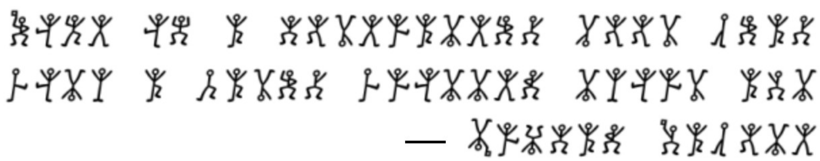
STA3020 Assignment 3.

[1]*School of Data Science, The Chinese University of Hong Kong, Shenzhen (CUHK-Shenzhen)*



1. UMVUE & Cramér-Rao Lower Bound

•*Exercise* 1.1. If δ_1 and δ_2 are UMVUE of $g_1(\theta)$ and $g_2(\theta)$, respectively, then please proof that $a_1\delta_1 + a_2\delta_2$ also has finite variance and is the UMVUE of $a_1g_1(\theta) + a_2g_2(\theta)$, for any real and fixed a_1 and a_2. Further, if $\delta_1\delta_2$ has finite variance, then $a_3\delta_1\delta_2$ is the UMVUE of $a_3\mathbb{E}\delta_1\delta_2$ for any real and fixed constant a_3.

•*Exercise* 1.2. Let $X_1, \dots, X_n$ be i.i.d random variables, where we have each $X_i \sim$Binomial(k, p) with k being known, n sufficiently large and p is the parameter of interests.

1. Please find the UMVUE for p^r, where $r \in \mathbb{N}^+$.
2. Please find the UMVUE for $\mathbb{P}(X_1 = x)$, where $x \in \mathbb{N}, 0 \le x \le k$.
3. Please find the UMVUE for e^{-p}.
4. Please find the UMVUE of $\mathbb{P}(X_1 + \cdots + X_m = X_{m+1} + \cdots + X_{2m})$ for some positive integer m such that $2m \le n$.

•*Exercise* 1.3. Let $X_1, \dots, X_n \overset{i.i.d.}{\sim} U(\theta_1 - \theta_2, \theta_1 + \theta_2)$ with $\theta_1 \in \mathbb{R}$, $\theta_2 > 0$ and n being sufficiently large, please find the UMVUE of θ_1, θ_2 and θ_1/θ_2.

•*Exercise* 1.4. Let $X_1, \dots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma^2)$ with unknown μ and a known σ^2. Please prove that, for an arbitrary polynomial of μ, i.e.,

$$f(\mu) = \sum_{r=0}^{k} a_r \mu^{k-r} \tag{1.1}$$

with known and fixed integer k and constants $a_0, a_1, \cdots, a_k$, the UMVUE of $f(\mu)$ is a polynomial of $\bar{X}$ and σ, i.e., please prove that the UMVUE of $f(\mu)$

can be represented as

$$\tilde{f}(\bar{X},\sigma)=\sum_{i=0}^{k}\sum_{j=0}^{k} b_{ij}\cdot\left(\bar{X}\right)^{i}\cdot\left(\sigma\right)^{j},$$

for some coefficients $\{b_{ij},0\le i,j\le k\}$, where the coefficients $\{b_{ij},0\le i,j\le k\}$ depends on $\{a_0,a_1,\cdots,a_k\}$ but not $\bar{X}$ and σ.

Hint.
1. First look at the UMVUE of μ^k for some fixed k, to see whether it can be represented as a polynomial of $\bar{X}$ and σ, denote it as $g_k(\bar{X},\sigma)=\sum_{i=0}^{k}\sum_{j=0}^{k} c_{ij}\cdot\left(\bar{X}\right)^{i}\cdot\left(\sigma\right)^{j}$.
2. Use mathematical induction on the order k and use the conclusion of Exercise.1.1.
3. Use the fact that, when define $Y=\bar{X}-\mu$, we have $Y\sim N(0,\sigma^2/n)$, and

$$\mathbb{E}\left(\bar{X}^k\right)=\sum_{r=0}^{k}\binom{k}{r}\mu^{k-r}\mathbb{E}(Y^r),$$

$$\text{where}\quad \mathbb{E}(Y^r)=\begin{cases}(r-1)!!\cdot\left(\dfrac{\sigma^2}{n}\right)^{r/2}, & \text{when } r \text{ is even.}\\ 0, & \text{when } r \text{ is odd.}\end{cases}$$

□

•*Exercise* 1.5. Suppose that X has the Poisson distribution truncated on the right at a, so that it has the conditional distribution of Y given $Y\le a$, where Y is distributed as $P(\lambda)$. Show that λ does not have an unbiased estimator.

•*Exercise* 1.6. Let $\mathcal{F}$ be the class of all univariate distribution functions F that have a probability density function f and finite mth moment.

(a) Let $X_1,\ldots,X_n$ be independently distributed with common distribution $F\in\mathcal{F}$. For $n\ge m$, find the UMVUE of ξ^m, where $\xi=\xi(F)=EX_i$.
(b) Show that for the case that $P(X_i=1)=p$, $\quad P(X_i=0)=q$, $\quad p+q=1$, the estimator of (a) reduced to $\delta(T)=\frac{T(T-1)\cdots(T-m+1)}{n(n-1)(n-m+1)}$, where $T=\sum X_i$.