$$x_* = \arg\min_{x \in \mathbb{R}^n} f(x)$$

Optimization Problems...

Exercise #1

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Distribution
Optimization Problems And Their Algorithms And Mathematics Students

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Contents

Nil fieri ex nihilo, in nihilum nil posse reverti.

- Lucretius
$99 - 55$ BC

Nothing can be produced from nothing,
nothing can be reduced to nothing.

Isolated systems conserve mass over time.

- Einstein
$1879 - 1955$

For, lo, each thing is quicker marred than made.

- Lucretius
$99 - 55$ BC

The entropy of an isolated system never decreases.

- Carnot
$1796 - 1832$

Optimization Problems, Algorithms, and Mathematics

NOTE: Exercises marked with a † are intended to be a bit more challenging than the others.

1.1 Metric Topology

EXERCISE 1: For $x, y \in \mathbb{R}$, define

$$
\begin{aligned}
d_1(x,y) &= (x-y)^2, \\
d_2(x,y) &= \sqrt{|x-y|}, \\
d_3(x,y) &= |x^2-y^2|, \\
d_4(x,y) &= |x-2y|, \\
d_5(x,y) &= \frac{|x-y|}{1+|x-y|}.
\end{aligned}
$$

Determine, for each of these, whether it is a metric or not[1].

EXERCISE 2: The statement of Taylor's theorem, FONC, SONC, and SOSC all begin with "Let $E \subset R$ be open ..." or "Let $E \subset R^n$ be open ...". Can you think of a reason for open sets in the hypothesis of these theorems?

1.2 Graphical Minimization

EXERCISE 3: Graphically determine a minimizer of the function

$$f(x) = (x_1-2)^2 + (x_2-1)^2,$$

subject to constraints

$$
\begin{aligned}
c_1 : &\qquad x_1^2 - x_2 \le 0, \\
c_2 : &\qquad x_1 + x_2 \le 2.
\end{aligned}
$$

Use Python, numpy, and matplotlib to draw the function and the constraints, determine a minimizer visually, and justify your answer. Is the minimizer *local* or *global*?

Continued on next page $\Longrightarrow$

[1] Cf. [2] Rudin, Exercise 2.11, p. 44.

1.3 Analytical Minimization

EXERCISE 4: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Answer the following questions relating to the point $t = [1,1]^T$:

1. Compute the gradient $\nabla f(t)$.
2. Compute the Hessian $\nabla^2 f(t)$.
3. Is FONC satisfied at t?
4. Is SONC satisfied at t?
5. Is SOSC satisfied at t?

What can you say about the point t?

1.4 Metric Topology and Analytical Minimization

EXERCISE 5: † Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = \|x\|^2$. Show that the sequence of iterates $\{x_k\}_{k=0}^{\infty}$ defined by

$$x_k = \left(1 + \frac{1}{2^k}\right) \begin{bmatrix} \cos(k) \\ \sin(k) \end{bmatrix}$$

satisfies $f(x_{k+1}) < f(x_k)$ for $k = 0, 1, 2, \ldots$. Show that every point on the unit circle $\{x \mid \|x\|^2 = 1\}$ is a limit point for $\{x_k\}_{k=0}^{\infty}$.
Hint: Every $\theta \in [0, 2\pi]$ is a limit point of the subsequence $\{\xi_k\}_{k=0}^{\infty}$ defined by

$$\xi_k = k \pmod{2\pi} = k - 2\pi \left\lfloor \frac{k}{2\pi} \right\rfloor,$$

where the operator $\lfloor . \rfloor$ denotes rounding down to the next integer[2].

[2] Cf. [1] Nocedal and Wright, Exercise 2.5, p. 28.

Bibliography

[1] Nocedal, J., and Wright, S. J. *Numerical Optimization, Second Edition*. Springer, New York, 2006.

[2] Walter Rudin. *Walter Rudin's Principles of Mathematical Analysis, Third Edition*. McGraw-Hill, Inc., Department of Mathematics, University of Wisconsin, Madison, 1976.