Tier 1 Analysis

Put into LaTeX by
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Abstract

I was passed on an extensive collection of handwritten notes – totaling nearly 600 pages – for the Tier 1 Analysis Exam at Indiana University. Here I am translating it into LaTeX, adding my own solutions, and adding any material that I view neceessary.

I currently do not know who the original author is.

Contents

1 Problems

Theorem 1 (Finite Intersection Property [1]). *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Note that this may be any collection of compact sets. No countability is assumed.

Proof of Theorem 1. Suppose that $\bigcap K_\alpha$ is empty. Fix an arbitary set $K \in \{K_\alpha\}$. Because $\bigcap K_\alpha$ is empty, for all $x \in K$ we have that $x \in K_\alpha^c$ for some α. Thus, $\bigcup (K_\alpha^c)$ is an oper cover of K.

Because K is compact there is a finite subcover, $K_{\alpha_1}^c, \ldots, K_{\alpha_n}^c$. That is, $K \subset \bigcup_{i=1}^n \left(K_{\alpha_i}^c\right)$, so

$$K \cap \left(\bigcap_{i=1}^n K_\alpha \right) = \emptyset$$

This contradicts the hypothesis that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty. So we conclude that $\bigcap K_\alpha$ is nonempty. □

Problem 2. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *proper* if it is continuous and $f^{-1}(B)$ is compact for each compact subset $B \subset \mathbb{R}^n$; f is *closed* if it is continuous and $f(A)$ is closed for each closed subset $A \subset \mathbb{R}^m$.

(a) Prove that any proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.

(b) Prove that every one-to-one closed map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.

Supposedly this question is from August 1997, but I could find no record of this when combing through previous exams. Note that Tier 1 August 2010, #2 touches on the same topic of proper maps.

Proof of Problem 2 (a). Let $A \subset \mathbb{R}^m$ be closed. To show that $f(A)$ is closed, it is sufficient to show that $f(A)^c$ is open.

Let $q \in f(A)^c$ and consider $\overline{B}_\epsilon(q)$ the closed ball of radius ϵ about q. $\overline{B}_\epsilon(q)$ is closed and bounded in $\mathbb{R}^n$, so it is compact. Because f is proper, we have that $f^{-1}\left(\overline{B}_\epsilon(q)\right)$ is also compact. Define the following sequence of decreasing compact sets,

$$U_j = A \cap f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right),$$

where $\epsilon_0 = \epsilon$ and $\epsilon_{j+1} < \epsilon_j$. Note that each U_j is indeed compact, because $f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right)$ is compact for all j and A is closed.

$$\begin{aligned} \bigcap U_j &= A \cap \bigcap f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right) \\ &= A \cap f^{-1}(q) \\ &= \emptyset \end{aligned}$$

The above intersection is empty by construction, as $q \in f(A)^c$.

By Theorem 1 – the finite intersection property for compact sets – because the intersection over all U_j's is empty, there must be some finite subcollection of $\{U_j\}$ with empty intersection. Write this finite subcollection as $U_1, \ldots, U_n$. Let N be the smallest such n and note because the U_j's are nested, $\bigcap_{j=1}^N U_j = U_N$.

$$\begin{aligned} \bigcap^N U_j &= U_N \\ &= A \cap f^{-1}\left(\overline{B}_{\epsilon_N}(q)\right) \\ &= \emptyset \end{aligned}$$

Thus $f(A) \cap B_r(q) = \emptyset$, where $r < \epsilon_N$. So $f(A)^c$ is open and we are done. □

[1] See [1] p.38

Proof of Problem 2 (b). Let $B \subset \mathbb{R}^n$ be compact. Because $\mathbb{R}^n$ is Hausdorff and B is compact, B is also closed. By the continuity of f, $f^{-1}(B)$ is also closed. To further show that $f^{-1}(B)$ is closed – and thus f is proper – we must now show that $f^{-1}(B)$ is bounded.

Let $K_j = \{x \in \mathbb{R}^m : ||x|| \geq j\}$. K_j is closed in $\mathbb{R}^m$ for all j. Because f is a closed map, $f(K_j)$ is likewise closed for all j. Thus $f(K_j) \cap B$ is compact, and $\{f(K_j) \cap B\}_{n=1}^{\infty}$ is a decreasing sequence of compact sets. Additionally,

$$\begin{aligned}\bigcap (f(K_j) \cap B) &= B \cap \bigcap f(K_j) \\ &= B \cap f\left(\bigcap K_j\right)\end{aligned}$$

We achieve the final equality above by using the fact that f is injective, so $\bigcap f(K_j) = f\left(\bigcap K_j\right)$. Further, $\bigcap K_j = \emptyset$, so $B \cap f\left(\bigcap K_j\right) = \emptyset$. Again by Theorem 1 there must be some finite collection of $\{f(K_j) \cap B\}_{n=1}^{\infty}$ with empty intersection, and because these sets are nested there must be some index N such that

$$f(K_N) \cap B = \emptyset$$

Using injectivity once again and looking at the preimages we then see,

$$K_N \cap f^{-1}(B) = \emptyset$$

So for all $x \in f^{-1}(B)$, $||x|| < N$. That is, $f^{-1}(B)$ bounded and thus compact. So f is then proper. □

Problem 3. Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following two conditions:

1. $f(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact.
2. If $\{K_n\}$ is a decreasing sequence of compact subset of $\mathbb{R}^n$, then

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n)$$

Prove that f is continuous.

Proof of Problem 3. Let $y \in \mathbb{R}^n$, $\epsilon > 0$, and $K_n = \overline{B}_{1/n}(y)$. $\{K_n\}$ is a sequence of decreasing compact sets. By Item 1, $f(K_n)$ is also compact for each n. So $\{f(K_n)\}$ is likewise a sequence of decreasing compact sets.

Define $A = \mathbb{R}^n \setminus B_\epsilon(f(y))$. A is closed and each $f(K_n)$ is compact, so $A \cap f(K_n)$ is also compact. By Item 2, $\bigcap f(K_n) = f\left(\bigcap K_n\right)$. So,

$$\begin{aligned}\bigcap (A \cap f(K_n)) &= A \cap \bigcap f(K_n) \\ &= A \cap f\left(\bigcap K_n\right) \\ &= A \cap f(y) \\ &= \emptyset\end{aligned}$$

Note that the final equality above holds, as $f(y) \notin A$ by construction of A. Thus by Theorem 1 there exists an index N such that $A \cap f(K_N) = \emptyset$. Thus, $f(K_N) \subset A^c = B_\epsilon(f(y))$. So for $x \in B_\delta(y)$ we have $f(x) \in B_\epsilon(f(y))$ for $\delta < \frac{1}{N}$. That is, f continuous at y, and because $y \in \mathbb{R}^n$ was arbitary f is continuous everywhere. □

Theorem 4 (Tier 1 August 2015, #10). [2]
Suppose K is compact, and

[2] See [1] p.150 and p.516

1. *$\{f_n\}$ is a sequence of continuous functions on K,*
2. *$\{f_n\}$ converges pointwise to a continuous function f on K,*
3. *$f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n \in \mathbb{N}$.*

Then $f_n \to f$ uniformly on K.

Proof of Theorem 4. Define $g_n = f_n - f$. Note we get entirely analagous conditions on g_n as were given for f. That is, Items 1 to 3 give us,

1. $\{g_n\}$ is a sequence of continuous functions on K,
2. $\{g_n\}$ converges pointwise to 0 on K,
3. $g_n(x) \geq g_{n+1}(x)$ for all $x \in K, n \in \mathbb{N}$.

To prove the desired statement, it is sufficient to show that $g_n \overset{\text{unif.}}{\to} 0$. Let $\epsilon > 0$ be given and let $E_n = g_n^{-1}([\epsilon, \infty))$. Because g_n is continuous, E_n is closed for all n. Further, E_n a closed subset of K and K is compact, so E_n compact. Note that E_n is a decreasing sequence of compact sets because of Item 3.

Because $g_n \overset{\text{p.w.}}{\to} 0$ on K, we have $\bigcap E_n = \emptyset$. So by Theorem 1, there is some index N such that $E_N = g_N^{-1}([\epsilon, \infty)) = \emptyset$. So for $x \in K$, we have $0 \leq g_N(x) < \epsilon$. Further, by Item 3 we have $g_n(x) \in [0, \epsilon)$ for $n \geq N$, $x \in K$. So $g_n \overset{\text{unif.}}{\to} 0$, and thus $f_n \overset{\text{unif.}}{\to} f$. □

Problem 5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that $F(0) = 0$. Assume that

$$\sum_{j,k=1}^{n} \left| \frac{\partial F_j}{\partial x_k}(0) \right|^2 = c < 1.$$

Prove that there is a ball $B \subset \mathbb{R}^n$ with center 0 such that $F(B) \subset B$.

Proof of Problem 5. Because F is differentiable on $\mathbb{R}^n$, it is in particular differentiable at 0. So there exists a linear transformation $DF(0)$ [3] – i.e. the Jacobian matrix evaluated at 0 – such that,

$$\begin{aligned} 0 &= \lim_{h \to 0} \frac{|F(h) - F(0) - DF(0)(h)|}{|h|} \\ &= \lim_{h \to 0} \frac{|F(h) - DF(0)(h)|}{|h|} \end{aligned}$$

Define $E(h) = F(h) - DF(0)(h)$ where $\lim_{h \to 0} \frac{|E(h)|}{|h|}$ [4]. We may think of this as the error term when considering the derivative as an approxiamtion of F in a neighborhood of 0. Thus,

$$F(h) = DF(0)(h) + E(h)$$

By the triangle inequality of $|\cdot|_{\mathbb{R}^n}$,

$$\begin{aligned} |F(h)| &\leq |DF(0)(h)| + |E(h)| \\ &\leq \sup\{|DF(0)(x)| : |x| = 1\}|h| + |E(h)| \end{aligned}$$

[3] This notation may be a little verbose, but I am choosing it here to be clear that the derivative of F is a map that assigns to each point p in the domain of F a linear transformation $DF(p)$. Another way to write this is $DF|_p$, however this notation didn't play nice with the absolute value bars in the above TeX.

[4] Another notational note: In this particular case, the norm of both the numerator and denominator is $|\cdot|_{\mathbb{R}^n}$. However in generality, the numerator and denominator do not have the same norm on them. The denominator belongs to the domain of F and the numerator belongs to the codomain. Even further, it may be more clear to indicate these norms with $||\cdot||_{\mathbb{R}^n}$, but I am not in the habit of writing this.

Using the substitution $A = \sup\{|DF(0)(x)| : |x| = 1\}$, we then rewrite this as,

$$\frac{|F(h)|}{|h|} \leq A + \frac{|E(h)|}{|h|} \tag{1}$$

Let $x \in \mathbb{R}^n, |x| = 1$ and consider the coordinates of $DF(0)(x)$.

$$(DF(0)(x))_i = \sum_{k=1}^{n} \frac{\partial F_i}{\partial x_k}(0) x_k$$

So by the triangle inequality,

$$\begin{aligned}
|(DF(0)(x))_i| &\leq \sum_{k=1}^{n} \left| \frac{\partial F_i}{\partial x_k}(0) x_k \right| \\
&\overset{\text{C.S. 5}}{\leq} \left(\sum \left| \frac{\partial F_i}{\partial x_k}(0) \right|^2 \right) \left(\sum |x_k|^2 \right) \\
&= \left(\sum \left| \frac{\partial F_i}{\partial x_k}(0) \right|^2 \right) |x| \\
&= \left(\sum \left| \frac{\partial F_i}{\partial x_k}(0) \right|^2 \right)
\end{aligned}$$

We then use this to find the norm of $DF(0)(x)$,

$$\begin{aligned}
|DF(0)(x)| &= \sum_i \sum_k \left| \frac{\partial F_i}{\partial x_k}(0) \right|^2 \\
&= c \\
&< 1
\end{aligned}$$

Thus, $A = \sup\{|DF(0)(x)| : |x| = 1\} \leq \sqrt{c} < 1$. Because $\frac{|E(h)|}{|h|} \to 0$ as $h \to 0$, there exists $\delta > 0$ such that for $|h| < \delta$ we have $\frac{|E(h)|}{|h|} < 1 - \sqrt{c}$. Taking B to be the ball of radius δ about 0, we modify Eq. (1) appropriately as,

$$\begin{aligned}
\frac{|F(h)|}{|h|} &< \sqrt{c} + 1 - \sqrt{c} \\
&= 1
\end{aligned}$$

So $|F(h)| < |h|$ for $|h| < \delta$. That is to say, $F(B) \subset B$. $\square$

Problem 6. Suppose that $E \subset \mathbb{R}^n$ is open and that $f : E \to \mathbb{R}^n$ is C^2. Suppose also that $f''(x_0)$ is positive definite for some $x_0 \in E$. Prove that there is $r > 0$ such that $f''(x)$ is positive definite for $x \in N_r(x_0)$.

Proof of Problem 6. Proceed by contradiction. Suppose not, then for all $r > 0$ there exists $x \in B_r(x_0)$ such that $f''(x)$ is not positive definite – that is, for each x there exists $y \neq 0$ such that $\langle y, y \rangle_{f''(x)} \leq 0$. [6]

[5] Cauchy-Schwartz Inequality

[6] I'm abusing notation a little here. $\langle y, y \rangle_{f''(x)}$ is meant to indicate that the Hessian can act as a bilinear form. We equivalently write this expression as $\langle y, y \rangle_{f''(x)} = \langle f''(x)y, y \rangle_{\mathbb{R}^n}$.

Create sequences following the above pattern: for each $r_n = \frac{1}{n}$ we have some $x_n \in B_{r_n}(x_0)$ such that there is some $y_n \neq 0$ where $\langle y_n, y_n \rangle_{f''(x_n)} \leq 0$. For convenience we may normalize y_n to be unit length. This in turn makes y_n a bounded sequence, as all entries in the sequence have length 1, so there is a convergent subsequence $y_{n_k} \to y$ where $|y| = 1$.

We wish now to show that $\langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})} \to \langle y, y \rangle_{f''(x_0)}$,

$$\begin{aligned}
|\langle y, y \rangle_{f''(x_0)} - \langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})}| &\leq |\langle y, y \rangle_{f''(x_0)} - \langle y, y_{n_k} \rangle_{f''(x_0)}| \\
&\quad + |\langle y, y_{n_k} \rangle_{f''(x_0)} - \langle y, y_{n_k} \rangle_{f''(x_{n_k})}| \\
&\quad + |\langle y, y_{n_k} \rangle_{f''(x_{n_k})} - \langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})}| \\
&= I_1 + I_2 + I_3, \text{ respectively}
\end{aligned}$$

$$\begin{aligned}
I_1 &= |\langle y, y \rangle_{f''(x_0)} - \langle y, y_{n_k} \rangle_{f''(x_0)}| \\
&= |\langle y, y - y_{n_k} \rangle_{f''(x_0)}| \\
&= |\langle f''(x_0)y, y - y_{n_k} \rangle_{\mathbb{R}^n}| \\
&\overset{\text{C.S.}}{\leq} |f''(x_0)y||y - y_{n_k}| \\
&\leq ||f''(x_0)||\,|y|\,|y - y_{n_k}| \\
&= ||f''(x_0)||\,|y - y_{n_k}|
\end{aligned}$$

where $||f''(x_0)|| = \sup\{|f''(x_0)(a)| : |a| = 1\}$ is the matrix norm. Further, $||f''(x_0)||$ is fixed and $y_{n_k} \to y$ as $k \to \infty$, so $I_1 \to 0$.

$$\begin{aligned}
I_2 &= |\langle y, y_{n_k} \rangle_{f''(x_0)} - \langle y, y_{n_k} \rangle_{f''(x_{n_k})}| \\
&= |\langle f''(x_0)y, y_{n_k} \rangle_{\mathbb{R}^n} - \langle f''(x_{n_k})y, y_{n_k} \rangle_{\mathbb{R}^n}| \\
&= |\langle (f''(x_0) - f''(x_{n_k})y, y_{n_k} \rangle_{\mathbb{R}^n}| \\
&\leq ||f''(x_0) - f''(x_{n_k}||\,|y|\,|y_{n_k}| \\
&= ||f''(x_0) - f''(x_{n_k}||
\end{aligned}$$

$I_2 \to 0$, as $x_{n_k} \to x_0$ as $k \to \infty$ and f is C^2 – in particular, f'' continuous – so $||f''(x_0) - f''(x_{n_k}||$ may be made arbitarily small.

$$\begin{aligned}
I_3 &= |\langle y, y_{n_k} \rangle_{f''(x_{n_k})} - \langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})}| \\
&\leq ||f''(x_{n_k})||\,|y - y_{n_k}| \\
&< M|y - y_{n_k}|
\end{aligned}$$

We achieve these results on I_3 via a similar argument to above, and noting that $||f''(x_{n_k})||$ bounded (above by M). Because $y_{n_k} \to y$, I_3 may be made arbitarily small. Combining this with the above,

$$\begin{aligned}
|\langle y, y \rangle_{f''(x_0)} - \langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})}| &\leq I_1 + I_2 + I_3 \\
&= \epsilon
\end{aligned}$$

For arbitarily small ϵ, so $\langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})} \to \langle y, y \rangle_{f''(x_0)}$. However, for sake of contradiction we assumed that $\langle y_{n_k}, y_{n_k} \rangle_{f''(x_{n_k})} \leq 0$ and it is given that $\langle y, y \rangle_{f''(x_0)} > 0$, as $y \neq 0$. Thus, the desired statement is true by contradiction.

□

Problem 7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f vanish at x. A critical point is *nondegenerate* if the $n \times n$ matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$ is non-singular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e. the nondegenerate critical points are isolated).

Proof of Problem 7. TODO $\square$

2 Exams Through August 2020

TODO: Uncomment when finished

References

[1] Rudin, Walter. *Principles of Mathematical Analysis.* 3d ed, McGraw-Hill, 1976.