

1. From the projective Hilbert space to state manifolds

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Jan Střeleček

1.1 Projective Hilbert space

Consider the Hilbert space \mathcal{H} to be a space of *bare states* and \mathcal{S} to be the space of *normalized bare states*. Physical observables are related to the *space of rays*, defined as $\mathcal{PH} := \mathcal{H}/U(1)$, for the factorization by elements of $U(1)$. This group consists of unitary transformations $e^{i\phi}$ for $\phi \in \mathbb{R}$, defining gauge symmetry between quantum states. \mathcal{PH} is then considered to be the *space of pure states*. For the sake of generality, let's not normalize our vectors yet, which would lead to the *space of unnormalized pure physical states*.

It can be shown, that \mathcal{PH} is of a Kähler structure, meaning it has two non-degenerate sesquilinear¹ 2-forms embedded along with operator complex unit

$$(J, G, \Omega),$$

such that

$$J^2 = \mathbb{I} \quad (1.1)$$

and any bracket of $|\psi_1\rangle, |\psi_2\rangle \in \mathcal{PH}$ can be decomposed into real and imaginary part[Ashtekar and Schilling]

$$\langle \psi_1 | \psi_2 \rangle = \frac{1}{2}G(\psi_1, \psi_2) - \frac{i}{2}\Omega(\psi_1, \psi_2). \quad (1.2)$$

From bracket sesquilinearity goes that G is symmetric and Ω antisymmetric form, thus they can be uniquely written into one 2-form called *Fubini-Study metric* with property

$$G = \text{Re}Q; \quad \Omega = \text{Im}Q. \quad (1.3)$$

Because $\langle \psi_1 | \psi_2 \rangle \in [0, 1]$ we say, that the metric is measuring the geodesic distance on the Bloch sphere. Here if we define

$$| \langle \psi_1 | \psi_2 \rangle | = \cos^2 \frac{\theta}{2}, \quad (1.4)$$

we get $d\theta = 2ds = 2\sqrt{|g_{\mu\nu}d\Lambda^\mu d\Lambda^\nu|}$, see Cheng.

To write the metric in a standard form, we need to realize how our space looks like. For finite $n+1$ -dimensional Hilbert space, one dimension is lost in the gauge transformation, leaving us with n -dimensional \mathcal{PH} . Another dimension is lost due to normalization, which is usually done by mapping to an n -dimensional complex sphere

$$CP^n = \left\{ \mathbb{Z} = (Z_0, Z_1, \dots, Z_n) \in \mathbb{C}^{n+1}/\{0\} \right\} / \{ \mathbb{Z} \sim c\mathbb{Z} \text{ for } c \in \mathbb{C} \}.$$

¹We are in physics, so complex conjugated is the first input of the 2-form.

Natural property of such complex spaces is splitting of its tangent space to holonomic and anholonomic part²

$$T^{1,0}\mathcal{M} = \text{Span} \left\{ \frac{\partial}{\partial Z_i} \right\}; \quad T^{0,1}\mathcal{M} = \text{Span} \left\{ \frac{\partial}{\partial \bar{Z}_i} \right\}.$$

Distance on \mathbb{C}^{n+1} is standardly defined using Hermitean metric³

$$ds^2 = d\bar{Z} \otimes dZ. \quad (1.5)$$

For *normalized states* in quantum mechanics is $dZ = ??????$, which plugged into Eq. 1.5 yields

$$ds^2 = 1 - |\langle \psi + \delta\psi | \psi \rangle|^2. \quad (1.6)$$

1.2 Restriction to eigenstate manifolds

In quantum mechanics, one can examine a Hamiltonian $\hat{H}(\Lambda)$, for some parameter $\Lambda \in \mathbb{R}^n$. At every point Λ we get projective Hilbert space $\mathcal{PH}(\Lambda)$. This creates a fibre structure space, in which there are some sections with interesting physical applications. Some of those sections are *eigenstate manifolds*, defined by setting only one non-zero coefficient Z_k in eigenbasis $|\psi\rangle = \sum_{k=0}^n Z_k |k\rangle$. From normalization goes automatically $Z_k = 1$. The distance is then

$$\begin{aligned} ds^2 &= 1 - \langle k + \delta k | k \rangle \langle k | k + \delta k \rangle = 1 - \langle k + \delta k | \left(1 - \sum_{j \neq k} |j\rangle \langle j| \right) |k + \delta k \rangle \\ &= \sum_{j \neq k} \langle k + \delta k | j \rangle \langle j | k + \delta k \rangle. \end{aligned} \quad (1.7)$$

Using the Schrödinger equation $\hat{H}|k\rangle = \mathcal{E}_k |k\rangle$, distributivity of derivative and projection to some state $\langle j|$, we get

$$\begin{aligned} \hat{H}|k\rangle &= \mathcal{E}_k |k\rangle \\ (\delta\hat{H})|k\rangle + \hat{H}|k + \delta k\rangle &= (\delta\mathcal{E}_k)|k\rangle + \mathcal{E}_k |k + \delta k\rangle \\ \langle j|(\delta\hat{H} - \delta\mathcal{E}_k)|k\rangle &= \langle j|(\mathcal{E}_k - \hat{H})|k + \delta k\rangle = \langle j|(\mathcal{E}_k - \mathcal{E}_j)|k + \delta k\rangle. \end{aligned} \quad (1.8)$$

We can set⁴ $\delta\mathcal{E}_k = 0$, leading for $j \neq k$ to

$$\frac{\langle j | \delta\hat{H} | k \rangle}{(\mathcal{E}_k - \mathcal{E}_j)^2} = \langle j | k + \delta k \rangle. \quad (1.9)$$

Plugging to Equation 1.7 and considering $\hat{H} = \hat{H}(\Lambda)$, we get metric on a ground state manifold

$$ds^2 = \text{Re} \sum_{j \neq k} \frac{\langle 0 | \partial_\mu \hat{H} | j \rangle \langle j | \partial_\nu \hat{H} | 0 \rangle}{(\mathcal{E}_k - \mathcal{E}_j)^2} d\Lambda^\mu d\Lambda^\nu \quad (1.10)$$

² $T^{p,q}\mathcal{M}$ means $p+q$ -cotravariant space (the space of vectors) on \mathcal{M} . The line over letter means complex conjugation.

³which is by definition sesquilinear, as one would expect in quantum mechanics later on

⁴Can it be done only for E_0 ? It does not make sense generally, because $E = E(\Lambda)$, even $E_0 = E_0(\Lambda)$

Definition of the k-state manifold is then

$$g_{\mu\nu}^{(k)} = \text{Re} \sum_{j \neq k} \frac{\langle k | \frac{\partial \hat{H}(\Lambda)}{\partial \lambda^\mu} | j \rangle \langle j | \frac{\partial \hat{H}(\Lambda)}{\partial \lambda^\nu} | k \rangle}{(E_k - E_j)^2}. \quad (1.11)$$

The Fubini-Study metric on the eigenstate manifold is sometimes called *Geometric tensor*. For the Lipkin-Meshkov Glick model, those can be seen on Fig. 1.2.

If we compare first eigenstate manifold \mathcal{M}_1 with difference in infidelity transport done along ground state geodesic and along straight line, we can see they have one similarity. The higher curvature of \mathcal{M}_1 degrades geodesic even faster, making the straight line more advantageous. This is caused by the wavefunction running out from \mathcal{M}_0 to \mathcal{M}_1

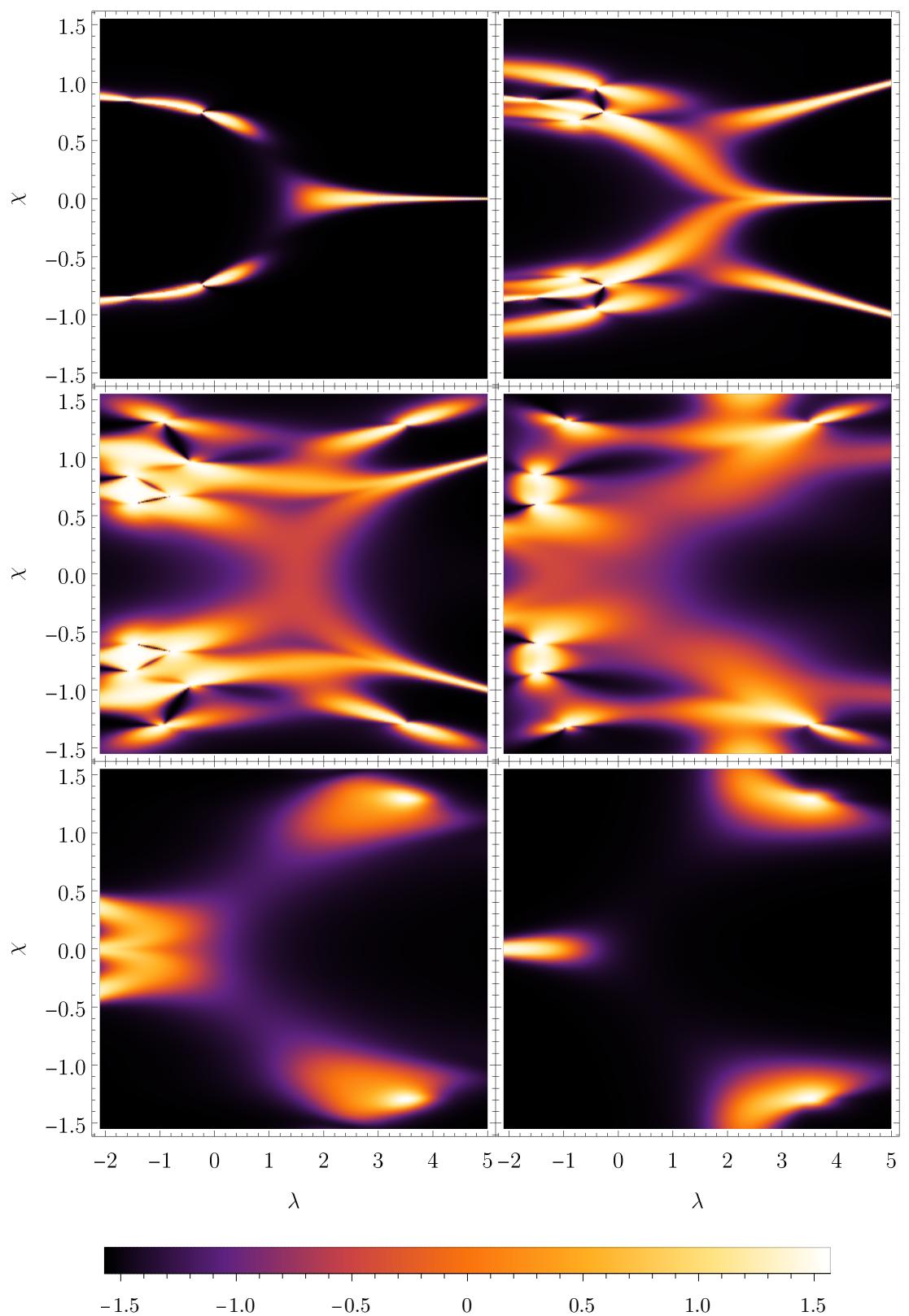


Figure 1.1: Arctangens of the metric tensor for higher state manifolds. By rows: $M_0, M_1; M_2, M_3; M_4, M_5$.

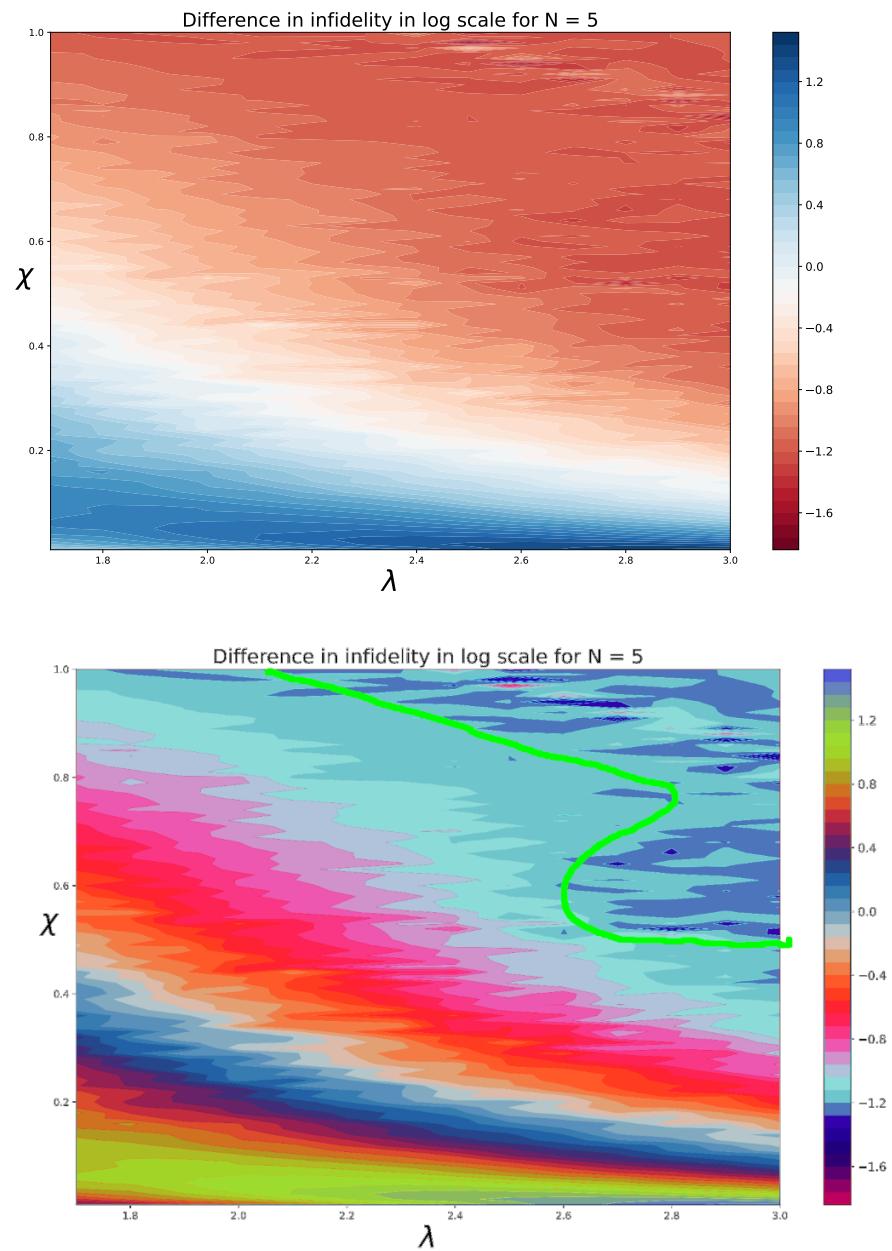


Figure 1.2: Infidelity difference between line and geodesic, above picture is original from Felipe, bottom one is edited in GIMP to see the difference more clearly.

2. The meaning of geodesics

2.1 Transport using quenches

Unifying the ground states $|o(\Lambda)\rangle$ over all points $\Lambda \in \mathbb{R}^n$ in the parameter space, we get the ground state manifold. Here the fidelity f and distance s are defined

$$ds^2 \equiv 1 - f^2 \equiv 1 - |\langle o(\Lambda + \delta\Lambda) | o(\Lambda) \rangle|^2. \quad (2.1)$$

The final fidelity of transport on \mathcal{M} is then

$$F = \iint g_{\mu\nu} d\lambda^\mu d\lambda^\nu = \int_{t_i}^{t_f} \underbrace{\int_{t_i}^\tau g_{\mu\nu} \frac{d\lambda^\mu}{dt} \frac{d\lambda^\nu}{dt} dt d\tau}_{\mathcal{L}(\lambda^\mu, \dot{\lambda}^\mu, \tau)}. \quad (2.2)$$

Using Euler-Lagrange equations for time-independent $g_{\mu\nu} = g_{\mu\nu}(\lambda^\mu)$, leads to

$$\int_{t_i}^\tau \left[g_{\mu\nu,\kappa} \dot{\lambda}^\mu \dot{\lambda}^\nu - \frac{d}{dt} \left[g_{\mu\nu} (\delta_\kappa^\mu \dot{\lambda}^\nu + \dot{\lambda}^\mu \delta_\kappa^\nu) \right] \right] dt = 0, \quad (2.3)$$

which needs to be zero for integration over any subset (t_i, τ) . This can be achieved for any path only if the integrand itself is zero, which happens if the geodesic equation holds.

The fidelity f measures transition probability between two neighboring eigenstates of two different Hamiltonians. Those two states belong to the same Fibre space $\mathcal{PH}(\Lambda) \times \mathbb{R}^n$ from which the coefficients (\mathbb{Z}, Λ) are taken. Because all $\mathcal{PH}(\Lambda)$ are canonically isomorphic, there is no problem in parallel transport from one space to another, which is needed for bracket¹.

The distance minimization runs into some interpretation problems. On one hand, minimalization of the distance is equivalent to maximalization of the sum of infinitesimal fidelities along the path (we say we *maximize the fidelity along the path*). On the other hand we are using only ground states in every step of the transport, therefore defining the fidelity to be one. There are actually two ways out of this confusion. *Perturbed adiabatic driving* and *Transport using quenches*.

Perturbed adiabatic driving

In the first case, we imagine at every point of transport, that the fidelity is small enough, such that in eigenbasis and some small parameters $\delta_i \in \mathbb{C}$ we get

$$|o(\Lambda_i)\rangle \equiv \begin{pmatrix} Z_0(\Lambda_i) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \xrightarrow{\text{transport } ds} |o(\Lambda_i + \delta\Lambda)\rangle \equiv \begin{pmatrix} Z_0(\Lambda_i + \delta\Lambda) \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ \delta_1(\Lambda_i + \delta\Lambda) \\ \vdots \\ \delta_n(\Lambda_i + \delta\Lambda) \end{pmatrix}}_{\Delta(\Lambda_i + \delta\Lambda)},$$

¹This procedure is done without thinking in the back part of our brains such, that we don't even think about it. That's how trivial it is.

where the last term is neglected using

$$\langle \Delta(\Lambda) | o(\Lambda + \delta\Lambda) \rangle \approx 0.$$

This might have interesting implication for slow transports, or small distance transports. For example when some slow thermalization is considered during a transport.

Transport using quenches

If we imagine $\delta\Lambda$ to be finite (not infinitely small, as the notation suggests), the **transport** means **doing a sequence of quenches and measuring the system after every quench**.

Some notion of the space of our Hamiltonian can be seen by quenching from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$, as can be seen in Figure 2.1.

In Figure 2.2 are marked equidistant points, meaning $\int_a^b ds = \text{const.}$ between every two neighboring points on curve. This means that if the system is measured periodically, the quenches jump smaller distances when closer to a singularity.

Decreasing time step Δt has no effect on the relative fidelity of quenches during the evolution but has an effect on their magnitude. As one would expect from *quantum Zeno effect*, when $\Delta t \rightarrow 0$, the transport becomes adiabatic, and the fidelity at any time will become 1. This can be observed in Figure 2.3, such that the shape of the point-like paths looks similar in the columns, and their magnitude decreases.

The quantum Zeno effect for this case can be shown directly by splitting the distance s to N equal pieces. The fidelity for N splits will then be

$$f(N) = (1 - \Delta s)^N = \left(1 - \left(\frac{s}{N}\right)^2\right)^N \xrightarrow{N \rightarrow \infty} 1, \quad (2.4)$$

meaning the consequent measurements will collapse the system to its instantenous eigenstate and the adiabatic condition for transport holds.

Such measurements can be achieved by fast thermalization of the system. If the finite speed thermalization with $N = T/\tau$ for the mean time between two measurements τ , we get

$$\begin{aligned} \log f(N) &= N \log \left(1 - \left(\frac{s}{N}\right)^2\right) = -\frac{s^2}{N} + o\left(\frac{s^4}{N^3}\right) \\ f(N) &= \exp\left(-s^2 \frac{\tau}{T} - \frac{s^4}{2N^3} \dots\right) = \exp\left(-s^2 \frac{\tau}{T}\right) \left(1 + o\left(\frac{s^4}{N^3}\right)\right) \end{aligned} \quad (2.5)$$

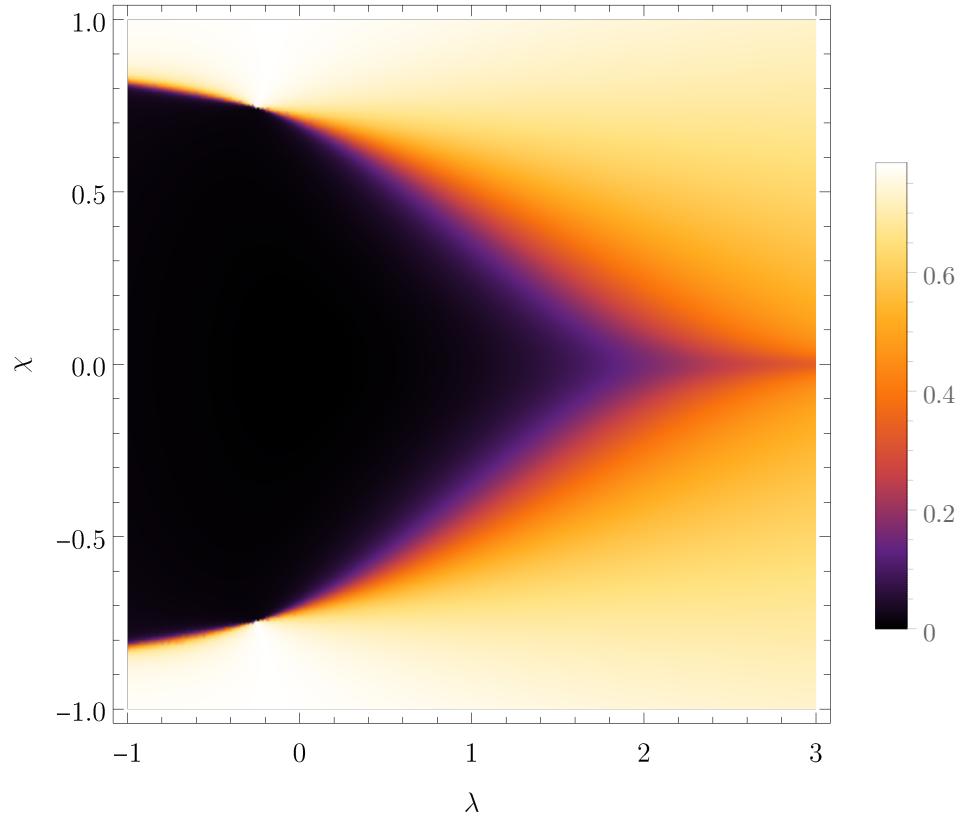


Figure 2.1: Arctangens of the fidelity of quenches from $(\lambda_i; \chi_i) = (0; 0)$ to $(\lambda; \chi)$.

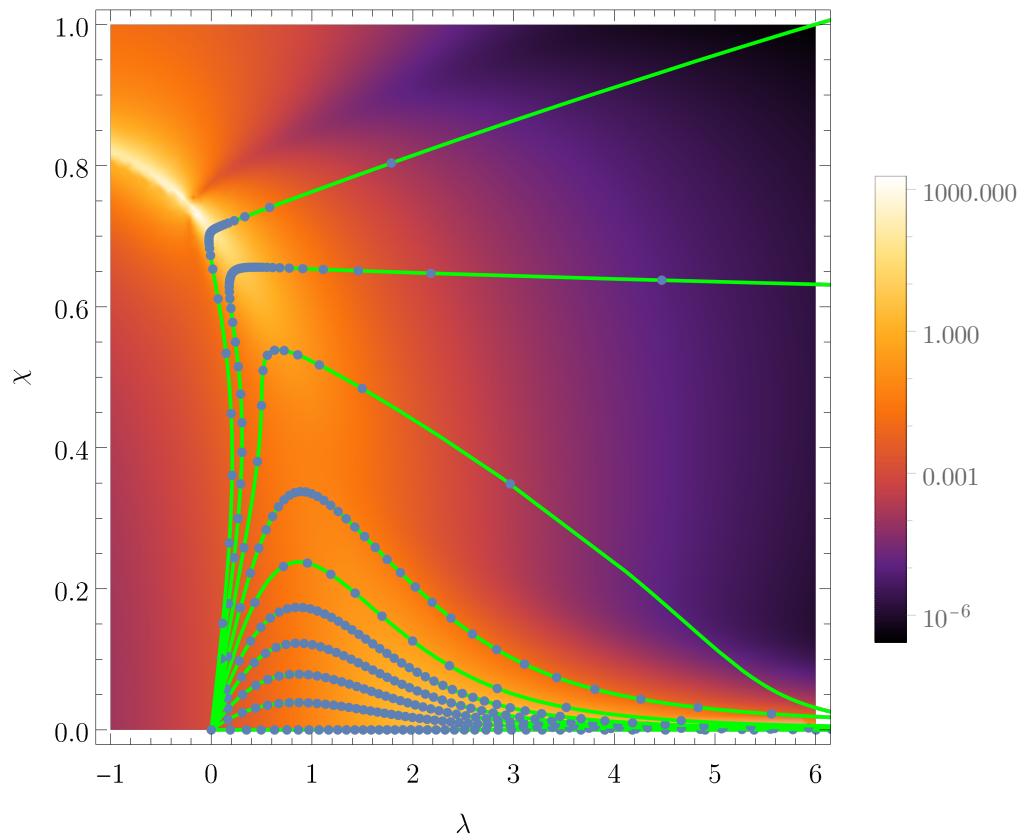


Figure 2.2: Equidistant points on geodesics of the ground state manifold.

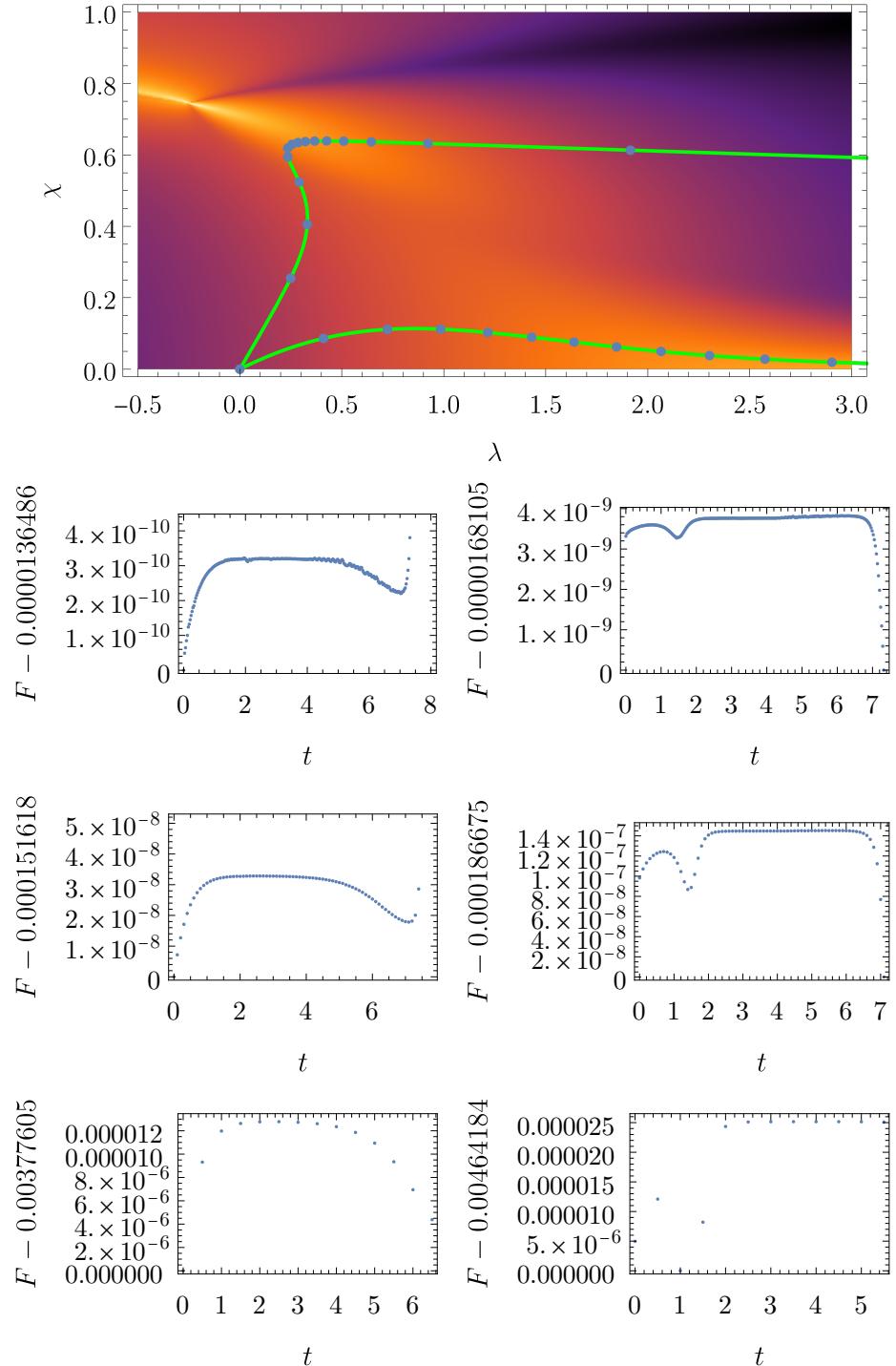


Figure 2.3: Fidelity for sequential quenches along geodesics (see green lines on top). Left (right) column corresponds to lower (upper) geodesic. Time steps from top are $\Delta t \in \{0.03, 0.1, 0.5\}$. Time difference between points in the plot on top is $\Delta t = 0.5$.

Bibliography

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