MIT 8.05 - Quantum Physics II

1 *Problem Set 1 - Solution to Problem 1*

Problem Statement A particle of mass m in a one-dimensional potential $V(x)$ has the wave function:

$$\psi(x) = Nx \exp\left(-\frac{1}{2}\alpha x^2\right), \quad \alpha > 0$$

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(a) Normalization and Expectation Values of x

1. Normalization:

To find N, we use the normalization condition $\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$:

$$|N|^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = 1$$

Using the standard Gaussian integral $\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx = \frac{\sqrt{\pi}}{2\alpha^{3/2}}$, we get:

$$|N|^2 \frac{\sqrt{\pi}}{2\alpha^{3/2}} = 1 \implies N = \left(\frac{4\alpha^3}{\pi}\right)^{1/4}$$

*2. Expectation value $\langle \hat{x} \rangle$:*ˆ

$$\langle \hat{x} \rangle = \int_{-\infty}^{\infty} \psi^* x \psi \, dx = |N|^2 \int_{-\infty}^{\infty} x^3 e^{-\alpha x^2} dx$$

Since the integrand is an odd function over a symmetric interval, $\langle \hat{x} \rangle = 0$.

*3. Expectation value $\langle \hat{x}^2 \rangle$:*** Using the integral $\int_{-\infty}^{\infty} x^4 e^{-\alpha x^2} dx = \frac{3\sqrt{\pi}}{4\alpha^{5/2}}$:

$$\langle \hat{x}^2 \rangle = |N|^2 \frac{3\sqrt{\pi}}{4\alpha^{5/2}} = \left(\frac{2\alpha^{3/2}}{\sqrt{\pi}}\right)\left(\frac{3\sqrt{\pi}}{4\alpha^{5/2}}\right) = \frac{3}{2\alpha}$$

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(b) Expectation Values of p

*1. Expectation value $\langle \hat{p} \rangle$:*** For any real wavefunction, $\langle \hat{p} \rangle = 0$. Mathematically, the integrand is odd.

*2. Expectation value $\langle \hat{p}^2 \rangle$:*** Using $\langle \hat{p}^2 \rangle = -\hbar^2 \int \psi \frac{d^2\psi}{dx^2} dx$:

$$\frac{d\psi}{dx} = N(1 - \alpha x^2) e^{-\frac{1}{2}\alpha x^2}$$

$$\frac{d^2\psi}{dx^2} = N(\alpha^2 x^3 - 3\alpha x) e^{-\frac{1}{2}\alpha x^2} = (\alpha^2 x^2 - 3\alpha)\psi(x)$$

Evaluating the integral:

$$\langle \hat{p}^2 \rangle = \frac{3}{2}\hbar^2 \alpha$$

(c) Eigenstate Verification
*Position:$\hat{x}\psi(x) = x\psi(x) \propto x^2 e^{-\cdots} \neq \lambda\psi(x)$. Not an eigenstate. *Momentum: $\hat{p}\psi(x) = -i\hbar\frac{d\psi}{dx} \propto (1 - \alpha x^2)e^{-\cdots} \neq \lambda\psi(x)$. Not an eigenstate.

(d) Expectation value of $\hat{H}$ if $V(x) = 0$ If $V(x) = 0$, then $\hat{H} = \frac{\hat{p}^2}{2m}$:

$$\langle \hat{H} \rangle = \frac{\langle \hat{p}^2 \rangle}{2m} = \frac{3\hbar^2\alpha}{4m}$$

(e) Determining Potential $V(x)$ and Energy E Assuming $\psi(x)$ is an energy eigenstate:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

Substituting $\frac{d^2\psi}{dx^2} = (\alpha^2 x^2 - 3\alpha)\psi$:

$$-\frac{\hbar^2}{2m}(\alpha^2 x^2 - 3\alpha) + V(x) = E$$

$$V(x) = E - \frac{3\hbar^2\alpha}{2m} + \frac{\hbar^2\alpha^2}{2m}x^2$$

Given $V(0) = 0$:

$$0 = E - \frac{3\hbar^2\alpha}{2m} \implies E = \frac{3\hbar^2\alpha}{2m}$$

Thus, the potential is a Harmonic Oscillator:

$$V(x) = \frac{\hbar^2\alpha^2}{2m}x^2$$

!!! Note: $\psi(x)$ is the first excited state ($n = 1$) because it has one node at $x = 0$.

2 *Part 2: Energy vs Potential Minimum*

!Theorem: For any normalizable state in a one-dimensional potential $V(x)$, the energy E must satisfy $E > V_{\min}$.

!Proof: Consider the time-independent Schrödinger equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x)$$

The expectation value of the Hamiltonian $\hat{H}$ is:

$$\langle H \rangle = \int_{-\infty}^{\infty} \psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi \, dx = E$$

Using integration by parts on the kinetic energy term:

$$\int_{-\infty}^{\infty} \psi^* \left(-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}\right) dx = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \left|\frac{d\psi}{dx}\right|^2 dx$$

Thus:

$$E = \frac{\hbar^2}{2m}\int_{-\infty}^{\infty}\left|\frac{d\psi}{dx}\right|^2 dx + \int_{-\infty}^{\infty} V(x)|\psi(x)|^2 dx$$

Since $|\psi(x)|^2$ is a probability density:

$$\int_{-\infty}^{\infty} V(x)|\psi(x)|^2 dx \geq V_{\min}\int_{-\infty}^{\infty} |\psi(x)|^2 dx = V_{\min}$$

And since $\int |d\psi/dx|^2 dx > 0$ (because a normalizable $\psi(x)$ cannot be constant everywhere):

$$E > V_{\min}$$

The equality $E = V_{\min}$ is only possible if the kinetic energy is zero, which implies $\psi(x)$ is constant, but a constant function is not normalizable over $(-\infty, \infty)$. Therefore, E must be strictly greater than $V_{\min}$.