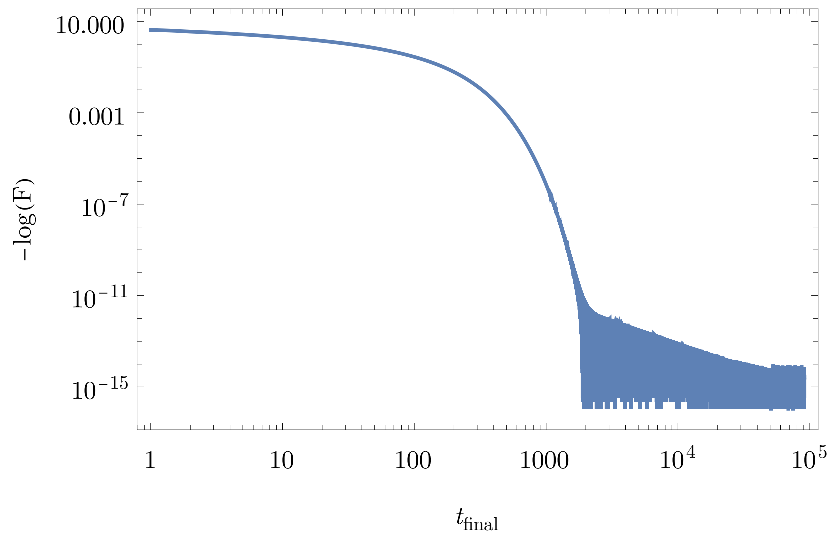

Figure 1.9: Final fidelity as a function of T_f.

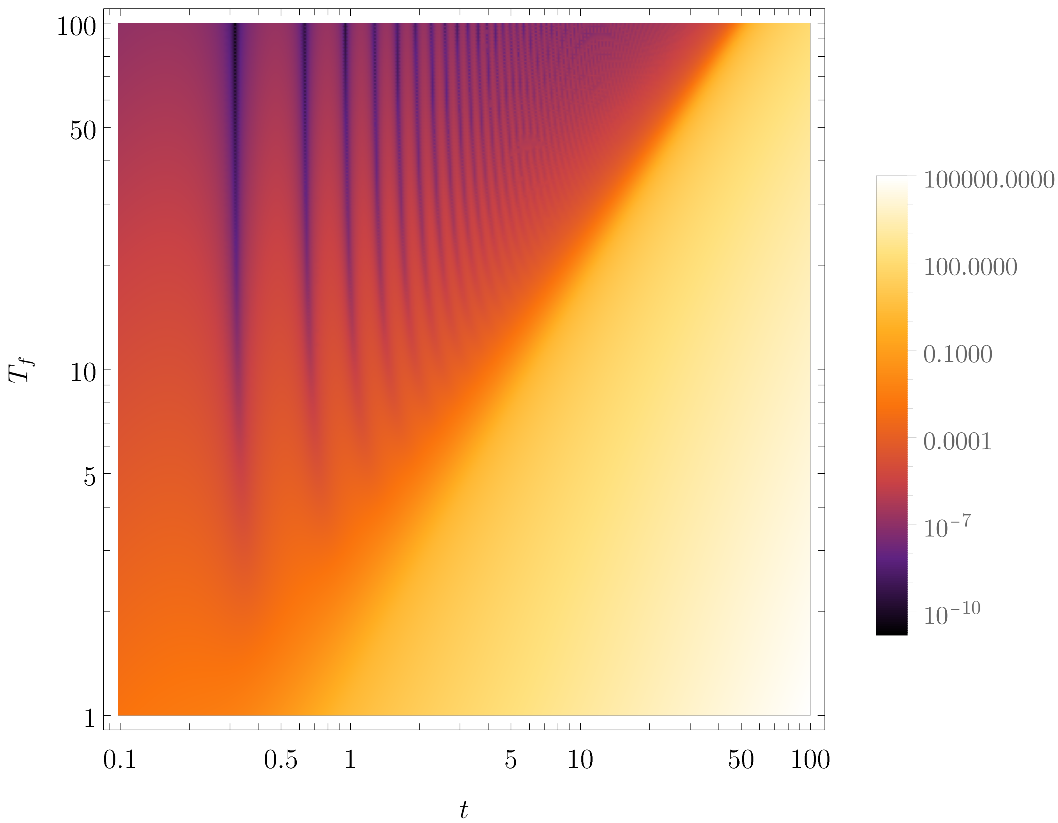

Figure 1.10: Energy variance for $\Omega = 0.2$ for linear driving.

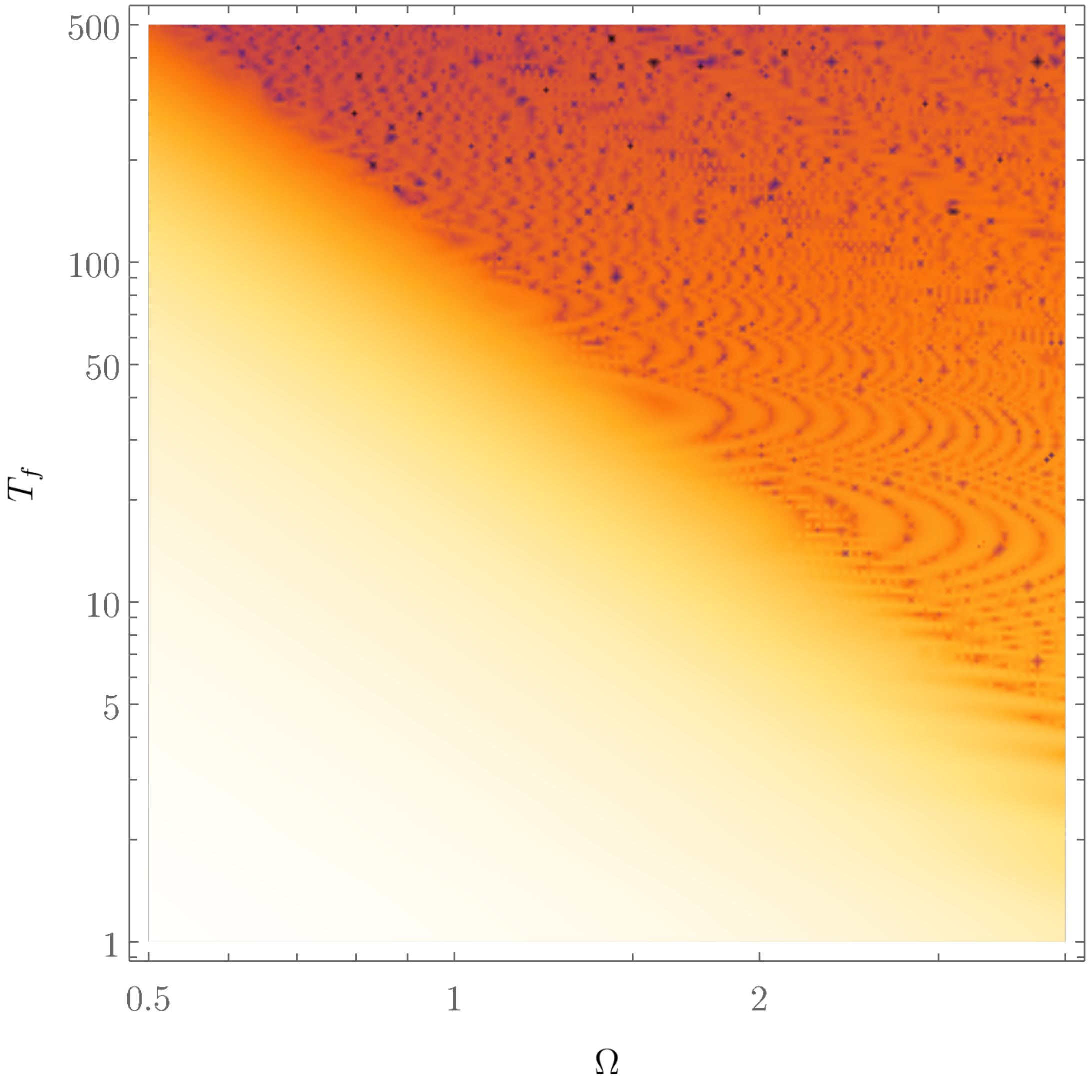

Figure 1.11: $-\log F$ has two regimes. Smooth and chaotic.

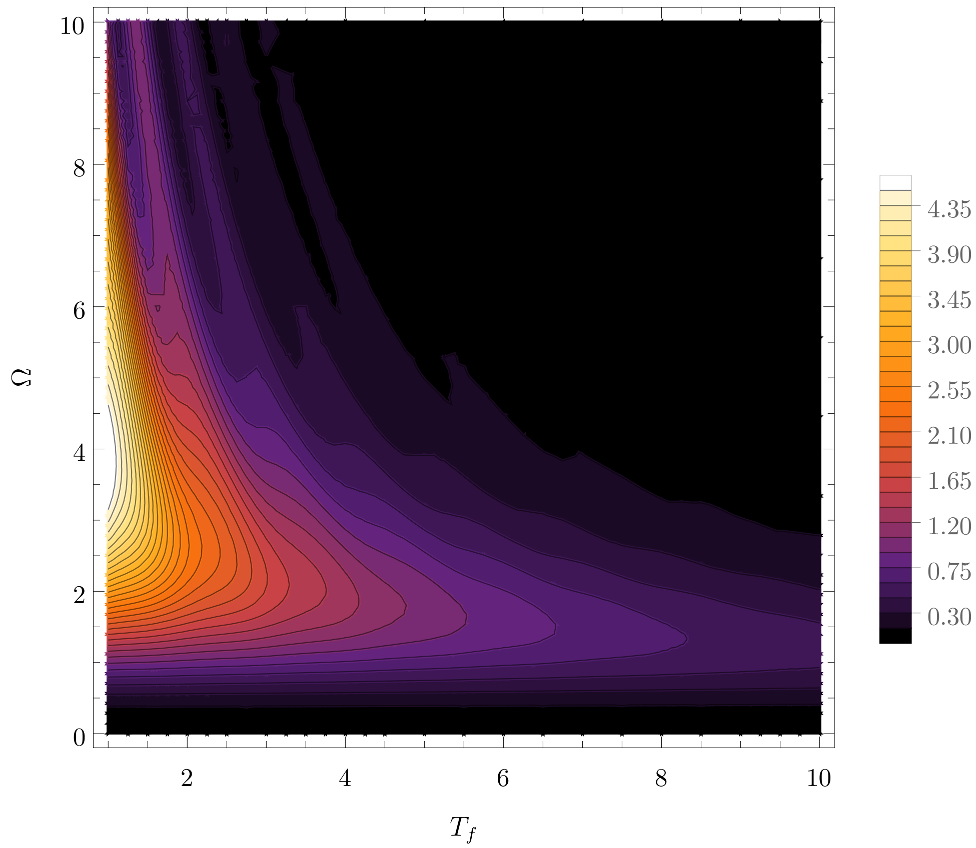

Figure 1.12: Energy variance for $t = T_f/2$ for linear driving.