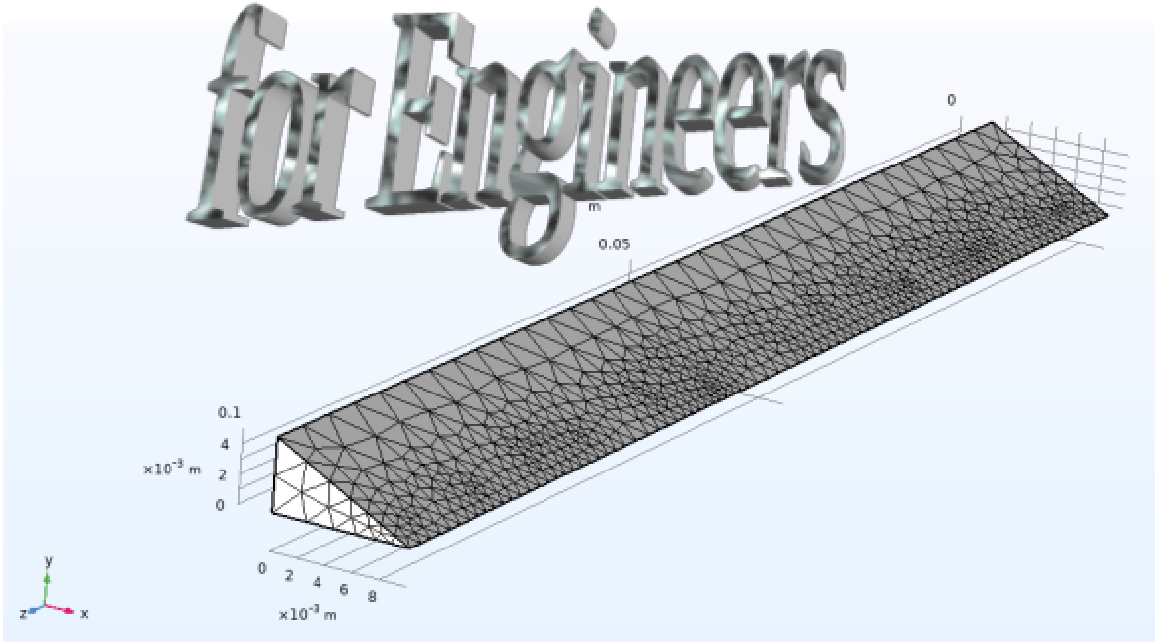
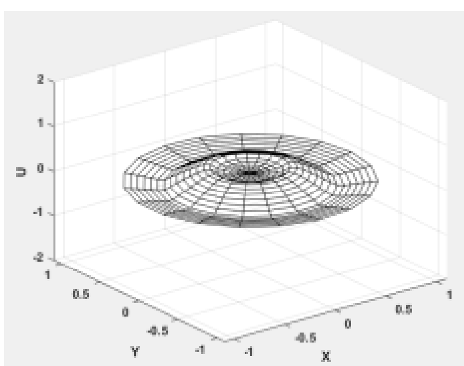
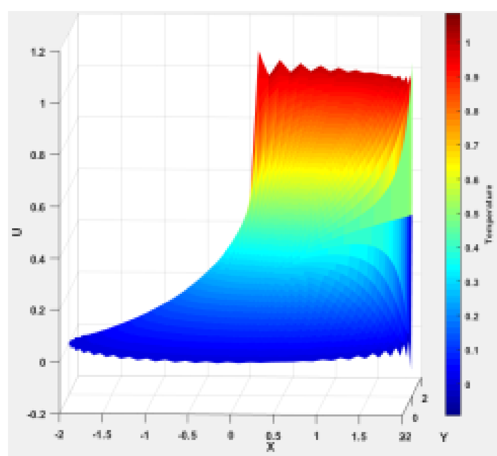
Mathematical Methods for Engineers







Stu Blair

Mighty Goat Press

WHEN IN DOUBT, MULTIPLY BOTH SIDES BY AN ORTHOGONAL FUNCTION AND INTEGRATE.

P.L. CHEBYSHEV

THE PURPOSE OF COMPUTING IS INSIGHT, NOT PICTURES

L.N. TREFETHEN

NEVER DO A CALCULATION UNTIL YOU ALREADY KNOW THE ANSWER.

J.A. WHEELER

UNITED STATES NAVAL ACADEMY

MATHEMATICAL METHODS FOR ENGINEERS

MIGHTY GOAT PRESS

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Preface

The purpose of this text is to provide a concise reference for engineering students who would like to strengthen their conceptual understanding and practical proficiency in analytical and numerical methods in engineering. The material is based on a sequence of two courses taught at the United States Naval Academy.

Analytical Methods

The first course focused on analytical methods for linear ordinary and partial differential equations. All students came into the course having taken a three-semester sequence of calculus along with a course in ordinary differential equations. The analytical methods portion quickly reviews methods for constant coefficient linear equations and proceeds to methods for non-constant coefficients including Cauchy-Euler equations, power series methods, and method of Frobenieus. After a review of Fourier Series methods and an introduction to Fourier-Legendre and Fourier-Bessel expansions we thoroughly explore solutions to second-order, linear, partial differential equations. Since many students are also studying nuclear engineering, there is a heavy focus on addressing boundary value problems in cylindrical and spherical coordinate systems that are applicable to other topics of interest such as reactor physics. There is also heavy emphasis on heat transfer applications that students will see later on in their undergraduate curriculum.

The materials presented are based heavily on Professor Dennis Zill's excellent book.[1] We lightly select from chapters 1-3 for review; chapter 5 for series solution methods; and chapters 12-14 for Fourier Series and solutions to linear boundary value problems. Material from that text is used throughout this book.

What distinguishes this course from Prof Zill's work is the incorporation of computational tools in the solution process. These "semi-analytical methods" are presented here in MATLAB[2] owing to the students preparation with that tool. Other open-source tools like Octave[3] and Python,[4] of course, could be used.

[1] Dennis G Zill. *Advanced engineering mathematics*. Jones & Bartlett Learning, 2020

[2] Inc. The Math Works. Matlab, v2022a, 2022. URL https://www.mathworks.com/

[3] John W. Eaton, David Bateman, Søren Hauberg, and Rik Wehbring. *GNU Octave version 5.2.0 manual: a high-level interactive language for numerical computations*, 2020. URL https://www.gnu.org/software/octave/doc/v5.2.0/

[4] Guido Van Rossum and Fred L. Drake. *Python 3 Reference Manual*. CreateSpace, Scotts Valley, CA, 2009. ISBN 1441412697

Numerical Methods

Part I

Introduction and Review

Lecture 1 - Introduction, Definitions and Terminology

Objectives

The objectives of this lecture are:

- Provide an overview of course content
- Define basic terms related to differential equations
- Provide examples of classification schemes for differential equations

Course Introduction

This course is intended as a one-semester introduction to partial differential equations. It is assumed that all students have a thorough background in single- and multi-variable calculus as well as differential equations. The first few lectures comprise a review of the portions of differential equations on which this course most heavily relies. This is followed by a treatment of power series methods and the method of Frobeneius. These are needed so that students will understand the origins of Legendre Polynomials and Bessel functions that will be used in the solution of boundary value problems in spherical and cylindrical coordinates respectively.

The main body of material deals with the solution of (mostly homogeneous) boundary value problems—wave equation, heat equation, and Laplace equation—in rectangular, polar/cylindrical, and spherical coordinate systems. For this a preparatory review of Fourier series expansions along with Fourier-Legendre and Fourier-Bessel expansions are introduced along with a levening of Sturm-Liouville theory in boundary value problems. The rest is a problem-by-problem tour of methods and analysis with heavy emphasis on heat transfer and nuclear engineering applications.

Classification of Differential Equations

IT IS IMPORTANT to be able to classify differential equations. In this class we will learn a variety of techniques to find the function that satisfies a differential equation along with its boundary or initial conditions.[5] The techniques we learn in this class are tailored for specific classes of problems; you classify the problem and that tells you what method to use. If you improperly classify the equation, you will likely use an inappropriate method and may have trouble figuring out why it is not working.

[5] Consider the differential equation: $\frac{du}{dx} = ux$. The variable u stands for the function, $u(x)$, that satisfies the equation; u is also referred to as the **dependent variable**. The variable x is the **independent variable**. By convention we will use the variables x, y, z and r, θ, ϕ as spatial independent variables and t as an independent variable for time dependent problems. We will use many other letters to denote dependent variables but most commonly u, v, and w.

Classification by Type and Order

WE SHALL START with the easiest classification categories: type and order. There are two *types* of differential equations that we will consider: ordinary differential equations; and partial differential equations.

IN AN ORDINARY differential equation, there is only one independent variable. In a *partial* differential equation, there are multiple independent variables and consequently derivatives of the depenent variable will partial derivatives.

Example ODE:

$$\frac{d^2u}{dt^2} + t\frac{du}{dt} = 3e^{-t}$$

There is one independent variable, t

Example PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

There are two independent variables, x, and y.

THE ORDER OF a differential equation is the order of the hightest derivative in the equation. This is typically not confusing for students. If anything needs to be added here it is to be mindful of the difference between a higher order derivative and an exponent. For example, in the second order, non-linear, ordinary differential equation shown below,

$$\frac{d^2u}{dx^2} + 5\left(\frac{du}{dx}\right)^3 - 4u = e^x$$

it isn't *too* hard to realize that the "3" is an exponent and the "2" denotes a second derivative. Still, be mindful.

Classification by Linearity

An n-th order ordinary differential equation is said to be *linear* when it can be written in the form shown in Equation 1:

$$a_n(x)u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_1(x)' + a_0(x)u = g(x) \qquad (1)$$

The key features that you should note in the form of Equation 1 are:

1. The *dependent* variable and *all of its derivatives* are of the first degree; that is, the power of each term involving u is 1.

2. The coefficients of each term, $a_n(x)$, depend at most on the *independent* variable.

A lot of students struggle with discriminating between linear and nonlinear ODEs but it really is as simple as checking these two things. If both conditions are satisfied; the equation is linear. If not, the equation is nonlinear. As examples, Equation 2 violates the first criterion; Equation 3 violates the second.

Why is this important? Most of the techniques we will learn in this course *depend* upon the fact tha the equation we are trying to solve is *linear*. In *"the wild"* you may be presented with (or, more likely *derive*) an equation and may not be explicitly told whether or not the equation is linear. If the equation is **not** linear you will find that most of the tools you learn in this course will not be applicable; you will most likely need to use a numerical method. You need to be able to tell the difference so you know what tools to use.

$$\frac{d^2u}{dx^2} + u^2 = 0 \tag{2}$$

$$\frac{d^3u}{dx^3} - 5u\frac{du}{dx} = x \tag{3}$$

Verification of an Explicit Solution

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an *explicit* solution. Otherwise, the solution is *implicit*.

Example explicit solution: $u(x) = f(x)$.
Example implicit solution: $G(x, u) = 0$

IN THIS CLASS we will mainly be interested in finding explicit solutions to differential equations that we are given or have derived. There are some cases, however, where we are given a function and we wish to verify that it is a solution to a given differential equation. To do this, we simply "plug" the equation into the differential operator and verify that an identity is derived.

Example: Verify that $u = \frac{6}{5} - \frac{6}{5}e^{-20t}$ is a solution to:

$$\frac{du}{dt} + 20u = 24$$

Solution: Since $\frac{du}{dt} = \frac{d}{dt}(\frac{6}{5} - \frac{6}{5}e^{-20t}) = 24e^{-20t}$, we can see that:

$$\begin{aligned}\frac{du}{dt} + 20u &= 24e^{-20t} + 20(\frac{6}{5} - \frac{6}{5}e^{-20t}) \\ &= 24e^{-20t} + 24 - 24e^{-20t} \\ &= 24\end{aligned}$$

which is the expected identity.

Lecture 2 - Separable and Linear 1st order Equations

Objectives

The objectives of this lecture are:

- Define and describe the solution procedure for *separable* first order equations
- Define and demonstrate the solution procedure for *linear* first order equations

Separable Equations

A first order differential equation of the form shown below

$$\frac{du}{dx} = g(u)h(x) \tag{4}$$

is said to be *separable* or have *separable variables.*

Note: there is **no** requirement that the 1st order equation be *linear*. This is one of the few techniques that we will study in this course that can be applied to nonlinear equations.

The solution method for separable equations is, in princple simple. For the separable differential equation given in Equation 4 we would separate and integrate:

$$\begin{aligned} \frac{du}{dx} &= g(u)h(x) \\ \frac{du}{g(u)} &= h(x)dx \\ \int \frac{1}{g(u)}\ du &= \int h(x)\ dx \end{aligned}$$

Note: there are at least two complications here.

1. The solution you thus derive may be either implicit or explicit. An implicit solution is, as a practical matter, fairly inconvenient to deal with; and
2. It may not be possible to actually carry out the integrals analytically.

Nonetheless, we shall carry on and give it a try anyway.

Generally speaking, one of your first checks for a first order equation should be: is it separable? If so, you should separate the variables and solve. The examples below are intended to illustrate the method. Note that in the final example, the integral cannot be done analytically.

Solve the following separable, first order differential equations .
Example 1:

$$
\begin{aligned}
\frac{du}{dx} &= \frac{u}{1+x} \\
\frac{du}{u} &= \frac{dx}{1+x} \\
\int \frac{d}{u} &= \int \frac{dx}{1+x} \\
\ln|u| + c_1 &= \ln|1+x| + c_2 \\
|u| &= e^{[\ln|1+x|+c_3]} \\
u(x) &= c|1+x|
\end{aligned}
$$

Example 2:

$$
\begin{aligned}
\frac{du}{dx} &= -\frac{x}{u} \\
\int u\ du &= -\int x\ dx \\
\frac{u^2}{2} &= -\frac{x^2}{2} + c \\
u(x) &= \sqrt{c - x^2}
\end{aligned}
$$

Example 3: Solve the first order initial value problem shown below:

$$\frac{du}{dx} = e^{-x^2}, \quad u(2) = 6, \quad 2 \le x < \infty \tag{5}$$

$$
\begin{aligned}
du &= e^{-x^2}\ dx \\
\int_2^x \frac{du}{dt}\ dt &= \int_2^x e^{-t^2}\ dt \\
u(x) - u(2) &= \int_2^x e^{-t^2}\ dt \\
u(x) &= 6 + \int_2^x e^{-t^2}\ dt
\end{aligned}
$$

where we have used the dummy variable t in the integrals; the last integral will need to be evaluated numerically.

Linear Equations

A first-order differential equation of the form:

$$a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{6}$$

is said to be a first order *linear equation* in the dependent variable u. When $g(x) = 0$, the first-order linear equation is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Note: it is sometimes customary to write the differential equation in *operator form* where the differential operator, $\mathcal{L} = a_1(x)\frac{d}{dx} + a_0(x)$, is applied to the function $u(x)$ to get $g(x)$; $\mathcal{L}u(x) = g(x)$

Notice that $g(x)$ is the only term in Equation 6 that does **not** include u or any of its derivatives.

WHEN SOLVING equations of this type it is useful to express it in the **standard form**:

$$\frac{du}{dx} + P(x)u = f(x) \tag{7}$$

The method for solving this equation makes use of the <u>*linearity*</u> property and express the solution in the following way: $u(x) = u_c(x) + u_p(x)$; plugging this into Equation 7 gives us:

When we say an operator is **linear**, what we mean is that the following relationships must hold:

1. $\mathcal{L}(\alpha u) = \alpha\mathcal{L}(u)$
2. $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

for functions u,v and scalar constant α. Think of this as a *definition* of linearity.

$$\begin{aligned}\frac{d}{dx}[u_c + u_p] + P(x)[u_c + u_p] &= \\ &\left[\frac{du_c}{dx} + P(x)u_c\right] + \left[\frac{du_p}{dx} + P(x)u_p\right] = f(x)\end{aligned} \tag{8}$$

where $u_c(x)$ is the solution to the *associated homogeneous problem*

$$\frac{du_c}{dx} + P(x)u_c = 0 \tag{9}$$

and $u_p(x)$ is the solution to:

$$\frac{du_p}{dx} + P(x)u_p = f(x) \tag{10}$$

The linear operator here is: $\mathcal{L} = \frac{d}{dx} + P(x)$. Equation 9 says $\mathcal{L}u_c = 0$; Equation 10 says $\mathcal{L}u_p = f(x)$; Equation 8 says that $\mathcal{L}(u_c + u_p) = 0 + f(x) = f(x)$.

What might trouble you now is: if we have u_p, is this not a solution to Equation 7? Why do we need u_c? The next thing that should trouble you is that if u_p is a solution, by the linearity property of $\mathcal{L}$, so is u_p plus *any* constant multiple of u_c. The solution is not *unique*

We can see that Equation 9 is separable:

$$\begin{aligned}\frac{du_c}{dx} + P(x)u_c &= 0 \\ \frac{du_c}{u_c} &= -P(x)\ dx \\ \ln u_c + C &= -\int_{P(x)\ dx} \\ u_c(x) &= e^{-\int P(x)\ dx + C_1} \\ u_c(x) &= e^{-\int P(x)\ dx}e^{C_1}u_c(x) \qquad = Ce^{-\int P(x)\ dx}\end{aligned}$$

where $C = e^{C_1}$.

This will all be resolved when we recall that u_c will have an arbitrary constant through which we will be able to say that $u = u_c + u_p$ is a function describing *all* possible solutions of Equation 7 and the arbitrary constant in u_c will be set so as to uniquely satisfy a given initial/boundary condition.

Note: At some point in time, I will desist in making such piddling distinctions between constants. C_1 is an arbitrary constant, e^{C_1} is still an arbitrary constant; there is no real difference between C_1 and C and, in this author's humble opinion, they do not rate different symbols.

WE NEED TO FIND a solution $u_p(x)$ to Equation 10. The technique we will use is called *variation of parameters*. It consists of looking for a solution in the form $y_p(x) = v(x)u_1(x)$, where $u_1(x) = e^{-\int P(x)\ dx}$ which is $u_c(x)$ with the arbitrary constant set to 1 and $v(x)$ might be thought of as some kind of weighting or *variational* function.

WE WILL INSERT this proposed form of $y_p(x)$ into Equation 10:

$$\frac{d(vu_1)}{dx} + P(x)(v(x)u_1(x)) = f(x)$$

We apply the product rule to the first term and re-arrange terms:

$$\begin{aligned} u_1(x)\frac{dv}{dx} + v(x)\frac{du_1}{dx} + P(x)(v(x)u_1(x)) &= f(x) \\ v(x)\underbrace{\left[\frac{du_1}{dx} + P(x)u_1(x)\right]}_{=0} + u_1(x)\frac{dv}{dx} &= f(x) \\ u_1(x)\frac{dv}{dx} &= f(x) \end{aligned}$$

In the last line we can observe that the equation is *separable* and thus solve:

$$\begin{aligned} v(x) &= \int \frac{f(x)}{u_1(x)}\,dx \\ &= \int e^{\int P(x)\,dx} f(x)\,dx \end{aligned}$$

Now that we know what $v(x)$ must be, we can combine this with $u_1(x)$ to get $u_p(x)$:

$$u_p(x) = e^{-\int P(x)\,dx}\left[\int e^{\int P(x)\,dx} f(x)\,dx\right] \tag{11}$$

Equation 11 is messy and perhaps a bit scary but given definitions of $P(x)$ and $f(x)$ we might hope we can solve it anyway. We now have expressions for both u_c and u_p; they can be combined into the solution for the first-order linear equation:

$$u(x) = Ce^{-\int P(x)\,dx} + e^{-\int P(x)\,dx}\left[e^{\int P(x)\,dx} f(x)\,dx\right] \tag{12}$$

Method of Solution

Once we have identified a problem to be first-order and linear, we will solve the problem using the following steps:

1. Write the equation in standard form (Equation 7)
2. Determine the integrating factor $\mu = e^{-\int P(x)\,dx}$.
3. Solve for the general solution $u(x)$ using Equation 12.
4. Apply initial/boundary condition if given.

Example: Solve the problem:

$$\frac{du}{dx} + u = x, \quad u(0) = 4$$

Solution:

Step 1: The equation is already in standard form, so this step is easy.

Step 2: Find the integrating factor μ.

$$mu = e^{-\int P(x)\ dx} = e^{-\int 1\ dx} = e^{-x}$$

Step 3: Solve for the general solution $u(x)$ using Equation 12

$$\begin{aligned} u(x) &= Ce^{-x} + e^{-x} \int e^{x} x\ dx \\ &= Ce^{-x} + e^{-x} \left[xe^{x} - e^{x} \right] \\ &= Ce^{-x} + x - 1 \end{aligned}$$

← For the integral $\int e^{x} x\ dx$ we need to use integration by parts.

Step 4: Apply initial/boundary conditions if given

$$\begin{aligned} u(0) &= Ce^{0} + 0 - 1 \\ &= C - 1 = 4 \\ \Rightarrow C &= 5 \\ u(x) &= 5e^{x} + x - 1 \end{aligned}$$

Assignment #1

State the order of the given ordinary differential equation and indicate if it is linear or non-linear.

1. $(1-x)\,u'' - 4xu' + 5u = \cos x$

2. $t^5 u^{(4)} - t^3 u'' + 6u = 0$

Verify the indicated function is an explicit solution of the given differential equation.

3. $2u' + u = 0, \quad u = e^{-x/2}$

4. $u'' - 6u' + 13u = 0, \quad u = e^{3x}\cos 2x$

Solve the given differential equation by separation of variables

5. $\frac{du}{dx} = \sin 5x$

6. $dx + e^{3x} du = 0$

7. $\frac{dS}{dr} = kS$

8. $\frac{du}{dx} = x\sqrt{1-u^2}$

Find an explicit solution of the given initial-value problem

9. $x^2 \frac{du}{dx} = u - xu, \quad u(-1) = -1$

Find the general solution of the given differential equation

10. $\frac{du}{dx} + u = e^{3x}$

11. $u' + 3x^2 u = x^2$

12. $x\frac{du}{dx} - u = x^2 \sin x$

Lecture 3 - Theory of Linear Equations

Objectives

The objectives of this lecture are:

- Introduce several theoretical concepts relevant to initial value problems and boundary value problems.
- Demonstrate use of the Wronskian to determine linear independence of solutions.
- Present some important theorems and definitions relevant to the theory of linear orinary differential equations.

Initial Value Problems

For a linear differential equation, an n$^{\text{th}}$-order initial value problem (IVP) is given by the following governing equation and initial conditions:

$$\text{Governing Equation: } a_n(x)\frac{d^n u}{dx^n} + a_{n-1}\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{13}$$

$$\text{Initial Conditions: } u(x_0) = u_0,\ u'(x_0) = u_1, \ldots, u^{(n-1)}(x_0) = u_{n-1} \tag{14}$$

Note: for an initial value problem, all of the initial conditions are provided at the same value of x; in accordance to custom we call this x_0. The name *initial* condition gives the implication that these conditions are at some "end" of the interval (beginnig, left side, whatever) and in most all examples and exercises this is indeed the case. It is <u>not</u> a requirement, however.
Generally for an n$^{\text{th}}$-order IVP you will need n conditions.

We seek a function defined on some interval containing x_0 that satisfies the differential equation with n conditions applied. The theorem below, which we will use by *citing* rather than *proving* gives us assurance that, subject some fairly reasonable assumptions, such a solution will exist.

Theorem 1 (Existence and Uniqueness for IVPs)
If $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ are continuous on an interval $\mathcal{I}$, and if $a_n(x) \neq 0$ for every $x \in \mathcal{I}$, and if x_0 is any point in this interval, then a solution $u(x)$ of the IVP exists on the interval and it is unique.

FOR THIS CLASS we will adopt a mostly operational definition of continuity: if you can draw the function throughout the specified interval without picking up your pencil or without diverging to infinity, then the function is continuous.

Consider, as an example, the following initial value problem:

$$u'' - 4u = 12x, \quad u(0) = 4, \; u'(0) = 1 \tag{15}$$

This IVP satisfies the conditions of Theorem 1 since all of the coefficients and $g(x)$ are continuous and a_1 is constant and nonzero; hence a unique solution exists on any interval and that solution is unique.

Take a moment to verify that $u(x) = 3e^{2x} + e^{-2x} - 3x$ satisfies both the governing equation and initial conditions and thus is *the* unique solution to this IVP.

Here is an IVP that does *not* satisfiy the criteria of Theorem 1:

$$x^2u'' - 2xu' + 2u = 6, \quad u(0) = 3, \; u'(0) = 1 \tag{16}$$

In this case, the coefficients and $g(x)$ are all continuous but $a_2(x)$ is equal to zero at $x = 0$. This might not be a problem—i.e. if $x = 0$ is not in the interval of interest for the IVP then we are okay—but since $x_0 = 0$, $x = 0$ *must* be in the domain for the theorem to apply. So we have no assurances that a solution exists or, if a solution does exist, it may not be unique.

You should take a moment to verify that $u = cx^2 + x + 3$ is a solution for *any* choice of parameter c.

Boundary Value Problems

For this section let us, without undue loss of generality, consider a 2^{nd}-order boundary value problem (BVP):

Almost all of the applications we will consider for this class will involve 2^{nd}-order operators. The way we derive important boundary-value problems from underlying physical laws like conservation of mass and consdervation of energy lead to them being 2^{nd}-order. Probably the most obvious counterexample is beam theory which involves a 4^{th}-order operator.

$$\text{Govering Equation: } a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{17}$$

$$\text{Boundary Conditions: } y(a) = y_0, \quad y(b) = y_1, \quad a \neq b \tag{18}$$

DEPENDING ON THE boundary conditions, BVPs may have no solutions, one unique solution, or infinitely many solutions.

Example: The equation $u'' + 16u = 0$ has the general solution $u(t) = c_1 \cos(4t) + c_2 \sin(4t)$. Consider the three different sets of boundary conditions provided below.

a) $x(0) = 0, \quad x(\pi/2) = 0$ Application of the first boundary condition gives us $c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$. The second boundary condition is $c_2 \sin(2\pi) = 0$, which is true for *any* value of c_2. Therefore there problem has infinitely many solutions.

b) $x(0) = 0, \quad x(\pi/8) = 0$ The first boundary condition again gives us $c_1 = 0$; the second condition $c_2 \sin(4\frac{\pi}{8})$ is only satisfied if $c_2 = 0$.

Thus $c_1 = c_2 = 0$; only the trivial solution, $u = 0$, satisfies both the differential equation and boundary conditions. This is not a very interesting solution but at least it *is a solution* so we will take this as an example of a BVP having a unique solution.

For applications, we will generally be only interested in *non-trivial* solutions; that is, solutions that are not identically equal to zero.

c) $x(0) = 0, \;\; x(\pi/2) = 1$ In this case, again $c_1 = 0$ from the first boundary condition. This leaves the second boundary condition: $c_2 \sin\left(4\frac{\pi}{2}\right) = c_2(0) = 1$ which cannot be satisfied for any value of c_2. In this case *no* solution exists.

Superposition and Linear Dependence

In this section some important theorems regarding IVPs and BVPs will be presented. No attempt will be made to prove these theorems; we will simply take these theorems as facts that are relevant for this course that you should try to understand as best you can.

Theorem 2 (Superposition Principle for Homogeneous Equations) *Let $u_1, u_2, \dots, u_k$ be solutions of a homogeneous n^{th}-order linear differential equation. Then any linear combination of those solutions*

$$u = c_1u_1 + c_2u_2 + \cdots + c_ku_k$$

where $c_1, c_2, \dots, c_k$ are arbitrary constants, is also a solution.

Note: It is essential that *both* the governing equation and given conditions (boundary or initial) for the linear differential equation are homogeneous. As a reminder, this means that *all* terms in the governing equation and boundary conditions must either a) involve the dependent variable or one of its derivatives; or b) be equal to zero.

As an example, If I denote the linear homogeneous differential equation as $\mathcal{L}$, then $\mathcal{L}(u_i) = 0$ for any $i \in [1, 2, \dots, k]$. By the linearity property of $\mathcal{L}$, for any constants α and β:

$$\begin{aligned}\mathcal{L}(\alpha u_i + \beta u_j) &= \alpha\mathcal{L}(u_i) + \beta\mathcal{L}(u_j)\\ &= \alpha(0) + \beta(0)\\ &= 0\end{aligned}$$

Theorem 3 (Linear Dependence / Independence of Functions) *A set of functions $f_1(x), f_2(x), \dots, f_k(x)$ is said to be* linearly dependent *on an interval $\mathcal{I}$ if there exist constants $c_1, c_2, \dots, c_k$,* not <u>all</u> of which are zero*, such that*

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_kf_k(x) = 0$$

for every $x \in \mathcal{I}$. If the set of functions is not *linearly dependent, it is linearly independent.*

What if a member of the set of functions is $f(x) = 0$?

Answer: The set will no longer be linearly independent. The trivial function $f(x) = 0$ is not linearly independent from *anything*.

Repeatedly throughout this course we will want to clarify whether or not, say, two functions are linearly independent of each other. I think most engineers have a general idea of what it is we *mean* when we say two functions are linearly independent or dependent but Theorem 3 specifies what these things mean *mathematically*.

We need a test to help us determine if the members of a set of functions are linearly independent or not. This will be especially important as we evaluate solutions to a linear homogeneous differential equation. Even if you are the sort of savant who can, by inspection, always detect linear dependence, you might have a hard time convincing your friends that your assessment is always correct. Luckily, there is a theorem that provides a suitable test that can serve as irrefutable evidence of the state of linear dependence/independence of functions.

Theorem 4 (Criterion for Linearly Independent Solutions)
Let $u_1, u_2, \ldots, u_n$ be solutions of a homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$. Then the set of solutions is linearly indepenent on the interval if and only if the Wronskian of the solution is non-zero for every $x \in \mathcal{I}$.

The Wronskian is a function that takes functions as arguments and returns a scalar numeric quantity.[6]

[6] Sometimes such functions are referred to as *functionals*.

$$W(u_1, u_2, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{19}$$

where $|\cdot|$ donetes the matrix determinant. For large values of n this is also difficult to calculate but, for the case $n = 2$, engineering students should be familiar with the formula:

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2 \tag{20}$$

Example: show that the functions $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ are linearly independent solutions to the homogeneous linear equation $u'' - 9u = 0$ for every $x \in (-\infty, \infty)$.
Solution: The Wronskian is given by:

The reader should verify that both $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ satisfy the given differential equation.

$$\begin{aligned} W &= \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} \\ &= e^{3x}\left(-3e^{-3x}\right) - 3e^{3x}\left(e^{-3x}\right) \\ &= 3e^{3x-3x} - 3e^{3x-3x} \\ &= 3 - 3 \\ &= 6 \end{aligned}$$

Since $6 \neq 0$ for all $x \in (-\infty, \infty)$ the solutions are linearly independent.

Definition 1 (Fundamental Set of Solutions)
Any set $u_1, u_2, \dots, u_n$ of n linearly independent solutions of the homogeneous linear n^{th}-order differential equation on an interval is said to be a fundamental set of solutions on an interval $\mathcal{I}$.

Theorem 5 (Existence of a Fundamental Set)
There exists a fundamental set of solutions for the homogeneous linear n^{th}-order differential equation on an interval $\mathcal{I}$.

Note: This is different than saying that a BVP or IVP has a solution. This theorem is only refering to the differential equation; not the boundary or initial conditions.

Definition 2 (General Solution—Homogeneous Equation)
Let $u_1, u_2, \dots, u_n$ be a fundamental set of solutions to the homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$, then the general solution is:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x)$$

IT IS IMPORTANT to understand from the above that:

- *any* possible solution to the homogeneous, linear, n^{th}-order differential equation can be constructed by setting the coefficients of the general solution; and
- there is **no** solution that can be constructed from functions that are linearly independent from the general solution.

General Solution for a Nonhomogeneous Problem

Recall: "nonhomogeneous" for a linear n^{th}-order differential equation means that $g(x) \neq 0$. if u_p is any particular solution to the nonhomogeneous, linear, n^{th}-order ODE on an interval $\mathcal{I}$ and $u_c = c_1 u_1(x) + c_2 u_2(x) + \cdots c_n u_n(x)$ is the general solution to the associated homogeneous ODE (called the *complementary* solution) then the general solution to the nonhomogeneous ODE is:

$$u = u_c + u_p$$

Example: By substitution it can be seen that $u_p = -\frac{11}{12} - \frac{1}{2}x$ is a particular solution to $u''' - 6u'' + 11u' - 6u = 3x$. The general solution to the associated homogeneous problem is $u_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. Consequently, the general solution to the linear nonhomogeneous problem is:

You are, again, strongly encouraged to verify that u_p satisfies the given equation and that u_c satisfies the associated homogeneous equation.

$$\begin{aligned} u(x) &= u_c + u_p \\ &= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x \end{aligned}$$

Lecture 4 - Homogeneous Linear Equations with Constant Coefficients

Objectives

The objectives of this lecture are:

- Review the solution methodology for homogeneous linear equations with constant coefficients.
- Illustrate this method with several examples.

Introduction

In this lecture we will review the well-trod ground of your differential equations class and remind ourselves how to solve linear, constant coefficient, homogeneous, n^{th}-order differential equations. These equations have the general form shown in Equation 21

$$c_n u^{(n)} + c_{n-1}u^{(n-1)} + \cdots + c_1 u' + c_0 u = 0 \tag{21}$$

where the coefficients are real and constant an $c_n \neq 0$.

THE BASIC STRATEGY is to assume the solution is of the form: $u(x) = e^{mx}$. For the case of 2^{nd}-order equations, we get:

$$\begin{aligned} c_2 m^2 e^{mx} + c_1 m e^{mx} + c_0 e^{mx} &= 0 \\ e^{mx}\left(c_2 m^2 + c_1 m + c_0\right) \end{aligned}$$

where the last line above is called the auxiliary equation:

Here we re-name the constants so Equation 22 takes a familiar form.

$$am^2 + bm + c = 0 \tag{22}$$

From the well-known quadratic equation, solutions are: $m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ Solution of this equation gives the following three cases:

1. **Distinct Real Roots** In this case $m_1 \neq m_2$ and the general solution is of the form:

$$u(x) = c_1 \underbrace{e^{m_1 x}}_{u_1(x)} + c_2 \underbrace{e^{m_2 x}}_{u_2(x)} \tag{23}$$

Using tools from the last lecture you should recognize that $u_1(x)$ and $u_2(x)$ are linearly independent for all $x \in (-\infty, \infty)$, thus form a fundamental set of solutions.

AN IMPORTANT SPECIAL CASE is when m_1 and m_2 are roots of a positive real number and thus $m_1 = -m_2$. This happens when the governing equation is of the form:

$$u'' - k^2 u = 0 \tag{24}$$

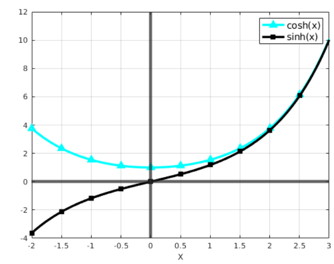

Figure 1: Plot of $\cosh x$ and $\sinh x$

The solutions are thus:

$$u(x) = c_1 e^{-kx} + c_2 e^{kx} \tag{25}$$

For reasons that will become clear later in the course, it is sometimes useful to re-express the solution shown in Equation 25 in terms of the functions cosh and sinh. These functions are defined as linear combinations of exponentials as shown below and plotted in Figure 1

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

2. **Real Repeated Roots** In this case $m_1 = m_2$. One solution is:

i.e. from the quadratic equation, $b^2 - 4ac = 0$

$$u_1(x) = e^{m_1 x} \tag{26}$$

The other solution so derived is, of course, the same and thus we do not have two linearly independent solutions as required to form a fundamental set of solutions for a 2^{nd}-order linear homogeneous equation.

IT CAN BE SHOWN that a second linearly independent solution can be formed by multiplying by the independent variable:

This is done using a technique referred to as *reduction of order*. We will not take the time to cover it in this class (or in this book) but is concisely described in section 3.2 of Zill. At a minimum you might at least confirm for yourself that a) $xu_1(x)$ is a solution to the equation; and b) use the Wronskian to confirm that it is linearly independent from $u_1(x)$.

$$u_2(x) = xu_1(x) = xe^{mx}$$

and thus the general solution for this case is:

$$u(x) = c_1 e^{mx} + c_2 x e^{mx} \tag{27}$$

3. **Conjugate Complex Roots** In this case the discriminant, $b^2 - 4ac$, is negative so its square root is imaginary. This results in m_1 and m_2 being complex conjugates which we will express as: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

The general solution is:

$$\begin{aligned} u(x) &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) \end{aligned}$$

The complex exponentials in the last equation can be re-expressed using the Euler Formula:

$$\begin{aligned} e^{i\beta x} &= \cos\beta x + i\sin\beta x \\ e^{-i\beta x} &= \cos\beta x - i\sin\beta x \end{aligned}$$

which is slighly more convenient insofar as the solutions are no longer expressed as complex exponentials but also by breaking each solution down into their real and complex parts. It can be shown that both the real and imaginary parts of the solution must satisfy the differential equation *independently*. This fact allows us to re-express the solution in a more simple form that does not involve complex numbers:

$$u(x) = e^{\alpha x}\left(c_1 \cos\beta x + c_2 \sin\beta x\right) \tag{28}$$

ANOTHER IMPORTANT special case is when the solution is *pure imaginary* (i.e. $\alpha = 0$) so the solution is:

$$u(x) = c_1 \cos\beta x + c_2 \sin\beta x \tag{29}$$

These solutions arise when the governing equation is of the shown in Equation 30:

$$u'' + k^2 u = 0 \tag{30}$$

The roots $m_{1,2} = \pm ik$ and the general solution is:

$$u(x) = c_1 \cos kx + c_2 \sin kx \tag{31}$$

This equation will be revisited throughout the course as it repeatedly comes up in applications.

Three Examples

The cases described above will be illustrated with three examples:

Example #1: Find the general solution to $2u'' - 5u' - 3u = 0$. Inserting $u = e^{mx}$ into the equation gives us the auxiliary equation:

$$2m^2 - 5m - 3 = (2m+1)(m-3)$$

with roots: $m_1 = -\frac{1}{2}$ and $m_2 = 3$. These are real, distinct roots so the general solution is:

$$u(x) = c_1 e^{-x/2} + c_2 e^{3x}$$

Example #2: Find the general solution to $u'' - 10u' + 25u = 0$. The auxiliary equation is:

$$m^2 - 10m + 25 = (m-5)(m-5)$$

with (repeated) roots: $m_1 = 5$ and $m_2 = 5$. These are real, repated roots so the general solution is:

$$u(x) = c_1 e^{5x} + c_2 x e^{5x}$$

Example #3: Find the general solution to $4u'' + 4u' + 17u = 0, \ \ u(0) = -1, \ u'(0) = 2$.

This is an initial value problem with continuous (and constant) coefficients. We know from Theorem 1 that a unique solution exists. We will first find the general solution, then apply the initial conditions to resolve the unknown coefficients to reveal the solution.

We can see that it must be an *initial* value problem because the conditions are both given at the same location, $x_0 = 0$.

The auxiliary equation is:

$$4m^2 + 4m + 17 = 0$$

using the quadratic equation, gives us:

$$\begin{aligned}\frac{-4 \pm \sqrt{16 - 4(4)(17)}}{2(4)} &= -\frac{1}{2} \pm \frac{\sqrt{-256}}{8} \\ &= -\frac{1}{2} \pm \frac{-16}{8} \\ &= -\frac{1}{2} \pm 2i\end{aligned}$$

This gives us complex conjugate roots and the general solution is:

$$u(x) = e^{-x/2}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

Applying the initial condition $u(0) = -1$ gives us:

$$\begin{aligned}u(0) &= e^0\left(c_1 \cos 0 + c_2 \sin 0\right) \\ &= 1(c_1(1) + c_2(0)) \\ &= c_1 = -1\end{aligned}$$

To apply the second initial condition we need to use the chain-rule and product rule to differentiate the general solution. This gives us:

$$\begin{aligned}u'(x) = -\frac{1}{2}e^{-x/2}c_1 \cos 2x - 2e^{-x/2}c_1 \sin 2x + \\ -\frac{1}{2}e^{-x/2}c_2 \sin 2x + 2e^{-x/2}c_2 \cos 2x\end{aligned}$$

Evaluating $u'(0)$ and substituting $c_1 = -1$ gives us:

$$\begin{aligned}
u'(0) &= -\frac{1}{2}(1)(-1)(1) + (1)(2)c_2(1) \\
&= \frac{1}{2} + 2c_2 = 2 \\
\Rightarrow 2c_2 &= \frac{3}{2} \\
c_2 &= \frac{3}{4}
\end{aligned}$$

Both constants are now known and the unique solution is:

$$u(x) = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

Part II

Power Series Methods

Part III

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Bibliography

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Appendices

Matlab Style Rules

1. **rule:** All scripts will start with the commands: **clear**, **clc**, and **close 'all'**

   rationale: No script should depend upon any data visible in the MATLAB workspace when the script starts. By omitting these commands, residual data within the workspace may hide errors.

2. **rule:** Your code must be documented with enough details such that a reader unfamiliar with your work will know what you are doing.

   rationale: Code documentation is a habit. For more significant projects readers may need help in deciding what the author of the code intended. For your own code, the most likely reader is you—a few months into the future.

3. **rule:** Function and variable names must be meaningful and reasonable in length.

   rationale: Failing to do either make code harder to read and maintain.

4. **rule:** All outputs from the code <u>**must**</u> be meaningful; numbers should be formatted, part of a sentence, and include units. Graphs should be readable and axis labels should make sense and include units.

   rationale: Code output is a form of communication. It is important that this communication be clear and unambiguous.

5. **rule:** Do not leave warnings from the Code Analyzer unaddressed.

   rationale: Sometimes Code Analyzer warnings can be safely ignored. Most of the time the warning points to a stylistic error that would be unacceptable in software that you use. Occasionally these warnings are indicative of a hidden error.

6. **rule:** Use the "smart indentation tool" to format the indentation of your code.

rationale: This tool improves code readability. It will also occasionally point out errors that you did not see before.

7. **rule:** Pre-allocate arrays; if possible initialize with **NaN** values.

   rationale: Pre-allocation improves performance and helps readability. Initialization with **NaN** helps avoid a range of potential logical errors.

8. **rule:** Avoid "magic numbers" — i.e. hard-coded constants.

   rationale: Constants included in your code tend to hide your program logic. Also, "magic numbers" make code maintenance more difficult and error prone.

9. **rule:** Only write one statement per line.

   rationale: Multi-statement-lines hurt code readability in almost all cases.

10. **rule:** Do not write excessively long lines of code; use the line continuation "..." and indentation to spread long expressions over several lines.

    rationale: Following this rule improves code readability.

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