EM424 Analytical Methods

Lecture 31 – Polar Coordinates with Radial Symmetry

Objectives:

- Carry out separation of variables process to solve the wave equation in polar coordinates.
- Quantitative example with MATLAB

Wave Equation in Polar Coordinates

PDE: $$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right) = \frac{\partial^2 u}{\partial t^2}$$

Apply radial symmetry to simplify the PDE. (be careful in making this assumption) $\Rightarrow \frac{\partial}{\partial \theta}, \frac{\partial^2}{\partial \theta^2} = 0$

Example: Radial Vibrations of a Circular Membrane

PDE: $$a^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{\partial^2 u}{\partial t^2} \qquad 0 < r < c, \quad t > 0$$

BCs: $u(c,t) = 0, \quad t > 0$

ICs: $u(r,0) = f(r), \quad \left.\frac{\partial u}{\partial t}\right|_{t=0} = g(r) \qquad 0 < r < c$

Step 1 – Assume a product solution

$$u = RT$$

Step 2 – Insert into PDE

$$a^2\left(R''T + \frac{1}{r}R'T\right) = RT''$$

Step 3 – Separate Variables

$$\frac{\cancel{a^2}\left(R''\cancel{T} + \frac{1}{r}R'\cancel{T}\right)}{\cancel{a^2}R\cancel{T}} = \frac{\cancel{R}T''}{a^2\cancel{R}T} \Rightarrow \underbrace{\frac{R'' + \frac{1}{r}R'}{R}}_{r \text{ only}} = \underbrace{\frac{T''}{a^2T}}_{t \text{ only}} = \underbrace{-\lambda}_{\text{const}}$$

$$rR'' + R' + \lambda r R = 0$$

$$T'' + a^2\lambda T = 0$$

Take an analysis short cut by assuming: oscillatory solution. Therefore $\lambda = \alpha^2, \alpha > 0$

Radial Eigenfunction:

$$rR'' + R' + \alpha^2 r R = 0$$

mult by r

$$r^2 R'' + rR' + \alpha^2 r^2 R = 0 \quad \text{Parametric Bessel eqn.}$$

$$\nu = 0$$

General Solution:

$$R = c_1 J_0(\alpha r) + c_2 Y_0(\alpha r)$$

Boundary condition for $r = 0$:

$$\lim_{r \to 0} R(r) < \infty$$

recall: Y_0 diverges to $-\infty$ as $r \to 0^+$

$$\Rightarrow c_2 = 0$$

Boundary condition for $r = c$

$$R(c) = c_1 J_0(\alpha c) = 0$$

$\Rightarrow$ αc must be a root of J_0

Resulting eigenvalues and eigenfunctions:

$$\alpha_n = \frac{k_{0,n}}{c} \quad \text{where } k_{0,n} \text{ is } n^{th} \text{ root of } J_0$$

$$R_n = c_n J_0(\alpha_n r)$$

Solution for $T(t)$:

$$T(t) = c_3 \cos a\alpha t + c_4 \sin a\alpha_n t$$

$\Rightarrow$ ~~$T_n(t)$~~

Product solution:

$$\boxed{u(r,t) = RT = \sum_{n=1}^{\infty} \left(A_n \cos a\alpha_n t + B_n \sin a\alpha_n t\right) J_0(\alpha_n r)}$$

Apply initial position:

$$u(r,0) = \sum_{n=1}^{\infty} \left(A_n \cdot 1 + B_n \cdot 0\right) J_0(\alpha_n r) = \sum_{n=1}^{\infty} A_n J_0(\alpha_n r) = f(r)$$

mult both sides by $J_0(\alpha_n r)$ and weight function $p(r) = r$
and integrate from 0 to c

$$\int_0^c A_n \underbrace{J_0(\alpha_m r) J_0(\alpha_n r)}_{0 \text{ if } m \neq n} r\,dr = \int_0^c f(r) J_0(\alpha_n r) r\,dr$$

$$\Rightarrow \boxed{A_n = \int_0^c f(r) J_0(\alpha_n r) r\,dr \bigg/ \int_0^c J_0(\alpha_n r)^2 r\,dr}$$

Apply initial velocity:

$$\frac{du}{dt} = \sum_{n=1}^{\infty} \left[-a\alpha_n A_n \sin(a\alpha_n t) + a\alpha_n B_n \cos(a\alpha_n t)\right] J_0(\alpha_n r)$$

$$\left.\frac{du}{dt}\right|_{t=0} = \sum_{n=1}^{\infty} \left[-a\alpha_n A_n \cdot 0 + a\alpha_n B_n \cdot 1\right] J_0(\alpha_n r) = g(r)$$

$$\sum_{n=1}^{\infty} a\alpha_n B_n J_0(\alpha_n r) = g(r)$$ same deal as above...

$$\boxed{B_n = \frac{1}{a\alpha_n} \frac{\int_0^c g(r) J_0(\alpha_n r) r\,dr}{\int_0^c J_0(\alpha_n r)^2 r\,dr}}$$

See MATLAB examples