03 – THE DOT PRODUCT

Contents

1. Definition

1.1. **Scalar (dot) product.** The **scalar** or **dot product** is a function $\cdot : F^n \times F^n \to F$. It takes two vectors and outputs an element of the field of scalars for the vector space. Given two vectors $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ it is defined as the sum product

$$\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^{n} u_i v_i.$$

1.2. **Examples.** Consider the vectors $\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$. Their dot product is then

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 2 \times (-3) + 1 \times 5 = -6 + 5 = -1.$$

We have a similar process for three-dimensional real vectors. Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$. Their dot product is

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = 1 \times 1 + 0 \times 0 + 1 \times (-1) = 1 + 0 + (-1) = 0.$$

1.2.1. *Exercise.* Show that $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$.

1.3. **Properties.** For vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ and scalars a and b, the dot product is

(1) Commutative: $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
(2) Distributive over vector addition: $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$.
(3) Compatible with scalar multiplication: $(a\mathbf{u}) \cdot (b\mathbf{v}) = ab(\mathbf{u} \cdot \mathbf{v})$.
(4) Not associative.

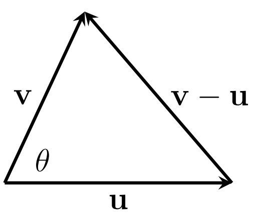
(5) Not cancellable: $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ does not imply that $\mathbf{v} = \mathbf{w}$.

1.3.1. *Exercise.* Prove the properties listed above.

2. Geometric interpretation

2.1. **Angle between vectors.** If we consider vectors in $\mathbb{R}^2$ or $\mathbb{R}^3$ it is easy for us to think about the angle between two vectors. What we're about to discuss however, works in any **Euclidean space** ($\mathbb{R}^n$). We're going to visualize it in two dimensions, but the proof holds in any Euclidean space.

Consider two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. They sit in the n-dimensional analogue of a plane. It is reasonable then to keep the picture below in our mind regardless of dimension.



2.1.1. *Exercise.* Use the Cosine rule to show that

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}| \cos\theta.$$

2.2. **Finding the angle between two vectors.** The result from the exercise above can be restated in terms of $\cos\theta$:

$$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}||\mathbf{v}|.}$$

Taking the arccosine of both sides then gives us a value for $\theta \in [0, \pi]$. This value is taken to be the angle between two vectors.

2.2.1. *Exercise.* Show that two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ are **orthogonal** (at $90°$ or $\frac{\pi}{2}$ radians) if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.