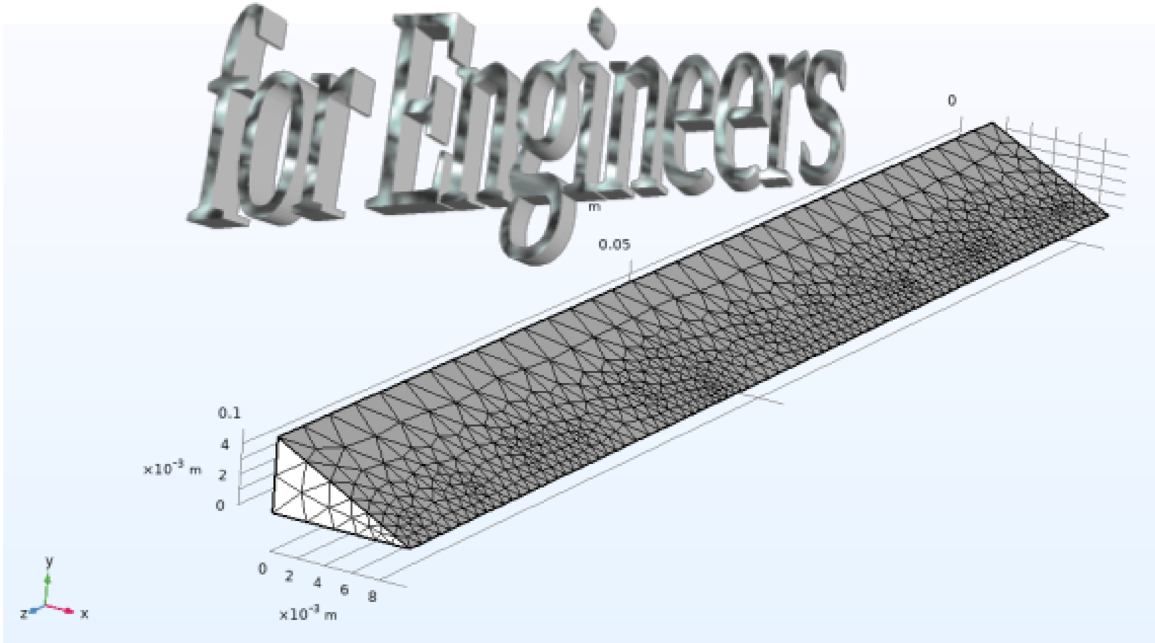
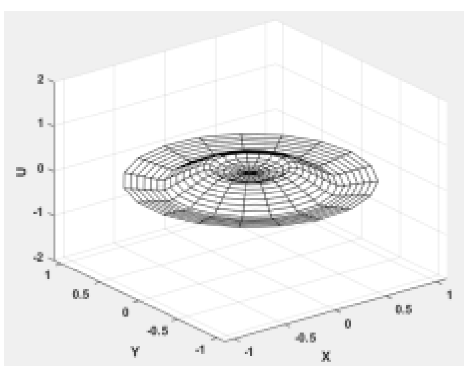
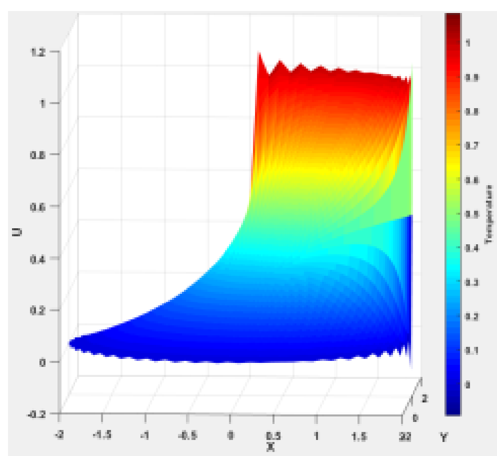
Mathematical Methods for Engineers







Stu Blair

Mighty Goat Press

WHEN IN DOUBT, MULTIPLY BOTH SIDES BY AN ORTHOGONAL FUNCTION AND INTEGRATE.

P.L. CHEBYSHEV

THE PURPOSE OF COMPUTING IS INSIGHT, NOT PICTURES

L.N. TREFETHEN

NEVER DO A CALCULATION UNTIL YOU ALREADY KNOW THE ANSWER.

J.A. WHEELER

UNITED STATES NAVAL ACADEMY

MATHEMATICAL METHODS FOR ENGINEERS

MIGHTY GOAT PRESS

PUBLISHED BY MIGHTY GOAT PRESS

First printing, May 2023

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Preface

The purpose of this text is to provide a concise reference for engineering students who would like to strengthen their conceptual understanding and practical proficiency in analytical and numerical methods in engineering. The material is based on a sequence of two courses taught at the United States Naval Academy.

Analytical Methods

The first course focused on analytical methods for linear ordinary and partial differential equations. All students came into the course having taken a three-semester sequence of calculus along with a course in ordinary differential equations. The analytical methods portion quickly reviews methods for constant coefficient linear equations and proceeds to methods for non-constant coefficients including Cauchy-Euler equations, power series methods, and method of Frobenieus. After a review of Fourier Series methods and an introduction to Fourier-Legendre and Fourier-Bessel expansions we thoroughly explore solutions to second-order, linear, partial differential equations. Since many students are also studying nuclear engineering, there is a heavy focus on addressing boundary value problems in cylindrical and spherical coordinate systems that are applicable to other topics of interest such as reactor physics. There is also heavy emphasis on heat transfer applications that students will see later on in their undergraduate curriculum.

The materials presented are based heavily on Professor Dennis Zill's excellent book.[1] We lightly select from chapters 1-3 for review; chapter 5 for series solution methods; and chapters 12-14 for Fourier Series and solutions to linear boundary value problems. Material from that text is used throughout this book.

What distinguishes this course from Prof Zill's work is the incorporation of computational tools in the solution process. These "semi-analytical methods" are presented here in MATLAB[2] owing to the students preparation with that tool. Other open-source tools like Octave[3] and Python,[4] of course, could be used.

[1] Dennis G Zill. *Advanced engineering mathematics*. Jones & Bartlett Learning, 2020

[2] Inc. The Math Works. Matlab, v2022a, 2022. URL https://www.mathworks.com/

[3] John W. Eaton, David Bateman, Søren Hauberg, and Rik Wehbring. *GNU Octave version 5.2.0 manual: a high-level interactive language for numerical computations*, 2020. URL https://www.gnu.org/software/octave/doc/v5.2.0/

[4] Guido Van Rossum and Fred L. Drake. *Python 3 Reference Manual*. CreateSpace, Scotts Valley, CA, 2009. ISBN 1441412697

Numerical Methods

Part I

Introduction and Review

Lecture 1 - Introduction, Definitions and Terminology

Objectives

The objectives of this lecture are:

- Provide an overview of course content
- Define basic terms related to differential equations
- Provide examples of classification schemes for differential equations

Course Introduction

This course is intended as a one-semester introduction to partial differential equations. It is assumed that all students have a thorough background in single- and multi-variable calculus as well as differential equations. The first few lectures comprise a review of the portions of differential equations on which this course most heavily relies. This is followed by a treatment of power series methods and the method of Frobeneius. These are needed so that students will understand the origins of Legendre Polynomials and Bessel functions that will be used in the solution of boundary value problems in spherical and cylindrical coordinates respectively.

The main body of material deals with the solution of (mostly homogeneous) boundary value problems—wave equation, heat equation, and Laplace equation—in rectangular, polar/cylindrical, and spherical coordinate systems. For this a preparatory review of Fourier series expansions along with Fourier-Legendre and Fourier-Bessel expansions are introduced along with a levening of Sturm-Liouville theory in boundary value problems. The rest is a problem-by-problem tour of methods and analysis with heavy emphasis on heat transfer and nuclear engineering applications.

Classification of Differential Equations

IT IS IMPORTANT to be able to classify differential equations. In this class we will learn a variety of techniques to find the function that satisfies a differential equation along with its boundary or initial conditions.[5] The techniques we learn in this class are tailored for specific classes of problems; you classify the problem and that tells you what method to use. If you improperly classify the equation, you will likely use an inappropriate method and may have trouble figuring out why it is not working.

[5] Consider the differential equation: $\frac{du}{dx} = ux$. The variable u stands for the function, $u(x)$, that satisfies the equation; u is also referred to as the **dependent variable**. The variable x is the **independent variable**. By convention we will use the variables x, y, z and r, θ, ϕ as spatial independent variables and t as an independent variable for time dependent problems. We will use many other letters to denote dependent variables but most commonly u, v, and w.

Classification by Type and Order

WE SHALL START with the easiest classification categories: type and order. There are two *types* of differential equations that we will consider: ordinary differential equations; and partial differential equations.

IN AN ORDINARY differential equation, there is only one independent variable. In a *partial* differential equation, there are multiple independent variables and consequently derivatives of the depenent variable will partial derivatives.

Example ODE:

$$\frac{d^2u}{dt^2} + t\frac{du}{dt} = 3e^{-t}$$

There is one independent variable, t

Example PDE:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

There are two independent variables, x, and y.

THE ORDER OF a differential equation is the order of the hightest derivative in the equation. This is typically not confusing for students. If anything needs to be added here it is to be mindful of the difference between a higher order derivative and an exponent. For example, in the second order, non-linear, ordinary differential equation shown below,

$$\frac{d^2u}{dx^2} + 5\left(\frac{du}{dx}\right)^3 - 4u = e^x$$

it isn't *too* hard to realize that the "3" is an exponent and the "2" denotes a second derivative. Still, be mindful.

Classification by Linearity

An n-th order ordinary differential equation is said to be *linear* when it can be written in the form shown in Equation 1:

$$a_n(x)u^{(n)} + a_{n-1}(x)u^{(n-1)} + \cdots + a_1(x)' + a_0(x)u = g(x) \qquad (1)$$

The key features that you should note in the form of Equation 1 are:

1. The *dependent* variable and *all of its derivatives* are of the first degree; that is, the power of each term involving u is 1.

2. The coefficients of each term, $a_n(x)$, depend at most on the *independent* variable.

A lot of students struggle with discriminating between linear and nonlinear ODEs but it really is as simple as checking these two things. If both conditions are satisfied; the equation is linear. If not, the equation is nonlinear. As examples, Equation 2 violates the first criterion; Equation 3 violates the second.

Why is this important? Most of the techniques we will learn in this course *depend* upon the fact tha the equation we are trying to solve is *linear*. In *"the wild"* you may be presented with (or, more likely *derive*) an equation and may not be explicitly told whether or not the equation is linear. If the equation is **not** linear you will find that most of the tools you learn in this course will not be applicable; you will most likely need to use a numerical method. You need to be able to tell the difference so you know what tools to use.

$$\frac{d^2u}{dx^2} + u^2 = 0 \tag{2}$$

$$\frac{d^3u}{dx^3} - 5u\frac{du}{dx} = x \tag{3}$$

Verification of an Explicit Solution

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is said to be an *explicit* solution. Otherwise, the solution is *implicit*.

Example explicit solution: $u(x) = f(x)$.
Example implicit solution: $G(x,u) = 0$

In this class we will mainly be interested in finding explicit solutions to differential equations that we are given or have derived. There are some cases, however, where we are given a function and we wish to verify that it is a solution to a given differential equation. To do this, we simply "plug" the equation into the differential operator and verify that an identity is derived.

Example: Verify that $u = \frac{6}{5} - \frac{6}{5}e^{-20t}$ is a solution to:

$$\frac{du}{dt} + 20u = 24$$

Solution: Since $\frac{du}{dt} = \frac{d}{dt}(\frac{6}{5} - \frac{6}{5}e^{-20t}) = 24e^{-20t}$, we can see that:

$$\begin{aligned}\frac{du}{dt} + 20u &= 24e^{-20t} + 20(\frac{6}{5} - \frac{6}{5}e^{-20t}) \\ &= 24e^{-20t} + 24 - 24e^{-20t} \\ &= 24\end{aligned}$$

which is the expected identity.

Lecture 2 - Separable and Linear 1st order Equations

Objectives

The objectives of this lecture are:

- Define and describe the solution procedure for *separable* first order equations
- Define and demonstrate the solution procedure for *linear* first order equations

Separable Equations

A first order differential equation of the form shown below

$$\frac{du}{dx} = g(u)h(x) \tag{4}$$

is said to be *separable* or have *separable variables.*

Note: there is **no** requirement that the 1st order equation be *linear*. This is one of the few techniques that we will study in this course that can be applied to nonlinear equations.

THE SOLUTION METHOD for separable equations is, in princple simple. For the separable differential equation given in Equation 4 we would separate and integrate:

$$\begin{aligned} \frac{du}{dx} &= g(u)h(x) \\ \frac{du}{g(u)} &= h(x)dx \\ \int \frac{1}{g(u)}\, du &= \int h(x)\, dx \end{aligned}$$

Note: there are at least two complications here.

1. The solution you thus derive may be either implicit or explicit. An implicit solution is, as a practical matter, fairly inconvenient to deal with; and
2. It may not be possible to actually carry out the integrals analytically.

Nonetheless, we shall carry on and give it a try anyway.

Generally speaking, one of your first checks for a first order equation should be: is it separable? If so, you should separate the variables and solve. The examples below are intended to illustrate the method. Note that in the final example, the integral cannot be done analytically.

Solve the following separable, first order differential equations .
Example 1:

$$\begin{aligned}
\frac{du}{dx} &= \frac{u}{1+x} \\
\frac{du}{u} &= \frac{dx}{1+x} \\
\int \frac{d}{u} &= \int \frac{dx}{1+x} \\
\ln|u| + c_1 &= \ln|1+x| + c_2 \\
|u| &= e^{[\ln|1+x|+c_3]} \\
u(x) &= c|1+x|
\end{aligned}$$

Example 2:

$$\begin{aligned}
\frac{du}{dx} &= -\frac{x}{u} \\
\int u\ du &= -\int x\ dx \\
\frac{u^2}{2} &= -\frac{x^2}{2} + c \\
u(x) &= \sqrt{c - x^2}
\end{aligned}$$

Example 3: Solve the first order initial value problem shown below:

$$\frac{du}{dx} = e^{-x^2},\ \ u(2) = 6,\ \ 2 \leq x < \infty$$

$$\begin{aligned}
du &= e^{-x^2}\ dx \\
\int_2^x \frac{du}{dt}\ dt &= \int_2^x e^{-t^2}\ dt \\
u(x) - u(2) &= \int_2^x e^{-t^2}\ dt \\
u(x) &= 6 + \int_2^x e^{-t^2}\ dt
\end{aligned}$$

where we have used the dummy variable t in the integrals; the last integral will need to be evaluated numerically.

Linear Equations

A first-order differential equation of the form:

$$a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{5}$$

is said to be a first order *linear equation* in the dependent variable u. When $g(x) = 0$, the first-order linear equation is said to be *homogeneous*; otherwise it is *nonhomogeneous*.

Note: it is sometimes customary to write the differential equation in *operator form* where the differential operator, $\mathcal{L} = a_1(x)\frac{d}{dx} + a_0(x)$, is applied to the function $u(x)$ to get $g(x)$; $\mathcal{L}u(x) = g(x)$

Notice that $g(x)$ is the only term in Equation 5 that does **not** include u or any of its derivatives.

WHEN SOLVING equations of this type it is useful to express it in the **standard form**:

$$\frac{du}{dx} + P(x)u = f(x) \tag{6}$$

The method for solving this equation makes use of the linearity property and express the solution in the following way: $u(x) = u_c(x) + u_p(x)$; plugging this into Equation 6 gives us:

When we say an operator is **linear**, what we mean is that the following relationships must hold:

1. $\mathcal{L}(\alpha u) = \alpha\mathcal{L}(u)$
2. $\mathcal{L}(u+v) = \mathcal{L}(u) + \mathcal{L}(v)$

for functions u,v and scalar constant α. Think of this as a *definition* of linearity.

$$\begin{aligned}\frac{d}{dx}[u_c + u_p] + P(x)[u_c + u_p] = \\ \left[\frac{du_c}{dx} + P(x)u_c\right] + \left[\frac{du_p}{dx} + P(x)u_p\right] = f(x)\end{aligned} \tag{7}$$

where $u_c(x)$ is the solution to the *associated homogeneous problem*

$$\frac{du_c}{dx} + P(x)u_c = 0 \tag{8}$$

The linear operator here is: $\mathcal{L} = \frac{d}{dx} + P(x)$. Equation 8 says $\mathcal{L}u_c = 0$; Equation 9 says $\mathcal{L}u_p = f(x)$; Equation 7 says that $\mathcal{L}(u_c + u_p) = 0 + f(x) = f(x)$.

and $u_p(x)$ is the solution to:

$$\frac{du_p}{dx} + P(x)u_p = f(x) \tag{9}$$

What might trouble you now is: if we have u_p, is this not a solution to Equation 6? Why do we need u_c? The next thing that should trouble you is that if u_p is a solution, by the linearity property of $\mathcal{L}$, so is u_p plus *any* constant multiple of u_c. The solution is not *unique*

We can see that Equation 8 is separable:

$$\begin{aligned}\frac{du_c}{dx} + P(x)u_c &= 0 \\ \frac{du_c}{u_c} &= -P(x)\ dx \\ \ln u_c + C &= -\int_{P(x)\ dx} \\ u_c(x) &= e^{-\int P(x)\ dx + C_1} \\ u_c(x) &= e^{-\int P(x)\ dx}e^{C_1}u_c(x) &&= Ce^{-\int P(x)\ dx}\end{aligned}$$

where $C = e^{C_1}$.

This will all be resolved when we recall that u_c will have an arbitrary constant through which we will be able to say that $u = u_c + u_p$ is a function describing *all* possible solutions of Equation 6 and the arbitrary constant in u_c will be set so as to uniquely satisfy a given initial/boundary condition.

Note: At some point in time, I will desist in making such piddling distinctions between constants. C_1 is an arbitrary constant, e^{C_1} is still an arbitrary constant; there is no real difference between C_1 and C and, in this author's humble opinion, they do not rate different symbols.

WE NEED TO FIND a solution $u_p(x)$ to Equation 9. The technique we will use is called *variation of parameters*. It consists of looking for a solution in the form $y_p(x) = v(x)u_1(x)$, where $u_1(x) = e^{-\int P(x)\ dx}$ which is $u_c(x)$ with the arbitrary constant set to 1 and $v(x)$ might be thought of as some kind of weighting or *variational* function.

WE WILL INSERT this proposed form of $y_p(x)$ into Equation 9:

$$\frac{d(vu_1)}{dx} + P(x)(v(x)u_1(x)) = f(x)$$

We apply the product rule to the first term and re-arrange terms:

$$\begin{aligned} u_1(x)\frac{dv}{dx} + v(x)\frac{du_1}{dx} + P(x)(v(x)u_1(x)) &= f(x) \\ v(x)\underbrace{\left[\frac{du_1}{dx} + P(x)u_1(x)\right]}_{=0} + u_1(x)\frac{dv}{dx} &= f(x) \\ u_1(x)\frac{dv}{dx} &= f(x) \end{aligned}$$

In the last line we can observe that the equation is *separable* and thus solve:

$$\begin{aligned} v(x) &= \int \frac{f(x)}{u_1(x)}\ dx \\ &= \int e^{\int P(x)\ dx} f(x)\ dx \end{aligned}$$

Now that we know what $v(x)$ must be, we can combine this with $u_1(x)$ to get $u_p(x)$:

$$u_p(x) = e^{-\int P(x)\ dx}\left[\int e^{\int P(x)\ dx} f(x)\ dx\right] \tag{10}$$

Equation 10 is messy and perhaps a bit scary but given definitions of $P(x)$ and $f(x)$ we might hope we can solve it anyway. We now have expressions for both u_c and u_p; they can be combined into the solution for the first-order linear equation:

$$u(x) = Ce^{-\int P(x)\ dx} + e^{-\int P(x)\ dx}\left[e^{\int P(x)\ dx} f(x)\ dx\right] \tag{11}$$

Method of Solution

Once we have identified a problem to be first-order and linear, we will solve the problem using the following steps:

1. Write the equation in standard form (Equation 6)
2. Determine the integrating factor $\mu = e^{-\int P(x)\ dx}$.
3. Solve for the general solution $u(x)$ using Equation 11.
4. Apply initial/boundary condition if given.

Example: Solve the problem:

$$\frac{du}{dx} + u = x, \quad u(0) = 4$$

Solution:

Step 1: The equation is already in standard form, so this step is easy.

Step 2: Find the integrating factor μ.

$$mu = e^{-\int P(x)\ dx} = e^{-\int 1\ dx} = e^{-x}$$

Step 3: Solve for the general solution $u(x)$ using Equation 11

$$\begin{aligned} u(x) &= Ce^{-x} + e^{-x} \int e^x x\ dx \\ &= Ce^{-x} + e^{-x} \left[xe^x - e^x \right] \\ &= Ce^{-x} + x - 1 \end{aligned}$$

← For the integral $\int e^x x\ dx$ we need to use integration by parts.

Step 4: Apply initial/boundary conditions if given

$$\begin{aligned} u(0) &= Ce^0 + 0 - 1 \\ &= C - 1 = 4 \\ \Rightarrow C &= 5 \\ u(x) &= 5e^x + x - 1 \end{aligned}$$

Assignment #1

State the order of the given ordinary differential equation and indicate if it is linear or non-linear.

1. $(1-x)\,u'' - 4xu' + 5u = \cos x$

2. $t^5 u^{(4)} - t^3 u'' + 6u = 0$

Verify the indicated function is an explicit solution of the given differential equation.

3. $2u' + u = 0, \quad u = e^{-x/2}$

4. $u'' - 6u' + 13u = 0, \quad u = e^{3x}\cos 2x$

Solve the given differential equation by separation of variables

5. $\frac{du}{dx} = \sin 5x$

6. $dx + e^{3x} du = 0$

7. $\frac{dS}{dr} = kS$

8. $\frac{du}{dx} = x\sqrt{1-u^2}$

Find an explicit solution of the given initial-value problem

9. $x^2 \frac{du}{dx} = u - xu, \quad u(-1) = -1$

Find the general solution of the given differential equation

10. $\frac{du}{dx} + u = e^{3x}$

11. $u' + 3x^2 u = x^2$

12. $x \frac{du}{dx} - u = x^2 \sin x$

Lecture 3 - Theory of Linear Equations

Objectives

The objectives of this lecture are:

- Introduce several theoretical concepts relevant to initial value problems and boundary value problems.
- Demonstrate use of the Wronskian to determine linear independence of solutions.
- Present some important theorems and definitions relevant to the theory of linear orinary differential equations.

Initial Value Problems

For a linear differential equation, an n$^{\text{th}}$-order initial value problem (IVP) is given by the following governing equation and initial conditions:

$$\text{Governing Equation: } a_n(x)\frac{d^n u}{dx^n} + a_{n-1}\frac{d^{n-1}u}{dx^{n-1}} + \cdots + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{12}$$

$$\text{Initial Conditions: } u(x_0) = u_0,\ u'(x_0) = u_1, \ldots, u^{(n-1)}(x_0) = u_{n-1} \tag{13}$$

Note: for an initial value problem, all of the initial conditions are provided at the same value of x; in accordance to custom we call this x_0. The name *initial* condition gives the implication that these conditions are at some "end" of the interval (beginnig, left side, whatever) and in most all examples and exercises this is indeed the case. It is not a requirement, however.
Generally for an n$^{\text{th}}$-order IVP you will need n conditions.

WE SEEK A function defined on some interval containing x_0 that satisfies the differential equation with n conditions applied. The theorem below, which we will use by *citing* rather than *proving* gives us assurance that, subject some fairly reasonable assumptions, such a solution will exist.

Theorem 1 (Existence and Uniqueness for IVPs)
If $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$ and $g(x)$ are continuous on an interval $\mathcal{I}$, and if $a_n(x) \neq 0$ for every $x \in \mathcal{I}$, and if x_0 is any point in this interval, then a solution $u(x)$ of the IVP exists on the interval and it is unique.

FOR THIS CLASS we will adopt a mostly operational definition of continuity: if you can draw the function throughout the specified interval without picking up your pencil or without diverging to infinity, then the function is continuous.

Consider, as an example, the following initial value problem:

$$u'' - 4u = 12x, \;\; u(0) = 4, \; u'(0) = 1 \tag{14}$$

This IVP satisfies the conditions of Theorem 1 since all of the coefficients and $g(x)$ are continuous and a_1 is constant and nonzero; hence a unique solution exists on any interval and that solution is unique.

Take a moment to verify that $u(x) = 3e^{2x} + e^{-2x} - 3x$ satisfies both the governing equation and initial conditions and thus is *the* unique solution to this IVP.

Here is an IVP that does *not* satisfiy the criteria of Theorem 1:

$$x^2u'' - 2xu' + 2u = 6, \;\; u(0) = 3, \; u'(0) = 1 \tag{15}$$

In this case, the coefficients and $g(x)$ are all continuous but $a_2(x)$ is equal to zero at $x = 0$. This might not be a problem—i.e. if $x = 0$ is not in the interval of interest for the IVP then we are okay—but since $x_0 = 0$, $x = 0$ *must* be in the domain for the theorem to apply. So we have no assurances that a solution exists or, if a solution does exist, it may not be unique.

You should take a moment to verify that $u = cx^2 + x + 3$ is a solution for *any* choice of parameter c.

Boundary Value Problems

For this section let us, without undue loss of generality, consider a 2nd-order boundary value problem (BVP):

$$\text{Govering Equation: } a_2(x)\frac{d^2u}{dx^2} + a_1(x)\frac{du}{dx} + a_0(x)u = g(x) \tag{16}$$

$$\text{Boundary Conditions: } y(a) = y_0, \;\; y(b) = y_1, \;\; a \neq b \tag{17}$$

Almost all of the applications we will consider for this class will involve 2nd-order operators. The way we derive important boundary-value problems from underlying physical laws like conservation of mass and consdervation of energy lead to them being 2nd-order. Probably the most obvious counterexample is beam theory which involves a 4th-order operator.

DEPENDING ON THE boundary conditions, BVPs may have no solutions, one unique solution, or infinitely many solutions.

Example: The equation $u'' + 16u = 0$ has the general solution $u(t) = c_1 \cos(4t) + c_2 \sin(4t)$. Consider the three different sets of boundary conditions provided below.

a) $x(0) = 0, \;\; x(\pi/2) = 0$ Application of the first boundary condition gives us $c_1(1) + c_2(0) = 0 \Rightarrow c_1 = 0$. The second boundary condition is $c_2 \sin(2\pi) = 0$, which is true for *any* value of c_2. Therefore there problem has infinitely many solutions.

b) $x(0) = 0, \;\; x(\pi/8) = 0$ The first boundary condition again gives us $c_1 = 0$; the second condition $c_2 \sin(4\frac{\pi}{8})$ is only satisfied if $c_2 = 0$.

Thus $c_1 = c_2 = 0$; only the trivial solution, $u = 0$, satisfies both the differential equation and boundary conditions. This is not a very interesting solution but at least it *is a solution* so we will take this as an example of a BVP having a unique solution.

For applications, we will generally be only interested in *non-trivial* solutions; that is, solutions that are not identically equal to zero.

c) $x(0) = 0, \;\; x(\pi/2) = 1$ In this case, again $c_1 = 0$ from the first boundary condition. This leaves the second boundary condition: $c_2 \sin\left(4\frac{\pi}{2}\right) = c_2(0) = 1$ which cannot be satisfied for any value of c_2. In this case *no* solution exists.

Superposition and Linear Dependence

In this section some important theorems regarding IVPs and BVPs will be presented. No attempt will be made to prove these theorems; we will simply take these theorems as facts that are relevant for this course that you should try to understand as best you can.

Theorem 2 (Superposition Principle for Homogeneous Equations) *Let $u_1, u_2, \ldots, u_k$ be solutions of a homogeneous n^{th}-order linear differential equation. Then any linear combination of those solutions*

$$u = c_1 u_1 + c_2 u_2 + \cdots + c_k u_k$$

where $c_1, c_2, \ldots, c_k$ are arbitrary constants, is also a solution.

Note: It is essential that *both* the governing equation and given conditions (boundary or initial) for the linear differential equation are homogeneous. As a reminder, this means that *all* terms in the governing equation and boundary conditions must either a) involve the dependent variable or one of its derivatives; or b) be equal to zero.

As an example, If I denote the linear homogeneous differential equation as $\mathcal{L}$, then $\mathcal{L}(u_i) = 0$ for any $i \in [1, 2, \ldots, k]$. By the linearity property of $\mathcal{L}$, for any constants α and β:

$$\begin{aligned} \mathcal{L}(\alpha u_i + \beta u_j) &= \alpha \mathcal{L}(u_i) + \beta \mathcal{L}(u_j) \\ &= \alpha(0) + \beta(0) \\ &= 0 \end{aligned}$$

Theorem 3 (Linear Dependence / Independence of Functions) *A set of functions $f_1(x), f_2(x), \ldots, f_k(x)$ is said to be* linearly dependent *on an interval $\mathcal{I}$ if there exist constants $c_1, c_2, \ldots, c_k$,* not <u>all</u> of which are zero*, such that*

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_k f_k(x) = 0$$

for every $x \in \mathcal{I}$. If the set of functions is not *linearly dependent, it is linearly independent.*

What if a member of the set of functions is $f(x) = 0$?

Answer: The set will no longer be linearly independent. The trivial function $f(x) = 0$ is not linearly independent from *anything*.

Repeatedly throughout this course we will want to clarify whether or not, say, two functions are linearly independent of each other. I think most engineers have a general idea of what it is we *mean* when we say two functions are linearly independent or dependent but Theorem 3 specifies what these things mean *mathematically*.

WE NEED A TEST to help us determine if the members of a set of functions are linearly independent or not. This will be especially important as we evaluate solutions to a linear homogeneous differential equation. Even if you are the sort of savant who can, by inspection, always detect linear dependence, you might have a hard time convincing your friends that your assessment is always correct. Luckily, there is a theorem that provides a suitable test that can serve as irrefutable evidence of the state of linear dependence/independence of functions.

Theorem 4 (Criterion for Linearly Independent Solutions)
Let $u_1, u_2, \ldots, u_n$ be solutions of a homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$. Then the set of solutions is linearly indepenent on the interval if and only if the Wronskian of the solution is non-zero for every $x \in \mathcal{I}$.

The Wronskian is a function that takes functions as arguments and returns a scalar numeric quantity.[6]

[6] Sometimes such functions are referred to as *functionals*.

$$W(u_1, u_2, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \vdots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix} \tag{18}$$

where $|\cdot|$ donetes the matrix determinant. For large values of n this is also difficult to calculate but, for the case $n = 2$, engineering students should be familiar with the formula:

$$W(u_1, u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1 u_2' - u_1' u_2 \tag{19}$$

Example: show that the functions $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ are linearly independent solutions to the homogeneous linear equation $u'' - 9u = 0$ for every $x \in (-\infty, \infty)$.
Solution: The Wronskian is given by:

The reader should verify that both $u_1 = e^{3x}$ and $u_2 = e^{-3x}$ satisfy the given differential equation.

$$\begin{aligned} W &= \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} \\ &= e^{3x}\left(-3e^{-3x}\right) - 3e^{3x}\left(e^{-3x}\right) \\ &= 3e^{3x-3x} - 3e^{3x-3x} \\ &= 3 - 3 \\ &= 6 \end{aligned}$$

Since $6 \neq 0$ for all $x \in (-\infty, \infty)$ the solutions are linearly independent.

Definition 1 (Fundamental Set of Solutions)
Any set $u_1, u_2, \ldots, u_n$ of n linearly independent solutions of the homogeneous linear n^{th}-order differential equation on an interval is said to be a fundamental set of solutions on an interval $\mathcal{I}$.

Theorem 5 (Existence of a Fundamental Set)
There exists a fundamental set of solutions for the homogeneous linear n^{th}-order differential equation on an interval $\mathcal{I}$.

Note: This is different than saying that a BVP or IVP has a solution. This theorem is only refering to the differential equation; not the boundary or initial conditions.

Definition 2 (General Solution—Homogeneous Equation)
Let $u_1, u_2, \ldots, u_n$ be a fundamental set of solutions to the homogeneous linear n^{th}-order differential equation defined on an interval $\mathcal{I}$, then the general solution is:

$$u(x) = c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x)$$

IT IS IMPORTANT to understand from the above that:

- *any* possible solution to the homogeneous, linear, n^{th}-order differential equation can be constructed by setting the coefficients of the general solution; and
- there is **no** solution that can be constructed from functions that are linearly independent from the general solution.

General Solution for a Nonhomogeneous Problem

Recall: "nonhomogeneous" for a linear n^{th}-order differential equation means that $g(x) \neq 0$. if u_p is any particular solution to the nonhomogeneous, linear, n^{th}-order ODE on an interval $\mathcal{I}$ and $u_c = c_1 u_1(x) + c_2 u_2(x) + \cdots c_n u_n(x)$ is the general solution to the associated homogeneous ODE (called the *complementary* solution) then the general solution to the nonhomogeneous ODE is:

$$u = u_c + u_p$$

Example: By substitution it can be seen that $u_p = -\frac{11}{12} - \frac{1}{2}x$ is a particular solution to $u''' - 6u'' + 11u' - 6u = 3x$. The general solution to the associated homogeneous problem is $u_c = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$. Consequently, the general solution to the linear nonhomogeneous problem is:

You are, again, strongly encouraged to verify that u_p satisfies the given equation and that u_c satisfies the associated homogeneous equation.

$$\begin{aligned} u(x) &= u_c + u_p \\ &= c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{11}{12} - \frac{1}{2}x \end{aligned}$$

Lecture 4 - Homogeneous Linear Equations with Constant Coefficients

Objectives

The objectives of this lecture are:

- Review the solution methodology for homogeneous linear equations with constant coefficients.
- Illustrate this method with several examples.

Introduction

In this lecture we will review the well-trod ground of your differential equations class and remind ourselves how to solve linear, constant coefficient, homogeneous, n$^{\text{th}}$-order differential equations. These equations have the general form shown in Equation 20

$$c_n u^{(n)} + c_{n-1} u^{(n-1)} + \cdots + c_1 u' + c_0 u = 0 \tag{20}$$

where the coefficients are real and constant an $c_n \neq 0$.

THE BASIC STRATEGY is to assume the solution is of the form: $u(x) = e^{mx}$. For the case of 2$^{\text{nd}}$-order equations, we get:

$$\begin{aligned} c_2 m^2 e^{mx} + c_1 m e^{mx} + c_0 e^{mx} &= 0 \\ e^{mx}\left(c_2 m^2 + c_1 m + c_0\right) \end{aligned}$$

where the last line above is called the auxiliary equation:

Here we re-name the constants so Equation 21 takes a familiar form.

$$am^2 + bm + c = 0 \tag{21}$$

From the well-known quadratic equation, solutions are: $m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ Solution of this equation gives the following three cases:

1. **Distinct Real Roots** In this case $m_1 \neq m_2$ and the general solution is of the form:

$$u(x) = c_1 \underbrace{e^{m_1 x}}_{u_1(x)} + c_2 \underbrace{e^{m_2 x}}_{u_2(x)} \tag{22}$$

Using tools from the last lecture you should recognize that $u_1(x)$ and $u_2(x)$ are linearly independent for all $x \in (-\infty, \infty)$, thus form a fundamental set of solutions.

AN IMPORTANT SPECIAL CASE is when m_1 and m_2 are roots of a positive real number and thus $m_1 = -m_2$. This happens when the governing equation is of the form:

$$u'' - k^2 u = 0 \tag{23}$$

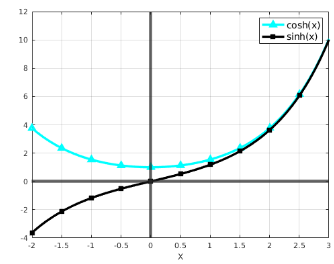

Figure 1: Plot of $\cosh x$ and $\sinh x$

The solutions are thus:

$$u(x) = c_1 e^{-kx} + c_2 e^{kx} \tag{24}$$

For reasons that will become clear later in the course, it is sometimes useful to re-express the solution shown in Equation 24 in terms of the functions cosh and sinh. These functions are defined as linear combinations of exponentials as shown below and plotted in Figure 1

$$\cosh x = \frac{e^x + e^{-x}}{2}$$
$$\sinh x = \frac{e^x - e^{-x}}{2}$$

2. **Real Repeated Roots** In this case $m_1 = m_2$. One solution is:

i.e. from the quadratic equation, $b^2 - 4ac = 0$

$$u_1(x) = e^{m_1 x} \tag{25}$$

The other solution so derived is, of course, the same and thus we do not have two linearly independent solutions as required to form a fundamental set of solutions for a 2nd-order linear homogeneous equation.

IT CAN BE SHOWN that a second linearly independent solution can be formed by multiplying by the independent variable:

This is done using a technique referred to as *reduction of order*. We will not take the time to cover it in this class (or in this book) but is concisely described in section 3.2 of Zill. At a minimum you might at least confirm for yourself that a) $xu_1(x)$ is a solution to the equation; and b) use the Wronskian to confirm that it is linearly independent from $u_1(x)$.

$$u_2(x) = xu_1(x) = xe^{mx}$$

and thus the general solution for this case is:

$$u(x) = c_1 e^{mx} + c_2 x e^{mx} \tag{26}$$

3. **Conjugate Complex Roots** In this case the discriminant, $b^2 - 4ac$, is negative so its square root is imaginary. This results in m_1 and m_2 being complex conjugates which we will express as: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

The general solution is:

$$\begin{aligned} u(x) &= c_1 e^{(\alpha+i\beta)x} + c_2 e^{(\alpha-i\beta)x} \\ &= e^{\alpha x}\left(c_1 e^{i\beta x} + c_2 e^{-i\beta x}\right) \end{aligned}$$

The complex exponentials in the last equation can be re-expressed using the Euler Formula:

$$\begin{aligned} e^{i\beta x} &= \cos\beta x + i\sin\beta x \\ e^{-i\beta x} &= \cos\beta x - i\sin\beta x \end{aligned}$$

which is slighly more convenient insofar as the solutions are no longer expressed as complex exponentials but also by breaking each solution down into their real and complex parts. It can be shown that both the real and imaginary parts of the solution must satisfy the differential equation *independently*. This fact allows us to re-express the solution in a more simple form that does not involve complex numbers:

$$u(x) = e^{\alpha x}\left(c_1\cos\beta x + c_2\sin\beta x\right) \tag{27}$$

Another important special case is when the solution is *pure imaginary* (i.e. $\alpha = 0$) so the solution is:

$$u(x) = c_1\cos\beta x + c_2\sin\beta x \tag{28}$$

These solutions arise when the governing equation is of the shown in Equation 29:

$$u'' + k^2 u = 0 \tag{29}$$

The roots $m_{1,2} = \pm ik$ and the general solution is:

$$u(x) = c_1\cos kx + c_2\sin kx \tag{30}$$

This equation will be revisited throughout the course as it repeatedly comes up in applications.

Three Examples

The cases described above will be illustrated with three examples:

Example #1: Find the general solution to $2u'' - 5u' - 3u = 0$. Inserting $u = e^{mx}$ into the equation gives us the auxiliary equation:

$$2m^2 - 5m - 3 = (2m+1)(m-3)$$

with roots: $m_1 = -\frac{1}{2}$ and $m_2 = 3$. These are real, distinct roots so the general solution is:

$$u(x) = c_1 e^{-x/2} + c_2 e^{3x}$$

Example #2: Find the general solution to $u'' - 10u' + 25u = 0$. The auxiliary equation is:

$$m^2 - 10m + 25 = (m-5)(m-5)$$

with (repeated) roots: $m_1 = 5$ and $m_2 = 5$. These are real, repated roots so the general solution is:

$$u(x) = c_1 e^{5x} + c_2 x e^{5x}$$

Example #3: Find the general solution to $4u'' + 4u' + 17u = 0, \quad u(0) = -1, \ u'(0) = 2$.

This is an initial value problem with continuous (and constant) coefficients. We know from Theorem 1 that a unique solution exists. We will first find the general solution, then apply the initial conditions to resolve the unknown coefficients to reveal the solution.

We can see that it must be an *initial* value problem because the conditions are both given at the same location, $x_0 = 0$.

The auxiliary equation is:

$$4m^2 + 4m + 17 = 0$$

using the quadratic equation, gives us:

$$\begin{aligned}\frac{-4 \pm \sqrt{16 - 4(4)(17)}}{2(4)} &= -\frac{1}{2} \pm \frac{\sqrt{-256}}{8} \\ &= -\frac{1}{2} \pm \frac{-16}{8} \\ &= -\frac{1}{2} \pm 2i\end{aligned}$$

This gives us complex conjugate roots and the general solution is:

$$u(x) = e^{-x/2}\left(c_1 \cos 2x + c_2 \sin 2x\right)$$

Applying the initial condition $u(0) = -1$ gives us:

$$\begin{aligned}u(0) &= e^0\left(c_1 \cos 0 + c_2 \sin 0\right) \\ &= 1(c_1(1) + c_2(0)) \\ &= c_1 = -1\end{aligned}$$

To apply the second initial condition we need to use the chain-rule and product rule to differentiate the general solution. This gives us:

$$\begin{aligned}u'(x) = -\frac{1}{2}e^{-x/2}c_1 \cos 2x - 2e^{-x/2}c_1 \sin 2x + \\ -\frac{1}{2}e^{-x/2}c_2 \sin 2x + 2e^{-x/2}c_2 \cos 2x\end{aligned}$$

Evaluating $u'(0)$ and substituting $c_1 = -1$ gives us:

$$\begin{aligned}
u'(0) &= -\frac{1}{2}(1)(-1)(1) + (1)(2)c_2(1) \\
&= \frac{1}{2} + 2c_2 = 2 \\
\Rightarrow 2c_2 &= \frac{3}{2} \\
c_2 &= \frac{3}{4}
\end{aligned}$$

Both constants are now known and the unique solution is:

$$u(x) = e^{-x/2}\left(-\cos 2x + \frac{3}{4}\sin 2x\right)$$

Lecture 5 - Nonhomogeneous Linear Equations with Constant Coefficients

Objectives

The objectives of this lecture are:

- Describe the Method of Undetermined Coefficients for solving nonhomogeneous linear equations with constant coefficients.
- Carry out some examples to illustrate the methods.

In this lecture we will review *a* method for finding solutions to nonhomogeneous linear equations with constant coefficients.

To be perfectly honest, we spend very little time in this class dealing with nonhomogeneous equations of any kind; many of those types of equations are beyond our ability to solve analytically so we turn to numerical methods instead. Nonetheless there is value in reminding ourselves how to construct solutions for those cases where we can.

Background

Consider the equation

$$a_n u^{(n)} + a_{n-1}u^{(n-1)} + \cdots + a_1 u' + a_0 u = g(x) \qquad (31)$$

where

- the coefficients $a_i,\ i \in [1, 2, \ldots, n]$ are constants; and
- the function $g(x)$ is a constant, a polynomial function, exponential function, sine or cosine, or finite sums or products of these functions.

The general solution, $u(x)$, can be constructed as $u_c(x) + u_p(x)$ where

- $u_c(x)$ is the complementary solution which, as you should recall, is the general solution to the associated homogeneous problem. (i.e. Equation 31 with $g(x) = 0$); and
- $u_p(x)$ is (any) particular solution—that is, a not-necessarily-unique function that satisfies Equation 31.

We spent the last lecture describing, effectively, how to find $u_c(x)$; the question this lecture will hope to answer is: "How do I find $u_p(x)$?"

Method of Undetermined Coefficients

One method for finding $u_p(x)$ is called the Method of Undetermined Coefficients.[7]

[7] Some people lovingly refer to this technique as "The Method of Guessing."

THERE ARE THREE parts to this technique

1. **Basic Rule:** based on the terms in $g(x)$, select the appropriate form for $u_p(x)$ using Table 1.

Term in $g(x)$	Choice for $u_p(x)$
$ke^{\gamma x}$	$Ae^{\gamma x}$
$kx^n, \ (n = 0, 1, \dots)$	$K_n x^n + K_{n-1}x^{n-1} + \cdots + K_1 x + K_0$
$k \cos \omega x$	$\left.\right\}$ $K \cos \omega x + M \sin \omega x$
$k \sin \omega x$	
$ke^{\alpha x} \cos \omega x$	$\left.\right\}$ $e^{\alpha x} \left(K \cos \omega x + M \sin \omega x \right)$
$ke^{\alpha x} \sin \omega x$	

Table 1: Forms of $u_p(x)$ for given terms in $g(x)$

2. **Modification rule:** if $u_p(x)$ obtained by the **Basic Rule** happens to be a solution to the associated homogeneous equation, multiply $u_p(x)$ from the table by x (or x^2 if needed).

3. **Sum rule:** if $g(x)$ is a linear combination of terms from the left-hand column, construct $u_p(x)$ from a linear combination of the corresponding entries in the right-hand column.

For the remainder of this lecture, we will practice applying these rules to some example problems.

Example: solve $u'' + 4u' - 2u = 2x^2 - 3x + 6$

Step #1: find the general solution to the associated homogeneous equation.

The auxiliary equation is: $m^2 + 4m - 2 = 0$; using the quadratic equation gives us:

Here you are expected to examine the associated homogeneous problem as $u'' + 4u' - 2u = 0$, identify it as constant coefficient and linear, and solve by assuming $u = e^{mx}$ and thus deriving the auxiliary equation shown without further prompting.

$$\begin{aligned} m &= \frac{-4 \pm \sqrt{16 - (4)(1)(-2)}}{2(1)} \\ &= -2 \pm \frac{\sqrt{24}}{2} \\ &= -2 \pm \sqrt{6} \end{aligned}$$

so $u_c(x) = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}$

Step #2: Apply the method of undetermined coefficients to construct a candidate $u_p(x)$

Since $g(x)$ is a second-order polynomial, the table tells us $u_p(x)$ is in the general form of a second-order polynomial.

$$u_p(x) = K_2x^2 + K_1x + K_0$$

We plug this into the governing equation and this gives us:

$$2K_2 + 4(2K_2x + K_1) - 2(K_2x^2 + K_1x + K_0) = 2x^2 - 3x + 6$$

Now we need to equate the coefficient for each power of x:

In general you cannot expect this to go so nicely. What you *can* hope for is that the, in this case, three equations you derive will have a unique solution. We could re-write the system in the form of a matrix-vector equation:

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & -2 & 8 \\ -2 & 4 & 2 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

If the solution of such a matrix cannot be done by inspection and simple algebra as it was in this case, we could use tools like MATLAB to solve the linear system of equations. This topic and much more is covered in the numerical methods portion of this text.

$$\begin{aligned} x^2&: & -2K_2 &= 2 \\ x&: & 8K_2 - 2K_1 &= -3 \\ 1&: & 2K_2 + 4K_1 - 2K_0 &= 6 \end{aligned}$$

Luckily for us, this system of equations is structured such that it can easily be solved. We see by inspection that $K_2 = 2/-2 = -1$; this can be plugged into th esecond equation to find $K_1 = -5/2$ and then we can solve the last equation to find that $K_0 = -9$.

Thus the particular solution is:

$$u_p(x) = -x^2 - \frac{5}{2}x - 9$$

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

We now have both the complementary solution and a particular solution; we form the general solution to the equation by adding them together.

$$\begin{aligned} u(x) &= u_c(x) + u_p(x) \\ &= c_1e^{(-2+\sqrt{6})x} + c_2e^{(-2-\sqrt{6})x} - x^2 - \frac{5}{2}x - 9 \end{aligned}$$

Why, again, do we need the constants c_1 and c_2?

Answer: Because we have not yet applied initial/boundary conditions. If those conditions are provided—two conditions for a 2nd-order problem—then we can resolve the constants.

Example: solve $u'' - 5u' + 4u = 8e^x$

Step #1: find the general solution to the associated homogeneous problem.

The auxiliary equation is $m^2 - m + 4 = 0$ the left side of which can easily be factored to give $(m-4)(m-1) = 0$; the roots of which are $m_1 = 4$, $m_2 = 1$. The comlementary solution is:

$$u_c(x) = c_1e^{4x} + c_2e^x$$

Step #2: Apply the method of undetermined coefficients to construct $u_p(x)$.

Inspecting Table 1 we see that $u_p(x)$ should be of the form Ae^x. If that function seems vaguely familiar it may be because e^x is part of the complementary solution.

Pop Quiz: if you plug Ae^x into your governing equation, without doing any calculations, what value should you get?

Answer: you will get 0! Why? Because e^x is one of the two linearly independent solutions to the associated homogeneous problem.

What do I do now?

Answer: invoke the Modification Rule—this is, after all, the reason why the rule exists—and multiply u_p by x. We now have $u_p(x) = Axe^x$.

We insert this proposed function for $u_p(x)$ into the equation and we get:

$$2Ae^x + Axe^x - 5(Ae^x + Axe^x) + 4Axe^x = 8e^x$$

Combine terms and solve for A:

$$\begin{aligned} 2Ae^x - 5Ae^x &= 8e^x \\ -3Ae^x &= 8e^x \\ A &= -\frac{8}{3} \end{aligned}$$

So the particular solution is:

$$u_p(x) = -\frac{8}{3}xe^x$$

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

$$\begin{aligned} u(x) &= u_c(x) + u_p(x) \\ &= c_1e^{4x} + c_2e^x - \frac{8}{3}xe^x \end{aligned}$$

Note: If any of this seems at all sketchy to you, the good news is that you need not worry if your proposed $u_p(x)$ is any good; you can just plug it into the differential equation and find out!

THIS LAST EXAMPLE illustrates the use of the Sum Rule; it also includes initial condition so the unique solution to the initial value problem can be found.

Example: solve the initial value problem: $u'' + u = 4x + 10\sin x$ with initial conditions $u(\pi) = 0,\ u'(\pi) = 2$.

Step #1: find the general solution to the associated homogeneous problem.

The auxiliary equation is: $m^2 + 1 = 0$, therefore $m = \pm i$ and $u_c(x)$ can be found as:

$$u_c(x) = c_1\cos x + c_2\sin x$$

Step #2: Apply the method of undetermined coefficients to construct $u_p(x)$.

For this problem, $g(x) = 4x + 10\sin x$ has two terms; so we will construct $u_p(x)$ using one term at a time; $u_{p_1}(x)$ using $4x$ and $u_{p_2}(x)$ using $10\sin x$

It's the linearity property of $\mathcal{L} = \frac{d^2}{dx} + 1$ that makes this possible. If $\mathcal{L}(u_{p_1}) = 4x$ and $\mathcal{L}(u_{p_2}) = 10\sin x$ then $\mathcal{L}(u_{p_1} + u_{p_2}) = 4x + 10\sin x$.

Step #2.a: find $u_{p_1}(x)$.

From Table 1, for $g(x) = 4x$ we should select $u_{p_1} = K_1 x + K_0$. Inserting this into the differential equation gives us: $K_1 x + K_0 = 4x$. By inspection we can see that $K_0 = 0$ and $K_1 = 4$ so $u_{p_1}(x) = 4x$.

Step #2.b: find $u_{p_2}(x)$.

From Table 1, for $g(x) = 10\sin x$ we should select $u_{p_2} = K\cos x + M\sin x$. Now that we have done this a couple of times we should be on the alert for portions of the complementary solution cropping up in our guesses for $u_p(x)$ so we immediately see that we must multiply u_{p_2} by x. If we do this and insert $Kx\cos x + Mx\sin x$ into the differential equation we get:

$$
\begin{aligned}
&(-2K - Mx)\sin x + (2M - Kx)\cos x + \dots \\
&\qquad\qquad\qquad\qquad Kx\cos x + Mx\sin x = 10\sin x
\end{aligned}
$$

Matching coefficients for $\sin x$ and $\cos x$ on both sides of the above equation leads us to conclude that $M = 0$ and $-2K = 10$. Therefore $K = -5$ and $u_{p_2}(x) = -5x\cos x$.

Again, there is no harm in testing your proposed $u_{p_2}(x)$ to see if it does indeed produce the expected result.

Step #3: construct the general solution: $u(x) = u_c(x) + u_p(x)$

$$
\begin{aligned}
u(x) &= u_c(x) + u_p(x) \\
&= u_c(x) + u_{p_1}(x) + u_{p_2}(x) \\
&= c_1\cos x + c_2\sin x + 4x - 5x\cos x
\end{aligned}
$$

All that remains is to apply the initial conditions.

$$
\begin{aligned}
u(\pi) &= c_1(-1) + c_2(0) + 4\pi - 5(\pi)(-1) \\
&= -c_1 + 9\pi = 0 \\
\Rightarrow &= c_1 = 9\pi
\end{aligned}
$$

Applying the initial condition $u' = 2$:

$$
u'(\pi) = -9\pi(0) + c_2(-1) + 4 - 5(-1) + 5\pi(0) = 2
$$

Solving for c_2 gives us: $c_2 = 7$; folding this into the general solution:

$$u(x) = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x$$

Assignment #2

The given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family (i.e. find the values for the constants c_1 and c_2) that is a solution of the initial-value problem.

1. $u = c_1 e^x + c_2 e^{-x}; \;\; u'' - u = 0, \; u(0) = 0, \; u'(0) = 1$

2. $u = c_1 x + c_2 x \ln x, \;\; (0, \infty), \; x^2 u'' - x u' = 0, \; u(1) = 3, \; u'(1) = -1$

The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine if a member of the family can be found that satisfies the boundary conditions.

3. $u = c_1 e^x \cos x + c_2 e^x \sin x; \;\; u'' - 2u' + 3u = 0$
   (a) $u(0) = 1, \; u'(\pi) = 0$
   (b) $u(0) = 1, \; u(\pi) = -1$
   (c) $u(0) = 1, \; u(\pi/2) = 1$
   (d) $u(0) = 0, \; u(\pi) = 0$

Determine if the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

4. $f_1(x) = x, \;\; f_2(x) = x^2, \;\; f_3(x) = 4x - 3x^2$

5. $f_1(x) = 1 + x, \;\; f_2(x) = x, \;\; f_3(x) = x^2$

Verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval.

6. $u'' - 7u' + 10u = 24e^x, \;\; u = c_1 e^{2x} + c_2 e^{5x} + 6e^x, \; (-\infty, \infty)$

Find the general solution to the given second-order differential equation.

7. $4u'' + u' = 0$

8. $u'' - u' - 6u = 0$

9. $u'' + 8u' + 16u = 0$

10. $u'' + 9u = 0$

Solve the given initial-value problem.

11. $u'' + 16u = 0, \;\; u(0) = 2, \; u'(0) = -2$

12. $u'' - 4u' - 5u = 0, \;\; u(1) = 0, \; u'(1) = 2$

13. $u'' + u = 0, \;\; u'(0) = 0, \; u'(\pi/2) = 0$

Solve the given differential equation using the Method of Undetermined Coefficients.

14. $u'' - 10u' + 25u = 30x + 3$

15. $u'' + 3u = -48x^2e^{3x}$

Solve the given initial-value problem.

16. $5u'' + u' - 6x, \;\; u(0) = 0, \; u'(0) = -10$

Solve the given boundary-value problem.

17. $u'' + u = x^2 + 1, \;\; u(0) = 5, \; u(1) = 0$

Solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint:* Solve the problem on two intervals and then find a solution so that u and u' are continuous at the boundary of the interval.]

18. $u'' + 4u = g(x), \;\; u(0) = 1, \; u'(0) = 2$

$$g(x) = \begin{cases} \sin x & 0 \le x \le \pi/2 \\ 0 & x > \pi/2 \end{cases}$$

Lecture 6 - Cauchy-Euler Equations

Objectives

The objectives of this lecture are:

- Introduce Cauchy-Euler equations and demonstrate a method of solution

- Carry out some examples to illustrate the methods for 2nd-order, homogeneous Cauchy-Euler equations.

Cauchy-Euler Equations

A linear differential equation of the form

$$a_n x^n \frac{d^n u}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + a_1 x \frac{du}{dx} + a_0 u = g(x) \tag{32}$$

is called a Cauch-Euler equation.

Note the relationship between the exponent of x is the coefficients and the order of the differential operators. This correspondence between the decreasing power of x in the coefficient and the decreasing order of the differential operator is characteristic of this type of equation and is the way you should recongize it.

It is the corresponding change in power/order that matters. $a\frac{d^2u}{dx^2} + \frac{1}{x^2}u = 0$ is also a Cauchy-Euler equation since the power of x in the coefficient goes from 0 to -2 while the order of the differential operator goes from 2nd to 0.

Note that this equation is *linear*; if $g(x) = 0$ it is homogeneous, otherwise it is nonhomogeneous. For this lecture we will focus our attention on the homogeneous, 2nd-order Cauchy-Euler equation:

$$ax^2 \frac{d^2u}{dx^2} + bx\frac{du}{dx} + cu = 0 \tag{33}$$

Note that the coefficient for the highest order derivative is 0 at $x = 0$; consequently we will restrict the interval of interest for these equations to $x \in (0, \infty)$.

THE BASIC STRATEGY in solving these equations is to try a solution in the form $u(x) = x^m$. When we substitute this solution into the equation we get:

$$am(m-1)x^2x^{m-2} + bmxx^{m-1} + cx^m = 0$$
$$x^m\left[am(m-1) + bm + c\right] = 0$$

That last part in the brackets is referred to as the "auxiliary equation":

$$am^2 + (b-a)m + c = 0 \tag{34}$$

We will look for values of m that satisfy this quadratic equation; that will be the exponent for our solution.

AS IS THE CASE for quadratic equation, there are three possible outcomes:

1. **Distinct Real Roots.** In this case $m_1 \neq m_2$ and the general soltuion is of the form

$$u(x) = c_1x^{m_1} + c_2x^{m_2} \tag{35}$$

Example: find the general solution for $x^2\frac{d^2u}{dx^2} - 2x\frac{du}{dx} - 4u = 0$

Referring to Equation 34, $a = 1,\ b = -2,\ c = -4$ so the auxiliary equation is:

$$m^2 - 3m - 4 = 0(m-4)(m+1) \qquad = 0$$

By inspection the roots are $m_1 = 4$ and $m_2 = -1$. The general solution is $u(x) = c_1x^4 + c_2x^{-1}$

Be careful with these coefficients; in contrast to the case with constant coefficient linear equations we do not plug these coefficients directly into the quadratic equation; we put them in the auxiliary equation and then solve *that* with the quadratic equation.

2. **Real Repeated Roots.** In this case, $m_1 = m_2$. We have one solution, $u_1(x) = c_1x^{m_1}$; clearly we need to take some kind of action if we hope to get another linearly independent solution. It can be shown that if we form the second solution by multiplying the first solution by $\ln x$—$u_2(x) = \ln(x)u_1(x)$—then $u_2(x)$ will satisfy the governing equation and also be linearly independent from $u_1(x)$.

The first one or two times you solve these problems, you should verify both of those assertions.

Example: find the general solution for $4x^2\frac{d^2u}{dx^2} + 8x\frac{du}{dx} + u = 0$.

The auxiliary equation in this case is: $4m^2 + 4m + 1 = 0$. This can be factored to give $(2m+1)^2 = 0$ so we have a case of repeated roots where $m_1 = m_2 = -\frac{1}{2}$.

The solution is: $u(x) = c_1x^{-1/2} + c_2x^{-1/2}\ln x$.

3. **Complex Conjugate Roots.** This case is completely analogous with the previous cases vis-à-vis linear constant coefficient equations. The roots are $m_{1,2} = \alpha \pm i\beta$ and the general solution is:

$$u(x) = x^{\alpha}\left[c_1 \cos\left(\beta \ln x\right) + c_2 \sin\left(\beta \ln x\right)\right] \tag{36}$$

Example: Solve: $4x^2u'' + 17u = 0, \;\; u(1) = -1, \; u'(1) = -1/2$

The auxiliary equation is $4m^2 - 4m + 17 = 0$. Using the quadratic formula the roots are found to be:

$$\begin{aligned} m_{1,2} &= \frac{4 \pm \sqrt{16 - 4(4)(17)}}{8} \\ &= \frac{1}{2} \pm \frac{\sqrt{-256}}{8} \\ &= \frac{1}{2} \pm \frac{16i}{8} \\ &= \underbrace{\frac{1}{2}}_{\alpha} \pm \underbrace{2}_{\beta} i \end{aligned}$$

So the general solution is:

$$u(x) = x^{1/2}\left[c_1 \cos\left(2 \ln x\right) + c_2 \sin\left(2 \ln x\right)\right]$$

We can apply the first boundary condition, $u(1) = -1$:

$$\begin{aligned} u(1) &= 1\left[c_1 \cos 0 + c_2 \sin 0\right] \\ &= c_1(1) + c_2(0) = -1 \\ \Rightarrow c_1 &= -1 \end{aligned}$$

The calculus is a bit more tedius for the second boundary condition:

$$\begin{aligned} u'(x) = -\frac{1}{2}x^{-1/2}\cos\left(2\ln x\right) + 2x^{-1/2}\sin\left(2\ln x\right) + ... \\ c_2\left[\frac{1}{2}x^{-1/2}\sin\left(2\ln x\right) + 2x^{-1/2}\cos\left(2\ln x\right)\right] \end{aligned}$$

Evaluating this at $x = 1$:

$$\begin{aligned} u'(1) &= -\frac{1}{2}(1)(1) + 2(1)(0) + c_2[0 + 2(1)(1)] \\ &= -\frac{1}{2} + 2c_2 = -\frac{1}{2} \\ \Rightarrow c_2 &= 0 \end{aligned}$$

So the solution is: $u(x) = -x^{1/2}\cos\left(2\ln x\right)$

Nonhomogeneous Cauchy-Euler Equations

Sadly, the Method of Undetermined Coefficients will not work with Cauchy-Euler equations; a limitation of that method is that the coefficients need to be constant. Interested students can investigate the method of Variation of Parameters that can be used to address this problem analytically. Otherwise, we will plan to use numerical methods to solve nonhomogeneous problems of this type.

Derivation of the Solution to Cauchy-Euler Equations

It would be hard not to notice the similarity in the solution methods of Cauchy-Euler equations and constant coefficient linear equations. This is not a coincidence. In this section I want to briefly show you that, through a change of variables, Cauchy-Euler equations are, in some sense, equivalent to constant coefficient equations.

Change of Independent Variable

What we will do, is change the independent variable from x to e^t.[8] If $x = e^t$, that means that $t = \ln x$ and $\frac{dt}{dx} = \frac{1}{x} = e^{-t}$.

If we consider, again, the 2nd-order Cauchy-Euler equation,

$$ax^2 \frac{d^2u}{dx^2} + bx\frac{du}{dx} + cu = 0$$

every appearance of x needs to be converted into its equivalent in terms of t and every derivative with respect to x needs to be converted into derivatives with respect to t.

It's easy enough to replace x with e^t; converting the derivatives takes a bit more work. We will use the chain rule as shown below:

$$\begin{aligned}\frac{du}{dx} &= \frac{du}{dt}\frac{dt}{dx} \\ &= u_t e^{-t}\end{aligned}$$

where we use the subscript notation to denote derivatives with respect to t and use the substitution $\frac{dt}{dx} = e^{-t}$ as determined above.

We do it again, to convert the second derivatives:

$$\begin{aligned}\frac{d^2u}{dx^2} &= \frac{d}{dx}\left(\frac{du}{dx}\right) \\ &= \frac{d}{dt}\left(\frac{du}{dx}\right)\frac{dt}{dx} \\ &= \frac{d}{dt}\left(u_t e^{-t}\right)e^{-t} \\ &= \left(u_{tt}e^{-t} - u_t e^{-t}\right)e^{-t} \\ &= e^{-2t}\left(u_{tt} - u_t\right)\end{aligned}$$

[8] Think of this as "streching" the x-axis.

We are now ready to make our substitutions into the differential equation:

$$a\underbrace{e^{2t}}_{x^2}\overbrace{e^{-2t}\left(u_{tt}-u_t\right)}^{\frac{d^2u}{dx^2}}+b\underbrace{e^{t}}_{x}\overbrace{u_te^{-t}}^{\frac{du}{dx}}+cu=0$$

Combining terms to simplify gives us Equation 37 which is now, under this change of variables, a 2nd-order linear constant coefficient equation.

$$au_{tt}+(b-a)u_t+cy=0 \tag{37}$$

If I solve this using our standard method, the resulting auxiliary equation is the same as what is shown in Equation 34.

In the case of constant coefficient linear equations, the solutions were of the form $u = e^{mx}$ which, according to the exponentiation rules, the same as $u = e^{x^m}$. But now, our independent variable is t, where $t = \ln x$. With this substitution:

$$\begin{aligned} u(t) &= e^{(\ln x)^m} \\ &= x^m \end{aligned}$$

which is the assumed form of solution for Cauchy-Euler equations.

Part II

Power Series Methods

Lecture 7 - Reveiw of Power Series

Objectives

The objectives of this lecture are:

- Review definitions and basic properties of power series.
- Illustrate important basic operations on power series

Introduction and Review

The methods that we have discussed so far have largely been a review of differential equations class. Sadly, even in the handful of lectures that we have had, our methods for solving equations are largely exhausted. We can solve constant coefficient linear equations, and variable coefficient linear equations *if* they happen to be Cauchy-Euler equations. We can solve many first-order linear equations but if the equation is nonlinear we are sunk unless they happen to be separable. This leaves out a lot of interesting equations. In this sequence of lectures we will discuss how to solve linear equations with variable coefficients (other than Cauchy-Euler equations). To do this we will need to use power series.

You learned about power series back in calculus class, but you weren't ready to use them for this imporant application. Now you are and now this is what we will do. We will begin this section with some definitions that will be needed as we describe the use power series in the solution of differential equations.

Definitions

Definition 3 (Sequence)
A sequence *is a list of numbers (or other mathematical objects, like functions) written in a definite order.*

$$\{c_0, c_1, c_2, c_3, \ldots, c_n\}$$

Definition 4 (Limit of a Sequence, convergence, divergence)
A sequence has a limit (L) *if we can make the terms c_n arbitrarily close to L by taking n sufficiently large. If* $\lim_{n\to\infty} c_n$ *exists, we say the sequence* converges*; otherwise, we say the sequence* diverges *or is* divergence*.*

There are various mathematical tools available for determining if an infinite sequence converges or diverges without needing to examine every element.

Definition 5 (Series, infinite series)
A series *is the sum of a sequence. For example, $S_0 = c_0$; $S_1 = c_0 + c_1$; $S_n = c_0 + c_1 + \cdots + c_n$. If the sequence is infinite, we call the sum an infinite series.*

Definition 6 (Series Convergence)
Given a series $\sum\limits_{n=0}^{\infty} s_i = s_1 + s_2 + \cdots + s_n + \cdots$, let s_n denote its n^{th} partial sum. If the sequence $\{s_n\}$ is convergent then the series is convergent to the same limit. Otherwise the series is divergent.

We will use notation such as $s_n \to \infty$ to indicate that the partial sum is unbounded.

Definition 7 (Power Series)
A series of the form $\sum\limits_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + \cdots$ is called a Power Series. The constant a is referred to as the "center" of the power series.

For almost all of the power series we will work with in this class, the series will be centered on $a = 0$ and will be denoted $\sum\limits_{n=0}^{\infty} c_n x^n$.

Definition 8 (Interval of Convergence, Radius of Convergence)
The set of all real numbers x for which the series converges. This interval can also be expressed as a radius of convergence. *(R); the series converges for all $a - R < x < a + R$.*

Ratio Test

We should have at least one test that we can use to decide whether or not a series, at least a power series, converges. The test we will use is called the *Ratio Test*; so named because it involves the ratio of the n^{th} and $(n+1)^{\text{th}}$ term in a power series. The Ratio Test is shown in Equation 38.

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{(n+1)}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L \tag{38}$$

The following cases are considered:

- if $L < 1$ then the series converges absolutely.
- if $L = 1$ then the test is inconclusive; some other test must be used; and
- if $L > 1$ then the series diverges.

Note: absolute convergence means that the series converges irrespective of the signs of each term. (i.e. whether or not all terms are positive, negative, or a mix of both positive and negative.)

Example: Find the radius of convergence and associated interval of convergence for the following power series:

1. $\sum\limits_{n=0}^{\infty} (-1)^n x^n$

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L < 1, \quad a = 0, \ c_n = (-1)^n$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| < 1$$

$$|x| \lim_{n\to\infty} |1| < 1$$

$$\Rightarrow |x| < 1$$

The radius of convergence $R = 1$ and the interval of convergence is $x \in (-1, 1)$.

Here I have purposely avoided analyzing the end-points to see if we could use a closed or partially-closed interval instead. Since we limited $L < 1$, that only gives the radius of absolute convergence. If we wanted to be picky, we could allow $L = 1$ and use some other test to determine if the series converges. If we did that in this case we would find that the series diverges at both endpoints.

2. $\sum\limits_{n=0}^{\infty} \frac{(-1)^n x^n}{n(n+1)}$

$$\lim_{n\to\infty} \left| \frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n} \right| = |x-a| \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right| = L < 1, \quad a = 0, \ c_n = \frac{(-1)^n}{n(n+1)}$$

$$\lim_{n\to\infty} \left| \frac{x^{n+1}}{(n+1)(n+1+1)} \frac{n(n+1)}{x^n} \right| < 1$$

$$|x| \lim_{n\to\infty} \left| \frac{n}{n+2} \right| < 1$$

$$\cancelto{1}{|x| \lim_{n\to\infty} \left| \frac{n}{n+2} \right|} < 1$$

$$|x| < 1$$

Once again, the radius of convergence $R = 1$ and the interval of convergence is $x \in (-1, 1)$.

3. $\sum\limits_{n=1}^{\infty} \frac{x^{2n}}{2^n n^2}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= L<1, \quad a=0,\ c_n=\frac{1}{2^n n^2}\\
\lim_{n\to\infty}\left|\frac{x^{2n+2}}{2^{n+1}(n+1)^2}\frac{2^n n^2}{x^{2n}}\right| &< 1\\
\lim_{n\to\infty}\left|\frac{x^2}{2}\frac{n^2}{(n+1)^2}\right| &< 1\\
\frac{|x^2|}{2}\cancelto{1}{\lim_{n\to\infty}\left|\frac{n^2}{(n+1)^2}\right|} &< 1\\
\left|x^2\right| &< 2\\
|x| &< \sqrt{2}
\end{aligned}$$

In this case the radius of convergence $R=\sqrt{2}$ and the interval of convergence is $x \in (-\sqrt{2}, \sqrt{2})$.

In this case, more detailed analysis shows that this series converges at both endpoints; a closed interval could be used instead.

4. $\sum\limits_{n=1}^{\infty} \frac{(x-2)^n}{3^n}$

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= L<1, \quad a=2,\ c_n=\frac{1}{3^n}\\
\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{3^{n+1}}\frac{3^n}{(x-2)^n}\right| &= L<1\\
\frac{|x-2|}{3}\lim_{n\to\infty}|1| &< 1\\
|x-2| &< 3
\end{aligned}$$

So for this example the radius of convergence is $R=3$ about the center at $x=2$; and the interval of convergence is $x \in (-1, 5)$

For the interested reader, it can be shown that this series is divergent at both endpoints so it should remain an open interval.

Properties of Convergent Series

Within the radius of convergence, a power series defines a function. Within the interval of convergence the function so defined is:

If x is not within the interval of convergence for a series or if the series is divergent then ***none*** of these are true. This is why it is important to be able to find the interval/radius of convergence.

- continuous
- differentiable (term-by-term); and
- integrable (term-by-term)

Definition 9 (Identity Property for a Power Series)
If $\sum_{n=0}^{\infty} c_n(x-a)^n = 0, \;\; R > 0,$ *for all numbers x in the interval of convergence then* $c_n = 0$ *for all n.*

Hopefully this definition seems obvious to you. You will find that most of what we do when using power series to solve homogeneous linear differential equations is carry out the necessary algebra to ensure that the coefficients for some series all are equal to zero.

Definition 10 (Analytic Function)
a function f is analytic *at a point a if it can be represented by a power series in* $x - a$ *with a positive radius of convergence.*

This is just a vocabulary term that you should know.

Some Common Power Series

You have probably had some exposure to power series in your previous mathematical courses. As a reminder, I've included the power series representations of some important/common functions.

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $\ln x = \frac{x-1}{x} + \frac{(x-1)^2}{2x^2} + \frac{(x-1)^3}{3x^3} + \cdots$
4. $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

Combining Power Series

This is a practical "utility skill" that you will need to master in order to be successful at this portion of the course. What we need to be able to do is combine multiple power series into a single expression.

For example, consider the two power series below that we want to write as a single power series:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1}$$

If I want to combine these series, I need to overcome two issues:

1. the powers of x in each term in both summations need to be "in phase" – that is the corresponding terms need to have the same power of x. The first term in the first summation is constant (x^0) while the first term in the second summation is linear (x^1); and
2. the first summation index starts at $n = 2$ while the second summation index starts at $n = 0$.

It is not only the first term that is important but if you can get the summations in phase for the first term, and if the power of x increases by one with each consecutive term, then if the first term is correct, they will all be correct.

We will address these issues one at a time, starting with the first one. We will leave the summation whose first term is highest order as-is; for all other summations (i.e. if there are more than two) we will "peel-off" any lower-order summation terms.

In this case that means we will "peel-off'; the constant term from the first summation:

$$\underbrace{(2)(1)c_2x^0}_{\text{constant term}} + \underbrace{\sum_{n=3}^{\infty}}_{\text{now } n=3} n(n-1)c_nx^{n-2} + \sum_{n=0}^{\infty} c_nx^{n+1}$$

Notice that now the summation index for the first summation starts at $n = 3$; this is because we've separated out the first term corresponding to $n = 2$. The two remaining summations are "in phase" since all of the terms now have the same power of x.

THE SECOND PROBLEM will be fixed by establishing a new common index, k, and re-write the existing indices (n for both summations) in terms of k. In each case we will set k equal to the exponent of x appearing in the summation.

- For the first summation—$\sum\limits_{n=3}^{\infty} n(n-1)c_nx^{n-2}$—we set $k = n-2$ because that is the exponent for x. We need to eliminate each occurance of n in the summation and replace it with it's equivalent expression in terms of k. From our definition of k for this summation, $n = k+2$. Our summation now can be written:

$$\sum_{k=1}^{\infty}(k+2)(k+2-1)c_{k+2}x^k$$

Everywere you see an n in the original summation, replace it with a $k+2$ and simplify.

- For the second summation—$\sum\limits_{n=0}^{\infty} c_nx^{n+1}$—we set $k = n+1$ because that is the exponent for x in this summation. This means $n = k-1$; substituting that expression in our summation gives us:

$$\sum_{k=1}^{\infty} c_{k-1}x^k$$

With these changes our original summation can be written:

Notice that I've made some obvious simplifications in the constant term and first summation.

$$2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} c_{k-1}x^k \tag{39}$$

The two summations are now ready to be joined into one as shown in Equation 40

$$2c_2 + \sum_{k=1}^{\infty}\left[(k+2)(k+1)c_{k+2} + c_{k-1}\right]x^k \tag{40}$$

Lecture 8 - Power Series Solutions at Ordinary Points

Objectives

The objectives of this lecture are:

- Introduce some definitions and concepts relevant for power series solutions of differential equations.
- Do some example problems.

Introduction

In this section we will restrict our attention to second-order, linear, homogeneous differential equations in standard form as shown in Equation 41.

$$u'' + P(x)u' + Q(x)u = 0 \tag{41}$$

Definition 11 (Ordinary Points and Singular Points)
A point x_0 is said to be an ordinary point *of a differential equation if both $P(x)$ and $Q(x)$ in the standard form are* analytic *at x_0. A point that is not an* ordinary point *is a* singular point*.*

Theorem 6 (Existence of Power Series Solutions)
If $x = x_0$ is an ordinary point *of the differential equation, we can always find two linearly independent solutions in the form of a power series centered at x_0. A series solution converges at least on some interval defined by $|x - x_0| < R$ where R is the distance from x_0 to the closest singular point.*

To clarify: this theorem applies only to second-order, linear, homogeneous differential equations.

The basic strategy we will use to find power series solutions for linear differential equations with variable coefficients where $P(x)$ and $Q(x)$ are analytic in the domain of interest is:

1. Find solutions in the form of a power series by substituting $u = \sum\limits_{n=0}^{\infty} c_n x^n$ into the differential equation.

2. Solve for the values of the coefficients by equating the coefficients on the left with those on the right (e.g. zero for homogeneous equations); and

3. The equations (often 2- or 3-term recurrence relations) for the series coefficents *defines* the function that is the solution of the differential equation.

Important: the series "solution" is only valid if the series so-derived has a non-zero radius of convergence.

Examples

It is not *required* that a problem have variable coefficients in order to use the Power Series method; only that $P(x)$ and $Q(x)$ are analytic on the domain of interest. Constants are always analytic over the entire real number line so you can always use the Power Series method on linear, constant-coefficient differential equations.

Example: Solve $u'' + u = 0$ using the power series method; compare with the known solution $u(x) = c_0 \cos x + c_1 \sin x$

IN ACCORDANCE WITH our strategy, we will assume that the solution is of the form: $u(x) = \sum_{n=0}^{\infty} c_n x^n$. This means that $u' = \sum_{n=1}^{\infty} n c_n x^{n-1}$ and $u'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$. Plugging this into our differential equation gives us:

Note that the $n = 0$ term in u' is omitted as is the $n = 0$ and $n = 1$ term in u''. These terms are zero due to having taken the first- and second-derivative on the constant (x^0) and linear (x^1) terms of the power series.

$$\begin{aligned} u'' + u &= 0 \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n &= 0 \end{aligned}$$

We need to combine the two summations. It is clear that the summation indexes start at different values but we are lucky in that the summations are already "in phase" since the first term in each summation is a constant (x^0) term.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} + \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{\substack{k=n \\ n=k}} = 0$$

For the first summation $k = n - 2$, so $n = k + 2$; we will use these defintions to re-write the first summation. For the second summation, $k = n$ so all we need to do for the second summation is replace all the n's with k's. The results of these substitutions and the combined summation are shown below:

$$\begin{aligned} \sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k + \sum_{k=0}^{\infty} c_k x^k &= 0 \\ \sum_{k=0}^{\infty} \underbrace{\left[(k+2)(k+1)c_{k+2} + c_k\right]}_{\text{coefficients for new power series}} x^k &= 0 \end{aligned}$$

THE EXPRESSION $[(k+2)(k+1)c_{k+2} + c_k]$ is now a formula for the coefficients of a new power series; this power series, according to the equation, is equal to *zero* so that means all of the coefficients must be equal to zero:

$$(k+2)(k+1)c_{k+2} + c_k = 0, \quad \text{for all } k \in [0, 2, 3, \dots]$$

This is called a *two-term recurrance* since the expression involves *two* terms; c_{k-2} and c_k.

By convention, we will re-write this recurrance relation to solve for the *higher*-index coefficients in terms of the *lower*-index coefficients. We do this in Equation 42

$$c_{k+2} = -\frac{c_k}{(k+2)(k+1)} \tag{42}$$

As we would expect the general solution for any other second order differential equation would have two unknown constants that can only be resolved by adding initial- or boundary-conditions.

As we should expect, there are two unknown constants in this general solution.The first value of k, from the summation in our solution is $k = 0$. Simplified equations for the first few coefficients are presented in the table below. Each cell in the table above is a formula

$k=0$	$k=2$	$k=4$
$c_2 = \frac{-c_0}{(1)(2)}$	$c_4 = \frac{-c_2}{(3)(4)} = \frac{c_0}{4!}$	$c_6 = \frac{-c_4}{(5)(6)} = \frac{-c_0}{6!}$
$k=1$	$k=3$	$k=5$
$c_3 = \frac{-c_1}{(2)(3)}$	$c_5 = \frac{-c_3}{(4)(5)} = \frac{c_1}{5!}$	$c_7 = \frac{-c_5}{(6)(7)} = \frac{-c_1}{7!}$

for the k^{th}-coefficient of our power series solution. Organizing this into a formula for our power series solution gives us:

Notice how the even-numbered coefficients are all dependent on c_0 and all of the odd-numbered coefficients are dependent on c_1.

$$\begin{aligned}
u(x) &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + c_7 x^7 + \cdots \\
u(x) &= c_0 \left(1 + \frac{c_2}{c_0}x^2 + \frac{c_4}{c_0}x^4 + \frac{c_6}{c_0}x^6 + \cdots\right) + c_1\left(x + \frac{c_3}{c_1}x^3 + \frac{c_5}{c_1}x^5 + \frac{c_7}{c_1}x^7 + \cdots\right) \\
u(x) &= c_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + c_1\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)
\end{aligned}$$

where in the last line we have substituted the formulas for coefficents c_2 through c_7 in terms of c_0 and c_1.

Recalling from the last lecture the power series representations of $\cos x$ and $\sin x$ and we should be able to see them again here.

In general you are not expected to, nor will you be able to, identify common functions from a power series solution. This is a special case.

$$\begin{aligned}
u(x) &= c_0 \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)}_{\cos x} + c_1 \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)}_{\sin x} \\
u(x) &= c_0 \cos x + c_1 \sin x
\end{aligned}$$

Which is exactly what we would have determined using our methods for constant coefficient linear equations.

Example: find the general solution to $u'' - xu = 0$

Notice first that while this equation is linear and homogeneous it is not constant-coefficient. It is also not a Cauchy-Euler equation. We will use the Power Series method to solve this problem. Assuming $u = \sum_{n=0}^{\infty} c_n x^n$ and inserting this into the governing equation gives us:

The equation is separable but, in this case, the solution is not so easy to obtain using that method either.

$$
\begin{aligned}
u'' - xu &= 0 \\
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - x\sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} &= 0
\end{aligned}
$$

We want to combine these summations and see that they are both "out of phase" as well as have different powers of x for their first terms.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{\text{for } n=2,\ \ x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\text{for } n=0,\ \ x^1} = 0$$

So we must separate out the first term in the first summation to get the summations in phase.

$$(2)(1)c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Next we must combine our indices using $k = n-2$ for the first summation and $k = n+1$ for the second summation respectively.

Reminder: we take our definition of k from the exponent for x in each summation term.

$$2c_2 x^0 + \underbrace{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2 \\ n=k+2}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{k=n+1 \\ n=k-1}} = 0$$

Doing this gives us:

$$2c_2 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_{k-1}\right] x^k = 0$$

Do not forget the minus sign in front of the second summation. It is easy to miss.

In order to solve the differential equation, the coefficient for every power of x needs to be zero. To do this:

$$\underbrace{2c_2}_{\Rightarrow c_2=0} + \sum_{k=1}^{\infty} [\underbrace{(k+2)(k+1)c_{k+2} - c_{k-1}}_{\text{must equal zero}}]x^k = 0$$

Our corresponding, two-term recurrence relation is:

Reminder: we should define our recurrence relation to give higher-order coefficients in terms of lower-order coefficients.

$$c_{k+2} = \frac{c_{k-1}}{(k+2)(k+1)}$$

$k=1$ $c_3 = \frac{c_0}{(3)(2)}$	$k=4$ $c_6 = \frac{c_3}{(6)(5)} = \frac{c_0}{(2)(3)(5)(6)}$
$k=2$ $c_4 = \frac{c_1}{(3)(4)}$	$k=5$ $c_7 = \frac{c_4}{(6)(7)} = \frac{c_1}{(3)(4)(6)(7)}$
$k=3$ $c_5 = \frac{c_2}{(5)(4)} = 0$	$k=6$ $c_8 = \frac{c_5}{(8)(7)} = 0$
$k=7$ $c_9 = \frac{c_6}{(9)(8)} = \frac{c_0}{(2)(3)(5)(6)(8)(9)}$	$k=8$ $c_{10} = \frac{c_7}{(10)(9)} = \frac{c_1}{(3)(4)(6)(7)(9)(10)}$

We expect two arbitrary constants, c_0 and c_1 and we know from the work above that $c_2 = 0$ so we will start solving for constants starting with $k=1$: Organizing the coefficients from the table into an equation we get:

$$
\begin{aligned}
u(x) &= c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + c_8x^8 + c_9x^9 + c_{10}x^{10} + \cdots \\
u(x) &= c_0\left(1 + \frac{c_3}{c_0}x^3 + \frac{c_6}{c_0}x^6 + \frac{c_9}{c_0}x^9 + \cdots\right) + c_1\left(x + \frac{c_4}{c_1}x^4 + \frac{c_7}{c_1}x^7 + \frac{c_{10}}{c_1}x^{10} + \cdots\right)
\end{aligned}
$$

which can be written:

$$
\begin{aligned}
u(x) = c_0\left(1 + \frac{x^3}{(2)(3)} + \frac{x^6}{(2)(3)(5)(6)} + \frac{x^9}{(2)(3)(5)(6)(8)(9)} + \cdots\right) + \\
c_1\left(x + \frac{x^4}{(3)(4)} + \frac{x^7}{(3)(4)(6)(7)} + \frac{x^{10}}{(3)(4)(6)(7)(9)(10)} + \cdots\right)
\end{aligned}
$$

The equation we solved is known as Airy's Equation. The power series solution is not pretty, but is a perfectly adequate representation of the function provided that we have the wherewithal to evaluate the function for a reasonable number of terms.

Lecture 9 Power Series Solutions with MATLAB

Objectives

The objectives of this lecture are:

- Illustrate the solution of a linear IVP (with a 3-term recurrence) using Power Series
- Demonstrate a way to analyze these solutions using MATLAB; and
- Demonstrate some expected elements of MATLAB style for this course

Solution of an IVP using Power Series

Consider the following IVP:

As before, we will assume $u = \sum_{n=0}^{\infty} c_n x^n$. This means that $u' = \sum_{n=1}^{\infty} n c_n x^{n-1}$, and $u'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}$.

$$\begin{aligned} \text{Governing Equation: } & u'' - (1+x)u = 0, \quad u \in [0,5] \\ \text{Initial Conditions: } & u(0) = 5, \quad u'(0) = 1 \end{aligned}$$

Inserting our assumed power series solution into the governing equation gives us:

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - (1+x)\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=0}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Note the effect of distributing $-(1-x)$ through the second summation.

We need to evaluate the order of x for the first term in each summation to determine if the summations are in phase:

$$\underbrace{\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}}_{x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^n}_{x^0} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{x^1} = 0$$

To get the three summations in phase we need to strip off the first terms in the first and second summations so that all three summations start at x^1. This gives us:

$$2c_2 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} - c_0 - \sum_{n=1}^{\infty} c_n x^n - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 - c_0 + \underbrace{\sum_{n=3}^{\infty} n(n-1)c_n x^{n-2}}_{\substack{k=n-2\\ n=k+2}} - \underbrace{\sum_{n=1}^{\infty} c_n x^n}_{\substack{k=n\\ n=k}} - \underbrace{\sum_{n=0}^{\infty} c_n x^{n+1}}_{\substack{k=n+1\\ n=k-1}} = 0$$

Substituting within each summation and combining the terms gives us:

$$(2c_2 - c_0)x^0 + \sum_{k=1}^{\infty} \left[(k+2)(k+1)c_{k+2} - c_k - c_{k-1}\right] x^k = 0$$

As usual, in order to satisfy this equation the coefficients for each power of x must be equal to zero. For x^0 this means $2c_2 - c_0 = 0$; For all the other powers of x, a *three-term recurrance* involving c_{k-1}, c_k, and c_{k+2} must be satisfied:

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+2)(k+1)}$$

We will help manage the complexity by adopting the following strategy:

- Case 1: Arbitrarily set $c_0 \neq 0$, set $c_1 = 0$ and derive a solution;
- Case 2: Arbitrarily set $c_0 = 0$, set $c_1 \neq 0$ and derive a second solution.

These two solutions are sure to be linearly independent since the first will not have a linear term (proportional to x) and the second equation will not have a constant term (proportional to 1).

Case 1: $c_0 \neq 0, \quad c_1 = 0$
Since $c_0 \neq 0$, we get $c_2 = \frac{c_0}{2}$. The coefficients derived for the first few values of k are shown in the table to the right.

The solution we thus derive is shown below.

$$u_1 = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots$$
$$u_1 = c_0 \left(1 + \frac{\cancelto{0}{c_1}}{c_0} x + \frac{c_2}{c_0} x^2 + \frac{c_3}{c_0} x^3 + \frac{c_4}{c_0} x^4 + \frac{c_5}{c_0} x^5 + \cdots\right)$$
$$u_1 = c_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \cdots\right)$$

Case 1:

$k = 1$	$k = 2$
$c_3 = \frac{c_0 + \cancelto{0}{c_1}}{(2)(3)} = \frac{c_0}{6}$	$c_4 = \frac{\cancelto{0}{c_1} + c_2}{(3)(4)} = \frac{c_2/2}{12} = \frac{c_0}{24}$
$k = 3$	
$c_5 = \frac{c_2 + c_3}{(4)(5)} = \frac{c_0/2 + c_0/6}{20} = \frac{c_0}{30}$	

Case 2: $c_0 = 0, \quad c_1 \neq 0$

Since $c_1 = 0$ and $c_2 = \frac{c_0}{2}$, $c_2 = 0$. The coefficients derived for the first few values of k are shown in the table.

Case 2:

$k = 1$	$k = 2$
$c_3 = \frac{\cancelto{0}{c_0} + c_1}{(2)(3)} = \frac{c_1}{6}$	$c_4 = \frac{c_1 + \cancelto{0}{c_2}}{(3)(4)} = \frac{c_1}{12}$
$k = 3$	
$c_5 = \frac{\cancelto{0}{c_2} + c_3}{(4)(5)} = \frac{c_1/6}{20} = \frac{c_1}{120}$	

The solution we thus derive is shown below:

$$
\begin{aligned}
u_2 &= \cancelto{0}{c_0} + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \cdots \\
u_2 &= c_1 \left(x + \frac{\cancelto{0}{c_2}}{c_1} x^2 + \frac{c_3}{c_1} x^3 + \frac{c_4}{c_1} x^4 + \frac{c_5}{c_1} x^5 \cdots \right) \\
u_2 &= c_1 \left(x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right)
\end{aligned}
$$

We now have two linearly independent solutions to the governing equation:

$$
\begin{aligned}
u(x) = u_1(x) + u_2(x) = c_0 \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \cdots \right) + \\
c_1 \left(x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right)
\end{aligned}
$$

We are now ready to apply the initial conditions.

$$
\begin{aligned}
u(0) &= c_0 = 5 \\
u'(0) &= c_1 = 1
\end{aligned}
$$

So the final solution is:

$$
\begin{aligned}
u(x) = 5 \left(1 + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{30} x^5 + \cdots \right) + \\
\left(x + \frac{1}{6} x^3 + \frac{1}{12} x^4 + \frac{1}{120} x^5 + \cdots \right)
\end{aligned}
$$

Part III

Back Matter

Bibliography

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Appendices

Matlab Style Rules

1. **rule:** All scripts will start with the commands: **clear**, **clc**, and **close 'all'**

   rationale: No script should depend upon any data visible in the MATLAB workspace when the script starts. By omitting these commands, residual data within the workspace may hide errors.

2. **rule:** Your code must be documented with enough details such that a reader unfamiliar with your work will know what you are doing.

   rationale: Code documentation is a habit. For more significant projects readers may need help in deciding what the author of the code intended. For your own code, the most likely reader is you—a few months into the future.

3. **rule:** Function and variable names must be meaningful and reasonable in length.

   rationale: Failing to do either make code harder to read and maintain.

4. **rule:** All outputs from the code <u>**must**</u> be meaningful; numbers should be formatted, part of a sentence, and include units. Graphs should be readable and axis labels should make sense and include units.

   rationale: Code output is a form of communication. It is important that this communication be clear and unambiguous.

5. **rule:** Do not leave warnings from the Code Analyzer unaddressed.

   rationale: Sometimes Code Analyzer warnings can be safely ignored. Most of the time the warning points to a stylistic error that would be unacceptable in software that you use. Occasionally these warnings are indicative of a hidden error.

6. **rule:** Use the "smart indentation tool" to format the indentation of your code.

rationale: This tool improves code readability. It will also occasionally point out errors that you did not see before.

7. **rule:** Pre-allocate arrays; if possible initialize with **NaN** values.

   rationale: Pre-allocation improves performance and helps readability. Initialization with **NaN** helps avoid a range of potential logical errors.

8. **rule:** Avoid "magic numbers" — i.e. hard-coded constants.

   rationale: Constants included in your code tend to hide your program logic. Also, "magic numbers" make code maintenance more difficult and error prone.

9. **rule:** Only write one statement per line.

   rationale: Multi-statement-lines hurt code readability in almost all cases.

10. **rule:** Do not write excessively long lines of code; use the line continuation "..." and indentation to spread long expressions over several lines.

    rationale: Following this rule improves code readability.

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