1) $A = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$

a) Sea $X = \begin{bmatrix} x \\ y \end{bmatrix}$ entonces:

$$X^T A X = (x \ \ y)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \left(x + \tfrac{1}{2}y \quad \tfrac{1}{2}x + \tfrac{1}{3}y\right)\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + \tfrac{1}{2}xy + \tfrac{1}{2}xy + \tfrac{1}{3}y^2 =$$

$$= x^2 + xy + \tfrac{1}{4}y^2 - \tfrac{1}{4}y^2 + \tfrac{1}{3}y^2 = \left(x + \tfrac{1}{2}y\right)^2 + \tfrac{1}{12}y^2 > 0 \quad \forall X \in \mathbb{R}^n - \{0\}$$

Además, como A es simétrica $(A = A^T)$ entonces A es definida positiva.

b) $k_1 = e_1 = (1,0)^T$

$$k_2 = e_2 - \mathrm{Proy}_{e_1}(e_2) = e_2 - \frac{\langle e_2, e_1\rangle}{\langle e_1, e_1\rangle} e_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{(1 \ \ 0)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}}{(1 \ \ 0)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$k_2 = (-1/2, 1)^T$

Normalizando:

$$v_1 = \frac{k_1}{\|k_1\|} = \frac{k_1}{\underbrace{(\langle k_1, k_1\rangle)^{1/2}}_{1}} = (1,0)^T$$

$$v_2 = \frac{k_2}{\|k_2\|} = \frac{k_2}{(\langle k_2, k_2\rangle)^{1/2}} = \frac{k_2}{(1/12)^{1/2}} = 2\sqrt{3}\, k_2 = (-\sqrt{3}, 2\sqrt{3})^T$$

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2) a) $x^{(k)} = A x^{(k-1)}$

$$\begin{bmatrix} f_{k+1} \\ f_k \end{bmatrix} = A \begin{bmatrix} f_k \\ f_{k-1} \end{bmatrix}$$

Como $f_{k+1} = f_k + f_{k-1}$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

b) $p(\lambda) = \det(A - \lambda I) = (1-\lambda)(-\lambda) - 1 = \lambda^2 - \lambda - 1$

Busco las raíces de $p(\lambda)$:

$$\frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \left\langle \begin{array}{l} \lambda_1 = \frac{1}{2}(1+\sqrt{5}) \\ \lambda_2 = \frac{1}{2}(1-\sqrt{5}) \end{array} \right\} \text{avas de } A$$

Calculo el ave asociado a λ_1:

$(A - \lambda_1 I)X = 0$

$$\left(\begin{array}{cc|c} \frac{1}{2}(1-\sqrt{5}) & 1 & 0 \\ 1 & -\frac{1}{2}(1+\sqrt{5}) & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} \frac{1}{2}(1-\sqrt{5}) & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = \frac{1}{2}(1+\sqrt{5})\alpha \\ x_2 = \alpha \end{cases} \Rightarrow v_1 = \begin{pmatrix} \frac{1}{2}(1+\sqrt{5}) \\ 1 \end{pmatrix}$$

Calculo el ave asociado a λ_2:

$(A - \lambda_2 I)X = 0$

$$\left(\begin{array}{cc|c} \frac{1}{2}(1+\sqrt{5}) & 1 & 0 \\ 1 & -\frac{1}{2}(1-\sqrt{5}) & 0 \end{array}\right)$$

$$\left(\begin{array}{cc|c} \frac{1}{2}(1+\sqrt{5}) & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{cases} x_1 = \frac{1}{2}(1-\sqrt{5})\alpha \\ x_2 = \alpha \end{cases} \Rightarrow v_2 = \begin{pmatrix} \frac{1}{2}(1-\sqrt{5}) \\ 1 \end{pmatrix}$$

A tiene $n=2$ avas distintos, entonces los ave correspondientes son l.i. Por lo tanto, A es diagonalizable.

c) Teníamos que:

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = A \cdot \begin{pmatrix} f_k \\ f_{k-1} \end{pmatrix} = A \cdot A \cdot \begin{pmatrix} f_{k-1} \\ f_{k-2} \end{pmatrix} = A \cdot A \cdot A \begin{pmatrix} f_{k-2} \\ f_{k-3} \end{pmatrix} = \cdots\cdots = A^{k-1} \begin{pmatrix} f_{k-(k-2)} \\ f_{k-(k-1)} \end{pmatrix} = A^{k-1} \begin{pmatrix} f_2 \\ f_1 \end{pmatrix}$$

$$\begin{pmatrix} f_{k+1} \\ f_k \end{pmatrix} = A^{k-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow f_k = \text{fila 2 de } A^{k+1} \text{ por el vector } (1,1)^T.$$

Como $A = XDX^{-1}$ entonces $A^{k-1} = X D^{k-1} X^{-1}$ con

$$X = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{bmatrix}$$

$$D^{k-1} = \begin{bmatrix} \left[\frac{1}{2}(1+\sqrt{5})\right]^{k-1} & 0 \\ 0 & \left[\frac{1}{2}(1-\sqrt{5})\right]^{k-1} \end{bmatrix}$$

$$X^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix}$$

Luego:

$$A^{k-1} = \begin{bmatrix} \frac{1}{2}(1+\sqrt{5}) & \frac{1}{2}(1-\sqrt{5}) \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} \left[\frac{1}{2}(1+\sqrt{5})\right]^{k-1} & 0 \\ 0 & \left[\frac{1}{2}(1-\sqrt{5})\right]^{k-1} \end{bmatrix} \cdot \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix} \right)$$

$$A^{k-1} = \begin{bmatrix} \frac{1}{2^k}(1+\sqrt{5})^k & \frac{1}{2^k}(1-\sqrt{5})^k \\ \frac{1}{2^{k-1}}(1+\sqrt{5})^{k-1} & \frac{1}{2^{k-1}}(1-\sqrt{5})^{k-1} \end{bmatrix} \cdot \left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix} \right)$$

$$A^{k-1} = \frac{1}{\sqrt{5}\, 2^{k-1}} \begin{bmatrix} \frac{1}{2}(1+\sqrt{5})^k & \frac{1}{2}(1-\sqrt{5})^k \\ (1+\sqrt{5})^{k-1} & (1-\sqrt{5})^{k-1} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{2}(1-\sqrt{5}) \\ -1 & \frac{1}{2}(1+\sqrt{5}) \end{bmatrix}$$

$$A^{k-1} = \frac{1}{\sqrt{5}\cdot 2^{k-1}} \begin{bmatrix} \frac{1}{2}\left[(1+\sqrt{5})^k - (1-\sqrt{5})^k\right] & -\frac{1}{4}\left[(1-\sqrt{5})(1+\sqrt{5})^k - (1+\sqrt{5})(1-\sqrt{5})^k\right] \\ (1+\sqrt{5})^{k-1} - (1-\sqrt{5})^{k-1} & -\frac{1}{2}\left[(1-\sqrt{5})(1+\sqrt{5})^{k-1} - (1+\sqrt{5})(1-\sqrt{5})^{k-1}\right] \end{bmatrix}$$

Luego:

$$f_k = \frac{1}{\sqrt{5}\, 2^{k-1}} \left[(1+\sqrt{5})^{k-1}\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right) + (1-\sqrt{5})^{k-1}\left(-\frac{1}{2} + \frac{\sqrt{5}}{2}\right) \right] = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k \right]$$