$$x_* = \arg\min_{x \in \mathbb{R}^n} f(x)$$

Optimization Problems…

Exercise #1

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Distribution
Optimization Problems And Their Algorithms And Mathematics Students

This document contains 7 (seven) pages.

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Contents

Nil fieri ex nihilo, in nihilum nil posse reverti.

- Lucretius
$99 - 55$ BC

Nothing can be produced from nothing,
nothing can be reduced to nothing.

Isolated systems conserve mass over time.

- Einstein
$1879 - 1955$

For, lo, each thing is quicker marred than made.

- Lucretius
$99 - 55$ BC

The entropy of an isolated system never decreases.

- Carnot
$1796 - 1832$

Optimization Problems, Algorithms, and Mathematics

NOTE: Exercises marked with a † are intended to be a bit more challenging than the others.

1.1 Metric Topology

EXERCISE 1: For $x, y \in \mathbb{R}$, define

$$\begin{aligned} d_1(x,y) &= (x-y)^2, \\ d_2(x,y) &= \sqrt{|x-y|}, \\ d_3(x,y) &= |x^2-y^2|, \\ d_4(x,y) &= |x-2y|, \\ d_5(x,y) &= \frac{|x-y|}{1+|x-y|}. \end{aligned}$$

Determine, for each of these, whether it is a metric or not[1].

EXERCISE 2: The statement of Taylor's theorem, FONC, SONC, and SOSC all begin with "Let $E \subset R$ be open ..." or "Let $E \subset R^n$ be open ...". Can you think of a reason for open sets in the hypothesis of these theorems?

1.2 Graphical Minimization

EXERCISE 3: Graphically determine a minimizer of the function

$$f(x) = (x_1-2)^2 + (x_2-1)^2,$$

subject to constraints

$$\begin{aligned} c_1: &\quad x_1^2 - x_2 \le 0, \\ c_2: &\quad x_1 + x_2 \le 2. \end{aligned}$$

Use Python, numpy, and matplotlib to draw the function and the constraints, determine a minimizer visually, and justify your answer. Is the minimizer *local* or *global*?

Continued on next page $\Longrightarrow$

[1] Cf. [2] Rudin, Exercise 2.11, p. 44.

1.3 Analytical Minimization

EXERCISE 4: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2.$$

Answer the following questions relating to the point $t = [1, 1]^T$:

1. Compute the gradient $\nabla f(t)$.
2. Compute the Hessian $\nabla^2 f(t)$.
3. Is FONC satisfied at t?
4. Is SONC satisfied at t?
5. Is SOSC satisfied at t?

What can you say about the point t?

1.4 Metric Topology and Analytical Minimization

† EXERCISE 1: Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x) = \|x\|^2$. Show that the sequence of iterates $\{x_k\}_{k=0}^{\infty}$ defined by

$$x_k = \left(1 + \frac{1}{2^k}\right) \begin{bmatrix} \cos(k) \\ \sin(k) \end{bmatrix}$$

satisfies $f(x_{k+1}) < f(x_k)$ for $k = 0, 1, 2, \ldots$. Show that every point on the unit circle $\{x \mid \|x\|^2 = 1\}$ is a limit point for $\{x_k\}_{k=0}^{\infty}$.
Hint: Every $\theta \in [0, 2\pi]$ is a limit point of the subsequence $\{\xi_k\}_{k=0}^{\infty}$ defined by

$$\xi_k = k \pmod{2\pi} = k - 2\pi \left\lfloor \frac{k}{2\pi} \right\rfloor,$$

where the operator $\lfloor . \rfloor$ denotes rounding down to the next integer[2].

[2] Cf. [1] Nocedal and Wright, Exercise 2.5, p. 28.

Bibliography

[1] NOCEDAL, J., AND WRIGHT, S. J. *Numerical Optimization, Second Edition*. Springer, New York, 2006.

[2] WALTER RUDIN. *Walter Rudin's Principles of Mathematical Analysis, Third Edition*. McGraw-Hill, Inc., Department of Mathematics, University of Wisconsin, Madison, 1976.