02 – VECTOR SPACES

Contents

1. Definition and examples

1.1. **Fields.** A **field** is a set F together with two binary operations: **addition** $+ : F \times F \to F$ and **multiplication** $\cdot : F \times F \to F$. Addition and multiplication are bound by the axioms listed below, for all $a, b, c \in F$.

(1) Addition and multiplication are associative: $(a + b) + c = a + (b + c)$, and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(2) Addition and multiplication are commutative: $a + b = b + a$, and $a \cdot b = b \cdot a$.
(3) Additive and multiplicative identities: there are two elements $0, 1 \in F$ such that $a + 0 = a$ and $a \cdot 1 = a$.
(4) Additive inverses: For every $a \in F$ there exists an element $(-a)$ such that $a + (-a) = 0$.
(5) Multiplicative inverses: For every $a \neq 0 \in F$ there exists an element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
(6) Distributivity of multiplication over addition: $a \cdot (b + c) = a \cdot b + a \cdot c$.

Hopefully, you have observed that these axioms are satisfied by the usual addition and multiplication on the rationals ($\mathbb{Q}$), the reals ($\mathbb{R}$) and the complex numbers ($\mathbb{C}$).

1.2. **Vector spaces.** A **vector space** over a field F is a set V together with two binary operations:

addition: $+ : V \times V \to V$, $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$, and
scalar multiplication: $\cdot : F \times V \to V$, $(a, \mathbf{v}) \mapsto a \cdot \mathbf{v}$.

The elements of V are called **vectors**, and the elements of F are called **scalars**.

The operations of addition and scalar multiplication satisfy the axioms listed below for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $a, b \in F$.

(1) Associativity of addition: $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$.
(2) Commutativity of addition: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

(3) Additive identity: There exists an element $\mathbf{0} \in V$ such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$.
(4) Additive inverses: For every $\mathbf{v}$ there exists an element $(-\mathbf{v}) \in V$ such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
(5) Compatibility of scalar multiplication with field multiplication: $a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$.
(6) Identity element of scalar multiplication: $1 \cdot \mathbf{v} = \mathbf{v}$, where $1 \in F$ is the multiplicative identity of F.
(7) Distributivity of scalar multiplication with respect to vector addition: $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$.
(8) Distributivity of scalar multiplication with respect to field addition: $(a + b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$.

The two and three-dimensional vectors with real components described in Lesson 01 form a vector space. They are examples of **coordinate spaces**. More generally, given a field F, all n-tuples

$$(a_1, a_2, \ldots, a_n)$$

where $a_1, a_2, \ldots, a_n \in F$ form a vector space over F, usually denoted by F^n. Addition and scalar multiplication are defined component-wise:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n),$$

and

$$k(a_1, a_2, \ldots a_n) = (ka_1, ka_2, \ldots, ka_n).$$