Mathematics 2, Part 4, Homework 2 **June 3rd 2025**

Duals and duals and duals

Consider the following two LPs.

$$\begin{aligned} \max\ & c^T x \\ & Ax \le b \qquad \text{(P)} \\ & x \ge 0 \end{aligned} \qquad\qquad \begin{aligned} \min\ & b^T y \\ & A^T y \ge c \qquad \text{(D)} \\ & y \ge 0 \end{aligned}$$

The *standard form* (sometimes also known as the production form) of a LP is the pattern (maximization, less-than constraints, nonnegativity) of (P). We have shown that the problem (D) is the *dual* of (P).

1. Show that (P) is the dual of (D). You can do that by rewriting (D) in standard form and dualizing it according to the above pattern.
2. Show that problem [1, (1), problem (D)] is the dual of [1, (2), problem (P)]. You can rewrite one of them in standard form and dualize it.

Implement the interior-point algorithm

Let us be precise and not as enthusiastic. Consider the problem

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ x > 0, s &> 0 \end{aligned} \tag{P_μ}$$

Assuming that you are given a solution (x, y, s, μ) to (P_μ), compute the next solution (x', y', s', μ') to $(P_{\mu'})$, where

$$\mu' = \left(1 - \frac{1}{6\sqrt{m}}\right)\mu.$$

3. Implement an algorithm computing the next-step solution (x', y', s', μ') from (x, y, s, μ). You are of course allowed to use library routines for solving linear systems.

Observe the following linear problem.

$$\begin{aligned} \min\ & -3x_1 - 4x_2 \\ & 3x_1 + 3x_2 + 3x_3 = 4 \\ & 3x_1 + x_2 + x_4 = 3 \\ & x_1 + 4x_2 + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0, \end{aligned} \tag{problem X}$$

5. Compute the dual problem of (problem X).

6. Show that the vectors

$$x = \left(\frac{2}{5}, \frac{8}{15}, \frac{2}{5}, \frac{19}{15}, \frac{22}{15}\right)^T$$
$$y = \left(-\frac{4}{5}, -\frac{4}{5}, -\frac{2}{3}\right)^T$$
$$s = \left(\frac{37}{15}, \frac{28}{15}, \frac{12}{5}, \frac{4}{5}, \frac{2}{3}\right)^T$$

are strictly feasible solutions of both (problem X) and its dual.

7. Show that above vectors form a good starting solution for a corresponding problem (P_μ) What is your initial choice of μ? Is $\mu = 1$ an appropriate possibility?

8. Iterate your next-step algorithm until it converges (or at least stabilizes on most of the digits).

9. Can you speed up the convergence? How does μ' relate to μ? Can you do better? Can you choose a smaller μ', possibly adaptively, so that the iterative invariants are still satisfied?

10. Heuristically decide when to stop. If your current μ is sufficiently small, then for every i the product $x_i s_i$ is small. For every i make a decision and set either $x_i = 0$ or $s_i = 0$, depending on which of x_i, s_i is closer to zero. This gives an extra collection of m scalar equations.

11. Test whether your choice is correct. Exactly compute the solution of

$$Ax = b$$
$$A^T y + s = c$$

assuming the above m scalar equations, and test if the solutions are indeed feasible and optimal.

Commercial solver

Look for commercial (open-source) optimization solvers that allow the use of both simplex and interior-point methods.

12. Test the solver of your choice on (problem X). Use both the combinatorial (simplex) method and interior-point method.

Analytic center

A LP can have many optimal solutions. For example, the set of points on a playing die that lies furthest from the top of a table is a facet (and not a single vertex).

A combinatorial approach (simplex algorithm) for finding an optimum will terminate in a vertex solution. The interior point method converges to the *analytic center* of the set of optimal solutions. Let us give the proper definition, see also[1] [2].

[1] Also available here.

Let

$$Ax \leq b \tag{1}$$

be a system of n linear inequalities, and let $\Phi = \{x; Ax \leq b\}$ be the set of its *feasible solutions.* We denote $s(x) = b - Ax$. Let $I \subseteq \{1, \ldots, n\}$ be the set of coordinates/indices, for which there exists $x \in \Phi$, so that $(Ax)_i < b_i$ or equivalently $s(x)_i > 0$.[2]

The *unique*[3] vector $x \in \Phi$ which maximizes

$$\prod_{i \in I} s(x)_i$$

is called the *analytic center* of a system of linear inequalities (1).

13. Show that there exists $x \in \Phi$ so that for all $i \in I$ we have $s(x)_i > 0$.
14. Show that the analytic center optimization problem is equivalent to a strictly convex optimization problem.
15. Show that the analytic center is unique.
16. Find the analytic center for the following system of linear inequalities:

$$\begin{aligned} 2x_1 + 1x_2 &\leq 6 \\ 6x_1 + 1x_2 &\leq 15 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

17. Let us add an additional constraint $2x_1 + 1x_2 \leq 18$ to the above system of inequalities. Clearly this constraint does not change the set of feasible solutions. What happens with the analytic center?
18. Let us add an additional constraint $4x_1 + 2x_2 \leq 12$ to the above system — this one suspiciously resembles one of the original constraints. What happens with the analytic center in this case?

References

[1] Kurt Mehlhorn and Sanjeev Saxena. A still simpler way of introducing the interior-point method for linear programming (ver. 8dec21). *CoRR*, abs/1510.03339, 2015.

[2] Imre Pólik and Tamás Terlaky. *Interior Point Methods for Nonlinear Optimization*, pages 215–276. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

Bottom line

Upload your solution in a single .zip archive which contains the source-code (I would be most pleased with a Jupyter notebook in Python - nevertheless you can use any programming platform/language according to your personal preferences) and a .pdf.

[2] x_i is the i-th coordinate of $x \in \mathbb{R}^n$.

[3] You have to show this in the first place.