Exercise solutions should be typed in LaTeX(or neatly handwritten) and included with your notes document.

1 Vector Spaces

1. Consider $S = \{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$, where
$$|v_1\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad |v_3\rangle = \begin{bmatrix} 0 \\ i \end{bmatrix}$$
   (a) Give a linear combination of the vectors in S.
   (b) Determine if $\begin{bmatrix} 1+i \\ 200-i \end{bmatrix}$ is in $\mathrm{Span}(S)$.
   (c) Describe $\mathrm{Span}(S)$ "geometrically."

2. Find the condition under which the following two vectors are linearly independent
$$|v_1\rangle = \begin{bmatrix} x \\ y \\ 3 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 2 \\ x-y \\ 1 \end{bmatrix} \in \mathbb{R}^3.$$

3. Show that the set formed by the following vectors is a basis for $\mathbb{C}^3$.
$$|v_1\rangle = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad |v_3\rangle = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

4. Let
$$|x\rangle = \begin{bmatrix} 1 \\ i \\ 2+i \end{bmatrix}, \quad |y\rangle = \begin{bmatrix} 2-i \\ 1 \\ 2+i \end{bmatrix}, \quad |z\rangle = \frac{\sqrt{2+\sqrt{5}}}{2+\sqrt{5}} |x\rangle.$$
   Find $\||x\rangle\|$, $\langle x|y\rangle$, $\langle y|x\rangle$, and $\||z\rangle\|$.

5. Prove that for all $|u\rangle, |v\rangle \in \mathbb{C}^n$
$$\langle u|v\rangle = \langle v|u\rangle^*.$$

6. Prove that the inner product on $\mathbb{C}^n$ is a **sesquilinear form**. That is, show that the inner product is linear in the "ket component" and conjugate linear in the "bra component."

7. Let $\mathcal{B} = \{|b_1\rangle, |b_2\rangle\}$, where
$$|b_1\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, |b_2\rangle = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$
   (a) Show that $\mathcal{B}$ is an orthonormal basis for $\mathbb{C}^2$. This basis is the basis obtained by putting the screen at a 45° angle for the incoming photon in the polarization experiment.
   (b) Find the **coordinates** (or components) of $|x\rangle = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ relative to basis $\mathcal{B}$, i.e., find the scalars $c_1, c_2 \in \mathbb{C}$ such that $c_1 |b_1\rangle + c_2 |b_2\rangle = |x\rangle$.
   (c) Find the coordinates of $|v\rangle = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ relative to $\mathcal{B}$ using the appropriate projection operators.

8. Use the Gram-Schmidt Process on $\{|v_1\rangle, |v_2\rangle, |v_3\rangle\}$ to produce an orthonormal basis $\mathcal{B}$ for $\mathbb{C}^3$, where
$$|v_1\rangle = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad |v_3\rangle = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}.$$
   Find the coordinates of $|u\rangle = \begin{bmatrix} 1 \\ -2 \\ 7 \end{bmatrix}$ relative to $\mathcal{B}$.

9. Use the Gram-Schmidt Process to produce an orthonormal basis $\mathcal{B}$ for $M = \text{Span}(\{|v_1\rangle, |v_2\rangle\})$, where

$$|v_1\rangle = \begin{bmatrix} 1 \\ i \\ 1 \end{bmatrix}, \quad |v_2\rangle = \begin{bmatrix} 3 \\ 1 \\ i \end{bmatrix}$$

2 Matrices

Exercise 2.1 Let $A, B \in M_n$ and $c \in \mathbb{C}$ be given. Show that

(a) $(cA)^\dagger = c^* A^\dagger$

(b) $(A + B)^\dagger = A^\dagger + B^\dagger$

(c) $(AB)^\dagger = B^\dagger A^\dagger$

(d) If A is invertible, then $(A^{-1})^\dagger = (A^\dagger)^{-1}$. ♦

HINT FOR 2.1(A),(B),(C): You could prove this writing the matrices *entry-wise*. Alternatively, you could use the fact that, given $A \in M_n$, the matrix $A^\dagger \in M_n$ is the unique matrix such that $\langle Ax|y\rangle = \langle x|A^\dagger y\rangle$ for all $|x\rangle, |y\rangle \in \mathbb{C}^n$.

Exercise 2.2 Let

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1+i \\ 1-i & 0 \end{bmatrix}$$

(a) Find the eigenvalues and corresponding normalized eigenvectors for A.

(b) Show that the eigenvectors are mutually orthogonal.

(c) Show the projection operators associated to the mutually orthogonal eigenvectors satisfy the *completeness relation*, i.e., if $|u_1\rangle, |u_2\rangle$ are mutually orthogonal unit eigenvectors for A, then $I = \sum_{i=1}^{2} |u_i\rangle \langle u_i|$.

(d) Find a unitary matrix which diagonalizes A, i.e., find a unitary matrix U and a diagonal matrix D such that $A = UDU^\dagger$. *Such a decomposition is called a unitary diagonalization for A.* ♦

Exercise 2.3 Prove the following:

(a) Suppose A is *skew-Hermitian*, i.e., $A^\dagger = -A$. Show that the eigenvalues of A are pure imaginary, i.e., if λ is an eigenvalue for A, then $\lambda = bi$ for some $b \in \mathbb{R}$.

(b) Let U be a unitary matrix. Show that the eigenvalues of U are *unimodular*, i.e., if λ is an eigenvalue of U, then $|\lambda| = 1$.

(c) Let A be a normal matrix. Show that A is Hermitian if and only if all the eigenvalues of A are real. ♦

HINT FOR 2.3: Given $A \in M_n$, the matrix $A^\dagger \in M_n$ is the unique matrix such that $\langle Ax|y\rangle = \langle x|A^\dagger y\rangle$ for all $|x\rangle, |y\rangle \in \mathbb{C}^n$.

Exercise 2.4 Let

$$U = \begin{bmatrix} 0 & 0 & i \\ 0 & i & 0 \\ i & 0 & 0 \end{bmatrix}$$

Find the eigenvalues for U (without calculation if possible) and its corresponding eigenvectors. ♦

Exercise 2.5 Let H be a Hermitian matrix.

(a) Show that $(I - iH)$ is invertible.

(b) Show that $U = (I + iH)(I - iH)^{-1}$ is unitary. This transformation is called the *Cayley transformation.* ⧫

Exercise 2.6 Suppose a 2×2 matrix A has eigenvalues $-1, 3$ and corresponding eigenvectors

$$|b_1\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ i \end{bmatrix}, \quad |b_2\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix},$$

respectively. Find A. ⧫

3 Framework of Quantum Mechanics

Exercise 3.1 Let

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(a) Explain why $\mathbb{C}^2$ must have an orthonormal basis of eigenvectors for A.

(b) Find the eigenvalues for A and the corresponding normalized eigenvectors.

(c) Find the spectral decomposition of A.

(d) Compute $\exp(i\alpha A)$. ⧫

Exercise 3.2 Suppose that A is a Hermitian matrix with spectral decomposition $A = \sum_{j=1}^{n} \lambda_j |u_j\rangle\langle u_j|$ and let $|\psi\rangle = \sum_{j=1}^{n} c_j |u_j\rangle$ be a state. Prove that $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle$, i.e., prove that $\langle\psi|A|\psi\rangle = \sum_{j=1}^{n} \lambda_j |c_j|^2$.

Hint/remark: Recall that for any complex number z, we have $z^*z = zz^* = |z|^2$.

Exercise 3.3 Consider a physical system with Hamiltonian

$$H = -\frac{\hbar}{2}\omega\sigma_y$$

and suppose the initial state of the system is

$$|\psi(0)\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(a) Find the wave function $|\psi(t)\rangle$ for $t > 0$.

(b) Find the probability for the system to have the outcome $+1$ upon measurement of σ_z at $t > 0$.

(c) Find the expectation value after many measurements of σ_z.

(d) Find the probability for the system to have the outcome $+1$ upon measurement of σ_x at $t > 0$.

(e) Find the expectation value after many measurements of σ_x. ⧫

Exercise 3.4 Suppose H is the Hamiltonian for a physical system with initial state $|\psi(0)\rangle$.

(a) Prove that $U(t) = \exp(-itH/\hbar)$ is a unitary matrix.

> HINT: Suppose H has the spectral decomposition $H = \sum_{i=1}^{n} \lambda_i |u_i\rangle \langle u_i|$. Express $U(t) = \exp(-itH/\hbar)$ using the functional calculus. Compute $U(t)^\dagger$ using your functional calculus expression for $U(t)$ and the properties of the conjugate transpose you proved in Exercise 2.1. Then do the thing to show $U(t)$ is unitary.
>
> HINT 2.0: When you compute $U(t)^\dagger$, you will need to use $(e^{i\alpha})^* = e^{-i\alpha}$ for any $\alpha \in \mathbb{R}$. You should verify this for yourself.

(b) Suppose that A is a Hermitian matrix that corresponds to an observable of the system. Define $\alpha_t(A) = U(t)^\dagger A U(t) \in M_n$ for $t > 0$. Prove that

$$\langle A \rangle_\psi = \langle \psi(0) | \alpha_t(A) | \psi(0) \rangle ,$$

i.e., prove that $\langle \psi(t)|A|\psi(t)\rangle = \langle \psi(0)|\alpha_t(A)|\psi(0)\rangle$. This is called the *Heisenberg picture*. ♦

> HINT: AGAIN, you might use the fact that, given $B \in M_n$, the matrix $B^\dagger \in M_n$ is the unique matrix such that $\langle Bx|y\rangle = \langle x|B^\dagger y\rangle$ for all $|x\rangle, |y\rangle \in \mathbb{C}^n$.

Exercise 3.5 Let $\lambda \in \mathbb{C}$, $A \in M_{mn}$, $B \in M_{pq}$, $C \in M_{nr}$, and $D \in M_{qs}$. Show that

1. $\lambda(A \otimes B) = \lambda A \otimes B = A \otimes \lambda B$
2. $\lambda(A \otimes B) \neq \lambda A \otimes \lambda B$
3. $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.
4. $A \otimes (B + C) = A \otimes B + A \otimes C$.
5. $(A \otimes B) + (C \otimes D) \neq (A + C) \otimes (B + D)$
6. $A \otimes B \neq B \otimes A$
7. $(A \otimes B)^\dagger = A^\dagger \otimes B^\dagger$
8. $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$

whenever the matrix operations are well-defined. ♦

4 Separable States and the SVD

Exercise 4.1 Show that if $|\psi_1\rangle \in \mathbb{C}^n$ and $|\psi_2\rangle \in \mathbb{C}^m$ are states (unit vectors), then $|\psi_1\rangle \otimes |\psi_2\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ is a state (unit vector). ♦

Exercise 4.2 Let $A \in M_{mn}$ be given. Show that $A^\dagger A$ is positive semidefinite. ♦

Exercise 4.3 Find the SVD of $A = \begin{bmatrix} 1 & 0 & i \\ i & 0 & 1 \end{bmatrix}$. ♦

Exercise 4.4 Consider the state $|\psi\rangle = \frac{1}{2}(1, 0, i, i, 0, 1)^t \in \mathbb{C}^6$.

(a) Find a Schmidt decomposition for $|\psi\rangle$ in $\mathbb{C}^2 \otimes \mathbb{C}^3$.

HINT: Note that $|\psi\rangle = \frac{1}{2}\left(|e_{1,1}\rangle \otimes |e_{2,1}\rangle + i\,|e_{1,1}\rangle \otimes |e_{2,3}\rangle + i\,|e_{1,2}\rangle \otimes |e_{2,1}\rangle + |e_{1,2}\rangle \otimes |e_{2,3}\rangle\right) \in \mathbb{C}^2 \otimes \mathbb{C}^3$. You might be able to use work from a previous exercise...

(b) Find a Schmidt decomposition for $|\psi\rangle$ in $\mathbb{C}^3 \otimes \mathbb{C}^2$.

Remark: This problem illustrates that the components of the bipartite system must be specified in order to write out a Schmidt decomposition. ♦

This concludes the homework to complete for the first LaTeX notes check on March 4, 2022.

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5 Density Matrices

Exercise 5.1 Suppose the state of a system is in a pure state of $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$. Find the density matrix associated to $|\psi\rangle$.

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{2}(|0\rangle - |1\rangle)(\langle 0| - \langle 1|) = \frac{1}{2}(|0\rangle\langle 0| - |0\rangle\langle 1| - |1\rangle\langle 0| + |1\rangle\langle 1| = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

♦

Exercise 5.2 Show that density matrices are positive semi-definite, i.e., show that a matrix written as $\rho = \sum_{i=1}^{N} p_i^2 |\psi_i\rangle\langle\psi_i|$ such that each $p_i \in \mathbb{R}$ and $\||\psi_i\rangle\| = 1$ is positive semi-definite. ♦

Exercise 5.3 Show that the following are equivalent:

(a) A state is pure, i.e., the density matrix associated to the state of the system is a rank-one projection $\rho = |\psi\rangle\langle\psi|$, where $\||\psi\rangle\| = 1$.

(b) $\rho^2 = \rho$.

(c) $\mathrm{tr}(\rho^2) = 1$. ♦

Exercise 5.4 Suppose an observable A of a quantum system represented by $\mathbb{C}^n$ has spectral decomposition $A = \sum_j \lambda_j |\lambda_j\rangle\langle\lambda_j| =: \sum_j \lambda_j P_j$, where $\lambda_j \in \mathbb{R}$ for all j. If the system is in state $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, prove that the probability of obtaining outcome λ_m for A is

$$\langle\lambda_m|\rho|\lambda_m\rangle = \sum_i p_i |\langle\lambda_m|\psi_i\rangle|^2 = \mathrm{tr}(P_m\rho).$$

Exercise 5.5 Prove or disprove: If $|\psi\rangle \in \mathbb{C}^n \otimes \mathbb{C}^m$ is a separable state, then $\rho := |\psi\rangle\langle\psi| \in M_n \otimes M_m$ is uncorrelated. ♦

Exercise 5.6 Let $|\psi\rangle := \frac{1}{\sqrt{2}} = (|00\rangle + |11\rangle) \in \mathbb{C}^2 \otimes \mathbb{C}^2$ (so $H_A = \mathbb{C}^2$ and $H_B = \mathbb{C}^2$). Compute the corresponding (pure state) density matrix, and determine if it is uncorrelated, separable, or inseparable. ♦

6 Qubits

Exercise 6.1 Find the expectation value of $\sigma_x \otimes \sigma_z$ measured in each of the *Bell states*: $|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, $|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$, $|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, $|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. ♦

Exercise 6.2 Show that $|0\rangle$ is represented on the Bloch sphere by the spherical coordinate $(1, 0, 0)$ and $|1\rangle$ is represented on the Bloch sphere by the spherical coordinate $(1, 0, \pi)$. ♦

Exercise 6.3 Explain why $\beta|1\rangle$ is represented by the *same* spherical coordinate on the Bloch sphere as $|1\rangle$ for all complex numbers β such that $|\beta|^2 = 1$. THEN, explain (mathematically) how/why a state $|\psi\rangle \in \mathbb{C}^2$ is represented by the same point on the Bloch sphere as $c|\psi\rangle$ for any $c \in \mathbb{C}$. ♦

7 Quantum Gates

Exercise 7.1 For each of the following quantum gates, graph the images $|\psi_1\rangle$ and $|\psi_2\rangle$ of $|0\rangle$ and $|1\rangle$, respectively, by finding θ_j and ϕ_j such that

$$|\psi_j\rangle = \cos\left(\frac{\phi_j}{2}\right)|0\rangle + e^{i\theta_j}\sin\left(\frac{\phi_j}{2}\right)|1\rangle$$

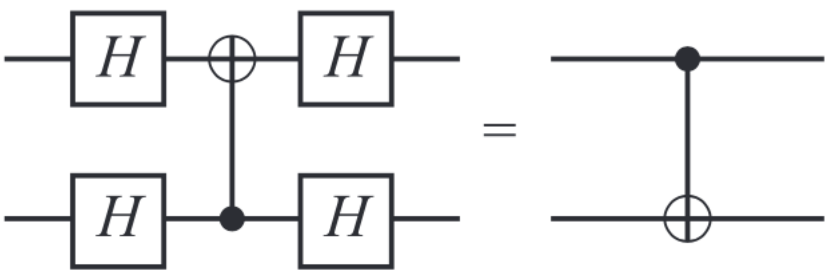
for each $j = 1, 2$.

$$\begin{aligned} I &: |0\rangle \to |0\rangle, |1\rangle \to |1\rangle \Rightarrow I = I_2 \\ X &: |0\rangle \to |1\rangle, |1\rangle \to |0\rangle \Rightarrow X = \sigma_x \\ Y &: |0\rangle \to -|1\rangle, |1\rangle \to |0\rangle \Rightarrow Y = -i\sigma_y = XZ \\ Z &: |0\rangle \to |0\rangle, |1\rangle \to -|1\rangle \Rightarrow Z = \sigma_z \end{aligned}$$

♦

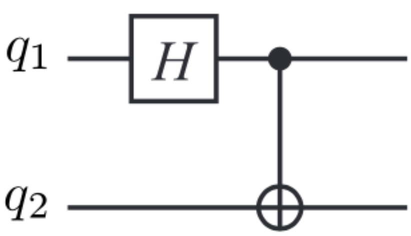
Exercise 7.2 Find the quantum gate U which carries the standard orthonormal basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ to the Bell basis for $\mathbb{C}^2 \otimes \mathbb{C}^2$. ♦

Exercise 7.3 Show that the two circuits below are equivalent:



Explain, using tensor products, what this says about the relationship between the `CNOT` gate with the Hadamard gate H. ♦

Exercise 7.4 Find the outputs of the qubits $|00\rangle$, $|01\rangle$, $|10\rangle$,$|11\rangle$ when passed through the circuit



Exercise 7.5 Prove that, given any density matrix ρ (possibly a mixed state), the evolution of ρ through a unitary gate U is still a density matrix. ♦

This concludes the homework to complete for the last LaTeX notes check on April 28, 2022.
