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Plane Poiseuille Flow

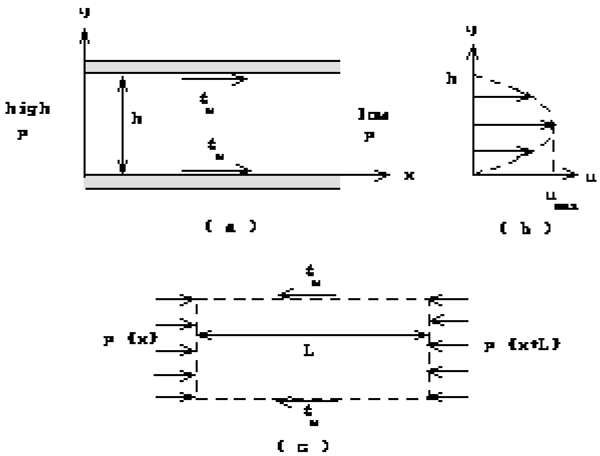


Figure 6.5: Above, (a) shows that in plane Poiseuille flow between parallel walls, the fluid flows from high to low pressure, exerting a shear stress on the walls in the direction of flow. In (b), the velocity profile is parabolic, and in (c) the shear stress acting on the fluid in a control volume is balanced by the pressure gradient.

The steady laminar viscous flow in a channel or tube from a region of higher pressure to one of lower pressure is called *Poiseuille flow*. The flow between parallel plates separated by a fixed distance **h**, illustrated in figure 6.5(a), is called *plane Poiseuille flow*. Assuming the higher pressure region is on the left and the lower pressure on the right, the pressure gradient $\partial p^* / \partial x$ would be negative while the fluid velocity is to the right, *i.e.*, the fluid is moving from a region of higher pressure $(-x)$ to one of lower pressure $(+x)$. Like plane Couette flow, the velocity $\mathbf{V}$ lies in the **x** direction and is a function of **y** alone, but the pressure gradient, which also lies in the **x** direction, is not zero but a constant:

$$\mathbf{V} = u\{y\}\,\mathbf{i}_x; \qquad \nabla p^* = \frac{\partial p^*}{\partial x}\,\mathbf{i}_x = constant$$

Substituting this form of the velocity and pressure fields in the Navier-Stokes equation 6.12 and multiplying by ρ, we find:

$$\begin{aligned} 0 &= -\frac{\partial p^*}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right) \\ \mu\frac{d^2 u}{dy^2} &= \frac{dp^*}{dx} \end{aligned}$$

where we have used total derivatives to indicate that **u** and p^* depend only upon **y** and **x**, respectively. Integrating twice upon **y**,

$$\mu u = \left(\frac{dp^*}{dx}\right)\frac{y^2}{2} + c_1 y + c_2$$

By applying the boundary conditions of zero velocity at the wall, $(u\{0\} = 0;\ u\{h\} = 0)$, the constants of integration (c_1, c_2) can be determined and the velocity $u\{y\}$ found to be:

$$u = \frac{1}{2\mu}\left(-\frac{dp^*}{dx}\right) y(h-y) \qquad (6.27)$$

which is plotted in figure 6.5(b). (We keep the negative sign inside the parenthesis enclosing dp^*/dx to remind us that the fluid velocity is positive when dp^*/dx is negative.) The velocity distribution of figure 6.5(b) is parabolic in **y**, the maximum velocity u_{max} occurring at the center of the channel $(y = h/2)$:

$$u_{max} = \frac{h^2}{8\mu}\left(-\frac{dp^*}{dx}\right) \qquad (6.28)$$

The volumetric flow rate **Q** through a channel cross-section of area **Wh** is found by integrating equation 6.27:

$$\begin{aligned} Q = W\int_0^h u\,dy &= \frac{W}{2\mu}\left(-\frac{dp^*}{dx}\right)\int_0^h y(h-y)\,dy = \frac{W}{2\mu}\left(-\frac{dp^*}{dx}\right)\left|\frac{y^2 h}{2} - \frac{y^3}{3}\right|_0^h \\ \frac{Q}{W} &= \frac{h^3}{12\mu}\left(-\frac{dp^*}{dx}\right) \qquad (6.29) \end{aligned}$$

The mean flow velocity $\bar{V}$ is:

$$\bar{V} = \frac{Q}{Wh} = \frac{h^2}{12\mu}\left(-\frac{dp^*}{dx}\right) = \frac{2}{3}u_{max} \qquad (6.30)$$

The shear stress τ^w acting on the lower wall (and the upper wall as well) may be found by evaluating the velocity gradient $\partial u/\partial y$ at the wall $(y = 0)$:

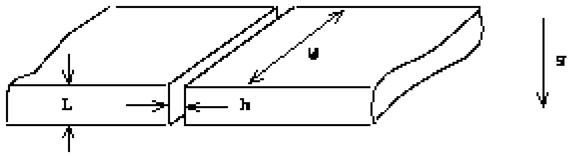
$$\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0} \mathbf{i}_x = \mu \frac{h}{2\mu} \left(-\frac{dp^*}{dx} \right) \mathbf{i}_x = \frac{h}{2} \left(-\frac{dp^*}{dx} \right) \mathbf{i}_x \qquad (6.31)$$

Notice that the wall shear stress τ_w does not depend explicitly upon the fluid viscosity (equation 6.31) but that, in contrast to plane Couette flow, the fluid velocity does (equation 6.27) . In plane Poiseuille flow, the flow speed adjusts until the wall shear stress reaches a value that balances the imposed pressure gradient, as in equation 6.31.

The relation 6.31 between wall shear stress and pressure gradient expresses the force balance on a control volume that encloses the fluid within the channel, as shown in figure 6.5(c). Assuming the length of this volume is **L** and applying the linear momentum theorem 5.11, noting that $(\dot{m}\mathbf{V})_{out} = (\dot{m}\mathbf{V})_{in}$ and $\mathbf{F}_s = 0$:

$$\begin{aligned} I\frac{d}{dt}\iiint_{\mathcal{V}} \rho \mathbf{V}\, d\mathcal{V} + (\dot{m}\mathbf{V})_{out} - (\dot{m}\mathbf{V})_{in} \\ = \iint_S (-p\mathbf{n})\, dS + \iint_S \tau\, dS + \iiint_{\mathcal{V}} \rho \mathbf{g}\, d\mathcal{V} + \Sigma \mathbf{F}_{sx} \\ 0 = Wh\left[p^*\{x\} - p^*\{x+L\}\right] \mathbf{i}_x - 2(\tau_w\, \mathbf{i}_x) LW \\ \tau_w = \frac{h}{2L}\left[p^*\{x\} - p^*\{x+L\}\right] = \frac{h}{2}\left(-\frac{dp^*}{dx}\right) \end{aligned}$$

where the pressure and gravity terms of 5.11 have been combined by use of p^* in place of **p**.



Example 6.5 A flat roof of a building is constructed of precast concrete slabs of width $W = 1.0\,m$ and depth $L = 0.1\,m$, as shown in figure E 6.5. Through an oversight, an end joint between two slabs was not sealed, leaving a crack of width $h = 1.0\,mm$. When it rains, the crack fills with water which leaks into the interior of the building. Calculate the volume flow rate of rainwater through the crack assuming steady laminar viscous (plane Poiseuille) flow with $\mu = 1.13E(-3)\,Pa\,s$.

Solution 6.5 *The volumetric flow rate through the crack is that of equation 6.29:*

$$Q = \frac{h^3 W}{12\mu}\left(-\frac{dp^*}{dx}\right)$$

Measuring x vertically downward from the upper surface, p^* is:

$$p^* = p - \rho\mathbf{g}\cdot\mathbf{R} = p - \rho(g\,\mathbf{i}_x)\cdot(x\,\mathbf{i}_x) = p - \rho g x$$

and, assuming atmospheric pressure above and below the roof, $-dp^*/dx$ becomes:

$$-\frac{dp^*}{dx} = -\frac{(p_a - \rho g L) - p_a}{L} = \rho g$$

Consequently, the volume flow rate Q is:

$$\begin{aligned} Q &= \frac{\rho g h^3 W}{12\mu} \\ &= \frac{(1.0E(3)\,kg/m^3)(9.807\,m/s^2)(1.0E(-3)\,m)^3)(1.0\,m)}{12 \times 1.13E(-3)\,Pa\,s} = 7.232E(-4)\,m^3/s \\ &= 0.7232\,l/s \end{aligned}$$

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