EM425 Assignment #4

Problem Statements

1. (Based on 4.10) Determine the LU decomposition of the matrix $a = \begin{pmatrix} 2 & 4 & 6 \\ 3 & 5 & 1 \\ 6 & -2 & 2 \end{pmatrix}$ by hand.

2. (Based on 4.15) Carry out (by hand) the first three iterations of the solution of the following system of equations using the Gauss-Seidel iterative method. For the first guess of the solution, take the value of all the unknowns to be zero.

$$\begin{aligned} 8x_1 + 2x_2 + 3x_3 &= 51 \\ 2x_1 + 5x_2 + x_3 &= 23 \\ -3x_1 + x_2 + 6x_3 &= 20 \end{aligned}$$

3. Consider the linear system given below.

$$\begin{bmatrix} 4 & 0 & 1 & 0 & 1 \\ 2 & 5 & -1 & 1 & 0 \\ 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 0 & 4 & -2 \\ 1 & 0 & -1 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 32 \\ 19 \\ 14 \\ -2 \\ 41 \end{bmatrix}$$

a) Find the solution using the built-in MATLAB function [L,U,P] = lu(A) – be sure to get and use the permutation matrix P.

b) Find the relative residual for your solution where the

relative residual $= \dfrac{\| r \|_2}{\| b \|_2} = \dfrac{\| Ax - b \|_2}{\| b \|_2}$ where x is the solution you found in part a) and

$\| \cdot \|_2$ is the vector 2-norm.

c) Find the inverse of the matrix, A^{-1}, using the built-in MATLAB function inv(A), and compute the 2-norm of A and A^{-1} using the built-in MATLAB function norm(A,p), where p should be set to 2 for the 2-norm.

d) Calculate the size of the error bound presented in Lecture 10, shown in the equation below:

$$\frac{1}{\|[a]\| \left\|[a]^{-1}\right\|} \frac{\|[r]\|}{\|[b]\|} \le \frac{\|[e]\|}{\|[x_{TS}]\|} \le \left\|[a]^{-1}\right\| \|[a]\| \frac{\|[r]\|}{\|[b]\|}$$

where $\| a^{-1} \| \| a \|$ is the condition number.

e) Repeat steps a)-d) for the test matrix nos5.mtx provided. Once you have read the matrix into MATLAB, convert the matrix to a “full” (non-sparse) format by using the MATLAB built-in function A_full = full(A_sparse). For the “right hand side” vector b, use a vector of all ones. Note how different the error bound is in this case.

4. Using Jacobi, Gauss-Seidel, and SOR $(\omega = 1.4)$ iterative methods, write and run code to solve the following linear system of equations:

$$\begin{bmatrix} 7 & 3 & -1 & 2 \\ 3 & 8 & 1 & -4 \\ -1 & 1 & 4 & -1 \\ 2 & -4 & -1 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

The stopping criteria should be the relative change of the estimated solution in the 2-norm:

$$\text{tolerance} = \frac{\| x^{k+1} - x^k \|_2}{\| x^k \|_2}$$

with a tolerance set to 10^{-9}. Compare the number of iterations required in each case.

5. For this problem we will compare two iterative methods to solve nos5—a test matrix derived from a structural analysis of a three-story building. For the "right hand side" vector b, use a vector of all ones.

a) Use the SOR method with an estimated relative error tolerance of 10^{-4}. Vary the relaxation parameter over the range $\omega \in [1.5, 1.96]$. Make a plot of the number of iterations required as a function of ω and approximate the best value of ω for this system.

b) Use the MATLAB built-in function gmres without preconditioner to solve this problem.

c) Use gmres again but with an incomplete LU preconditioner. Use opts_ilu.type='ilutp' and opts_ilu.droptol=dtol and vary the value of dtol over the range $\text{dtol} \in [10^{-7}, 10^{-3}]$. What happens as dtol gets larger? ***Hint:*** *Use the built-in function spy(L) to show the sparsity pattern of L and note the number of non-zeros (nnz) of L. How does nnz(L) change as dtol is changed?*

d) This is a relatively small sparse system and can be solved using direct methods. Use the MATLAB built-in function timeit to estimate the time required to solve the system of equations using the built-in mldivide ("backslash") function.

Organize your findings from part a) - d) into a short document. A couple of paragraphs should be enough. Include the plot you created from part a) and any other graphics/tables that you find helpful to communicate what you observed in parts b) through d). Upload the document in PDF format for your homework submission.

Numeric Answers

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Problem #1

   1.0000         0         0
   1.5000    1.0000         0
   3.0000   14.0000    1.0000

    2     4     6
    0    -1    -8
    0     0    96
```

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Problem #2

After iteration 1:
    6.3750    2.0500    6.1792

After iteration 2:
    3.5453    1.9460    4.7816

After iteration 3:
    4.0954    2.0055    5.0468
```

```
Problem #3

lower error bound: 3.67942e-17
upper error bound: 4.20131e-16
Repeating for nos5.mtx
lower error bound: 7.7844e-18
upper error bound: 9.42348e-10
```

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Problem #4

Jacobi Solver iterations: 143
Gauss-Seidel iterations: 75
SOR iterations: 24
```

Your answers for Problem #3 may vary somewhat.

Problem #5:

I leave most of this to you as I want you to spend some time experimenting with the different SOR relaxation parameters and MATLAB built-in pre-conditioner settings.