4.4 Potencjał grawitacyjny

$$\frac{d^2\Phi}{dx^2} = 4\pi G \rho(x)$$

$$\Phi(0) = 5$$

$$\Phi(3) = 4$$

$$\rho(x) = \begin{cases} 0 & \text{dla } x \in [0,1] \\ 1 & \text{dla } x \in (1,2] \\ 0 & \text{dla } x \in (2,3] \end{cases}$$

Niech v - funkcja testowa, $v \in V$, $v(0)=v(3)=0$

$$\int_0^3 \Phi''(x) v(x) dx = \int_0^3 4\pi G \rho(x) v(x) dx$$

$$\Phi'(x) v(x) \Big|_0^3 - \int_0^3 \Phi'(x) v'(x) dx = \int_0^3 4\pi G \rho(x) v(x) dx$$

$$-\int_0^3 \Phi'(x) v'(x) dx = \int_0^3 4\pi G \rho(x) v(x) dx$$

$$\Phi(x) = \tilde{\Phi}(x) + w(x), \quad w \in V, \; w(0) = w(3) = 0$$

$$\tilde{\Phi}(x) = \frac{4-5}{3-0} x + 5 = 5 - \frac{x}{3}$$

$$\Phi'(x) = w'(x) - \frac{1}{3}$$

$$-\int_0^3 \left(w'(x) - \frac{1}{3}\right) v'(x) dx = \int_0^3 4\pi G \rho(x) v(x) dx$$

Po podstawieniu $\rho(x)$ i uporządkowaniu

$$-\int_0^3 w'(x) v'(x) dx = 4\pi G \int_1^2 v(x) dx - \frac{1}{3} \int_0^3 v'(x) dx$$

$$B(w, v) = L(v)$$

gdzie

$$B(w,v) = -\int_0^3 w'(x) v'(x) \, dx$$

$$L(v) = 4\pi G \int_1^2 v(x) \, dx - \frac{1}{3} \int_0^3 v'(x) \, dx$$

Stosując metodę Galerkina

$$w \approx w_h = \sum_{i=0}^{n} w_i e_i$$

gdzie

$$w_h \in V_h \subset V$$

n - liczba elementów skończonych

e_i - f. bazowe generujące przestrzeń V_h

$$e_i = \begin{cases} \dfrac{x - x_{i-1}}{x_i - x_{i-1}}, & x \in (x_{i-1}, x_i] \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i}, & x \in (x_i, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}$$

$$w(0) = w(3) = 0 \Rightarrow w_0 = w_n = 0$$

$$w_h = \sum_{i=1}^{n-1} w_i e_i \quad , \quad V_h = \langle e_1, e_2, \ldots, e_{n-1} \rangle$$

$$B\left(\sum_{i=1}^{n-1} w_i e_i, v\right) = L(v) \quad \forall_{v \in V}$$

$$\sum_{i=1}^{n-1} w_i B(e_i, v) = L(v) \quad \forall_{v \in V}$$

$$\sum_{i=1}^{n-1} w_i B(e_i, e_j) = L(e_j), \quad j = 1, 2, \dots, n-1$$

$$\begin{bmatrix} B(e_1,e_1) & \cdots & B(e_{n-1},e_1) \\ \vdots & \ddots & \vdots \\ B(e_1,e_{n-1}) & \cdots & B(e_{n-1},e_{n-1}) \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_{n-1} \end{bmatrix} = \begin{bmatrix} L(e_1) \\ \vdots \\ L(e_{n-1}) \end{bmatrix}$$

$$\Phi(x) = \hat{\Phi}(x) + w(x) = 5 - \frac{x}{3} + \sum_{i=1}^{n-1} w_i e_i$$

Do wyliczenia całek (korzystając z kwadratur Gauss-Legendre), rozwiązania układu równań i narysowania wykresu rozwiązania $\Phi(x)$ wykorzystam program napisany w języku MATLAB.

~~Zauważam, że~~

$$\cancel{e_i' - e_j' = \begin{cases} \frac{1}{h^2}, & i = j \wedge x \in (x_{i-1}, x_{i+1}) \\ -\frac{1}{h^2}, & |i-j| = 1 \wedge x \in (x_i, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}}$$

~~gdzie~~ $\cancel{h = x_i - x_{i-1}, \quad i = 1, 2, \dots, n}$

Niech $h = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$

$h = \frac{3}{n}$, $x_i = i \cdot h$

$$e_i = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in (x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h}, & x \in (x_i, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}$$

$$e_i' = \begin{cases} \frac{1}{h}, & x \in (x_{i-1}, x_i] \\ -\frac{1}{h}, & x \in (x_i, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}$$

$$e_i' \cdot e_i' = \begin{cases} \frac{1}{h^2}, & x \in (x_{i-1}, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}$$

$$e_i' \cdot e_{i+1}' = \begin{cases} -\frac{1}{h^2}, & x \in (x_i, x_{i+1}) \\ 0 & \text{w.p.p.} \end{cases}$$

$$e_i' \cdot e_j' = 0, \quad |i-j| > 1$$