(Ask ... change coordinate system)

Analogous to the changes in independent variable that we did to transform eqns into Bessel's Eqn. in Chapter 5.

EM424 Analytical Methods

Lecture 30 – Laplacian in Polar Coordinates

Objectives:

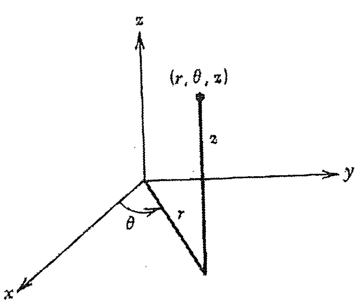
- Discuss derivation of the Laplacian Operator in polar coordinates
- Example solution of the Laplace Equation in polar coordinates

Laplacian in Polar Coordinates

$r^2 = x^2 + y^2$ $\theta = \tan^{-1}(y/x)$ $z = z$	or	$x = r\cos\theta$ $y = r\sin\theta$ $z = z$

{ Need to ~~ex~~ ~~get derivatives~~ express $u_{xx} + u_{yy}$ ~~but~~ in in new ~~ba~~ independent vars. }

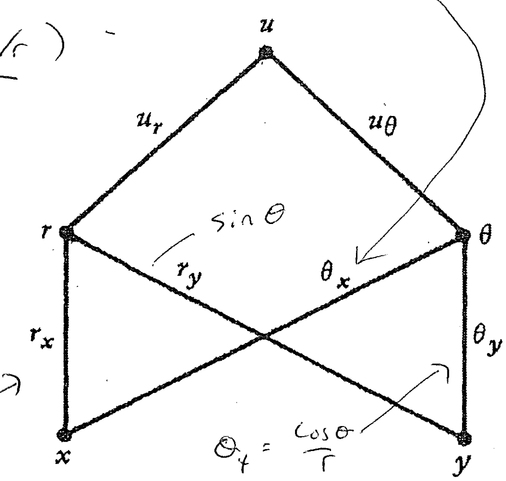
$$\theta_x = \frac{d}{dx}\tan^{-1}(y/x) = -\frac{y}{x^2}\cdot\frac{1}{1+y^2/x^2} = -\frac{y}{x^2+y^2} = \frac{-r\sin\theta}{r^2} = -\frac{\sin\theta}{r}$$



$$u_x = u_r r_x + u_\theta \theta_x$$
$$= u_r(\cos\theta) - u_\theta(\sin\theta/r)$$

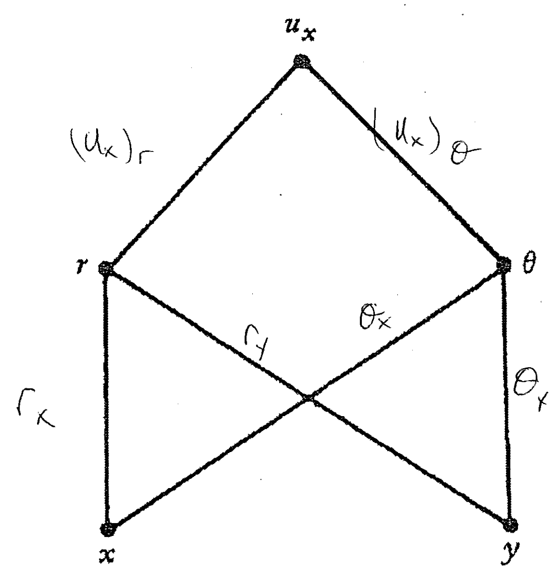
$$u_y = u_r r_y + u_\theta \theta_y$$
$$= u_r \sin\theta + u_\theta \frac{\cos\theta}{r}$$

$$r = (x^2+y^2)^{1/2}$$
$$r_x = \frac{dr}{dx} = \frac{1}{2}(x^2+y^2)^{-1/2}\,2x$$
$$= \frac{1}{\cancel{2}(r^2)^{1/2}}\cancel{2}r\cos\theta = \cos\theta$$



example

$$u_{xx} = (u_x)_x$$
$$= (u_x)_r r_x + (u_x)_\theta \theta_x$$
$$= (u_r\cos\theta - u_\theta \sin\theta/r)_r \cos\theta$$
$$- (u_r\cos\theta - u_\theta\sin\theta/r)_\theta \sin\theta/r$$



$$\boxed{\text{Same process for } u_{yy}}$$

{ Balance math tedium w/ conceptual understanding but, like many hazing rituals, deriving ∇^2 for polar coordinates has some redeeming beneficial effects. }

After much tedious work and simplification, you can show that:

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

Example – Steady-state Temperature in a Circular Plate

PDE: $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad 0 < r < c, \quad 0 < \theta < 2\pi$

BC:

$u(c,\theta) = f(\theta)$, $\left\{ \text{implicit: } \lim_{r\to 0} u(r,\theta) < \infty \quad u(r, \theta + 2\pi) = u(r,\theta) \text{ 2π-periodic} \right\}$

Step 1: assume a product solution

$u = R\Theta$ {use prime notation}

Step 2: insert into differential equation $r^2 R_{rr}\Theta + r R_r \Theta + R\Theta_{\theta\theta}$

Step 3: separate variables

$$\frac{r^2 R_{rr}\cancel{\Theta}}{R\cancel{\Theta}} + \frac{r R_r \cancel{\Theta}}{R\cancel{\Theta}} + \frac{\cancel{R}\Theta_{\theta\theta}}{\cancel{R}\Theta} = 0 \qquad \frac{r^2 R_{rr} + r R_r}{R} = -\frac{\Theta_{\theta\theta}}{\Theta} = \lambda$$

{for convenience sake}

$r^2 R'' + r R' - \lambda R = 0, \quad \Theta'' + \lambda\Theta$

Step 4: form product solutions depending on value of separation constant

$\lambda = 0$

$\Theta = c_1 + c_2\theta$ $\quad$ Θ must be 2-π periodic $\Rightarrow$ only true if $c_2 = 0$

$\Rightarrow \Theta = c_1$ for $\lambda = 0$

$r^2 R'' + r R' = 0$ Cauchy-Euler eqn $\quad r^m[m(m-1) + m] = 0$

$m^2 - m + m = 0 \Rightarrow m^2 = 0 \Rightarrow m = 0$

$R(r) = c_1 + c_2 \ln r$

$c_2 = 0$ for $\lim_{r\to 0} R(r) < \infty \Rightarrow R = c$

$c_1 \cdot c = A_0$ eigenfunction $= 1$

$\lambda = -\alpha^2 < 0$

$$\Theta'' - \alpha^2 \Theta = 0$$

$$\Theta(\theta) = c_1 \cosh \alpha\theta + c_2 \sinh \alpha\theta$$

Non-periodic for $c_1, c_2 \neq 0 \Rightarrow c_1 = c_2 = 0$

{ ⇒ No need to check R }

$\lambda = \alpha^2 > 0$

$$\Theta'' + \alpha^2 \Theta = 0$$

$\sin x$, $\cos x$ are 2π-periodic functions.

$0 < \theta < 2\pi$

$$\Theta(\theta) = c_1 \cos\alpha\theta + c_2 \sin\alpha\theta$$

periodic for $\alpha = n$

$$\Rightarrow \Theta_n = c_1 \cos n\theta + c_2 \sin n\theta$$ ← eigenfunctions. eigenvalues

$$r^2 R'' + r R' - n^2 R = 0 \qquad r^m[m(m-1) + m - n^2] = 0$$

$$m^2 - n = 0 \Rightarrow m = \pm n$$

$$R(r) = c_3 r^n + c_4 r^{-n} \qquad c_4 = 0 \text{ for } \lim_{r\to 0} R(r) < \infty$$

Product solution:

$$u(r,\theta) = \underbrace{A_0}_{\text{for } \lambda = 0} + \underbrace{\sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)}_{\text{for } \lambda = 1}$$

Use boundary condition $f(\theta)$ to determine unknown coefficients

$$\cancel{u(c,\theta) = f(\theta) + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta)}$$

$$u(c,\theta) = A_0 + \sum_{n=1}^{\infty} c^n (A_n \cos n\theta + B_n \sin n\theta) = f(\theta)$$

orth functions
1
$\cos\theta$
$\sin\theta$
$\cos 2\theta$
$\sin 2\theta$
$\cos 3\theta$
$\sin 3\theta$
⋮

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta)$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

$c^n \cancel{\frac{2}{4}}\ \pi$

$$A_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta, \quad B_n = \frac{1}{c^n \pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$$

See MATLAB for plot