Intrinsic Verification of Parsers in Dependent Lambek Calculus

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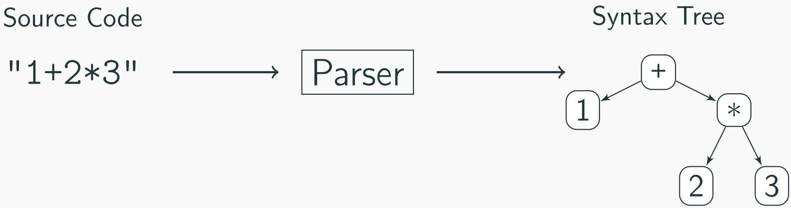
Parsing is Everywhere!



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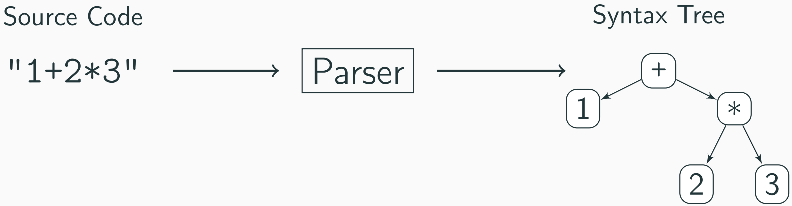

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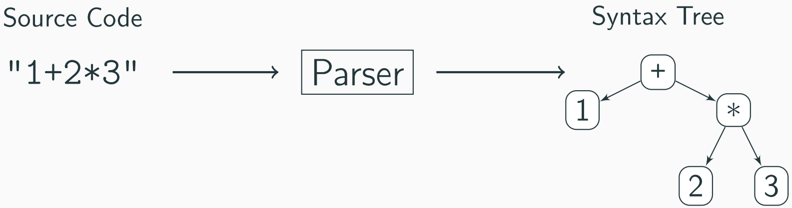

- Incorrect parsers jeopardize verified developments
- Early versions of CompCert
    - 0 middle or backend bugs, 6 parser bugs (Yang et al., 2011)

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Our Approach

Define a domain-specific language in which all parsers are **sound-by-construction**

1. A DSL for Verified Parsing

2. An Example Parser

3. Implementation

4. Future Work

Domain-Specific Language for Verified Parsers

Dependent Lambek Calculus

- Non-commutative linear logic

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Grammar `A`	Linear type `A`

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Implemented in Agda and instantiated with verified parsers for regular expressions and selected context-free grammars

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These restrictions ensure parsers are **sound-by-construction**

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  - `!` in linear logic, $\square$ in separation logic

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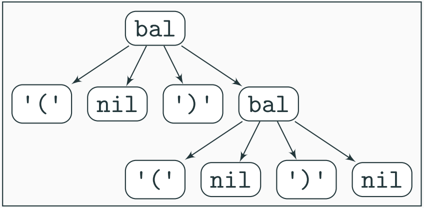
$$\texttt{"()()"} \vdash \texttt{bal l1 nil r1 (bal l2 nil r2) : Dyck}$$



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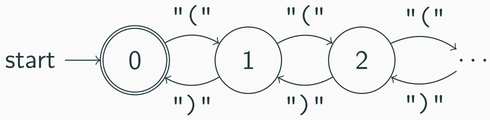
Need an automaton to define the parser!

Dyck Traces



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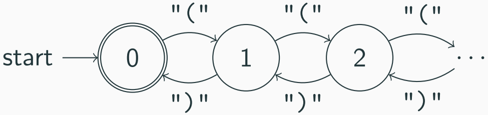


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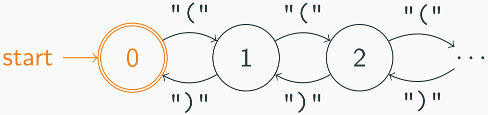


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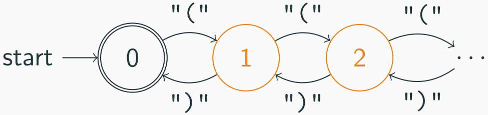


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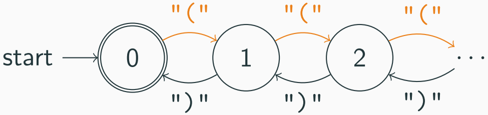


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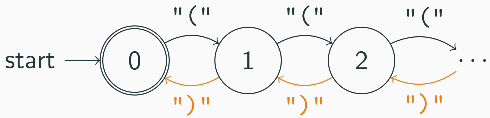


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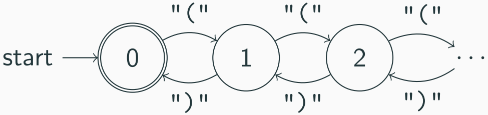


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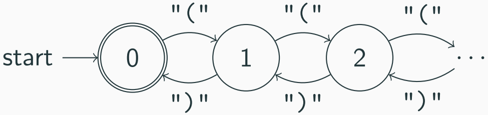


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Deterministic Automaton

$$\texttt{String} \cong \texttt{Trace}\ \texttt{true}\ 0 \oplus \texttt{Trace}\ \texttt{false}\ 0$$

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A Parser for Dyck Traces

Looks like an ordinary functional program

```
parse :
  ↑(String ⊸ &[ n ∈ ℕ ] ⊕[ b ∈ Bool ] Trace b n)
parse nil zero = σ true eof
parse nil (suc n) = σ false leftovers
parse (cons (σ '(' a) w) n =
  let σ b tr = parse w (suc n) in
  σ b (push a tr)
parse (cons (σ ')' a) w) zero =
  σ false (unexpected a _)
parse (cons (σ ')' a) w) (suc n) =
  let σ b tr = parse w n in
  σ b (pop a tr)
```

Semantics and Implementation

Denotational semantics

$$[\![\texttt{A}]\!] : \texttt{String} \rightarrow \texttt{Set} \qquad [\![\texttt{A}]\!]\ \texttt{w} = \{\text{parse trees of } \texttt{w} \text{ for } \texttt{A}\}$$

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Implementation of Dependent Lambek Calculus shallowly embedded in Cubical Agda

- $\mathtt{LinTy} := \mathtt{String} \to \mathtt{Set}$
- **Pros:** reuses the Agda typechecker
- **Cons:** poor performance and enforced combinatory style

Parsers Implemented in Agda

Dyck grammar (LL(0))

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Arithmetic expressions with a binary operation (LL(1))

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Regular expressions

- Equivalences between NFAs, DFAs, and regexes
- Thompson's construction, powerset construction

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Grammars are types! Automata are types!

Language and examples implemented in Cubical Agda

- github.com/maxsnew/grammars-and-semantic-actions

Preprint of this work

- maxsnew.com/docs/lambek.pdf