Tier 1 Analysis

Put into LaTeX by
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Abstract

I was passed on an extensive collection of handwritten notes – totaling nearly 600 pages – for the Tier 1 Analysis Exam at Indiana University. Here I am translating it into LaTeX, adding my own solutions, and adding any material that I view neceessary.

I currently do not know who the original author is.

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1 Problems

Theorem 1 (Finite Intersection Property [1]). *If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.*

Note that this may be any collection of compact sets. No countability is assumed.

Proof of Theorem 1. Suppose that $\bigcap K_\alpha$ is empty. Fix an arbitary set $K \in \{K_\alpha\}$. Because $\bigcap K_\alpha$ is empty, for all $x \in K$ we have that $x \in K_\alpha^c$ for some α. Thus, $\bigcup (K_\alpha^c)$ is an oper cover of K.

Because K is compact there is a finite subcover, $K_{\alpha_1}^c, \ldots, K_{\alpha_n}^c$. That is, $K \subset \bigcup_{i=1}^n \left(K_{\alpha_i}^c\right)$, so

$$K \cap \left(\bigcap_{i=1}^n K_\alpha\right) = \emptyset$$

This contradicts the hypothesis that the intersection of every finite subcollection of $\{K_\alpha\}$ is nonempty. So we conclude that $\bigcap K_\alpha$ is nonempty. □

Problem 2. A map $f : \mathbb{R}^m \to \mathbb{R}^n$ is *proper* if it is continuous and $f^{-1}(B)$ is compact for each compact subset $B \subset \mathbb{R}^n$; f is *closed* if it is continuous and $f(A)$ is closed for each closed subset $A \subset \mathbb{R}^m$.

(a) Prove that any proper map $f : \mathbb{R}^m \to \mathbb{R}^n$ is closed.

(b) Prove that every one-to-one closed map $f : \mathbb{R}^m \to \mathbb{R}^n$ is proper.

Supposedly this question is from August 1997, but I could find no record of this when combing through previous exams. Note that Tier 1 August 2010, #2 touches on the same topic of proper maps.

Proof of Problem 2 (a). Let $A \subset \mathbb{R}^m$ be closed. To show that $f(A)$ is closed, it is sufficient to show that $f(A)^c$ is open.

Let $q \in f(A)^c$ and consider $\overline{B}_\epsilon(q)$ the closed ball of radius ϵ about q. $\overline{B}_\epsilon(q)$ is closed and bounded in $\mathbb{R}^n$, so it is compact. Because f is proper, we have that $f^{-1}\left(\overline{B}_\epsilon(q)\right)$ is also compact. Define the following sequence of decreasing compact sets,

$$U_j = A \cap f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right),$$

where $\epsilon_0 = \epsilon$ and $\epsilon_{j+1} < \epsilon_j$. Note that each U_j is indeed compact, because $f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right)$ is compact for all j and A is closed.

$$\begin{aligned}\bigcap U_j &= A \cap \bigcap f^{-1}\left(\overline{B}_{\epsilon_j}(q)\right)\\ &= A \cap f^{-1}(q)\\ &= \emptyset\end{aligned}$$

The above intersection is empty by construction, as $q \in f(A)^c$.

By Theorem 1 – the finite intersection property for compact sets – because the intersection over all U_j's is empty, there must be some finite subcollection of $\{U_j\}$ with empty intersection. Write this finite subcollection as $U_1, \ldots, U_n$. Let N be the smallest such n and note because the U_j's are nested, $\bigcap_{j=1}^N U_j = U_N$.

$$\begin{aligned}\bigcap^N U_j &= U_N\\ &= A \cap f^{-1}\left(\overline{B}_{\epsilon_N}(q)\right)\\ &= \emptyset\end{aligned}$$

Thus $f(A) \cap B_r(q) = \emptyset$, where $r < \epsilon_N$. So $f(A)^c$ is open and we are done. □

[1] See [1] p.38

Proof of Problem 2 (b). TODO □

Problem 3. Let the function $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfy the following two conditions:

(i) $f(K)$ is compact whenever $K \subset \mathbb{R}^n$ is compact.

(ii) If $\{K_n\}$ is a decreasing sequence of compact subset of $\mathbb{R}^n$, then

$$f\left(\bigcap_{n=1}^{\infty} K_n\right) = \bigcap_{n=1}^{\infty} f(K_n)$$

Prove that f is continuous.

Theorem 4 (Tier 1 August 2015, #10). [2]
Suppose K is compact, and

(i) $\{f_n\}$ is a sequence of continuous functions on K,

(ii) $\{f_n\}$ converges pointwise to a continuous function f on K,

(iii) $f_n(x) \geq f_{n+1}(x)$ for all $x \in K, n \in \mathbb{N}$.

Then $f_n \to f$ uniformly on K.

Proof of Theorem 4. TODO □

Problem 5. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable map such that $F(0) = 0$. Assume that

$$\sum_{j,k=1}^{n} \left| \frac{\partial F_j}{\partial x_k}(0) \right|^2 = c < 1.$$

Prove that there is a ball $B \subset \mathbb{R}^n$ with center 0 such that $F(B) \subset B$.

Proof of Problem 5. TODO □

Problem 6. Suppose that $E \subset \mathbb{R}^n$ is open an that $f : E \to \mathbb{R}^n$ is C^2. Suppose also that $f''(x_0)$ is positive definite for some $x_0 \in E$. Prove that there is $r > 0$ such that $f''(x)$ is positive definite for $x \in N_r(x_0)$.

Proof of Problem 6. TODO □

Problem 7. Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^2. A point $x \in \mathbb{R}^n$ is a *critical point* of f if all the partial derivatives of f vanish at x. A critical point is *nondegenerate* if the $n \times n$ matrix $\left[\frac{\partial^2 f}{\partial x_i \partial x_j}(x)\right]$ is non-singular.

Let x be a nondegenerate critical point of f. Prove that there is an open neighborhood of x which contains no other critical points (i.e. the nondegenerate critical points are isolated).

[2]See [1] p.150 and p.516

2 Exams Through August 2020

TODO: Uncomment when finished

References

[1] Rudin, Walter. *Principles of Mathematical Analysis.* 3d ed, McGraw-Hill, 1976.