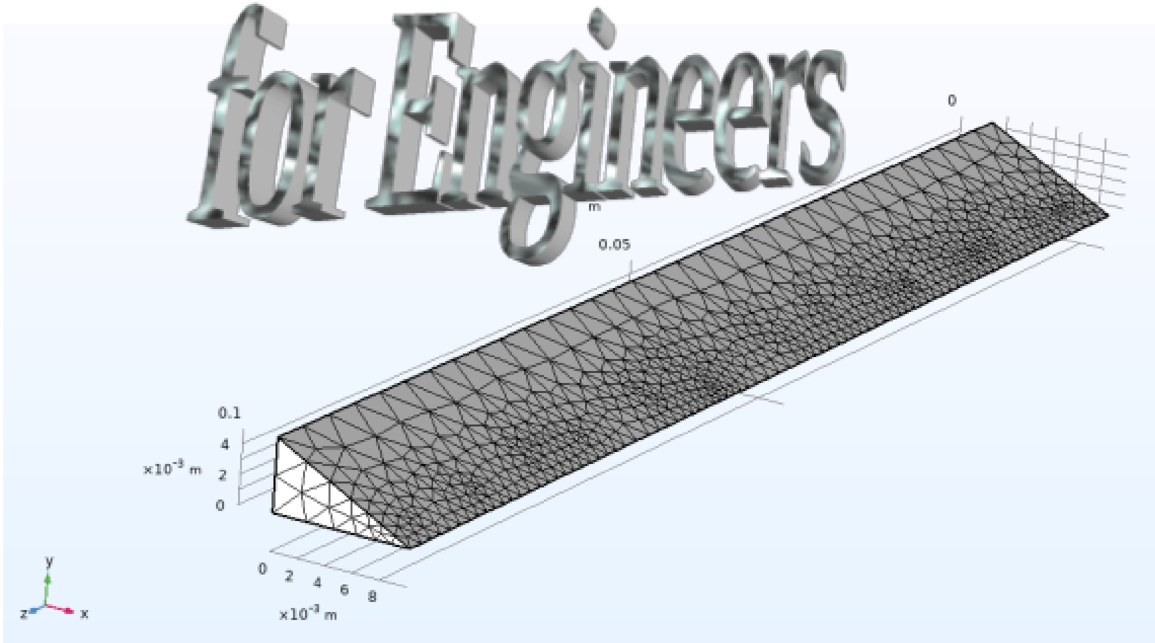
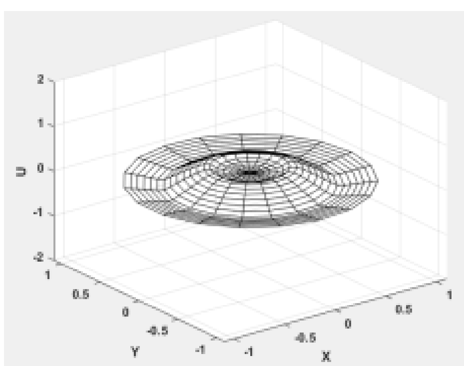
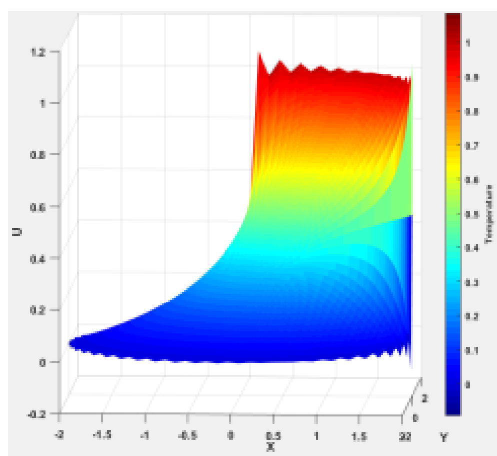
Mathematical Methods for Engineers







Stu Blair

Mighty Goat Press

WHEN IN DOUBT, MULTIPLY BOTH SIDES BY AN ORTHOGONAL FUNCTION AND INTEGRATE.

P.L. CHEBYSHEV

THE PURPOSE OF COMPUTING IS INSIGHT, NOT PICTURES

L.N. TREFETHEN

NEVER DO A CALCULATION UNTIL YOU ALREADY KNOW THE ANSWER.

J.A. WHEELER

UNITED STATES NAVAL ACADEMY

MATHEMATICAL METHODS FOR ENGINEERS

MIGHTY GOAT PRESS

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Preface

The purpose of this text is to provide a concise reference for engineering students who would like to strengthen their conceptual understanding and practical proficiency in analytical and numerical methods in engineering. The material is based on a sequence of two courses taught at the United States Naval Academy.

Analytical Methods

The first course focused on analytical methods for linear ordinary and partial differential equations. All students came into the course having taken a three-semester sequence of calculus along with a course in ordinary differential equations. The analytical methods portion quickly reviews methods for constant coefficient linear equations and proceeds to methods for non-constant coefficients including Cauchy-Euler equations, power series methods, and method of Frobenieus. After a review of Fourier Series methods and an introduction to Fourier-Legendre and Fourier-Bessel expansions we thoroughly explore solutions to second-order, linear, partial differential equations. Since many students are also studying nuclear engineering, there is a heavy focus on addressing boundary value problems in cylindrical and spherical coordinate systems that are applicable to other topics of interest such as reactor physics. There is also heavy emphasis on heat transfer applications that students will see later on in their undergraduate curriculum.

The materials presented are based heavily on Professor Dennis Zill's excellent book.[1] We lightly select from chapters 1-3 for review; chapter 5 for series solution methods; and chapters 12-14 for Fourier Series and solutions to linear boundary value problems. Material from that text is used throughout this book.

What distinguishes this course from Prof Zill's work is the incorporation of computational tools in the solution process. These "semi-analytical methods" are presented here in MATLAB[2] owing to the students preparation with that tool. Other open-source tools like Octave[3] and Python,[4] of course, could be used.

[1] Dennis G Zill. *Advanced Engineering Mathematics*. Jones & Bartlett Learning, 2020

[2] Inc. The Math Works. Matlab, v2022a, 2022. URL `https://www.mathworks.com/`

[3] John W. Eaton, David Bateman, Søren Hauberg, and Rik Wehbring. *GNU Octave version 5.2.0 manual: a high-level interactive language for numerical computations*, 2020. URL `https://www.gnu.org/software/octave/doc/v5.2.0/`

[4] Guido Van Rossum and Fred L. Drake. *Python 3 Reference Manual*. CreateSpace, Scotts Valley, CA, 2009. ISBN 1441412697

Part I

Introduction and Review

Lecture 1 - Introduction, Definitions and Terminology

Objectives

The objectives of this lecture are:

- Provide an overview of course content
- Define basic terms related to differential equations
- Provide examples of classification schemes for differential equations

Course Introduction

This course is intended as a one-semester introduction to partial differential equations. It is assumed that all students have a thorough background in single- and multi-variable calculus as well as differential equations. The first few lectures comprise a review of the portions of differential equations on which this course most heavily relies. This is followed by a treatment of power series methods and the method of Frobenius. These are needed so that students will understand the origins of Legendre Polynomials and Bessel functions that will be used in the solution of boundary value problems in spherical and cylindrical coordinates respectively.

The main body of material deals with the solution of (mostly homogeneous) boundary value problems—wave equation, heat equation, and Laplace equation—in rectangular, polar/cylindrical, and spherical coordinate systems. For this a preparatory review of Fourier series expansions along with Fourier-Legendre and Fourier-Bessel expansions are introduced along with a leavening of Sturm-Liouville theory in boundary value problems. The rest is a problem-by-problem tour of methods and analysis with heavy emphasis on heat transfer and nuclear engineering applications.

Classification of Differential Equations

It is important to be able to classify differential equations. In this class we will learn a variety of techniques to find the function that satisfies a differential equation along with its boundary or initial conditions.[5] The techniques we learn in this class are tailored for specific classes of problems; you classify the problem and that tells you what method to use. If you improperly classify the equation, you will likely use an inappropriate method and may have trouble figuring out why it is not working.

[5] Consider the differential equation: $\frac{du}{dx} = ux$. The variable u stands for the function, $u(x)$, that satisfies the equation; u is also referred to as the **dependent variable**. The variable x is the **independent variable**. By convention we will use the variables x, y, z and r, θ, ϕ as spatial independent variables and t as an independent variable for time dependent problems. We will use many other letters to denote dependent variables but most commonly u, v, and w.

Classification by Type and Order

We shall start with the easiest classification categories: type and order. There are two *types* of differential equations that we will consider: ordinary differential equations; and partial differential equations.

In an ordinary differential equation, there is only one independent variable. In a *partial* differential equation, there are multiple independent variables and consequently derivatives of the dependent variable will partial derivatives.

Example ODE:

$$\frac{d^2u}{dt^2} + t\frac{du}{dt} = 3e^{-t}$$

There is one independent variable, t

Example PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

There are two independent variables, x, and y.

The order of a differential equation is the order of the highest derivative in the equation. This is typically not confusing for students. If anything needs to be added here it is to be mindful of the difference between a higher order derivative and an exponent. For example, in the second order, non-linear, ordinary differential equation shown below,

$$\frac{d^2u}{dx^2} + 5\left(\frac{du}{dx}\right)^3 - 4u = e^x$$

it isn't *too* hard to realize that the "3" is an exponent and the "2" denotes a second derivative. Still, be mindful.

Classification by Linearity

An n-th order ordinary differential equation is said to be *linear* when it can be written in the form shown in Equation 1:

$$a_n(x)u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_1(x)' + a_0(x)u = g(x) \qquad (1)$$

The key features that you should note in the form of Equation 1 are:

1. The *dependent* variable and *all of its derivatives* are of the first degree; that is, the power of each term involving u is 1.

2. The coefficients of each term, $a_n(x)$, depend at most on the *independent* variable.

A lot of students struggle with discriminating between linear and nonlinear ODEs but it really is as simple as checking these two things. If both conditions are satisfied; the equation is linear. If not, the equation is nonlinear. As examples, Equation 2 violates the first criterion; Equation 3 violates the second.

Why is this important? Most of the techniques we will learn in this course *depend* upon the fact that the equation we are trying to solve is *linear*. In *"the wild"* you may be presented with (or, more likely *derive*) an equation and may not be explicitly told whether or not the equation is linear. If the equation is **not** linear you will find that most of the tools you learn in this course will not be applicable; you will most likely need to use a numerical method. You need to be able to tell the difference so you know what tools to use.

$$\frac{d^2u}{dx^2} + u^2 = 0 \tag{2}$$

$$\frac{d^3u}{dx^3} - 5u\frac{du}{dx} = x \tag{3}$$

Verification of an Explicit Solution

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an *explicit* solution. Otherwise, the solution is *implicit*.

Example explicit solution: $u(x) = f(x)$.
Example implicit solution: $G(x, u) = 0$

In this class we will mainly be interested in finding explicit solutions to differential equations that we are given or have derived. There are some cases, however, where we are given a function and we wish to verify that it is a solution to a given differential equation. To do this, we simply "plug" the equation into the differential operator and verify that an identity is derived.

Example: Verify that $u = \frac{6}{5} - \frac{6}{5}e^{-20t}$ is a solution to:

$$\frac{du}{dt} + 20u = 24$$

Solution: Since $\frac{du}{dt} = \frac{d}{dt}(\frac{6}{5} - \frac{6}{5}e^{-20t}) = 24e^{-20t}$, we can see that:

$$\begin{aligned}\frac{du}{dt} + 20u &= 24e^{-20t} + 20(\frac{6}{5} - \frac{6}{5}e^{-20t}) \\ &= 24e^{-20t} + 24 - 24e^{-20t} \\ &= 24\end{aligned}$$

which is the expected identity.

Lecture 2 - Separable and Linear 1st order Equations

Objectives

The objectives of this lecture are:

- Define and describe the solution procedure for *separable* first order equations
- Define and demonstrate the solution procedure for *linear* first order equations

Separable Equations

A first order differential equation of the form shown below

$$\frac{du}{dx} = g(u)h(x) \tag{4}$$

is said to be *separable* or have *separable variables.*

Note: there is **no** requirement that the 1st order equation be *linear*. This is one of the few techniques that we will study in this course that can be applied to nonlinear equations.

The solution method for separable equations is, in principle simple. For the separable differential equation given in Equation 4 we would separate and integrate:

$$\begin{aligned}\frac{du}{dx} &= g(u)h(x)\\ \frac{du}{g(u)} &= h(x)dx\\ \int \frac{1}{g(u)}\,du &= \int h(x)\,dx\end{aligned}$$

Note: there are at least two complications here.

1. The solution you thus derive may be either implicit or explicit. An implicit solution is, as a practical matter, fairly inconvenient to deal with; and
2. It may not be possible to actually carry out the integrals analytically.

Nonetheless, we shall carry on and give it a try anyway.

Generally speaking, one of your first checks for a first order equation should be: is it separable? If so, you should separate the variables and solve. The examples below are intended to illustrate the method. Note that in the final example, the integral cannot be done analytically.

Solve the following separable, first order differential equations .
Example 1:

$$\begin{aligned}
\frac{du}{dx} &= \frac{u}{1+x} \\
\frac{du}{u} &= \frac{dx}{1+x} \\
\int \frac{d}{u} &= \int \frac{dx}{1+x} \\
\ln|u| + c_1 &= \ln|1+x| + c_2 \\
|u| &= e^{[\ln|1+x|+c_3]} \\
u(x) &= c|1+x|
\end{aligned}$$

Example 2:

$$\begin{aligned}
\frac{du}{dx} &= -\frac{x}{u} \\
\int u\ du &= -\int x\ dx \\
\frac{u^2}{2} &= -\frac{x^2}{2} + c \\
u(x) &= \sqrt{c - x^2}
\end{aligned}$$

Example 3: Solve the first order initial value problem shown below:

$$\frac{du}{dx} = e^{-x^2}, \;\; u(2) = 6, \;\; 2 \leq x < \infty$$

$$\begin{aligned}
du &= e^{-x^2}\ dx \\
\int_2^x \frac{du}{dt}\ dt &= \int_2^x e^{-t^2}\ dt \\
u(x) - u(2) &= \int_2^x e^{-t^2}\ dt \\
u(x) &= 6 + \int_2^x e^{-t^2}\ dt
\end{aligned}$$

where we have used the dummy variable t in the integrals; the last integral will need to be evaluated numerically.

Linear Equations

A first-order differential equation of the form:

$$a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{5}$$

is said to be a first order *linear equation* in the dependent variable u. When $g(x) = 0$, the first-order linear equation is said to be *homogeneous*; otherwise it is *non-homogeneous*.

Note: it is sometimes customary to write the differential equation in *operator form* where the differential operator, $\mathcal{L} = a_1(x)\frac{d}{dx} + a_0(x)$, is applied to the function $u(x)$ to get $g(x)$; $\mathcal{L}u(x) = g(x)$

Notice that $g(x)$ is the only term in Equation 5 that does **not** include u or any of its derivatives.

WHEN SOLVING equations of this type it is useful to express it in the **standard form**:

$$\frac{du}{dx} + P(x)u = f(x) \tag{6}$$

The method for solving this equation makes use of the linearity property and express the solution in the following way: $u(x) = u_c(x) + u_p(x)$; plugging this into Equation 6 gives us:

When we say an operator is **linear**, what we mean is that the following relationships must hold:

1. $\mathcal{L}(\alpha u) = \alpha\mathcal{L}(u)$
2. $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

for functions u,v and scalar constant α. Think of this as a *definition* of linearity.

$$\begin{aligned}\frac{d}{dx}[u_c + u_p] + P(x)[u_c + u_p] = \\ \left[\frac{du_c}{dx} + P(x)u_c\right] + \left[\frac{du_p}{dx} + P(x)u_p\right] = f(x)\end{aligned} \tag{7}$$

where $u_c(x)$ is the solution to the *associated homogeneous problem*

$$\frac{du_c}{dx} + P(x)u_c = 0 \tag{8}$$

and $u_p(x)$ is the solution to:

$$\frac{du_p}{dx} + P(x)u_p = f(x) \tag{9}$$

The linear operator here is: $\mathcal{L} = \frac{d}{dx} + P(x)$. Equation 8 says $\mathcal{L}u_c = 0$; Equation 9 says $\mathcal{L}u_p = f(x)$; Equation 7 says that $\mathcal{L}(u_c + u_p) = 0 + f(x) = f(x)$.

What might trouble you now is: if we have u_p, is this not a solution to Equation 6? Why do we need u_c? The next thing that should trouble you is that if u_p is a solution, by the linearity property of $\mathcal{L}$, so is u_p plus *any* constant multiple of u_c. The solution is not *unique*

This will all be resolved when we recall that u_c will have an arbitrary constant through which we will be able to say that $u = u_c + u_p$ is a function describing *all* possible solutions of Equation 6 and the arbitrary constant in u_c will be set so as to uniquely satisfy a given initial/boundary condition.

We can see that Equation 8 is separable:

$$\begin{aligned}\frac{du_c}{dx} + P(x)u_c &= 0 \\ \frac{du_c}{u_c} &= -P(x)\ dx \\ \ln u_c + C &= -\int_{P(x)\ dx} \\ u_c(x) &= e^{-\int P(x)\ dx + C_1} \\ u_c(x) &= e^{-\int P(x)\ dx}e^{C_1}u_c(x) \qquad = Ce^{-\int P(x)\ dx}\end{aligned}$$

where $C = e^{C_1}$.

Note: At some point in time, I will desist in making such piddling distinctions between constants. C_1 is an arbitrary constant, e^{C_1} is still an arbitrary constant; there is no real difference between C_1 and C and, in this author's humble opinion, they do not rate different symbols.

WE NEED TO FIND a solution $u_p(x)$ to Equation 9. The technique we will use is called *variation of parameters*. It consists of looking for a solution in the form $y_p(x) = v(x)u_1(x)$, where $u_1(x) = e^{-\int P(x)\ dx}$ which is $u_c(x)$ with the arbitrary constant set to 1 and $v(x)$ might be thought of as some kind of weighting or *variational* function.

WE WILL INSERT this proposed form of $y_p(x)$ into Equation 9:

$$\frac{d(vu_1)}{dx} + P(x)(v(x)u_1(x)) = f(x)$$

We apply the product rule to the first term and re-arrange terms:

$$\begin{aligned} u_1(x)\frac{dv}{dx} + v(x)\frac{du_1}{dx} + P(x)(v(x)u_1(x)) &= f(x) \\ v(x)\underbrace{\left[\frac{du_1}{dx} + P(x)u_1(x)\right]}_{=0} + u_1(x)\frac{dv}{dx} &= f(x) \\ u_1(x)\frac{dv}{dx} &= f(x) \end{aligned}$$

In the last line we can observe that the equation is *separable* and thus solve:

$$\begin{aligned} v(x) &= \int \frac{f(x)}{u_1(x)}\, dx \\ &= \int e^{\int P(x)\, dx} f(x)\, dx \end{aligned}$$

Now that we know what $v(x)$ must be, we can combine this with $u_1(x)$ to get $u_p(x)$:

$$u_p(x) = e^{-\int P(x)\, dx}\left[\int e^{\int P(x)\, dx} f(x)\, dx\right] \tag{10}$$

Equation 10 is messy and perhaps a bit scary but given definitions of $P(x)$ and $f(x)$ we might hope we can solve it anyway. We now have expressions for both u_c and u_p; they can be combined into the solution for the first-order linear equation:

$$u(x) = Ce^{-\int P(x)\, dx} + e^{-\int P(x)\, dx}\left[e^{\int P(x)\, dx} f(x)\, dx\right] \tag{11}$$

Method of Solution

Once we have identified a problem to be first-order and linear, we will solve the problem using the following steps:

1. Write the equation in standard form (Equation 6)
2. Determine the integrating factor $\mu = e^{-\int P(x)\, dx}$.
3. Solve for the general solution $u(x)$ using Equation 11.
4. Apply initial/boundary condition if given.

Example: Solve the problem:

$$\frac{du}{dx} + u = x, \quad u(0) = 4$$

Solution:

Step 1: The equation is already in standard form, so this step is easy.

Step 2: Find the integrating factor μ.

$$mu = e^{-\int P(x)\ dx} = e^{-\int 1\ dx} = e^{-x}$$

Step 3: Solve for the general solution $u(x)$ using Equation 11

$$\begin{aligned} u(x) &= Ce^{-x} + e^{-x}\int e^{x}x\ dx \\ &= Ce^{-x} + e^{-x}\left[xe^{x} - e^{x}\right] \\ &= Ce^{-x} + x - 1 \end{aligned}$$

← For the integral $\int e^{x}x\ dx$ we need to use integration by parts.

Step 4: Apply initial/boundary conditions if given

$$\begin{aligned} u(0) &= Ce^{0} + 0 - 1 \\ &= C - 1 = 4 \\ \Rightarrow C &= 5 \\ u(x) &= 5e^{x} + x - 1 \end{aligned}$$

Assignment #1

State the order of the given ordinary differential equation and indicate if it is linear or non-linear.

1. $(1-x)\,u'' - 4xu' + 5u = \cos x$

2. $t^5 u^{(4)} - t^3 u'' + 6u = 0$

Verify the indicated function is an explicit solution of the given differential equation.

3. $2u' + u = 0, \quad u = e^{-x/2}$

4. $u'' - 6u' + 13u = 0, \quad u = e^{3x}\cos 2x$

Solve the given differential equation by separation of variables

5. $\frac{du}{dx} = \sin 5x$

6. $dx + e^{3x} du = 0$

7. $\frac{dS}{dr} = kS$

8. $\frac{du}{dx} = x\sqrt{1-u^2}$

Find an explicit solution of the given initial-value problem

9. $x^2 \frac{du}{dx} = u - xu, \quad u(-1) = -1$

Find the general solution of the given differential equation

10. $\frac{du}{dx} + u = e^{3x}$

11. $u' + 3x^2 u = x^2$

12. $x\frac{du}{dx} - u = x^2 \sin x$

Lecture 3 - Theory of Linear Equations

Objectives

The objectives of this lecture are:

- Introduce several theoretical concepts relevant to initial value problems and boundary value problems.
- Demonstrate use of the Wronskian to determine linear independence of solutions.
- Present some important theorems and definitions relevant to the theory of linear ordinary differential equations.

Initial Value Problems

For a linear differential equation, an n$^{\text{th}}$-order initial value problem (IVP) is given by the following governing equation and initial conditions:

$$\text{Governing Equation: } a_n(x)\frac{d^n u}{dx^n} + a_{n-1}\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{12}$$

$$\text{Initial Conditions: } u(x_0) = u_0,\ u'(x_0) = u_1, \ldots, u^{(n-1)}(x_0) = u_{n-1} \tag{13}$$

Note: for an initial value problem, all of the initial conditions are provided at the same value of x; in accordance to custom we call this x_0. The name *initial* condition gives the implication that these conditions are at some "end" of the interval (beginning, left side, whatever) and in most all examples and exercises this is indeed the case. It is not, however, a requirement.

Generally for an n$^{\text{th}}$-order IVP you will need n conditions.

We seek a function defined on some interval containing x_0 that satisfies the differential equation with n conditions applied. The theorem below, which we will use by *citing* rather than *proving*, gives us assurance that, subject some fairly reasonable assumptions, such a solution will exist.

Theorem 1 (Existence and Uniqueness for IVPs)
If $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ are continuous on an interval $\mathcal{I}$, and if $a_n(x) \neq 0$ for every $x \in \mathcal{I}$, and if x_0 is any point in this interval, then a solution $u(x)$ of the IVP exists on the interval and it is unique.

For this class we will adopt a mostly operational definition of continuity: if you can draw the function throughout the specified interval without picking up your pencil or without diverging to infinity, then the function is continuous.

Consider, as an example, the following initial value problem:

$$u'' - 4u = 12x, \;\; u(0) = 4, \; u'(0) = 1 \tag{14}$$

This IVP satisfies the conditions of Theorem 1 since all of the coefficients and $g(x)$ are continuous and a_1 is constant and nonzero; hence a unique solution exists on any interval and that solution is unique.

Take a moment to verify that $u(x) = 3e^{2x} + e^{-2x} - 3x$ satisfies both the governing equation and initial conditions and thus is *the* unique solution to this IVP.

Here is an IVP that does *not* satisfy the criteria of Theorem 1:

$$x^2u'' - 2xu' + 2u = 6, \;\; u(0) = 3, \; u'(0) = 1 \tag{15}$$

In this case, the coefficients and $g(x)$ are all continuous but $a_2(x)$ is equal to zero at $x = 0$. This might not be a problem—i.e. if $x = 0$ is not in the interval of interest for the IVP then we are okay—but since $x_0 = 0$, $x = 0$ *must* be in the domain for the theorem to apply. So we have no assurances that a solution exists or, if a solution does exist, it may not be unique.

You should take a moment to verify that $u(x) = cx^2 + x + 3$ is a solution for the IVP given in Equation 15 for *any* choice of parameter c.

Boundary Value Problems

For this section let us, without undue loss of generality, consider a 2nd-order boundary value problem (BVP):

$$\text{Governing Equation: } a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{16}$$

$$\text{Boundary Conditions: } y(a) = y_0, \;\; y(b) = y_1, \;\; a \neq b \tag{17}$$

Almost all of the applications we will consider for this class will involve 2nd-order operators. The way we derive important boundary-value problems from underlying physical laws like conservation of mass and conservation of energy lead to them being 2nd-order. You should think about this while you are sitting in your fluid dynamics class and equations are being derived for conservation of mass and momentum for viscous incompressible fluid flow or when you are sitting in heat transfer class and the heat equation is being derived from conservation of energy principles. Probably the most obvious counterexample is beam theory which involves a 4th-order operator.

Depending on the boundary conditions, BVPs may have no solutions, one unique solution, or infinitely many solutions.

Example: The equation $u'' + 16u = 0$ has the general solution $u(t) = c_1 \cos(4t) + c_2 \sin(4t)$. Consider the three different sets of boundary conditions provided below.

a) $x(0) = 0, \;\; x(\pi/2) = 0$ Application of the first boundary condition gives us $c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$. The second boundary condition is $c_2 \sin(2\pi) = 0$, which is true for *any* value of c_2. Therefore there problem has infinitely many solutions.

b) $x(0) = 0, \;\; x(\pi/8) = 0$ The first boundary condition again gives us $c_1 = 0$; the second condition $c_2 \sin(4\frac{\pi}{8}) = 0$ is only satisfied if

$c_2 = 0$. Thus $c_1 = c_2 = 0$; only the trivial solution, $u = 0$, satisfies both the differential equation and boundary conditions. This is not a very interesting solution but at least it *is a solution* so we will take this as an example of a BVP having a unique solution.

For applications, we will generally be only interested in *non-trivial* solutions; that is, solutions that are not identically equal to zero.

c) $x(0) = 0, \;\; x(\pi/2) = 1$ In this case, again $c_1 = 0$ from the first boundary condition. This leaves the second boundary condition: $c_2 \sin\left(4\frac{\pi}{2}\right) = c_2(0) = 1$ which cannot be satisfied for any value of c_2. In this case *no* solution exists.

Superposition and Linear Dependence

In this section some important theorems regarding IVPs and BVPs will be presented. No attempt will be made to prove these theorems; we will simply take these theorems as facts that are relevant for this course that you should try to understand as best you can.

Theorem 2 (Superposition Principle for Homogeneous Equations) *Let $u_1, u_2, \ldots, u_k$ be solutions of a homogeneous n^{th}-order linear differential equation. Then any linear combination of those solutions*

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants, is also a solution.

Note: It is essential that *both* the governing equation and given conditions (boundary or initial) for the linear differential equation are homogeneous. As a reminder, this means that *all* terms in the governing equation and boundary conditions must either a) involve the dependent variable or one of its derivatives; or b) be equal to zero.

As an example, If I denote the linear homogeneous differential equation as $\mathcal{L}$, then $\mathcal{L}(u_i) = 0$ for any $i \in [1, 2, \ldots, k]$. By the linearity property of $\mathcal{L}$, for any constants α and β:

$$\begin{aligned} \mathcal{L}(\alpha u_i + \beta u_j) &= \alpha \mathcal{L}(u_i) + \beta \mathcal{L}(u_j) \\ &= \alpha(0) + \beta(0) \\ &= 0 \end{aligned}$$

Theorem 3 (Linear Dependence / Independence of Functions) *A set of functions $f_1(x), f_2(x), \ldots, f_k(x)$ is said to be* linearly dependent *on an interval $\mathcal{I}$ if there exist constants $c_1, c_2, \ldots, c_k$,* not <u>all</u> of which are zero*, such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0$$

for every $x \in \mathcal{I}$. If the set of functions is not *linearly dependent, it is linearly independent.*

What if a member of the set of functions is $f(x) = 0$?

Answer: The set will no longer be linearly independent. The trivial function $f(x) = 0$ is not linearly independent from *anything*.

Repeatedly throughout this course we will want to clarify whether or not two or more functions are linearly independent of each other. I think most engineers have a general idea of what it is we *mean* when we say two functions are linearly independent or dependent but Theorem 3 specifies what these things mean *mathematically*.

WE NEED A TEST to help us determine if the members of a set of functions are linearly independent or not. This will be especially important as we evaluate solutions to a linear homogeneous differential equation. Even if you are the sort of savant who can, by inspection, always detect linear dependence, you might have a hard time convincing your friends that your assessment is always correct. Luckily, there is a theorem that provides a suitable test that can serve as irrefutable evidence of the state of linear dependence/independence of functions.

Theorem 4 (Criterion for Linearly Independent Solutions)
Let $u_1, u_2, \ldots, u_n$ be solutions of a homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$. Then the set of solutions is linearly independent on the interval if and only if the Wronskian of the solution is non-zero for every $x \in \mathcal{I}$.

The Wronskian is a function that takes functions as arguments and returns a scalar numeric quantity.[6]

[6] Sometimes such functions are referred to as *functionals*.

$$W(u_1, u_2, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{18}$$

where $|\cdot|$ denotes the matrix determinant. For large values of n this is also difficult to calculate but, for the case $n = 2$, engineering students should be familiar with the formula:

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2 \tag{19}$$

Example: show that the functions $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ are linearly independent solutions to the homogeneous linear equation $u'' - 9u = 0$ for every $x \in (-\infty, \infty)$.
Solution: The Wronskian is given by:

The reader should verify that both $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ satisfy the given differential equation.

$$\begin{aligned} W &= \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} \\ &= e^{3x}\left(-3e^{-3x}\right) - 3e^{3x}\left(e^{-3x}\right) \\ &= 3e^{3x-3x} - 3e^{3x-3x} \\ &= 3 - 3 \\ &= 6 \end{aligned}$$

Since $6 \neq 0$ for all $x \in (-\infty, \infty)$ the solutions are linearly independent.

Definition 1 (Fundamental Set of Solutions)
Any set $u_1, u_2, \ldots, u_n$ of n linearly independent solutions of the homogeneous linear n^{th}-order differential equation on an interval is said to be a fundamental set of solutions on an interval $\mathcal{I}$.

Theorem 5 (Existence of a Fundamental Set)
There exists a fundamental set of solutions for the homogeneous linear n^{th}-order differential equation on an interval $\mathcal{I}$.

Note: This is different than saying that a BVP or IVP has a solution. This theorem is only referring to the differential equation; not the boundary or initial conditions.

Definition 2 (General Solution—Homogeneous Equation)
Let $u_1, u_2, \ldots, u_n$ be a fundamental set of solutions to the homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$, then the general solution is:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x)$$

It is important to understand from the above that:

- *any* possible solution to the homogeneous, linear, n$^{\text{th}}$-order differential equation can be constructed by setting the coefficients of the general solution; and
- there is **no** solution that can be constructed from functions that are linearly independent from the general solution.

General Solution for a Non-homogeneous Problem

Recall: “non-homogeneous” for a linear n$^{\text{th}}$-order differential equation means that $g(x) \neq 0$. if u_p is any particular solution to the non-homogeneous, linear, n$^{\text{th}}$-order ODE on an interval $\mathcal{I}$ and $u_c = c_1 u_1(x) + c_2 u_2(x) + \cdots c_n u_n(x)$ is the general solution to the associated homogeneous ODE (called the *complementary* solution) then the general solution to the non-homogeneous ODE is:

$$u = u_c + u_p$$

Example: By substitution it can be seen that $u_p = -\frac{11}{12} - \frac{1}{2}x$ is a particular solution to $u''' - 6u'' + 11u' - 6u = 3x$. The general solution to the associated homogeneous problem is $u_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. Consequently, the general solution to the linear non-homogeneous problem is:

You are, again, strongly encouraged to verify that u_p satisfies the given equation and that u_c satisfies the associated homogeneous equation.

$$\begin{aligned} u(x) &= u_c + u_p \\ &= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x \end{aligned}$$

Lecture 4 - Homogeneous Linear Equations with Constant Coefficients

Objectives

The objectives of this lecture are:

- Review the solution methodology for homogeneous linear equations with constant coefficients.
- Illustrate this method with several examples.

Introduction

In this lecture we will review the well-trod ground of your differential equations class and remind ourselves how to solve linear, constant coefficient, homogeneous, n$^{\text{th}}$-order differential equations. These equations have the general form shown in Equation 20

$$c_n u^{(n)} + c_{n-1} u^{(n-1)} + \cdots + c_1 u' + c_0 u = 0 \tag{20}$$

where the coefficients are real and constant an $c_n \neq 0$.

The basic strategy is to assume the solution is of the form: $u(x) = e^{mx}$. For the case of 2$^{\text{nd}}$-order equations, we get:

$$\begin{gathered} c_2 m^2 e^{mx} + c_1 m e^{mx} + c_0 e^{mx} = 0 \\ e^{mx}\left(c_2 m^2 + c_1 m + c_0\right) \end{gathered}$$

where the last line above is called the auxiliary equation:

Here we re-name the constants so Equation 21 takes a familiar form.

$$am^2 + bm + c = 0 \tag{21}$$

From the well-known quadratic equation, solutions are: $m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ Solution of this equation gives the following three cases:

1. **Distinct Real Roots** In this case $m_1 \neq m_2$ and the general solution is of the form:

$$u(x) = c_1 \underbrace{e^{m_1 x}}_{u_1(x)} + c_2 \underbrace{e^{m_2 x}}_{u_2(x)} \tag{22}$$

Using tools from the last lecture you should recognize that $u_1(x)$ and $u_2(x)$ are linearly independent for all $x \in (-\infty, \infty)$, thus form a fundamental set of solutions.

AN IMPORTANT SPECIAL CASE is when m_1 and m_2 are roots of a positive real number and thus $m_1 = -m_2$. This happens when the governing equation is of the form:

$$u'' - k^2 u = 0 \tag{23}$$

The solutions are thus:

$$u(x) = c_1 e^{-kx} + c_2 e^{kx} \tag{24}$$

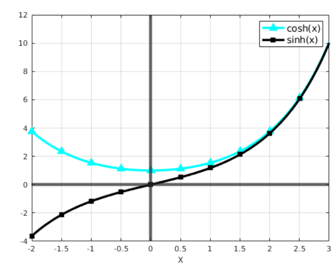


Figure 1: Plot of $\cosh x$ and $\sinh x$

For reasons that will become clear later in the course, it is sometimes useful to re-express the solution shown in Equation 24 in terms of the functions cosh and sinh. These functions are defined as linear combinations of exponentials as shown below and plotted in Figure 1

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

2. **Real Repeated Roots** In this case $m_1 = m_2$. One solution is:

i.e. from the quadratic equation, $b^2 - 4ac = 0$

$$u_1(x) = e^{m_1 x} \tag{25}$$

The other solution so derived is, of course, the same and thus we do not have two linearly independent solutions as required to form a fundamental set of solutions for a 2nd-order linear homogeneous equation.

IT CAN BE SHOWN that a second linearly independent solution can be formed by multiplying by the independent variable:

This is done using a technique referred to as *reduction of order*. We will not take the time to cover it in this class (or in this book) but is concisely described in section 3.2 of Zill. At a minimum you might at least confirm for yourself that a) $xu_1(x)$ is a solution to the equation; and b) use the Wronskian to confirm that it is linearly independent from $u_1(x)$.

$$u_2(x) = xu_1(x) = xe^{mx}$$

and thus the general solution for this case is:

$$u(x) = c_1 e^{mx} + c_2 x e^{mx} \tag{26}$$

3. **Conjugate Complex Roots** In this case the discriminant, $b^2 - 4ac$, is negative so its square root is imaginary. This results in m_1 and m_2 being complex conjugates which we will express as: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

The general solution is:

$$\begin{aligned} u(x) &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) \end{aligned}$$

The complex exponentials in the last equation can be re-expressed using the Euler Formula:

$$\begin{aligned} e^{i\beta x} &= \cos\beta x + i\sin\beta x \\ e^{-i\beta x} &= \cos\beta x - i\sin\beta x \end{aligned}$$

which is slightly more convenient insofar as the solutions are no longer expressed as complex exponentials but also by breaking each solution down into their real and complex parts. It can be shown that both the real and imaginary parts of the solution must satisfy the differential equation *independently*. This fact allows us to re-express the solution in a more simple form that does not involve complex numbers:

$$u(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right) \tag{27}$$

ANOTHER IMPORTANT special case is when the solution is *pure imaginary* (i.e. $\alpha = 0$) so the solution is:

$$u(x) = c_1\cos\beta x + c_2\sin\beta x \tag{28}$$

These solutions arise when the governing equation is of the shown in Equation 29:

$$u'' + k^2 u = 0 \tag{29}$$

The roots $m_{1,2} = \pm ik$ and the general solution is:

$$u(x) = c_1\cos kx + c_2\sin kx \tag{30}$$

This equation will be revisited throughout the course as it repeatedly comes up in applications.

Three Examples

The cases described above will be illustrated with three examples:

Example #1: Find the general solution to $2u'' - 5u' - 3u = 0$. Inserting $u = e^{mx}$ into the equation gives us the auxiliary equation:

$$2m^2 - 5m - 3 = (2m+1)(m-3)$$

with roots: $m_1 = -\frac{1}{2}$ and $m_2 = 3$. These are real, distinct roots so the general solution is:

$$u(x) = c_1 e^{-x/2} + c_2 e^{3x}$$

Example #2: Find the general solution to $u'' - 10u' + 25u = 0$. The auxiliary equation is:

$$m^2 - 10m + 25 = (m-5)(m-5)$$

with (repeated) roots: $m_1 = 5$ and $m_2 = 5$. These are real, repeated roots so the general solution is:

$$u(x) = c_1 e^{5x} + c_2 x e^{5x}$$

Example #3: Find the general solution to $4u'' + 4u' + 17u = 0, \quad u(0) = -1, \; u'(0) = 2$.

This is an initial value problem with continuous (and constant) coefficients. We know from Theorem 1 that a unique solution exists. We will first find the general solution, then apply the initial conditions to resolve the unknown coefficients to reveal the solution.

We can see that it must be an *initial* value problem because the conditions are both given at the same location, $x_0 = 0$.

The auxiliary equation is:

$$4m^2 + 4m + 17 = 0$$

using the quadratic equation, gives us:

$$\begin{aligned} \frac{-4 \pm \sqrt{16 - 4(4)(17)}}{2(4)} &= -\frac{1}{2} \pm \frac{\sqrt{-256}}{8} \\ &= -\frac{1}{2} \pm \frac{-16}{8} \\ &= -\frac{1}{2} \pm 2i \end{aligned}$$

This gives us complex conjugate roots and the general solution is:

$$u(x) = e^{-x/2}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

Applying the initial condition $u(0) = -1$ gives us:

$$\begin{aligned} u(0) &= e^0\left(c_1 \cos 0 + c_2 \sin 0\right) \\ &= 1(c_1(1) + c_2(0)) \\ &= c_1 = -1 \end{aligned}$$

To apply the second initial condition we need to use the chain-rule and product rule to differentiate the general solution. This gives us:

$$\begin{aligned} u'(x) = -\frac{1}{2}e^{-x/2}c_1 \cos 2x - 2e^{-x/2}c_1 \sin 2x + \\ -\frac{1}{2}e^{-x/2}c_2 \sin 2x + 2e^{-x/2}c_2 \cos 2x \end{aligned}$$

Evaluating $u'(0)$ and substituting $c_1 = -1$ gives us:

$$\begin{aligned} u'(0) &= -\frac{1}{2}(1)(-1)(1) + (1)(2)c_2(1) \\ &= \frac{1}{2} + 2c_2 = 2 \\ \Rightarrow 2c_2 &= \frac{3}{2} \\ c_2 &= \frac{3}{4} \end{aligned}$$

Both constants are now known and the unique solution is:

$$u(x) = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

Lecture 5 - Non-homogeneous Linear Equations with Constant Coefficients

Objectives

The objectives of this lecture are:

- Describe the Method of Undetermined Coefficients for solving non-homogeneous linear equations with constant coefficients.
- Carry out some examples to illustrate the methods.

In this lecture we will review *a* method for finding solutions to non-homogeneous linear equations with constant coefficients.

To be perfectly honest, we spend very little time in this class dealing with non-homogeneous equations of any kind; many of those types of equations are beyond our ability to solve analytically so we turn to numerical methods instead. Nonetheless there is value in reminding ourselves how to construct solutions for those cases where we can.

Background

Consider the equation

$$a_n u^{(n)} + a_{n-1} u^{(n-1)} + \cdots + a_1 u' + a_0 u = g(x) \tag{31}$$

where

- the coefficients $a_i,\ i \in [1, 2, \ldots, n]$ are constants; and
- the function $g(x)$ is a constant, a polynomial function, exponential function, sine or cosine, or finite sums or products of these functions.

The general solution, $u(x)$, can be constructed as $u_c(x) + u_p(x)$ where

- $u_c(x)$ is the complementary solution which, as you should recall, is the general solution to the associated homogeneous problem. (i.e. Equation 31 with $g(x) = 0$); and
- $u_p(x)$ is (any) particular solution—that is, a not-necessarily-unique function that satisfies Equation 31.

We spent the last lecture describing, effectively, how to find $u_c(x)$; the question this lecture will hope to answer is: "How do I find $u_p(x)$?"

Method of Undetermined Coefficients

One method for finding $u_p(x)$ is called the Method of Undetermined Coefficients.[7]

[7] Some people lovingly refer to this technique as "The Method of Guessing."

THERE ARE THREE parts to this technique

1. **Basic Rule:** based on the terms in $g(x)$, select the appropriate form for $u_p(x)$ using Table 1.

Term in $g(x)$	Choice for $u_p(x)$
$ke^{\gamma x}$	$Ae^{\gamma x}$
$kx^n,\ (n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$\}\ K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\}\ e^{\alpha x} (K \cos \omega x + M \sin \omega x)$
$ke^{\alpha x} \sin \omega x$	

Table 1: Forms of $u_p(x)$ for given terms in $g(x)$

2. **Modification rule:** if $u_p(x)$ obtained by the **Basic Rule** happens to be a solution to the associated homogeneous equation, multiply $u_p(x)$ from the table by x (or x^2 if needed).

3. **Sum rule:** if $g(x)$ is a linear combination of terms from the left-hand column, construct $u_p(x)$ from a linear combination of the corresponding entries in the right-hand column.

For the remainder of this lecture, we will practice applying these rules to some example problems.

Example: solve $u'' + 4u' - 2u = 2x^2 - 3x + 6$

Step #1: find the general solution to the associated homogeneous equation.

The auxiliary equation is: $m^2 + 4m - 2 = 0$; using the quadratic equation gives us:

Here you are expected to examine the associated homogeneous problem as $u'' + 4u' - 2u = 0$, identify it as constant coefficient and linear, and solve by assuming $u = e^{mx}$ and thus deriving the auxiliary equation shown without further prompting.

$$\begin{aligned} m &= \frac{-4 \pm \sqrt{16 - (4)(1)(-2)}}{2(1)} \\ &= -2 \pm \frac{\sqrt{24}}{2} \\ &= -2 \pm \sqrt{6} \end{aligned}$$

so $u_c(x) = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$

Step #2: Apply the method of undetermined coefficients to construct a candidate $u_p(x)$

Since $g(x)$ is a second-order polynomial, the table tells us $u_p(x)$ is in the general form of a second-order polynomial.

$$u_p(x) = K_2x^2 + K_1x + K_0$$

We plug this into the governing equation and this gives us:

$$2K_2 + 4(2K_2x + K_1) - 2(K_2x^2 + K_1x + K_0) = 2x^2 - 3x + 6$$

Now we need to equate the coefficient for each power of x:

$$\begin{aligned} x^2&: & -2K_2 &= 2 \\ x&: & 8K_2 - 2K_1 &= -3 \\ 1&: & 2K_2 + 4K_1 - 2K_0 &= 6 \end{aligned}$$

In general you cannot expect this to go so nicely. What you *can* hope for is that the, in this case, three equations you derive will have a unique solution. We could re-write the system in the form of a matrix-vector equation:

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 4 & 2 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

If the solution of such a matrix cannot be done by inspection and simple algebra as it was in this case, we could use tools like MATLAB to solve the linear system of equations. This topic and much more is covered in the numerical methods portion of this text.

Luckily for us, this system of equations is structured such that it can easily be solved. We see by inspection that $K_2 = 2/-2 = -1$; this can be plugged into the second equation to find $K_1 = -5/2$ and then we can solve the last equation to find that $K_0 = -9$.

Thus the particular solution is:

$$u_p(x) = -x^2 - \frac{5}{2}x - 9$$

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

We now have both the complementary solution and a particular solution; we form the general solution to the equation by adding them together.

$$\begin{aligned} u(x) &= u_c(x) + u_p(x) \\ &= c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9 \end{aligned}$$

Why, again, do we need the constants c_1 and c_2?

Answer: Because we have not yet applied initial/boundary conditions. If those conditions are provided—two conditions for a 2nd-order problem—then we can resolve the constants.

Example: solve $u'' - 5u' + 4u = 8e^x$

Step #1: find the general solution to the associated homogeneous problem.

The auxiliary equation is $m^2 - m + 4 = 0$ the left side of which can easily be factored to give $(m-4)(m-1) = 0$; the roots of which are $m_1 = 4$, $m_2 = 1$. The complementary solution is:

$$u_c(x) = c_1e^{4x} + c_2e^x$$

Step #2: Apply the method of undetermined coefficients to construct $u_p(x)$.

Inspecting Table 1 we see that $u_p(x)$ should be of the form Ae^x. If that function seems vaguely familiar it may be because e^x is part of the complementary solution.

Pop Quiz: if you plug Ae^x into your governing equation, without doing any calculations, what value should you get?

Answer: you will get 0! Why? Because e^x is one of the two linearly independent solutions to the associated homogeneous problem.

What do I do now?

Answer: invoke the Modification Rule—this is, after all, the reason why the rule exists—and multiply u_p by x. We now have $u_p(x) = Axe^x$.

We insert this proposed function for $u_p(x)$ into the equation and we get:

$$2Ae^x + Axe^x - 5(Ae^x + Axe^x) + 4Axe^x = 8e^x$$

Combine terms and solve for A:

$$\begin{aligned} 2Ae^x - 5Ae^x &= 8e^x \\ -3Ae^x &= 8e^x \\ A &= -\frac{8}{3} \end{aligned}$$

So the particular solution is:

$$u_p(x) = -\frac{8}{3}xe^x$$

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

$$\begin{aligned} u(x) &= u_c(x) + u_p(x) \\ &= c_1e^{4x} + c_2e^x - \frac{8}{3}xe^x \end{aligned}$$

Note: If any of this seems at all sketchy to you, the good news is that you need not worry if your proposed $u_p(x)$ is any good; you can just plug it into the differential equation and find out!

THIS LAST EXAMPLE illustrates the use of the Sum Rule; it also includes initial condition so the unique solution to the initial value problem can be found.

Example: solve the initial value problem: $u'' + u = 4x + 10\sin x$ with initial conditions $u(\pi) = 0,\ u'(\pi) = 2$.

Step #1: find the general solution to the associated homogeneous problem.

The auxiliary equation is: $m^2 + 1 = 0$, therefore $m = \pm i$ and $u_c(x)$ can be found as:

$$u_c(x) = c_1\cos x + c_2\sin x$$

Step #2: Apply the method of undetermined coefficients to construct $u_p(x)$.

For this problem, $g(x) = 4x + 10\sin x$ has two terms; so we will construct $u_p(x)$ using one term at a time; $u_{p_1}(x)$ using $4x$ and $u_{p_2}(x)$ using $10\sin x$

It's the linearity property of $\mathcal{L} = \frac{d^2}{dx} + 1$ that makes this possible. If $\mathcal{L}(u_{p_1}) = 4x$ and $\mathcal{L}(u_{p_2}) = 10\sin x$ then $\mathcal{L}(u_{p_1} + u_{p_2}) = 4x + 10\sin x$.

Step #2.a: find $u_{p_1}(x)$.

From Table 1, for $g(x) = 4x$ we should select $u_{p_1} = K_1 x + K_0$. Inserting this into the differential equation gives us: $K_1 x + K_0 = 4x$. By inspection we can see that $K_0 = 0$ and $K_1 = 4$ so $u_{p_1}(x) = 4x$.

Step #2.b: find $u_{p_2}(x)$.

From Table 1, for $g(x) = 10\sin x$ we should select $u_{p_2} = K\cos x + M\sin x$. Now that we have done this a couple of times we should be on the alert for portions of the complementary solution cropping up in our guesses for $u_p(x)$ so we immediately see that we must multiply u_{p_2} by x. If we do this and insert $Kx\cos x + Mx\sin x$ into the differential equation we get:

$$
\begin{aligned}
&(-2K - Mx)\sin x + (2M - Kx)\cos x + \dots \\
&\qquad\qquad\qquad\qquad Kx\cos x + Mx\sin x = 10\sin x
\end{aligned}
$$

Matching coefficients for $\sin x$ and $\cos x$ on both sides of the above equation leads us to conclude that $M = 0$ and $-2K = 10$. Therefore $K = -5$ and $u_{p_2}(x) = -5x\cos x$.

Again, there is no harm in testing your proposed $u_{p_2}(x)$ to see if it does indeed produce the expected result.

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

$$
\begin{aligned}
u(x) &= u_c(x) + u_p(x) \\
&= u_c(x) + u_{p_1}(x) + u_{p_2}(x) \\
&= c_1\cos x + c_2\sin x + 4x - 5x\cos x
\end{aligned}
$$

All that remains is to apply the initial conditions.

$$
\begin{aligned}
u(\pi) &= c_1(-1) + c_2(0) + 4\pi - 5(\pi)(-1) \\
&= -c_1 + 9\pi = 0 \\
\Rightarrow &= c_1 = 9\pi
\end{aligned}
$$

Applying the initial condition $u' = 2$:

$$u'(\pi) = -9\pi(0) + c_2(-1) + 4 - 5(-1) + 5\pi(0) = 2$$

Solving for c_2 gives us: $c_2 = 7$; folding this into the general solution:

$$u(x) = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

Assignment #2

The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family (i.e. find the values for the constants c_1 and c_2) that is a solution of the initial-value problem.

1. $u = c_1 e^x + c_2 e^{-x}; \;\; u'' - u = 0, \; u(0) = 0, \; u'(0) = 1$

2. $u = c_1 x + c_2 x \ln x, \;\; (0, \infty), \; x^2 u'' - x u' = 0, \; u(1) = 3, \; u'(1) = -1$

The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine if a member of the family can be found that satisfies the boundary conditions.

3. $u = c_1 e^x \cos x + c_2 e^x \sin x; \;\; u'' - 2u' + 3u = 0$
   (a) $u(0) = 1, \; u'(\pi) = 0$
   (b) $u(0) = 1, \; u(\pi) = -1$
   (c) $u(0) = 1, \; u(\pi/2) = 1$
   (d) $u(0) = 0, \; u(\pi) = 0$

Determine if the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

4. $f_1(x) = x, \;\; f_2(x) = x^2, \;\; f_3(x) = 4x - 3x^2$

5. $f_1(x) = 1 + x, \;\; f_2(x) = x, \;\; f_3(x) = x^2$

Verify that the given two-parameter family of functions is the general solution of the non-homogeneous differential equation on the indicated interval.

6. $u'' - 7u' + 10u = 24e^x, \;\; u = c_1 e^{2x} + c_2 e^{5x} + 6e^x, \; (-\infty, \infty)$

Find the general solution to the given second-order differential equation.

7. $4u'' + u' = 0$

8. $u'' - u' - 6u = 0$

9. $u'' + 8u' + 16u = 0$

10. $u'' + 9u = 0$

Solve the given initial-value problem.

11. $u'' + 16u = 0, \;\; u(0) = 2, \; u'(0) = -2$

12. $u'' - 4u' - 5u = 0, \;\; u(1) = 0, \; u\prime(1) = 2$

13. $u'' + u = 0, \;\; u'(0) = 0, \; u'(\pi/2) = 0$

Solve the given differential equation using the Method of Undetermined Coefficients.

14. $u'' - 10u' + 25u = 30x + 3$

15. $u'' + 3u = -48x^2e^{3x}$

Solve the given initial-value problem.

16. $5u'' + u' - 6x, \;\; u(0) = 0, \; u'(0) = -10$

Solve the given boundary-value problem.

17. $u'' + u = x^2 + 1, \;\; u(0) = 5, \; u(1) = 0$

Solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint:* Solve the problem on two intervals and then find a solution so that u and u' are continuous at the boundary of the interval.]

18. $u'' + 4u = g(x), \;\; u(0) = 1, \; u'(0) = 2$

$$g(x) = \begin{cases} \sin x & 0 \le x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}$$

Lecture 6 - Cauchy-Euler Equations

Objectives

The objectives of this lecture are:

- Introduce Cauchy-Euler equations and demonstrate a method of solution
- Carry out some examples to illustrate the methods for 2$^{\text{nd}}$-order, homogeneous Cauchy-Euler equations.

Cauchy-Euler Equations

A linear differential equation of the form

$$a_n x^n \frac{d^n u}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_1 x \frac{du}{dx} + a_0 u = g(x) \tag{32}$$

is called a Cauchy-Euler equation.

NOTE THE RELATIONSHIP between the exponent of x is the coefficients and the order of the differential operators. This correspondence between the decreasing power of x in the coefficient and the decreasing order of the differential operator is characteristic of this type of equation and is the way you should recognize it.

It is the corresponding change in power/order that matters. $a\frac{d^2u}{dx^2} + \frac{1}{x^2}u = 0$ is also a Cauchy-Euler equation since the power of x in the coefficient goes from 0 to -2 while the order of the differential operator goes from 2$^{\text{nd}}$ to 0.

NOTE THAT this equation is *linear*; if $g(x) = 0$ it is homogeneous, otherwise it is non-homogeneous. For this lecture we will focus our attention on the homogeneous, 2$^{\text{nd}}$-order Cauchy-Euler equation:

$$ax^2 \frac{d^2 u}{dx^2} + bx \frac{du}{dx} + cu = 0 \tag{33}$$

NOTE THAT the coefficient for the highest order derivative is 0 at $x = 0$; consequently we will restrict the interval of interest for these equations to $x \in (0, \infty)$.

THE BASIC STRATEGY in solving these equations is to try a solution in the form $u(x) = x^m$. When we substitute this solution into the equation we get:

$$am(m-1)x^2x^{m-2} + bmxx^{m-1} + cx^m = 0$$
$$x^m\left[am(m-1) + bm + c\right] = 0$$

That last part in the brackets is referred to as the "auxiliary equation":

$$am^2 + (b-a)m + c = 0 \tag{34}$$

We will look for values of m that satisfy this quadratic equation; that will be the exponent for our solution.

AS IS THE CASE for quadratic equation, there are three possible outcomes:

1. **Distinct Real Roots.** In this case $m_1 \neq m_2$ and the general solution is of the form

$$u(x) = c_1x^{m_1} + c_2x^{m_2} \tag{35}$$

   Example: find the general solution for $x^2\frac{d^2u}{dx^2} - 2x\frac{du}{dx} - 4u = 0$

   Referring to Equation 34, $a = 1,\ b = -2,\ c = -4$ so the auxiliary equation is:

$$m^2 - 3m - 4 = 0(m-4)(m+1) \qquad = 0$$

   By inspection the roots are $m_1 = 4$ and $m_2 = -1$. The general solution is $u(x) = c_1x^4 + c_2x^{-1}$

Be careful with these coefficients; in contrast to the case with constant coefficient linear equations we do not plug these coefficients directly into the quadratic equation; we put them in the auxiliary equation and then solve *that* with the quadratic equation.

2. **Real Repeated Roots.** In this case, $m_1 = m_2$. We have one solution, $u_1(x) = c_1x^{m_1}$; clearly we need to take some kind of action if we hope to get another linearly independent solution. It can be shown that if we form the second solution by multiplying the first solution by $\ln x$—$u_2(x) = \ln(x)u_1(x)$—then $u_2(x)$ will satisfy the governing equation and also be linearly independent from $u_1(x)$.

The first one or two times you solve these problems, you should verify both of those assertions.

   Example: find the general solution for $4x^2\frac{d^2u}{dx^2} + 8x\frac{du}{dx} + u = 0$.

   The auxiliary equation in this case is: $4m^2 + 4m + 1 = 0$. This can be factored to give $(2m+1)^2 = 0$ so we have a case of repeated roots where $m_1 = m_2 = -\frac{1}{2}$.

   The solution is: $u(x) = c_1x^{-1/2} + c_2x^{-1/2}\ln x$.

3. **Complex Conjugate Roots.** This case is completely analogous with the previous cases vis-à-vis linear constant coefficient equations. The roots are $m_{1,2} = \alpha \pm i\beta$ and the general solution is:

$$u(x) = x^{\alpha}\left[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)\right] \tag{36}$$

Example: Solve: $4x^2u'' + 17u = 0, \;\; u(1) = -1, \; u'(1) = -1/2$

The auxiliary equation is $4m^2 - 4m + 17 = 0$. Using the quadratic formula the roots are found to be:

$$\begin{aligned} m_{1,2} &= \frac{4 \pm \sqrt{16 - 4(4)(17)}}{8} \\ &= \frac{1}{2} \pm \frac{\sqrt{-256}}{8} \\ &= \frac{1}{2} \pm \frac{16i}{8} \\ &= \underbrace{\frac{1}{2}}_{\alpha} \pm \underbrace{2}_{\beta} i \end{aligned}$$

So the general solution is:

$$u(x) = x^{1/2}\left[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)\right]$$

We can apply the first boundary condition, $u(1) = -1$:

$$\begin{aligned} u(1) &= 1\left[c_1 \cos 0 + c_2 \sin 0\right] \\ &= c_1(1) + c_2(0) = -1 \\ \Rightarrow c_1 &= -1 \end{aligned}$$

The calculus is a bit more tedious for the second boundary condition:

$$\begin{aligned} u'(x) = -\frac{1}{2}x^{-1/2}\cos(2\ln x) + 2x^{-1/2}\sin(2\ln x) + ... \\ c_2\left[\frac{1}{2}x^{-1/2}\sin(2\ln x) + 2x^{-1/2}\cos(2\ln x)\right] \end{aligned}$$

Evaluating this at $x = 1$:

$$\begin{aligned} u'(1) &= -\frac{1}{2}(1)(1) + 2(1)(0) + c_2[0 + 2(1)(1)] \\ &= -\frac{1}{2} + 2c_2 = -\frac{1}{2} \\ \Rightarrow c_2 &= 0 \end{aligned}$$

So the solution is: $u(x) = -x^{1/2}\cos(2\ln x)$

Non-homogeneous Cauchy-Euler Equations

Sadly, the Method of Undetermined Coefficients will not work with Cauchy-Euler equations; a limitation of that method is that the coefficients need to be constant. Interested students can investigate the method of Variation of Parameters that can be used to address this problem analytically. Otherwise, we will plan to use numerical methods to solve non-homogeneous problems of this type.

Derivation of the Solution to Cauchy-Euler Equations

It would be hard not to notice the similarity in the solution methods of Cauchy-Euler equations and constant coefficient linear equations. This is not a coincidence. In this section I want to briefly show you that, through a change of variables, Cauchy-Euler equations are, in some sense, equivalent to constant coefficient equations.

Change of Independent Variable

What we will do, is change the independent variable from x to e^t.[8] If $x = e^t$, that means that $t = \ln x$ and $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$.

If we consider, again, the 2nd-order Cauchy-Euler equation,

$$ax^2 \frac{d^2u}{dx^2} + bx\frac{du}{dx} + cu = 0$$

every appearance of x needs to be converted into its equivalent in terms of t and every derivative with respect to x needs to be converted into derivatives with respect to t.

It's easy enough to replace x with e^t; converting the derivatives takes a bit more work. We will use the chain rule as shown below:

$$\begin{aligned}\frac{du}{dx} &= \frac{du}{dt}\frac{dt}{dx} \\ &= u_t e^{-t}\end{aligned}$$

where we use the subscript notation to denote derivatives with respect to t and use the substitution $\frac{dt}{dx} = e^{-t}$ as determined above.

We do it again, to convert the second derivatives:

$$\begin{aligned}\frac{d^2u}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}\right) \\ &= \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx} \\ &= \frac{d}{dt}\left(u_t e^{-t}\right)e^{-t} \\ &= \left(u_{tt}e^{-t} - u_t e^{-t}\right)e^{-t} \\ &= e^{-2t}\left(u_{tt} - u_t\right)\end{aligned}$$

[8] Think of this as "stretching" the x-axis.

We are now ready to make our substitutions into the differential equation:

$$a \underbrace{e^{2t}}_{x^2} \overbrace{e^{-2t}\left(u_{tt} - u_t\right)}^{\frac{d^2u}{dx^2}} + b \underbrace{e^t}_{x} \overbrace{u_t e^{-t}}^{\frac{du}{dx}} + cu = 0$$

Combining terms to simplify gives us Equation 37 which is now, under this change of variables, a 2nd-order linear constant coefficient equation.

$$au_{tt} + (b-a)u_t + cy = 0 \tag{37}$$

If I solve this using our standard method, the resulting auxiliary equation is the same as what is shown in Equation 34.

IN THE CASE of constant coefficient linear equations, the solutions were of the form $u = e^{mx}$ which, according to the exponentiation rules, the same as $u = e^{x^m}$. But now, our independent variable is t, where $t = \ln x$. With this substitution:

$$\begin{aligned} u(t) &= e^{(\ln x)^m} \\ &= x^m \end{aligned}$$

which is the assumed form of solution for Cauchy-Euler equations.

Part II

Power Series Methods

Lecture 7 - Review of Power Series

Objectives

The objectives of this lecture are:

- Review definitions and basic properties of power series.
- Illustrate important basic operations on power series

Introduction and Review

The methods that we have discussed so far have largely been a review of differential equations class. Sadly, even in the handful of lectures that we have had, our methods for solving equations are largely exhausted. We can solve constant coefficient linear equations, and variable coefficient linear equations *if* they happen to be Cauchy-Euler equations. We can solve many first-order linear equations but if the equation is nonlinear we are sunk unless they happen to be separable. This leaves out a lot of interesting equations. In this sequence of lectures we will discuss how to solve linear equations with variable coefficients (other than Cauchy-Euler equations). To do this we will need to use power series.

You learned about power series back in calculus class, but you weren't ready to use them for this important application. Now you are and now this is what we will do. We will begin this section with some definitions that will be needed as we describe the use power series in the solution of differential equations.

Definitions

Definition 3 (Sequence)
A sequence *is a list of numbers (or other mathematical objects, like functions) written in a definite order.*

$$\{c_0, c_1, c_2, c_3, \ldots, c_n\}$$

Definition 4 (Limit of a Sequence, convergence, divergence)
A sequence has a limit (L) *if we can make the terms c_n arbitrarily close to L by taking n sufficiently large. If* $\lim_{n\to\infty} c_n$ *exists, we say the sequence* converges*; otherwise, we say the sequence* diverges *or is* divergence*.*

There are various mathematical tools available for determining if an infinite sequence converges or diverges without needing to examine every element.

Definition 5 (Series, infinite series)
A series *is the sum of a sequence. For example,* $S_0 = c_0$; $S_1 = c_0 + c_1$; $S_n = c_0 + c_1 + \cdots + c_n$. *If the sequence is infinite, we call the sum an infinite series.*

Definition 6 (Series Convergence)
Given a series $\sum_{n=0}^{\infty} s_i = s_1 + s_2 + \cdots + s_n + \cdots$, *let* s_n *denote its* n^{th} *partial sum. If the sequence* $\{s_n\}$ *is convergent then the series is convergent to the same limit. Otherwise the series is divergent.*

We will use notation such as $s_n \to \infty$ to indicate that the partial sum is unbounded.

Definition 7 (Power Series)
A series of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + \cdots$ *is called a Power Series. The constant a is referred to as the "center" of the power series.*

For almost all of the power series we will work with in this class, the series will be centered on $a = 0$ and will be denoted $\sum_{n=0}^{\infty} c_n x^n$.

Definition 8 (Interval of Convergence, Radius of Convergence)
The set of all real numbers x for which the series converges. This interval can also be expressed as a radius of convergence. *(R); the series converges for all* $a - R < x < a + R$.

Ratio Test

We should have at least one test that we can use to decide whether or not a series, at least a power series, converges. The test we will use is called the *Ratio Test*; so named because it involves the ratio of the n^{th} and $(n+1)^{\text{th}}$ term in a power series. The Ratio Test is shown in Equation 38.

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{(n+1)}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L \tag{38}$$

The following cases are considered:

- if $L < 1$ then the series converges absolutely.
- if $L = 1$ then the test is inconclusive; some other test must be used; and
- if $L > 1$ then the series diverges.

Note: absolute convergence means that the series converges irrespective of the signs of each term. (i.e. whether or not all terms are positive, negative, or a mix of both positive and negative.)

Example: Find the radius of convergence and associated interval of convergence for the following power series:

1. $\sum_{n=0}^{\infty} (-1)^n x^n$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= L < 1, \quad a = 0, \; c_n = (-1)^n \\
\lim_{n\to\infty}\left|\frac{x^{n+1}}{x^n}\right| &< 1 \\
|x| \lim_{n\to\infty} |1| &< 1 \\
\Rightarrow |x| &< 1
\end{aligned}
$$

The radius of convergence $R = 1$ and the interval of convergence is $x \in (-1, 1)$.

Here I have purposely avoided analyzing the end-points to see if we could use a closed or partially-closed interval instead. Since we limited $L < 1$, that only gives the radius of absolute convergence. If we wanted to be picky, we could allow $L = 1$ and use some other test to determine if the series converges. If we did that in this case we would find that the series diverges at both endpoints.

2. $\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n(n+1)}$

$$
\begin{aligned}
\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a| \lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= L < 1, \quad a = 0, \; c_n = \frac{(-1)^n}{n(n+1)} \\
\lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)(n+1+1)} \frac{n(n+1)}{x^n}\right| &< 1 \\
|x| \lim_{n\to\infty}\left|\frac{n}{n+2}\right| &< 1 \\
|x| \cancelto{1}{\lim_{n\to\infty}\left|\frac{n}{n+2}\right|} &< 1 \\
|x| &< 1
\end{aligned}
$$

Once again, the radius of convergence $R = 1$ and the interval of convergence is $x \in (-1, 1)$.

3. $\sum_{n=1}^{\infty} \frac{x^{2n}}{2^n n^2}$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L < 1, \quad a = 0,\ c_n = \frac{1}{2^n n^2}$$

$$\lim_{n\to\infty}\left|\frac{x^{2n+2}}{2^{n+1}(n+1)^2}\frac{2^n n^2}{x^{2n}}\right| < 1$$

$$\lim_{n\to\infty}\left|\frac{x^2}{2}\frac{n^2}{(n+1)^2}\right| < 1$$

$$\frac{\left|x^2\right|}{2}\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|^{1} < 1$$

$$\left|x^2\right| < 2$$

$$|x| < \sqrt{2}$$

In this case the radius of convergence $R = \sqrt{2}$ and the interval of convergence is $x \in (-\sqrt{2}, \sqrt{2})$.

In this case, more detailed analysis shows that this series converges at both endpoints; a closed interval could be used instead.

4. $\sum_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L < 1, \quad a = 2,\ c_n = \frac{1}{3^n}$$

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{3^{n+1}}\frac{3^n}{(x-2)^n}\right| = L < 1$$

$$\frac{|x-2|}{3}\lim_{n\to\infty}|1| < 1$$

$$|x-2| < 3$$

So for this example the radius of convergence is $R = 3$ about the center at $x = 2$; and the interval of convergence is $x \in (-1, 5)$

For the interested reader, it can be shown that this series is divergent at both endpoints so it should remain an open interval.

Properties of Convergent Series

Within the radius of convergence, a power series defines a function. Within the interval of convergence the function so defined is:

- continuous
- differentiable (term-by-term); and
- integrable (term-by-term)

If x is not within the interval of convergence for a series or if the series is divergent then ***none*** of these are true. This is why it is important to be able to find the interval/radius of convergence.

Definition 9 (Identity Property for a Power Series)
If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0$, $R > 0$, *for all numbers* x *in the interval of convergence then* $c_n = 0$ *for all* n.

Hopefully this definition seems obvious to you. You will find that most of what we do when using power series to solve homogeneous linear differential equations is carry out the necessary algebra to ensure that the coefficients for some series all are equal to zero.

Definition 10 (Analytic Function)
a function f *is* analytic *at a point* a *if it can be represented by a power series in* $x - a$ *with a positive radius of convergence.*

This is just a vocabulary term that you should know.

Some Common Power Series

You have probably had some exposure to power series in your previous mathematical courses. As a reminder, I've included the power series representations of some important/common functions.

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $\ln x = \frac{x-1}{x} + \frac{(x-1)^2}{2x^2} + \frac{(x-1)^3}{3x^3} + \cdots$
4. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

Combining Power Series

This is a practical "utility skill" that you will need to master in order to be successful at this portion of the course. What we need to be able to do is combine multiple power series into a single expression.

FOR EXAMPLE, consider the two power series below that we want to write as a single power series:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

If I want to combine these series, I need to overcome two issues:

1. the powers of x in each term in both summations need to be "in phase" – that is the corresponding terms need to have the same power of x. The first term in the first summation is constant (x^0) while the first term in the second summation is linear (x^1); and
2. the first summation index starts at $n = 2$ while the second summation index starts at $n = 0$.

It is not only the first term that is important but if you can get the summations in phase for the first term, and if the power of x increases by one with each consecutive term, then if the first term is correct, they will all be correct.

WE WILL ADDRESS these issues one at a time, starting with the first one. We will leave the summation whose first term is highest order as-is; for all other summations (i.e. if there are more than two) we will "peel-off" any lower-order summation terms.

In this case that means we will "peel-off'; the constant term from the first summation:

$$\underbrace{(2)(1)c_2x^0}_{\text{constant term}} + \underbrace{\sum_{n=3}^{\infty}}_{\text{now } n=3} n(n-1)c_nx^{n-2} + \sum_{n=0}^{\infty} c_nx^{n+1}$$

Notice that now the summation index for the first summation starts at $n = 3$; this is because we've separated out the first term corresponding to $n = 2$. The two remaining summations are "in phase" since all of the terms now have the same power of x.

THE SECOND PROBLEM will be fixed by establishing a new common index, k, and re-write the existing indices (n for both summations) in terms of k. In each case we will set k equal to the exponent of x appearing in the summation.

- For the first summation—$\sum\limits_{n=3}^{\infty} n(n-1)c_nx^{n-2}$—we set $k = n-2$ because that is the exponent for x. We need to eliminate each occurrence of n in the summation and replace it with it's equivalent expression in terms of k. From our definition of k for this summation, $n = k+2$. Our summation now can be written:

  $$\sum_{k=1}^{\infty}(k+2)(k+2-1)c_{k+2}x^k$$

Everywhere you see an n in the original summation, replace it with a $k+2$ and simplify.

- For the second summation—$\sum\limits_{n=0}^{\infty} c_nx^{n+1}$—we set $k = n+1$ because that is the exponent for x in this summation. This means $n = k-1$; substituting that expression in our summation gives us:

  $$\sum_{k=1}^{\infty} c_{k-1}x^k$$

With these changes our original summation can be written:

Notice that I've made some obvious simplifications in the constant term and first summation.

$$2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k \tag{39}$$

The two summations are now ready to be joined into one as shown in Equation 40

$$2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} + c_{k-1}\right]x^k \tag{40}$$

Lecture 8 - Power Series Solutions at Ordinary Points

Objectives

The objectives of this lecture are:

- Introduce some definitions and concepts relevant for power series solutions of differential equations.
- Do some example problems.

Introduction

In this section we will restrict our attention to second-order, linear, homogeneous differential equations in standard form as shown in Equation 41.

$$u'' + P(x)u' + Q(x)u = 0 \tag{41}$$

Definition 11 (Ordinary Points and Singular Points)
A point x_0 is said to be an ordinary point *of a differential equation if both $P(x)$ and $Q(x)$ in the standard form are* analytic *at x_0. A point that is not an* ordinary point *is a* singular point*.*

Theorem 6 (Existence of Power Series Solutions)
If $x = x_0$ is an ordinary point *of the differential equation, we can always find two linearly independent solutions in the form of a power series centered at x_0. A series solution converges at least on some interval defined by $|x - x_0| < R$ where R is the distance from x_0 to the closest singular point.*

To clarify: this theorem applies only to second-order, linear, homogeneous differential equations.

The basic strategy we will use to find power series solutions for linear differential equations with variable coefficients where $P(x)$ and $Q(x)$ are analytic in the domain of interest is:

1. Find solutions in the form of a power series by substituting $u = \sum\limits_{n=0}^{\infty} c_n x^n$ into the differential equation.

2. Solve for the values of the coefficients by equating the coefficients on the left with those on the right (e.g. zero for homogeneous equations); and

3. The equations (often 2- or 3-term recurrence relations) for the series coefficients *defines* the function that is the solution of the differential equation.

Important: the series "solution" is only valid if the series so-derived has a non-zero radius of convergence.

Examples

It is not *required* that a problem have variable coefficients in order to use the Power Series method; only that $P(x)$ and $Q(x)$ are analytic on the domain of interest. Constants are always analytic over the entire real number line so you can always use the Power Series method on linear, constant-coefficient differential equations.

Example: Solve $u'' + u = 0$ using the power series method; compare with the known solution $u(x) = c_0 \cos x + c_1 \sin x$

In accordance with our strategy, we will assume that the solution is of the form: $u(x) = \sum_{n=0}^{\infty} c_n x^n$. This means that $u' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $u'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$. Plugging this into our differential equation gives us:

Note that the $n = 0$ term in u' is omitted as is the $n = 0$ and $n = 1$ term in u''. These terms are zero due to having taken the first- and second-derivative on the constant (x^0) and linear (x^1) terms of the power series.

$$\begin{aligned} u'' + u &= 0 \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n &= 0 \end{aligned}$$

We need to combine the two summations. It is clear that the summation indexes start at different values but we are lucky in that the summations are already "in phase" since the first term in each summation is a constant (x^0) term.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\substack{k=n \\ n=k}} = 0$$

For the first summation $k = n - 2$, so $n = k + 2$; we will use these definitions to re-write the first summation. For the second summation, $k = n$ so all we need to do for the second summation is replace all the n's with k's. The results of these substitutions and the combined summation are shown below:

$$\begin{aligned} \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} \underbrace{\left[(k+2)(k+1)c_{k+2} + c_k\right]}_{\text{coefficients for new power series}} x^k &= 0 \end{aligned}$$

The expression $[(k+2)(k+1)c_{k+2} + c_k]$ is now a formula for the coefficients of a new power series; this power series, according to the equation, is equal to *zero* so that means all of the coefficients must be equal to zero:

$$(k+2)(k+1)c_{k+2} + c_k = 0, \quad \text{for all } k \in [0, 2, 3, \dots]$$

This is called a *two-term recurrence* since the expression involves *two* terms; c_{k-2} and c_k.

By convention, we will re-write this recurrence relation to solve for the *higher*-index coefficients in terms of the *lower*-index coefficients. We do this in Equation 42

$$c_{k+2} = -\frac{c_k}{(k+2)(k+1)} \tag{42}$$

As we would expect the general solution for any other second order differential equation would have two unknown constants that can only be resolved by adding initial- or boundary-conditions.

As we should expect, there are two unknown constants in this general solution.The first value of k, from the summation in our solution is $k = 0$. Simplified equations for the first few coefficients are presented in the table below. Each cell in the table above is a formula

$k=0$ $c_2 = \frac{-c_0}{(1)(2)}$	$k=2$ $c_4 = \frac{-c_2}{(3)(4)} = \frac{c_0}{4!}$	$k=4$ $c_6 = \frac{-c_4}{(5)(6)} = \frac{-c_0}{6!}$
$k=1$ $c_3 = \frac{-c_1}{(2)(3)}$	$k=3$ $c_5 = \frac{-c_3}{(4)(5)} = \frac{c_1}{5!}$	$k=5$ $c_7 = \frac{-c_5}{(6)(7)} = \frac{-c_1}{7!}$

for the k^{th}-coefficient of our power series solution. Organizing this into a formula for our power series solution gives us:

Notice how the even-numbered coefficients are all dependent on c_0 and all of the odd-numbered coefficients are dependent on c_1.

$$u(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \cdots$$
$$u(x) = c_0\left(1 + \frac{c_2}{c_0}x^2 + \frac{c_4}{c_0}x^4 + \frac{c_6}{c_0}x^6 + \cdots\right) + c_1\left(x + \frac{c_3}{c_1}x^3 + \frac{c_5}{c_1}x^5 + \frac{c_7}{c_1}x^7 + \cdots\right)$$
$$u(x) = c_0\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

where in the last line we have substituted the formulas for coefficients c_2 through c_7 in terms of c_0 and c_1.

Recalling from the last lecture the power series representations of $\cos x$ and $\sin x$ and we should be able to see them again here.

In general you are not expected to, nor will you be able to, identify common functions from a power series solution. This is a special case.

$$u(x) = c_0 \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos x} + c_1 \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\sin x}$$
$$u(x) = c_0 \cos x + c_1 \sin x$$

Which is exactly what we would have determined using our methods for constant coefficient linear equations.

Example: find the general solution to $u'' - xu = 0$

Notice first that while this equation is linear and homogeneous it is not constant-coefficient. It is also not a Cauchy-Euler equation. We will use the Power Series method to solve this problem. Assuming $u = \sum_{n=0}^{\infty} c_n x^n$ and inserting this into the governing equation gives us:

The equation is separable but, in this case, the solution is not so easy to obtain using that method either.

$$\begin{aligned} u'' - xu &= 0 \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} &= 0 \end{aligned}$$

We want to combine these summations and see that they are both "out of phase" as well as have different powers of x for their first terms.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\text{for } n=2,\ \ x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\text{for } n=0,\ \ x^1} = 0$$

So we must separate out the first term in the first summation to get the summations in phase.

$$(2)(1)c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Next we must combine our indices using $k = n-2$ for the first summation and $k = n+1$ for the second summation respectively.

Reminder: we take our definition of k from the exponent for x in each summation term.

$$2c_2 x^0 + \underbrace{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{k=n+1 \\ n=k-1}} = 0$$

Doing this gives us:

$$2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_{k-1}\right] x^k = 0$$

Do not forget the minus sign in front of the second summation. It is easy to miss.

In order to solve the differential equation, the coefficient for every power of x needs to be zero. To do this:

$$\underbrace{2c_2}_{\Rightarrow c_2=0} + \sum_{k=1}^{\infty} [\underbrace{(k+2)(k+1)c_{k+2} - c_{k-1}}_{\text{must equal zero}}] x^k = 0$$

Our corresponding, two-term recurrence relation is:

Reminder: we should define our recurrence relation to give higher-order coefficients in terms of lower-order coefficients.

$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}$$

$k=1$ $c_3 = \frac{c_0}{(3)(2)}$	$k=4$ $c_6 = \frac{c_3}{(6)(5)} = \frac{c_0}{(2)(3)(5)(6)}$
$k=2$ $c_4 = \frac{c_1}{(3)(4)}$	$k=5$ $c_7 = \frac{c_4}{(6)(7)} = \frac{c_1}{(3)(4)(6)(7)}$
$k=3$ $c_5 = \frac{c_2}{(5)(4)} = 0$	$k=6$ $c_8 = \frac{c_5}{(8)(7)} = 0$
$k=7$ $c_9 = \frac{c_6}{(9)(8)} = \frac{c_0}{(2)(3)(5)(6)(8)(9)}$	$k=8$ $c_{10} = \frac{c_7}{(10)(9)} = \frac{c_1}{(3)(4)(6)(7)(9)(10)}$

We expect two arbitrary constants, c_0 and c_1 and we know from the work above that $c_2 = 0$ so we will start solving for constants starting with $k=1$: Organizing the coefficients from the table into an equation we get:

$$u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots$$

$$u(x) = c_0 \left(1 + \frac{c_3}{c_0} x^3 + \frac{c_6}{c_0} x^6 + \frac{c_9}{c_0} x^9 + \cdots\right) + c_1 \left(x + \frac{c_4}{c_1} x^4 + \frac{c_7}{c_1} x^7 + \frac{c_{10}}{c_1} x^{10} + \cdots\right)$$

which can be written:

$$u(x) = c_0 \left(1 + \frac{x^3}{(2)(3)} + \frac{x^6}{(2)(3)(5)(6)} + \frac{x^9}{(2)(3)(5)(6)(8)(9)} + \cdots\right) + c_1 \left(x + \frac{x^4}{(3)(4)} + \frac{x^7}{(3)(4)(6)(7)} + \frac{x^{10}}{(3)(4)(6)(7)(9)(10)} + \cdots\right)$$

The equation we solved is known as Airy's Equation. The power series solution is not pretty, but is a perfectly adequate representation of the function provided that we have the wherewithal to evaluate the function for a reasonable number of terms.

Assignment #3

Find the general solution to the following differential equations

1. $x^2u'' - 2u = 0$

2. $xu'' + u' = 0$

3. $x^2u'' - 3xu' - 2u = 0$

Find the solution to the given initial value problem

4. $x^2u'' + 3xu' = 0, \;\; u(1) = 0, \; u'(1) = 4$
   Use MATLAB to plot the solution for $x \in [1, 10]$.

5. A very long cylindrical shell is formed by two concentric circular cylinders of different radii. A chemically reactive fluid fills the space between the concentric cylinders. The inner cylinder has a radius of 1 and is thermally insulated, while the outer cylinder has a radius of 2 and is maintained at a constant temperature T_0. The rate of heat generation in the fluid due to the chemical reaction is proportional to T/r^2, where $T(r)$ is the temperature of the fluid within the space bounded between the cylinders defined by $1 < r < 2$. Under these conditions the temperature of the fluid is defined by the following boundary value problem:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} = -\frac{1}{r}T, \;\; 1 < r < 2,$$

$$\left.\frac{dT}{dr}\right|_{r=1} = 0, \; T(2) = T_0$$

Solve the boundary value problem to find the temperature of the fluid within the cylindrical shell.

For the following problems, find the radius and interval of convergence. For the intervals of convergence, you do not need to check the endpoints (unless you want to!).

6. $\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$

7. $\sum_{n=1}^{\infty} \frac{(-1)^n}{10^n} (x-5)^n$

Rewrite the given expression as a single power series whose general term involves x^k.

8. $\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$

Find two power series solutions of the given differential equation

9. $u'' - 2xu' + u = 0$

Lecture 9 - Power Series Solutions with MATLAB

Objectives

The objectives of this lecture are:

- Illustrate the solution of a linear IVP (with a 3-term recurrence) using Power Series
- Demonstrate a way to analyze these solutions using MATLAB; and
- Demonstrate some expected elements of MATLAB style for this course

Solution of an IVP using Power Series

Consider the following IVP:

$$\text{Governing Equation: } u'' - (1+x)u = 0, \quad u \in [0,5]$$
$$\text{Initial Conditions: } u(0) = 5, \quad u'(0) = 1$$

As before, we will assume $u = \sum\limits_{n=0}^{\infty} c_n x^n$. This means that $u' = \sum\limits_{n=1}^{\infty} n c_n x^{n-1}$, and $u'' = \sum\limits_{n=2}^{\infty} n(n-1)c_n x^{n-2}$.

Inserting our assumed power series solution into the governing equation gives us:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (1+x)\sum_{n=0}^{\infty} c_n x^n = 0$$
$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Note the effect of distributing $-(1-x)$ through the second summation.

We need to evaluate the order of x for the first term in each summation to determine if the summations are in phase:

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{x^1} = 0$$

To get the three summations in phase we need to strip off the first terms in the first and second summations so that all three summations start at x^1. This gives us:

$$2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - c_0 - \sum_{n=1}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 - c_0 + \underbrace{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2\\n=k+2}} - \underbrace{\sum_{n=1}^{\infty} c_n x^n}_{\substack{k=n\\n=k}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{k=n+1\\n=k-1}} = 0$$

Substituting within each summation and combining the terms gives us:

$$(2c_2 - c_0)x^0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_k - c_{k-1}\right] x^k = 0$$

As usual, in order to satisfy this equation the coefficients for each power of x must be equal to zero. For x^0 this means $2c_2 - c_0 = 0$; For all the other powers of x, a *three-term recurrence* involving c_{k-1}, c_k, and c_{k+2} must be satisfied:

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}$$

We will help manage the complexity by adopting the following strategy:

- Case 1: Arbitrarily set $c_0 \neq 0$, set $c_1 = 0$ and derive a solution;
- Case 2: Arbitrarily set $c_0 = 0$, set $c_1 \neq 0$ and derive a second solution.

These two solutions are sure to be linearly independent since the first will not have a linear term (proportional to x) and the second equation will not have a constant term (proportional to 1).

Case 1: $c_0 \neq 0, \;\; c_1 = 0$
Since $c_0 \neq 0$, we get $c_2 = \frac{c_0}{2}$. The coefficients derived for the first few values of k are shown in the table to the right.

The solution we thus derive is shown below.

$$u_1 = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$
$$u_1 = c_0 \left(1 + \frac{\cancelto{0}{c_1}}{c_0} x + \frac{c_2}{c_0} x^2 + \frac{c_3}{c_0} x^3 + \frac{c_4}{c_0} x^4 + \frac{c_5}{c_0} x^5 + \cdots\right)$$
$$u_1 = c_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \cdots\right)$$

Case 1:

$k=1$	$k=2$
$c_3 = \frac{c_0 + \cancelto{0}{c_1}}{(2)(3)} = \frac{c_0}{6}$	$c_4 = \frac{\cancelto{0}{c_1} + c_2}{(3)(4)} = \frac{c_2/2}{12} = \frac{c_0}{24}$
$k=3$	
$c_5 = \frac{c_2 + c_3}{(4)(5)} = \frac{c_0/2 + c_0/6}{20} = \frac{c_0}{30}$	

Case 2: $c_0 = 0, \;\; c_1 \neq 0$

Since $c_1 = 0$ and $c_2 = \frac{c_0}{2}$, $c_2 = 0$. The coefficients derived for the first few values of k are shown in the table.

Case 2:

$k=1$	$k=2$
$c_3 = \frac{\cancelto{0}{c_0} + c_1}{(2)(3)} = \frac{c_1}{6}$	$c_4 = \frac{c_1 + \cancelto{0}{c_2}}{(3)(4)} = \frac{c_1}{12}$
$k=3$	
$c_5 = \frac{\cancelto{0}{c_2} + c_3}{(4)(5)} = \frac{c_1/6}{20} = \frac{c_1}{120}$	

The solution we thus derive is shown below:

$$
\begin{aligned}
u_2 &= \cancelto{0}{c_0} + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
u_2 &= c_1 \left(x + \frac{\cancelto{0}{c_2}}{c_1} x^2 + \frac{c_3}{c_1} x^3 + \frac{c_4}{c_1} x^4 + \frac{c_5}{c_1} x^5 \cdots \right) \\
u_2 &= c_1 \left(x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right)
\end{aligned}
$$

We now have two linearly independent solutions to the governing equation:

$$
\begin{aligned}
u(x) = u_1(x) + u_2(x) = c_0 &\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots \right) + \\
c_1 &\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots \right)
\end{aligned}
$$

We are now ready to apply the initial conditions.

$$
\begin{aligned}
u(0) &= c_0 = 5 \\
u'(0) &= c_1 = 1
\end{aligned}
$$

So the final solution is:

$$
\begin{aligned}
u(x) = 5 &\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots \right) + \\
&\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots \right)
\end{aligned}
$$

Generating and Plotting Solutions with MATLAB

The point of doing all of this math is to gain insight. In many cases systems of interest are subject to physical laws that are expressed in the form of differential equations; we solve these differential equations to better understand how the systems will perform.

THERE ARE TWO PROBLEMS that I hope to address with this section.

1. It is both tedious and error-prone to generate the coefficients for the series solution. we worked hard to construct a small portion of the power series solutions and we hope that we did it without any errors. With a modest amount of programming, we will construct a script that can build a series solution with as many terms as we would like. With conscientious debugging we can be sure that, floating point round-off errors aside, the calculations are done quickly and correctly.

2. We may gain considerably more insight from the solution if we can create a plot. If the plot can easily be made with the same

computing tools used to generate the solution, we can get this insight with very little extra work.

We will use MATLAB to generate and plot the solutions. We start our MATLAB script in the same way we will start *all* of our MATLAB scripts: with the following three lines:

This is done in accordance with the MATLAB Style Rules for this course. These rules are listed in the Appendices and I will try to exemplify the rules in the code I provide as examples in the lectures.

```
1 clear
2 clc
3 close 'all'
```

Next we will specify the order of the solution we will construct, and thus the number of coefficients that we need to compute for each solution and construct arrays to store the coefficients.

We will use two arrays to store the coefficients; one for $u_1(x)$ and the other for $u_2(x)$.

```
4 n=25;
5 C1 = nan(1,n); % coefficients for u1(x)
6 C2 = nan(1,n); % coefficients for u2(x)
```

Recall that for $u_1(x)$ we applied our strategy for "case 1" in which we assumed that $c_0 \neq 0$ and $c_1 = 0$. This implied that $c_2 = c_1/2$ while the recurrence relation was used for all of the other coefficients. When we applied the initial conditions we found that $c_0 = 5$ for $u_1(x)$. The code snippet below accomplishes these tasks.

Note how each line of MATLAB in this listing is terminated with a semicolon. This is to suppress the output to the command line that would otherwise happen each time we make an assignment to a variable. While this output might be helpful during debugging, during any other time it is distracting (drowning out other, more useful output) and slows code execution.

```
7 C1_0 = 5; % c_0 for u1(x), handled separatly since MATLAB array
8 % indices start at 1
9 C1(1) = 0; % c_1 for u1(x)
10 C1(2) = C1_0/2; % c_2 for u1(x)
11
12 % handle the k=1 case separately since it involves the term C1_0
13 k = 1;
14 C1(k+2) = (C1(k) + C1_0)/((k+1)*(k+2));
15
16 for k=2:(n-2)
17   C1(k+2) = (C1(k) + C1(k-1))/((k+1)*(k+2));
18 end
```

Now we have calculated all of the desired coefficients, we are ready to construct the first solution.

Here $u_1(x)$ is created as an *anonymous* function. We begin with the constant term on line 28 and build up the function term-by-term within the *for* loop on line 30.

```
19 u1 = @(x) C1_0;
20 for k = 1:n
21   u1 = @(x) u1(x) + C1(k)*x.^k;
22 end
```

We continue in this same vein to construct $u_2(x)$ as we did in our "case 2" strategy and construct the solution $u(x) = u_1(x) + u_2(x)$.

```
23 C2_0 = 0; % c_0 for u2(x)
24 C2(1) = 1; % from the initial condition
25 C2(2) = C2_0/2; % just adding for consistency's sake
26
27 k=1;
28 C2(k+2) = (C2(k) + C2_0)/((k+1)*(k+2));
29 for k = 2:(n-2)
30   C2(k+2) = (C2(k) + C2(k-1))/((k+1)*(k+2));
```

```
end

u2 = @(x) C2_0;
for k = 1:n
  u2 = @(x) u2(x) + C2(k)*x.^k;
end

u = @(x) u1(x) + u2(x);
```

Now that we have a MATLAB representation of the solution, let us create a plot. One way to make such a plot is shown in the listing below; the output is shown in Figure 2.:

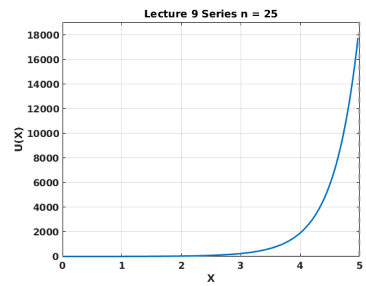


Figure 2: Power series solution to $u'' - (1+x)u = 0,\ u(0) = 5,\ u'(0) = 1$.

```
xMin = 0; xMax = 5;
figure(1)
fplot(u,[xMin, xMax],'linewidth',2);
title_str = sprintf('Lecture 9 Series n = %d',n);
title(title_str,'fontsize',18,'fontweight','bold');
xlabel('X','fontsize',16,'fontweight','bold');
ylabel('U(X)','fontsize',16,'fontweight','bold');
set(gca,'fontsize',12,'fontweight','bold');
grid on
```

Note: pay particular attention to the formatting details of the plot. There is a title along with axis-labels for both the x- and y-axis; fonts are bold and sized in a particular way and grid-lines are used. These details have all been added in observance of MATLAB Style rule #4.

A few more details are worth noting.

1. We know our solution is inexact since we truncated the infinite power series and used finite-precision arithmetic while calculating the coefficients for those terms we *did* bother to include. Still, we might want to know *how wrong* the solution is.

2. Taking a more positive tack, we might ask how much *better* the solution gets when we add more terms to our solution.

To answer either of these questions, we will need access to the solution of the IVP. For this lecture, we will take a numeric solution generated using MATLAB's built-in IVP-solving tool *ODE45* as "the solution."

Use of tools such as *ODE45* will be treated in the numerical methods section.

Power series results for various values of n are compared to the numerical solution in Figure 3. Some things to notice:

1. The solution gets worse the further one gets from zero; and

2. The solution gets better for larger values of n.

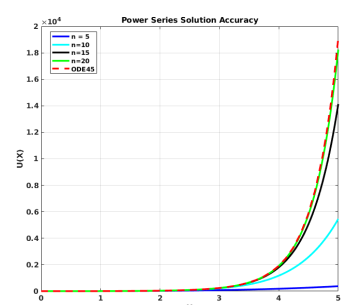


Figure 3: Power series solution with different values of n.

Neither of these observations should be particularly surprising but there is value to seeing it in your results; it adds confidence to the proposition that your (approximate) solution is correct.

As a last note it should be pointed out that, while plots like that shown in Figure 3 gives a good qualitative feel for how the solution is improving as the number of power series terms increases, a quantitative measure for correctness is preferable. In Figure 4 a quantitative measure—the relative error in the 2-norm—is used to quantify the difference between different power series solutions and the solution generated using *ODE45*. Details of this error measure will be discussed in future lectures.

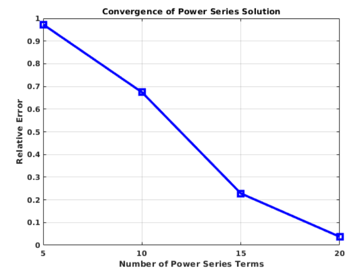


Figure 4: Convergence of the power series solution to the numeric solution.

Lecture 10 - Legendre's Equation

Objectives

The objectives of this lecture are:

- Illustrate the use of the power series method to solve Legendre's equation; and
- Introduce some of the properties of Legendre polynomials

Legendre's Equation

The following 2^{nd}-order linear, homogeneous ODE is known as Legendre's equation:

$$\left(1-x^2\right)u'' - 2xu' + m(m+1)u = 0 \tag{43}$$

where m is a constant.

THE FIRST THING we will do is to put Equation 43 into standard form:

$$u'' - \frac{2x}{(1-x^2)}u' + \frac{m(m+1)}{(1-x^2)}u = 0$$

We should immediately note that $P(x) = \frac{2x}{(1-x^2)}$ and $Q(x) = \frac{m(m+1)}{(1-x^2)}$ are singular (and thus not analytic) at $x = \pm 1$. Recall from Theorem 6 that $P(x)$ and $Q(x)$ must be analytic for power series solutions to exist.

WE WILL RESTRICT our attention to the interval $x \in (-1, 1)$ and use the power series method to find a solution. Inserting our assumed power series solution into Equation 43 gives us:

$$\left(1-x^2\right)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2x\sum_{n=1}^{\infty} nc_n x^{n-1} + m(m+1)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{x^0} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{x^2} - \underbrace{\sum_{j=1}^{\infty} 2nc_n x^n}_{x^1} + \underbrace{m(m+1)\sum_{n=0}^{\infty} c_n x^n}_{x^0} = 0$$

where we see that all terms with order lower than x^2 need to be pulled outside of their summations so all four can be in phase.

$$(2)(1)c_2 + (3)(2)c_3 x + \underbrace{\sum_{n=4}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2\\ n=k+2}} - \underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^n}_{\substack{k=n\\ n=k}} - (2)(1)c_1 x - \underbrace{\sum_{n=2}^{\infty} 2nc_n x^n}_{\substack{k=n\\ n=k}} + \ldots$$

$$m(m+1)\,(c_0 + c_1 x) + m(m+1) \underbrace{\sum_{n=2}^{\infty} c_n x^n}_{\substack{k=n\\ n=k}} = 0$$

Combining terms outside of the summations and making the indicated substitutions to combine the summations we get:

$$[m(m+1)c_0 + 2c_2] + [\underbrace{(m(m+1)-2)}_{(m-1)(m+2)} c_1 + 6c_3]x + \ldots$$

$$\sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} \underbrace{- k(k-1)c_k - 2kc_k + m(m+1)c_k}_{(m-k)(m+k+1)c_k}]x^k = 0$$

Applying the indicated algebraic simplifications leads us finally to:

Note: Obviously this is a tedious business. Be careful and make sure you understand each manipulation.

$$[m(m+1)c_0 + 2c_2] + [(m+1)(m+2)c_1 + 6c_3]x + \cdots$$

$$\sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} + (m-k)(m+k+1)c_k]x^k = 0$$

The next steps are to find formulas for the power series coefficients c_n so that the combined coefficient for each power of x in the equation above equals zero. For the constant term (x^0) we have:

$$c_2 = \frac{-m(m+1)c_0}{2}$$

For the linear term (x^1) we get

$$c_3 = \frac{-(m-1)(m+2)}{6} c_1$$

For all other powers of x, we get a 2-term recurrence:

$$c_{k+1} = \frac{-(m-k)(m+k+1)}{(k+2)(k+1)} c_k$$

$k = 2$
$c_4 = \frac{-(m-2)(m+3)}{(4)(3)} c_2 = \frac{(m-2)(m+3)m(m+1)}{(4)(3)(2)} c_0$
$k = 3$
$c_5 = \frac{-(m-3)(m+4)}{(5)(4)} c_3 = \frac{(m-3)(m+4)(m-1)(m+2)}{(5)(4)(3)(2)} c_1$

Organizing these into two solutions we get:

$$\begin{aligned} u_1(x) &= c_0 \left[1 + \frac{c_2}{c_0} x^2 + \frac{c_4}{c_0} x^4 + \cdots \right] \\ &= c_0 \left[1 - \frac{m(m+1)}{2!} x^2 + \frac{(m-2)(m+3)m(m+1)}{4!} x^4 + \cdots \right] \\ u_2(x) &= c_1 \left[x + \frac{c_3}{c_1} x^3 + \frac{c_5}{c_1} x^5 + \cdots \right] \\ &= c_1 \left[x - \frac{(m-1)(m+2)}{3!} x^3 + \frac{(m-3)(m+4)(m-1)(m+2)}{5!} x^5 + \cdots \right] \end{aligned}$$

So far what we have is messy but when dealing with power series solutions, messiness is the order of the day. One point that we have quietly left to the side is whether or not we expect this (so called) power series solution to converge.

Reminder: if the power series that we purport to be a solution to the differential equation is divergent than we really have nothing.

One way that we can permanently leave these questions to the side is if m is an integer. Notice that if $m = 0$ or is an even integer, $u_1(x)$ terminates with a finite number of terms. Similarly with $u_2(x)$ in the case that m is an odd integer.

i.e. if m is even then $u_1(x)$ is a polynomial.

These polynomial solutions, where m is an integer, are referred to as Legendre Polynomials. Legendre Polynomials have several important applications; we will need to use them when solving equations in spherical coordinate systems.

Important Properties of Legendre Polynomials

As a recap, Legendre polynomials are solutions to Legendre's equation where m is an integer:

$$\left(1 - x^2\right) u'' - 2xu' + m(m+1)u = 0$$

By convention the leading coefficients are chosen such that Legendre polynomials have a maximum value of 1 on the interval $x \in [-1, 1]$.

$P_0(x) = 1$	$P_1(x) = x$
$P_2(x) = \frac{1}{2}(3x^2 - 1)$	$P_3(x) = \frac{1}{2}(5x^3 - 3x)$

Table 2: The first four Legendre Polynomials

The first few Legendre Polynomials are shown in the table. Higher order Legendre polynomials can be constructed using a three-term recurrence relation shown in Equation 44

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \tag{44}$$

Some other properties include:

$$
\begin{aligned}
P_n(-x) &= (-1)^n P_n(x) \\
P_n(1) &= 1 \\
P_n(-1) &= (-1)^n \\
P_n(0) &= 0 \text{ for } n \in \text{odd} \\
P_n'(0) &= 0 \text{ for } n \in \text{even}
\end{aligned}
$$

A plot of the first several Legendre Polynomials is shown in Figure 5

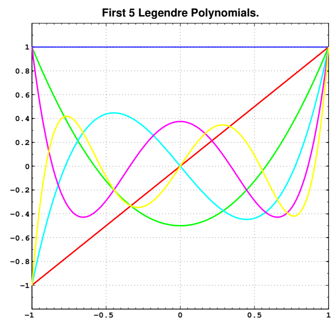


Figure 5: Legendre Polynomials of order 0 through 5.

THE LAST PROPERTY that we will mention here, and that we will make use of extensively in this course, is the *orthogonality* property of Legendre Polynomials. Legendre Polynomials are orthogonal over the interval $x \in [-1, 1]$. This mean that Equation 45 holds:

$$\int_{-1}^{1} P_n(x) P_m(x) \, dx = \begin{cases} 0, & n \neq m \\ \frac{2}{n+1}, & n = m \end{cases} \tag{45}$$

Orthogonality of functions is analogous to orthogonality of vectors. Two functions, $f_1(x)$ and $f_2(x)$ are orthogonal on an interval $x \in [a, b]$ if $\int_a^b f_1(x) f_2(x) \, dx = 0$.

Lecture 11 - Solutions about Singular Points

Objectives

The objectives of this lecture are:

- Define regular and irregular singular points and give examples of their classification.
- Describe the Extended Power Series Method (Method of Frobenius)
- Do an example problem

Definitions

Consider a linear, homogeneous, second-order differential equation in standard form as shown below:

$$u'' + P(x)u' + Q(x)u = 0$$

Definition 12 (Singular Point)
A singular point*, x_0, is a point where $P(x)$ or $Q(x)$ are not analytic.*

Definition 13 (Regular/Irregular Singular Point)
A singular point x_0 is said to be a regular *singular point of the differential equation if the functions $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. If a singular point is not regular, it is* irregular.

Example: Classify the singular points of $(x^2 - 4)^2 u'' + 3(x-2)u' + 5u = 0$

In standard form, $P(x) = \frac{3(x-2)}{(x^2-4)^2} = \frac{(3(x-2)}{(x+2)^2(x-2)^2}$; and $Q(x) = \frac{5}{(x+2)^2(x-2)^2}$. There are two singular points: -2 and 2. Work is shown in the margin for $p(x)$. From the work in the margin it should be clear for this problem that $q(x)$ is analytic at both $x = -2$ and $x = 2$ but since $p(x)$ is not analytic at $x_0 = -2$, $x_0 = -2$ is an irregular singular point and $x_0 = 2$ is a regular singular point.

$x_0 = 2: \quad p(x) = \frac{\cancel{(x-2)}3\cancel{(x-2)}}{(x+2)^2\cancel{(x-2)^2}}$, so $p(x) = \frac{3}{(x+2)^2}$ which is analytic at $x = 2$.

$x_0 = -2: \quad p(x) = \frac{(x+2)3(x-2)}{(x+2)^2(x-2)^2}$ so $p(x) = \frac{3(x-2)}{(x+2)(x-2)}$.

Theorem 7 (Frobenius' Theorem)
If $x = x_0$ is a regular *singular point then at least one non-zero solution of the form:*

$$u(x) = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

where r is to be determined. The series will converge at least on some radius of convergence defined by: $0 < x - x_0 < R$.

Example: find a series solution to: $3xu'' + u' - u = 0$
Per Theorem 7 this equation should have a solution of the form $u = \sum_{n=0}^{\infty} c_n x^{n+r}$ where r is constant. Taking the first and second derivatives we get:

You should verify that this equation has a regular singularity at $x_0 = 0$.

$$u' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$
$$u'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

Notice in this case that the summations for u' and u'' start at $n = 0$ while for the power series solution method the starting index for u' was $n = 1$ and the starting index for u'' was $n = 2$. The reason for the difference is the factor x^r; there most likely are not any constant terms to the series so taking derivatives does not make any terms zero.

We insert these expressions into the differential equation and will combine the summations to derive recurrence relations.

$$3x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$
$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1) c_n x^{n+r-1} + \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$x^r \left[\underbrace{\sum_{n=0}^{\infty} 3(n+r)(n+r-1) c_n x^{n-1}}_{\substack{\text{for } n=0 \\ x^{-1}}} + \underbrace{\sum_{n=0}^{\infty} (n+r) c_n x^{n-1}}_{\substack{\text{for } n=0 \\ x^{-1}}} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\substack{\text{for } n=0 \\ x^{0}}} \right] = 0$$

We can see now that one term must be "peeled off" from the first two summations in order to get the summations in phase.

$$x^r \left[\sum_{n=1}^{\infty} 3(n+r)(n+r-1) c_n x^{n-1} + \sum_{n=1}^{\infty} (n+r) c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n \right] + \cdots$$
$$\underbrace{x^r \left[3r(r-1) + r \right] c_o x^{-1}}_{\substack{\text{first terms from} \\ \text{first two summations}}} = 0 \quad (46)$$

Let us focus for a moment on the last term on the left-hand side of Equation 46. We know from our experience with the power series solution process that, in order to *solve* the equation, the coefficient for each power of x needs to be zero. Consider now specifically the coefficient for x^{r-1}. It needs to be zero; there are a couple of ways that can happen which are shown in the margin note.

$x^r\left[3r(r-1)+r\right]c_o x^{-1} = 0$

Option #1 set $c_0 = 0$;

Option #2 set r to a root of $3r(r-1) + r = 0$.

THE CUSTOMARY PROCEDURE for Method of Frobenius dictates that we go with option #2. We refer to—$3r(r-1)+r = 0$—as the *indicial equation* and the roots of the indicial equation are known as the *indicial roots*.[9]

[9] For second-order problems, the form of the indicial equation will be a quadratic. The details will be different for different problems but the indicial equation will always be second-order.

FOR THIS CASE, the indicial equation can be factored:

$$\begin{aligned} f(x) &= 3r(r-1)+r \\ &= 3r^2-3r+r \\ &= 3r^2-2r \\ &= r(3r-2) = 0 \end{aligned}$$

so the roots are $r_1 = 0$, and $r_2 = \frac{2}{3}$. So long as r is chosen to be one of those values, the coefficient for x^{r-1} will be zero. Let us refocus our attention on the remaining terms:

$$x^r\left[\sum_{n=1}^{\infty} 3(n+r)(n+r-1)c_n x^{n-1} + \sum_{n=1}^{\infty}(n+r)c_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n\right] = 0$$

The summations are all in phase—recall that is how we obtained the indicial equation—but we need to combine the three summations under a common index.

$$x^r\left[\underbrace{\sum_{n=1}^{\infty} 3(n+r)(n+r-1)c_n x^{n-1}}_{\substack{k=n-1\\ n=k+1}} + \underbrace{\sum_{n=1}^{\infty}(n+r)c_n x^{n-1}}_{\substack{k=n-1\\ n=k+1}} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\substack{k=n\\ n=k}}\right] = 0$$

Making the indicated substitution in each summation gives us:

$$x^r\left\{\sum_{k=0}^{\infty}\underbrace{\left[\underbrace{3(k+1+r)(k+r)c_{k+1} + (k+1+r)c_{k+1}}_{(k+1+r)(3(k+r)+1)c_{k+1}} - c_k\right]}_{(k+1+r)(3(k+r)+1)c_{k+1}-c_k=0} x^k\right\} = 0$$

The resulting two-term recurrence relation for the coefficient for x^{k+r} is:

$$c_{k+1} = \frac{c_k}{(k+1+r)(3k+3r+1)}$$

We have two cases: one for $r = 0$; the other for $r = 2/3$.

$r = 0$:

$$c_{k+1} = \frac{c_k}{(k+1)(3k+1)}$$

Coefficients are shown in the table in the margin; the resulting solution is:

case 1, r = 0
$k = 0$ $c_1 = \frac{c_0}{(1)(1)} = c_0$
$k = 1$ $c_2 = \frac{c_1}{(2)(4)} = \frac{c_0}{8}$
$k = 2$ $c_3 = \frac{c_2}{(3)(7)} = \frac{c_0}{(3)(7)(8)}$

$$\begin{aligned} u_1(x) &= x^0 \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \right) \\ &= c_0 \left(1 + \frac{c_1}{c_0} x + \frac{c_2}{c_0} x^2 + \frac{c_3}{c_0} x^3 + \cdots \right) \\ &= c_0 \left(1 + x + \frac{1}{8} x^2 + \frac{1}{168} x^3 + \cdots \right) \end{aligned}$$

$r = 2/3$:

$$\begin{aligned} c_{k+1} &= \frac{c_k}{(k+1+\frac{2}{3})(3k+3\left(\frac{2}{3}\right)+1)} \\ &= \frac{c_k}{\left(k+\frac{5}{3}\right)(3k+3)} \\ &= \frac{c_k}{(3k+5)(k+1)} \end{aligned}$$

Coefficients are shown in the table in the margin; the resulting solution is:

case 2, r = 2/3
$k = 0$ $c_1 = \frac{c_0}{(5)(1)} = \frac{c_0}{5}$
$k = 1$ $c_2 = \frac{c_1}{(8)(2)} = \frac{c_0}{(2)(5)(8)}$
$k = 2$ $c_3 = \frac{c_2}{(11)(3)} = \frac{c_0}{(2)(3)(5)(8)(11)}$

$$\begin{aligned} u_2(x) &= x^{2/3} \left(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \right) \\ &= c_0 x^{2/3} \left(1 + \frac{c_1}{c_0} x + \frac{c_2}{c_0} x^2 + \frac{c_3}{c_0} x^3 + \cdots \right) \\ &= c_0 x^{2/3} \left(1 + \frac{1}{5} x + \frac{1}{80} x^2 + \frac{1}{264} x^3 + \cdots \right) \end{aligned}$$

A QUICK INSPECTION of $u_1(x)$ and $u_2(x)$ should be sufficient to convince you that the solutions are linearly independent. The general solution to the differential equation comprises a linear combination of $u_1(x)$ and $u_2(x)$.

Indicial Equation

It turns out that we could have determined the indicial roots before set out upon the Method of Frobenius. Recall that we use the Method of Frobenius on differential equations with regular singular points; also recall that a regular singular point is one where $p(x) = xP(x)$ and $q(x) = x^2Q(x)$ are both analytic. If $p(x)$ and $q(x)$ are analytic, that means that they can be represented as a convergent power series.

Suppose we did that, and expressed $p(x)$ and $q(x)$ as a power series; if we did they could be written as:

$$p(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$
$$q(x) = \sum_{n=0}^{\infty} b_n x^n = b_0 + b_1 x + b_2 x^2 + \cdots$$

It can be shown that the indicial equation that we derive from the Method of Frobenius will be equal to:

$$r(r-1) + a_0 r + b_0 = 0$$

where $a_0 = p(0)$ and $b_0 = q(0)$. Applying this equation to our last example where $p(x) = xP(x) = 1/3$ and $q(x) = x^2 Q(x) = -x/3$. We can see $a_0 = p(0) = 1/3$ and $b_0 = q(0) = 0$. Inserting these numbers into the indicial equation gives us:

$$\begin{aligned} r(r-1) + \frac{1}{3}r + 0 &= 0 \\ r^2 - r + \frac{1}{3}r &= 0 \\ r^2 - \frac{2}{3}r &= 0 \\ r(r - \frac{2}{3}) &= 0 \end{aligned}$$

which has the roots: $r = 0$, and $r = 2/3$.

Example: Use the indicial equation to determine the indicial roots to:

$$2xu'' - (3 + 2x)u' + u = 0$$

We see that $P(x) = -\frac{(3-2x)}{2x}$, so $p(x) = xP(x) = -\frac{(3-2x)}{2}$, and $p(0) = -3/2$. By inspection $q(x) = x^2 Q(x) = \frac{x}{2}$, so $q(0) = 0$. The indicial equation is:

$$\begin{aligned} r(r-1) - \frac{3}{2}r + 0 &= 0 \\ r^2 - r - \frac{3}{2}r &= 0 \\ r^2 - \frac{5}{2}r &= 0 \\ r\left(r - \frac{5}{2}\right) &= 0 \end{aligned}$$

so the indicial roots are: $r = 0$, and $r = 5/2$.

In general, of course the indicial equation is just a quadratic equation, so the roots may be real and repeated, real and distinct and complex conjugates. There are three cases that will be of immediate interest to us:

1. Two distinct roots that do *not* differ by an integer. In this case it can be shown that there exist two linearly independent solutions: $u_1 = \sum_{n=0}^{\infty} c_n x^{n+r_1}$, and $u_2 = \sum_{n=0}^{\infty} c_n x^{n+r_2}$.

In both of these cases we implicitly assume that $c_0 \neq 0$.

2. Two distinct roots that differ by an integer. In this case there exists two linearly independent solutions of the form:

$$u_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$
$$u_2(x) = C u_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0$$

In this case, the constant C *might* be zero.

Note: The goal in this treatment of Method of Frobenius is not to make you a "Frobenius Genius". The goal is to provide a sufficiently thorough introduction so that you can understand where Bessel functions and other such mathematical objects come from. For this reason, we will emphasize systems that fall into case 1.

3. If $r_1 = r_2$ then there exist two linearly independent solutions of the form:

$$u_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1}, \quad c_0 \neq 0$$
$$u_2(x) = u_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2}, \quad b_0 \neq 0$$

Additional Notes:

- When the difference between the indicial roots is equal to an integer, find the solution with the smaller root first.
- The indicial equation could, in principle, have complex roots. We will avoid those cases for this class.
- If x_0 is an irregular singular point, the Frobenius theorem does not apply and we may not be able to find any solution to the differential equation using this method.

For ordinary differential equations that fall into the last two categories, do not fret: numerical methods are always available that are more than adequate for finding solutions to the equations.

Assignment #4

Find two power series solutions of the given differential equation.

1. $u'' + x^2u' + xu = 0$

2. $u'' - (x+1)u' - u = 0$

3. Solve the given initial value problem. Use MATLAB to represent the power series. Make 2 plots; for the first plot compare the partial sum of the power series with 5 terms to the exact solution which is $u = 8x - 2e^x$; for the second plot compare the partial sum of the power series with 15 terms to the exact solution. Submit the "published" version of your MATLAB script (PDF format) along with your written solution.

$$(x-1)u'' - xu' + u = 0, \;\; u(0) = -2, \; u'(0) = 6$$

Determine the singular points for the differential equation. Classify each singular point as irregular or regular.

4. $\left(x^2-9\right)^2 u'' + (x+3)u' + 2u = 0$

5. $x^3\left(x^2-25\right)(x-2)^2 u'' + 3x(x-2)u' + 7(x+5)u = 0$

Use the general form of the indicial equation to find the indicial roots.

6. $x^2u'' + \left(\frac{5}{3}x + x^2\right)u' - \frac{1}{3}u = 0$

Use the method of Frobenius to obtain two linearly independent series solutions:

7. $3xu'' + (2-x)u' - u = 0$

Lecture 12 - Bessel's Equation and Bessel Functions

Objectives

The objectives of this lecture are:

- Introduce Bessel's equation and solve it using the method of Frobenius.
- Discuss Bessel Functions of the 1st and 2nd Kind and use them to solve instances of Bessel's Equation.

Bessel's Equation

Bessel's equation is given in Equation 47

$$x^2u'' + xu' + \left(x^2 - \nu^2\right)u = 0 \tag{47}$$

where ν is a constant.[10] You should spend a moment to verify that this equation has a singular point at $x_0 = 0$ and that it is a regular singular point. Therefore we should use the method of Frobenius to find solutions; that is what we will do.

[10] Let me warn you for the first time here that, if $\nu = 0$, Bessel's equation bears a striking resemblance to the Cauchy-Euler equation. Notice the difference and try not to fall into that trap.

As a reminder, we will make the following substitutions into Equation 47:

$$\begin{aligned} u(x) &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ u'(x) &= \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} \\ u''(x) &= \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} \end{aligned}$$

which gives us:

$$x^2 \sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r-2} + x\sum_{n=0}^{\infty}(n+r)c_n x^{n+r-1} + \left(x^2-\nu^2\right)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

or, if we distribute terms through the sums:

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r+2} - \sum_{n=0}^{\infty} \nu^2 c_n x^{n+r} = 0$$

Let us inspect the first term in each summation and see what needs to be done to get the summations in phase.

$$\underbrace{\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n x^{n+r}}_{n=0,\ x^r} + \underbrace{\sum_{n=0}^{\infty}(n+r)c_n x^{n+r}}_{n=0,\ x^r} + \underbrace{\sum_{n=0}^{\infty} c_n x^{n+r+2}}_{n=0,\ x^{r+2}} - \underbrace{\sum_{n=0}^{\infty} \nu^2 c_n x^{n+r}}_{n=0,\ x^r} = 0$$

For reasons that (hopefully) will become apparent, we are going to go through this process in two steps. For $n = 0$, we will separate out all terms that are proportional to x^r.

$$\begin{aligned} r(r-1)c_0 x^r + x^r \sum_{n=1}^{\infty}(n+r)(n+r-1)c_n x^n + rc_0 x^r + x^r \sum_{n=1}^{\infty}(n+r)c_n x^n + \cdots \\ x^r \sum_{n=0}^{\infty} c_n x^{n+2} - \nu^2 c_0 x^r - x^r \sum_{n=1}^{\infty} \nu^2 c_n x^n = 0 \end{aligned}$$

Now let us collect the terms outside of the summations and re-write the equation:

$$\begin{aligned} &\overbrace{\left[r(r-1) + r - \nu^2\right]}^{\text{indicial equation}} c_0 x^r + \cdots \\ &x^r \sum_{n=1}^{\infty} \underbrace{\left[(n+r)(n+r-1) + (n+r) - \nu^2\right]}_{\text{combined 3 of 4 summations}} c_n x^n + x^r \sum_{n=0}^{\infty} c_n x^{n+2} = 0 \end{aligned}$$

The first line of the equation is the indicial equation for this problem; we use it to determine allowable values of ν. In the second line of the equation we have combined the first, second, and fourth summation because they were in phase and had a common index. The remaining summation needs to be put in phase yet before we can combine everything under a single summation.

From the indicial equation:

$$\begin{aligned} r(r-1) + r - \nu^2 &= 0 \\ r^2 - r + r - \nu^2 &= 0 \\ r^2 - \nu^2 &= 0 \\ (r-\nu)(r+\nu) &= 0 \end{aligned}$$

Reminder: we need to ensure the coefficient for x^r is equal to zero. By convention we assume $c_0 \neq 0$. We *could* allow $c_0 = 0$ but then we would just need to derive another indicial equation for some other power of x. We adopt the convention $c_0 \neq 0$ so that our choices for indicial roots will be unique.

we see that, to ensure the coefficient for $x^r = 0$, $r = \pm\nu$. To simplify the discussion to follow, let us take $r = \nu$ and continue with the solution. Our equation is now:

$$\begin{aligned} x^{\nu} \sum_{n=1}^{\infty} \left[\underbrace{(n+\nu)(n+\nu-1) + (n+\nu) - \nu^2}_{n^2+2n\nu+\nu^2-\nu^2}\right] c_n x^n + x^{\nu} \sum_{n=0}^{\infty} c_n x^{n+2} &= 0 \\ x^{\nu} \sum_{n=1}^{\infty} \underbrace{\left[n(n+2\nu)\right] c_n x^n}_{n=1,\ x^1} + x^{\nu} \sum_{n=0}^{\infty} \underbrace{c_n x^{n+2}}_{n=0,\ x^2} &= 0 \end{aligned}$$

The two summations are out of phase, so we need to separate out the first term of the first summation.

$$\underbrace{(1)(1+2\nu)c_1}_{\text{coefficient for } x^{\nu+1}} x^{\nu+1} + \underbrace{x^{\nu}\sum_{n=2}^{\infty} c_n[n(n+2\nu)]x^n}_{n=2,\ x^{\nu+2}} + \underbrace{x^{\nu}\sum_{n=0}^{\infty} c_n x^{n+2}}_{n=0,\ x^{\nu+2}} = 0$$

to the best of your ability, In order to satisfy the equation, we need the coefficient for $x^{\nu+1}$ to be equal to zero; the only way to do this is to set $c_1 = 0$.[11]

The summations in the equation above are in-phase so we need to combine under a common index.

$$\underbrace{x^{\nu}\sum_{n=2}^{\infty} c_n[n(n+2\nu)]x^n}_{\substack{k=n\\ n=k}} + \underbrace{x^{\nu}\sum_{n=0}^{\infty} c_n x^{n+2}}_{\substack{k=n+2\\ n=k-2}} = 0$$

$$x^{\nu}\sum_{k=2}^{\infty} \underbrace{[k(k+2\nu)c_k + c_{k-2}]}_{\text{coefficient for } x^{\nu+k}} x^k = 0$$

[11] Remember: we do not control what ν is; that is part of the equation.

In order to set the coefficients for $x^{\nu+k}$ to zero, we derive the following two-term recurrence:

$$c_k = \frac{-c_{k-2}}{k(k+2\nu)}, \quad k = 2,3,4,\ldots \tag{48}$$

Since we have already determined that $c_1 = 0$, Equation 48 tells us that $c_3 = c_5 = \cdots = 0$; all the odd-numbered coefficients must be zero. To simplify the notation further, we will thus assume that $k = 2n$ and re-write our recurrence as:

$$\begin{aligned} c_{2n} &= \frac{-c_{2n-2}}{2n(2n+2\nu)}, \quad n = 1,2,3,\ldots \\ &= \frac{-c_{2n-2}}{2^2 n(n+\nu)}, \quad n = 1,2,3,\ldots \end{aligned}$$

Expressions for the first few terms is given in Table 3. From this pattern you should be able to see that the general form of the coefficients is as shown in Equation 49

$n=1$ $c_2 = \frac{-c_0}{2^2(1)(1+\nu)}$
$n=2$ $c_4 = \frac{-c_2}{2^2(2)(2+\nu)} = \frac{c_0}{2^4(1)(2)(1+\nu)(2+\nu)}$
$n=3$ $c_6 = \frac{-c_4}{2^2(3)(3+\nu)} = \frac{-c_0}{2^6(1)(2)(3)(1+\nu)(2+\nu)(3+\nu)}$

Table 3: First few coefficients in solution to Bessel's Equation.

$$c_{2n} = \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \quad n = 1,2,3,\ldots \tag{49}$$

It may be worthwhile to take a step back and summarize what we have found so far. We are solving Bessel's equation; we looked for solutions of the form $u(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$. We found that r must be equal to $\pm\nu$ and, for the case $r = \nu$, derived a perfectly acceptable expression for the coefficients in this solution in Equation 49. What follows is a bit of, what we will call, "mathematical grooming" which we will do so that we can derive solutions to Bessel's equation in a form that appears elsewhere in the literature and that, indeed, you will use for the remainder of this course.

We will deal with the case $r = -\nu$, albeit in a perfunctory manner, below.

Gamma Function

The Gamma function is defined in Equation 50.

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt \tag{50}$$

One property of the Gamma function is that $\Gamma(x+1) = x\Gamma(x)$ for any real argument x. If x is an integer, this makes the Gamma function equivalent to a factorial:

Note: the Gamma function is also defined when a complex argument is used, but that is beyond the scope of this class.

$$\begin{aligned}\Gamma(1) &= \int_0^\infty t^{1-1}e^{-t}\, dt \\ &= \int_0^\infty e^{-t}\, dt \\ &= -e^{-t}\Big|_0^\infty \\ &= -[0-1] \\ &= 1\end{aligned}$$

Put differently, the Gamma function is a *generalization* of a factorial.

$$\begin{aligned}\Gamma(2) &= \Gamma(1+1) = 1\Gamma(1) = 1 \\ \Gamma(3) &= \Gamma(2+1) = 2\Gamma(2) = 2 \\ \Gamma(4) &= \Gamma(3+1) = 3\Gamma(3) = 6\end{aligned}$$

In general for $x \in \mathcal{I}$, $\Gamma(x) = (x-1)!$.

IN THE CONTEXT of our solution to Bessel's equation, we use the Gamma function to compactly represent the term $(1+\nu)(2+\nu)\cdots(n+\nu)$ in the denominator of Equation 49:

$$\begin{aligned}\Gamma(1+\nu+1) &= (1+\nu)\Gamma(1+\nu) \\ \Gamma(1+\nu+2) &= (2+\nu)\Gamma(2+\nu) = (2+\nu)(1+\nu)\Gamma(1+\nu) \\ \Gamma(1+\nu+3) &= (3+\nu)\Gamma(3+\nu) = (3+\nu)(2+\nu)(1+\nu)\Gamma(1+\nu) \\ &\vdots \\ \Gamma(1+\nu+n) &= (n+\nu)\cdots(1+\nu)\Gamma(1+\nu)\end{aligned}$$

Bessel Function of the First Kind of order ν

We will use everything that we have done thus far to define a Bessel Function of the First Kind of order ν. We will start with our series solution $u(x) = \sum_{n=0}^{\infty} c_n x^{n+\nu}$ and the formula for the non-zero (even) coefficients given in Equation 49 and take a couple of steps:

1. We will set $c_0 = \frac{1}{2^{\nu}\Gamma(1+\nu)}$.

$$\begin{aligned} c_{2n} &= \frac{(-1)^n c_0}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)}, \quad n = 1,2,3,\ldots \\ &= \frac{(-1)^n}{2^{2n} n!(1+\nu)(2+\nu)\cdots(n+\nu)} \frac{1}{2^{\nu}\Gamma(1+\nu)} \\ &= \frac{(-1)^n}{2^{2n+\nu} n!(1+\nu)\cdots(n+\nu)\Gamma(1+\nu)} \\ &= \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)} \end{aligned}$$

Remember c_0 is just an arbitrary constant. This decision allows us, with the help of Gamma functions, to express the coefficients to the solution in a compact form. The resulting solution can then be multiplied by *another* arbitrary constant if needed to satisfy a given initial/boundary condition.

2. Combining this new expression for c_{2n} into the solution gives us the standard definition for a Bessel Function of the First Kind:

$$J_{\nu}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \tag{51}$$

It can be shown that if $\nu \geq 0$ the series converges for all x.

We can similarly handle the case where $r = -\nu$:

$$J_{-\nu} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1-\nu+n)} \left(\frac{x}{2}\right)^{2n-\nu}$$

We were sly about it, but we quietly added the $n = 0$ term to the summation. A more verbose expression would be:

$$\begin{aligned} u(x) &= c_0 x^0 + \sum_{n=1}^{\infty} c_{2n} x^{2n+\nu} \\ &= \frac{1}{2^{\nu}\Gamma(1+\nu)} + \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)} x^{2n+\nu} \\ &= \frac{1}{2^{2(0)+\nu} 0!\Gamma(1+\nu+0)} + \cdots \\ &\cdots \sum_{n=1}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)} x^{2n+\nu} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+\nu} n!\Gamma(1+\nu+n)} x^{2n+\nu} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu} \end{aligned}$$

We will not prove this, but if ν is *not* an integer, then J_{ν} and $J_{-\nu}$ are linearly independent. In that case, the solution (at long last) to Bessel's equation is just a linear combination of $J_{\nu}(x)$ and $J_{-\nu}(x)$ as shown in Equation 52.

$$u(x) = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x) \tag{52}$$

Bessel Function of the Second Kind of order ν

If $\nu \in \mathcal{I}$, then $J_{\nu}(x)$ and $J_{-\nu}(x)$ are not linearly independent so I need to find another solution to Bessel's equation. To this end, we define the Bessel Function of the second kind of order ν, given in Equation 53.

$$Y_{\nu}(x) = \frac{\cos \nu\pi J_{\nu}(x) - J_{-\nu}(x)}{\sin \nu\pi} \tag{53}$$

which is linearly independent of J_{ν} even if ν is an integer. The solution to Bessel's equation can thus alternately be expressed:

$$u(x) = c_1 J_{\nu}(x) + c_2 Y_{\nu}(x)$$

I recommend that you always use J_{ν} and Y_{ν}. It's not hard to decide if ν is an integer or not but consistency has its benefits.

NOW THAT WE KNOW a pair of linearly independent solutions to Bessel's equation, we no longer need to go through the rigmarole of *actually solving* the equation; we can simply use the solution we have derived.

Example: find the general solution to:

$$x^2u'' + xu' + \left(x^2 - \frac{1}{9}\right) u = 0$$

We recognize the given equation as Bessel's equation of order $\nu = 1/3$. The general solution is:

$$u(x) = c_1 J_{1/3}(x) + c_2 Y_{1/3}(x)$$

Alternatively we could, of course, have used: $u(x) = c_1 J_{1/3}(x) + c_2 J_{-1/3}(x)$.

Example: find the general solution to:

$$xu'' + u' + xu = 0$$

To the uninitiated this may not look like Bessel's equation but with practice you will learn to automatically see the above equation as:

$$\begin{aligned} xu'' + u' + xu &= 0, \quad \text{multiply by x} \\ x^2u'' + xu' + x^2u &= 0 \\ x^2u'' + xu' + \left(x^2 - 0^2\right) u &= 0 \end{aligned}$$

and recognize it to be Bessel's equation of order zero. The general solution is:

$$u(x) = c_1 J_0(x) + c_2 Y_0(x)$$

Lecture 13 - Solving ODEs Reducible to Bessel's Equation

Objectives

Demonstrate reducing and ODE to Bessel's Equation by:

- changing the dependent variable;
- changing the independent variable; and
- changing both the dependent and independent variables

If we have learned one thing over the course of the last couple of lectures it is that using the method of Frobenius—whether we are solving Bessel's equation or some other differential equation with regular singular points—is tedious and error-prone. The good news, at least for Bessel's equation, is that if we see that it is Bessel's equation we are trying to solve, we can simply write down the solution in terms of Bessel functions.[12]

[12] Let me reiterate that this was the point to learning how to solve Bessel's equation.

In this and the next lecture we will learn some techniques by which a broad range of differential equations can be transformed into or expressed as Bessel's equation. The best way to learn is by doing, so we will simply start with the examples.

Example: Find the general solution to the following differential equation by applying the given transformation to the dependent variable: $u = {}^{v}/_{x^2}$.

You can think of this as cleverly distorting the y-axis in order to make the problem easier.

$$xu'' + 5u' + xu = 0$$

We need to replace all appearances of u with the equivalent in terms of v.

Using the product rule:

$$\begin{aligned} u &= \frac{v}{x^2} \\ u' &= \frac{-2v}{x^3} + \frac{v'}{x^2} \\ u'' &= \frac{6v}{x^4} \underbrace{- \frac{2v'}{x^3} - \frac{2v'}{x^2}}_{\frac{-4v'}{x^3}} + \frac{v''}{x^2} \end{aligned}$$

where ${}^{dv}/_{dx} = v'$.

$$\begin{aligned} xu'' &= \frac{v''}{x} - \frac{4v'}{x^2} + \frac{6v}{x^3} \\ 5u' &= \frac{5v'}{x^2} - \frac{10v}{x^3} \\ xu &= \frac{v}{x} \end{aligned}$$

Combining these terms together gives us:

Seeing the necessary transformations comes with practice; that is what homework is for.

$$\begin{aligned} xu'' + 5u' + xu &= 0 \\ \frac{v''}{x} - \frac{4v'}{x^2} + \frac{6v}{x^3} + \frac{5v'}{x^2} - \frac{10v}{x^3} + \frac{v}{x} &= 0, \quad \text{combine like terms} \\ \frac{v''}{x} + \frac{v'}{x^2} + \left(\frac{1}{x} - \frac{4}{x^3}\right) v &= 0, \quad \text{multiply by } x^3 \\ x^2 v'' + xv' + \left(x^2 - 4\right) v &= 0 \end{aligned}$$

where on the last line we recognize the ODE as Bessel's equation of order $\nu = 2$. The solution is:

$$v(x) = c_1 J_2(x) + c_2 Y_2(x)$$

Of course, we were trying to solve for $u(x)$ so we must undo the transformation to the dependent variable:

$$u(x) = \frac{v(x)}{x^2} = \frac{1}{x^2}\left[c_1 J_2(x) + c_2 Y_2(x)\right]$$

Example: Find the general solution to the following differential equation by applying the given transformation to the independent variable: $\sqrt{x} = z$:

This is like cleverly distorting the x-axis with the goal of making the problem easier.

$$4xu'' + 4u' + u = 0$$

In this case we need to change occurrences of x into its equivalent in terms of z and we need to change all derivatives with respect to x to derivatives with respect to z. We are given $\sqrt{x} = z$ which is, of course, equivalent to $x = z^2$. For the derivatives we have:

Note: it is important that you purge all expressions including x out of these derivatives. For example, when computing the equivalent of u' it was essential that we make the substitution $x^{-1/2} = z^{-1}$. When we used that result in calculating u'' and took derivatives with respect to z, any occurrence of x needs to be replaced with its equivalent in z or the derivative would have been wrong.

$$\begin{aligned} u' &= \frac{du}{dx} = \frac{du}{dz}dzdx = u_z \frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}\underbrace{x^{-1/2}}_{z^{-1}} u_z \\ &= \frac{1}{2z}u_z \\ u'' &= \frac{d}{dx}\left(\frac{du}{dx}\right) = \frac{d}{dz}\left(\frac{du}{dx}\right)\frac{dz}{dx} \\ &= \frac{d}{dz}\left[\frac{1}{2z}u_z\right]\frac{1}{2z} = \left[-\frac{1}{2z^2}u_z + \frac{1}{2z}u_{zz}\right]\frac{1}{2z} \\ &= \frac{1}{4z^2}u_{zz} - \frac{1}{4z^3}u_z \end{aligned}$$

We now use these results to make substitutions in the original equation:

$$\begin{aligned} 4xu'' &= 4z^2\left[\frac{1}{4z^2}u_{zz} - \frac{1}{4z^3}u_z\right] \\ 4u' &= 4\left[\frac{1}{2z}u_z\right] \end{aligned}$$

So the transformed equation is:

$$\begin{aligned}
u_{zz} - \frac{1}{z}u_z + \frac{2}{z}u_z + u &= 0, \quad \text{combine like terms} \\
u_{zz} + \frac{1}{z}u_z + u &= 0, \quad \text{multiply by } z^2 \\
z^2 u_{zz} + z u_z + \underbrace{z^2}_{(z^2-0^2)} u &= 0
\end{aligned}$$

and we can immediately recognize this as Bessel's equation of order $\nu = 0$ and the general solution is:

$$\begin{aligned}
u(z) &= c_1 J_0(z) + c_2 Y_0(z), \quad \text{undo transformation: } z \to x \\
u(x) &= c_1 J_0(\sqrt{x}) + c_2 Y_0(\sqrt{x})
\end{aligned}$$

Example: Find the general solution to the equation below by transforming the dependent variable $u = v\sqrt{x}$, and the independent variable $\sqrt{x} = z$.

In this case we are distorting *both* the x- and y-axis to "simplify" the problem.

$$x^2 u'' + \frac{1}{4}\left(x + \frac{3}{4}\right)u = 0$$

We will first transform the dependent variable: $u = v\sqrt{x} = x^{1/2}v$. As before we will replace all appearances of u with the equivalent in terms of v. Using the product rule:

$$\begin{aligned}
u &= x^{1/2}v \\
u' &= \frac{1}{2}x^{-1/2}v + x^{1/2}v' \\
u'' &= -\frac{1}{4}x^{-3/2}v + \underbrace{\frac{1}{2}x^{-1/2}v' + \frac{1}{2}x^{-1/2}v'}_{x^{-1/2}v'} + x^{1/2}v''
\end{aligned}$$

and inserting into our equation gives us:

$$x^2\left[x^{1/2}v'' + x^{-1/2}v' - \frac{1}{4}x^{-3/2}v\right] + \frac{1}{4}\left[x + \frac{3}{4}\right]x^{1/2}v = 0$$

Now we transform the independent variable $\sqrt{x} = z$, which is the same transformation that we did for the last example so we will not repeat the work. and we will substitute variously in the equation: $x^{1/2} = z,\ x = z^2$, and $x^2 = z^4$.

From the last example:

$$\begin{aligned}
v' &= \frac{1}{2z}v_z \\
v'' &= \frac{1}{4z^2}v_{zz} - \frac{1}{4z^3}v_z
\end{aligned}$$

$$z^4\left[z\left(\frac{1}{4z^2}v_{zz} - \frac{1}{4z^3}v_z\right) + \frac{1}{z}\left(\frac{1}{2z}v_z\right) - \frac{1}{4}z^{-3}v\right] + \cdots$$

$$\cdots \frac{1}{4}\left(z^3 + \frac{3}{4}\right)zv = 0$$

Distributing the z's and grouping terms gives us:

$$\begin{aligned}
\frac{z^3}{4}v_{zz} + \left(-\frac{z^2}{4} + \frac{z^2}{2}\right)v_z + \left(-\frac{z}{4} + \frac{z^3}{4} + \frac{3z}{16}\right)v = 0, &\text{ combining like terms} \\
\frac{z^3}{4}v_{zz} + \frac{z^2}{4}v_z + \left(\frac{z^3}{4} - \frac{z}{16}\right) = 0, &\text{ multiply by } 4/z \\
z^2 v_{zz} + z v_z + \left(z^2 - \frac{1}{4}\right)v = 0 &
\end{aligned}$$

which, at long last, we recognize as Bessel's equation of order $\nu = 1/2$.

The solution, by inspection, is:

$$\begin{aligned}
&v(z) = c_1 J_{1/2}(z) + c_2 Y_{1/2}(z), \quad \text{un-transform the dependent variable.} \\
&u(z) = \sqrt{x}\left(c_1 J_{1/2}(z) + c_2 Y_{1/2}(z)\right), \quad \text{un-transform the independent variable.} \\
&u(x) = \sqrt{x}\left(c_1 J_{1/2}(\sqrt{x}) + c_2 Y_{1/2}(\sqrt{x})\right)
\end{aligned}$$

Notes:

- Obviously, one would need to have spectacular insight to know in advance what transformations should be made in order to convert a given differential equation into Bessel's equation.
- In the next lecture we will make use of some tools that have been developed to simplify these transformations.

Lecture 14 - Modified Bessel Function and Parametric Modified Bessel Function

Objectives

- Show how to use the Parametric Bessel Equation of order ν.
- Describe the Modified Bessel Equation and, their solutions, Modified Bessel Functions.
- Introduce and illustrate a tool for solving second-order ODEs in terms of Bessel Functions.

Many differential equations can be solved in terms of Bessel functions. This lecture will introduce some relatively simple and powerful tools for doing so.

Parametric Bessel Equation of Order ν

The parametric Bessel equation of order ν as the form given in Equation 54

$$x^2u'' + xu' + \left(\alpha^2x^2 - \nu^2\right)u = 0 \tag{54}$$

Rather than state the solution outright, let us take a different shortcut and apply the well-known transformation that will convert Equation 54 into Bessel's equation that we can solve, by inspection, with Bessel functions.

The transformation is $t = \alpha x$. This, of course, means that $x = t/\alpha$ and the derivatives of u with respect to t are shown in the margin. Applying these substitutions gives us:

$$\begin{aligned} x^2u'' + xu' + \left(\alpha^2x^2 - \nu^2\right)u &= 0 \\ \frac{t^2}{\alpha^2}\alpha^2u_{tt} + \frac{t}{\alpha}\alpha u_t + \left(\alpha^2\frac{t^2}{\alpha^2} - \nu^2\right)u &= 0 \\ t^2u_{tt} + tu_t + \left(t^2 - \nu^2\right)u &= 0 \end{aligned}$$

The derivatives of u with respect to t:

$$\begin{aligned} \frac{dt}{dx} &= \alpha \\ u' &= \frac{du}{dx} = \frac{du}{dt}\frac{dt}{dx} = \alpha u_t \\ u'' &= \frac{d}{dx}\left(\frac{du}{dx}\right) = \cdots \\ &= \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx} = \cdots \\ &= \frac{d}{dt}\left(\alpha u_t\right)\alpha = \alpha^2u_{tt} \end{aligned}$$

The last line is, of course, Bessel's equation and the solution is:

$$u(t) = c_1 J_\nu(t) + c_2 Y_\nu(t)$$

Undoing the change of independent variables to express the answer in terms of x gives us the solution shown in Equation 55.

$$u(x) = c_1 J_\nu(\alpha x) + c_2 Y_\nu(\alpha x) \tag{55}$$

Example: us the parametric Bessel equation to find the general solution to:

$$x^2 u'' + x u' + \left(36x^2 - \frac{1}{4}\right) u = 0$$

We recognize the equation as a parametric Bessel equation; the parameter $\alpha = \sqrt{36} = 6$ and $\nu^2 = 1/4 \Rightarrow \nu = 1/2$. The solution is:

$$u(x) = c_1 J_{1/2}(6x) + c_2 Y_{1/2}(6x)$$

At this point in the course you should be developing a list, of sorts, of differential equations that you recognize and know how to analyze. Call it something like *"A Field Guide to Differential Equations I Know How To Solve"*. This list should include:

- first-order linear equations
- separable equations
- linear constant-coefficient equations
- linear equations with variable coefficients, including:
  - Cauchy-Euler equations
  - Legendre's equation
  - Bessel's equation; and now
  - parametric Bessel's equation.

For these last problem types you "solve" them by recognizing the equation and writing down the solution. **Note:** for the example, instead of using $Y_{1/2}(6x)$ as the second linearly independent solution, we could have used $J_{-1/2}(6x)$; it is entirely up to you.

Modified Bessel Equations and Bessel Functions

A subtle but non-trivial variation to Bessel's equation is when we flip one crucial sign as shown in Equation 56

$$x^2 u'' + x u' - \left(x^2 + \nu^2\right) u = 0 \tag{56}$$

This equation can be converted into Bessel's equation by transforming the dependent variable—$u = i^{-\nu} v$—and independent variable—$t = ix$. We will omit these details and instead simply give the solution as shown in Equation 57 which includes modified Bessel functions of the first[13] and second kind[14]:

$$u(x) = c_1 I_\nu(x) + c_2 K_\nu(x) \tag{57}$$

The modified Bessel's equation also has a parametric form as shown in Equation 58

$$x^2 u'' + x u' - \left(\alpha^2 x^2 + \nu^2\right) u = 0 \tag{58}$$

with the general solution given in Equation 59.

$$u(x) = c_1 I_\nu(\alpha x) + c_2 K_\nu(\alpha x) \tag{59}$$

[13] Modified Bessel functions of the first kind are defined as:

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$

[14] Modified Bessel functions of the second kind, analogous to Bessel functions of the second kind, are defined in terms of modified Bessel functions of the first kind:

$$K_\nu(x) = \frac{\pi}{2} \frac{I_\nu(x) - I_\nu(x)}{\sin \nu\pi}$$

Tool for Solving Second-Order ODEs

A more general-purpose tool for solving linear, homogeneous, second-order ODEs in terms of Bessel functions is presented in [15] and shown below in Equation 60.

[15] Dennis G Zill. *Advanced Engineering Mathematics*. Jones & Bartlett Learning, 2020

$$u'' + \frac{1-2a}{x}u' + \left(b^2c^2x^{2c-2} + \frac{a^2-p^2c^2}{x^2}\right)u = 0, \quad p \geq 0 \tag{60}$$

The general solution for equations of this form is given in Equation 61

$$u = x^a\left[c_1 J_p\left(bx^c\right) + c_2 Y_p\left(bx^c\right)\right] \tag{61}$$

Using this tool requires you to solve four non-linear equations as shown below:

Probably the most challenging or, at least, error-prone part of this process is writing a given ODE in the form of Equation 60.

$$u'' + \frac{\overset{①}{\boxed{1-2a}}}{x}u' + \left(\underset{②}{\boxed{b^2c^2}}x^{\overset{③}{\boxed{2c-2}}} + \frac{\overset{④}{\boxed{a^2-p^2c^2}}}{x^2}\right)u = 0, \quad p \geq 0$$

Example: Use Equation 60 to find the general solution to the following differential equation:

$$x^2u'' + \left(x^2-2\right)u = 0$$

Re-writing the equation in the form of Equation 60 gives us:

$$\begin{aligned} x^2u'' + \left(x^2-2\right)u &= 0 \\ u'' + \frac{x^2-2}{x^2}u &= 0 \\ u'' + 0u' + \left(1x^0 + \frac{-2}{x^2}\right)u &= 0 \end{aligned}$$

Now we solve the four equations:

- ① $1-2a = 0 \Rightarrow a = 1/2$
- ③ $2c-2 = 0 \Rightarrow c = 1$
- ② $b^2c^2 = 1, \;\; b^2(1) = 1, \;\Rightarrow b = 1$
- ④ $a^2 - p^2c^2 = -2$

$$\begin{aligned} \left(\frac{1}{2}\right)^2 - p^2(1)^2 &= -2 \\ p^2 &= \frac{1}{4} + 2 = \frac{9}{4} \\ \Rightarrow p &= \frac{3}{2} \end{aligned}$$

Using Equation 61 the general solution is:

$$u(x) = x^{1/2}\left[c_1 J_{3/2}(x) + c_2 Y_{3/2}(x)\right]$$

Assignment #5

Find the general solution to the following differential equations in terms of Bessel Functions:

1. $4x^2u'' + 4xu' + \left(4x^2 - 25\right) u = 0$

2. $x^2u'' + xu' + \left(9x^2 - 4\right) u = 0$

3. $x^2u'' + xu' - \left(16x^2 + \frac{4}{9}\right) u = 0$

Use the indicated change of variables to find the general solution of the given differential equation.

4. $x^2u'' + 2xu' + \alpha^2x^2u = 0, \quad u = x^{-1/2}v(x)$

Use Equation 60 to find the general solution of the following differential equation in terms of Bessel functions.

5. $xu'' + 2u' + 4u = 0$

Review #1

Solve the following differential equations:

1. $6x^2u'' + 5xu' - u = 0$

2. $x^2u'' - 7xu' + 12u = 0, \;\; u(0) = 0, \; u(1) = 0$

Use the method of power series to solve the following initial value problem:

3. $u'' + xu' + 2u = 0, \;\; u(0) = 3, \; u'(0) = -2$

Use the method of Frobenius to solve the following differential equation:

4. $2xu'' + u' + u = 0$

Find the general solution of the given differential equation in terms of Bessel functions:

5. $4x^2u'' + 4xu' + \left(64x^2 - 9\right) u = 0$

Part III

Orthogonal Functions and Fourier Series

Lecture 15 - Introduction to Orthogonal Functions

Objectives

- Define orthogonal functions, weighted orthogonality, function norms, and complete sets of orthogonal functions.
- Provide analogies of these concepts as applied to vectors and functions.

In previous lectures we were able to solve some differential equations by representing the solution in the form of an infinite series. For second-order, homogeneous, linear, variable coefficient ODEs where $P(x)$ and $Q(x)$ are analytic throughout the domain of interest, we used power series:

$$u(x) = \sum_{n=0}^{\infty} c_n x^n$$

Recall that ODEs of this type can be expressed in standard form as:

$$u'' + P(x)u' + Q(x)u = 0$$

For equations with regular singularities in the domain of interest, we used the method of Frobenius and expressed the solutions:

$$u(x) = \sum_{n=0}^{\infty} c_n x^{n+r}$$

It should be stressed that, if you calculate the coefficients (c_n) in exact arithmetic and if you sum *all* of the terms $(n \to \infty)$, the representation is exact. Each term in the series is linearly independent from all other terms, so as we keep adding terms to the representation of $u(x)$, greater accuracy is achieved.

Our next idea is to generalize this approach by representing our solution, $u(x)$, as a linear combination of *orthogonal functions.*

Inner Product and Orthogonality of Functions

From previous courses in calculus, you should be familiar with the concept of orthogonality of vectors. We test for orthogonality by taking the "dot-product"; if the dot-product is equal to zero, the vectors are orthogonal, otherwise they are not.

Vector dot product:

$$\left(\vec{a},\vec{b}\right) = \sum_{i=1}^{n}(a_i)(b_i)$$
$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \cdots a_nb_n$$

Orthogonality can also be defined for functions. Consider two functions $u_1(x)$ and $u_2(x)$ defined on an interval $x \in [a,b]$. The inner product of $u_1(x)$ and $u_2(x)$ is defined in Equation 62.

$$(u_1(x), u_2(x)) = \int_a^b u_1(x)u_2(x)\ dx \tag{62}$$

If $(u_1(x), u_2(x)) = 0$ then $u_1(x)$ and $u_2(x)$ are said to be *orthogonal* on interval $x \in [a,b]$.

Example: show that the functions $u_1(x) = x^2$ and $u_2(x) = x^3$ are orthogonal on the interval $x \in [-1,1]$.

$$\begin{aligned}(u_1,u_2) &= \int_{-1}^{1} x^2x^3\ dx \\ &= \int_{-1}^{1} x^5\ dx = \frac{1}{6}x^6\Big|_{-1}^{1} \\ &= \frac{1}{6} - \frac{1}{6} = 0\end{aligned}$$

A SLIGHT GENERALIZATION is *weighted* orthogonality, where we apply a *weight function* to the inner product:

$$(u_1,u_2) = \int_a^b u_1(x)u_2(x)w(x)\ dx \tag{63}$$

where if $(u_1,u_2) = 0$ then we say they are orthogonal with respect to weight function $w(x)$.

WE CAN ASSEMBLE sets of orthogonal functions on a specified interval. If $\{\phi_1,\phi_2,\phi_3,\ldots,\phi_n\}$ is a set of orthogonal functions on the interval $x \in [a,b]$, then:

$$\left(\phi_i,\phi_j\right) = \int_a^b \phi_i\phi_j\ dx = 0, \ \text{ if } i \neq j$$

We can then use a linear combination of the members of this set of orthogonal functions to represent practically *any* continuous, or piecewise-continuous function on the interval. This concept will be used *extensively* later in this course.

WHEN DEALING WITH vectors, it is sometimes the case that we want to work with *unit vectors.* Even if we are not in need of unit vectors it is often the case that we need some standard definition of the *size* of a vector. Such a measure is referred to as a *norm*. Norms are often

Example *unit vectors* include the classic Cartesian basis vectors of $\hat{i} = (1,0,0)$, $\hat{j} = (0,1,0)$ and $\hat{k} = (0,0,1)$.

A norm is a functional that assigns a measure to a mathematical object like a vector or a function. To qualify as a norm, the functional must satisfy three basic properties:

1. $||f|| \geq 0$, and $||f|| = 0$ if and only if $f = 0$
2. $||\alpha f|| = \alpha||f||$ for any constant α; and
3. $||f+g|| \geq ||f|| + ||g||$

where f and g are mathematical objects subject to the norm.

denoted $||\cdot||$—i.e. the norm of $f(x)$ is $||f(x)||$—and several types of norms have been defined for vectors, matrices, and functions. The norm we will use for this class is defined in Equation 64.

$$||f(x)||^2 = \int_a^b f(x)^2 \, dx \tag{64}$$

Example: find the norm of the functions $f_0(x) = 1$ and $f_n(x) = \cos nx$ on the interval $[-\pi, \pi]$.

$$\begin{aligned} ||f_0(x)||^2 &= \int_{-\pi}^{\pi} (1)(1) \, dx = x\Big|_{-\pi}^{\pi} \\ &= 2\pi \\ \Rightarrow ||f_0|| &= \sqrt{2\pi} \end{aligned}$$

$$\begin{aligned} ||f_n(x)||^2 &= \int_{-\pi}^{\pi} \cos^2 nx \, dx \\ &= \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx) \, dx \\ &= \frac{1}{2}x + \frac{1}{2n}\sin 2nx\Big|_{-\pi}^{\pi} \\ &= \pi \\ \Rightarrow ||f_n|| &= \sqrt{\pi} \end{aligned}$$

Recall the "double-angle" identity: $\cos 2x = 2\cos^2 x - 1$.

We can apply norms to define *orthonormal* sets of functions in which $\{\phi_0, \phi_1, \ldots \phi_n\}$ are orthonormal if the following is true:

$$(\phi_n, \phi_m) = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

Now, instead of expanding $u(x)$ in a power series or an extended power series, we could expand $u(x)$ in terms of orthonormal functions:

$$u(x) = \sum_{n=0}^{\infty} c_n \phi_n = c_0\phi_0 + c_1\phi_1 + \cdots$$

were $\phi_n(x)$ are members of an orthogonal set of functions.[16] Suppose we wished to expand $f(x)$ in terms of an infinite set of orthogonal functions $\{\phi_0, \phi_1, \ldots,\}$ on the interval $x \in [a, b]$:

$$u(x) = c_0\phi_0 + c_1\phi_1 + c_2\phi_2 + \cdots + c_n\phi_n + \cdots$$

This is analogous to a (possibly) familiar operation in vector analysis. Suppose the vector u is expanded as a linear combination of three orthogonal vectors v_1, v_2, and v_3:

$$u = c_1 v_1 + c_2 v_2 + c_3 v_3$$

Suppose we know u and know v_1, v_2, and v_3; we merely wish to find the coefficients c_1, c_2, and c_3. We can find them by using the inner product for vectors:

$$\begin{aligned} (u, v_1) &= c_1 \overbrace{(v_1, v_1)}^{||v_1||^2} + c_2 \cancelto{0}{(v_2, v_1)} + c_3 \cancelto{0}{(v_3, v_1)} \\ \Rightarrow c_1 &= \frac{(u, v_1)}{||v_1||^2} \end{aligned}$$

Generalizing for all three coefficients:

$$u = \sum_{n=1}^{3} \frac{(u, v_n)}{||v_n||^2} v_n$$

[16] The orthogonal set of functions may be—in fact, in many cases *is*—infinite as is indicated here.

to get individual values c_n, take the inner product—i.e. multiply both sides by the orthogonal function c_n and integrate:

$$
\begin{aligned}
(u, \phi_n) &= \int_a^b u(x)\phi_n(x)\ dx \\
&= \int_a^b c_0 \cancelto{0}{\phi_0\phi_n} + c_1 \cancelto{0}{\phi_1\phi_n} + \cdots + c_n \overbrace{\phi_n\phi_n}^{||\phi_n||^2} + \cdots \\
&= c_n||\phi_n||^2
\end{aligned}
$$

Therefore we can construct or expansion as shown in Equation 65.

$$
u(x) = \sum_{n=0}^{\infty} \frac{(u, \phi_n)}{||\phi_n||^2}\phi_n \tag{65}
$$

As was the case with power series and extended power series: subject to some fairly lenient restrictions on $u(x)$, the expansion shown in Equation 65 is *exact*. Sadly, some practical matters will sully this pristine mathematical paradise. The obvious example is that we will not *actually* be able to sum all of the terms and we will not be able to calculate all of the coefficients, c_n, exactly. In particular, we will favor the use of numeric integration to compute the inner products specified in Equation 65.

It takes a while to add an infinite number of terms.

Lecture 16 - Fourier Series

Objectives

- Review trigonometric Series.
- Derive/show the formulas for expansion of a function as a Fourier series.
- Discuss periodic extensions of non-periodic functions, sine/cosine expansions, and convergence behavior.

Review of Fourier Series

In the last lecture we learned about orthogonal functions and sets of orthogonal functions. We stated that most functions can be expressed as a linear combination of those orthogonal functions:

$$u(x) = \sum_{n=0}^{\infty} c_n \phi_n$$

where ϕ_n are members of a set of orthogonal functions and c_n are determined by:

$$c_n = \frac{(u, \phi_n)}{||\phi_n||^2}\phi_n$$

You should already have experience with expansions such as this from your previous classes in differential equations in the form of Fourier series expansions. In this case the orthogonal functions, $\phi_n(x)$, are:

$$\left\{1, \cos\frac{\pi x}{p}, \cos\frac{2\pi x}{p}, \dots, \sin\frac{\pi x}{p}, \sin\frac{2\pi x}{p}, \dots\right\}$$

The function $\phi(x) = 1$ could also be written: $\cos\frac{0\pi x}{p}$.

where p indicates the *period*.[17]

[17] Reminder that a function, $f(x)$, is periodic with period p if $f(x+p) = f(x)$.

That members of this set of functions are all mutually orthogonal can be directly shown; we will demonstrate this for the members of the form $\phi_n(x) = \cos{}^{n\pi x}/_{p}$; other cases are left for homework exercises.

Example: Show that functions of the form $\phi_n(x) = \cos\frac{n\pi x}{p}$ are orthogonal over the interval $x \in [-p, p]$:

Consider two functions, $\phi_n(x)$ and $\phi_m(x)$ where m, n are integers and $m \neq n$. The functions are orthogonal on the interval $x \in [-p, p]$ if $(\phi_n, \phi_m) = 0$. From the definition of the inner product:

Here we use the identity: $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$. So that

$$\begin{aligned} \cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\ +\cos(\alpha-\beta) &= \cos\alpha\cos\beta + \sin\alpha\sin\beta \\ &= 2\cos\alpha\cos\beta \end{aligned}$$

and therefore

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta) + \cos(\alpha-\beta)\right]$$

$$\begin{aligned} (\phi_n, \phi_m) &= \int_{-p}^{p} \cos\frac{n\pi x}{p}\cos\frac{m\pi x}{p}\,dx \\ &= \frac{1}{2}\int_{-p}^{p} \cos(n+m)\frac{\pi x}{p} + \cos(n-m)\frac{\pi x}{p}\,dx \\ &= \frac{1}{2}\left[\frac{1}{n+m}\frac{p}{\pi}\sin(n+m)\frac{\pi x}{p}\Big|_{-p}^{p} + \frac{1}{n-m}\frac{p}{\pi}\sin(n-m)\frac{\pi x}{p}\Big|_{-p}^{p}\right] \\ &= 0 \end{aligned}$$

where the last terms are zero since we are evaluating the sine function at integer multiples of π. This shows, at least, all of the cosine members are orthogonal. For the case $m = n$ we get:

Rather than derive this rigorously, we will combine a tabulated result of standard integrals: $\int \cos^2 u\,du = \frac{1}{2}u + \frac{1}{4}\sin 2u + C$, with u substitution.

$$\begin{aligned} (\phi_n, \phi_n) &= \int_{-p}^{p} \cos\left(\frac{n\pi x}{p}\right)^2 \\ &= \frac{p}{n\pi}\left[\frac{1}{2}\frac{n\pi x}{p} + \frac{1}{4}\sin\frac{2n\pi x}{p}\right]\Big|_{-p}^{p} \\ &= \frac{p}{2} - \frac{-p}{2} \\ &= p \end{aligned}$$

WE CAN USE this infinite set of orthogonal functions to represent any other continuous function over the interval $[-p, p]$. In your differential equations class you were taught to do this by using Equation 66:

You might be wondering at this point why you would ever want to represent a function $f(x)$ as a linear combination of orthogonal functions. The answer is that the members of the set of orthogonal functions are solutions to a linear homogeneous boundary value problem and the function $f(x)$ will be a boundary condition for a partial differential equation that we are trying to solve.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{p} + b_n\sin\frac{n\pi x}{p}\right] \tag{66}$$

We can solve for the coefficients a_0, a_n and b_n one at a time by multiplying both sides of Equation 66 by the corresponding orthogonal function, and integrating.[18] The orthogonal function corresponding to a_0 is 1; so to find a_0 we multiply both sides of Equation 66 by 1

[18] In more formal mathematical terms: we take the *inner product* of both sides with respect to the orthogonal function, but of course that means the same thing.

and integrate:

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right] \\
\int_{-p}^{p} f(x)(1)\ dx &= \int_{-p}^{p}\frac{a_0}{2}(1)\ dx + \underbrace{\int_{-p}^{p}\sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right](1)\ dx}_{=0 \text{ due to orthogonality}} \\
\int_{-p}^{p} f(x)\ dx &= \frac{a_0}{2}2p + 0 \\
\Rightarrow a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\ dx
\end{aligned}
$$

To get the value of a_1, we multiply both sides by $\cos {}^{\pi x}/_{p}$ and integrate:

$$
\begin{aligned}
\int_{-p}^{p} f(x)\cos\frac{\pi x}{p}\ dx &= \frac{a_0}{2}\cancelto{0}{\int_{-p}^{p}(1)\cos\frac{\pi x}{p}\ dx} + \cdots \\
& \underbrace{a_1\int_{-p}^{p}\cos\frac{\pi x}{p}^2\ dx}_{=p} + b_1\cancelto{0}{\int_{-p}^{p}\sin\frac{\pi x}{p}\cos\frac{\pi x}{p}\ dx} + a_2\cancelto{0}{\int_{-p}^{p}\cos\frac{2\pi x}{p}\cos\frac{\pi x}{p}\ dx} + \cdots
\end{aligned}
$$

Solving for a_1 we get:

$$
\begin{aligned}
\int_{-p}^{p} f(x)\cos\frac{\pi x}{p}\ dx &= a_1 p \\
\Rightarrow a_1 &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{\pi x}{p}\ dx
\end{aligned}
$$

We repeat the process for b_1 by multiplying both sides by $\sin {}^{\pi x}/_{p}$; for a_n we use $\cos {}^{n\pi x}/_{p}$ and for b_n, $\sin {}^{n\pi x}/_{p}$. The resulting formulas for the coefficients are given in Equation 67.

$$
\begin{aligned}
a_0 &= \frac{1}{p}\int_{-p}^{p} f(x)\ dx \\
a_n &= \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\ dx \\
b_n &= \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\ dx
\end{aligned}
\tag{67}
$$

Since this is an infinite series, we need to concern ourselves with convergence. The theorem below provides us assurance of convergence for continuous and piece-wise continuous functions on the interval $[-p, p]$.

Theorem 8

Convergence of Fourier Series If f and ${}^{df}/_{dx}$ are piece-wise continuous on an interval $[-p, p]$ then for all x in the interval $[-p, p]$ the Fourier

In coming lectures and when doing assignments you will see that issues of continuity of f and ${}^{df}/_{dx}$ have obvious visible influence on the convergence behavior of Fourier Series.

Series converges to f at points where the function is continuous; at points of discontinuity, the Fourier series converges to

$$\frac{f(x^-) + f(x^+)}{2}$$

where $f(x^-)$ and $f(x^+)$ denote the limit of $f(x)$ from the left and right at the point of discontinuity.

In the next few lectures we will define other orthogonal function expansions similar to the Fourier Series. Nonetheless, for periodic functions defined on a finite interval, the Fourier series provides the best representation of a function. There are some special cases, however, where we can take advantage of structural properties of $f(x)$ to reduce the amount of work we need to do in carrying out the Fourier series expansions.

When we say "best" representation, we (more or less) mean two things:

1. $||\tilde{f}_n - f||$, where $\tilde{f}_n$ is the power series representation of f up to n terms, gets smaller with fewer terms than other series expansions; and
2. calculation of the coefficients a_n and b_n can be carried out with greater numeric stability than for other expansions.

Even Functions and Odd Functions

When doing a Fourier series expansion it is sometimes helpful to consider whether a function is *even* or *odd*.

Definition 14 (Even Function)
A function is even if, for all real values x, $f(-x) = f(x)$.

An example of an even function is shown in Figure 6.

Definition 15 (Odd Function)
A function is even if, for all real values x, $f(-x) = -f(x)$.

An example of an odd function is shown in Figure 7.

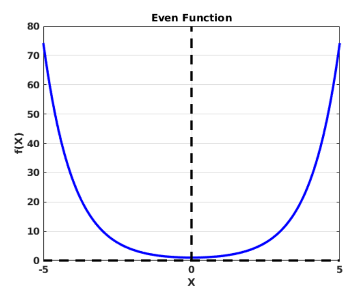


Figure 6: An example even function.

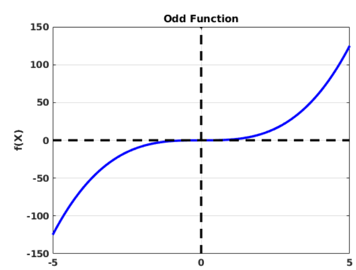


Figure 7: An example odd function.

SOME PROPERTIES of even an odd functions include:[19]

1. an even function times an even function results in an even function;
2. an odd function times an odd function results in an even function;
3. an even function times an odd function results in an odd function;
4. adding or subtracting two even functions results in an even function;
5. adding or subtracting two odd functions results in an odd function;
6. $\int_{-p}^{p} f_{\text{even}}(x)\,dx = 2\int_0^p f_{\text{even}}(x)\,dx$
7. $\int_{-p}^{p} f_{\text{odd}}(x)\,dx = 0$.

[19] Students are welcome to prove these assertions.

The "even-ness" or "odd-ness" of a function is relevant to Fourier series expansions. If you expand an *even* function in a Fourier series you will find that all of the b_n coefficients are zero; if you expand an *odd* function in a Fourier series you will find that a_0 and a_n terms are all zero.

You can still use the formulas presented in Equation 67 when expanding even or odd functions. Alternatively, you can use the formulas for the Cosine expansion or Sine expansion below for even or odd functions respectively.

Cosine series:

The cosine and sine series expansions are sometimes referred to as "half-wave" expansions since the calculations, as shown in the formulas, only involve the portion of the wave in the interval [0,p]—the positive half-wave.

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{p} \\ a_0 &= \frac{2}{p}\int_0^p f(x)\,dx \\ a_n &= \frac{2}{p}\int_0^p f(x)\cos\frac{n\pi x}{p}\,dx \end{aligned} \tag{68}$$

Sine series:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{p} \\ b_n &= \frac{2}{p}\int_0^p f(x)\,\sin\frac{n\pi x}{p}\,dx \end{aligned} \tag{69}$$

Lecture 17 - Generating and Plotting Fourier Series in MATLAB

Objectives

- Demonstrate how to carry out Fourier series expansions using MATLAB.
- Give a qualitative demonstration of convergence behavior of Fourier series.
- Demonstrate Cosine and Sine series expansions.

In this lecture, we will illustrate the process of Fourier series expansions with three examples.

Example #1: Carry out the Fourier series expansion of the function given in Equation 70, illustrated in Figure 8

$$f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ \pi - x, & 0 \le x \le \pi \end{cases} \tag{70}$$

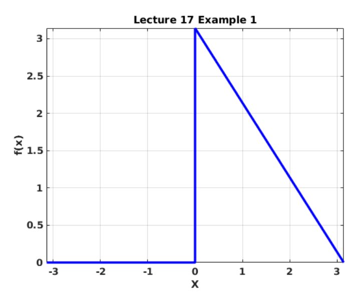


Figure 8: Example #1 $f(x)$.

We wish to represent this function as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{n\pi x}{p} + b_n \sin\frac{n\pi x}{p}\right]$$

where, in this case, $p = \pi$. Even though we are only interested in the function in the interval $[-\pi, \pi]$, since the Fourier series represents the function in terms of a constant and an infinite linear combination of of periodic functions, we should think of the function that we are representing as periodic.[20]

[20] This outlook will help us understand the convergence behavior of the Fourier series.

We have everything we need; it is just a matter of calculating the coefficients from Equation 67. Rather than carrying out the calculations with pencil and paper we will use MATLAB. In the listing below I will describe the code necessary to calculate Fourier coefficients through N=5.

```
clear
clc
close 'all'

N = 5; % specify number of coefficients

f = @(x) ex1(x);
p = pi; % specifiy period
```

We start, as always, by clearing out the workspace memory and command-prompt output and closing any open figures. We also need to represent $f(x)$ in MATLAB; we will do this with a local function named ex1(x).[21]

The integrals needed to determine the Fourier coefficients will be evaluated numerically using the MATLAB built-in function integral () .[22] Let us start with a_0 which, as a reminder is computed by:

$$a_0 = \frac{1}{p}\int_{-p}^{p} f(x)\ dx$$

[21] Since inline functions must appear *after* all of the other code in a MATLAB script file, I will provide the code for ex1(x) last.

[22] This function has default signature Q = integral(FUN,A,B) approximates the integral of function FUN over the interval A to B using global adaptive quadrature. The error tolerances for this numeric integration algorithm can be specified by the user; in most cases we will use default values. Students interested in how this works should take the Numerical Methods elective.

```
a0 = (1/p)*integral(f,-p,p);
FF = @(x) a0/2;
```

The first line of the listing numerically evaluates a_0; the second line creates an anonymous function and initializes it to the first term in the Fourier expansion.

We will use a loop to construct the remaining terms in the Fourier expansion.

Recall:

$$an = \frac{1}{p}\int_{-p}^{p} f(x)\cos\frac{n\pi x}{p}\ dx$$
$$bn = \frac{1}{p}\int_{-p}^{p} f(x)\sin\frac{n\pi x}{p}\ dx$$

```
for n = 1:N
an = (1/p)*integral(@(x) f(x).*cos(n*pi*x/p),-p,p);
bn = (1/p)*integral(@(x) f(x).*sin(n*pi*x/p),-p,p);
FF = @(x) FF(x) + an*cos(n*pi*x/p) + bn*sin(n*pi*x/p);
end
```

Note how you can practically read the equation directly from the MATLAB code.

Note in the last line where we append the newly computed terms to the Fourier series expansion FF(x). Now we have a function, FF(x) that represents the Fourier series expansion with $N = 5$ terms. In the next listing we add the code to plot the function and verify that it makes sense.

```
Nx = 1000;
X = linspace(-p,p,Nx);

plot(X,f(X),'-b',...
    X,FF(X),'--r',...
    'LineWidth',3)
title_str = sprintf('Example 1, n = %d',n);❶
title(title_str,'FontSize',16,...
    'FontWeight','bold');
xlabel('X','FontSize',14,... ❷
    'FontWeight','bold');
ylabel('f(X)','FontSize',14,...
    'FontWeight','bold');
```

Referring to the annotations:

❶ Using sprintf () allows us to combine the variable n in the title string.

❷ Optional keyword-argument pairs such as 'Linewidth',3, 'FontSize',16, and 'FontWeight','bold' help make the plot and labels more readable.

❸ Make a habit of using legends for graphs that include multiple data series. Once again, this makes the plot more readable.

❹ The keyword gca means “get current axis”. Calling the set () function with keyword-argument pairs 'FontSize',10 and 'FontSize','bold' sets the font size and weight for the axis markings.

In general it is important that your plots look good.

```
grid on
legend('f(x)','FF(x)')❸
set(gca,'FontSize',12,...
    'FontWeight','bold');❹
```

The resulting Fourier series expansion is shown in Figure 9. If I increase the number of Fourier series terms in my expansion, I need only change N; Figure 10 shows the series expansion with N=15 terms.

Some things to note about the resulting Fourier series representation of f(x):

1. As n increases, FF(x) generally "looks more like" f(x).

2. At the discontinuity in f(x), the Fourier series representation appears to be converging on the midpoint between $f(x^-)$ and $f(x^+)$ as the theory says it should; and

3. The Fourier series representation near the point of discontinuity has "wiggliness" that doesn't go away as n increases.

4. In particular, note the undershoot and overshoot of $f(x)$ to the left and right respectively of $f(0)$. This is called "Gibbs phenomena" and it does not go away as n increases but it moves closer to the point of discontinuity.

As Figure 11 shows, as N is increased, we can make FF(x) arbitrarily close to f(x) with the exception of the perturbations at the point of discontinuity.

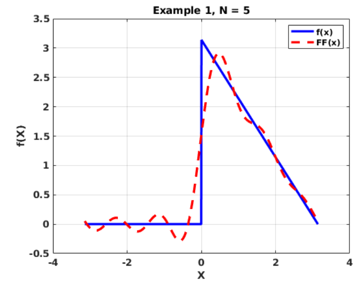


Figure 9: Fourier series expansion with N=5.

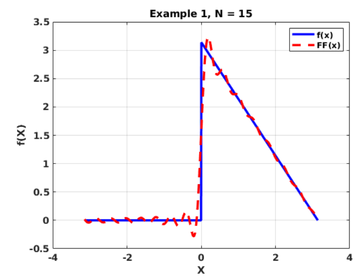


Figure 10: Fourier series expansion with N=15.

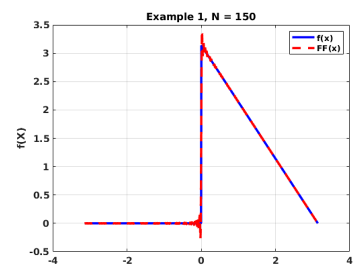


Figure 11: Fourier series expansion with N=150.

AN IMPORTANT MATTER that we have not yet dealt with is how to represent piece-wise continuous functions like $f(x)$ in MATLAB.[23] As stated previously, we will use a *local function* to do this. The code is shown in the listing below.

[23] For whatever reason, piece-wise continuous functions are intensively used in *textbooks* on partial differential equations—I am not so sure that they are as important in real-life applications.

```
%% Local functions
function y = ex1(x)
[m,n] = size(x); ❶
y = nan(m,n); ❷
for i = 1:length(x) ❸
    if (x(i) >= -pi) && (x(i) < 0)
        y(i) = 0;
    elseif (x(i) >= 0) && (x(i) <= pi) ❹
        y(i) = pi - x(i);
    end
end
end
```

Some notes on the annotations for this listing:

❶ We use the MATLAB built-in function size() to get the dimensions of the input vector. The return values [m,n] give the number of rows and columns of x respectively. For this function we are implicitly expecting x to be a vector, but it can be either a row-vector or a column vector.

❷ We construct the output vector y using the nan() function. This line creates the vector y to be the same size and shape as the input vector x.

❸ We use the built-in function length() to get the number of elements of x. This is a bit of a hack since, if x were ***not*** a vector, ex1(x) would no longer work properly.[24]

[24] It would be a good idea to verify that the input x is actually a vector. MATLAB, like most other languages, includes features to enforce assumptions like this. The code: assert(min(size(x))==1,'x must be a vector') would raise an error in MATLAB if the minimum dimension of x is anything other than 1. That would be one way to ensure x is a vector.

❹ The symbol && is "element-wise and". Pay attention to use of >= and <= operators to get the details of the intended function correct.

Example #2: Carry out the Fourier series expansion of the function given in Equation 71, illustrated in Figure 12.

$$f(x) = \begin{cases} -1, & -\pi \leq x < 0 \\ 1, & 0 \leq x \leq \pi \end{cases} \tag{71}$$

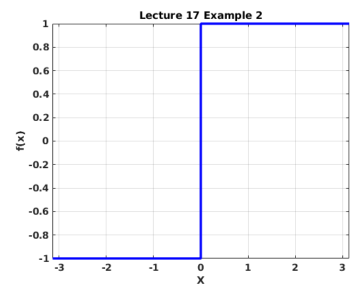


Figure 12: Example #2 $f(x)$.

Fourier series expansions of this function are shown in Figures 13 through 15.

Some notes:

1. As with the first example, the function has the Gibbs phenomena near the discontinuity at $x = 0$.

2. Also, as with the first example, the Gibbs phenomena does not go away as N increases, but it gets more "peaked" and closer to the origin.

3. Unlike the first example, we get the Gibbs phenomena and wiggliness at the ends also. This is because the Fourier series representation is periodic; the periodic extension of this function has discontinuities at the endpoints since $f(-\pi) \neq f(\pi)$.

4. You should also note that this function is *even*. That means we expect a_0 and all values of a_n to be equal to zero. If I modify the for-loop to output values for the a_n coefficients I get all zeros.

```
for n = 1:N
an = (1/p)*integral(@(x) f(x).*cos(n*pi*x/p),-p,p);
fprintf('a_%d = %g \n',n,an);
bn = (1/p)*integral(@(x) f(x).*sin(n*pi*x/p),-p,p);
FF = @(x) FF(x) + an*cos(n*pi*x/p) + bn*sin(n*pi*x/p);
end
```

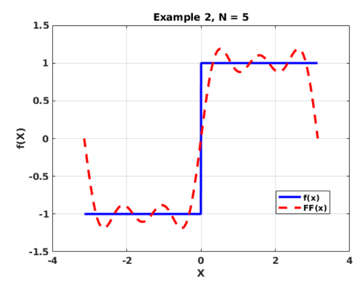


Figure 13: Fourier series expansion with N=5.

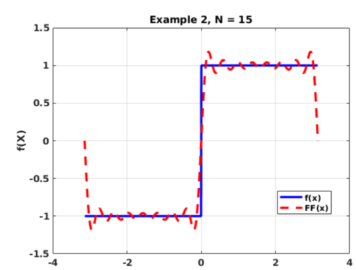


Figure 14: Fourier series expansion with N=15.

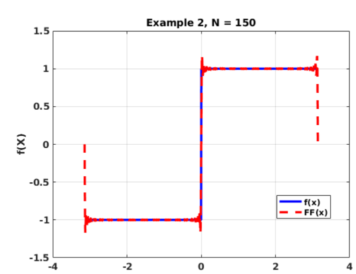
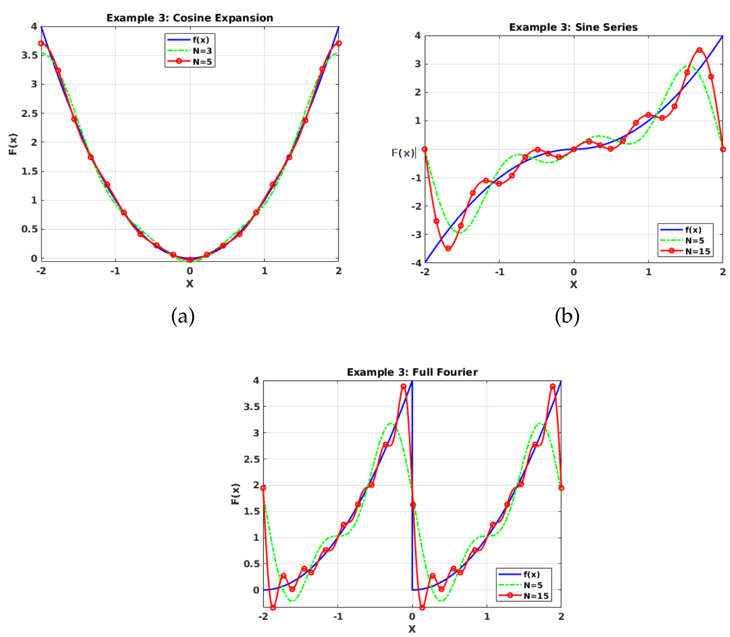


Figure 15: Fourier series expansion with N=150.

Example #3: Construct the Fourier series expansion of the function given in Equation 72.

$$f(x) = x^2, \quad x \in [0, 2] \tag{72}$$

This function is not periodic and, unlike the previous examples, does not even span a symmetric interval about the origin. In this case we will still use the same Fourier series formulas but we will construct a "reflection" about the origin. This reflection can be *even-*, *odd-*, or it can be an *identity-reflection* with respect to the y-axis; these correspond to the Cosine expansion, Sine expansion and the full Fourier series expansions. These different expansion options are shown in Figure 16.



(a) (b) (c)

Figure 16: Even-, odd- and identity-reflection for $f(x) = x^2$.

Note that the convergence behavior for the Fourier expansion is different for each case.

As you can see, in cases where you can choose which expansion you use, some choices are good and some are bad. As we will see in coming chapters, we often do not have a choice in which set of orthogonal functions we will use to do our expansion; so we cannot pick one that we think will be best. What we *can* do is analyze the expansion that we *do* get and understand the convergence behavior by examining the continuity of the functions and derivatives of functions that we are representing.

- For the even-reflection Cosine expansion convergence is very rapid. Both $f(x)$ and $f'(x)$ are continuous throughout the interval $x \in (-2, 2)$. The function itself is continuous at the end-points but notice that the derivative is not. If you were to draw an additional period on the left and right-hand side of the cosine expansion, $f'(x)$ would have a discontinuity; that explains the (relatively) poor convergence of the series at the end-points.
- For the odd-reflection Sine expansion the function and derivative is continuous throughout the domain. The derivative of the func-

tion is continuous at the periodic end-points but $f(x)$ itself is not. This explains why the Sine series expansion converges to zero at both end-points.

- The identity-reflection full Fourier series has discontinuities in $f(x)$ and $f'(x)$ at both endpoints and at $x = 0$. The convergence behavior is correspondingly bad.

Assignment #6

Show that the given functions are orthogonal on the given interval.

1. $f_1(x) = e^x,\ f_2(x) = xe^{-x} - e^{-x},\ \ x \in [0, 2]$

2. $f_1(x) = x,\ f_2(x) = \cos 2x,\ \ x \in [-\pi/2, \pi/2]$

Show that the given set of functions is orthogonal on the indicated interval. Find the norm of each function in the set.

3. $\{\sin x, \sin 3x, \sin 5x, \dots\},\ \ x \in [0, \pi/2]$

Use MATLAB to verify by numeric integration that the functions are orthogonal with respect to the indicated weight function on the given interval:

4. $H_0(x) = 1,\ H_1(x) = 2x,\ H_2(x) = 4x^2 - 2;\ \ w(x) = e^{-x^2},\ \ x \in (-\infty, \infty)$
   Note: use the built-in function `integral()` to do the numeric integration. In MATLAB, $-\infty$ and ∞ are represented by `-inf` and `inf` respectively.

Use MATLAB to construct the Fourier series expansion of the given function $f(x)$ on the given interval. For each problem create a plot that shows: a) $f(x)$ along with b) the truncated Fourier series of $f(x)$ with N =5 and c) N=15 terms. Also give the number to which the Fourier series expansion converges at any point(s) of discontinuity in $f(x)$.

5. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$

6. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \le x < \pi \end{cases}$

7. $f(x) = \begin{cases} 0, & -2 < x < 0 \\ -2, & -1 \le x < 0 \\ 1, & 0 \le x < 1 \\ 0, & 1 \le x < 2 \end{cases}$

Determine whether the given function is even, odd, or neither.

8. $f(x) = x^2 + x$

9. $f(x) = \begin{cases} x^2, & -1 < x < 0 \\ -x^2, & 0 \le x < 1 \end{cases}$

Use MATLAB to expand the given function in an appropriate cosine or sine series. For each function create a plot showing: a) $f(x)$ along with b) the truncated Fourier series of $f(x)$ with N=5 and; c) N=15 terms.

10. $f(x) = |x|, \quad -\pi < x\pi$

11. $f(x) = \begin{cases} x - 1, & -\pi < x < 0 \\ x + 1, & 0 \le x < \pi \end{cases}$

12. $f(x) = \begin{cases} 1, & -2 < x < -1 \\ -x, & -1 \le x < 0 \\ x, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \end{cases}$

13. $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 \le x < \pi \end{cases}$

Lecture 18 - Sturm-Liouville Problems

Objectives

- Define Regular/Singular Sturm-Liouville eigenvalue problems and give properties of their solutions.
- Do an example problem for finding eigenvalues and eigenfunctions.
- Do an example problem for transforming a linear, second-order, homogeneous boundary value problem into self-adjoint form.

Regular Sturm-Liouville Eigenvalue Problem

We have solved several differential equations in this class. All of the problems that we have solved so far are special cases of a more general problem That is what we wish to discuss in this lecture.

For the Regular Sturm-Liouville Eigenvalue problem we hope to solve:

$$\frac{d}{dx}\left[r(x)u'\right] + \left(q(x) + \lambda p(x)\right)u = 0, \quad x \in (a,b) \tag{73}$$

Note: for Equation 73, $r(x)$, $r'(x)$, $q(x)$, and $p(x)$ must be real-valued and continuous on the interval $x \in (a,b)$. Also $p(x) > 0$ and $r(x) > 0$ for all $x \in (a,b)$. These are important conditions that should be verified each time you encounter a new problem. The constant λ is referred to as an *eigenvalue*.

subject to the boundary conditions:

$$\begin{aligned} A_1 u(a) + B_1 u'(a) = 0, &\text{ where } A_1, \text{ and } B_1 \text{ are not both zero.} \\ A_2 u(b) + B_2 u'(b) = 0, &\text{ where } A_2, \text{ and } B_2 \text{ are not both zero.} \end{aligned}$$

Note that these boundary conditions are referred to as *homogeneous*. The same rule that we use to decide if a differential equation is homogeneous apply in the same way to the boundary conditions. For a boundary value problem to be homogeneous, *both* the differential equation *and* boundary conditions must be homogeneous.

For ODEs that we solved in earlier lectures, we routinely dealt with problems having non-homogeneous boundary conditions. As we go forward to solve linear partial differential equations using separation of variables, it will be *essential* that the boundary conditions are homogeneous. So you should be sure that you know how to check/verify that condition.

Properties of the Regular Sturm-Liouville problem:

1. There exist an infinite number of real eigenvalues that can be arranged in increasing order. (e.g. $\lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n < \cdots$)

2. For each eigenvalue, λ_n, there is exactly one eigenfunction, $u_n(x)$, that is a solution to the problem.
3. eigenfunctions corresponding to different eigenvalues are linearly independent.
4. The set of eigenfunctions is orthogonal with respect to $p(x)$ on the interval $[a, b]$. In other words: $\int_a^b u_n(x)u_m(x)p(x)\,dx = 0$ if $n \neq m$.
5. The set of eigenfunctions is complete on the interval $[a, b]$. In other words, for any (reasonable) $f(x)$, we can represent $f(x)$ as a linear combination of those eigenfunctions: $f(x) = \sum_{n=0}^{\infty} c_n u_n(x)$.[25]

[25] Another way of saying this is that no function $v(x)$ can be orthogonal to *all* of the eigenfunctions, $u_n(x)$ on the interval $[a, b]$.
To obtain values for the coefficients c_n, we need only take the inner product with the corresponding eigenfunction, u_n. i.e. multiply both sides by an orthogonal function and integrate.

IF $r(x)$ IN Equation 73 at either boundary the problem is said to be a singular boundary value problem. If $r(a) = r(b)$, with suitable boundary conditions, the problem is said to be a periodic boundary value problem.

Example: find the eigenvalues and eigenfunctions of the following boundary value problem:

Note that this problem is not presented in self-adjoint form. Have faith that it is, indeed, a Sturm-Liouville eigenvalue problem and could be presented in self-adjoint form. We will practice making this transformation later in the lecture.

$$\begin{aligned} \text{Equation:} &\qquad u'' + \lambda u = 0, \quad x \in [0, 1] \\ \text{BCs:} &\qquad u(0) = 0, \quad u(1) + u'(1) = 0 \end{aligned}$$

To fully analyze this problem we will have to consider three cases for λ: $\lambda <= 0$, $\lambda = 0$, and $\lambda > 0$.

$\underline{\lambda = 0}$: In this case, the differential equation reduces to:

$$u'' = 0$$

with general solution: $u(x) = c_1(x) + c_2$. If we apply the boundary condition $u(0) = 0$, this implies that $u(0) = c_1(x) + c_2 = c_2 = 0$. So the solution is simplified to $u(x) = c_1(x)$. The second boundary condition: $u(1) + u'(1) = c_1(1) + c_1 = 2c_1 = 0 \Rightarrow c_1 = 0$. The only solution that satisfies the equation and boundary conditions for $\lambda = 0$ is the trivial solution $u(x) = 0$.[26]

[26] The trivial solution, $u(x) = 0$ will always satisfy a homogeneous boundary value problem and, in general, is of little interest to us. What we take from this part of the analysis is that we will rule out $\lambda = 0$ as there are no *interesting* solutions for that case.

$\underline{\lambda < 0}$: For this case we will assume $\lambda = -\alpha^2$ where $\alpha > 0$. The differential equation reduces to:

$$u'' - \alpha^2 u = 0$$

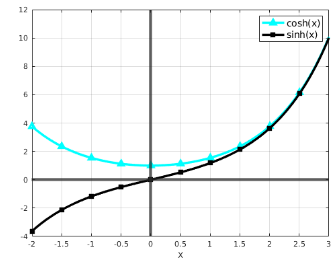
This equation has the general solution of:

$$\begin{aligned} &u(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x} \\ &\text{or:} \\ &u(x) = c_1 \cosh \alpha x + c_2 \sinh \alpha x \end{aligned}$$

Recall that these two solutions are equivalent. We will generally use the first form on *unbounded* intervals; the second form on *bounded* intervals.

Since this problem is posed on a bounded interval, we will choose the second form above. Applying the first boundary condition gives us: $u(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$. Applying the second boundary condition to the current solution gives us: $u(1) + u'(1) = c_2 \sinh 1 + c_2 \cosh 1 = 0$.

We recall that both $\sinh x$ and $\cosh x$ are strictly positive on $x \in (0, 1)$ so the only way the second boundary condition can be met is for $c_2 = 0$. Consequently only the trivial solution $u(x) = 0$ satisfies the governing equation and boundary conditions for the case that $\lambda < 0$.



$\underline{\lambda > 0}$: For this case we will assume $\lambda = \alpha^2$ where $\alpha > 0$. The differential equation reduces to:

$$u'' + \alpha^2 u = 0$$

This equation has the general solution of $u(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$. Applying the first boundary condition gives us: $u(0) = c_1 \cos 0 + c_2 \sin 0 = c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$. Applying the second boundary condition t the current solution gives us: $u(1) + u'(1) = c_2 \sin \alpha + \alpha c_2 \cos \alpha = 0$, or:

$$u(x) = c_2 [\sin \alpha + \alpha \cos \alpha] = 0 \tag{74}$$

This equation can be satisfied simply by setting $c_2 = 0$, but we will resist that temptation since that would then imply that there are *no* values of λ that admit a non-trivial solution for this problem. Instead we will look for values of α such that:

$$\sin \alpha + \alpha \cos \alpha = 0 \tag{75}$$

We can see from Figure 17 that there are values of α that satisfy this condition; we will denote these eigenvalues $\alpha_1^2 = \lambda_1$, $\alpha_2^2 = \lambda_2$, $\dots$, $\alpha_n^2 = \lambda_n$ and the corresponding eigenfunctions are denoted: $u_n(x) = \sin \alpha_n x$.

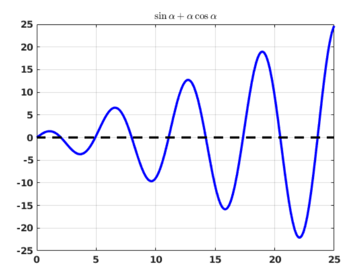


Figure 17: Plot of $\sin \alpha + \alpha \cos \alpha$.

We will defer, for the moment, the problem of finding the roots to Equation 75. Suffice it to say that there are infinitely many distinct roots yielding the infinitely many eigenvalues to go with the infinitely many eigenfunctions. They can be found with a non-linear equation solver ("root-finder") for which there are several reliable algorithms.

Transforming Equations to Self-Adjoint Form

APART FROM SHARING some theoretical tid-bits regarding Sturm-Liouville eigenvalue problems, the *point* of this lecture is to highlight: a) the eigenfunctions that solve the eigenvalue problem; and b) their property of weighted orthogonality. Recalling the last two lectures where we used an infinite set of trigonometric functions for functional expansion in a Fourier series, we will want to use *other* functions for such expansions. Those other functions will be the set of eigenfunctions associated with a Sturm-Liouville eigenvalue problem.

As previously mentioned, the eigenfunction solutions are linearly independent and orthogonal with respect to weight function $p(x)$. We need to know what that weight function is in order to carry out an orthogonal function expansion like Fourier series.

The sines and cosines used in Fourier series fit within this theory. It turns out that the weight function $p(x)$ in that case is $p(x) = 1$.

Consider the linear, homogeneous, second-order boundary value problem shown in Equation 76.

$$a(x)u'' + b(x)u' + (c(x) + \lambda d(x))u = 0 \tag{76}$$

where $a(x) \neq 0$ and $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are continuous. We will convert to the self-adjoint form: $\frac{d}{dx}\left[r(x)u'\right] + \left[q(x) + \lambda p(x)\right]u = 0$ by determining the functions $r(x)$, $q(x)$, and $p(x)$ as follows:

1. $r(x) = e^{\int b(x)/a(x)\ dx}$
2. $q(x) = \frac{c(x)}{a(x)}r(x)$
3. $p(x) = \frac{d(x)}{a(x)}r(x)$

Example: express the following equation, which has solutions $P_n(x)$ in self-adjoint form and give the orthogonality relation.

This is Legendre's equation that we solved in a previous lecture. $P_n(x)$ is standard notation for Legendre polynomials of order n.

$$\underbrace{\left(1 - x^2\right)}_{a(x)} u'' \underbrace{-2x}_{b(x)} u' + \overbrace{n(n+1)}^{\lambda} u = 0, \quad x \in (-1, 1)$$

From the equation, $a(x)$ and $b(x)$ are annotated; $c(x) = 0$ and $d(x) = 1$. We first compute $r(x)$:

$$\begin{aligned} r(x) &= e^{\int \frac{-2x}{(1-x^2)}\ dx} \\ &= e^{\int \frac{1}{u}\ du} \\ &= e^{\ln u} \\ &= u \\ &= 1 - x^2 \end{aligned}$$

Here we use a u-substitution:

$$\begin{aligned} u &= \left(1 - x^2\right) \\ du &= -2x\ dx \end{aligned}$$

so $e^{\int \frac{-2x}{(1-x^2)}\ dx} = e^{\int \frac{1}{u}\ du}$ following this substitution.

Now we compute $q(x)$:

$$
\begin{aligned}
q(x) &= \frac{c(x)}{a(x)} r(x) \\
&= \frac{0}{(1-x^2)} \left(1 - x^2\right) \\
&= 0
\end{aligned}
$$

Then $p(x)$:

$$
\begin{aligned}
p(x) &= \frac{d(x)}{a(x)} r(x) \\
&= \frac{1}{(1-x^2)} \left(1 - x^2\right) \\
&= 1
\end{aligned}
$$

So the boundary value problem in self-adjoint form is:

$$
\frac{d}{dx}\left[\left(1 - x^2\right) u'\right] + \lambda_n u = 0 \tag{77}
$$

where $\lambda_n = n(n+1)$. As given in the problem statement, the eigenfunctions are $P_n(x)$ and the weight function $p(x) = 1$. The orthogonality relation is:

$$
(P_m, P_n) = \int_{-1}^{1} P_m(x) P_n(x)(1)\, dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}
$$

Note: you will not be expected to know, by inspection, the value of (P_n, P_n) but it is provided here for your information.

Lecture 19 - Fourier-Bessel Series Expansions

Objectives

- Present the Parametric Bessel Equation as a Sturm-Liouville problem and derive the orthogonality relation.
- Do an example to show expansion of a function in terms of Bessel functions.
- Demonstrate use of the MATLAB function besselzero().

Parametric Bessel Equation

The parametric Bessel equation is a second-order linear, homogeneous differential equation that also fits within Sturm-Liouville theory. As a reminder, the equation is:

$$x^2u'' + xu' + \left(\alpha^2x^2 - \nu^2\right)u = 0$$

and the general solution is given by:

$$u(x) = c_1J_\nu(\alpha x) + c_2Y_\nu(\alpha x)$$

The solutions, $J_\nu(\alpha x)$ and $Y_\nu(\alpha x)$ are, of course, linearly independent but they also are orthogonal with respect to some weight function $p(x)$ and we can use them to construct an orthogonal function expansion in exactly the same way we did with Fourier series. That is what we will do in this lecture. To accomplish this we want to put the parametric Bessel equation in self-adjoint form and we will proceed in this effort just as we did in the last lecture.

Let us first put the parametric Bessel equation in standard form:

It may not be clear immediately that λ corresponds to values of α but that is the correct inference; when we do the orthogonal function expansion with Bessel functions it will be more clear why that is the case.

$$\begin{aligned} a(x)u'' + b(x)u' + [c(x) + \lambda d(x)]\,u &= 0 \\ x^2u'' + xu' + \left[-\nu^2 + \alpha^2x^2\right]u &= 0 \end{aligned}$$

so, $a(x) = x^2$, $b(x) = x$, $c(x) = -\nu^2$, and $d(x) = x^2$.

Next we will compute $r(x)$:

$$\begin{aligned} r(x) &= e^{\int \frac{b(x)}{a(x)}\,dx} \\ &= e^{\int \frac{x}{x^2}\,dx} \\ &= e^{\int \frac{1}{x}\,dx} \\ &= e^{\ln x} \\ &= x \end{aligned}$$

Now we compute $q(x)$:

$$\begin{aligned} q(x) &= \frac{c(x)}{a(x)} r(x) \\ &= \frac{-\nu^2}{x^2} x \\ &= -\frac{\nu^2}{x} \end{aligned}$$

Then $p(x)$:

$$\begin{aligned} p(x) &= \frac{d(x)}{a(x)} r(x) \\ &= \frac{x^2}{x^2} x \\ &= x \end{aligned}$$

So the self-adjoint form of the parametric Bessel equation is:

$$\frac{d}{dx}\left[xu'\right] + \left(-\frac{\nu^2}{x} + \alpha^2 x\right) u = 0$$

Admittedly, the real reason why we want to do this is to obtain the weight function $p(x)$ which, in this case is $p(x) = x$.

The corresponding orthogonality relation is shown in Equation 78

$$\int_a^b J_\nu(\alpha_n x) J_\nu(\alpha_m x) x \, dx = 0, \quad n \neq m \tag{78}$$

Like other Sturm-Liouville problems we will find that there are infinitely many distinct eigenvalues, λ_n, which for this equation we will refer to as α_n. Note the weight function x now appears in the inner product.

where a and b are the bounds of the interval on which orthogonality is expressed.

Remember that it is the *boundary conditions* that allow us to determine the eigenvalues.

Example: expand $f(x) = x$, $0 < x < 3$, in a Fourier-Bessel series, using Bessel functions of order $\nu = 1$ that satisfy the boundary condition $J_1(3\alpha) = 0$.

So what we want is:

$$f(x) = x = \sum_{n=1}^{\infty} c_n J_1(\alpha_n x)$$

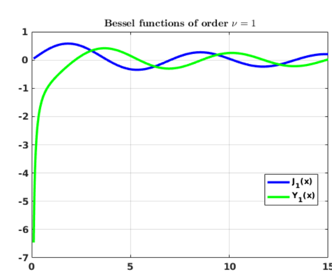


Figure 18: Bessel functions of order 1.

Note that we omit Bessel functions of the second kind, $Y_n(x)$, because as is shown in the figure, they diverge to negative infinity as x goes to zero. This *implicit boundary condition* where one solution of the differential equation diverges at the problem boundary.

The other boundary condition applies at $x = 3$: $J_1(3\alpha_n) = 0$ or, put differently, we select the values of α_n such that $3\alpha_n$ is a root of $J_1(x)$. While our plot of $J_1(x)$ does not extend out to infinity, it turns out that $J_1(x)$ has infinitely many roots and we need to find them.

Interlude on Open-Source Software

At some point in time in your life as an engineer it is inevitable that a problem will arise that you are not prepared to tackle yourself. The tools you have been given to do your job do not fully answer to the task at hand. This is one such occasion. We need to find the roots of $J_1(x)$. *You* know those values exist but you don't know what they are and it turns out that MATLAB does not (at this time) have any built-in functions to give you the roots of such functions.[27]

Some options available to you include:

1. Go to the library and check out a book that tabulates some roots of $J_1(x)$—possibly also including scads of additional Bessel function lore[28]—and enter the desired roots by hand into your MATLAB code.

2. Implement an algorithm to find the roots of $J_1(x)$, possibly using a root-finding tool in MATLAB such as `fzero()` or `fsolve()`.

3. Find a third-party function or library that has already been written that solves the problem.

In this case we will take the last option since, it turns out, someone has already solved this problem for us and it is a safe bet that they did a better job than what we would be prepared to do. The MATLAB file exchange is an online repository where people can obtain and share code that they find useful.[29]

REGARDING OPEN-SOURCE software: We have a lot of experience with proprietary software; from operating systems like Microsoft Windows or Apple's IOs to office productivity tools like Microsoft Word or Excel, to valuable and important engineering tools like MATLAB or COMSOL. We also have experience with free software, such as many applications that you download onto your smartphones. I want to write a few words in hopes of dispelling any negative connotations that you may have developed in relation to open-source software in comparison to proprietary software.

- Scientists and engineers of all types—not just computer scientists—write and share software in an open-source framework. Online

[27] MATLAB *does* have built-in functions to represent several types of Bessel functions; $J_\nu(x)$ and $Y_\nu(x)$ are represented, respectively, by `besselj(nu,x)` and `bessely(nu,x)`. We will learn about more Bessel functions in future lectures.

[28] Frank Bowman. *Introduction to Bessel functions*. Courier Corporation, 2012

The Fourier-Bessel expansion that we are learning about in this lecture is a standard element in the analytical methods repertoire; *of course* someone else has already figured out how to find the roots of Bessel functions.

[29] Note that a (free) MathWorks account is required to use the MATLAB file exchange.

repositories like GitHub and GitLab are meant expressly for developing code in an open and collaborative way and then sharing the results freely.

- "open-source" ensures the source code is available. Sometimes the code is also free but that is not the essential part.[30]
- Open-source software is a *hugely* important contribution to science. Free and open-source tools like
  - the programming languages that have been a part of the scientific computing landscape for generations. Examples include Python, C++, Java and FORTRAN among others.
  - OpenMC[31] - a powerful particle transport simulation tool similar to MCNP.
  - MOOSE - Multi-physics Object-Oriented Simulation Environment[32] combines the open-source finite element library libMesh[33] and the Portable, Extensible Toolkit for Scientific Computation (PETSc)[34] along with a host of other free, open-source libraries to create an enormously powerful and flexible tool-set that is used to create the majority of all new multi-physics nuclear analysis codes in the United States.[35]
- As the previous item should help illustrate, open-source software can be of very high quality. The developers of MOOSE-based applications at the Department of Energy labs are highly trained scientists following nuclear quality assurance standards to ensure that the resulting software tools work correctly and do what they are supposed to do.

The LaTeX tools and almost all of the other software on the computer used to prepare this manuscript, including the Linux operating system, are free and open-source software.[36] If you have any interest in scientific computing, now is a good time to develop an interest in open-source software.

[30] Think: "free speech", not "free beer."

[31] Openmc: A state-of-the-art monte carlo code for research and development. *Annals of Nuclear Energy*, 82: 90–97, 2015

[32] Alexander D. Lindsay et al. 2.0 - MOOSE: Enabling massively parallel multiphysics simulation. *SoftwareX*, 20: 101202, 2022. ISSN 2352-7110

[33] Benjamin S Kirk, John W Peterson, Roy H Stogner, and Graham F Carey. libmesh: a c++ library for parallel adaptive mesh refinement/coarsening simulations. *Engineering with Computers*, 22:237–254, 2006

[34] Balay et al. PETSc/TAO users manual. Technical Report ANL-21/39 - Revision 3.19, Argonne National Laboratory, 2023

[35] For a list of current applications tracked by the MOOSE development team see: `https://mooseframework.inl.gov/application_usage/tracked_apps.html`. Not all of these codes are open-source, but they have all been created with open-source tools.

[36] MATLAB is a notable exception to this list. There is a free and open-source alternative called Octave. `https://octave.org/`

Back to the Example

We want to expand $f(x) = x$ for $0 < x < 3$ in a Fourier-Bessel series expansion using Bessel functions of order 1 that satisfy the boundary condition $J_1(3\alpha_n) = 0$. We will use MATLAB along with the function besselzero() that we obtained from the MATLAB file exchange to carry out this task. In particular we will compute the truncated expansion with $N = 15$ terms:

$$f(x) = x = \sum_{n=1}^{15} c_n J_1(\alpha_n x)$$

1. Use besselzero() to get $\alpha_1, \alpha_2, \ldots, \alpha_N$ for our expansion.

```
clear
clc
close 'all'

N = 15; % number of eigenvalues
a = 0; b = 3; % bounds of the domain
nu = 1; kind = 1;
k = besselzero(nu,N,kind); % get roots    ❶
alpha = k/b;  ❷
```

❶ besselzero() takes up to three arguments; the first, ν, is the order of the Bessel function; the second is the number of roots requested; the third is to indicate the *kind*—first or second—of Bessel function for which you want the roots.

❷ since $J_1(\alpha_n 3) = k_n$ where k_n is the n[th] root of J_1, α_n must be equal to $k_n/3$.

Now we have the first N=15 values of α_n.

2. Compute the coefficients of the expansion c_n. As with the Fourier series, we do this by multiplying both sides of our equation by an orthogonal function *and the weight function* $p(x) = x$ and integrating. For example, to get c_1, we do the following:

$$f(x) = x = c_1 J_1(\alpha_1 x) + c_2 J_1(\alpha_2 x) + \cdots$$

$$\int_0^3 x J_1(\alpha_1 x)\, x\, dx = c_1 \int_0^3 J_1(\alpha_1 x)^2\, x\, dx + c_2 \underbrace{\int_0^3 J_1(\alpha_2 x)\, J_1(\alpha_1 x)\, x\, dx}_{=0 \text{ by orthogonality}} + \cdots$$

$$\Rightarrow c_1 = \frac{\int_0^3 x J_1(\alpha_1 x)\, x\, dx}{\int_0^3 J_1(\alpha_1 x)^2\, x\, dx}$$

where we recall that the weight function for the orthogonality relation for the Bessel equation is $p(x) = x$. For the calculation of c_1 all of the remaining terms are zero due to the weighted orthogonality of the eigenfunctions $J_1(\alpha_n x)$. We repeat the process for all values of c_n and, in MATLAB, we implement this process in the form of a loop.

```
f = @(x) x;
cn = nan(N,1); % store the coefficients (optional)

FB = @(x) 0; % initialize the Fourier-Bessel expansion
for n = 1:N
    % compute the i-th coefficient
    cn(n) =...
      integral(@(x) f(x).*besselj(nu,alpha(n)*x).*x,a,b)./...  ❸
      integral(@(x) x.*besselj(nu,alpha(n)*x).^2,a,b);
    % update the Fourier-Bessel expansion
    FB = @(x) FB(x) + cn(n)*besselj(nu,alpha(n)*x);
end
end
```

❸ these three lines are actually one long line of MATLAB that calculates the coefficients:

$$c_n = \frac{\int_0^3 x J_1(\alpha_1 x)\, x\, dx}{\int_0^3 J_1(\alpha_1 x)^2\, x\, dx}$$

We are now ready to plot the resulting Fourier expansion.

```
Nx = 1000;
X = linspace(a,b,Nx); ❹

figure(1)
plot(X,FB(X),'-b','LineWidth',3);
```

❹ Create a vector to represent the x-axis.

```
xlabel('X','fontsize',14,'fontweight','bold');
ylabel('f(X)','fontsize',14,'fontweight','bold');
titlestr = sprintf('Fourier-Bessel expansion, N = %d',N);
title(titlestr,'fontsize',16,'fontweight','bold');
grid on
set(gca,'fontsize',12,'fontweight','bold');
```

The Fourier-Bessel expansion of $f(x) = x$ with $N = 15$ is shown in the figure. Note that the expansion for $N = 15$ looks pretty rough; all of the wiggles through the domain and dropping suddenly to zero as the function approaches $x = 3$. The reason for this is that *it had to.* We are building the expansion with orthogonal functions that are all equal to zero at $x = 3$. Of course $f(x) = x$ is equal to 3 at $x = 3$ so something had to give.

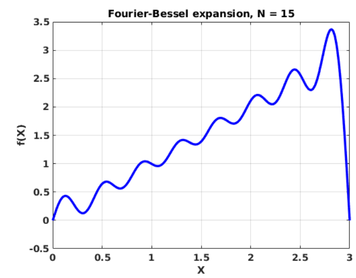


Figure 19: Fourier-Bessel expansion of $f(x) = x$.

We can improve the quality of the expansion by taking more terms. Luckily, since we are using a computer, it is no problem at all to simply increase N; the computer does the same thing, just more of it. The result is shown in Figure 20 where the wiggliness remains—including the Gibbs phenomena we saw with Fourier series—but overall the representation is much more exact.

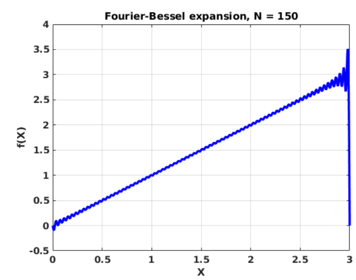


Figure 20: Fourier-Bessel expansion of $f(x) = x$.

Measuring Expansion Accuracy

There is a straight-forward way to be more precise when we speak of the accuracy of an orthogonal function expansion. A frequently used relative error measure is shown in Equation 79.

$$\text{Relative error} = \frac{(f(x) - FB(x), f(x) - FB(x))}{(f(x), f(x))} = \cdots$$
$$\frac{\int_a^b \left(f(x) - FB(x)\right)^2 \, dx}{\int_a^b f(x)^2 \, dx} \tag{79}$$

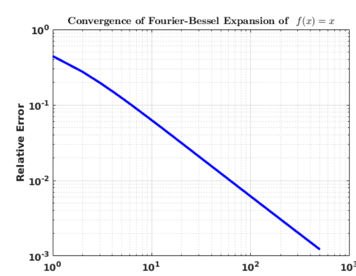


Figure 21: Convergence of the Fourier-Bessel expansion of $f(x) = x$.

MATLAB code for quantitatively measuring the relative error as the number of terms increases is shown below. From Figure 21 we can see that, as expected, the relative error steadily goes down.[37]

```
clear
clc
close 'all'

N = 500; % number of eigenvalues
a = 0; b = 3; % bounds of the domain
nu = 1; kind = 1;
k = besselzero(nu,N,kind); % get roots
alpha = k/b;

f = @(x) x;
cn = nan(N,1); % store the coefficients (optional)
rel_err = nan(N,1);

```

[37] Note that it is conventional to show convergence graphs such as this on a log-log plot. Eventually, we should expect errors in the determination of Bessel function roots and/or errors in carrying out the numeric integration to prevent further reduction in relative error.

```
FB = @(x) 0; % initialize the Fourier-Bessel expansion
for n = 1:N
    % compute the i-th coefficient
    cn(n) =...
        integral(@(x) f(x).*besselj(nu,alpha(n)*x).*x,a,b)./...
        integral(@(x) x.*besselj(nu,alpha(n)*x).^2,a,b);
    % update the Fourier-Bessel expansion
    FB = @(x) FB(x) + cn(n)*besselj(nu,alpha(n)*x);

    % calculate sqruare norm of the relative "error"
    err_fn = @(x) FB(x) - f(x);
    rel_err(n) = integral(@(x) err_fn(x).^2,a,b)./...
        integral(@(x) f(x).^2,a,b);
end

figure(1)
loglog(1:N,rel_err,'-b',...
    'LineWidth',3);
title('\textbf{Convergence of Fourier-Bessel Expansion of } $$f(
    x)=x$$',...
    'Interpreter','latex');
ylabel('Relative Error','FontSize',14,...
    'FontWeight','bold');
xlabel('Number of Terms','FontSize',14,...
    'FontWeight','bold');
grid on
set(gca,'FontSize',12,'FontWeight','bold');
```

Lecture 20 - Fourier-Legendre Series Expansion

Objectives

- Recap the Legendre equation as a Sturm-Liouville problem and give its orthogonality relation.
- Give an example to show the expansion of a function in terms of Legendre polynomials.

Orthogonality with Legendre Polynomials

We have some experience with Legendre's equation and their solutions Legendre Polynomials. As a recap, however, Legendre's equation is shown in Equation 80.

$$\left(1-x^2\right)u'' - 2xu' + n(n+1)u = 0, \quad x \in (-1,1) \tag{80}$$

The general solution is $u(x) = c_n P_n(x)$ where $P_n(x)$ is the Legendre polynomial of order n.

As a reminder the first few Legendre polynomials are: $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = \frac{1}{2}\left(3x^2-1\right)$, and $P_3(x) = \frac{1}{2}\left(5x^3-3x\right)$.

As demonstrated in Lecture 18, the self-adjoint form for Legendre's equation is given in Equation 81.

$$\frac{d}{dx}\left[\left(1-x^2\right)u'\right] + \overbrace{n(n+1)}^{\lambda}u = 0 \tag{81}$$

The orthogonality relation is shown below:

$$\int_{-1}^{1} P_m(x)P_n(x)(1)\ dx = \begin{cases} 0, & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

Recall that for Legendre's equation, the weight function $p(x)$ is equal to 1. Also recall that $(P_n, P_n) = 2/2n+1$.

If we need to represent a function $f(x)$ in terms of Legendre polynomials, we can carry out a *Fourier-Legendre* expansion as shown below:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x) \tag{82}$$

As usual we can derive the formulas for the coefficients c_n of Equation 83 by multiplying both sides of Equation 82 by $P_n(x)$ and integrating.

where:

$$c_n = \frac{(f(x), P_n(x))}{(P_n(x), P_n(x))} = \frac{\int_{-1}^{1} f(x)P_n(x)\ dx}{2/2n+1} \tag{83}$$

The convergence of Fourier-Legendre series expansions behave similarly to the Fourier series expansion using trigonometric polynomials. This behavior is recapitulated in the next theorem.

Theorem 9 (Convergence of Fourier-Legendre Series)
Let $f(x)$ and $f'(x)$ be piece-wise continuous on the interval $[-1,1]$. Then for all x in the interval, the Fourier-Legendre series of f converges to $f(x)$ at a point where $f(x)$ is continuous and to the average:

$$\frac{f(x^+) + f(x^-)}{2}$$

at points where $f(x)$ is discontinuous.

This also happens to be true for Fourier-Bessel expansions.

Example: construct the Fourier-Legendre expansion of:

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ 1, & 0 \leq x < 1 \end{cases}$$

Since the tools we need to use have largely be introduced already, I will simply present the necessary MATLAB code in a single listing.

```
clear
clc
close 'all'

f = @(x) ex1(x);

N = 15; % number of terms
a = -1; b = 1; % boundaries

% handle P0 coefficient separately
c0 = (1/2)*integral(@(x) f(x),a,b);   ❶
cn = nan(N-1,1);
error_norm = nan(N,1);

FL = @(x) c0;

% calculate relative error
err_fn = @(x) FL(x) - f(x);
error_norm(1) = integral(@(x) err_fn(x).^2,a,b)./...
        integral(@(x) f(x).^2,a,b);

for n = 1:(N-1)
    % compute the n'th coefficient ❷
    cn(n) = ((2*n+1)/2)*integral(@(x) f(x).*legendreP(n,x),a,b);
    FL = @(x) FL(x) + cn(n)*legendreP(n,x); %update the
     expansion

    % compute the error.
    err_fn = @(x) FL(x) - f(x);
    error_norm(n+1) = integral(@(x) err_fn(x).^2,a,b)./...
        integral(@(x) f(x).^2,a,b); % normalize error by size of
      function.
end
```

❶ Recall that $P_0(x) = 1$ and, according to our formula,

$$\begin{aligned}(P_0(x), P_0(x)) &= \frac{2}{2n+1} \\ &= \frac{2}{2(0)+1} \\ &= 2.\end{aligned}$$

Hence:

$$\begin{aligned}c_0 &= \frac{(f(x), P_0)}{(P_0, P_0)} \\ &= \frac{\int_{-1}^{1} f(x)(1)\, dx}{2}\end{aligned}$$

❷ In addition to using the formula for $(P_n(x), P_n(x))$ we use the built-in MATLAB function for constructing $P_n(x)$: legendreP(n,x).

```
%% Plot the result
Nx = 1000;
X = linspace(a,b,Nx);

figure(1)
plot(X,FL(X),'-g',...
    X,f(X),'--b',...
    'LineWidth',3);
grid on
xlabel('X','fontsize',14,'fontweight','bold');
ylabel('f(X)','fontsize',14,'fontweight','bold');
titlestr = ...
    sprintf('Fourier-Legendre expansion, N = %d',N);
title(titlestr,'fontsize',16,'fontweight','bold');
set(gca,'fontsize',12,'fontweight','bold');

%% Plot the error
figure(2)
loglog(1:N,error_norm,'-ok','linewidth',3);
title('Convergence behavior','fontsize',16,'fontweight','bold');
grid on
xlabel('Number of Fourier-Legendre Terms','fontsize',14,'
    fontweight','bold');
ylabel('Relative Error','fontsize',14,'fontweight','bold');
set(gca,'fontsize',12,'fontweight','bold');


%% Local functions
function y = ex1(x)
[m,n] = size(x);
% expect vector inputs.
assert(min(m,n) == 1,'Bad input for ex1');  ❸
% construct y so that it has the same shape as x
y = nan(m,n);
for i = 1:length(x)
    if (x(i) > -1) && (x(i) < 0)
        y(i) = 0;
    elseif (x(i) >= 0) && (x(i) < 1)
        y(i) = 1;
    end
end
end
```

❸ Here we used an assert() function to enforce the requirement that inputs to ex1(x) be scalars or vectors but not a matrix. Any time you write a piece of code that relies on some kind of assumption—the input x must be a vector, for example—you really should add something like this assert() function to ensure that your assumption really is true. For larger software projects this sort of small-scale testing is essential for code reliability and maintainability.

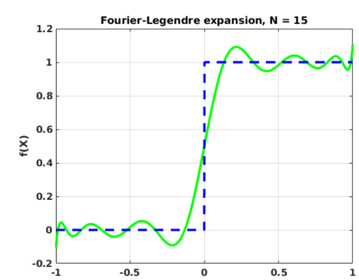


Figure 22: Fourier-Legendre expansion with $N = 15$.

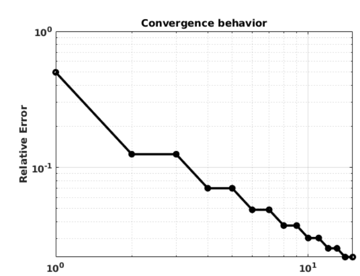


Figure 23: Convergence of Fourier-Legendre expansion

A PLOT OF the Fourier-Legendre expansion is shown in Figure 22 and the convergence behavior is shown in Figure 23. Several things should be noted.

1. Clearly we can see from Figure 22 that the Fourier-Legendre expansion is converging to the average value at the point of discontinuity at $x = 0$.

2. Like other Fourier expansions, perturbations ("wiggliness") is introduced by that discontinuity and this is something that we should learn to expect.

3. Also note that from Figure 23 we see that the expansion improves when we add the c_0 term, the c_1 term, c_3 term, and all

odd-numbered terms but the relative error does not change for the even-numbered terms $c_2, c_4, \ldots, c_{14}$. Looking at $f(x)$ is should be apparent that, in some sense anyway, the function is *odd*—or at least "odd-ish"; you could make it odd by subtracting out a constant term (i.e. $f(x) - 0.5$ is odd and the c_0 coefficient is equal to that 0.5). The even-order Legendre polynomials are even and orthogonal to $f(x)$.

Assignment #7

Find the eigenfunctions and the equation that defines the eigenvalues for the boundary-value problem. Use MATLAB to estimate the first 4 eigenvalues $\lambda_1, \lambda_2, \lambda_3$, and λ_4. Give the eigenfunctions corresponding to these eigenvalues and find the square norm of each eigenfunction.

1. $u'' + \lambda u = 0, \;\; u'(0) = 0, \;\; u(1) + u'(1) = 0$

2. Consider $u'' + \lambda u = 0$ subject to $u'(0) = 0, \; u'(L) = 0$. Show that the eigenfunctions are:

$$\left\{ 1, \cos\frac{\pi x}{L}, \cos\frac{2\pi x}{L}, \ldots \right\}$$

   This set, which is orthogonal on $x \in [0, L]$, is the basis for the Fourier cosine series.

3. Consider the following boundary value problem:

$$\begin{aligned} &x^2 u'' + x u' + \lambda u = 0, \quad x \in (1, 5) \\ &u(1) = 0, \quad u(5) = 0 \end{aligned}$$

   (a) Find the (non-trivial) eigenvalues and eigenfunctions of the boundary value problem. Note: this is a Cauchy-Euler equation with solutions of the form $u = x^m$.

   (b) Put the differential equation into self-adjoint form.

   (c) Give the orthogonality relation. Use MATLAB to verify the orthogonality relation for the first two eigenfunctions.

4. Consider Laguerre's differential equation defined on the semi-infinite interval $x \in (0, \infty)$:

$$x u'' + (1 - x) u' + \overset{\lambda}{n}\, u = 0, \quad n = 0, 1, 2, \ldots$$

   This equation has polynomial solutions $L_n(x)$. Put the equation into self-adjoint form and give an orthogonality relation.

For the next two problems, please use MATLAB along with the provided function besselzero(nu,n,kind) as shown in class.

5. Find the first four $\alpha_n > 0$ defined by $J_1(3\alpha) = 0$.

6. Expand $f(x) = 1,\ 0 < x < 2$, in a Fourier-Bessel series using Bessel functions of order zero that satisfy the boundary condition: $J_0(2\alpha) = 0$. Make a plot in MATLAB of the given function and the Fourier-Bessel expansion of the function with the first four terms.

For the next problem, use the MATLAB built-in function legendreP(n,x) to represent Legendre Polynomials for Fourier-Legendre expansions.

7. Use MATLAB to calculate and print out the value of the first five non-zero terms in the Fourier-Legendre expansion of the given function. Make a plot in MATLAB of the given function and the Fourier-Legendre partial sum with five (non-zero) terms.

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$$

Part IV

Partial Differential Equations in Rectangular Coordinates

Lecture 21 - Introduction to Separable Partial Differential Equations

Objectives

- Review description of linear second-order, Partial Differential Equations (PDEs).
- Introduce a classification scheme for second-order linear PDEs.
- Illustrate the use of separation of variables to find solutions to some PDEs.

Linear Partial Differential Equations

Consider the linear, second-order, partial differential equation in two independent variables shown in Equation 84

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \tag{84}$$

where $A \rightarrow G$ are constants or functions of the *independent* variables x and/or y only.[38] If $G = 0$ then the equation is *homogeneous*, otherwise the equation is *non-homogeneous*.

[38] If the coefficients are functions of the dependent variable u or any of its partial derivatives, the equation would, of course, be non-linear.

Classification of Linear 2^nd-Order PDEs

The solution of a PDE is a *function* of two (or more) independent variables that satisfies the PDE and boundary/initial conditions in some region of the space defined by the independent variables. Some important qualitative features of the solutions can be anticipated by using the following classification scheme for linear second-order PDEs.

Hyperbolic: $B^2 - 4AC > 0$

Hyperbolic differential equations are characteristic of wave-type phenomena. In the linear homogeneous case, waves travel through the domain without distortion until a boundary is encountered. We will examine problems such as vibrating strings and membranes that are governed by hyperbolic PDEs and will exhibit this wave-type behavior.

There is an important first-order PDE that does not conform to this classification scheme but is considered hyperbolic; a typical example is the scalar linear advection equation:

$$u_t + a \cdot \nabla u = f(x, y)$$

This equation exhibits similar wave-type behavior.

A common non-linear variation is:

$$u_t + \nabla \cdot f(u) = 0$$

where $f(u)$ is called a *flux function*. This equation plays a role in modeling a variety of physical conservation laws often associated with transport phenomena. These equations are known for being capable of producing shocks; discontinuities in the solution even when the initial data is smooth.

Parabolic: $B^2 - 4AC = 0$

Parabolic differential equations are characteristic of *diffusive* phenomena like transient heat conduction. The time evolution of the solution of these equations typically has a "smoothing" behavior. Even if the initial data is only piece-wise smooth, as time evolves the solution tends to diffuse into a smooth function.

Elliptic: $B^2 - 4AC < 0$

Elliptic differential equations are characteristic of *steady-state* phenomena like static electrical potential and the steady-state heat equation.

Example: Classify the following linear partial differential equations.

1. $3\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y}$

2. $\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial y^2}$

3. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Separation of Variables

The basic technique we will use to solve second-order, linear, homogeneous PDEs is called separation of variables. Once again, I will illustrate this method by way of doing an example.

Example: use separation of variables to find product solutions of:

$$\frac{\partial^2 u}{\partial x^2} = 4\frac{\partial u}{\partial y} \tag{85}$$

Step #1: Assume a solution can be expressed as a product of functions—one function for each independent variable.

$$u(x, y) = F(x)G(y)$$

Step #2: Insert the proposed solution into the governing equation.

Here we will use subscript notation to denote partial derivatives.

$$\frac{\partial^2}{\partial x^2}\left[F(x)G(y)\right] = 4\frac{\partial}{\partial y}\left[F(x)G(y)\right]$$
$$F_{xx}G = 4FG_y$$

Step #3: Separate variables and introduce separation constant. In this example we will separate variables by dividing both sides of the equation by $4FG$.

$$\frac{F_{xx}G}{4FG} = \frac{4FG_y}{4FG}$$
$$\frac{F_{xx}}{4F} = \frac{G_y}{G}$$

In this last equation, the terms on the left are only a function of x; the terms on the right are only a function of y. The left- and right-hand side of the equality must be the same for *all* values of x and y. The only way this can be expected to be true is if *both* sides are equal to a constant. We will denote this constant: $-\lambda$.

This bit of reasoning is a key element of separation of variables.

$$\frac{F_{xx}}{4F} = \frac{G_y}{G} = -\lambda$$

We can now decompose the partial differential equation in two independent variables into two ordinary differential equations:

You might wonder why we chose $-\lambda$ rather than λ. In honesty there is no good answer to this question; let us chalk it up to a bias towards having a plus-sign in the separated equations.

$$F_{xx} + 4\lambda F = 0$$
$$G_y + \lambda G = 0$$

Step #4: Form product solutions for all possible values of λ.

The "possible values" of λ can be put into three familiar categories: λ can be *positive*, *negative*, or *zero*.

$\lambda = 0$:

Note how in this case and the cases to follow, we will simply write down the general solution to the separated ODEs with little/no to-do over deriving that solution. By this point in the course you *need* to be able to quickly recognize those equations. In most cases you should be able to write down the solutions by inspection.

$$F_{xx} = 0 \Rightarrow F(x) = c_1 + c_2 x$$
$$G_y = 0 \Rightarrow G(x) = c_3$$

$$u(x,y) = F(x)G(x) = (c_1 + c_2 x)\, c_3$$
$$= A_1 + B_1 x$$

$\lambda < 0$: For this case we will let $\lambda = -\alpha^2, \;\; \alpha > 0$.

We will assume, for this problem, that the x-dimension is bounded and thus it is convenient to use the $\cosh 2\alpha x$ and $\sinh 2\alpha x$ form of the solution. If the domain is unbounded you would use $e^{2\alpha x}$ and $e^{-2\alpha x}$. It will be up to you to make this determination.

$$F_{xx} - 4\alpha^2 F = 0 \Rightarrow F(x) = c_1 \cosh 2\alpha x + c_2 \sinh 2\alpha x$$
$$G_y - \alpha^2 G = 0 \Rightarrow G(y) = c_3 e^{\alpha^2 y}$$

$$u(x,y) = F(x)G(y) = (c_1 \cosh 2\alpha x + c_2 \sinh 2\alpha x)\, c_3 e^{\alpha^2 y}$$
$$= (A_2 \cosh 2\alpha x + B_2 \sinh 2\alpha x)\, e^{\alpha^2 y}$$

$\underline{\lambda > 0}$: For this case we will let $\lambda = \alpha^2, \;\; \alpha > 0$.

$$F_{xx} + 4\alpha^2 F = 0 \Rightarrow F(x) = c_1 \cos 2\alpha x + c_2 \sin 2\alpha x$$
$$G_y + \alpha^2 G = 0 \Rightarrow G(y) = c_3 e^{-\alpha^2 y}$$

$$\begin{aligned} u(x,y) = F(x)G(y) &= (c_1 \cos 2\alpha x + c_2 \sin 2\alpha x)\, c_3 e^{-\alpha^2 y} \\ &= (A_3 \cos 2\alpha x + B_3 \sin 2\alpha x)\, e^{-\alpha^2 y} \end{aligned}$$

Notes:

- There is no assurance that a linear 2nd-order PDE will be separable. We will spend a lot of time in this course in dealing with equations that happen to be separable. In reality, many are not, in particular if the problem is non-homogeneous. It is a good idea to check to see if an equation is homogeneous before launching down the separation-of-variables path.
- This example is a bit of an anomaly. We will usually not attempt to find *general* solutions to PDEs, but only *particular* solutions. Therefore a problem statement will not be fully meaningful without boundary/initial conditions by which we will be able to derive particular solutions.
- Specific values of λ that result in non-trivial solutions will depend on the boundary conditions.
- Since the PDEs are linear, the *superposition principle* will apply. That is, if $u_1, u_2, \ldots, u_k$ are solutions of a linear homogeneous PDE (including boundary conditions) then a linear combination:

  $$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

  is also a solution.

In this equation c_i are constants.

Part V

Back Matter

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Appendices

Matlab Style Rules

1. **rule:** All scripts will start with the commands: **clear**, **clc**, and **close 'all'**

   rationale: No script should depend upon any data visible in the MATLAB workspace when the script starts. By omitting these commands, residual data within the workspace may hide errors.

2. **rule:** Your code must be documented with enough details such that a reader unfamiliar with your work will know what you are doing.

   rationale: Code documentation is a habit. For more significant projects readers may need help in deciding what the author of the code intended. For your own code, the most likely reader is you—a few months into the future.

3. **rule:** Function and variable names must be meaningful and reasonable in length.

   rationale: Failing to do either make code harder to read and maintain.

4. **rule:** All outputs from the code **<u>must</u>** be meaningful; numbers should be formatted, part of a sentence, and include units. Graphs should be readable and axis labels should make sense and include units.

   rationale: Code output is a form of communication. It is important that this communication be clear and unambiguous.

5. **rule:** Do not leave warnings from the Code Analyzer unaddressed.

   rationale: Sometimes Code Analyzer warnings can be safely ignored. Most of the time the warning points to a stylistic error that would be unacceptable in software that you use. Occasionally these warnings are indicative of a hidden error.

6. **rule:** Use the "smart indentation tool" to format the indentation of your code.

rationale: This tool improves code readability. It will also occasionally point out errors that you did not see before.

7. **rule:** Pre-allocate arrays; if possible initialize with **NaN** values.

   rationale: Pre-allocation improves performance and helps readability. Initialization with **NaN** helps avoid a range of potential logical errors.

8. **rule:** Avoid "magic numbers" — i.e. hard-coded constants.

   rationale: Constants included in your code tend to hide your program logic. Also, "magic numbers" make code maintenance more difficult and error prone.

9. **rule:** Only write one statement per line.

   rationale: Multi-statement-lines hurt code readability in almost all cases.

10. **rule:** Do not write excessively long lines of code; use the line continuation "..." and indentation to spread long expressions over several lines.

    rationale: Following this rule improves code readability.

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