Cheatsheet - Course Name

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1 Sequences

1.1 Convergence

A sequence $(a_n)_{n\in\mathbb{N}}$ converges to L if and only if

$$\lim_{n\to\infty} a_n = L$$

if and only if

$$\forall\epsilon > 0,\ \exists N_\epsilon,\ \forall n \geq N_\epsilon :\ |a_n - L| < \epsilon$$

We may assume (without loss of generality) that ϵ is bounded by a constant $C \in \mathbb{R}$. Additionally, the following holds:

- Convergent $\implies$ bounded, but not vice versa.
- (a_n) is convergent $\iff$ (a_n) is bounded **and**

$$\liminf a_n = \limsup a_n$$

Limit Superior & Inferior

$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty}\left(\inf_{m\geq n} x_m\right)$$

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty}\left(\sup_{m\geq n} x_m\right)$$

Squeeze Theorem (Sandwich Theorem)

If $\lim_{n\to\infty} a_n = \alpha$, $\lim_{n\to\infty} b_n = \alpha$, and $a_n \leq c_n \leq b_n$ for all $n \geq k$, then $\lim_{n\to\infty} c_n = \alpha$.

Weierstrass Theorem

If a_n is monotonically increasing and bounded above, then a_n converges with the limit $\lim_{n\to\infty} a_n = \sup\{a_n : n \geq 1\}$.
If a_n is monotonically decreasing and bounded below, then a_n converges with the limit $\lim_{n\to\infty} a_n = \inf\{a_n : n \geq 1\}$.

Cauchy Criterion

The sequence a_n is convergent if and only if $\forall\epsilon > 0,\ \exists N \geq 1$ such that $|a_n - a_m| < \epsilon \quad \forall n, m \geq N$.

1.1.1 Subsequence

A subsequence of a_n is a sequence b_n where $b_n = a_{l(n)}$ and l is a function with $l(n) < l(n+1) \quad \forall n \geq 1$ (e.g., $l = 2n$ for every even index).

1.1.2 Bolzano-Weierstrass Theorem

Every bounded sequence has a convergent subsequence.

1.2 Strategy - Convergence of Sequences

1. For fractions: Simplify by the highest power of n. Eliminate all fractions of the form $\frac{a}{n^a}$, as they approach 0.
2. For roots in the sum in the denominator: Multiply the denominator and numerator by the difference of the sum in the denominator (e.g., multiply $(a+b)$ by $(a-b)$).
3. For recursive sequences: Apply the Weierstrass theorem for monotone convergence.
4. Apply the Squeeze Theorem (Sandwich Theorem).
5. Compare with a known sequence.
6. Determine the limit by simple transformation.
7. Show the limit using the definition of convergence.
8. Apply the Cauchy Criterion.
9. Find a convergent majorant.
10. Cry and skip the problem.

1.3 Strategy - Divergence of Sequences

1. Find a divergent comparison sequence.
2. For alternating sequences: Show that subsequences do not become equal, i.e., $\lim_{n\to\infty} a_{p_1(n)} \neq \lim_{n\to\infty} a_{p_2(n)}$ (e.g., even/odd subsequences).

1.4 Tricks for Limits

1.4.1 Binomials

$$\lim_{x\to\infty}\left(\sqrt{x+4} - \sqrt{x-2}\right) = \lim_{x\to\infty}\frac{(x+4)-(x-2)}{\sqrt{x+4}+\sqrt{x-2}}$$

1.4.2 Substitution

$$\lim_{x\to\infty} x^2\left(1 - \cos\left(\frac{1}{x}\right)\right)$$

Substitute $u = \frac{1}{x}$:

$$\lim_{u\to 0}\frac{1-\cos(u)}{u^2} = \lim_{u\to 0}\frac{\sin(u)}{2u} = \lim_{u\to 0}\frac{\cos(u)}{2} = \frac{1}{2}$$

1.4.3 Inductive Sequences (Induction Trick)

1. Show monotonicity (increasing/decreasing).
2. Show boundedness.
3. Use the Weierstrass theorem, i.e., the sequence must converge to a limit.
4. Use the induction trick:

If the sequence converges, every subsequence has the same limit. Consider the subsequence $l(n) = n+1$ for $d_{n+1} = \sqrt{3d_n - 2}$:

$$d = \lim_{n\to\infty} d_n = \lim_{n\to\infty} d_{n+1} = \sqrt{\lim_{n\to\infty} 3d_n - 2} = \sqrt{3d-2}$$

Rearrange to $d^2 = 3d - 2 \Rightarrow d \in \{1, 2\}$. Now, we can take $d = 2$ and show boundedness with $d = 2$ using induction.