Math 10

Lesson 1-7 Negative Exponents/Reciprocals

I. Negative exponents

Consider the table to the right. Notice the patterns that emerge. As we decrease the power of 2 from 4 to 3 to 2 to 1, we eventually have a power of 0 and then negative powers –1 to –2 to –3 to –4. In addition, note that as the negative powers get larger the actual number gets smaller and smaller.

Number	Evaluate Number	Number in exponent form
$2\cdot2\cdot2\cdot2$	16	2^4
$2\cdot2\cdot2$	8	2^3
$2\cdot2$	4	2^2
2	2	2^1
1	1	2^0
$\frac{1}{2}$	$\frac{1}{2}$	2^{-1}
$\frac{1}{2\cdot2}$	$\frac{1}{4}$	2^{-2}
$\frac{1}{2\cdot2\cdot2}$	$\frac{1}{8}$	2^{-3}
$\frac{1}{2\cdot2\cdot2\cdot2}$	$\frac{1}{16}$	2^{-4}

Now consider the following fraction

$$\frac{1}{3^4}$$

Another way to write it is

$$3^{-4}$$

Likewise, we can rewrite

$$\frac{1}{5^{-3}}$$

as

$$5^3$$

The general rule can be written as

$$x^{-n}=\frac{1}{x^n} \quad \text{and} \quad \frac{1}{x^{-n}}=x^n,\ x\neq 0$$

Another way to think about it is that x^{-n} **is the reciprocal of** x^n. For example, the expression

$$\left(\frac{3}{2}\right)^{-5}$$

can be rewritten as

$$\left(\frac{2}{3}\right)^{5}$$

The key is to flip the numerator and denominator and change the sign of the power.

Example 1 Evaluate each power.

$$7^{-2} \qquad \left(\frac{10}{3}\right)^{-3} \qquad (-1.5)^{-3}$$

Solution:

$$\begin{array}{lll}
7^{-2} & \left(\frac{10}{3}\right)^{-3} & (-1.5)^{-3} \\
=\frac{1}{7^2} & =\left(\frac{3}{10}\right)^{3} & =-0.2963 \leftarrow \text{use calculator} \\
=\frac{1}{49} & =\frac{3^3}{10^3} & \\
 & =\frac{27}{1000} &
\end{array}$$

Question 1

Evaluate each power

$$7^{-2} \qquad \frac{1}{4^{-2}} \qquad \left(\frac{-6}{5}\right)^{-3}$$

We can now combine negative exponents with the rational exponents that we learned about in Lesson 1-6. Consider, for example, $6^{-\frac{2}{3}}$. The meaning of each part of the exponent is shown to the right.

reciprocal

$$6^{-\frac{2}{3}} \quad \begin{array}{l} 2 \leftarrow \text{power} \\ 3 \leftarrow \text{root} \end{array}$$

Since the exponent $-\frac{2}{3}$ is the product $(-1)\left(-\frac{1}{3}\right)(2)$ and the order of multiplication does not matter, **we can apply the three operations of reciprocal, root and power in any order**. Consider the following example.

Example 2 Evaluate $\left(\frac{9}{16}\right)^{-\frac{3}{2}}$.

We can apply the operations of reciprocal, power and root in any order. Therefore, we can do this problem in several ways:

$$\begin{aligned}\left(\frac{9}{16}\right)^{-\frac{3}{2}} &= \left(\frac{16}{9}\right)^{\frac{3}{2}} && \text{reciprocal}\\ &= \left(\left(\frac{16}{9}\right)^{\frac{1}{2}}\right)^{3} && \text{square root}\\ &= \left(\frac{4}{3}\right)^{3} && \text{cube}\\ &= \frac{64}{27}\end{aligned}$$

$$\begin{aligned}\left(\frac{9}{16}\right)^{-\frac{3}{2}} &= \left(\left(\frac{9}{16}\right)^{\frac{1}{2}}\right)^{-3} && \text{square root}\\ &= \left(\frac{3}{4}\right)^{-3} && \text{cube}\\ &= \left(\frac{27}{64}\right)^{-1} && \text{reciprocal}\\ &= \frac{64}{27}\end{aligned}$$

$$\begin{aligned}\left(\frac{9}{16}\right)^{-\frac{3}{2}} &= \left(\frac{16}{9}\right)^{\frac{3}{2}} && \text{reciprocal}\\ &= \left(\left(\frac{16}{9}\right)^{3}\right)^{\frac{1}{2}} && \text{cube}\\ &= \left(\frac{4096}{729}\right)^{\frac{1}{2}} && \text{square root}\\ &= \frac{64}{27}\end{aligned}$$

The order that you decide on for a particular problem should be based on what looks easiest to do to <u>you</u>. There is <u>not</u> one way or one order of steps that works well for <u>all</u> problems.

Question 2

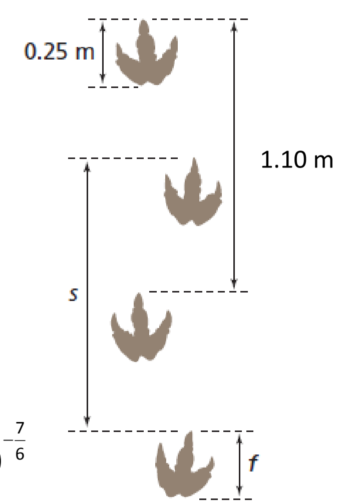
Evaluate each power

$16^{-\frac{5}{4}}$ $\qquad$ $\left(\frac{125}{216}\right)^{-\frac{2}{3}}$

Example 3 Palaeontologists use measurements from fossilised dinosaur tracks and the formula

$$v = 0.155 s^{\frac{5}{3}} f^{-\frac{7}{6}}$$

to estimate the speed at which a dinosaur travelled. In the formula, v is the speed in metres per second, s is the stride length, and f is the foot length in metres.
Use the measurements in the diagram to the right to estimate the speed of the dinosaur.



Solution:
Substitute s = 1.10 and f = 0.25 into $v = 0.155 s^{\frac{5}{3}} f^{-\frac{7}{6}}$

$v = 0.155(1.10)^{\frac{5}{3}}(0.25)^{-\frac{7}{6}}$

$v = 0.9156359...$

v = 0.92

Example 4

Food manufacturers use a beneficial bacterium called *Lactobacillus bulgaricus* to make yoghurt and cheese. The growth of bacteria can be modelled using the formula

$$N = N_o(2)^{\frac{h}{42}}$$

where N_o is the initial number of bacteria and N is the number of bacteria after h hours. If the initial number of bacteria was 10 000

a) How many bacteria are present after 42 h?
b) How many more bacteria are present after 2 h?
c) How many bacteria are present after 105 h?

Solution

a) Substitute the values N_o = 10 000 and h = 42 into the formula and evaluate.

$$N = N_o(2)^{\frac{h}{42}}$$
$$N = 10000(2)^{\frac{42}{42}}$$
$$N = 10000(2)^{1}$$
$$N = 20000$$

There are 20 000 bacteria after 42 h.

b) Substitute the values N_o = 10 000 and h = 2 into the formula and evaluate.

$$N = N_o(2)^{\frac{h}{42}}$$
$$N = 10000(2)^{\frac{2}{42}}$$
$$N = 10335.58...$$

10 336 – 10 000 = 336

There are approximately 336 more bacteria after 2 h.

c) Substitute the values N_o = 10 000 and h = 105 into the formula and evaluate.

$$N = N_o(2)^{\frac{h}{42}}$$
$$N = 10000(2)^{\frac{105}{42}}$$
$$N = 56568.54...$$

There are approximately 56 569 bacteria after 105 h.

Question 3

Using the formula given in Example 3, calculate the speed of a dinosaur that has a stride length of 1.5 m and a foot length of 0.30 m.

II. Assignment

1. Complete each equation.

   a) $\frac{1}{5^4} = 5^{\square}$ d) $\frac{1}{4^{-2}} = 4^{\square}$

2. Evaluate the powers in each pair without a calculator.

   a) 4^2 and 4^{-2} b) 2^4 and 2^{-4}

   Describe what is similar about the answers, and what is different.

3. Given that $2^{10} = 1024$, what is 2^{-10}?

4. Write each power with a positive exponent.

   a) $\left(\frac{1}{2}\right)^{-2}$ c) $\left(-\frac{6}{5}\right)^{-4}$

5. Evaluate each power without using a calculator.

   a) 3^{-2} b) 2^{-4} c) $(-2)^{-5}$

   d) $\left(\frac{1}{3}\right)^{-3}$ e) $\left(-\frac{2}{3}\right)^{-2}$ f) $\frac{1}{5^{-3}}$

6. Evaluate each power without using a calculator.

   a) $4^{-\frac{1}{2}}$ b) $0.09^{-\frac{1}{2}}$

   c) $27^{-\frac{1}{3}}$ d) $(-64)^{-\frac{1}{3}}$

7. Use a power with a negative exponent to write an equivalent form for each number.

   a) $\frac{1}{9}$ b) $\frac{1}{5}$ c) 4 d) -3

8. When you save money in a bank, the bank pays you *interest*. This interest is added to your investment and the resulting amount also earns interest. We say the interest *compounds*. Suppose you want an amount of $3000 in 5 years. The interest rate for the savings account is 2.5% compounded annually. The money, *P* dollars, you must invest now is given by the formula:

   $P = 3000(1.025)^{-5}$.

   How much must you invest now to have $3000 in 5 years?

9. Evaluate each power without using a calculator.

   a) $27^{-\frac{4}{3}}$ b) $16^{-1.5}$ c) $32^{-0.4}$

   d) $\left(-\frac{8}{27}\right)^{-\frac{2}{3}}$ e) $\left(\frac{81}{16}\right)^{-\frac{3}{4}}$ f) $\left(\frac{9}{4}\right)^{-\frac{5}{2}}$

10. Here is a student's solution for evaluating a power. Identify any errors in the solution. Write a correct solution.

$$\left(-\frac{64}{125}\right)^{-\frac{5}{3}} = \left(\frac{64}{125}\right)^{\frac{5}{3}}$$
$$= \left(\sqrt[3]{\frac{64}{125}}\right)^{5}$$
$$= \left(\frac{4}{5}\right)^{5}$$
$$= \frac{1024}{3125}$$

11. Michelle wants to invest enough money on January 1st to pay her nephew \$150 at the end of each year for the next 10 years. The savings account pays 3.2% compounded annually. The money, P dollars, which Michelle must invest today, is given by the formula
$$P = \frac{150\left[1-1.032^{-10}\right]}{0.032}$$
How much must Michelle invest on January 1st?

12. The intensity of light at its source is 100%. The intensity, I, at a distance d centimetres from the source is given by the formula $I = 100d^{-2}$. Use the formula to determine the intensity of the light 23 cm from the source.

13. Which is greater, 2^{-5} or 5^{-2}? Verify your answer.

14.
   a) Identify the patterns in this list.
      $16 = 2^4$
      $8 = 2^3$
      $4 = 2^2$
   b) Extend the patterns in part a) downward. Write the next 5 rows in the pattern.
   c) Explain how this pattern shows that $a^{-n} = \frac{1}{a^n}$.

15. How many times as great as 3^{-5} is 3^3? Express your answer as a power and in standard form.

16. What do you know about the sign of the exponent in each case? Justify your answers.
   a) $3^x > 1$ b) $3^x < 1$ c) $3^x = 1$

17. A number is raised to a negative exponent. Is it always true that the value of the power will be less than 1? Use an example to explain.

18. There is a gravitational force F between Earth and the moon. This force is given by the formula $F = (6.67 \times 10^{-11})Mmr^{-2}$ where M is the mass of Earth in kilograms, m is the mass of the moon in kilograms, and r is the distance between Earth and the moon in metres. The mass of Earth is approximately 5.9736×10^{24} kg. The mass of the moon is approximately 7.349×10^{22} kg. The mean distance between them is approximately 382 260 000 m. What is the gravitational force between Earth and the moon?

19. Julia is a veterinarian who needs to determine the remaining concentration of a particular drug in a horse's bloodstream. She can model the concentration using the formula

$$C = C_o\left(\frac{1}{2}\right)^{\frac{t}{4}}$$

where C is an estimate of the remaining concentration of drug in the bloodstream in milligrams per millilitre of blood, C_0 is the initial concentration, and t is the time in hours that the drug is in the bloodstream. At 10:15 a.m. the concentration of drug in the horse's bloodstream was 40 mg/mL.

a) If only a single dose of the drug is given, what will the approximate concentration of the drug be 6 h later?

b) Julia needs to administer a second dose of the drug when the concentration in the horse's bloodstream is down to 10 mg/mL. Estimate after how many hours this would occur.

