Tier 1 Analysis

Put into LaTeX by
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Abstract

I was passed on an extensive collection of handwritten notes – totaling nearly 600 pages – for the Tier 1 Analysis Exam at Indiana University. Here I am translating it into LaTeX, adding my own solutions, and adding any material that I view neceessary.

I currently do not know who the original author is.

Theorem 1 (Finite Intersection Property [1]). *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Note that this may be any collection of compact sets. No countability is assumed.

Proof of Theorem 1. TODO □

Problem 2. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *proper* if it is continuous and $f^{-1}(b)$ is compact for each compact subset $B \subset \mathbb{R}^n$; f is *closed* if it is continuous and $f(A)$ is closed for each closed subset $A \subset \mathbb{R}^m$.

(a) Prove that any proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.

(b) Prove that every one-to-one closed map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.

Supposedly this question is from August 1997, but I could find no record of this when combing through previous exams. Note that Tier 1 August 2010, #2 touches on the same topic of proper maps.

Proof of Problem 2 (a). TODO □

Proof of Problem 2 (b). TODO □

Problem 3. Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following two conditions:

(i) $f(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact.

(ii) If $\{K_n\}$ is a decreasing sequence of compact subset of $\mathbb{R}^n$, then

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n)$$

Prove that f is continuous.

Theorem 4 (Tier 1 August 2015, #10). [2]
Suppose K is compact, and

(i) $\{f_n\}$ is a sequence of continuous functions on K,

(ii) $\{f_n\}$ converges pointwise to a continuous function f on K,

(iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n \in \mathbb{N}$.

Then $f_n \to f$ uniformly on K.

Proof of Theorem 4. TODO □

Problem 5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that $F(0) = 0$. Assume that

$$\sum_{j,k=1}^{n} \left| \frac{\partial F_j}{\partial x_k}(0) \right|^2 = c < 1.$$

Prove that there is a ball $B \subset \mathbb{R}^n$ with center 0 such that $F(B) \subset B$.

Proof of Problem 5. TODO □

Problem 6. Suppose that $E \subset \mathbb{R}^n$ is open an that $f : E \to \mathbb{R}^n$ is C^2. Suppose also that $f''(x_0)$ is positive definite for some $x_0 \in E$. Prove that there is $r > 0$ such that $f''(x)$ is positive definite for $x \in N_r(x_0)$.

Proof of Problem 6. TODO □

Problem 7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f vanish at x. A critical point is *nondegenerate* if the $n \times n$ matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$ is non-singular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e. the nondegenerate critical points are isolated).

[1] See [1] p.38
[2] See [1] p.150 and p.516

References

[1] Rudin, Walter. *Principles of Mathematical Analysis.* 3d ed, McGraw-Hill, 1976.