Course Project - Step 1: Solving the problem in a deterministic manner

Author: Sebastian Troitzsch
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1 Introduction

This work considers the combined optimal operation problem of a district-level energy system with 1) air-conditioned buildings as flexible loads (FLs), 2) a district cooling system (DCS) as thermal grid and 3) an electric distribution grid. In the FLs, thermal power demand p_t^{th} arises due to the cooling demand for ensuring thermal comfort. Along with this, active / reactive power demand p_t, q_t arises for the pumping and ventilation equipment of the air-conditioning system as well as other appliances. The thermal power demand is supplied from the DCS, whereas the active / reactive power demand is supplied from the electric distribution grid. The DCS, which is assumed to be located at the electric grid source node, consumes active / reactive power from the electric grid. The electric grid source node draws active / reactive power from the transmission system at the cleared electricity wholesale market price c_t^{ref}.

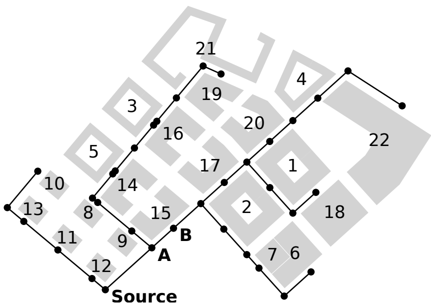

Figure 1: Test case with 22 commercial buildings as FLs. The thermal and electric grid layout is congruent, i.e. thermal and electric grid nodes coincide.

The considered test case is based on a neighbourhood in Singapore and was developed as part of [1]. The test case consists of thermal and electric grids with identical layout, i.e. thermal and electric grid nodes coincide according to fig. 1, and 22 commercial buildings modelled as FLs. The electricity wholesale market price c_t^{ref} is derived from one day of the Universal Singapore Energy Price (USEP) [1].

[1] https://www.emcsg.com/MarketData/PriceInformation

1.1 Notation

Let $\mathbb{R}$ be the domain of real numbers. Non-bold letters x, X denote scalars $\mathbb{R}^1$, bold lowercase letters $\boldsymbol{x}$ denote vectors $\mathbb{R}^n$ and bold uppercase letters $\boldsymbol{X}$ denote matrices $\mathbb{R}^{n,m}$. Bold numbers $\mathbf{0}$ and $\mathbf{1}$ denote vectors or matrices of zeros and ones of appropriate sizes. The transpose of a vector or matrix is denoted by $()^\intercal$. Prices are in Singapore Dollar (SGD) which is denoted by S$.

1.2 Flexible load (FL) model

The FL model is expressed in state space form as:

$$\boldsymbol{x}_{f,t^0} = \boldsymbol{x}_f^0 \quad : \boldsymbol{\lambda}_f^{x^0} \quad \forall f \in \mathcal{F} \tag{1a}$$

$$\boldsymbol{x}_{f,t+1} = \boldsymbol{A}_f \boldsymbol{x}_{f,t} + \boldsymbol{B}_f^c \boldsymbol{c}_{f,t} + \boldsymbol{B}_f^d \boldsymbol{d}_{f,t} \quad : \boldsymbol{\lambda}_{f,t}^x \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T}/t^N \tag{1b}$$

$$\boldsymbol{y}_{f,t} = \boldsymbol{C}_f \boldsymbol{x}_{f,t} + \boldsymbol{D}_f^c \boldsymbol{c}_{f,t} + \boldsymbol{D}_f^d \boldsymbol{d}_{f,t} \quad : \boldsymbol{\lambda}_{f,t}^y \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{1c}$$

$$\boldsymbol{y}_{f,t}^- \leq \boldsymbol{y}_{f,t} \leq \boldsymbol{y}_{f,t}^+ \quad : \boldsymbol{\mu}_{f,t}^{y^+}, \boldsymbol{\mu}_{f,t}^{y^-} \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{1d}$$

$$p_{f,t}^{th} = \boldsymbol{M}_f^{p^{th},y} \boldsymbol{y}_{f,t} \ : \lambda_{f,t}^{p^{th}} \quad p_{f,t} = \boldsymbol{M}_f^{p,y} \boldsymbol{y}_{f,t} \ : \lambda_{f,t}^{p} \quad q_{f,t} = \boldsymbol{M}_f^{q,y} \boldsymbol{y}_{f,t} \ : \lambda_{f,t}^{p} \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{1e}$$

The vectors $\boldsymbol{y}_{f,t}$, $\boldsymbol{x}_{f,t}$, $\boldsymbol{c}_{f,t}$ and $\boldsymbol{d}_{f,t}$ are the output, state, control and disturbance vectors for FL f at time step t. The matrices $\boldsymbol{A}_f$, $\boldsymbol{C}_f$ are the state and output matrix, and $\boldsymbol{B}_f^c$, $\boldsymbol{D}_f^c$, $\boldsymbol{B}_f^d$, $\boldsymbol{D}_f^d$ are the control and feed-through matrices, on the control and disturbance vectors respectively. The vector $\boldsymbol{x}^0$ is the initial state vector at the initial time step t^0 in $\mathcal{T}$. The symbol t^N denotes the final time step in $\mathcal{T}$. The matrices $\boldsymbol{M}_f^{p^{th},y}, \boldsymbol{M}_f^{p,y}, \boldsymbol{M}_f^{q,y}$ are the mapping matrices for relating the output vector $\boldsymbol{y}_{f,t}$ to the thermal, active and reactive power demand $\boldsymbol{p}_t^{th}, \boldsymbol{p}_t, \boldsymbol{q}_t \in \mathbb{R}^f$. Note that $\boldsymbol{x}^0$ and $\boldsymbol{d}_{f,t}$ are fixed parameter vectors. For the chosen test case, air-conditioned buildings serve as FLs and a fixed power factor, i.e. a fixed ratio between active and apparent power, is assumed. The detailed formulation of the state space model is omitted for brevity, but can be obtained from [2].

1.3 Thermal grid model

The thermal grid, i.e. the DCS, is modelled with a linear approximate model as:

$$\boldsymbol{h}^- \leq \boldsymbol{h}^{ref} + \boldsymbol{M}^{h,p^{th}} (\boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref}) \quad : \boldsymbol{\mu}_t^{h^-} \quad \forall t \in \mathcal{T} \tag{2a}$$

$$\boldsymbol{v}^{ref} + \boldsymbol{M}^{v,p^{th}} (\boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref}) \leq \boldsymbol{v}^+ \quad : \boldsymbol{\mu}_t^{v^+} \quad \forall t \in \mathcal{T} \tag{2b}$$

$$\frac{1}{\eta^{ch}} \left(p_t^{th,src} - \mathbf{1}^\intercal \boldsymbol{p}_t^{th} \right) = p^{pm,ref} + \boldsymbol{M}^{p^{pm},p^{th}} (\boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref}) \quad : \lambda_t^{p^{pm}} \quad \forall t \in \mathcal{T} \tag{2c}$$

The vectors $\boldsymbol{h}_t \in \mathbb{R}^{n^{th}}$, $\boldsymbol{v}_t \in \mathbb{R}^{b^{th}}$ are the pressure head at thermal grid nodes $n^{th} \in \mathcal{N}^{th}$ and the branch volume flow at thermal grid branches $b^{th} \in \mathcal{B}^{th}$ for time step $t \in \mathcal{T}$. The scalar p_t^{pm} denotes the total electric distribution pumping power demand for time step t. The reference point for each property is denoted by $()^{ref}$, which in the following is chosen to be the nominal operation point of the thermal grid. The matrices

$\boldsymbol{M}^{h,p^{th}} \in \mathbb{R}^{n^{th},f}$, $\boldsymbol{M}^{v,p^{th}} \in \mathbb{R}^{b^{th},f}$, $\boldsymbol{M}^{p^{pm},p^{th}} \in \mathbb{R}^{1,f}$ are the sensitivity matrices for the change of the respective properties to the thermal power change, which in turn is defined as $\Delta \boldsymbol{p}_t^{th} = \boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref}$. The vector $\boldsymbol{p}_t^{th}, \in \mathbb{R}^f$ is the absolute thermal power demand and the vector $\boldsymbol{p}_t^{th,ref}$ is the thermal power demand reference which is chosen to be the nominal load, i.e. peak load, of the FLs f. The derivation of sensitivity matrices is omitted for the sake of brevity, but is discussed in detail in [3].

1.4 Electric grid model

The electric grid, i.e. the power flow in the distribution grid, is modelled with a linear approximate model as:

$$\boldsymbol{u}^- \leq \boldsymbol{u}^{ref} + \boldsymbol{M}^{u,p}(\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{u,q}(\boldsymbol{q}_t - \boldsymbol{q}^{ref}) \leq \boldsymbol{u}^+ \quad : \boldsymbol{\mu}_t^{u^+}, \boldsymbol{\mu}_t^{u^-} \quad \forall t \in \mathcal{T} \tag{3a}$$

$$|\boldsymbol{s}^{f,ref}| + \boldsymbol{M}^{s^f,p}(\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{s^f,q}(\boldsymbol{q}_t - \boldsymbol{q}^{ref}) \leq |\boldsymbol{s}^{f,+}| \quad : \boldsymbol{\mu}_t^{s^{f,+}} \quad \forall t \in \mathcal{T} \tag{3b}$$

$$|\boldsymbol{s}^{t,ref}| + \boldsymbol{M}^{s^t,p}(\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{s^t,q}(\boldsymbol{q}_t - \boldsymbol{q}^{ref}) \leq |\boldsymbol{s}^{t,+}| \quad : \boldsymbol{\mu}_t^{s^{t,+}} \quad \forall t \in \mathcal{T} \tag{3c}$$

$$p_t^{src} - \boldsymbol{1}^\intercal \boldsymbol{p}_t = p^{ls,ref} + \boldsymbol{M}^{p^{ls},p}(\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{p^{ls},q}(\boldsymbol{q}_t - \boldsymbol{q}^{ref}) \quad : \lambda_t^{p^{ls}} \quad \forall t \in \mathcal{T} \tag{3d}$$

$$q_t^{src} - \boldsymbol{1}^\intercal \boldsymbol{q}_t = q^{ls,ref} + \boldsymbol{M}^{q^{ls},p}(\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{q^{ls},q}(\boldsymbol{q}_t - \boldsymbol{q}^{ref}) \quad : \lambda_t^{q^{ls}} \quad \forall t \in \mathcal{T} \tag{3e}$$

The vectors $\boldsymbol{u}_t \in \mathbb{R}^{n^{el}}$, $|\boldsymbol{s}_t^f|, |\boldsymbol{s}_t^t| \in \mathbb{R}^{b^{el}}$ are the voltage magnitude at electric grid nodes $n^{el} \in \mathcal{N}^{el}$ and the branch power flow magnitude in “from" and “to" direction at electric grid branches $b^{el} \in \mathcal{B}^{el}$ for time step $t \in \mathcal{T}$. The scalars p_t^{ls} and q_t^{ls} denote the total active and reactive loss for time step t. The reference point for each property is denoted by $()^{ref}$, which in the following is chosen to be the nominal operation point of the electric grid. The matrices $\boldsymbol{M}^{u,p}, \boldsymbol{M}^{u,q} \in \mathbb{R}^{n^{el},f}$, $\boldsymbol{M}^{s^f,p}, \boldsymbol{M}^{s^f,q}, \boldsymbol{M}^{s^t,p}, \boldsymbol{M}^{s^t,q} \in \mathbb{R}^{b^{el},f}$, $\boldsymbol{M}^{p^{ls},p}, \boldsymbol{M}^{p^{ls},q}, \boldsymbol{M}^{q^{ls},p}, \boldsymbol{M}^{q^{ls},q} \in \mathbb{R}^{1,f}$ are the sensitivity matrices for the change of the respective properties to the active and reactive power change, which in turn are defined as $\Delta \boldsymbol{p}_t = \boldsymbol{p}_t - \boldsymbol{p}^{ref}$, $\Delta \boldsymbol{q}_t = \boldsymbol{q}_t - \boldsymbol{q}^{ref}$. The vectors $\boldsymbol{p}_t, \boldsymbol{q}_t \in \mathbb{R}^f$ are the absolute active and reactive power demand of the FLs f. The vectors $\boldsymbol{p}_t^{ref}, \boldsymbol{q}_t^{ref}$ are the active and reactive power demand reference which is chosen to be the nominal load, i.e. peak load, of the FLs f. The derivation of sensitivity matrices is omitted for the sake of brevity, but can be obtained from [4].

2 Primal problem

The primal problem addresses the economic dispatch of FLs subject to the operational constraints of the FLs, the thermal grid and the electric grid. Thus, it consists of the combination of the FL, thermal grid and electric grid models eqs. (1) to (3) with the added objective for minimization of overall energy cost, expressed as:

$$\min_{\substack{\boldsymbol{x}_{f,t}, \boldsymbol{c}_{f,t}, \boldsymbol{y}_{f,t}, \\ \boldsymbol{p}_t^{th}, \boldsymbol{p}_t, \boldsymbol{q}_t, \\ p_t^{th,src}, p_t^{src}, q_t^{src}}} \quad \sum_{t \in \mathcal{T}} c_t^{ref} \left(p_t^{src} + \frac{1}{\eta^{ch}} p_t^{th,src} \right) \tag{4a}$$

$$\text{s.t.} \quad (1), (2), (3) \tag{4b}$$

3 KKT conditions

To formulate the KKT conditions for the optimization problem, the Lagrangian function is first derived in appendix A, eq. (8). Then, the KKT stationarity, primal feasibility and complimentary slackness conditions can be deduced. The KKT stationarity condition $\frac{\delta L}{\delta x} = 0$ yields:

$$\mathbf{0} = \left(\boldsymbol{\lambda}_f^{x^0}\right)^\intercal - \left(\boldsymbol{\lambda}_{f,t}^x\right)^\intercal \boldsymbol{A}_f - \left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal \boldsymbol{C}_f \quad \forall f \in \mathcal{F},\ t = t^0 \tag{5a}$$

$$\mathbf{0} = \left(\boldsymbol{\lambda}_{f,t-1}^x\right)^\intercal - \left(\boldsymbol{\lambda}_{f,t}^x\right)^\intercal \boldsymbol{A}_f - \left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal \boldsymbol{C}_f \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T}/t^N \tag{5b}$$

$$\mathbf{0} = \left(\boldsymbol{\lambda}_{f,t-1}^x\right)^\intercal - \left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal \boldsymbol{C}_f \quad \forall f \in \mathcal{F},\ t = t^N \tag{5c}$$

$$\mathbf{0} = -\left(\boldsymbol{\lambda}_{f,t}^x\right)^\intercal \boldsymbol{B}_f^c - \left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal \boldsymbol{D}_f^c \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T}/t^N \tag{5d}$$

$$\mathbf{0} = -\left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal \boldsymbol{D}_f^c \quad \forall f \in \mathcal{F},\ \forall t = t^N \tag{5e}$$

$$\mathbf{0} = \left(\boldsymbol{\lambda}_{f,t}^y\right)^\intercal - \left(\boldsymbol{\mu}_{f,t}^{y^-}\right)^\intercal + \left(\boldsymbol{\mu}_{f,t}^{y^+}\right)^\intercal - \lambda_{f,t}^{p^{th}} \boldsymbol{M}_f^{p^{th},y} - \lambda_{f,t}^{p} \boldsymbol{M}_f^{p,y} - \lambda_{f,t}^{q} \boldsymbol{M}_f^{q,y} \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{5f}$$

$$\mathbf{0} = \left(\boldsymbol{\lambda}_t^{p^{th}}\right)^\intercal - \left(\boldsymbol{\mu}_t^{h^-}\right)^\intercal \boldsymbol{M}^{h,p^{th}} + \left(\boldsymbol{\mu}_t^{v^+}\right)^\intercal \boldsymbol{M}^{v,p^{th}} - \lambda_t^{p^{pm}} \left(\frac{1}{\eta^{ch}} \mathbf{1} + \boldsymbol{M}^{p^{pm},p^{th}}\right) \quad \forall t \in \mathcal{T} \tag{5g}$$

$$\begin{aligned}\mathbf{0} = (\boldsymbol{\lambda}_t^p)^\intercal - (\boldsymbol{\mu}_t^{u^-})^\intercal \boldsymbol{M}^{u,p} + \left(\boldsymbol{\mu}_t^{u^+}\right)^\intercal \boldsymbol{M}^{u,p} + \left(\boldsymbol{\mu}_t^{s^{f,+}}\right)^\intercal \boldsymbol{M}^{s^f,p} + \left(\boldsymbol{\mu}_t^{s^{t,+}}\right)^\intercal \boldsymbol{M}^{s^t,p} \\ - \lambda_t^{p^{ls}} \left(\mathbf{1}^\intercal + \boldsymbol{M}^{p^{ls},p}\right) - \lambda_t^{q^{ls}} \boldsymbol{M}^{q^{ls},p}\end{aligned} \quad \forall t \in \mathcal{T} \tag{5h}$$

$$\begin{aligned}\mathbf{0} = (\boldsymbol{\lambda}_t^q)^\intercal - (\boldsymbol{\mu}_t^{u^-})^\intercal \boldsymbol{M}^{u,q} + \left(\boldsymbol{\mu}_t^{u^+}\right)^\intercal \boldsymbol{M}^{u,q} + \left(\boldsymbol{\mu}_t^{s^{f,+}}\right)^\intercal \boldsymbol{M}^{s^f,q} + \left(\boldsymbol{\mu}_t^{s^{t,+}}\right)^\intercal \boldsymbol{M}^{s^t,q} \\ - \lambda_t^{p^{ls}} \boldsymbol{M}^{p^{ls},q} - \lambda_t^{q^{ls}} \left(\mathbf{1}^\intercal + \boldsymbol{M}^{q^{ls},q}\right)\end{aligned} \quad \forall t \in \mathcal{T} \tag{5i}$$

$$0 = c_t^{ref} + \lambda_t^{p^{pm}}, \quad 0 = c_t^{ref} + \lambda_t^{p^{ls}}, \quad 0 = \lambda_t^{q^{ls}} \quad \forall t \in \mathcal{T} \tag{5j}$$

The KKT primal feasibility condition $h(x) = 0$ yields that primal equality constraints eqs. (1a) to (1c), (1e), (2c), (3d) and (3e) have to hold. The KKT complementary slackness condition $0 \leq -g(x) \perp \mu \geq 0$ yields:

$$\mathbf{0} \leq -\left(\boldsymbol{y}_{f,t}^- - \boldsymbol{y}_{f,t}\right) \perp \boldsymbol{\mu}_{f,t}^{y^-} \geq \mathbf{0} \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{6a}$$

$$\mathbf{0} \leq -\left(\boldsymbol{y}_{f,t} - \boldsymbol{y}_{f,t}^+\right) \perp \boldsymbol{\mu}_{f,t}^{y^+} \geq \mathbf{0} \quad \forall f \in \mathcal{F},\ \forall t \in \mathcal{T} \tag{6b}$$

$$\mathbf{0} \leq -\left(\boldsymbol{h}^- - \boldsymbol{h}^{ref} - \boldsymbol{M}^{h,p^{th}} (\boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref})\right) \perp \boldsymbol{\mu}_t^{h^-} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6c}$$

$$\mathbf{0} \leq -\left(\boldsymbol{v}^{ref} + \boldsymbol{M}^{v,p^{th}} (\boldsymbol{p}_t^{th} - \boldsymbol{p}^{th,ref}) - \boldsymbol{v}^+\right) \perp \boldsymbol{\mu}_t^{v^+} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6d}$$

$$\mathbf{0} \leq -\left(\boldsymbol{u}^- - \boldsymbol{u}^{ref} - \boldsymbol{M}^{u,p} (\boldsymbol{p}_t - \boldsymbol{p}^{ref}) - \boldsymbol{M}^{u,q} (\boldsymbol{q}_t - \boldsymbol{q}^{ref})\right) \perp \boldsymbol{\mu}_t^{u^-} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6e}$$

$$\mathbf{0} \leq -\left(\boldsymbol{u}^{ref} + \boldsymbol{M}^{u,p} (\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{u,q} (\boldsymbol{q}_t - \boldsymbol{q}^{ref}) - \boldsymbol{u}^+\right) \perp \boldsymbol{\mu}_t^{u^+} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6f}$$

$$\mathbf{0} \leq -\left(|\boldsymbol{s}^{f,ref}| + \boldsymbol{M}^{s^f,p} (\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{s^f,q} (\boldsymbol{q}_t - \boldsymbol{q}^{ref}) - |\boldsymbol{s}^{f,+}|\right) \perp \boldsymbol{\mu}_t^{s^{f,+}} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6g}$$

$$\mathbf{0} \leq -\left(|\boldsymbol{s}^{t,ref}| + \boldsymbol{M}^{s^t,p} (\boldsymbol{p}_t - \boldsymbol{p}^{ref}) + \boldsymbol{M}^{s^t,q} (\boldsymbol{q}_t - \boldsymbol{q}^{ref}) - |\boldsymbol{s}^{t,+}|\right) \perp \boldsymbol{\mu}_t^{s^{t,+}} \geq \mathbf{0} \quad \forall t \in \mathcal{T} \tag{6h}$$

4 Dual problem

The dual problem is the maximization problem with respect to the dual variables of the minimization of the Langrangian function eq. (8) with respect to the primal variables, subject to the KKT stationarity condition eq. (5). Therefore, the dual problem is derived as:

$$
\begin{aligned}
\max_{\substack{\boldsymbol{\lambda}_f^{x^0}, \boldsymbol{\lambda}_{f,t}^{x}, \boldsymbol{\lambda}_{f,t}^{y}, \\ \boldsymbol{\mu}_{f,t}^{y^-} \geq \mathbf{0}, \boldsymbol{\mu}_{f,t}^{y^+} \geq \mathbf{0}, \\ \boldsymbol{\lambda}_t^{p^{th}}, \boldsymbol{\lambda}_t^{p}, \boldsymbol{\lambda}_t^{q} \\ \boldsymbol{\mu}_t^{h^-} \geq \mathbf{0}, \boldsymbol{\mu}_t^{v^+} \geq \mathbf{0}, \\ \lambda_t^{p^{pm}}, \boldsymbol{\mu}_t^{u^-} \geq \mathbf{0}, \\ \boldsymbol{\mu}_t^{u^+} \geq \mathbf{0}, \boldsymbol{\mu}_t^{s^{f,+}} \geq \mathbf{0}, \\ \boldsymbol{\mu}_t^{s^{t,+}} \geq \mathbf{0}, \lambda_t^{p^{ls}}, \\ \lambda_t^{q^{ls}}}} \quad & \sum_{f \in \mathcal{F}} - \left(\boldsymbol{\lambda}_f^{x^0}\right)^\intercal \boldsymbol{x}_f^0 - \sum_{t \in \mathcal{T}/t^N} \sum_{f \in \mathcal{F}} \left(\boldsymbol{\lambda}_{f,t}^{x}\right)^\intercal \boldsymbol{B}_f^d \boldsymbol{d}_{f,t} - \sum_{t \in \mathcal{T}} \sum_{f \in \mathcal{F}} \left(\boldsymbol{\lambda}_{f,t}^{y}\right)^\intercal \boldsymbol{D}_f^d \boldsymbol{d}_{f,t} \qquad \text{(7a)} \\
& + \sum_{t \in \mathcal{T}} \left(\sum_{f \in \mathcal{F}} \left(\boldsymbol{\mu}_{f,t}^{y^-}\right)^\intercal \boldsymbol{y}_{f,t}^{-} - \sum_{f \in \mathcal{F}} \left(\boldsymbol{\mu}_{f,t}^{y^+}\right)^\intercal \boldsymbol{y}_{f,t}^{+} + \left(\boldsymbol{\mu}_t^{h^-}\right)^\intercal \left(\boldsymbol{h}^- - \boldsymbol{h}^{ref} + \boldsymbol{M}^{h,p^{th}} \boldsymbol{p}^{th,ref}\right) \right) \\
& + \sum_{t \in \mathcal{T}} \left(\left(\boldsymbol{\mu}_t^{v^+}\right)^\intercal \left(\boldsymbol{v}^{ref} - \boldsymbol{M}^{v,p^{th}} \boldsymbol{p}^{th,ref} - \boldsymbol{v}^+\right) + \lambda_t^{p^{pm}} \left(-p^{pm,ref} + \boldsymbol{M}^{p^{pm},p^{th}} \boldsymbol{p}^{th,ref}\right) \right) \\
& + \sum_{t \in \mathcal{T}} \left(\boldsymbol{\mu}_t^{u^-}\right)^\intercal \left(\boldsymbol{u}^- - \boldsymbol{u}^{ref} + \boldsymbol{M}^{u,p} \boldsymbol{p}^{ref} + \boldsymbol{M}^{u,q} \boldsymbol{q}^{ref}\right) \\
& + \sum_{t \in \mathcal{T}} \left(\boldsymbol{\mu}_t^{u^+}\right)^\intercal \left(\boldsymbol{u}^{ref} - \boldsymbol{M}^{u,p} \boldsymbol{p}^{ref} - \boldsymbol{M}^{u,q} \boldsymbol{q}^{ref} - \boldsymbol{u}^+\right) \\
& + \sum_{t \in \mathcal{T}} \left(\boldsymbol{\mu}_t^{s^{f,+}}\right)^\intercal \left(|\boldsymbol{s}^{f,ref}| - \boldsymbol{M}^{s^f,p} \boldsymbol{p}^{ref} - \boldsymbol{M}^{s^f,q} \boldsymbol{q}^{ref} - |\boldsymbol{s}^{f,+}|\right) \\
& + \sum_{t \in \mathcal{T}} \left(\boldsymbol{\mu}_t^{s^{t,+}}\right)^\intercal \left(|\boldsymbol{s}^{t,ref}| - \boldsymbol{M}^{s^t,p} \boldsymbol{p}^{ref} - \boldsymbol{M}^{s^t,q} \boldsymbol{q}^{ref} - |\boldsymbol{s}^{t,+}|\right) \\
& + \sum_{t \in \mathcal{T}} \lambda_t^{p^{ls}} \left(-p^{ls,ref} + \boldsymbol{M}^{p^{ls},p} \boldsymbol{p}^{ref} + \boldsymbol{M}^{p^{ls},q} \boldsymbol{q}^{ref}\right) \\
& + \sum_{t \in \mathcal{T}} \lambda_t^{q^{ls}} \left(-q^{ls,ref} + \boldsymbol{M}^{q^{ls},p} \boldsymbol{p}^{ref} + \boldsymbol{M}^{q^{ls},q} \boldsymbol{q}^{ref}\right) \\
\text{s.t.} \quad & (5) \qquad \text{(7b)}
\end{aligned}
$$

5 Results

The solution of the primal, dual and KKT problems are compared in the following, where KKT problem refers to the solution of the equation system eqs. (1a) to (1c), (1e), (2c), (3d), (3e), (5) and (6). The primal, dual and KKT problems are implemented in Python, where CVXPY[2] is used as optimization modelling framework and FLEDGE[3] is used to obtain the underlying FL, thermal grid and electric grid model matrices. Gurobi is used as the solver and computations are evaluated on a machine with Intel i7 8th Gen processor and 16 GB RAM.

Table 1: Solution statistics for primal, dual and KKT problems.

	Primal problem	Dual problem	KKT problem
Objective value [S$]	67483.4	67483.4	67483.4 [4]
Variable count	12725	34150	72150
Solution time [s]	1.17	1.81	4.93

[2] https://www.cvxpy.org/

[3] https://github.com/TUMCREATE-ESTL/fledge

[4] Since the KKT problem does not have an objective, the primal objective was evaluated based on the KKT solution to obtain this value.

Table 1 highlights that all three problem solutions arrived at the same final objective value, thereby confirming the expected equivalence between the different problem formulations. The variable count is exceeds 10 thousand in all problems, where the dual problem has a higher variable count than the primal problem. However, neither problem was thoroughly reformulated to eliminate auxiliary variables, which may enable reduction of the variable count. The KKT problem exhibits the highest variable counts, since it contains both primal and dual variables in addition to binary variables to represent its complementarity constraints eq. (6). The solution time correlates with the variable count, but also increases further for the KKT problem due to the introduction of binary constraints.

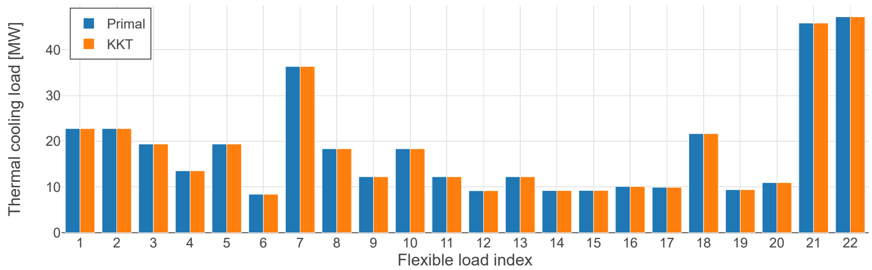

Figure 2: Thermal cooling load p_t^{th} at t = 12pm.

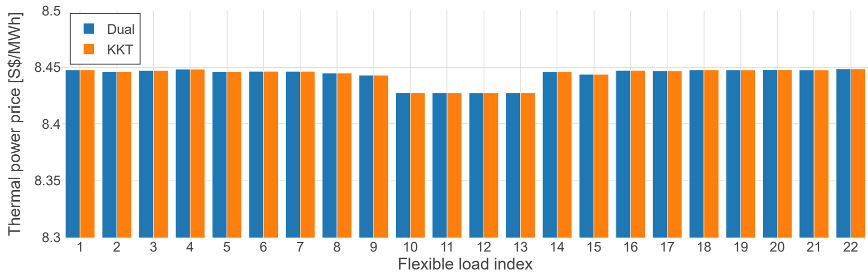

Figure 3: Dual variable $\lambda_t^{p^{th}}$ of the thermal power balance eq. (1e) at t = 12pm.

Figures 2 and 3 highlight that the solutions of the primal / dual problem for the exemplary variables p_t^{th} / $\lambda_t^{p^{th}}$ conincide with the solutions of the KKT problem. To this end, the dual variable $\lambda_t^{p^{th}}$ of the thermal power balance eq. (1e) can be interpreted as the distribution locational marginal price (DLMP) of the thermal grid. The y-axis range of fig. 2 was modified to highlight the slight differences in $\lambda_t^{p^{th}}$. In the chosen test case, no branch, node head or node voltage limits are binding in either the thermal or the electric grid. Thus the value differences in $\lambda_t^{p^{th}}$ are due to each FL causing pumping losses depending on its location in the thermal grid.

6 Conclusions

This work presented the primal, dual and KKT problem formulations for the combined optimal operation problem of a district-level energy system. The solutions for all problem formulations were shown to be equivalent. Therefore, whichever problem problem formulation enables the best performance for a particular use case may be solved in place of the other formulations.

Appendix A: Lagrangian function

$$
\begin{aligned}
L = &\sum_{t\in\mathcal{T}} c_t^{ref}\left(p_t^{src}+\frac{1}{\eta^{ch}}p_t^{th,src}\right)\\
&+\sum_{f\in\mathcal{F}}\left(\boldsymbol{\lambda}_f^{x^0}\right)^\intercal\left(\boldsymbol{x}_{f,t^0}-\boldsymbol{x}_f^0\right)\\
&+\sum_{t\in\mathcal{T}/t^N}\sum_{f\in\mathcal{F}}\left(\boldsymbol{\lambda}_{f,t}^{x}\right)^\intercal\left(\boldsymbol{x}_{f,t+1}-\boldsymbol{A}_f\boldsymbol{x}_{f,t}-\boldsymbol{B}_f^c\boldsymbol{c}_{f,t}-\boldsymbol{B}_f^d\boldsymbol{d}_{f,t}\right)\\
&+\sum_{t\in\mathcal{T}}\sum_{f\in\mathcal{F}}\left(\boldsymbol{\lambda}_{f,t}^{y}\right)^\intercal\left(\boldsymbol{y}_{f,t}-\boldsymbol{C}_f\boldsymbol{x}_{f,t}-\boldsymbol{D}_f^c\boldsymbol{c}_{f,t}-\boldsymbol{D}_f^d\boldsymbol{d}_{f,t}\right)\\
&+\sum_{t\in\mathcal{T}}\sum_{f\in\mathcal{F}}\left(\boldsymbol{\mu}_{f,t}^{y^-}\right)^\intercal\left(\boldsymbol{y}_{f,t}^- -\boldsymbol{y}_{f,t}\right)\\
&+\sum_{t\in\mathcal{T}}\sum_{f\in\mathcal{F}}\left(\boldsymbol{\mu}_{f,t}^{y^+}\right)^\intercal\left(\boldsymbol{y}_{f,t}-\boldsymbol{y}_{f,t}^+\right)\\
&+\sum_{t\in\mathcal{T}}\sum_{f\in\mathcal{F}}\left(\lambda_{f,t}^{p^{th}}\left(p_{f,t}^{th}-\boldsymbol{M}_f^{p^{th},y}\boldsymbol{y}_{f,t}\right)+\lambda_{f,t}^{p}\left(p_{f,t}-\boldsymbol{M}_f^{p,y}\boldsymbol{y}_{f,t}\right)+\lambda_{f,t}^{q}\left(q_{f,t}-\boldsymbol{M}_f^{q,y}\boldsymbol{y}_{f,t}\right)\right)\\
&+\sum_{t\in\mathcal{T}}\left(\boldsymbol{\mu}_t^{h^-}\right)^\intercal\left(\boldsymbol{h}^- -\boldsymbol{h}^{ref}-\boldsymbol{M}^{h,p^{th}}(\boldsymbol{p}_t^{th}-\boldsymbol{p}^{th,ref})\right)\\
&+\sum_{t\in\mathcal{T}}\left(\boldsymbol{\mu}_t^{v^+}\right)^\intercal\left(\boldsymbol{v}^{ref}+\boldsymbol{M}^{v,p^{th}}(\boldsymbol{p}_t^{th}-\boldsymbol{p}^{th,ref})-\boldsymbol{v}^+\right)\\
&+\sum_{t\in\mathcal{T}}\lambda_t^{p^{pm}}\left(\frac{1}{\eta^{ch}}\left(p_t^{th,src}-\boldsymbol{1}^\intercal\boldsymbol{p}_t^{th}\right)-p^{pm,ref}-\boldsymbol{M}^{p^{pm},p^{th}}(\boldsymbol{p}_t^{th}-\boldsymbol{p}^{th,ref})\right)\\
&+\sum_{t\in\mathcal{T}}(\boldsymbol{\mu}_t^{u^-})^\intercal\left(\boldsymbol{u}^- -\boldsymbol{u}^{ref}-\boldsymbol{M}^{u,p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})-\boldsymbol{M}^{u,q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})\right)\\
&+\sum_{t\in\mathcal{T}}\left(\boldsymbol{\mu}_t^{u^+}\right)^\intercal\left(\boldsymbol{u}^{ref}+\boldsymbol{M}^{u,p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})+\boldsymbol{M}^{u,q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})-\boldsymbol{u}^+\right)\\
&+\sum_{t\in\mathcal{T}}\left(\boldsymbol{\mu}_t^{s^{f,+}}\right)^\intercal\left(|\boldsymbol{s}^{f,ref}|+\boldsymbol{M}^{s^f,p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})+\boldsymbol{M}^{s^f,q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})-|\boldsymbol{s}^{f,+}|\right)\\
&+\sum_{t\in\mathcal{T}}\left(\boldsymbol{\mu}_t^{s^{t,+}}\right)^\intercal\left(|\boldsymbol{s}^{t,ref}|+\boldsymbol{M}^{s^t,p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})+\boldsymbol{M}^{s^t,q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})-|\boldsymbol{s}^{t,+}|\right)\\
&+\sum_{t\in\mathcal{T}}\lambda_t^{p^{ls}}\left(p_t^{src}-\boldsymbol{1}^\intercal\boldsymbol{p}_t-p^{ls,ref}-\boldsymbol{M}^{p^{ls},p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})-\boldsymbol{M}^{p^{ls},q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})\right)\\
&+\sum_{t\in\mathcal{T}}\lambda_t^{q^{ls}}\left(q_t^{src}-\boldsymbol{1}^\intercal\boldsymbol{q}_t-q^{ls,ref}-\boldsymbol{M}^{q^{ls},p}(\boldsymbol{p}_t-\boldsymbol{p}^{ref})-\boldsymbol{M}^{q^{ls},q}(\boldsymbol{q}_t-\boldsymbol{q}^{ref})\right)
\end{aligned}
\tag{8}
$$

Bibliography

[1] S. Troitzsch, B. K. Sreepathi, T. P. Huynh, A. Moine, S. Hanif, J. Fonseca, and T. Hamacher, "Optimal electric-distribution-grid planning considering the demand-side flexibility of thermal building systems for a test case in Singapore," en, *Applied Energy*, vol. 273, 2020, ISSN: 03062619. DOI: 10.1016/j.apenergy.2020.114917.

[2] S. Troitzsch and T. Hamacher, "Control-oriented Thermal Building Modelling," en, in *IEEE PES General Meeting*, 2020.

[3] S. Troitzsch, M. Grussmann, K. Zhang, and T. Hamacher, "Distribution Locational Marginal Pricing for Combined Thermal and Electric Grid Operation," in *IEEE PES Innovative Smart Grid Technologies Conference Europe*, 2020. DOI: 10.36227/techrxiv.11918712.

[4] S. Hanif, K. Zhang, C. M. Hackl, M. Barati, H. B. Gooi, and T. Hamacher, "Decomposition and equilibrium achieving distribution locational marginal prices using trust-region method," *IEEE Trans. Smart Grid*, vol. 10, no. 3, pp. 3269–3281, 2019, ISSN: 1949-3053. DOI: 10.1109/TSG.2018.2822766.