Math 10

Lesson 1-6 Radicals and Fractional Exponents

I. Fractional (rational) exponents

In previous grades we learned that rather than write 4 · 4 · 4 · 4 · 4 we could write it as 4^5. In addition we learned that when we multiply two powers with the same base we add the exponents together.

$$4^5 \cdot 4^4 = 4^{5+4} = 4^9$$

In general

$$x^n \cdot x^m = x^{n+m}$$

Further, we can evaluate 4^9 in a calculator. $4^9 = 262144$
Now, what if we evaluate numbers that involve a fractional exponent like ½ for example? Using your calculator, calculate $4^{\frac{1}{2}}$, $9^{\frac{1}{2}}$ and $16^{\frac{1}{2}}$.

$$4^{\frac{1}{2}} = 2 \qquad 9^{\frac{1}{2}} = 3 \qquad 16^{\frac{1}{2}} = 4$$

Recall that

$$\sqrt{4} = 2 \qquad \sqrt{9} = 3 \qquad \sqrt{16} = 4$$

It turns out that a **fractional exponent is related to the root of a number**.

$$\sqrt{4} = 4^{\frac{1}{2}} \qquad \sqrt{9} = 9^{\frac{1}{2}} \qquad \sqrt{16} = 16^{\frac{1}{2}}$$

Other examples are

$$\sqrt{6} = 6^{\frac{1}{2}} \qquad \sqrt[4]{15} = 15^{\frac{1}{4}} \qquad \sqrt[3]{7} = 7^{\frac{1}{3}}$$

In general

$$x^{\frac{1}{n}} = \sqrt[n]{x}$$

which is how it is written on your formula sheet.

Example 1 Write the following radicals in exponent form.

$$\sqrt{5} \qquad 4\sqrt[3]{7} \qquad 3\sqrt[4]{23}$$

The index of each radical becomes the denominator of the fraction.

$$\sqrt{5} = 5^{\frac{1}{2}} \qquad 4\sqrt[3]{7} = 4 \cdot 7^{\frac{1}{3}} \qquad 3\sqrt[4]{23} = 3 \cdot 23^{\frac{1}{4}}$$

Example 2 Write the following radicals in exponent form.

$$\sqrt{3^3} \qquad 2\left(\sqrt[3]{6}\right)^2 \qquad 3\sqrt[4]{11}^7$$

The index of each radical becomes the denominator of the rational number and the power becomes the numerator of the rational number.

$$\sqrt{3^3} = 3^{\frac{3}{2}} \qquad 2\left(\sqrt[3]{6}\right)^2 = 2\cdot 6^{\frac{2}{3}} \qquad 3\sqrt[4]{11}^7 = 3\cdot 11^{\frac{7}{4}}$$

Example 3 Write the following exponents in radical form.

$$5^{\frac{1}{2}} \qquad 2\cdot 3^{\frac{1}{4}} \qquad 6\cdot 7^{\frac{5}{3}} \qquad 13\cdot 9^{\frac{2}{5}}$$

The denominator of the rational number becomes the index of the radical and the numerator of the rational number becomes the power.

Note there are many ways to write the exponent: inside the radical, outside the radical, or using brackets around the radical.

$$5^{\frac{1}{2}} = \sqrt{5} \qquad 2\cdot 3^{\frac{1}{4}} = 2\sqrt[4]{3} \qquad 6\cdot 7^{\frac{5}{3}} = 6\left(\sqrt[3]{7}\right)^5 \qquad 13\cdot 9^{\frac{2}{5}} = 13\sqrt[5]{9}^2$$

In general, when we combine the fraction rule $x^{\frac{1}{n}} = \sqrt[n]{x}$ with the power rule $\left(x^m\right)^n = x^{m\cdot n}$ we get the following

$$x^{\frac{m}{n}} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m = \left(x^m\right)^{\frac{1}{n}} = \left(x^{\frac{1}{n}}\right)^m$$

In other words we can write roots of powers and powers of roots in a multitude of ways.

Question 1

a) Write $\sqrt[3]{18}$ in exponent form.

b) Write $14^{\frac{1}{4}}$ as a radical.

c) Write $3\sqrt[4]{6}$ in exponent form.

b) Write $5\cdot 4^{\frac{1}{3}}$ as a radical.

e) Write $\sqrt[3]{3^2}$ in exponent form.

f) Write $5^{0.75}$ as a radical.

Example 4 Evaluate each of the following.

$25^{\frac{1}{2}}$ $\quad 16^{\frac{3}{4}}$ $\quad \left(\frac{25}{49}\right)^{\frac{3}{2}}$

When evaluating a number with a rational exponent we separate the root (denominator) from the power (numerator).

$$25^{\frac{1}{2}} = \mathbf{5}$$

$$16^{\frac{3}{4}}$$

First, rewrite the fraction as two parts: a root and a power.

$$= \left(16^{\frac{1}{4}}\right)^3$$

The fourth root of 16 is 2.

$$= 2^3$$

2 to the 3rd power is 8.

$$= \mathbf{8}$$

$$\left(\frac{25}{49}\right)^{\frac{3}{2}}$$

First, rewrite the fraction as two parts: a root and a power.

$$= \left(\left(\frac{25}{49}\right)^{\frac{1}{2}}\right)^3$$

$$= \left(\frac{25^{\frac{1}{2}}}{49^{\frac{1}{2}}}\right)^3$$

Find the roots of the numerator and denominator separately.

$$= \left(\frac{5}{7}\right)^3$$

$$= \frac{5^3}{7^3}$$

Evaluate the powers of the numerator and denominator separately.

$$= \mathbf{\frac{125}{343}}$$

Example 5 Evaluate $\left(-\frac{27}{8}\right)^{\frac{2}{3}}$

Solution:

$$\left(-\frac{27}{8}\right)^{\frac{2}{3}} = \left(\left(\frac{-27}{8}\right)^{\frac{1}{3}}\right)^2$$

Rewrite the fraction as two parts: a root and a power.

$$= \left(\frac{(-27)^{\frac{1}{3}}}{8^{\frac{1}{3}}}\right)^2$$

Find the roots of the numerator and denominator separately.

$$= \left(\frac{-3}{2}\right)^2$$

$$= \frac{(-3)^2}{2^2}$$

Evaluate the powers of the numerator and denominator separately.

$$= \mathbf{\frac{9}{4}}$$

<u>Important note</u>:
The minus sign in front of the fraction can be left where it is or it can be moved to the numerator or denominator, whichever is most convenient. The following versions all mean the same thing:

$$\left(-\frac{27}{8}\right)^{\frac{2}{3}} = \left(\frac{-27}{8}\right)^{\frac{2}{3}} = \left(\frac{27}{-8}\right)^{\frac{2}{3}}$$

Question 2

Evaluate each of the following.

$100^{\frac{1}{2}}$ $1000^{\frac{1}{3}}$ $\left(\frac{4}{9}\right)^{\frac{1}{2}}$ $\left(\frac{9}{16}\right)^{\frac{3}{2}}$

II. Nasty question of the day

Consider $\sqrt[3]{343^4}$

Rewrite this expression in four other ways and check to see if they all evaluate to the same result.

III. Assignment

1. Evaluate each power without using a calculator.
   a) $16^{\frac{1}{2}}$ c) $64^{\frac{1}{3}}$
   d) $32^{\frac{1}{5}}$ f) $(-1000)^{\frac{1}{3}}$

2. Evaluate each power without using a calculator.
   a) $100^{0.5}$ b) $81^{0.25}$
   c) $1024^{0.2}$ d) $(-32)^{0.2}$

3. Write each power as a radical.
   a) $36^{\frac{1}{3}}$ b) $48^{\frac{1}{2}}$ c) $(-30)^{\frac{1}{5}}$

4. Write each radical as a power.
   a) $\sqrt{39}$ b) $\sqrt[4]{90}$
   c) $\sqrt[3]{29}$ d) $\sqrt[5]{100}$

5. Evaluate each power without using a calculator.
   a) 8^0 b) $8^{\frac{1}{3}}$ c) $8^{\frac{2}{3}}$
   d) $8^{\frac{3}{3}}$ e) $8^{\frac{4}{3}}$ f) $8^{\frac{5}{3}}$

6. Write each power as a radical.
   a) $4^{\frac{2}{3}}$ b) $(-10)^{\frac{3}{5}}$ c) $2.3^{\frac{3}{2}}$

7. A cube has a volume of 350 cm^3. Write the edge length of the cube as a radical and as a power.

8. Write each power as a radical.

d) $0.75^{0.75}$ e) $\left(-\frac{5}{9}\right)^{\frac{2}{5}}$ f) $1.25^{1.5}$

9. Write each radical as a power.

a) $\sqrt{3.8^3}$ b) $=\left(\sqrt[3]{-1.5}\right)^2$ c) $\sqrt[4]{\left(\frac{9}{5}\right)^5}$

10. Evaluate each power without using a calculator.

a) $9^{\frac{3}{2}}$ b) $\left(\frac{27}{8}\right)^{\frac{2}{3}}$ c) $(-27)^{\frac{2}{3}}$

11. Write an equivalent form for each number using a power with exponent 1/2, then write the answer as a radical.
a) 2 b) 4 c) 10 d) 3 e) 5

12. Write an equivalent form for each number using a power with exponent 1/3, then write the answer as a radical.
a)-1 b) 2 c) 3 d)-4 e) 4

13. Arrange these numbers in order from least to greatest. Describe your strategy.

$\sqrt[3]{4}, 4^{\frac{3}{2}}, 4^2, \left(\frac{1}{4}\right)^{\frac{3}{2}}$

14. The height, *h* metres, of a certain species of fir tree can be estimated from the formula $h = 35d^{\frac{2}{3}}$, where *d* is the diameter at the base in metres. Use the formula to determine the approximate height of a fir tree with base diameter 3.2 m.

15. Here is a student's solution for evaluating a power.

$$\begin{aligned} 1.96^{\frac{3}{2}} &= \left(\sqrt[3]{1.96}\right)^2 \\ &= (1.2514)^2 \\ &= 1.5661... \end{aligned}$$

Identify the errors the student made. Write a correct solution.

16. A formula for the approximate surface area, *SA* in square metres, of a person's body is *SA* = $0.096m^{0.7}$, where *m* is the person's mass in kilograms. Calculate the surface area of a child with mass 40 kg.

17. Two students discussed the meaning of the statement $3.2^{4.2} = 132.3213...$
Luc said: It means 3.2 multiplied by itself 4.2 times is about 132.3213.
Karen said: No, you can't multiply a number 4.2 times. $3.2^{4.2}$ can be written as $3.2^{\frac{42}{10}}$. So the statement means that 42 factors, each equal to the tenth root of 3.2, multiplied together will equal about 132.3213.
Which student is correct? Explain.