① Prove f is convex $\Leftrightarrow$ For each $x_1, \dots x_k \in D$ and $\alpha_1, \dots, \alpha_k \in [0,1]$; $\sum^k \alpha_i = 1$ we have

$$f\left(\sum_{i=1}^{k} \alpha_i x_i\right) \leq \sum \alpha_i f(x_i)$$

Induction:

① $k=2$: $f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2)$ Holds by definition of convexity.

② Assume it holds for n. Prove that it holds for $n+1$:

Let $z = \frac{1}{\beta} \sum_{i=1}^{n} \alpha_i x_i$ and $\beta = \sum^n \alpha_i$

Rewrite for $n+1$:

$$\sum_{i=1}^{n+1} \alpha_i x_i = \beta z + \alpha_{n+1} x_{n+1} \quad ; \quad \beta + \alpha_{n+1} = 1$$

BY CONVEXITY:

$$f(\beta z + \alpha_{n+1} x_{n+1}) \leq \beta f(z) + \alpha_{n+1} f(n+1)$$

Then we have (by hypothesis) $\left(\sum^n \frac{\alpha_i}{\beta} = 1\right)$

$$f(z) = f\left(\sum_{i=1}^{n} \frac{\alpha_i}{\beta} x_i\right) \leq \sum^n \frac{\alpha_i}{\beta} f(x_i)$$

$$f\left(\sum_{i=1}^{n+1} \alpha_i x_i\right) \leq \beta\left(\sum^n \frac{\alpha_i}{\beta} f(x_i)\right) + \alpha_{n+1} X_{n+1} = \sum_{i=1}^{n+1} \alpha_i f(x_i)$$

The reverse implication holds, since the above holds for all $n \in \mathbb{N}$,

including $n=2$ □

② $f(x,y) = x^2 + e^x + y^2 - xy \quad \in [-2,2] \times [-2,2]$

$$\frac{\partial f}{\partial x} = 2x + e^x - y \qquad \frac{\partial f}{\partial y} = 2y - x$$

$$\nabla f = (2x + e^x - y,\ 2y - x)$$

$$2x + e^x - y = 0 \quad \to \quad 4y + e^{2y} - y = 0$$

$$2y - x = 0 \quad x = 2y \qquad 3y + e^{2y} = 0$$

$$\log 3y + 2y = 0$$

$$\frac{\partial^2 f}{\partial^2 x} = 2 + e^x \qquad \frac{\partial^2 f}{\partial^2 y} = 2 \qquad \frac{\partial^2 f}{\partial x \partial y} = -1$$

$$\nabla^2 f = \begin{bmatrix} 2+e^x & -1 \\ -1 & 2 \end{bmatrix}$$

a) Convex: ($\nabla^2 f$ PSD)

$\Rightarrow 2 + e^x \geq 0 \quad \forall x \in [-2,2] \quad (2 > \frac{1}{e})$

$\det(\nabla^2 f) = 4 + 2e^x - 1 = 3 + 2e^x \qquad 2 \cdot \frac{1}{e} < 3 \Rightarrow \geq 0$

$\Rightarrow$ $\nabla^2 f$ has positive eigenvalues thus f is convex.

b) β ... given by largest eigenvalue $\qquad \alpha$ ... given by smallest eigenvalue

$$|\nabla^2 f - \lambda I| = \begin{vmatrix} 2+e^x-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (2+e^x-\lambda)(2-\lambda) - 1 = 4 - 2\lambda + 2e^x - e^x\lambda - 2\lambda + \lambda^2 - 1 = 2e^x - e^x\lambda - 4\lambda + \lambda^2 + 3 = 0$$

$$\lambda^2 + \lambda(-e^x - 4) + (2e^x + 3) = 0$$

$$\lambda_{1,2} = \frac{(e^x+4) \pm \sqrt{(e^x+4)^2 - 4 \cdot (2e^x+3)}}{2}$$

$$e^{2x} + \cancel{8e^x} + 16 - \cancel{8e^x} - 12 = e^{2x} + 4$$

$$\lambda_{1,2} = \frac{e^x + 4 \pm \sqrt{e^{2x}+4}}{2}$$

e^x is increasing $\Rightarrow x = 2$

$$\Rightarrow \lambda_{max} = \frac{e^2 + 4 + \sqrt{e^4+4}}{2} \approx 9{,}52 = \beta$$

$$\Rightarrow \lambda_{MIN} = \frac{e^{-2} + 4 - \sqrt{e^{-4}+4}}{2} \approx 1{,}065 = \alpha$$

$x = -2$

$$\nabla f = (2x + e^x - y,\ 2y - x)$$

$$\|\nabla f\| = \sqrt{(2x+e^x-y)^2 + (2y-x)^2} = \sqrt{(2x+e^x-y)(2x+e^x-y) + (2y-x)^2}$$

$$= \sqrt{4x^2 + 2e^x x - 2xy + 2e^x x + e^{2x} - e^x y - 2xy - e^x y + y^2 + 4y^2 - 4xy + x^2}$$

$$= \sqrt{5x^2 - 8xy + 4e^x x + e^{2x} - 2e^x y + 5y^2} \quad \text{(MAXIMIZE ON K)}$$

→ Eval at borders to get max $\|\nabla f\|$

$(2,2)$: $\sqrt{20 - 32 + 8e^2 + e^4 - 4e^2 + 20} = \sqrt{8 + 4e^2 + e^4} = 9{,}6$

$(2,-2)$: $\sqrt{20 + 32 + 8e^2 + e^4 + 4e^2 + 20} = \sqrt{72 - 4e^2 + e^4} = 14{,}67$

$(-2,2)$: $\sqrt{20 + 32 - 8e^{-2} + e^{-4} - 4e^{-2} + 20} < \|\nabla f(2,-2)\|$

$(-2,-2)$: $\sqrt{20 - 32 - 8e^{-2} + e^{-4} + 4e^{-2} + 20} < \|\nabla f(2,-2)\|$

$\underline{\underline{L = 14{,}67}}$

$\alpha = 1{,}07$

$\beta = 9{,}52$

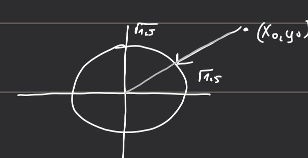
$L = 14{,}67$

③ $\text{proj}_K(x_0, y_0) \Rightarrow \min \sqrt{(x-x_0)^2 + (y-y_0)^2}$

$x^2 + y^2 \leq 1.5$

I) $(x_0, y_0) \in K$: $\text{proj}_K(x_0, y_0) = (x_0, y_0)$

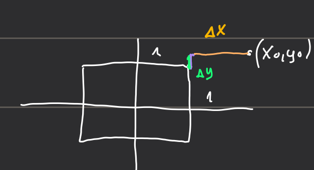
II) Project on circle band:



$(0,0) \to (x_0, y_0)$ + normalized

$\Rightarrow \text{proj}_K(x_0, y_0) = \left(\frac{x_0}{\sqrt{x_0^2+y_0^2}},\ \frac{y_0}{\sqrt{x_0^2+y_0^2}}\right)$ (DIRECTION)

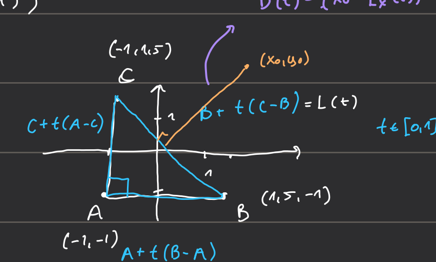
$\Rightarrow \text{proj}_K(x_0, y_0) = \left(\frac{\sqrt{1.5}\, x_0}{\sqrt{x_0^2+y_0^2}},\ \frac{\sqrt{1.5}\, y_0}{\sqrt{x_0^2+y_0^2}}\right)$ (FULL PROJECTION)



$K = [-1,1] \times [-1,1]$ If $(x_0, y_0) \notin K$, clip to $[-1,1] \times [-1,1]$

$$\text{proj}_K(x_0, y_0) = (\underbrace{\text{sign}(x_0)}_{\text{Which side of K is closer}} \cdot \underbrace{\min(|x_0|, 1)}_{\in K},\ \text{sign}(y_0) \cdot \min(|y_0|, 1))$$

$D(t) = (x_0 - L_x(t))^2 + (y_0 - L_y(t))^2$



K = triangle $((-1,-1),\ (1{,}5,-1),\ (-1, 1{,}5))$

$D(t) = (x_0 - L_x(t))^2 + (y_0 - L_y(t))^2$

$= ((x_0 - B_x) - t(C_x - B_x))^2 + ((y_0 - B_y) - t(C_y - B_y))^2$

$= (x_0 - B_x)^2 - 2(x_0 - B_x)t(C_x - B_x) + t^2(C_x - B_x)^2 + (y_0 - B_y)^2 - 2t(y_0 - B_y)(C_y - B_y) + t^2(C_y - B_y)^2$

$\frac{\partial D}{\partial t} = -2(x_0 - B_x)(C_x - B_x) + 2t(C_x - B_x)^2 - 2(y_0 - B_y)(C_y - B_y) + 2t(C_y - B_y)^2 = 0$

$2t((C_x - B_x)^2 + (C_y - B_y)^2) - 2((x_0 - B_x)(C_x - B_x) + (y_0 - B_y)(C_y - B_y))$

$\cancel{2}t((C_x - B_x)^2 + (C_y - B_y)^2) = \cancel{2}((x_0 - B_x)(C_x - B_x) + (y_0 - B_y)(C_y - B_y))$

$$t = \frac{((x_0 - B_x)(C_x - B_x) + (y_0 - B_y)(C_y - B_y))}{(C_x - B_x)^2 + (C_y - B_y)^2} = \frac{(C-B) \cdot ((x_0, y_0) - B)}{\|C - B\|^2}$$

⇒ for all sides

⇒ plug in pts.

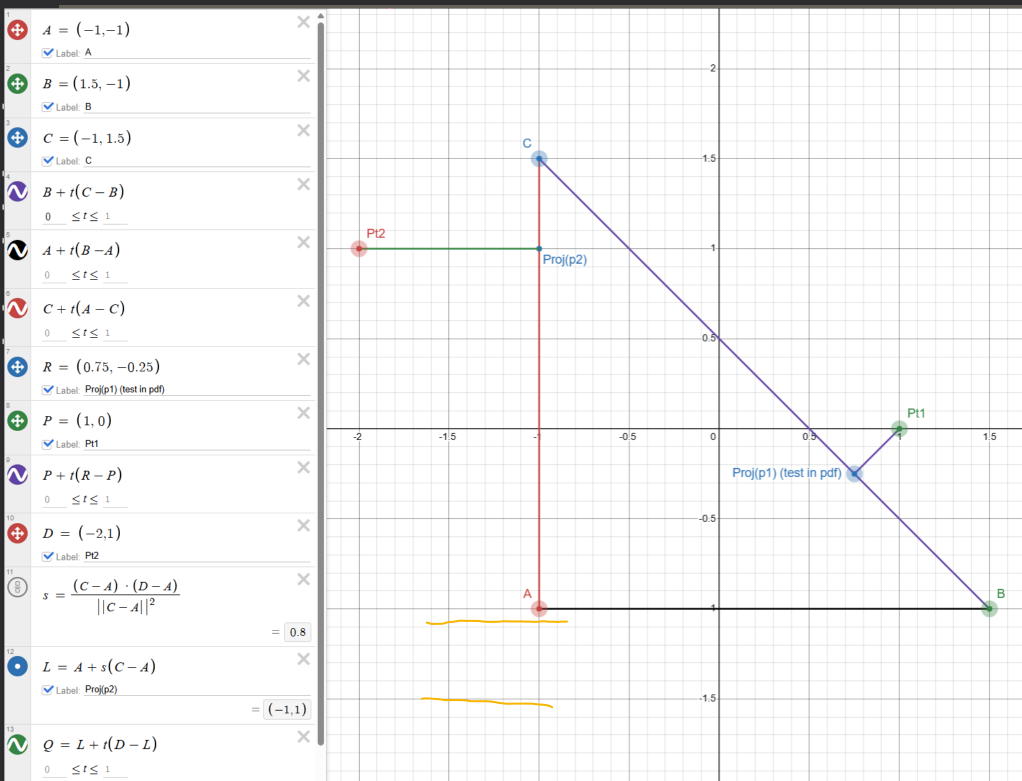
$(x_0, y_0) = (1, 0)$
$C = (-1, 1{,}5)$
$B = (1{,}5, -1)$

$$\Rightarrow t = \frac{\left(\begin{bmatrix}-1\\1{,}5\end{bmatrix} - \begin{bmatrix}1{,}5\\-1\end{bmatrix}\right) \cdot (-0{,}5,\ 1)}{(-2{,}5)^2 + (2{,}5)^2} = \frac{[-2{,}5,\ 2{,}5]\begin{bmatrix}-0{,}5\\1\end{bmatrix}}{12{,}5} = \frac{1{,}25 + 2{,}5}{12{,}5} = \frac{3{,}75}{12{,}5} = 0{,}3$$

$\text{proj}_K(0,1) = (1{,}5, -1) + 0{,}3\,(-2{,}5, 2{,}5) = (1{,}5, -1) + (-0{,}75, 0{,}75)$

$= (0{,}75, -0{,}25)$ ✓

VISUALIZATION



$$\mathrm{proj}_{A,B}(x_0,y_0) = \begin{cases} A; & t \le 0 \\ B; & t \ge 1 \\ A + t(B-A); & t \in [0,1] \end{cases}$$

$$t = \frac{(B-A)^T \cdot ((x_0,y_0) - A)}{\|B-A\|^2}$$

$$\mathrm{proj}_K(x_0,y_0) = \begin{cases} \mathrm{proj}_{A,B}(x_0,y_0); & y_0 \le -1; \ y_0 < x_0 \\ \mathrm{proj}_{A,C}(x_0,y_0); & x_0 \le -1, \ x_0 < y_0 \\ \mathrm{proj}_{B,C}(x_0,y_0) & \text{otherwise} \end{cases}$$

4) $f(x,y) = x^2 + 2y^2 \qquad x_1 = (1,1)$

$\nabla f(x,y) = (2x, 4y) \qquad \nabla f(1,1) = (2,4)$

$\Rightarrow \quad x_2 = (1,1) - \gamma \nabla f(1,1) = (1,1) - \gamma(2,4) = (1-2\gamma, 1-4\gamma)$

$f(x_2) = (1-2\gamma)^2 + 2(1-4\gamma)^2$

$= 1 - 4\gamma + 4\gamma^2 + 2 - 16\gamma + 32\gamma^2 = 36\gamma^2 - 20\gamma + 3 \quad \Rightarrow \text{MIN}$

$\frac{\partial f(x_2)}{\partial \gamma} = 72\gamma - 20$

$\gamma = \frac{20}{72} = \frac{10}{36} = \underline{\underline{\frac{5}{18}}}$

$x_2 = \left(1 - 2 \cdot \frac{5}{18},\ 1 - 4\frac{5}{18}\right) = \left(\frac{4}{9}, -\frac{1}{9}\right)$

$f(x_2) = \frac{16}{81} + \frac{2}{81} = \frac{18}{81} = \frac{2}{9} = \underline{\underline{0{,}22}}$

$f_{min}: \quad \left.\begin{array}{l} 2x = 0 \Rightarrow x = 0 \\ 4y = 0 \Rightarrow y = 0 \end{array}\right\} \quad \begin{array}{l} x_{min} = (0,0) \\ f_{min}(x_{min}) = 0 \end{array}$

b) MIN $d(x^*, x_2)$:

$d(x^*, x_2)^2 = x^{*2} + x_2^2 = \underline{x_2^2} = (1-2\gamma)^2 + (1-4\gamma)^2$

$\frac{\partial d(x_2)^2}{\partial x_2} = -4(1-2\gamma) - 8(1-4\gamma) = 0$

$-4 + 8\gamma - 8 + 32\gamma = 0$

$40\gamma = 12$

$\gamma = \frac{12}{40} = \frac{6}{20} = \frac{3}{10}$

$x_2 = \left(1 - \frac{6}{10},\ 1 - \frac{12}{10}\right) = (0{,}4, -0{,}2)$

$\|x_2\| = \sqrt{0{,}4^2 + 0{,}2^2} = \sqrt{0{,}16 + 0{,}04} = \sqrt{0{,}2} \approx \underline{0{,}447}$

(6) $f(z_1, z_2, z_3) = -\sum_{i=1}^{4} c_i \, e^{\left(-\sum_{j=1}^{3} a_{ij} (z_j - p_{ij})^2\right)}$

$$\frac{\partial f}{\partial z_1} = -\sum^{4} c_i e^{-\sum^{3} a_{ij} (z_j - p_{ij})^2} \left(-a_{i1} (z_1 - p_{i1}) \cdot 2\right)$$

$$\frac{\partial f}{\partial z_2} = -\sum_{i=1}^{4} \left(c_i e^{-\sum^{3} a_{ij} (z_j - p_{ij})^2} - (a_{i2} (z_2 - p_{i2}) \cdot 2)\right)$$

$$\frac{\partial f}{\partial z_3} = -\sum^{4} \left(c_i e^{-\sum^{3} a_{ij} (z_j - p_{ij})^2} (-a_{i3} (z_3 - p_{i3}) \cdot 2)\right)$$

$$\Rightarrow \frac{\partial f}{\partial z_k} = -\sum^{4} -2 c_i a_{ik} (z_k - p_{i,k}) \, e^{-\sum_{j=1}^{3} a_{ij} (z_j - p_{ij})^2}$$