02 – VECTOR SPACES

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1. Definition and examples

1.1. **Fields.** A **field** is a set F together with two binary operations: **addition** $+ : F \times F \to F$ and **multiplication** $\cdot : F \times F \to F$. Addition and multiplication are bound by the axioms listed below, for all $a, b, c \in F$.

(1) Addition and multiplication are associative: $(a+b)+c = a+(b+c)$, and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(2) Addition and multiplication are commutative: $a+b = b+a$, and $a \cdot b = b \cdot a$.
(3) Additive and multiplicative identities: there are two elements $0, 1 \in F$ such that $a + 0 = a$ and $a \cdot 1 = a$.
(4) Additive inverses: For every $a \in F$ there exists an element $(-a)$ such that $a + (-a) = 0$.
(5) Multiplicative inverses: For every $a \neq 0 \in F$ there exists an element a^{-1} such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
(6) Distributivity of multiplication over addition: $a \cdot (b+c) = a \cdot b + a \cdot c$.

Hopefully, you have observed that these axioms are satisfied by the usual addition and multiplication on the rationals ($\mathbb{Q}$), the reals ($\mathbb{R}$) and the complex numbers ($\mathbb{C}$).

1.2. **Vector spaces.** A **vector space** over a field F is a set V together with two binary operations:

addition: $+ : V \times V \to V$, $(\mathbf{u}, \mathbf{v}) \mapsto \mathbf{u} + \mathbf{v}$, and
scalar multiplication: $\cdot : F \times V \to V$, $(a, \mathbf{v}) \mapsto a \cdot \mathbf{v}$.

The elements of V are called **vectors**, and the elements of F are called **scalars**.

The operations of addition and scalar multiplication satisfy the axioms listed below for all vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, and scalars $a, b \in F$.

(1) Associativity of addition: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
(2) Commutativity of addition: $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
(3) Additive identity: There exists an element $\mathbf{0} \in V$ such that $\mathbf{v}+\mathbf{0}=\mathbf{v}$.
(4) Additive inverses: For every $\mathbf{v}$ there exists an element $(-\mathbf{v}) \in V$ such that $\mathbf{v}+(-\mathbf{v})=\mathbf{0}$.
(5) Compatibility of scalar multiplication with field multiplication: $a \cdot (b \cdot \mathbf{v}) = (ab) \cdot \mathbf{v}$.
(6) Identity element of scalar multiplication: $1 \cdot \mathbf{v} = \mathbf{v}$, where $1 \in F$ is the multiplicative identity of F.
(7) Distributivity of scalar multiplication with respect to vector addition: $a \cdot (\mathbf{u}+\mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$.
(8) Distributivity of scalar multiplication with respect to field addition: $(a+b) \cdot \mathbf{v} = a \cdot \mathbf{v} + b \cdot \mathbf{v}$.

The two and three-dimensional vectors with real components described in Lesson 01 form a vector space. They are examples of **coordinate spaces**. More generally, given a field F, all n-tuples

$$(a_1, a_2, \ldots, a_n)$$

where $a_1, a_2, \ldots, a_n \in F$ form a vector space over F, usually denoted by F^n. Addition and scalar multiplication are defined component-wise:

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n),$$

and

$$k(a_1, a_2, \ldots a_n) = (ka_1, ka_2, \ldots, ka_n).$$

2. Linear dependence

2.1. **Linear combinations.** A **linear combination** of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ in a vector space V over a field F is a sum

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3,$$

where $k_1, k_2, k_3 \in F$.

For example, let $\mathbf{u} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, and $\mathbf{v} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. The vector $\begin{pmatrix} 1 \\ 17 \end{pmatrix}$ is the linear combination

$$2\mathbf{u} + 3\mathbf{v} = \begin{pmatrix} 4 \\ 8 \end{pmatrix} + \begin{pmatrix} -3 \\ 9 \end{pmatrix}.$$

2.2. **Linear dependence.** A set of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ is **linearly dependent** if one of the vectors can be written as a linear combination of the others, or equivalently, if there exist scalars $a_1, a_2, \ldots, a_k$, not all zero, such that

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}.$$

If no vector can be written as a linear combination of the others, or all the scalars in the expression above must be zero, then the vectors are said to be **linear independent**.

2.2.1. *Exercise.* Show the equivalence of the two statements describing linear dependence above.