① $f(x) = (x+x^3)^3 - x^3$

$$f'(x) = 3(x+x^3)^2 \cdot (1+3x^2) - 3x^2$$
$$= 3(x^2+2x^4+x^6)(1+3x^2) - 3x^2$$
$$= 3(\cancel{x^2} + 2x^4 + x^6 + 3x^4 + 6x^6 + 3x^8) - \cancel{3x^2}$$
$$= 5x^4 + 7x^6 + 3x^8$$

② $f(x) = x^2 + 3x$

a) $f(x_0+\delta) = f(x_0) + \frac{df}{dx}\Big|_{x_0} \cdot \delta$ $\qquad$ $f'(x) = 2x+3$

$= 28 + 11 \cdot \delta$ $\qquad$ $f'(4) = 11$

$= 28 + 11(x - x_0)$

$= 28 + 11x - 44 = \underline{\underline{11x - 16}}$

③ $f(\vec{x}(\vec{p}_0 + \vec{\delta})) \approx f(\vec{x}(\vec{p}_0)) + \left[(\text{grad } f)\, J_F\right]_{p_0} \cdot \vec{\delta}$

$f(x_1(\vec{p}), x_2(\vec{p})) = x_1(\vec{p}) + x_2^3(\vec{p})$

$\Rightarrow \text{grad } f = \left[1, 3x_2^2(\vec{p})\right]$

$x_1(\vec{p}) = 2p_2$

$x_2(\vec{p}) = p_1 + p_2^2$ $\quad \Rightarrow J = \begin{bmatrix} 0 & 2 \\ 1 & 2p_2 \end{bmatrix}$

$$\Rightarrow f(x_1(\vec{p}+\vec{\delta}), x_2(\vec{p}+\vec{\delta})) = f(x_1(\vec{p}), x_2(\vec{p})) + \left[1, 3x_2^2(\vec{p})\right] \begin{bmatrix} 0 & 2 \\ 1 & 2p_2 \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$
$$= f(x_1(\vec{p}), x_2(\vec{p})) + \left[3x_2^2, 2 + 6p_2x_2^2\right] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} \qquad (x_2 = p_1 + p_2^2)$$
$$= f(x_1(\vec{p}), x_2(\vec{p})) + \left[3(p_1+p_2^2)^2, 2 + 6p_2(p_1+p_2^2)^2\right] \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix}$$
$$= f(x_1(\vec{p}), x_2(\vec{p})) + \left(3(p_1+p_2^2)^2\delta_1 + 2\delta_2 + 6p_2(p_1+p_2^2)^2\delta_2\right)$$

④ $x \in \mathbb{R}^d$ $\quad A \in \mathbb{R}^{d \times d}$ $\quad$ PD

$f(\vec{x}) = 2\vec{x}^T A \vec{x} - 10\vec{x}$

$$\frac{\partial f}{\partial \vec{x}} = 2\vec{x}^T \overbrace{(A + A^T)}^{2A} - 10\vec{x}^T = \underline{\underline{4\vec{x}^T A - 10\vec{x}^T}}$$