Coincidence and disparity of fractal dimensions

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Dedicated to everyone who supported me along the way.

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"For a moment, nothing happened. Then, after a second or so, nothing continued to happen." — Douglas Adams, The Hitchhiker's Guide to the Galaxy.

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Publications and collaboration

The following five papers, three of which were collaborations, form the core around which this thesis is built.

- S. A. Burrell. On the dimension and measure of inhomogeneous attractors. *Real Anal. Exchange*, **44** (1), 199–216 (2019).

- S. A. Burrell and J. M. Fraser. The dimensions of inhomogeneous self-affine sets. *Ann. Acad. Sci. Fenn. Math.*, **45**, 313–324 (2020).

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- S. A. Burrell. Dimensions of fractional Brownian images. *Preprint* (2020).

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Abstract

We investigate the dimension and structure of four fractal families: inhomogeneous attractors, fractal projections, fractional Brownian images, and elliptical polynomial spirals. For each family, particular attention is given to the relationships between different notions of dimension. This may take the form of determining conditions for them to coincide, or, in the case they differ, calculating the spectrum of dimensions interpolating between them. Material for this thesis is drawn from the papers [6, 7, 8, 9, 10].

First, we develop the dimension theory of inhomogeneous attractors for non-linear and affine iterated function systems. In both cases, we find natural quantities that bound the upper box-counting dimension from above and identify sufficient conditions for these bounds to be obtained. Our work improves and unifies previous theorems on inhomogeneous self-affine carpets, while providing inhomogeneous analogues of Falconer's seminal results on homogeneous self-affine sets.

Second, we prove that the intermediate dimensions of the orthogonal projection of a Borel set $E \subset \mathbb{R}^n$ onto a linear subspace V are almost surely independent of the choice of subspace. Similar methods identify the almost sure value of the dimension of Borel sets under index-α fractional Brownian motion. Various applications are given, including a surprising result that relates the *box* dimension of the Hölder images of a set to the *Hausdorff* dimension of the preimages.

Finally, we investigate fractal aspects of elliptical polynomial spirals; that is, planar spirals with differing polynomial rates of decay in the two axis directions. We give a full dimensional analysis, computing explicitly their intermediate, box-counting and Assouad-type dimensions. Relying on this, we bound the Hölder regularity of maps that deform one spiral into another, generalising the 'winding problem' of when spirals are bi-Lipschitz equivalent to a line segment. A novel feature is the use of fractional Brownian motion and dimension profiles to bound the Hölder exponents.

Contents

Chapter 1

Foundations

1.1 Motivation

Natural forms often exhibit a complexity and detail that lies outside the scope of classical geometry. Imagine the the rugged outline of mountain landscapes, the self-similarity of branching trees, or the intricate structure of the central nervous system. Mathematically, we view shapes such as these and their abstract analogues in higher dimensions as subsets of Euclidean space.

Fractal geometry provides a framework for the rigorous study of such sets and gained momentum in the twentieth century due to the popular works of Mandelbrot [52, 53]. Its development as a mathematical field has been fuelled by numerous connections with other domains, such as dynamical systems, number theory and stochastic processes [17]. Across wider science, applications have been found in areas from financial modelling and computer graphics, to cosmology and the study of fluid turbulence [17, 27, 40].

Several prominent strands run through the literature on fractals, and those of particular relevance to this thesis include the theory of attractors and iterated function systems, their associated self-similar and self-affine sets, and projection. Common to the study of all is the concept of 'fractal dimension' that associates a positive number $d \geq 0$ with a set $F \subset \mathbb{R}^n$ and quantifies the irregularity of F at small scales. It is a natural way to classify fractals and a useful invariant when considering problems such as bi-Lipschitz equivalence. There is not, however, a unique definition of dimension and a variety of notions exist, each sensitive to different geometric properties. Consequently, two notions of dimension may take distinct values for complex sets.

Understanding the structural properties of sets that lead to disparities between dimensions often provides a feedback loop of information; we learn more about the sets in question, and, in certain circumstances, the dimensions themselves. These relationships are the unifying theme that runs throughout our study of four families of fractals: inhomogeneous attractors, fractal projections, fractional Brownian images, and elliptical polynomial spirals. We consider when dimensions coincide or probe the manner in which they differ via the emerging field of dimension interpolation. In the next section, we elaborate on these questions and provide a macroscopic overview of each chapter.

1.2 Overview

The remaining sections of this chapter introduce foundational material, such as formal definitions of fractal dimension and dimension interpolation. The start of each subsequent chapter begins with an introduction to that topic, surveying relevant literature and setting the scene with topic-specific definitions and notation. Throughout, we highlight related open questions and suggest potential lines of enquiry.

Chapter 2 is derived from the papers [6, 10] and considers the class of inhomogeneous attractors, introduced in 1985 independently by Barnsley [4] and Hata [41]. The dimension theory of these sets is well understood for iterated function systems containing similarities, and we develop a theory for nonlinear and affine systems. The central question explored in recent literature asks in what circumstances do the Hausdorff and box-counting dimensions coincide. To answer this question in the nonlinear case we introduce a quantity termed upper Lipschitz dimension that bounds the box-counting dimension from above. Further conditions determine when this upper bound is sharp and coincides with the Hausdorff dimension. In the affine case, we prove that the affinity dimension of Falconer [14] plays a similar role. This unifies previous results on inhomogeneous self-affine carpets, while providing inhomogeneous analogues of Falconer's seminal results on homogeneous self-affine sets.

Chapter 3 revisits classical theorems on the dimensions of projections and stochastic images for the intermediate dimensions that interpolate between the Hausdorff and box-counting dimensions. Theorems on projection have a long history, dating back to seminal work on the Hausdorff dimension of projections by Marstrand in 1954 [54]. This was extended to the box-counting dimensions by Falconer and Howroyd [23] through the introduction of 'dimension profiles', which in turn lead Xiao [68] to adapt the methodology to study the dimensions of fractional Brownian images, forming a link between the two topics.

We generalise these results by proving that the intermediate dimensions of orthogonal projections are almost surely independent of the choice of linear subspace. Then, following the tradition of Xiao, we show how similar methods identify the almost sure value of the intermediate dimensions of fractional Brownian images.

Our approach is based on a capacity theoretic formulation of dimension profiles, building on recent work of Falconer [18, 19] that re-examined the box-counting dimensions of projections and stochastic images using this methodology. By adapting the strategy of

Falconer for a new family of kernels, we show that the intermediate dimensions and their associated profiles may be defined in terms of capacities, a significant step towards our main results.

To conclude the chapter we consider a few applications. This includes bounds on the dimensions of exceptional sets and a surprising result that relates the *box-counting* dimensions of a Hölder image to the *Hausdorff* dimension of the preimage. Of course, this applies to projections and fractional Brownian images, yet more generally too.

In Chapter 4 we investigate fractal aspects of elliptical polynomial spirals; that is, planar spirals with differing polynomial rates of decay in the two axis directions. These generalise traditional polynomial spirals, as recently studied in [31]. We give a full dimensional analysis of these spirals, computing explicitly their intermediate, box-counting and Assouad-type dimensions, which turn out to be typically distinct. Together, these calculations provide a complete and continuous spectrum between the two extremes of the dimensional repertoire. An exciting feature is that these spirals exhibit two phase transitions within the Assouad spectrum, the first natural class of fractals known to have this property. The location of these phase transitions points to a surprising and subtle interaction between the two parameters controlling the rates of decay.

The final part of this chapter applies dimensional information to obtain bounds on the Hölder regularity of maps that deform one spiral into another, generalising the 'winding problem' of when spirals are bi-Lipschitz equivalent to a line segment. A novel feature is the use of fractional Brownian motion and dimension profiles to bound the Hölder exponents.

1.3 Fractal dimensions

In this section we lay the groundwork for later chapters by stating the formal definitions of various fractal dimensions. However, we expect a basic familiarity with fractal geometry, and direct the reader to the classic text [17] for a thorough introduction.

Throughout, let $F \subset \mathbb{R}^n$ be bounded and non-empty. The Hausdorff dimension of F may be defined in terms of Hausdorff measure in a natural way. For $0 \leq s \leq n$ and $\delta > 0$, the s-dimensional δ-approximate Hausdorff measure of F is

$$\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s : F \subseteq \bigcup_{i=1}^{\infty} U_i, 0 < |U_i| < \delta \right\},$$

where $|U|$ denotes the diameter of a set $U \subset \mathbb{R}^n$, and

$$\mathcal{H}^s(F) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$$

is the s-dimensional Hausdorff measure. Then, the *Hausdorff dimension* of F, denoted $\dim_{\mathrm{H}} F$, may be expressed as

$$\dim_{\mathrm{H}} F = \inf\{s : \mathcal{H}^s(F) = 0\} = \sup\{s : \mathcal{H}^s(F) = \infty\}.$$

In other words, the Hausdorff dimension of a set is the critical value of s at which a phase transition occurs in the s-dimensional Hausdorff measure.

A coarser notion of dimension, known as the *box-counting* dimension is also common in the literature on fractals. Comparatively simplistic in nature, the box-counting dimension is derived from the growth rate of the size of covers of F as the diameter of the covering sets tends to 0.

Formally, the *upper* and *lower* box-counting dimensions are defined as

$$\overline{\dim}_{\mathrm{B}} F = \limsup_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \tag{1.3.1}$$

and

$$\underline{\dim}_{\mathrm{B}} F = \liminf_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}, \tag{1.3.2}$$

respectively, where $N_\delta(F)$ denotes the minimum cardinality of a cover of F by hypercubes of diameter δ. Equivalently, $N_\delta(F)$ may defined in terms of covers by hypercubes of sidelength δ or balls of diameter δ. We use these different formulations interchangeably depending on which is most convenient in a given context. If (1.3.1) and (1.3.2) coincide we say the set has box-counting dimension equal to the common value and denote this by $\dim_{\mathrm{B}} F$. Such a definition applies equally well in the setting of general metric spaces (X, d) that we will meet in Chapter 2.

While the Hausdorff and box-counting dimensions describe the *average* local irregularity of a set, it may also be desirable to quantify the *extremal* irregularity. This is done by the more obscure Assouad-type dimensions that have been gaining popularity in recent years, see [33] for an overview. The *Assouad dimension* of F, denoted $\dim_{\mathrm{A}} F$, is defined as

$$\dim_{\mathrm{A}} F = \inf \big\{ \alpha \geq 0 : \ \exists\, C > 0 \text{ such that, } \forall\ 0 < r < R < 1 \text{ and } x \in F, \\ N_r\big(B(x,R) \cap F\big) \ \leq \ C(R/r)^\alpha \ \big\},$$

where $B(x, R)$ denotes the ball centred at x of radius R. This notion of dimension has seen a wide array of applications in fields such as embedding theory, number theory, probability and functional analysis [33]. We shall see the Assouad dimension feature in our work on elliptical polynomial spirals, where we compute it by way of the Assouad spectrum, a form of dimension interpolation.

1.4 Dimension interpolation

An emerging new perspective within dimension theory seeks to *interpolate* between dimensions [32]. Rather than viewing the existing notions of dimension as discrete entities, we embed them within a unifying framework.

Suppose you are given two notions of dimension $\dim_X$ and $\dim_Y$ with $\dim_X F \leq \dim_Y F$ for all $F \subset \mathbb{R}^n$. An interpolation between $\dim_X$ and $\dim_Y$ is a continuum of dimensions, parametrised by $\theta \in [0,1]$ and denoted $\dim_\theta$, such that

$$\dim_X F = \dim_0 F \leq \dim_\theta F \leq \dim_1 F = \dim_Y F$$

for all $F \subset \mathbb{R}^n$. Dimension interpolation provides finer geometric information than dimension alone, such as increasing discriminatory power when studying the bi-Lipschitz equivalence of two sets.

It is immediate from the definitions that for bounded $F \subset \mathbb{R}^n$

$$\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \dim_{\mathrm{A}} F,$$

and this ordering gives rise to the two interpolations we consider. The first inequality gives rise to the intermediate dimensions, and the second to the Assouad spectrum.

Intermediate dimensions were introduced by Falconer, Fraser and Kempton in [21] to interpolate between the Hausdorff and box-counting dimensions. The lower and upper intermediate dimensions of a set $F \subset \mathbb{R}^n$ are denoted $\underline{\dim}_\theta F$ and $\overline{\dim}_\theta F$, respectively. Like other notions of dimension they may be defined using covers, and the parameter $\theta \in [0,1]$ plays of the role of determining which covers are permissible. Through this dependence on θ, they reflect the range of diameters of sets needed to construct efficient covers at different scales.

Formally, for bounded $F \subset \mathbb{R}^n$ and $0 < \theta \leq 1$, the *lower intermediate dimension* of F may be defined as

$$\begin{aligned}\underline{\dim}_{\,\theta} F = \inf \big\{ s \geq 0 :\ &\text{for all } \epsilon > 0 \text{ and all } r_0 > 0, \text{ there exists} \\ &0 < r \leq r_0 \text{ and a cover } \{U_i\} \text{ of } F \text{ such that} \\ &r^{1/\theta} \leq |U_i| \leq r \text{ and } \textstyle\sum |U_i|^s \leq \epsilon \big\},\end{aligned} \tag{1.4.1}$$

and the corresponding *upper intermediate dimension* by

$$\begin{aligned}\overline{\dim}_{\,\theta} F = \inf \big\{ s \geq 0 :\ &\text{for all } \epsilon > 0, \text{ there exists } r_0 > 0 \text{ such that} \\ &\text{for all } 0 < r \leq r_0, \text{ there is a cover } \{U_i\} \text{ of } F \\ &\text{such that } r^{1/\theta} \leq |U_i| \leq r \text{ and } \textstyle\sum |U_i|^s \leq \epsilon \big\}.\end{aligned} \tag{1.4.2}$$

When $\theta = 0$ we take (1.4.1) and (1.4.2) with no lower bounds on the diameters of covering sets, recovering the Hausdorff dimension in both cases. If (1.4.1) and (1.4.2) coincide we say the set has θ-intermediate dimension equal to the common value and denote this by $\dim_\theta F$. When $\theta = 1$ all covering sets are forced to have the same diameter and we recover the lower and upper box-counting dimensions, respectively. Note that, for $0 < \theta \leq 1$, it is often convenient to use an equivalent definition based on the restriction $r \leq |U| \leq r^\theta$.

Various properties of intermediate dimensions are established in [21]. In particular $\underline{\dim}_{\,\theta} F$ and $\overline{\dim}_{\,\theta} F$ are monotonically increasing in $\theta \in [0, 1]$, are continuous except perhaps at $\theta = 0$, and are invariant under bi-Lipschitz mappings. Intermediate dimensions are of interest for sets which have differing Hausdorff and box-counting dimensions, such as sequence sets of the form $\{0\} \cup \{n^{-p} : n = 1, 2, \ldots\}$ for $p > 0$, self-affine carpets, and many other examples. Since their initial development, they have seen further attention in a variety of contexts, see [2, 7, 8, 9, 20, 51, 65].

Alongside the intermediate dimensions, we also consider the Assouad spectrum in Chapter 4, a family of dimensions indexed by $\theta \in [0,1)$ and introduced in [39]. The limit of the Assouad spectrum as $\theta \to 1$ is known as the *quasi-Assouad* dimension and often coincides with the Assouad dimension. In such instances, the Assouad spectrum may be thought of as providing a genuine interpolation between the upper box-counting and Assouad dimensions. This is proven to be the case for elliptical polynomial spirals in Chapter 4, Theorem 4.2.7.

Formally, the *Assouad spectrum* is the function $\theta \mapsto \dim_{\mathrm{A}}^{\theta} F$ defined by

$$\begin{aligned} \dim_{\mathrm{A}}^{\theta} F = \inf \big\{ \alpha \geq 0 : \ \exists\, C > 0 \text{ such that, for all } 0 < r < 1 \text{ and } x \in F, \\ N_r\big(B(x, r^{\theta}) \cap F\big) \ \leq \ C(r^{\theta}/r)^{\alpha} \ \big\}. \end{aligned} \tag{1.4.3}$$

One of the key motivations for this definition is that, in contrast to the Assouad dimension, an explicit formula in terms of θ provides information on which set of scales $0 < r < R$ witness the maximum exponential growth rate of $N_r(B(x,R) \cap F)$. For a thorough treatment of its properties and applications, we direct the reader to [33, 39].

Chapter 2

Inhomogeneous attractors

2.1 Introduction

Let (X, d) be a compact metric space. A map $S : X \to X$ is a *contraction* on X if there exists a $c \in (0, 1)$ such that

$$d(S(x), S(y)) \leq cd(x, y)$$

for all $x, y \in X$, and a *similarity* with ratio c if $\leq$ may be replaced with $=$. We call a finite collection $\mathbb{I} = \{S_i\}_{i=1}^N$ of contractions on X an *iterated function system* (IFS). In practice, further conditions are often put on the maps S_i to establish various subcategories of IFS.

In this chapter, we deal with two varieties of IFS. In Section 2.2 we consider general bi-Lipschitz mappings, that is, those contractions for which there also exists a $c' > 0$ with

$$d(S(x), S(y)) \geq c'd(x, y)$$

for all $x, y \in X$. In Section 2.3, we consider systems that contain affine mappings, a family of IFS that has received significant attention since seminal work of Bedford, McMullen and Falconer in the 1980s [5, 14, 15, 59]. When dealing with affine systems we specialise to a compact subset of the metric space $\mathbb{R}^n$ equipped with the Euclidean norm. Recall that a map $S : \mathbb{R}^n \to \mathbb{R}^n$ is *affine* if it can be written

$$S(x) = Ax + b$$

for some $A \in \mathrm{GL}(\mathbb{R}, n)$ and translation vector $b \in \mathbb{R}^n$. Bedford and McMullen pioneered a grid based approach resulting in *affine carpets*, while we follow the tradition of Falconer and consider systems containing generic affine maps.

The connection to fractal geometry comes from the study of sets that are in some sense invariant under an IFS and often exhibit fine local geometry. The existence of such sets follows from a classic application of Banach's contraction mapping theorem, which shows there is a unique non-empty compact set F, called a *homogeneous* attractor, such that

$$F = \bigcup_{i=1}^{N} S_i(F).$$

Our focus is a related family of fractals known as inhomogeneous attractors that were introduced independently by Barnsley [4] and Hata [41] in 1985. If we fix a compact set $C \subseteq X$, then there exists a unique non-empty compact set F_C such that

$$F_C = \bigcup_{i=1}^{N} S_i(F_C) \cup C.$$

F_C is called an *inhomogeneous* attractor with *condensation set* C [4]. In the affine setting, we say F_C is an inhomogeneous *self-affine* set.

It is possible to express F_C in a more explicit way with some symbolic notation. Hereafter, let $\mathbb{I} = \{S_i\}_{i=1}^{N}$ denote an IFS and $\mathcal{I} = \{1, \dots, N\}$. We write $S_{\mathbf{i}} = S_{i_1} \circ \cdots \circ S_{i_k}$ for

$\mathbf{i} = (i_1, i_2, \ldots, i_k) \in \mathcal{I}^k$. Furthermore, let

$$\mathcal{I}^* = \bigcup_{k=1}^{\infty} \mathcal{I}^k$$

denote the set of finite words over $\mathcal{I}$. An elegant formula for F_C, seen in [4, 61], is

$$F_C = F_\emptyset \cup \mathcal{O},$$

where $F_\emptyset$ is the homogeneous attractor corresponding to $C = \emptyset$, and $\mathcal{O}$ is the *orbital set* defined by

$$\mathcal{O} = C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C).$$

Intuitively, $\mathcal{O}$ is the union of all images of C built via composition of maps from $\mathbb{I}$.

Since their introduction in 1985, inhomogeneous attractors have received further attention in, for example, [1, 6, 29, 30, 46, 61, 64]. A natural question explored in recent work concerns the relationship between the dimensions of F_C, C and $F_\emptyset$. In particular, one may wonder in what situations

$$\dim F_C = \max\left\{\dim F_\emptyset, \dim C\right\}, \tag{2.1.1}$$

where dim denotes some notion of dimension. For dimensions satisfying countable stability, such as the Hausdorff or packing dimensions, this is immediate. Consequently, the recent focus has been on if or when the box-counting dimension, a popular example of a dimension that is not countably stable, satisfies (2.1.1) and so coincides with the Hausdorff dimension. In the case of lower box-counting dimension, (2.1.1) fails to hold generally even for self-similar systems satisfying the strong separation condition [29]. Thus, subsequent works have focussed solely on the upper box-counting dimension.

In [1, 29, 61, 64], (2.1.1) is proven to hold in various situations for the upper box-counting dimension in the case when $\mathbb{I}$ consists of similarity mappings. However, (2.1.1) may still fail for self-similar sets with overlaps [1] and specific self-affine settings [30]. In the nonlinear setting, we provide bounds on $\overline{\dim}_{\mathrm{B}} F_C$ for systems containing arbitrary bi-Lipschitz maps in Section 2.2. Corollaries of this result establish (2.1.1) for some low-dimensional affine systems and those satisfying bounded distortion, such as conformal systems (see [24] for definitions). We then consider affine systems of arbitrary dimension in Section 2.3.

The typical strategy used to approach (2.1.1), introduced in [29], is to establish bounds of the form

$$\max\left\{\overline{\dim}_{\mathrm{B}} F_{\emptyset}, \overline{\dim}_{\mathrm{B}} C\right\} \leq \overline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}, \tag{2.1.2}$$

where $s \in \mathbb{R}$ is a natural estimate for $\overline{\dim}_{\mathrm{B}} F_{\emptyset}$, such as similarity dimension in the self-similar case [29]. This exploits existing literature on the equality of s and $\overline{\dim}_{\mathrm{B}} F_{\emptyset}$, which may then determine precise conditions for equality depending on context. In the first of our two settings, we introduce a quantity that arises in various forms throughout the literature to serve as s, which we call upper Lipschitz dimension. For affine systems, one may suspect the affinity dimension of Falconer [14] is a sensible choice of s, and we prove this to be the case in Theorem 2.3.4. Since the affinity dimension springs up in both the non-linear and affine sections, we conclude this section with its definition.

The affinity dimension is derived from Falconer's singular value function that was introduced in [14]. The singular values of $A \in \mathrm{GL}(\mathbb{R}, n)$ are written $\alpha_j(A)$ (or simply α_j) and correspond to the lengths of the mutually perpendicular principal axes of $A(B)$, where B denotes a ball of unit diameter in $\mathbb{R}^n$ [14]. Alternatively, they are the positive square roots of the eigenvalues of AA^T. We adopt the convention $1 > \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n > 0$.

For $0 \leq s \leq n$, the singular value function of $A \in \mathrm{GL}(\mathbb{R}, n)$ is given by

$$\phi^s(A) = \alpha_1(A)\alpha_2(A)\cdots\alpha_m(A)^{s-m+1},$$

where $m \in \mathbb{Z}$ satisfies $m - 1 < s \leq m$. As in [14], we define $\phi^s(A) = (\det A)^{s/n}$ for $s > n$ and set $\phi^s(S) = \phi^s(A)$, where A is the linear component of an affine map S.

Then, for each $k \in \mathbb{N}$, define s_k to be the solution of

$$\sum_{\mathbf{i} \in \mathcal{I}^k} \phi^{s_k}(S_{\mathbf{i}}) = 1,$$

for which the corresponding limit

$$s := \lim_{k \to \infty} s_k$$

exists and is known as the *affinity dimension* associated with $\mathbb{I}$.

2.2 Nonlinear iterated function systems

Throughout this section, let $\mathbb{I} = \{S_i\}_{i=1}^N$ be an IFS consisting of bi-Lipschitz maps and $C \subseteq X$ be compact. To obtain (2.1.2) for general classes of maps we first construct an appropriate analogue of similarity dimension. For $S : X \to X$, let

$$\mathrm{Lip}^+(S) = \sup_{\substack{x,y \in X \\ x \neq y}} \frac{d(S(x), S(y))}{d(x, y)}$$

and

$$\mathrm{Lip}^-(S) = \inf_{\substack{x,y \in X \\ x \neq y}} \frac{d(S(x), S(y))}{d(x, y)}$$

denote the upper and lower Lipschitz constants respectively. Since S is a bi-Lipschitz contraction, recall that $0 < \mathrm{Lip}^-(S) \leq \mathrm{Lip}^+(S) < 1$.

We proceed in a similar way as for the affinity dimension, but instead define s_k to be the solution of

$$\sum_{\mathbf{i}\in\mathcal{I}_k} \mathrm{Lip}^+(S_{\mathbf{i}})^{s_k} = 1 \tag{2.2.1}$$

for each $k \in \mathbb{N}$. We then call the corresponding limit,

$$s = \lim_{k\to\infty} s_k,$$

the *upper Lipschitz dimension.* A similar, but not identical, construction may be found in work of Edgar and Golds [12]. The existence of this limit follows by considering the pressure function $P(t) = \lim_{k\to\infty} P_k(t)$ where

$$P_k(t) = \frac{1}{k} \log \sum_{\mathbf{i}\in\mathcal{I}^k} \mathrm{Lip}^+(S_{\mathbf{i}})^t.$$

Subadditivity and Fekete's lemma imply $P(t)$ exists for all $t \geq 0$. Moreover, it is well known P is continuous, monotonically decreasing and has a unique zero. Since $P_k \to P$ pointwise, it follows that the upper Lipschitz dimension exists and is equal to the zero of P. For further details on pressure functions and techniques from thermodynamic formalism we direct the reader to [16, Chapter 5] and the references therein.

2.2.1 Dimension

Our main result of this section establishes bounds on the upper box-counting dimension of F_C for general IFSs consisting of bi-Lipschitz contractions. The methodology of Fraser relies heavily on the multiplicativity of the pressure function for similarities, which presents complications in the general case. To overcome this, we show that dimension is invariant under passing to a derived system that may be chosen to have desirable properties relating to the quantity s_1 defined above.

Lemma 2.2.1. *For all $k \in \mathbb{N}$, the IFS given by $\mathbb{I}_k = \{S_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{I}^k}$ satisfies*

$$\overline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C^k,$$

where F_C^k denotes the inhomogeneous attractor associated with $\mathbb{I}_k$ and C.

Proof. Fix $k \in \mathbb{N}$ and observe that

$$F_\emptyset = \bigcup_{i \in \mathcal{I}} S_i(F_\emptyset) = \bigcup_{i_1 \in \mathcal{I}} \bigcup_{i_2 \in \mathcal{I}} S_{i_1}(S_{i_2}(F_\emptyset)) = \cdots = \bigcup_{\mathbf{i} \in \mathcal{I}^k} S_{\mathbf{i}}(F_\emptyset)$$

and so $F_\emptyset = F_\emptyset^k$, where $F_\emptyset^k$ denotes the unique homogeneous attractor associated with $\mathbb{I}_k$. Recall that $F_C = F_\emptyset \cup \mathcal{O}$ and $\mathcal{O} = C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C)$. Hence,

$$F_C = F_\emptyset \cup C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C)$$

and

$$F_C^k = F_\emptyset \cup C \cup \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} S_{\mathbf{i}}(C),$$

where $(\mathcal{I}^k)^*$ denotes all finite concatenations of length k words over $\mathcal{I}$. Thus, by finite stability of box-counting dimension, it suffices to show

$$\overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C) \leq \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} S_{\mathbf{i}}(C), \tag{2.2.2}$$

since the opposite inequality follows immediately by monotonicity. Observe

$$\overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C) = \max_{t=1,\dots,k} \overline{\dim}_{\mathrm{B}} \bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ |\mathbf{i}| = nk+t \\ n \geq 0}} S_{\mathbf{i}}(C),$$

and let m be the value of t that realises the maximum. First, note that

$$\overline{\dim}_{\mathrm{B}} C = \overline{\dim}_{\mathrm{B}} S_{\mathbf{i}}(C) \leq \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} S_{\mathbf{i}}(C) \tag{2.2.3}$$

for all $\mathbf{i} \in \mathcal{I}^*$, since $S_{\mathbf{i}}$ is bi-Lipschitz.

Hence

$$\begin{aligned}
\overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C) &= \overline{\dim}_{\mathrm{B}} \bigcup_{n=0}^{\infty} \bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ |\mathbf{i}| = nk+m}} S_{\mathbf{i}}(C) \\
&= \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{j} \in \mathcal{I}^m} \left(S_{\mathbf{j}}(C) \cup \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} (S_{\mathbf{ji}}(C)) \right) \\
&= \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{j} \in \mathcal{I}^m} S_{\mathbf{j}} \left(C \cup \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} (S_{\mathbf{i}}(C)) \right) \\
&= \max_{\mathbf{j} \in \mathcal{I}^m} \overline{\dim}_{\mathrm{B}} S_{\mathbf{j}} \left(C \cup \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} (S_{\mathbf{i}}(C)) \right) \\
&\leq \max\{\overline{\dim}_{\mathrm{B}} C, \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} S_{\mathbf{i}}(C)\} \\
&\leq \overline{\dim}_{\mathrm{B}} \bigcup_{\mathbf{i} \in (\mathcal{I}^k)^*} S_{\mathbf{i}}(C)
\end{aligned}$$

by (2.2.3). □

This has the following corollary that is fundamental to our approach. In the following few results, recall the definition of s_k given by (2.2.1).

Corollary 2.2.2. *For $t > \max\{s, \overline{\dim}_{\mathrm{B}} C\}$, there exists a $K \in \mathbb{N}$ such that $t > s_k$ for all $k > K$, and each IFS given by $\mathbb{I}_k = \{S_{\mathbf{i}}\}_{\mathbf{i} \in \mathcal{I}^k}$ satisfies*

$$\overline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} F_C^k.$$

Proof. Since $s_k \to s$, there exists $K \in \mathbb{N}$ such that $|s - s_k| \leq \frac{t-s}{2}$ for all $k > K$. The result then follows immediately from Lemma 2.2.1. $\square$

The next Lemma is analogous to [29, Lemma 3.2] and illustrates the motivation for Corollary 2.2.2.

Lemma 2.2.3. *If $t > s_1$, then there exists a constant b_t such that*

$$\sum_{\mathbf{i} \in \mathcal{I}^*} \mathrm{Lip}^+(S_{\mathbf{i}})^t = b_t < \infty.$$

Proof. Observe that $t > s_1$ implies

$$\sum_{i \in \mathcal{I}} \mathrm{Lip}^+(S_i)^t < 1.$$

Hence

$$\begin{aligned}
\sum_{\mathbf{i} \in \mathcal{I}^*} \mathrm{Lip}^+(S_{\mathbf{i}})^t &= \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in \mathcal{I}^k} \mathrm{Lip}^+(S_{\mathbf{i}})^t \\
&\leq \sum_{k=1}^{\infty} \left(\sum_{i \in \mathcal{I}} \mathrm{Lip}^+(S_i)^t \right)^k \\
&< \infty,
\end{aligned}$$

by convergence of the geometric series. $\square$

A natural way to construct efficient δ-covers is to consider the finite set of cylinders $S_{\mathbf{i}}(X)$ such that $\mathrm{Lip}^+(S_{\mathbf{i}}) < \delta$ and $\mathrm{Lip}^+(S_{\mathbf{i}_p}) \geq \delta$ for any prefix $\mathbf{i}_p$ of $\mathbf{i}$. For $\mathbf{i} = (i_1, ..., i_k) \in \mathcal{I}^*$ we let $\mathbf{i}_- = (i_1, ..., i_{k-1})$ and write $|\mathbf{i}|$ to denote the length of the string $\mathbf{i}$. If $\delta \in (0, 1]$, we define the *δ-stopping*, denoted $\mathcal{I}(\delta)$, by

$$\mathcal{I}(\delta) = \{\mathbf{i} \in \mathcal{I}^* : \mathrm{Lip}^+(S_{\mathbf{i}}) < \delta \leq \mathrm{Lip}^+(S_{\mathbf{i}_-})\},$$

and assume for convenience that $\mathrm{Lip}^+(S_\omega) = 1$, where ω denotes the empty word. If $\mathbf{i} \in \mathcal{I}^*$ satisfies $\mathrm{Lip}^+(S_\mathbf{i}) < \delta$, then it is clear there exists a prefix $\mathbf{i}_p$ such that $\mathbf{i}_p \in \mathcal{I}(\delta)$. To establish a bound on $|\mathcal{I}(\delta)|$ in our next lemma, it is useful to set

$$L_{\min} = \min_{i \in \mathcal{I}} \mathrm{Lip}^-(S_i) > 0.$$

Lemma 2.2.4. *If $t > s_1$, then*

$$|\mathcal{I}(\delta)| \leq b_t L_{\min}^{-t} \delta^{-t}$$

for all $\delta \in (0, 1]$.

Proof. For $\mathbf{i} \in \mathcal{I}(\delta)$, we have

$$\mathrm{Lip}^+(S_\mathbf{i}) \geq \mathrm{Lip}^+(S_{\mathbf{i}_-}) L_{\min} \geq \delta L_{\min} > 0. \tag{2.2.4}$$

Hence

$$b_t \geq \sum_{\mathbf{i} \in \mathcal{I}(\delta)} \mathrm{Lip}^+(S_\mathbf{i})^t \geq \sum_{\mathbf{i} \in \mathcal{I}(\delta)} (\delta L_{\min})^t = |\mathcal{I}(\delta)| (\delta L_{\min})^t$$

and the desired inequality follows immediately. $\square$

This yields an alternative and succinct proof of the well-known result that the dimension of the homogeneous attractor is bounded above by the upper Lipschitz dimension.

Lemma 2.2.5. $\overline{\dim}_\mathrm{B} F_\emptyset \leq s$, *where s denotes the upper Lipschitz dimension of $\mathbb{I}$.*

Proof. Fix $\delta \in (0, 1]$ and let $t > s$ be arbitrary. By Corollary 2.2.2, we may assume $t > s_1$ without loss of generality. This is because $F_\emptyset$ may be replaced with a set of equal dimension $F_\emptyset^k$ for some sufficiently large k with $s_k < t$. The result then follows from Lemma 2.2.4, since the cylinder sets $\{S_{\mathbf{i}}(X) : \mathbf{i} \in \mathcal{I}(\delta)\}$ form a δ-cover of $F_\emptyset$, and so $N_\delta(F_\emptyset) \le |\mathcal{I}(\delta)| \le b_t L_{\min}^{-t} \delta^{-t}$. □

For clarity in our later calculation, we provide one further lemma.

Lemma 2.2.6. *For $\delta \in (0, 1]$, we have*

$$\bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) < \delta}} S_{\mathbf{i}}(C) \subseteq \bigcup_{\mathbf{i} \in \mathcal{I}(\delta)} S_{\mathbf{i}}(X).$$

Proof. If

$$x \in \bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) < \delta}} S_{\mathbf{i}}(C)$$

there exists some $\mathbf{i} = (i_1, i_2, \dots, i_n) \in \mathcal{I}^*$ with $\mathrm{Lip}^+(S_{\mathbf{i}}) < \delta$ and a $c \in C$ such that $x = S_{\mathbf{i}}(c)$. Let $\mathbf{i}_p = (i_1, i_2, \dots, i_p)$ denote the prefix of $\mathbf{i}$ with $\mathbf{i}_p \in \mathcal{I}(\delta)$, then $x = S_{\mathbf{i}_p}(S_{(i_{p+1}, i_{p+2}, \dots, i_n)}(c)) \in S_{\mathbf{i}_p}(X)$, as required. □

Theorem 2.2.7. *Let (X, d) be a compact metric space and $\mathbb{I} = \{S_i\}_{i=1}^N$ denote an IFS consisting of bi-Lipschitz maps with compact condensation set $C \subseteq X$. We have*

$$\max\{\overline{\dim}_B F_\emptyset, \overline{\dim}_B C\} \le \overline{\dim}_B F_C \le \max\left\{s, \overline{\dim}_B C\right\},$$

where s is equal to the upper Lipschitz dimension.

Proof. Monotonicity of upper box-counting dimension implies

$$\max\{\overline{\dim}_B F_\emptyset, \overline{\dim}_B C\} \le \overline{\dim}_B F_C,$$

since $F_\emptyset \cup C \subseteq F_\emptyset \cup \mathcal{O} = F_C$. Moreover, by finite stability of upper box-counting dimension we have

$$\overline{\dim}_B F_C \leq \max\{\overline{\dim}_B F_\emptyset, \overline{\dim}_B \mathcal{O}\}.$$

Hence, since $\overline{\dim}_{\mathrm{B}} F_\emptyset \leq s$ by Lemma 2.2.5, it suffices to show

$$\overline{\dim}_B \mathcal{O} \leq \max\{s, \overline{\dim}_B C\},$$

where s denotes the upper Lipschitz dimension.

Let $t > \max\{s, \overline{\dim}_{\mathrm{B}} C\}$. By Corollary 2.2.2, since our interest is in $\overline{\dim}_{\mathrm{B}} F_C$, we can assume hereafter that $t > s_1$ by passing to a derived system F_C^K with $t > s_K$ for some sufficiently large $K \in \mathbb{N}$, without loss of generality.

The definition of box-counting dimension implies that there exists a constant c_t such that

$$N_\delta(C) \leq c_t \delta^{-t} \tag{2.2.5}$$

for all $\delta \in (0,1]$. Further, since X is compact, $N_1(X)$ is a finite constant that does not depend on t. Hence

$$\begin{aligned}
N_\delta(\mathcal{O}) &= N_\delta\left(C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_{\mathbf{i}}(C)\right) \\
&\leq N_\delta(C) + N_\delta\left(\bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) \geq \delta}} S_{\mathbf{i}}(C)\right) + N_\delta\left(\bigcup_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) < \delta}} S_{\mathbf{i}}(C)\right) \\
&\leq N_\delta(C) + \sum_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) \geq \delta}} N_\delta(S_{\mathbf{i}}(C)) + N_\delta\left(\bigcup_{\mathbf{i} \in \mathcal{I}(\delta)} S_{\mathbf{i}}(X)\right) \quad \text{(by Lemma 2.2.6)} \\
&\leq N_\delta(C) + \sum_{\substack{\mathbf{i} \in \mathcal{I}^* \\ \mathrm{Lip}^+(S_{\mathbf{i}}) \geq \delta}} N_\delta(S_{\mathbf{i}}(C)) + \sum_{\mathbf{i} \in \mathcal{I}(\delta)} N_\delta(S_{\mathbf{i}}(X))
\end{aligned}$$

$$
\begin{aligned}
&\le N_\delta(C) + \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \mathrm{Lip}^+(S_\mathbf{i})\ge\delta}} N_{\delta/\mathrm{Lip}^+(S_\mathbf{i})}(C) + \sum_{\mathbf{i}\in\mathcal{I}(\delta)} N_{\delta/\mathrm{Lip}^+(S_\mathbf{i})}(X) \\
&\le c_t\delta^{-t} + \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \mathrm{Lip}^+(S_\mathbf{i})\ge\delta}} c_t(\delta/\mathrm{Lip}^+(S_\mathbf{i}))^{-t} + \sum_{\mathbf{i}\in\mathcal{I}(\delta)} N_1(X) \\
&\le c_t\delta^{-t} + c_t\delta^{-t}\sum_{\mathbf{i}\in\mathcal{I}^*} \mathrm{Lip}^+(S_\mathbf{i})^t + |\mathcal{I}(\delta)|N_1(X) \\
&\le c_t\delta^{-t} + c_t\delta^{-t}b_t + b_t L_{\min}^{-t}\delta^{-t}N_1(X) \quad \text{(by Lemmas 2.2.3 and 2.2.4)} \\
&\le \delta^{-t}(c_t + c_t b_t + b_t L_{\min}^{-t}N_1(X)).
\end{aligned}
$$

□

If we make a few further assumptions we are able to obtain some stronger corollaries in popular contexts, such as conformal and low dimensional affine systems. For example, we may wish to consider applications in which $s = \overline{\dim}_{\mathrm{B}} F_\emptyset$, that is, where (2.1.1) is satisfied. One such scenario involves the notion of *bounded distortion*. An IFS $\mathbb{I} = \{S_i\}_{i=1}^N$ satisfies the property of bounded distortion if there exists some uniform constant $L > 1$ such that

$$\frac{\mathrm{Lip}^+(S_\mathbf{i})}{\mathrm{Lip}^-(S_\mathbf{i})} < L,$$

for all $\mathbf{i} \in \mathcal{I}^*$. Lemma 2.2.5 and a simple modification of [17, Proposition 9.7] imply that bounded distortion together with the SOSC force $s = \overline{\dim}_{\mathrm{B}} F_\emptyset$. This immediately yields the following corollary of Theorem 2.2.7.

Corollary 2.2.8. *Let (X, d) be a compact metric space and $\mathbb{I} = \{S_i\}_{i=1}^N$ denote an IFS satisfying bounded distortion with compact condensation set $C \subseteq X$. If $\mathbb{I}$ satisfies the SOSC, then*

$$\overline{\dim}_B F_C = \max\left\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_B C\right\}.$$

Moreover, Theorem 2.2.7 provides an extremely succinct proof of (2.1.2) with s equal to the affinity dimension if $s \leq 1$. This is because, if the affinity dimension is less than or equal to one, then it coincides with the upper Lipschitz dimension, since $\mathrm{Lip}^+(S)$ corresponds to the largest singular value of the linear component of S.

Corollary 2.2.9. *Let $\mathbb{I} = \{S_i\}_{i=1}^N$ be an affine IFS with compact condensation set $C \subseteq X$ and affinity dimension s. If $s \leq 1$, then*

$$\max\left\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_B C\right\} \leq \overline{\dim}_B F_C \leq \max\left\{s, \overline{\dim}_B C\right\}.$$

In particular, Corollary 2.2.9 implies that if the affinity dimension is less than or equal to one and equals $\overline{\dim}_{\mathrm{B}} F_\emptyset$, then (2.1.1) is satisfied. Falconer shows in [14] that the affinity and upper box-counting dimensions coincide almost surely upon randomizing the translations, even if the SOSC fails, and it follows from recent results of Bárány, Hochman and Rapaport [3] that mild assumptions are sufficient to force equality in the plane. For a more detailed discussion, see [10]. However, it is worth noting that (2.1.1) does not always hold in the affine setting. In particular, from the results of Fraser [30], it is possible to construct simple examples of inhomogeneous self-affine sets with affinity dimension $s < 1$ satisfying

$$\max\left\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\right\} < \overline{\dim}_{\mathrm{B}} F_C < \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}.$$

2.2.2 Measure

It has been of historical interest (e.g. [17]) to compute $\mathcal{H}^t(F)$ when $t = \dim_H F$. It may well be zero, finite or infinite. Recall that countable stability and monotonicity of

Hausdorff dimension [17] readily imply that

$$\dim_{\mathrm{H}} F_C = \max\{\dim_{\mathrm{H}} F_\emptyset, \dim_{\mathrm{H}} C\},$$

and we investigate $\mathcal{H}^t(F_C)$ in each case. Although we omit the details, the following results extend to any family of measures satisfying the scaling property and their associated dimension, such as packing measures.

It turns out similar methods to those in the last section may be used. First, we prove two technical lemmas that mirror the strategy Lemma 2.2.2 allowed for dimension.

Lemma 2.2.10. *Let $t \geq 0$ and suppose $\mathcal{H}^t(C) < \infty$. For all $K \in \mathbb{N}$ and $1 \leq a, b \leq K$, we have*

$$\mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{a+nK}} S_{\mathbf{i}}(C)\right) < \infty \iff \mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{b+nK}} S_{\mathbf{i}}(C)\right) < \infty.$$

Proof. Fix $K \in \mathbb{N}$ and let

$$L_k = \max_{\mathbf{i} \in \mathcal{I}^k} \mathrm{Lip}^+(S_{\mathbf{i}})$$

for $k \in \mathbb{N}$. If $0 < a, b \leq K$ are distinct, we have

$$\begin{aligned}
&\mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{a+nK}} S_{\mathbf{i}}(C)\right) \\
&\leq \mathcal{H}^t\left(\bigcup_{\mathbf{u} \in \mathcal{I}^a} S_{\mathbf{u}}(C) \cup \bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{K-b+a}} \bigcup_{\mathbf{j} \in \mathcal{I}^{b+nK}} S_{\mathbf{ij}}(C)\right) \\
&\leq \sum_{\mathbf{u} \in \mathcal{I}^a} \mathcal{H}^t(S_{\mathbf{u}}(C)) + \sum_{\mathbf{i} \in \mathcal{I}^{K-b+a}} \mathcal{H}^t\left(S_{\mathbf{i}}\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{j} \in \mathcal{I}^{b+nK}} S_{\mathbf{j}}(C)\right)\right) \\
&\leq N^a L_a^t \mathcal{H}^t(C) + N^{K-b+a} L_{K-b+a}^t \mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{j} \in \mathcal{I}^{b+nK}} S_{\mathbf{j}}(C)\right),
\end{aligned}$$

by numerous applications of the scaling property of Hausdorff measure. The result follows since a and b were arbitrary and may be interchanged. $\square$

Lemma 2.2.11. *Let $t \geq 0$ and suppose $\mathcal{H}^t(C) < \infty$. Then $\mathcal{H}^t(\mathcal{O})$ is finite if and only if*

$$\mathcal{H}^t\left(\bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{nK}} S_{\mathbf{i}}(C)\right)$$

is finite for some $K \in \mathbb{N}$.

Proof. Let $K \in \mathbb{N}$ and observe

$$\begin{aligned} \mathcal{H}^t(\mathcal{O}) &= \mathcal{H}^t\left(C \cup \bigcup_{m=1}^{K} \bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{nK+m}} S_{\mathbf{i}}(C)\right) \\ &\leq \mathcal{H}^t(C) + \sum_{m=1}^{K} \mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{nK+m}} S_{\mathbf{i}}(C)\right), \end{aligned}$$

and so if

$$\mathcal{H}^t\left(\bigcup_{n=0}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{nK+K}} S_{\mathbf{i}}(C)\right) = \mathcal{H}^t\left(\bigcup_{n=1}^{\infty} \bigcup_{\mathbf{i} \in \mathcal{I}^{nK}} S_{\mathbf{i}}(C)\right) < \infty,$$

then Lemma 2.2.10 implies $\mathcal{H}^s(\mathcal{O})$ is also finite. The opposite implication follows by monotonicity. $\square$

Note that if $\mathcal{H}^t(C) = 0$ or $\mathcal{H}^t(C) = \infty$, then $\mathcal{H}^t(F_C) = \mathcal{H}^t(F_\emptyset)$ and $\mathcal{H}^t(F_C) = \infty$, respectively, recalling that $t = \dim_{\mathrm{H}} F_C$. Thus, our main theorem deals with the case where $\mathcal{H}^t(C)$ is positive and finite. For the problem to be tractable, some separation conditions are required. A natural choice is the *condensation open set condensation* (COSC), a modification of the SOSC adapted for the inhomogeneous case, as utilised in [10, 46, 61, 64].

An IFS satisfies the COSC if there exists an open set U with

$$C \subset U \setminus \bigcup_{i=1}^{N} \overline{S_i(U)},$$

such that $S_i(U) \subset U$ for $i = 1, \ldots, N$, and $i \neq j \implies S_i(U) \cap S_j(U) = \emptyset$.

Theorem 2.2.12. *Let (X, d) be a compact metric space and $\mathbb{I} = \{S_i\}_{i=1}^{N}$ denote an IFS with compact condensation set $C \subseteq X$ and upper Lipschitz dimension s. Suppose $t = \overline{\dim}_{\mathrm{B}} F_C$ and $0 < \mathcal{H}^t(C) < \infty$. It follows that*

i) if $t > s$, then $0 < \mathcal{H}^t(F_C) < \infty$;

ii) if $\mathbb{I}$ satisfies the COSC, then

$$\mathcal{H}^t(F_C) \geq \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(C)\left(1 + \sum_{k=1}^{\infty}\left(\sum_{i \in \mathcal{I}} \mathrm{Lip}^-(S_i)^t\right)^k\right)$$

and

$$\mathcal{H}^t(F_C) \leq \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(C)\left(1 + \sum_{k=1}^{\infty}\left(\sum_{i \in \mathcal{I}} \mathrm{Lip}^+(S_i)^t\right)^k\right).$$

Proof. Let $\mathbb{I} = \{S_i\}_{i=1}^{N}$ be an IFS and $C \subseteq X$ be compact.

(i) We first note that

$$\mathcal{H}^t(F_\emptyset) = 0,$$

since $s < t$ implies $\dim_{\mathrm{H}} F_\emptyset \leq \overline{\dim}_{\mathrm{B}} F_\emptyset \leq s < t$. Hence,

$$\mathcal{H}^t(F_C) \leq \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(\mathcal{O}) = \mathcal{H}^t(\mathcal{O}).$$

Thus, it suffices to show that

$$\mathcal{H}^t\left(\bigcup_{n=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{I}^{nK}} S_{\mathbf{i}}(C)\right) < \infty$$

for some $K \in \mathbb{N}$ by Lemma 2.2.11. Since $t > s$, it is possible to choose $K \in \mathbb{N}$ such that $t > s_K$ (see (2.2.1)), implying

$$\sum_{\mathbf{i}\in\mathcal{I}^K} \mathrm{Lip}^+(S_{\mathbf{i}})^t < \sum_{\mathbf{i}\in\mathcal{I}^K} \mathrm{Lip}^+(S_{\mathbf{i}})^{s_K} = 1.$$

It follows that

$$\begin{aligned}
\mathcal{H}^t\left(\bigcup_{n=1}^{\infty}\bigcup_{\mathbf{i}\in\mathcal{I}^{nK}} S_{\mathbf{i}}(C)\right) &\leq \sum_{n=1}^{\infty}\sum_{\mathbf{i}\in\mathcal{I}^{nK}} \mathcal{H}^t\left(S_{\mathbf{i}}(C)\right) \\
&\leq \mathcal{H}^t(C)\sum_{n=1}^{\infty}\sum_{\mathbf{i}\in\mathcal{I}^{nK}} \mathrm{Lip}^+(S_{\mathbf{i}})^t \\
&\leq \mathcal{H}^t(C)\sum_{n=1}^{\infty}\left(\sum_{\mathbf{i}\in\mathcal{I}^{K}} \mathrm{Lip}^+(S_{\mathbf{i}})^t\right)^n
\end{aligned}$$

which is a convergent geometric series and so finite, as required.

(ii) Suppose $\mathbb{I}$ satisfies the COSC, then

$$\mathcal{H}^t(S_{\mathbf{i}}(C) \cap S_{\mathbf{j}}(C)) = 0 \tag{2.2.6}$$

and

$$\mathcal{H}^t(F_\emptyset \cap S_{\mathbf{i}}(C)) = 0$$

for every $\mathbf{i} \neq \mathbf{j} \in \mathcal{I}^*$. Hence

$$\mathcal{H}^t(F_C) = \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(\mathcal{O})$$

$$
\begin{aligned}
&= \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(C) + \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in \mathcal{I}^k} \mathcal{H}^t(S_{\mathbf{i}}(C)) \\
&\geq \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(C) \left(1 + \sum_{k=1}^{\infty} \sum_{\mathbf{i} \in \mathcal{I}^k} \mathrm{Lip}^-(S_{\mathbf{i}})^t \right) \\
&\geq \mathcal{H}^t(F_\emptyset) + \mathcal{H}^t(C) \left(1 + \sum_{k=1}^{\infty} \left(\sum_{i \in \mathcal{I}} \mathrm{Lip}^-(S_i)^t \right)^k \right),
\end{aligned}
$$

and the corresponding inequality with $\mathrm{Lip}^+(S_{\mathbf{i}})$ follows similarly. $\square$

Theorem 2.2.12 yields a pleasing closed form expression for inhomogeneous self-similar sets, as studied in [29, 46, 64].

Corollary 2.2.13. *Let (X, d) be a compact metric space and $\mathbb{I} = \{S_i\}_{i=1}^N$ denote an IFS consisting of similarities satisfying the COSC with compact condensation set $C \subseteq X$ and similarity dimension s. If $t = \dim_{\mathrm{H}} F_C > s$ and $0 < \mathcal{H}^t(C) < \infty$, then*

$$
\mathcal{H}^t(F_C) = \frac{\mathcal{H}^t(C)}{1 - \sum_{i \in \mathcal{I}} \mathrm{Lip}(S_i)^t}.
$$

Proof. For a similarity S, we have $\mathrm{Lip}^+(S) = \mathrm{Lip}^-(S)$, and the result follows immediately from Theorem 2.2.12 (ii), since the upper Lipschitz and similarity dimensions coincide. $\square$

We hope the above may prompt future work. In particular, it would be interesting to discover alternative conditions to the COSC that control the sensitive interaction between $F_\emptyset$ and $\mathcal{O}$ while yielding similar results.

2.3 Affine iterated function systems

The purpose of this section is to establish (2.1.2) for affine systems with s equal to the affinity dimension and establish settings where (2.1.1) holds, drawing on work from [10]. This improves and unifies previous results on inhomogeneous self-affine carpets [30], and may be considered an inhomogeneous analogue of Falconer's seminal result on homogeneous self-affine sets. Throughout this section, we fix a compact ball $B \subset \mathbb{R}^n$ such that $S_{\mathbf{i}}(B) \subset B$ for $i = 1, \dots N$ and $C \subseteq B$. Such a ball always exists and without loss of generality, we may assume that B has unit diameter.

In Section 2.2 we saw that if the affinity dimension s is less than or equal to one and coincides with $\overline{\dim}_{\mathrm{B}} F_{\emptyset}$, then

$$\overline{\dim}_{\mathrm{B}} F_C = \max\left\{\overline{\dim}_{\mathrm{B}} F_{\emptyset}, \overline{\dim}_{\mathrm{B}} C\right\}. \tag{2.3.1}$$

This is an immediate corollary of Theorem 2.2.7. Otherwise, if the affinity dimension is greater than one it is elementary to see that it is strictly less than the upper Lipschitz dimension. Thus, establishing (2.1.2) for affinity dimension constitutes a natural and strictly improved bound for affine systems in comparison to the universal bound from Section 2.2.

We begin some technical lemmas, starting with a minor variation on Lemma 2.2.6. Here, and throughout, we require the definition of *m-δ-stoppings* that generalise δ-stoppings to the affine setting. For each $1 \leq m \leq n$ and $\delta \in (0, 1]$, define the m-δ-stopping to be

$$\mathcal{I}_m(\delta) = \{\mathbf{i} \in \mathcal{I}^* : \alpha_m(S_{\mathbf{i}}) < \delta \leq \alpha_m(S_{\mathbf{i}_-})\},$$

where $\mathbf{i}_- = (i_1, \dots, i_{k-1})$ for $\mathbf{i} = (i_1, \dots, i_k)$.

Lemma 2.3.1. *For $\delta \in (0,1]$ and $1 \leq m \leq n$, we have*

$$\bigcup_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \delta>\alpha_m(S_\mathbf{i})}} S_\mathbf{i}(C) \subseteq \bigcup_{\mathbf{i}\in\mathcal{I}_m(\delta)} S_\mathbf{i}(B). \tag{2.3.2}$$

Proof. For

$$x \in \bigcup_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \delta>\alpha_m(S_\mathbf{i})}} S_\mathbf{i}(C),$$

there exists some $\mathbf{i} = (i_1, \ldots, i_k) \in \mathcal{I}^*$ such that $x \in S_\mathbf{i}(C)$ and $\delta > \alpha_m(S_\mathbf{i})$. Since $\delta > \alpha_m(S_\mathbf{i})$, there also exists some prefix $\mathbf{i}_p$ of $\mathbf{i}$ with $\mathbf{i}_p \in \mathcal{I}_m(\delta)$, and so let us consider the concatenation $\mathbf{i} = \mathbf{i}_p\mathbf{j}$. If $\mathbf{j} = \emptyset$, then $\mathbf{i} \in \mathcal{I}_m(\delta)$. Else, there exists some c such that $x = S_\mathbf{i}(c) = S_{\mathbf{i}_p}(S_\mathbf{j}(c)) \in S_{\mathbf{i}_p}(B)$ as required. $\square$

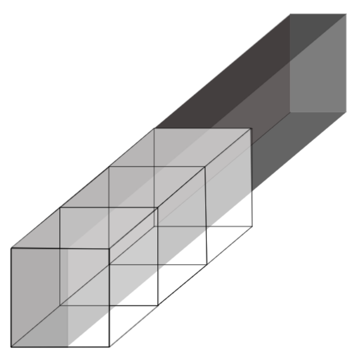

Figure 2.1: Covering a cuboid of sidelengths $a > b > c$ in $\mathbb{R}^3$ with cubes of sidelength b.

For our next lemma, the following simple geometric observation may aid the reader less familiar with the classical arguments on self-affine sets found in [14] or [17]. Consider an ellipsoid E with principal axes of lengths $l_1, \ldots, l_n$. For dimension calculations, we are interested in obtaining an estimate of the number of hypercubes of a given sidelength required to cover such ellipsoids. Constants are typically inconsequential, so often a coarse estimate suffices. The minimum number of hypercubes of sidelength l_m required to cover E is at most

$$\left(\frac{l_1}{l_m}+1\right)\cdots\left(\frac{l_{m-1}}{l_m}+1\right) \leq 2^n \frac{l_1}{l_m}\frac{l_2}{l_m}\cdots\frac{l_{m-1}}{l_m} = 2^n l_1 l_2 \cdots l_{m-1} l_m^{-m+1}. \tag{2.3.3}$$

This can be seen by first covering E by a minimal hypercuboid of sidelengths equal to the principal axes of E and then covering this optimally. Figure 2.1 illustrates this for a cuboid of sidelengths $a > b > c$ in $\mathbb{R}^3$. Specifically, we see that $2a/b$ cubes of sidelength b would suffice, whereas we would require a single cube of sidelength a or at most $2^2(a/c)(b/c)$ cubes of sidelength c.

Lemma 2.3.2. *Fix $1 \leq m \leq n$ and let $\mathbf{i} \in \mathcal{I}^*$ be such that $\alpha_m(S_\mathbf{i}) < \delta$. Then*

$$N_\delta(S_\mathbf{i}(B)) \leq 2^n \frac{\alpha_1(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})}\frac{\alpha_2(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})}\cdots\frac{\alpha_{m-1}(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})}.$$

Proof. First note that $S_\mathbf{i}(B)$ is an ellipsoid with principal axes having lengths equal to the singular values of $S_\mathbf{i}$. The result then follows follows immediately from the geometric observation described by equation (2.3.3). □

Lemma 2.3.3. *Let $\overline{\dim}_\mathrm{B} C \leq t \leq n$ and $m \in \mathbb{Z}$ be the integer satisfying $m-1 < t \leq m$. If $\mathbf{i} \in \mathcal{I}^*$ is such that $\alpha_m(S_\mathbf{i}) \geq \delta$, then*

$$N_\delta(S_\mathbf{i}(C)) \leq 2^n A_t \delta^{-t} \phi^t(S_\mathbf{i}),$$

where A_t is a constant depending only on t.

Proof. The image under $S_\mathbf{i}$ of a cover of C by balls of diameter $\delta/\alpha_m(S_\mathbf{i})$ is a cover of $S_\mathbf{i}(C)$ by ellipsoids with the m largest principal axes of lengths

$$\alpha_i(S_\mathbf{i})\left(\frac{\delta}{\alpha_m(S_\mathbf{i})}\right) = \delta\frac{\alpha_i(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})}$$

for $i = 1, \dots, m$, the smallest of which has length δ. Each such ellipsoid can be covered by at most

$$\frac{2\delta\frac{\alpha_1(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}}{\delta}\frac{2\delta\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}}{\delta}\cdots\frac{2\delta\frac{\alpha_{m-1}(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}}{\delta} \leq 2^n \frac{\alpha_1(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\cdots\frac{\alpha_{m-1}(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}$$

hypercubes of sidelength δ. Hence

$$\begin{aligned} N_\delta(S_{\mathbf{i}}(C)) &\leq N_{\delta/\alpha_m(S_{\mathbf{i}})}(C)\left(2^n \frac{\alpha_1(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\cdots\frac{\alpha_{m-1}(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\right) \\ &\leq A_t\left(\frac{\delta}{\alpha_m(S_{\mathbf{i}})}\right)^{-t}\left(2^n \frac{\alpha_1(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\cdots\frac{\alpha_{m-1}(S_{\mathbf{i}})}{\alpha_m(S_{\mathbf{i}})}\right) \\ &= 2^n A_t \delta^{-t}\phi^t(S_{\mathbf{i}}) \end{aligned}$$

as required. □

This prepares us to state and prove our main result of this section. It may be considered an inhomogeneous analogue of Falconer's result on homogeneous self-affine sets [14] that established the affinity dimension as an upper bound on $\overline{\dim}_{\mathrm{B}} F_\emptyset$.

Theorem 2.3.4. *Let $F_C \subset \mathbb{R}^n$ be an inhomogeneous self-affine set with compact condensation set $C \subset \mathbb{R}^n$. Then*

$$\max\left\{\overline{\dim}_{\mathrm{B}} F_\emptyset,\, \overline{\dim}_{\mathrm{B}} C\right\} \leq \overline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\},$$

where s is the affinity dimension associated with the underlying IFS.

Proof. Let $\mathbb{I} = \{S_i\}_{i=1}^N$ be an affine IFS and $C \subseteq B$ be compact. Denote the affinity dimension of $\mathbb{I}$ by s and assume $s \leq n$, since if $s > n$ the result is trivial. It follows immediately from the definition of box-counting dimension that for $t > \overline{\dim}_{\mathrm{B}} C$ there exists a constant A_t satisfying

$$N_\delta(C) \leq A_t \delta^{-t} \tag{2.3.4}$$

for all $\delta \in (0,1]$. In addition, if $t > s$, then

$$B_t := \sum_{\mathbf{i} \in \mathcal{I}^*} \phi^t(S_\mathbf{i}) < \infty \tag{2.3.5}$$

by [14, Proposition 4.1 (c)], where B_t depends only on t. We fix a constant $b \in \mathbb{R}$ satisfying

$$0 < b < \min_{i=1,\dots,N} \alpha_n(S_i) < 1,$$

and note for any $\delta \in (0,1]$, $1 \leq m \leq n$ and $\mathbf{i} \in \mathcal{I}_m(\delta)$, we have

$$\delta \geq \alpha_m(S_\mathbf{i}) \geq \alpha_m(S_{\mathbf{i}_-})b \geq \delta b. \tag{2.3.6}$$

Monotonicity and finite stability of upper box-counting dimension imply

$$\max\left\{\overline{\dim}_\mathrm{B} F_\emptyset, \overline{\dim}_\mathrm{B} C\right\} \leq \overline{\dim}_\mathrm{B} F_C \leq \max\left\{\overline{\dim}_\mathrm{B} F_\emptyset, \overline{\dim}_\mathrm{B} \mathcal{O}\right\}$$

and so it suffices to show that

$$\overline{\dim}_\mathrm{B} \mathcal{O} \leq \max\left\{s, \overline{\dim}_\mathrm{B} C\right\},$$

since it is well known (see [17, Theorem 9.12]) that $s \geq \overline{\dim}_\mathrm{B} F_\emptyset$. Fix $\delta \in (0,1]$ and $t > \max\{s, \overline{\dim}_\mathrm{B} C\}$. If $\max\{s, \overline{\dim}_\mathrm{B} C\} \geq n$ then the result is trivial, so we may assume $t \leq n$. For $m \in \mathbb{Z}$ satisfying $m - 1 < t \leq m$, we have

$$\delta^t N_\delta(\mathcal{O}) = \delta^t N_\delta\left(C \cup \bigcup_{\mathbf{i} \in \mathcal{I}^*} S_\mathbf{i}(C)\right)$$

$$
\begin{aligned}
&\le A_t + \delta^t N_\delta \left(\bigcup_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})\ge\delta}} S_\mathbf{i}(C) \right) + \delta^t N_\delta \left(\bigcup_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})<\delta}} S_\mathbf{i}(C) \right) \quad \text{(using (2.3.4))} \\
&\le A_t + \delta^t \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})\ge\delta}} N_\delta(S_\mathbf{i}(C)) + \delta^t \sum_{\mathbf{i}\in\mathcal{I}_m(\delta)} N_\delta\left(S_\mathbf{i}(B)\right) \quad \text{(by Lemma 2.3.1)} \\
&\le A_t + \delta^t \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})\ge\delta}} 2^n A_t \delta^{-t} \phi^t(S_\mathbf{i}) \\
&\quad + \delta^t \sum_{\mathbf{i}\in\mathcal{I}_m(\delta)} 2^n \frac{\alpha_1(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \frac{\alpha_2(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \cdots \frac{\alpha_{m-1}(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \quad \text{(by Lemmas 2.3.2 and 2.3.3)} \\
&\le A_t + 2^n A_t \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})\ge\delta}} \phi^t(S_\mathbf{i}) \\
&\quad + 2^n \sum_{\mathbf{i}\in\mathcal{I}_m(\delta)} \frac{\alpha_1(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \frac{\alpha_2(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \cdots \frac{\alpha_{m-1}(S_\mathbf{i})}{\alpha_m(S_\mathbf{i})} \frac{\alpha_m(S_\mathbf{i})^t}{b^t} \quad \text{(using (2.3.6))} \\
&\le A_t + 2^n A_t \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ \alpha_m(S_\mathbf{i})\ge\delta}} \phi^t(S_\mathbf{i}) + \frac{2^n}{b^t} \sum_{\mathbf{i}\in\mathcal{I}_m(\delta)} \phi^t(S_\mathbf{i}) \\
&\le A_t + 2^n B_t \left(A_t + b^{-t}\right) \quad \text{(using (2.3.5)).}
\end{aligned}
$$

Hence,

$$
\frac{\log N_\delta(\mathcal{O})}{-\log\delta} \le t + \frac{\log\left(A_t + 2^n B_t\left(A_t + b^{-t}\right)\right)}{-\log\delta},
$$

from which the result follows as $\delta \to 0$. □

The following corollary is immediate.

Corollary 2.3.5. *Let $F_C \subset \mathbb{R}^n$ be an inhomogeneous self-affine set with compact condensation set $C \subset \mathbb{R}^n$ and let s be the associated affinity dimension. Then*

1. *if $\overline{\dim}_\mathrm{B} F_\emptyset = s$, then $\overline{\dim}_\mathrm{B} F_C = \max\left\{\overline{\dim}_\mathrm{B} F_\emptyset, \overline{\dim}_\mathrm{B} C\right\}$,*

2. *if $\overline{\dim}_\mathrm{B} C \ge s$, then $\overline{\dim}_\mathrm{B} F_C = \overline{\dim}_\mathrm{B} C$.*

Establishing precise conditions for the affinity dimension to coincide with $\overline{\dim}_{\mathrm{B}} F_\emptyset$ is a major open problem in fractal geometry and has been the focus of considerable amounts of work, for example [3, 14, 15, 28, 30, 35, 43, 45]. Therefore there are numerous explicit and non-explicit situations where Corollary 2.3.5 provides a precise result, and an affirmative solution to (2.1.1) in the self-affine setting. For example, a well-known result by Falconer [14] states that $s = \overline{\dim}_{\mathrm{B}} F_\emptyset = \dim_{\mathrm{H}} F_\emptyset$ almost surely if one randomises the translation vectors associated with the affine maps, provided the linear parts all have norm strictly bounded above by $1/2$, see also [45]. Falconer proved in a subsequent paper that if $F_\emptyset \subset \mathbb{R}^2$ satisfies some separation conditions and contains a connected component not contained in a straight line, then $s = \overline{\dim}_{\mathrm{B}} F_\emptyset$ holds, see [15, Corollary 5]. In addition, the aforementioned result of Bárány, Hochman and Rapaport [3] proves $s = \overline{\dim}_{\mathrm{B}} F_\emptyset = \dim_{\mathrm{H}} F_\emptyset$ in the planar case assuming only strong separation, together with mild non-compactness and irreducibility assumptions on the linear components of the maps S_i.

The next result explores the case where $\overline{\dim}_{\mathrm{B}} F_C > \max\{\overline{\dim}_{\mathrm{B}} F_\emptyset, \overline{\dim}_{\mathrm{B}} C\}$, that is when (2.1.1) fails. This is an exploration of conditions under which C compensates for dimension drop between s and $\overline{\dim}_{\mathrm{B}} F_\emptyset$.

Theorem 2.3.6. *Let $\mathbb{I} = \{S_i\}_{i=1}^N$ denote an affine IFS with affinity dimension $s \leq n$ and condensation set $C \subset \mathbb{R}^n$ satisfying the COSC. If $\underline{\dim}_{\mathrm{B}} C \geq n-1$ and there exists $\kappa > 0$ such that for all $\delta \in (0,1]$ and $\mathbf{i} \in \mathcal{I}_n(\delta)$ we have*

$$N_\delta(S_{\mathbf{i}}(C)) \geq \kappa N_\delta(S_{\mathbf{i}}(B)),$$

then

$$\overline{\dim}_{\mathrm{B}} F_C = \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}$$

and

$$\max\left\{s, \underline{\dim}_{\mathrm{B}} C\right\} \leq \underline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}.$$

Proof. Fix $\delta \in (0,1)$ and recall that s denotes the affinity dimension of $\mathbb{I}$. It is stated in [15] that for $t < s$ there exists $c_t > 0$ with

$$\sum_{\mathcal{I}_n(\delta)} \phi^t(S_{\mathbf{i}}) \geq c_t \tag{2.3.7}$$

for some constant c_t that does not depend on δ. This follows immediately from [14, Proposition 4.1 (a)]. Since we assume $\underline{\dim}_{\mathrm{B}} C \geq n-1$, if $s \leq n-1$, then Theorem 2.3.4 implies that $\overline{\dim}_{\mathrm{B}} F_C = \overline{\dim}_{\mathrm{B}} C = \max\{s, \overline{\dim}_{\mathrm{B}} C\}$, and also $\underline{\dim}_{\mathrm{B}} F_C \geq \underline{\dim}_{\mathrm{B}} C = \max\{s, \underline{\dim}_{\mathrm{B}} C\}$. Thus, henceforth we assume that $n-1 < t < s \leq n$.

Let U denote the open set satisfying the COSC. Compactness of C implies that there exists some constant $\eta > 0$ with

$$\inf \left\{ |x-y| : x \in C,\ y \in \bigcup_{i=1}^{N} S_{\mathbf{i}}(U) \cup (\mathbb{R}^n \setminus U) \right\} = 2\eta.$$

Let $B(C,\eta)$ denote a closed η-neighborhood of C and E be a hypercube of sidelength δ in a minimal δ-cover of $\mathcal{O}$. For $\mathbf{i} \in \mathcal{I}_n(\delta)$, we have $S_{\mathbf{i}}(B(C,\eta))$ is a neighborhood of $S_{\mathbf{i}}(C)$ satisfying

$$S_{\mathbf{i}}(B(C,\eta)) \cap F_C = S_{\mathbf{i}}(C)$$

and

$$\inf\{|x-y| : x \in S_{\mathbf{i}}(C), y \notin S_{\mathbf{i}}(B(C,\eta))\} \geq \alpha_n(S_{\mathbf{i}})\eta > b\delta\eta,$$

implying

$$\inf\{|x-y| : x \in S_{\mathbf{i}}(C), y \in S_{\mathbf{j}}(C) \text{ such that } \mathbf{i},\mathbf{j} \in \mathcal{I}_n(\delta), \mathbf{i} \neq \mathbf{j}\} > 2b\delta\eta.$$

Let V_n denote the constant such that the volume of an n-sphere of radius $2b\eta\delta$ is $V_n\delta^n$. For the sets in $\{S_{\mathbf{i}}(C) : \mathbf{i} \in \mathcal{I}_n(\delta)\}$ that intersect E we can associate pairwise disjoint open sets in E of volume at least $V_n\delta^n/2^n$ (with this lower bound obtained at the vertices) and it therefore follows by a simple volume argument that E can intersect at most

$$\frac{\delta^n}{\frac{1}{2^n}V_n\delta^n} = (2^{-n}V_n)^{-1}$$

of the sets $\{S_{\mathbf{i}}(C) : \mathbf{i} \in \mathcal{I}_n(\delta)\}$.

Hence

$$N_\delta(\mathcal{O}) \geq 2^{-n}V_n \sum_{\mathbf{i}\in\mathcal{I}_n(\delta)} N_\delta(S_{\mathbf{i}}(C)). \tag{2.3.8}$$

Our assumption on C implies that for $\mathbf{i} \in \mathcal{I}_n(\delta)$ we have

$$\begin{aligned} N_\delta(S_{\mathbf{i}}(C)) &\geq \kappa N_\delta(S_{\mathbf{i}}(B)) \\ &\geq \kappa b^n N_{b\delta}(S_{\mathbf{i}}(B)) \\ &\geq \kappa b^n N_{\alpha_n(S_{\mathbf{i}})}(S_{\mathbf{i}}(B)) \\ &\geq \kappa b^n c \frac{\alpha_1(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})}\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})}\cdots\frac{\alpha_{n-1}(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})} \end{aligned} \tag{2.3.9}$$

for some constant $c > 0$ only depending on n. This yields

$$\begin{aligned} N_\delta(\mathcal{O}) &\geq 2^{-n}V_n \sum_{\mathbf{i}\in\mathcal{I}_n(\delta)} N_\delta(S_{\mathbf{i}}(C)) \quad \text{(using (2.3.8))} \\ &\geq 2^{-n}V_n \sum_{\mathbf{i}\in\mathcal{I}_n(\delta)} \kappa b^n c \frac{\alpha_1(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})}\frac{\alpha_2(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})}\cdots\frac{\alpha_{n-1}(S_{\mathbf{i}})}{\alpha_n(S_{\mathbf{i}})} \quad \text{(using (2.3.9))} \\ &= \kappa b^n c 2^{-n} V_n \sum_{\mathbf{i}\in\mathcal{I}_n(\delta)} \phi^t(S_{\mathbf{i}})\alpha_n(S_{\mathbf{i}})^{-t} \\ &\geq \kappa b^n c 2^{-n} V_n \delta^{-t} \sum_{\mathbf{i}\in\mathcal{I}_n(\delta)} \phi^t(S_{\mathbf{i}}) \\ &\geq \kappa b^n c 2^{-n} V_n c_t \delta^{-t} \quad \text{(by (2.3.7))}. \end{aligned}$$

Hence $\underline{\dim}_{\mathrm{B}}\mathcal{O} \geq t$, from which it follows that $\overline{\dim}_{\mathrm{B}} F_C \geq \underline{\dim}_{\mathrm{B}} F_C \geq \underline{\dim}_{\mathrm{B}}\mathcal{O} \geq s$, proving the theorem. $\square$

Note that the condition of the theorem is independent of the choice of ball B, although the constant κ may change. The fact that we only get bounds for the lower box-counting dimension of F_C should not come as a surprise and one should not expect to be able to improve these bounds in general, see [29]. Note that if, in the setting of Theorem 2.3.6, the box-counting dimension of C exists, then so does the box-counting dimension of F_C.

The assumption in Theorem 2.3.6 arises in quite natural circumstances. For example, the setting of the following proposition, an inhomogeneous analogue of Falconer's [15, Proposition 4], requires only that C be in some sense robust under projection onto subspaces. Let $\mathcal{L}^k$ denote k-dimensional Lebesgue measure and P_k denote the set of orthogonal projections onto k-dimensional subspaces of $\mathbb{R}^n$.

Proposition 2.3.7. *Let $F_C \subset \mathbb{R}^n$ be an inhomogeneous self-affine set with compact condensation set $C \subset \mathbb{R}^n$ satisfying the COSC and let $s \le n$ be the associated affinity dimension. If*

$$\inf_{\pi \in P_{n-1}} \mathcal{L}^{n-1}(\pi C) > 0,$$

then

$$\overline{\dim}_{\mathrm{B}} F_C = \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}$$

and

$$\max\left\{s, \underline{\dim}_{\mathrm{B}} C\right\} \le \underline{\dim}_{\mathrm{B}} F_C \le \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}.$$

Proof. Let $\mathbb{I} = \{S_i\}_{i=1}^N$ denote an affine IFS with compact condensation set $C \subset \mathbb{R}^n$ satisfying the COSC. Moreover, suppose

$$\inf_{\pi \in P_{n-1}} \mathcal{L}^{n-1}(\pi C) > 0.$$

By Theorem 2.3.6 it suffices to show that there exists $\kappa > 0$ such that for all $\delta > 0$ and

$\mathbf{i} \in \mathcal{I}_n(\delta)$ we have

$$N_\delta(S_\mathbf{i}(C)) \geq \kappa N_\delta(S_\mathbf{i}(B)).$$

Therefore, in order to reach a contradiction, assume that for arbitrarily small $\kappa > 0$ we can find $\delta > 0$ and $\mathbf{i} \in \mathcal{I}_n(\delta)$ such that

$$N_\delta(S_\mathbf{i}(C)) < \kappa N_\delta(S_\mathbf{i}(B)) \leq \kappa 2^n \frac{\alpha_1(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})} \frac{\alpha_2(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})} \cdots \frac{\alpha_{n-1}(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})},$$

where the final inequality comes from Lemma 2.3.2. Let $\{E_j\}_j$ be an optimal cover of $S_\mathbf{i}(C)$ by hypercubes of sidelength δ and place each E_j inside a ball B_j of diameter $\sqrt{n}\delta$ and consider $\{S_\mathbf{i}^{-1} B_j\}_j$, which is a cover of C by ellipsoids with axes of length $\sqrt{n}\delta/\alpha_1(S_\mathbf{i}), \dots, \sqrt{n}\delta/\alpha_n(S_\mathbf{i})$. Note that, for all j, the longest axes of each of these ellipsoids are all parallel (by the singular value decomposition theorem, for example) and let π denote projection onto the $(n-1)$-dimensional hyperplane orthogonal to the common direction of the longest axes of the ellipsoids $\{S_\mathbf{i}^{-1} B_j\}_j$. It follows that $\{\pi S_\mathbf{i}^{-1} B_j\}_j$ is a cover of $\pi(C)$ by sets, each of which is easily seen to have $(n-1)$-volume at most

$$n^{(n-1)/2} \frac{\delta}{\alpha_1(S_\mathbf{i})} \frac{\delta}{\alpha_2(S_\mathbf{i})} \cdots \frac{\delta}{\alpha_{n-1}(S_\mathbf{i})}$$

and therefore we can bound the $(n-1)$-volume of $\pi(C)$ above by

$$\begin{aligned} &\kappa 2^n \frac{\alpha_1(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})} \frac{\alpha_2(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})} \cdots \frac{\alpha_{n-1}(S_\mathbf{i})}{\alpha_n(S_\mathbf{i})} \times n^{(n-1)/2} \frac{\delta}{\alpha_1(S_\mathbf{i})} \frac{\delta}{\alpha_2(S_\mathbf{i})} \cdots \frac{\delta}{\alpha_{n-1}(S_\mathbf{i})} \\ &\leq \kappa 2^n n^{(n-1)/2} b^{-(n-1)}, \end{aligned}$$

using (2.3.6). This contradicts the assumption that $\inf_{\pi \in P_{n-1}} \mathcal{L}^{n-1}(\pi C) > 0$ since we can choose κ arbitrarily small. $\square$

The robustness assumption on C in Proposition 2.3.7 forces $\underline{\dim}_\mathrm{B} C \geq n-1$ and so this result only yields new information when $s > n-1$. Moreover, observe that the projection of a connected set in $\mathbb{R}^2$ which is not contained in a line onto a line contains an interval with length uniformly bounded away from 0. This observation yields the following corollary of Proposition 2.3.7.

Corollary 2.3.8. *Let $F_C \subset \mathbb{R}^2$ be an inhomogeneous self-affine set with compact condensation set $C \subset \mathbb{R}^2$ satisfying the COSC and affinity dimension $s \leq 2$. If C has a connected component not contained in a line, then*

$$\overline{\dim}_{\mathrm{B}} F_C = \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}$$

and

$$\max\left\{s, \underline{\dim}_{\mathrm{B}} C\right\} \leq \underline{\dim}_{\mathrm{B}} F_C \leq \max\left\{s, \overline{\dim}_{\mathrm{B}} C\right\}.$$

The reader may find it interesting to notice the parallels between this result and Falconer's [15, Corollary 5], which concerns the equality of $\overline{\dim}_{\mathrm{B}} F_\emptyset$ and s under similar conditions concerning the robustness of connected components under projection. In some sense our inhomogeneous analogue is easier to use than the homogeneous result of Falconer. Our result requires a connectedness condition on C, which is given, whereas the homogeneous result requires one to check a connectedness condition on $F_\emptyset$, which depends delicately on the IFS. Moreover, the separation assumption makes it difficult for $F_\emptyset$ to be connected at all. For example, the strong separation condition forces $F_\emptyset$ to be totally disconnected, but our result can still apply in this setting.

The above results provide new families of inhomogeneous attractors where (2.1.1) fails for the upper (and lower) box-counting dimension. We illustrate this by example. Let $n = 2$ and $\mathbb{I} = \{S_1, S_2\}$, where S_1, S_2 are the linear maps associated with the matrices

$$\begin{bmatrix} 1/2 & 0 \\ 1/2 & 1/2 \end{bmatrix}, \qquad \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

respectively. It is clear that the affinity dimension of this system is strictly greater than one and that $F_\emptyset$ is just a single point at the origin. Let C be the boundary of a circle centred at $(3/4, 3/4)$ with radius $1/5$. It is also clear that the COSC is satisfied by taking $U = (0,1)^2$ and that C is connected but not contained in a line, see Figure 2.2. It follows from Corollary 2.3.8 that

$$\overline{\dim}_{\mathrm{B}} F_C = \underline{\dim}_{\mathrm{B}} F_C = s > 1 = \max\left\{\dim_{\mathrm{B}} F_\emptyset, \dim_{\mathrm{B}} C\right\}.$$

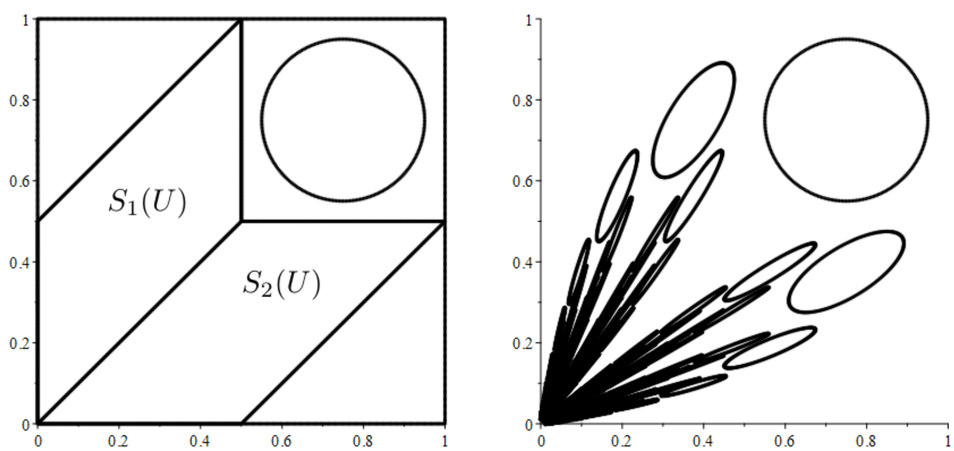

Figure 2.2: A bouquet of ovals: the condensation set together with the two images of the open rectangle $U = (0,1)^2$ (left) and the corresponding inhomogeneous self-affine set (right).

This is the first counter example to (2.1.1) where $F_\emptyset$ is a single point and the OSC is satisfied. Moreover, it was shown in [1, Corollary 4.9] that for planar inhomogeneous self-*similar* sets one always has

$$\overline{\dim}_{\mathrm{B}} F_C \leq \max\left\{\overline{\dim}_{\mathrm{B}} C,\ \overline{\dim}_{\mathrm{B}} F_\emptyset + \overline{\dim}_{\mathrm{B}} C - \frac{\overline{\dim}_{\mathrm{B}} F_\emptyset \overline{\dim}_{\mathrm{B}} C}{s}\right\},$$

where s is the similarity dimension. In particular this shows that when $\overline{\dim}_{\mathrm{B}} F_\emptyset = 0$ the formula (2.1.1) cannot fail. The example presented above shows that this phenomenon does *not* extend to the self-affine case. It was also shown in [1, Corollary 4.8] that, in the self-similar setting, if $\max\{\overline{\dim}_{\mathrm{B}} F_{\emptyset}, \overline{\dim}_{\mathrm{B}} C\} < s$, then $\overline{\dim}_{\mathrm{B}} F_C < s$. The above example also demonstrates that this does not extend to the self-affine setting.

The assumption in Proposition 2.3.7 is by no means necessary, and advancements in the homogeneous setting may illuminate further the capacity for C to mitigate dimension drop. Excitingly, we suggest the natural interplay between these questions may allow further study of inhomogeneous attractors to translate into novel conditions relating to dimension drop in the homogeneous case.

Chapter 3

Projections and fractional Brownian images

3.1 Introduction

Theorems on dimensions of projections of fractals in Euclidean space have a long history. In 1954 Marstrand [54] proved that the Hausdorff dimension of the orthogonal projections of a Borel set $E \subset \mathbb{R}^2$ onto linear subspaces was almost surely constant. More specifically,

$$\dim_{\mathrm{H}} \pi_V E = \min\{\dim_{\mathrm{H}} E, 1\},$$

for almost all one-dimensional subspaces V, where π_V denotes orthogonal projection onto V. Kaufman gave a potential-theoretic proof of Marstrand's result [49], and in 1975 Mattila extended it to Borel sets $E \subset \mathbb{R}^n$ and almost-all subspaces V in the Grassmannian $G(n, m)$ with respect to the natural invariant probablity measure [55]. These seminal results set in motion a sustained interest in the behaviour of dimension under projections, see [17, 56] for basic expositions and [22, 58, 63] for recent surveys.

It is natural to seek projection results for the various other dimensions that occur throughout fractal geometry. For example, Järvenpää showed that for the box-counting dimension an exact analogue of the Marsrand-Mattila result could not hold [44]. However, in 1997 Falconer and Howroyd showed that the upper and lower box-counting dimensions of the projections of a set are almost surely constant and given by what they termed a 'dimension profile' [23, 42], reflecting how a set in $\mathbb{R}^n$ appears when viewed from an m-dimensional perspective for $m \in \{1, \ldots, n\}$. The link to stochastic processes then came from Xiao [68], who used dimension profiles almost immediately after their introduction to consider the almost-sure value of the dimensions of fractional Brownian images, a connection also explored in [18, 50].

However, the dimension profiles were, in their original form, implicitly defined and somewhat awkward to work with, leading to a recent re-working of the theory using a potential-theoretic approach [18, 19]. In this chapter, we build on the methodology of [18, 19] to study the intermediate dimensions (see Section 1.4), first to give a definition of these dimensions in terms of capacities with respect to certain kernels, and then to consider projections and fractional Brownian images using the associated dimension profiles. To conclude, some observations and applications are given.

3.2 Capacities and dimension profiles

In this section we introduce a notion of dimension derived from capacities that is closely related to the intermediate dimensions and which is amenable to studying projections and fractional Brownian images. The first step in defining potential theoretic concepts such as the capacity of a set is to choose an appropriate kernel. Throughout, let $\theta \in (0, 1]$ and $0 < t \leq n$.

For $0 \leq s \leq t$ and $0 < r < 1$, define the potential kernels

$$\phi_{r,\theta}^{s,t}(x) = \begin{cases} 1 & 0 \leq |x| < r \\ \left(\frac{r}{|x|}\right)^s & r \leq |x| < r^\theta \\ \frac{r^{\theta(t-s)+s}}{|x|^t} & r^\theta \leq |x| \end{cases} \qquad (x \in \mathbb{R}^n). \tag{3.2.1}$$

When $s = t$, this becomes

$$\phi_{r,\theta}^{t,t}(x) = \begin{cases} 1 & 0 \leq |x| < r \\ \left(\frac{r}{|x|}\right)^t & r \leq |x| \end{cases} \qquad (x \in \mathbb{R}^n), \tag{3.2.2}$$

and so corresponds to the kernel $\phi_r^t(x)$ used in [18, 19] in the context of box-counting dimension if $t \in \mathbb{N}$. As one would expect, this kernel is also recovered when $\theta = 1$ where $\phi_{r,\theta}^{s,t}$ is independent of s. Note that $\phi_{r,\theta}^{s,t}(x)$ is continuous in x and monotonically decreasing in $|x|$. Letting $\mathcal{M}(E)$ denote the set of Borel probability measures supported on E, we say that the *energy* of $\mu \in \mathcal{M}(E)$ with respect to $\phi_{r,\theta}^{s,t}$ is

$$\int\int \phi_{r,\theta}^{s,t}(x-y)\, d\mu(x)d\mu(y)$$

and the *potential* of μ at $x \in \mathbb{R}^n$ is

$$\int \phi_{r,\theta}^{s,t}(x-y)\, d\mu(y).$$

We define the *capacity* $C_{r,\theta}^{s,t}(E)$ of E to be the reciprocal of the minimum energy achieved by probability measures on E, that is

$$C_{r,\theta}^{s,t}(E) = \left(\inf_{\mu \in \mathcal{M}(E)} \int\int \phi_{r,\theta}^{s,t}(x-y)\, d\mu(x)d\mu(y)\right)^{-1}.$$

Since $\phi_{r,\theta}^{s,t}(x)$ is continuous in x, strictly positive and E is compact, $C_{r,\theta}^{s,t}(E)$ is positive and finite. For bounded sets that are not closed, we take the capacity to be that of the closure.

A measure that obtains the infimum in the definition of capacity is known as an *equilibrium measure*. The existence of such measures and the relationship between the minimal energy and the corresponding potentials is standard in classical potential theory. We state this in a convenient form; it is easily proved for continuous kernels, see, for example, [19, Lemma 2.1].

Lemma 3.2.1. *Let $E \subset \mathbb{R}^n$ be compact, $0 < t \leq n$, $0 \leq s \leq t$, $\theta \in (0, 1]$ and $0 < r < 1$. Then there exists an equilibrium measure $\mu \in \mathcal{M}(E)$ such that*

$$\int \int \phi_{r,\theta}^{s,t}(x - y) d\mu(x) d\mu(y) = \frac{1}{C_{r,\theta}^{s,t}(E)} =: \beta.$$

Moreover,

$$\int \phi_{r,\theta}^{s,t}(x - y) d\mu(y) \geq \beta$$

for all $x \in E$, with equality for μ-almost all $x \in E$.

As we will see, these capacities are closely related to the sums considered in Section 3.3. The following lemma, which parallels Lemma 3.3.1, enables us to define 'intermediate dimension profiles'.

Lemma 3.2.2. *Let $E \subset \mathbb{R}^n$ be compact, $0 < t \leq n$, $\theta \in (0, 1]$ and $E \subset \mathbb{R}^n$. If $0 < r < 1$, then for all $0 \leq s' \leq s \leq t$,*

$$-(s - s') \leq \left(\frac{\log C_{r,\theta}^{s,t}(E)}{-\log r} - s \right) - \left(\frac{\log C_{r,\theta}^{s',t}(E)}{-\log r} - s' \right) \leq -\theta(s - s'). \tag{3.2.3}$$

Moreover, there is a unique $\underline{s} \in [0, t]$ such that $\liminf_{r \to 0} \frac{\log C_{r,\theta}^{\underline{s},t}(E)}{-\log r} = \underline{s}$ and a unique $\overline{s} \in [0, t]$ such that $\limsup_{r \to 0} \frac{\log C_{r,\theta}^{\overline{s},t}(E)}{-\log r} = \overline{s}$.

Proof. By comparison of the kernels it is easily checked that, for $0 \le s' \le s \le t$,

$$\phi^{s,t}_{r,\theta}(x) \le \phi^{s',t}_{r,\theta}(x) \le r^{(s'-s)(1-\theta)}\phi^{s,t}_{r,\theta}(x) \qquad (x \in \mathbb{R}^n).$$

Using the definition of capacity and that an equilibrium measure on E for the kernel $\phi^{s,t}_{r,\theta}$ is a candidate for an equilibrium measure for $\phi^{s',t}_{r,\theta}$ and vice-versa, we obtain

$$C^{s,t}_{r,\theta}(E) \ge C^{s',t}_{r,\theta}(E) \ge r^{(s-s')(1-\theta)}C^{s,t}_{r,\theta}(E).$$

Taking logarithms and rearranging gives (3.2.3).

The inequalities (3.2.3) remain true on taking lower limits of the quotients, so

$$\liminf_{r\to 0} \frac{\log C^{s,t}_{r,\theta}(E)}{-\log r} - s$$

is strictly monotonic decreasing and continuous in $s \in [0,t]$.

Next, we show

$$\liminf_{r\to 0} \frac{\log C^{t,t}_{r,\theta}(E)}{-\log r} - t \le 0,$$

or, equivalently,

$$C^{t,t}_{r,\theta}(E) \le cr^{-t} \tag{3.2.4}$$

for some fixed $c > 0$ depending only on E and t. Let $0 < r < 1$ and μ be the equilibrium measure associated with $\phi^{t,t}_{r,\theta}$. Since E is bounded, there exists a constant $B > 1$ such that

$$|x - y| \le B$$

for all $x, y \in E$.

Then, directly from the definition,

$$\phi^{t,t}_{r,\theta}(x-y) = \begin{cases} 1 & 0 \le |x-y| < r \\ \left(\frac{r}{|x-y|}\right)^t & r \le |x-y| \end{cases}$$
$$\ge B^{-t} r^t$$

for all $x, y \in E$. Hence,

$$\int\int \phi^{t,t}_{r,\theta}(x-y)\, d\mu(x) d\mu(y) \ge B^{-t} r^t,$$

from which (3.2.4) follows.

Since the kernels are bounded above by 1, $C^{0,t}_{r,\theta}(E) \ge 1$, so

$$\liminf_{r\to 0} \frac{\log C^{0,t}_{r,\theta}(E)}{-\log r} - 0 \ge 0.$$

We conclude that there is a unique $\underline{s} \in [0,t]$ such that $\liminf_{r\to 0} \frac{\log C^{\underline{s},t}_{r,\theta}(E)}{-\log r} = \underline{s}$, and similarly argue for the upper limits and $\overline{s}$. $\square$

Thus, for $t \in (0,n]$, we define the *lower intermediate dimension profile* of $E \subset \mathbb{R}^n$ as

$$\underline{\dim}^t_\theta E = \left(\text{the unique } s \in [0,t] \text{ such that } \liminf_{r\to 0} \frac{\log C^{s,t}_{r,\theta}(E)}{-\log r} = s \right) \tag{3.2.5}$$

and the *upper intermediate dimension profile* as

$$\overline{\dim}^t_\theta E = \left(\text{the unique } s \in [0,t] \text{ such that } \limsup_{r\to 0} \frac{\log C^{s,t}_{r,\theta}(E)}{-\log r} = s \right). \tag{3.2.6}$$

When the context is clear, we may write lower dimension profile and upper dimension profile, for brevity.

Lemma 3.2.3. *The intermediate dimension profiles are increasing in m, that is, for compact E, $\theta \in (0,1]$ and $1 \le t_1 \le t_2 \le n$*

$$\underline{\dim}^{t_1}_{\theta} E \le \underline{\dim}^{t_2}_{\theta} E \qquad \textit{and} \qquad \overline{\dim}^{t_1}_{\theta} E \le \overline{\dim}^{t_2}_{\theta} E.$$

Proof. This follows immediately noting that the kernels $\phi^{s,t}_{r,\theta}(x)$ are clearly *decreasing* in t. $\square$

The next section concerns the relationship between the intermediate dimensions of a set E, defined in terms of the sums over restricted covers of E, and intermediate dimension profiles, defined in terms of capacities. We will see that the dimension profiles recover the intermediate definitions when $t = n$, that is $\underline{\dim}_{\theta} E = \underline{\dim}^{n}_{\theta} E$ for $E \subset \mathbb{R}^n$.

3.3 Capacities and intermediate dimensions

For our purposes it is convenient to work with equivalent definitions of the intermediate dimensions in terms of limits of logarithms of sums over covers. For bounded and non-empty $E \subset \mathbb{R}^n$, $\theta \in (0,1]$ and $s \in [0,n]$, define

$$S^s_{r,\theta}(E) := \inf\Big\{ \sum_i |U_i|^s : \{\mathrm{U_i}\}_\mathrm{i} \text{ is a cover of E with } \mathrm{r} \le |\mathrm{U_i}| \le \mathrm{r}^\theta \text{ for all i}\Big\}. \tag{3.3.1}$$

We claim

$$\underline{\dim}_{\theta} E = \left(\text{the unique } s \in [0,n] \text{ such that } \liminf_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = 0 \right) \tag{3.3.2}$$

and

$$\overline{\dim}_{\theta} E = \left(\text{the unique } s \in [0,n] \text{ such that } \limsup_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = 0 \right). \tag{3.3.3}$$

It is easy to see from (1.4.1) and (1.4.2) that $\underline{\dim}_\theta E$ and $\overline{\dim}_\theta E$ are the infima of s for which these lower and upper limits equal 0; that there are unique such values follows from the following lemma.

Lemma 3.3.1. *Let $\theta \in (0,1]$ and $E \subset \mathbb{R}^n$. For each $0 < r < 1$,*

$$-(s-t) \leq \frac{\log S^s_{r,\theta}(E)}{-\log r} - \frac{\log S^t_{r,\theta}(E)}{-\log r} \leq -\theta(s-t) \qquad (0 \leq t \leq s \leq n). \tag{3.3.4}$$

Moreover, there is a unique $\underline{s} \in [0,n]$ such that $\liminf_{r\to 0} \frac{\log S^{\underline{s}}_{r,\theta}(E)}{-\log r} = 0$ and a unique $\overline{s} \in [0,n]$ such that $\limsup_{r\to 0} \frac{\log S^{\overline{s}}_{r,\theta}(E)}{-\log r} = 0$.

Proof. For a cover $\{U_i\}$ of E satisfying $r \leq |U_i| \leq r^\theta$ and $0 \leq t \leq s \leq n$,

$$\sum_i |U_i|^t r^{s-t} \leq \sum_i |U_i|^s \leq \sum_i |U_i|^t r^{\theta(s-t)}.$$

Taking infima over all such covers yields

$$r^{s-t} S^t_{r,\theta}(E) \leq S^s_{r,\theta}(E) \leq r^{\theta(s-t)} S^t_{r,\theta}(E),$$

from which (3.3.4) follows. These inequalities carry over on taking lower limits of the quotients, so in particular

$$\liminf_{r\to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r}$$

is strictly monotonically decreasing and continuous for $s \in [0,n]$. Since $S^0_{r,\theta}(E)$ is bounded below by the box-counting number of E at scale r^θ, it follows that

$$\liminf_{r\to 0} \frac{\log S^0_{r,\theta}(E)}{-\log r} \geq \theta\, \underline{\dim}_{\mathrm{B}} E \geq 0.$$

Also, $S^n_{r,\theta}(E)$ is bounded above by the n-dimensional volume of a ball containing E, so

$$\liminf_{r\to 0} \frac{\log S^n_{r,\theta}(E)}{-\log r} \leq 0.$$

Continuity now gives a unique $\underline{s} \in [0,n]$ such that $\liminf_{r\to 0} \frac{\log S^{\underline{s}}_{r,\theta}(E)}{-\log r} = 0$. A similar argument holds for upper limits. $\square$

Next, we see how to characterise the intermediate dimensions of sets $E \subset \mathbb{R}^n$ in terms of dimension profiles which we have defined in terms of capacities $C^{s,n}_{r,\theta}(E)$ with respect to the kernels $\phi^{s,n}_{r,\theta}$. We begin with two lemmas that relate the capacity of a set to sums over restricted covers. Throughout, we may assume that E is compact since the intermediate dimensions are stable under taking closures for $\theta > 0$, see [21].

Lemma 3.3.2. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, $0 < r < 1$ and $0 \leq s \leq n$. Then*

$$r^s C^{s,n}_{r,\theta}(E) \leq S^s_{r,\theta}(E). \tag{3.3.5}$$

Proof. By Lemma 3.2.1 there exists an equilibrium measure $\mu \in \mathcal{M}(E)$ and a set E_0 with $\mu(E_0) = 1$ such that

$$\int \phi^{s,n}_{r,\theta}(x-y)d\mu(y) = \frac{1}{C^{s,n}_{r,\theta}(E)} =: \beta$$

for all $x \in E_0$. Let $r \leq \delta \leq r^\theta$ and $x \in E_0$. Then

$$\beta = \int \phi^{s,n}_{r,\theta}(x-y)d\mu(y) \geq \int \left(\frac{r}{\delta}\right)^s 1_{B(0,\delta)}(x-y)d\mu(y) \geq \left(\frac{r}{\delta}\right)^s \mu(B(x,\delta)). \tag{3.3.6}$$

Let $\{U_i\}_i$ be a finite cover of E by sets of diameters $r \leq |U_i| \leq r^\theta$ and define $\mathcal{I} = \{i : U_i \cap E_0 \neq \emptyset\}$. Then for each $i \in \mathcal{I}$, there exists $x_i \in U_i \cap E_0$ so that $U_i \subset B(x_i, |U_i|)$.

Hence

$$1 = \mu(E_0) \le \sum_{i\in\mathcal{I}} \mu(U_i) \le \sum_{i\in\mathcal{I}} \mu(B(x_i, |U_i|)) \le r^{-s}\beta \sum_{i\in\mathcal{I}} |U_i|^s$$

by (3.3.6), and so

$$\sum_i |U_i|^s \ge r^s C^{s,n}_{r,\theta}(E),$$

which yields the desired result upon taking the infimum over all such covers. Note that $C^{s,t}_{r,\theta}(E) \le C^{s,n}_{r,\theta}(E)$ for $t \le n$, so (3.3.5) implies $r^s C^{s,t}_{r,\theta}(E) \le S^s_{r,\theta}(E)$. □

In the next proof, we use potential estimates to find a Besicovitch cover of E by balls of relatively large measure. The Besicovitch covering lemma gives a bounded number of families of disjoint such balls with their union covering E. The balls with diameters between r and r^θ, together with covers of any larger balls by balls of diameters at most r^θ, provide efficient covers for estimating the sums $S^s_{r,\theta}(E)$.

Lemma 3.3.3. *Let $E \subset \mathbb{R}^n$ be compact, $0 \le s \le n$ and $\theta \in (0,1]$. If there exists a measure $\mu \in \mathcal{M}(E)$ and $\beta > 0$ such that*

$$\int \phi^{s,n}_{r,\theta}(x-y)d\mu(y) \ge \beta \tag{3.3.7}$$

for all $x \in E$, then there is a number $r_0 > 0$ such that for all $0 < r \le r_0$,

$$S^s_{r,\theta}(E) \le a_n \lceil \log_2(|E|/r) + 1 \rceil \frac{r^s}{\beta}$$

where the constant a_n depends only on n. In particular,

$$S^s_{r,\theta}(E) \le a_n \lceil \log_2(|E|/r) + 1 \rceil C^{s,n}_{r,\theta}(E) r^s.$$

Proof. To avoid ambiguity we will assume that $\theta \in (0,1)$, though the proof is virtually the same when $\theta = 1$, essentially by taking $M = 0$; this 'box-counting dimension' case is also covered in [19].

Let $D = \lceil \log_2(|E|/r) \rceil$ and let M be the integer satisfying

$$2^{M-1} r < r^{\theta} \leq 2^M r. \tag{3.3.8}$$

We choose r_0 sufficiently small to ensure that $2 \leq M \leq D-2$ for all $0 < r \leq r_0$. For $x \in E$, using (3.3.7) and estimating the kernel $\phi^{s,n}_{r,\theta}(x-y)$ given by (3.2.1) over consecutive annuli $B(x, 2^k r) \setminus B(x, 2^{k-1} r)$ $(1 \leq k \leq D)$,

$$\begin{aligned}
\beta &\leq \int \phi^{s,n}_{r,\theta}(x-y) d\mu(y) \\
&\leq \mu(B(x,r)) + \sum_{k=1}^{D} \int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} \phi^{s,n}_{r,\theta}(x-y) d\mu(y) \\
&\leq \mu(B(x,r)) + \sum_{k=1}^{M} \int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} 2^{-(k-1)s} d\mu(y) \\
&\quad + \sum_{k=M+1}^{D} \int_{B(x,2^k r)\setminus B(x,2^{k-1}r)} r^{\theta(n-s)+s} (2^{k-1} r)^{-n} d\mu(y) \\
&\leq \sum_{k=0}^{M-2} 2^s \mu(B(x, 2^k r)) 2^{-ks} + \sum_{k=M-1}^{M} 2^s \mu(B(x, 2^k r)) 2^{-ks} \\
&\quad + r^{(\theta-1)(n-s)} \sum_{k=M+1}^{D} \mu(B(x, 2^k r)) 2^{-(k-1)n}.
\end{aligned}$$

Hence, for each $x \in E$, there exists some integer $0 \leq k(x) \leq D$ such that one of the above summands is at least the arithmetic mean of the sum. There are three cases. We will use that there are numbers d_n depending only on n such that every ball of radius ρ in $\mathbb{R}^n$ may be covered by at most $\lambda^{-n} d_n$ balls of diameter $\lambda\rho$ for all $0 < \lambda \leq 1$ ($d_n = 3^n n^{n/2}$ will certainly do).

(i) If $0 \leq k(x) \leq M-2$ then

$$\frac{\beta}{D+1} \leq 2^s \mu(B(x, 2^{k(x)}r))2^{-k(x)s} = 4^s \mu(B(x, 2^{k(x)}r))|B(x, 2^{k(x)}r)|^{-s} r^s,$$

so

$$|B(x, 2^{k(x)}r)|^s \leq (D+1)\beta^{-1}4^s r^s \mu(B(x, 2^{k(x)}r)); \tag{3.3.9}$$

(ii) if $M-1 \leq k(x) \leq M$ then

$$\begin{aligned}\frac{\beta}{D+1} &\leq 2^s \mu(B(x, 2^{k(x)}r))2^{-k(x)s} \\ &\leq \mu(B(x, 2^{k(x)}r))2^s 2^{-(M-1)s} \\ &\leq \mu(B(x, 2^{k(x)}r))2^{2s} r^{(1-\theta)s},\end{aligned}$$

so

$$4^n d_n\, r^{\theta s} \leq 4^n 2^{2s}(D+1)\beta^{-1} d_n r^s \mu(B(x, 2^{k(x)}r)); \tag{3.3.10}$$

(iii) if $M+1 \leq k(x) \leq D$ then

$$\frac{\beta}{D+1} \leq r^{(\theta-1)(n-s)}\mu(B(x, 2^{k(x)}r))2^{-(k(x)-1)n},$$

so

$$d_n 2^{k(x)n} r^{(1-\theta)n} \leq 2^n (D+1)\beta^{-1} d_n r^{s(1-\theta)} \mu(B(x, 2^{k(x)}r)). \tag{3.3.11}$$

The cover of E by the balls $\mathcal{B} = \{B(x, 2^{k(x)}r) : x \in E\}$ is a Besicovitch cover, that is each point of E is at the centre of some ball in the collection. The Besicovitch covering theorem, see for example [55, Theorem 2.7], allows us to extract subcollections $\mathcal{C}_1, \ldots, \mathcal{C}_{c_n}$ of disjoint balls from $\mathcal{B}$ where c_n depends only on n and such that $E \subset \bigcup_i \bigcup_{B \in \mathcal{C}_i} B$. Let

$$\mathcal{E}_i = \{B(x, 2^{k(x)}r) \in \mathcal{C}_i : M-1 \leq k(x) \leq M\}$$

and

$$\mathcal{F}_i = \{B(x, 2^{k(x)}r) \in \mathcal{C}_i : M+1 \leq k(x) \leq D\}.$$

From (3.3.8) each $B \in \mathcal{C}_i \setminus (\mathcal{E}_i \cup \mathcal{F}_i)$ has diameter at most r^θ. Also, for each $B = B(x, 2^{k(x)}r) \in \mathcal{E}_i$ let $\mathcal{D}_B$ denote a collection of at most $(2^M r/r^\theta)^n d_n \leq 2^n d_n$ balls of diameter r^θ that cover B, and for each $B = B(x, 2^{k(x)}r) \in \mathcal{F}_i$ let $\mathcal{D}_B$ denote a collection of at most $\left(2^{k(x)} r/r^\theta\right)^n d_n$ balls of diameter r^θ that cover B.

For each $i = 1, \ldots, c_n$, we consider the cover

$$\widetilde{\mathcal{C}}_i := \big(\mathcal{C}_i \setminus (\mathcal{E}_i \cup \mathcal{F}_i)\big) \cup \bigcup_{B \in \mathcal{E}_i \cup \mathcal{F}_i} \mathcal{D}_B$$

of $\bigcup\limits_{B \in \mathcal{C}_i} B$. Then using (3.3.9) - (3.3.11),

$$\begin{aligned}
&\sum_{B \in \mathcal{C}_i \setminus (\mathcal{E}_i \cup \mathcal{F}_i)} |B|^s + \sum_{B \in \mathcal{E}_i} \sum_{B' \in \mathcal{D}_B} |B'|^s + \sum_{B \in \mathcal{F}_i} \sum_{B' \in \mathcal{D}_B} |B'|^s \\
&\qquad \leq 4^s (D+1) \frac{r^s}{\beta} \sum_{B \in \mathcal{C}_i \setminus (\mathcal{E}_i \cup \mathcal{F}_i)} \mu(B) + \sum_{B \in \mathcal{E}_i} 4^n d_n \, r^{\theta s} \\
&\qquad\quad + \sum_{B \in \mathcal{F}_i} d_n \left(\frac{2^{k(x)} r}{r^\theta} \right)^n r^{\theta s} \\
&\qquad \leq 4^s (D+1) \frac{r^s}{\beta} + \sum_{B \in \mathcal{E}_i} \frac{4^n 2^{2s} (D+1) d_n}{\beta} r^s \mu(B) \\
&\qquad\quad + \sum_{B \in \mathcal{F}_i} \frac{2^n (D+1) d_n}{\beta} r^{s(1-\theta)} r^{\theta s} \mu(B) \\
&\qquad \leq 4^s (D+1) \frac{r^s}{\beta} + \frac{4^n 2^{2s} (D+1) d_n}{\beta} r^s \sum_{B \in \mathcal{E}_i} \mu(B) \\
&\qquad\quad + \frac{2^n (D+1) d_n}{\beta} r^s \sum_{B \in \mathcal{F}_i} \mu(B) \\
&\qquad \leq (4^n + 2 \cdot 4^{2n} d_n)(D+1) \frac{r^s}{\beta},
\end{aligned}$$

where we have used that $\mathcal{C}_i$ is a disjoint collection of balls. Hence, writing $\mathcal{C} = \bigcup_i \widetilde{\mathcal{C}}_i$,

$$S^s_{r,\theta}(E) \leq \sum_{B \in \mathcal{C}} |B|^s \leq c_n(4^n + 2 \cdot 4^{2n} d_n)(D+1)\frac{r^s}{\beta} = a_n \lceil \log_2(|E|/r) + 1 \rceil \frac{r^s}{\beta}$$

on setting $a_n = c_n(4^n + 2 \cdot 4^{2n} d_n)$. $\square$

In the next section, Lemma 3.3.3 will be important when considering intermediate dimensions of projections and fractional Brownian images. We summarise the previous two results in the following proposition.

Proposition 3.3.4. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, and $0 \leq s \leq n$. Then there is a number $r_0 > 0$ such that for all $0 < r \leq r_0$,*

$$r^s C^{s,n}_{r,\theta}(E) \leq S^s_{r,\theta}(E) \leq a_n \lceil \log_2(|E|/r) + 1 \rceil r^s C^{s,n}_{r,\theta}(E), \tag{3.3.12}$$

where the number a_n depends only on n. Consequently

$$\liminf_{r \to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = -s + \liminf_{r \to 0} \frac{\log C^{s,n}_{r,\theta}(E)}{-\log r} \tag{3.3.13}$$

and

$$\limsup_{r \to 0} \frac{\log S^s_{r,\theta}(E)}{-\log r} = -s + \limsup_{r \to 0} \frac{\log C^{s,n}_{r,\theta}(E)}{-\log r}. \tag{3.3.14}$$

Proof. The left hand inequality of (3.3.12) follows from Lemma 3.3.2 and the right hand inequality from Lemma 3.3.3. Then, (3.3.13) and (3.3.14) are obtained by re-arranging and taking appropriate limits. $\square$

The fruit of this labour is now apparent; when the parameter of our dimension profile is equal to the topological dimension of the ambient space, they simply recover the intermediate dimensions.

Theorem 3.3.5. *Let $E \subset \mathbb{R}^n$ be bounded and $\theta \in (0,1]$. Then*

$$\underline{\dim}_{\theta} E = \underline{\dim}^n_{\theta} E$$

and

$$\overline{\dim}_{\theta} E = \overline{\dim}^n_{\theta} E.$$

Proof. This is an immediate consequence of Proposition 3.3.4, together with the definitions (3.3.2), (3.3.3), (3.2.5) and (3.2.6). $\square$

3.4 Projections and fractional Brownian images

In this section, we will see how the profiles may be thought of as viewing a set E from an m-dimensional viewpoint for $m \in \{1, \ldots, n\}$, and more generally provide information on the intermediate dimensions of fractional Brownian images.

Let us begin by briefly recalling the definition of index-α fractional Brownian motion ($0 < \alpha < 1$), which we denote $B_\alpha : \mathbb{R}^n \to \mathbb{R}^m$ for $m \leq n$. In particular, $B_\alpha = (B_{\alpha,1}, \ldots, B_{\alpha,m})$, where for each $B_{\alpha,i} : \mathbb{R}^n \to \mathbb{R}$:

i) $B_{\alpha,i}(0) = 0$;

ii) $B_{\alpha,i}$ is continuous with probability 1;

iii) the increments $B_{\alpha,i}(x) - B_{\alpha,i}(y)$ are normally distributed with with mean 0 and variance $|x-y|^{2\alpha}$ for all $x, y \in \mathbb{R}^n$.

It immediately follows that, for Borel $A \subset \mathbb{R}$,

$$\mathbb{P}(B_{\alpha,i}(x) - B_{\alpha,i}(y) \in A) = \frac{1}{\sqrt{2\pi}} \frac{1}{|x-y|^\alpha} \int_{t \in A} \exp\left(\frac{-t^2}{2|x-y|^{2\alpha}}\right) dt. \tag{3.4.1}$$

As a stochastic process, it enjoys many of the same properties as standard Brownian motion. For example, the process is self-affine, meaning the scaled processes $c^{-\alpha}B_\alpha(ct)$ have the same statistical distribution as $B_\alpha(t)$ for $c > 0$ [17]. The reader may enjoy the classical text of Kahane [47] for a more detailed account of index-α fractional Brownian motion.

Our first result establishes an upper bound on the intermediate dimensions of Hölder images using dimension profiles, motivated by the fact index-α fractional Brownian motion is almost surely $(\alpha - \varepsilon)$-Hölder for all $\varepsilon > 0$, while projection is 1-Hölder.

Theorem 3.4.1. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0, 1)$, $m \in \{1, \ldots, n\}$ and $f : E \to \mathbb{R}^m$. If there exists $c > 0$ and $0 < \alpha \leq 1$ such that*

$$|f(x) - f(y)| \leq c|x - y|^\alpha \tag{3.4.2}$$

for all $x, y \in E$, then

$$\underline{\dim}_\theta f(E) \leq \frac{1}{\alpha}\underline{\dim}^{m\alpha}_\theta E$$

and

$$\overline{\dim}_\theta f(E) \leq \frac{1}{\alpha}\overline{\dim}^{m\alpha}_\theta E.$$

Proof. To prove Theorem 3.4.1 we use Lemma 3.3.3. Intermediate dimension is invariant under scaling and thus we may assume the Hölder constant c in (3.4.2) equals one. First, note

$$\frac{r^s}{|x - y|^{\alpha s}} \leq \frac{r^{\theta(m-s)+s}}{|x - y|^{\alpha m}}$$

for $|x - y| \leq r^{\theta/\alpha}$. It then follows from the definition of $\phi^{s,m}_{r,\theta}$ that

$$\phi^{s,m}_{r,\theta}(f(x) - f(y)) = \min\left\{1, \frac{r^s}{|f(x) - f(y)|^s}, \frac{r^{\theta(m-s)+s}}{|f(x) - f(y)|^m}\right\}$$

$$
\begin{aligned}
&\geq \min\left\{1, \frac{r^s}{|x-y|^{\alpha s}}, \frac{r^{\theta(m-s)+s}}{|x-y|^{\alpha m}}\right\} \\
&= \begin{cases} 1 & |x-y| < r^{1/\alpha} \\ \left(r^{1/\alpha}/|x-y|\right)^{s\alpha} & r^{1/\alpha} \leq |x-y| \leq r^{\theta/\alpha} \\ (r^{1/\alpha})^{\theta(m\alpha - s\alpha)+s\alpha} / \left(|x-y|\right)^{m\alpha} & |x-y| > r^{\theta/\alpha} \end{cases} \\
&= \phi^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(x-y).
\end{aligned}
$$

By Lemma 3.2.1, for each $0 \leq s \leq m$ there exists a measure $\mu \in \mathcal{M}(E)$ such that for all $x \in E$

$$
\begin{aligned}
\frac{1}{C^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(E)} &\leq \int \phi^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(x-y)d\mu(y) \\
&\leq \int \phi^{s,m}_{r,\theta}(f(x)-f(y))d\mu(y) \\
&\leq \int \phi^{s,m}_{r,\theta}(f(x)-w)d(f\mu)(w),
\end{aligned}
$$

where $f\mu \in \mathcal{M}(E)$ is defined by $\int g(w)d(f\mu)(w) = \int g(f(x))d\mu(x)$ for all continuous functions g and by extension. This verifies that $f(E)$ supports a measure satisfying the condition of Lemma 3.3.3. Hence, for sufficiently small $r > 0$,

$$
S^s_{r,\theta}(f(E)) \leq a_m \lceil \log_2(|E|/r) + 1 \rceil r^s C^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(E)
$$

for all $0 \leq s \leq m$. This implies

$$
\liminf_{r\to 0} \frac{S^s_{r,\theta}(f(E))}{-\log r} \leq -s + \liminf_{r\to 0} \frac{C^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(E)}{-\alpha \log r^{1/\alpha}},
$$

and so

$$
\alpha \liminf_{r\to 0} \frac{S^s_{r,\theta}(f(E))}{-\log r} \leq -s\alpha + \liminf_{r\to 0} \frac{C^{s\alpha,m\alpha}_{r^{1/\alpha},\theta}(E)}{-\log r^{1/\alpha}}. \tag{3.4.3}
$$

Recall,

$$\frac{1}{\alpha}\underline{\dim}_\theta^{m\alpha} E \leq \frac{1}{\alpha} m\alpha = m,$$

and thus we may set $s\alpha = \underline{\dim}_\theta^{m\alpha} E$. It follows from the definition (3.2.5) and replacing $s\alpha$ by $\underline{\dim}_\theta^{m\alpha} E$ in (3.4.3) that

$$\liminf_{r\to 0} \frac{S_{r,\theta}^{\frac{1}{\alpha}\underline{\dim}_\theta^{m\alpha} E}(f(E))}{-\log r} \leq 0,$$

implying

$$\underline{\dim}_\theta f(E) \leq \frac{1}{\alpha}\underline{\dim}_\theta^{m\alpha} E.$$

The inequality for $\overline{\dim}_\theta f(E)$ follows by using a similar argument and taking upper limits.

□

It is interesting to note how the Hölder exponent dictates which profile appears in the bound. This would not have been immediately clear had we only considered the setting of projections, where the profile appearing in the upper-bound is determined solely by the topological dimension of the codomain, since projection is 1-Hölder.

Establishing non-trivial absolute lower bounds is not possible in general, but for certain families of mappings we are able to obtain almost-sure lower bounds. For this, we need to introduce a probability space $(\Omega, \mathcal{F}, \tau)$. Here, each $\omega \in \Omega$ corresponds to a $\sigma(\{F \times B : F \in \mathcal{F}, B \in \mathcal{B}\})$-measurable function $f_\omega : \mathbb{R}^n \to \mathbb{R}^m$, where $\mathcal{B}$ denotes the Borel subsets of $\mathbb{R}^n$. However, in order for this problem to be tractable some further conditions must be placed on the set of functions. Specifically, we need to assume that

$$\int 1_{[0,r]}(|f_\omega(x) - f_\omega(y)|)d\tau(\omega) = \tau\left(\{\omega : |f_\omega(x) - f_\omega(y)| \leq r\}\right) \tag{3.4.4}$$

is bounded above by the kernels (3.2.1), see (3.4.6).

A consequence of this assumption relates to a set of modified kernels, denoted $\widetilde{\phi}^s_{r,\theta}$: $\mathbb{R}^m \to \mathbb{R}$ for $0 < r < 1, \theta \in (0,1]$ and $0 \leq s \leq m$, that are defined by

$$\widetilde{\phi}^s_{r,\theta}(x) = \begin{cases} 1 & |x| < r \\ \left(\frac{r}{|x|}\right)^s & r \leq |x| \leq r^\theta \\ 0 & r^\theta < |x| \end{cases} \qquad (x \in \mathbb{R}^m). \tag{3.4.5}$$

The motivation for these kernels is that whilst $\widetilde{\phi}^s_{r,\theta}$ is of the same form as $\phi^{s,t}_{r,\theta}$ in the key region $|x| \leq r^\theta$, integrating $\widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y))$ over the probability space gives a kernel approximately bounded above by our original kernels. This is made precise in the following lemma, which is a critical component of why the profiles of higher dimensional sets relate to lower dimensional images.

Lemma 3.4.2. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, $\gamma > 0$, $m \in \{1, \dots, n\}$ and $0 \leq s < m \leq n$. If $\{f_\omega : E \to \mathbb{R}^m, \omega \in \Omega\}$ is a set of continuous $\sigma(\{F \times B : F \in \mathcal{F}, B \in \mathcal{B}\})$-measurable functions such that there exists $c > 0$ satisfying*

$$\tau\left(\{\omega : |f_\omega(x) - f_\omega(y)| \leq r\}\right) \leq c\phi^{m/\gamma, m/\gamma}_{r^\gamma, \theta}(x - y) \tag{3.4.6}$$

for all $x, y \in E$ and $r > 0$, then there exists $C_{s,m} > 0$ such that

$$\int \widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y)) d\tau(\omega) \leq C_{s,m}\phi^{s/\gamma, m/\gamma}_{r^\gamma, \theta}(x - y).$$

Proof. Let $\theta \in (0,1]$. To ease notation, define

$$\phi^{m/\gamma}_{r^\gamma}(x - y) := \phi^{m/\gamma, m/\gamma}_{r^\gamma, \theta}(x - y) = \begin{cases} 1 & |x - y| < r^\gamma \\ \left(\frac{r^\gamma}{|x-y|}\right)^{m/\gamma} & |x - y| \geq r^\gamma \end{cases},$$

since $\phi^{s,t}_{r,\theta}$ takes the same form on $[r, r^\theta]$ and (r^θ, ∞) when $s = t$.

First, suppose $|x| \leq r$. Then,

$$sr^s \int_{u=r}^{r^\theta} 1_{[0,u]}(|x|)u^{-(s+1)}du + r^{s(1-\theta)}1_{[0,r^\theta]}(|x|) = sr^s \left[\frac{u^{-s}}{-s}\right]_r^{r^\theta} + r^{s(1-\theta)}$$

$$= r^s(-r^{-\theta s} + r^{-s}) + r^{s(1-\theta)}$$

$$= 1. \tag{3.4.7}$$

On the other hand, if $r \leq |x| \leq r^\theta$, then

$$\begin{aligned}
&sr^s \int_{u=r}^{r^\theta} 1_{[0,u]}(|x|)u^{-(s+1)}du + r^{s(1-\theta)}1_{[0,r^\theta]}(|x|) \\
&= sr^s \left(\int_{u=r}^{|x|} 1_{[0,u]}(|x|)u^{-(s+1)}du + \int_{u=|x|}^{r^\theta} u^{-(s+1)}du \right) + r^{s(1-\theta)} \\
&= sr^s \left[\frac{u^{-s}}{-s}\right]_{|x|}^{r^\theta} + r^{s(1-\theta)} \\
&= \left(\frac{r}{|x|}\right)^s.
\end{aligned} \tag{3.4.8}$$

Finally, if $|x| > r^\theta$, then clearly

$$sr^s \int_{u=r}^{r^\theta} 1_{[0,u]}(|x|)u^{-(s+1)}du + r^{s(1-\theta)}1_{[0,r^\theta]}(|x|) = 0. \tag{3.4.9}$$

Hence, by (3.4.7), (3.4.8) and (3.4.9),

$$\widetilde{\phi}^s_{r,\theta}(x) = sr^s \int_{u=r}^{r^\theta} 1_{[0,u]}(|x|)u^{-(s+1)}du + r^{s(1-\theta)}1_{[0,r^\theta]}(|x|).$$

Consider this formula for the increment $f_\omega(x) - f_\omega(y)$. Integrating both sides yields

$$\int \widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y))d\tau(\omega) = sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)} \left[\int 1_{[0,u]}(|f_\omega(x) - f_\omega(y)|)d\tau(\omega)\right] du$$
$$+ r^{s(1-\theta)} \int 1_{[0,r^\theta]}(|f_\omega(x) - f_\omega(y)|)d\tau(\omega),$$

by an appication of Fubini's theorem. From (3.4.6),

$$\int 1_{[0,u]}(|f_\omega(x) - f_\omega(y)|)d\tau(\omega) \leq c\phi^{m/\gamma}_{u^\gamma}(x-y) \tag{3.4.10}$$

and

$$\int 1_{[0,r^\theta]}(|f_\omega(x) - f_\omega(y)|)d\tau(\omega) \leq c\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y). \tag{3.4.11}$$

Hence

$$\frac{1}{c}\int \widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y))d\tau(\omega) \leq sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)}\phi^{m/\gamma}_{u^\gamma}(x-y)\,du + r^{s(1-\theta)}\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y),$$

which must be evaluated in three cases.

Case 1: suppose $|x-y| \leq r^\gamma$.

Then

$$\phi^{m/\gamma}_{u^\gamma}(x-y) = 1$$

for all $r \leq u \leq r^\theta$, and

$$\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y) = 1.$$

Hence

$$\frac{1}{c}\int \widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y))d\tau(\omega) \leq sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)}\phi^{m/\gamma}_{u^\gamma}(x-y)\,du + r^{s(1-\theta)}\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y)$$

$$= sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)}\, du + r^{s(1-\theta)}$$

$$= 1.$$

Case 2: suppose $r^\gamma \leq |x-y| \leq r^{\theta\gamma}$.

Then

$$\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y) = 1.$$

Moreover, for $r \leq u \leq |x-y|^{1/\gamma}$ we have

$$\phi^{m/\gamma}_{u^\gamma}(x-y) = \frac{u^m}{|x-y|^{m/\gamma}}$$

while

$$\phi^{m/\gamma}_{u^\gamma}(x-y) = 1$$

for $|x-y|^{1/\gamma} \leq u \leq r^\theta$.

Hence

$$\begin{aligned}
&\frac{1}{c}\int \widetilde{\phi}^s_{r,\theta}(f_\omega(x) - f_\omega(y))d\tau(\omega) \\
&\leq sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)}\phi^{m/\gamma}_{u^\gamma}(x-y)\, du + r^{s(1-\theta)}\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y) \\
&= sr^s \int\limits_{u=r}^{r^\theta} u^{-(s+1)}\phi^{m/\gamma}_{u^\gamma}(x-y)\, du + r^{s(1-\theta)} \\
&= sr^s \int\limits_{u=r}^{|x-y|^{1/\gamma}} u^{-(s+1)}\frac{u^m}{|x-y|^{m/\gamma}}\, du + sr^s \int\limits_{u=|x-y|^{1/\gamma}}^{r^\theta} u^{-(s+1)}\, du + r^{s(1-\theta)} \\
&\leq \left(\frac{s}{m-s}+1\right)\left(\frac{r^\gamma}{|x-y|}\right)^{s/\gamma}.
\end{aligned}$$

Case 3: suppose $|x-y| \geq r^{\theta\gamma}$.

Then

$$\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y) = \frac{r^{\theta m}}{|x-y|^{m/\gamma}}$$

and

$$\phi^{m/\gamma}_{u^{\gamma}}(x-y) = \frac{u^{m}}{|x-y|^{m/\gamma}}$$

for $r \leq u \leq r^{\theta}$. Hence

$$\begin{aligned}
&\frac{1}{c}\int \widetilde{\phi}^{s}_{r,\theta}(f_\omega(x)-f_\omega(y))d\tau(\omega) \\
&\leq sr^{s}\int_{u=r}^{r^{\theta}} u^{-(s+1)}\phi^{m/\gamma}_{u^{\gamma}}(x-y)\,du + r^{s(1-\theta)}\phi^{m/\gamma}_{r^{\theta\gamma}}(x-y) \\
&= sr^{s}\int_{u=r}^{r^{\theta}} u^{-(s+1)}\frac{u^{m}}{|x-y|^{m/\gamma}}\,du + r^{s(1-\theta)}\frac{r^{\theta m}}{|x-y|^{m/\gamma}} \\
&= \left(\frac{s}{m-s}+1\right)\frac{(r^{\gamma})^{\theta(m/\gamma-s/\gamma)+s/\gamma}}{|x-y|^{m/\gamma}}.
\end{aligned}$$

To conclude, we deduce from Case 1, Case 2 and Case 3 that

$$\begin{aligned}
&\frac{1}{c}\int \widetilde{\phi}^{s}_{r,\theta}(f_\omega(x)-f_\omega(y))d\tau(\omega) \\
&\leq \begin{cases} 1 & |x-y| < r^{\gamma} \\ \left(\frac{s}{m-s}+1\right)\left(\frac{r^{\gamma}}{|x-y|}\right)^{s/\gamma} & r^{\gamma} \leq |x-y| \leq r^{\gamma\theta} \\ \left(\frac{s}{m-s}+1\right)\frac{(r^{\gamma})^{\theta(m/\gamma-s/\gamma)+s/\gamma}}{|x-y|^{m/\gamma}} & r^{\gamma\theta} < |x-y| \end{cases} \\
&\leq \left(\frac{s}{m-s}+1\right)\phi^{s/\gamma, m/\gamma}_{r^{\gamma},\theta}(x-y),
\end{aligned}$$

as required. $\square$

This is analogous to the upper bound in Matilla's result [56, Lemma 3.11], which covers the special case where $\theta = 0$, f_ω denote orthogonal projections and $\Omega = G(n,m)$, the Grassmanian of m dimensional subspaces of $\mathbb{R}^n$. Before obtaining a lower bound, we require one further lemma which is a variant of Lemma 3.3.2 for the modified kernels $\widetilde{\phi}^s_{r,\theta}$.

Lemma 3.4.3. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, $0 < r < 1$ and $0 \leq s \leq n$. If there exists $\mu \in \mathcal{M}(E)$ and a Borel set $F \subset E$ such that*

$$\int \widetilde{\phi}^s_{r,\theta}(x-y)d\mu(y) \leq \beta$$

for all $x \in F$, then

$$\mu(F) r^s \beta^{-1} \leq S^s_{r,\theta}(E),$$

where $S^s_{r,\theta}(E)$ is given by (3.3.1).

Proof. As in Lemma 3.3.2,

$$\beta \geq \int \widetilde{\phi}^s_{r,\theta}(x-y)d\mu(y) \geq \left(\frac{r}{\delta}\right)^s \mu(B(x,\delta))$$

for all $x \in F$ and $r \leq \delta \leq r^\theta$. Let $\{U_i\}_i$ be a cover of F by sets with $r \leq |U_i| \leq r^\theta$. We may assume that for each i there is some $x_i \in F \cap U_i$, so that $U_i \subset B(x_i, |U_i|)$. Hence

$$\mu(F) \leq \sum_i \mu(U_i) \leq \sum_i \mu(B(x_i,|U_i|)) \leq r^{-s}\beta \sum_i |U_i|^s,$$

so taking infima over all such covers,

$$S^s_{r,\theta}(E) \geq S^s_{r,\theta}(F) \geq \mu(F) r^s \beta^{-1}.$$

□

The machinery is now in place for us to state and prove an almost-sure lower bound that will coincide with the upper bounds for both projections and fractional Brownian images.

Theorem 3.4.4. *Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, $\gamma \geq 1$ and $m \in \{1, \ldots, n\}$. If $\{f_\omega : E \to \mathbb{R}^m, \omega \in \Omega\}$ is a set of continuous $\sigma(\{F \times B : F \in \mathcal{F}, B \in \mathcal{B}\})$-measurable functions such that there exists $c > 0$ satisfying*

$$\tau(\{\omega : |f_\omega(x) - f_\omega(y)| \leq r\}) \leq c\phi^{m/\gamma, m/\gamma}_{r^\gamma, \theta}(x - y) \tag{3.4.12}$$

for all $x, y \in E$ and $r > 0$, then

$$\underline{\dim}_\theta f_\omega(E) \geq \gamma \underline{\dim}^{m/\gamma}_\theta E$$

and

$$\overline{\dim}_\theta f_\omega(E) \geq \gamma \overline{\dim}^{m/\gamma}_\theta E$$

for τ-almost all $\omega \in \Omega$.

Proof. Let $E \subset \mathbb{R}^n$ be compact, $\theta \in (0,1]$, $\gamma \geq 1$, $m \in \{1, \ldots, n\}$ and $0 \leq s < m$. Choose a sequence $(r_k)_{k \in \mathbb{N}}$ such that $0 < r_k < 2^{-k}$ and

$$\limsup_{k \to \infty} \frac{C^{s,m}_{r_k^\gamma, \theta}(E)}{-\log r_k^\gamma} = \limsup_{r \to 0} \frac{C^{s,m}_{r,\theta}(E)}{-\log r}. \tag{3.4.13}$$

Moreover, define a sequence of constants β_k by

$$\beta_k := \frac{1}{C^{s/\gamma, m/\gamma}_{r_k^\gamma, \theta}(E)} = \int\int \phi^{s/\gamma, m/\gamma}_{r_k^\gamma, \theta}(x - y) d\mu^k(x)\mu^k(y),$$

where, using Lemma 3.2.1, μ^k is an equilibrium measure on E associated with $\phi^{s/\gamma, m/\gamma}_{r_k^\gamma, \theta}$.

Hence, by (3.4.12) and Lemma 3.4.2 we have

$$\begin{aligned}
&\int\int\int \widetilde{\phi}^s_{r_k,\theta}(f_\omega(x)-f_\omega(y))d\tau(\omega)d\mu^k(x)d\mu^k(y)\\
&\le C_{s,m}\int\int \phi^{s/\gamma,m/\gamma}_{r_k^\gamma,\theta}(x-y)d\mu^k(x)d\mu^k(y)\\
&\le C_{s,m}\beta_k.
\end{aligned}$$

Then, for each $\varepsilon > 0$,

$$\int\int\int \beta_k^{-1} r_k^\varepsilon \widetilde{\phi}^s_{r_k,\theta}(f_\omega(x)-f_\omega(y))d\tau(\omega)d\mu^k(x)d\mu^k(y) \le C_{s,m} r_k^\varepsilon$$

from which Fubini's theorem implies

$$\int \sum_{k=1}^{\infty}\left(\int\int \beta_k^{-1} r_k^\varepsilon \widetilde{\phi}^s_{r_k,\theta}(f_\omega(x)-f_\omega(y))d\mu^k(x)d\mu^k(y)\right)d\tau(\omega) \le C_{s,m}\sum_{k=1}^{\infty} r_k^\varepsilon < \infty$$

since $|r_k^\varepsilon| \le 2^{-k\varepsilon}$. Hence, for τ-almost all $\omega \in \Omega$, there exists $M_\omega > 0$ such that

$$\int\int \beta_k^{-1} r_k^\varepsilon \widetilde{\phi}^s_{r_k,\theta}(t-u)d\mu^k_\omega(t)d\mu^k_\omega(u) \le M_\omega < \infty$$

for all k, where μ^k_ω is the image of μ^k under f_ω. Thus,

$$\int\int \widetilde{\phi}^s_{r_k,\theta}(t-u)d\mu^k_\omega(t)d\mu^k_\omega(u) \le M_\omega \beta_k r_k^{-\varepsilon}$$

for all k. Hence, for each k there exists a set $F_k \subset f_\omega(E)$ with $\mu^k_\omega(F_k) \ge 1/2$ and

$$\int \widetilde{\phi}^s_{r^k,\theta}(t-u)d\mu^k_\omega(t) \le 2M_\omega \beta_k r_k^{-\varepsilon}$$

for all $u \in F_k$. Hence, by Lemma 3.4.3,

$$S^s_{r_k,\theta}(f_\omega(E)) \ge \frac{1}{2}(2M_\omega\beta_k)^{-1} r_k^{s+\varepsilon} = (4M_\omega\beta_k)^{-1} r_k^{s+\varepsilon},$$

and so

$$\limsup_{k\to\infty} \frac{\log S^s_{r_k,\theta}(f_\omega(E))}{-\log r_k} \geq \limsup_{k\to\infty} \frac{\log r_k^{s+\varepsilon}(4M_\omega\beta_k)^{-1}}{-\log r_k}$$
$$= \limsup_{k\to\infty} \frac{\log r_k^{s+\varepsilon} C^{s/\gamma,m/\gamma}_{r_k^\gamma,\theta}(E)}{-\log r_k}$$
$$= -(s+\epsilon) + \limsup_{k\to\infty} \frac{\log C^{s/\gamma,m/\gamma}_{r_k^\gamma,\theta}(E)}{-\log r_k}.$$

Hence

$$\frac{1}{\gamma} \limsup_{k\to\infty} \frac{\log S^s_{r_k,\theta}(f_\omega(E))}{-\log r_k} \geq -\frac{s+\varepsilon}{\gamma} + \limsup_{k\to\infty} \frac{\log C^{s/\gamma,m/\gamma}_{r_k^\gamma,\theta}(E)}{-\log r_k^\gamma}.$$

This is true for all $\epsilon > 0$, so using (3.4.13),

$$\frac{1}{\gamma} \limsup_{r\to 0} \frac{\log S^s_{r,\theta}(f_\omega(E))}{-\log r} \geq -\frac{s}{\gamma} + \limsup_{r\to 0} \frac{\log C^{s/\gamma,m/\gamma}_{r,\theta}(E)}{-\log r}$$

for all $s \in [0,m)$. Since the expressions on both sides of this inequality are continuous for $s \in [0,m]$ by Lemma 3.3.1 and Lemma 3.2.2, the inequality is valid for $s \in [0,m]$ and consequently $s/\gamma \in [0, m/\gamma]$. Hence, for $s/\gamma = \overline{\dim}^{m/\gamma}_\theta E$

$$\limsup_{r\to 0} \frac{\log S^s_{r,\theta}(f_\omega(E))}{-\log r} \geq 0,$$

implying $\overline{\dim}_\theta f_\omega(E) \geq s = \gamma \overline{\dim}^{m/\gamma}_\theta E$. The argument for $\underline{\dim}_\theta f_\omega E$ is similar, although it suffices to set $r_k = 2^{-k}$. □

To apply these results to projection, we first must recall a result of Mattila [56, Lemma 3.11]. The following version differs slightly from the original, which does not explicitly state the lower bound.

Lemma 3.4.5. *For $m \in \{1, \dots, n-1\}$, there exist constants $c_{n,m}, d_{n,m} > 0$ depending only on n and m such that for all $x \in \mathbb{R}^n$ and $r \in (0,1)$,*

$$c_{n,m}\phi_r^m(x) \leq \int 1_{[0,r]}(|\pi_V x|)d\gamma_{n,m}(V) \leq d_{n,m}\phi_r^m(x).$$

Proof. The right-hand inequality is given in [56, Lemma 3.11]. The left-hand inequality is obvious when $|x| \leq r$. Otherwise, we may adapt the proof of [56, Lemma 3.11] by using the estimate

$$\begin{aligned}\sigma^{n-1}&\left(\left\{y \in S^{n-1} : \Big(\sum_{i=m+1}^{n} y_i^2\Big)^{1/2} \leq r\right\}\right) \\ &\geq \ \alpha(n)^{-1}\mathcal{L}^n\left(\{y \in \mathbb{R}^n : |y_i| \leq 1/2 \text{ for } i \leq m, |y_i| \leq r/n \text{ for } i > m\}\right),\end{aligned}$$

where σ^{n-1} denotes the normalised surface measure on S^{n-1}, $\alpha(n)$ is the volume of the unit ball in $\mathbb{R}^n$ and $\mathcal{L}^n$ is n-dimensional Lebesgue measure. □

This shows that the family of projections indexed by $V \in G(n,m)$ (viewed as a probability space) satisfies the conditions of Lemma 3.4.2 and thus Theorem 3.4.4, obtaining an almost-sure lower bound. Since projection is Lipschitz and so 1-Hölder, Theorem 3.4.1 establishes the corresponding upper bound.

Theorem 3.4.6. *Let $E \subset \mathbb{R}^n$ be bounded. Then, for all $V \in G(n,m)$*

$$\underline{\dim}_\theta \pi_V E \leq \underline{\dim}_\theta^m E \quad \text{and} \quad \overline{\dim}_\theta \pi_V E \leq \overline{\dim}_\theta^m E \tag{3.4.14}$$

for all $\theta \in (0,1]$. Moreover, for $\gamma_{n,m}$-almost all $V \in G(n,m)$,

$$\underline{\dim}_\theta \pi_V E = \underline{\dim}_\theta^m E \quad \text{and} \quad \overline{\dim}_\theta \pi_V E = \overline{\dim}_\theta^m E \tag{3.4.15}$$

for all $\theta \in (0,1]$.

Proof. This follows immediately from Theorem 3.4.1, Lemma 3.4.2, Lemma 3.4.5 and Theorem 3.4.4. Note that for the second part, it suffices to prove the *a priori* weaker result where we first fix $\theta \in (0, 1]$ and then establish the result for almost all V. We can do this because the intermediate dimensions are continuous in $\theta \in (0, 1]$ and are therefore determined by their values on the rationals. $\square$

It is similarly straightforward to apply Theorem 3.4.1 and Theorem 3.4.4 to fractional Brownian motion, by using (3.4.1) to establish (3.4.12).

Theorem 3.4.7. *Let $\theta \in (0, 1]$, $B_\alpha : \mathbb{R}^n \to \mathbb{R}^m$ be index-α fractional Brownian motion $(0 < \alpha < 1)$ and $E \subset \mathbb{R}^n$ be compact. Then*

$$\underline{\dim}_\theta B_\alpha(E) = \frac{1}{\alpha} \underline{\dim}^{m\alpha}_\theta E$$

and

$$\overline{\dim}_\theta B_\alpha(E) = \frac{1}{\alpha} \overline{\dim}^{m\alpha}_\theta E$$

almost surely.

Proof. Let $\theta \in (0, 1]$ and $0 < \varepsilon < \alpha < 1$. By [18, Corollary 2.11] there exists, almost surely, $M > 0$ such that

$$|B_\alpha(x) - B_\alpha(y)| \leq M|x - y|^{\alpha-\varepsilon} \tag{3.4.16}$$

for all $x, y \in E$. In addition,

$$\begin{aligned} \mathbb{P}(|B_\alpha(x) - B_\alpha(y)| \leq r) &\leq \mathbb{P}(|B_{\alpha,i}(x) - B_{\alpha,i}(y)| \leq r \text{ for all } 1 \leq i \leq m) \\ &\leq \left(\frac{1}{\sqrt{2\pi}} \frac{1}{|x-y|^\alpha} \int_{|t| \leq r} \exp\left(\frac{-t^2}{2|x-y|^{2\alpha}} \right) dt \right)^m \end{aligned}$$

$$
\begin{aligned}
&\leq \left(\frac{1}{|x-y|^\alpha} \int\limits_{|t|\leq r} 1\, dt \right)^m \\
&= 2^m \left(\frac{r^{1/\alpha}}{|x-y|} \right)^{m\alpha} \qquad (3.4.17) \\
&= 2^n \phi_{r^\gamma,\theta}^{m/\gamma, m/\gamma}(x-y)
\end{aligned}
$$

for all $x, y \in E$ and $r > 0$, where $\gamma = 1/\alpha$. By applying Theorem 3.4.4 and Theorem 3.4.1,

$$
\frac{1}{\alpha} \underline{\dim}_\theta^{m\alpha} E \leq \underline{\dim}_\theta B_\alpha(E) \leq \frac{1}{\alpha - \varepsilon} \underline{\dim}_\theta^{m(\alpha-\varepsilon)} E \leq \frac{1}{\alpha - \varepsilon} \underline{\dim}_\theta^{m\alpha} E
$$

and

$$
\frac{1}{\alpha} \overline{\dim}_\theta^{m\alpha} E \leq \overline{\dim}_\theta B_\alpha(E) \leq \frac{1}{\alpha - \varepsilon} \overline{\dim}_\theta^{m(\alpha-\varepsilon)} E \leq \frac{1}{\alpha - \varepsilon} \overline{\dim}_\theta^{m\alpha} E
$$

almost surely, with the last inequality in each case holding since the profiles are monotonically increasing. Letting $\varepsilon \to 0$, the result follows. $\square$

3.5 Observations and applications

One of the most natural questions concerning the intermediate dimensions is that of continuity at $\theta = 0$, since they are known to be continuous elsewhere, see Section 1.4 or [21]. In such cases the intermediate dimensions form a complete continuous interpolation between the Hausdorff and box-counting dimensions, and we seek to identify classes of sets that witness this behaviour. For example, this was demonstrated in [21, Proposition 4.1] for Bedford-McMullen self-affine carpets, despite the absence of a precise formula for the intermediate dimensions. Theorem 3.4.6 yields another class of examples by showing continuity at 0 implies continuity at 0 for the projections almost surely.

Corollary 3.5.1. *Let $E \subset \mathbb{R}^n$ be a bounded set such that $\underline{\dim}_\theta E$ is continuous at $\theta = 0$. If $V \in G(n,m)$ is such that $\dim_{\mathrm{H}} \pi_V E = \min\{m, \dim_{\mathrm{H}} E\}$, then $\underline{\dim}_\theta \pi_V E$ is continuous at $\theta = 0$. In particular, $\underline{\dim}_\theta \pi_V E$ is continuous at $\theta = 0$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$. A similar result holds for the upper intermediate dimensions.*

Proof. If $m \le \dim_{\mathrm{H}} E$, then the result is immediate and so we may assume that $m > \dim_{\mathrm{H}} E$. Then, for $\theta \in (0,1)$, using (3.4.14), Lemma 3.2.3, Theorem 3.3.5, and the assumption that $\underline{\dim}_\theta E$ is continuous at $\theta = 0$, we get

$$\dim_{\mathrm{H}} E \le \dim_{\mathrm{H}} \pi_V E \le \underline{\dim}_\theta \pi_V E \le \underline{\dim}_\theta^m E \le \underline{\dim}_\theta^n E = \underline{\dim}_\theta E \to \dim_{\mathrm{H}} E$$

as $\theta \to 0$, which proves continuity of $\underline{\dim}_\theta \pi_V E$ at $\theta = 0$. The final part of the result, concerning almost sure continuity at 0, follows from the above result together with the Marstrand-Mattila projection theorems for Hausdorff dimension. □

Results in this vein also hold for fractional Brownian images and Bedford-McMullen carpets.

Corollary 3.5.2. *Let $E \subset \mathbb{R}^n$ be bounded and $B_\alpha : \mathbb{R}^n \to \mathbb{R}^m$ denote index-α fractional Brownian motion. If $\underline{\dim}_\theta E$ is continuous at $\theta = 0$, then $\underline{\dim}_\theta B_\alpha(E)$ is almost surely continuous at $\theta = 0$. Moreover, the analagous result holds for upper dimensions.*

Proof. From [47, Corollary, pp. 267],

$$\dim_{\mathrm{H}} B_\alpha(E) = \frac{1}{\alpha} \dim_{\mathrm{H}} E$$

almost surely, and so

$$\dim_{\mathrm{H}} E \le \alpha \underline{\dim}_\theta B_\alpha(E) \le \alpha \frac{1}{\alpha} \underline{\dim}_\theta^{m\alpha} E \le \underline{\dim}_\theta^n E = \underline{\dim}_\theta E$$

by monotonicity of the profiles. Hence, as $\theta \to 0$, continuity of $\underline{\dim}_{\theta} B_\alpha(E)$ at $\theta = 0$ is established, since $\underline{\dim}_{\theta} E \to \dim_{\mathrm{H}} E$ by definition. The proof for upper dimensions is similar. $\square$

Corollary 3.5.3. *Let $E \subset \mathbb{R}^2$ be a Bedford-McMullen carpet associated with a regular $a \times b$ grid for integers $b > a \geq 2$. Then $\underline{\dim}_{\theta} \pi_V E$ and $\overline{\dim}_{\theta} \pi_V E$ are continuous at $\theta = 0$ for $\gamma_{2,1}$-almost all $V \in G(2,1)$. Moreover, if $\log a / \log b \notin \mathbb{Q}$, then $\underline{\dim}_{\theta} \pi_V E$ and $\overline{\dim}_{\theta} \pi_V E$ are continuous at $\theta = 0$ for all $V \in G(2,1)$.*

Proof. The almost sure result follows immediately from Corollary 3.5.1 and [21, Proposition 4.1]. The upgrade from almost all to all follows by applying [25, Theorem 1.1], which proved there are *no* exceptions to Marstrand's projection theorem for Bedford-McMullen carpets of 'irrational type', apart from possibly the projections onto the coordinate axes. However, the coordinate projections are both self-similar sets and therefore the intermediate dimensions are automatically continuous at 0. $\square$

The converse implication in Corollary 3.5.1 does not necessarily hold, since continuity at 0 for all of the projections of E does not guarantee continuity at 0 for E. For example, let E be a set in the plane with $\dim_{\mathrm{H}} E = 1$ that satisfies $\underline{\dim}_{\theta} E = 2$ for all $\theta \in (0,1]$ and place it inside a circle. The existence of such an E follows easily from the following consequence of [21, Proposition 2.4]. Our capacity approach yields a simple proof, which we include for completeness.

Corollary 3.5.4. *If $E \subset \mathbb{R}^n$ is bounded and satisfies $\underline{\dim}_{\mathrm{B}} E = n$, then $\underline{\dim}_{\theta} E = \overline{\dim}_{\theta} E = n$ for all $\theta \in (0,1]$. Similarly, if $\overline{\dim}_{\mathrm{B}} E = n$, then $\overline{\dim}_{\theta} E = n$ for all $\theta \in (0,1]$.*

Proof. Observe that

$$\liminf_{r \to 0} \frac{\log C_{r,\theta}^{n,n}(E)}{-\log r} = \underline{\dim}_{\mathrm{B}} E = n$$

and so by (3.2.5) and Theorem 3.3.5 it follows $\overline{\dim}_\theta E \geq \underline{\dim}_\theta E = \underline{\dim}_\theta^n E = \underline{\dim}_{\mathrm{B}} E = n$. The result concerning $\overline{\dim}_\theta E$ alone follows similarly. $\square$

The following counter-intuitive result follows by piecing together Corollaries 3.5.1 and 3.5.4. This gives a concrete application of the intermediate dimensions to a question concerning only the box and Hausdorff dimensions.

Corollary 3.5.5. *Let $E \subset \mathbb{R}^n$ be a bounded set such that $\underline{\dim}_\theta E$ is continuous at $\theta = 0$. Then*

$$\underline{\dim}_{\mathrm{B}} \pi_V E = m$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$ if and only if

$$\dim_{\mathrm{H}} E \geq m.$$

A similar result holds for upper dimensions by replacing $\underline{\dim}_\theta E$ and $\underline{\dim}_{\mathrm{B}} E$ with $\overline{\dim}_\theta E$ and $\overline{\dim}_{\mathrm{B}} E$, respectively.

Proof. One direction is trivial, and holds without the continuity assumption, since, if $\dim_{\mathrm{H}} E \geq m$, then

$$m \geq \underline{\dim}_{\mathrm{B}} \pi_V E \geq \dim_{\mathrm{H}} \pi_V E \geq m$$

for $\gamma_{n,m}$-almost all $V \in G(n,m)$. The other direction is where the interest lies. Indeed, suppose $\underline{\dim}_{\mathrm{B}} \pi_V E = m$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$ but $\dim_{\mathrm{H}} E < m$. Then Corollary 3.5.4 implies that $\underline{\dim}_\theta \pi_V E = m$ for $\gamma_{n,m}$-almost all $V \in G(n,m)$ and all $\theta \in (0,1]$. Applying the Marstrand-Mattila projection theorem for Hausdorff dimension, it follows that for $\gamma_{n,m}$-almost all $V \in G(n,m)$ $\underline{\dim}_\theta \pi_V E$ is not continuous at $\theta = 0$, which contradicts Corollary 3.5.1. $\square$

To motivate Corollary 3.5.5 we give a couple of simple applications. If $E \subset \mathbb{R}^2$ is a Bedford-McMullen carpet satisfying $\dim_{\mathrm{H}} E < 1 \leq \dim_{\mathrm{B}} E$, then

$$\overline{\dim}_{\mathrm{B}} \pi_V E < 1 = \min\{\dim_{\mathrm{B}} E, 1\}$$

for $\gamma_{2,1}$-almost all $V \in G(2,1)$. This surprising application seems difficult to derive directly, noting that there is very little known about the *box* dimensions of projections of Bedford-McMullen carpets, aside from them being almost surely constant. Another, more accessible, example is provided by the sequence sets $F_p = \{n^{-p} : n \geq 1\}$ for fixed $p > 0$. It is well-known that $\dim_{\mathrm{B}} F_p = 1/(1+p)$ and therefore

$$\dim_{\mathrm{B}}(F_p \times F_p) = 2/(1+p) \tag{3.5.1}$$

which is at least 1 for $p \leq 1$ and approaches 2 as p approaches 0. Continuity at $\theta = 0$ for $\overline{\dim}_{\,\theta} F_p$ was established in [21, Proposition 3.1] and it is straightforward to extend this to $\overline{\dim}_{\,\theta}(F_p \times F_p)$. Therefore, since $\dim_{\mathrm{H}}(F_p \times F_p) = 0 < 1$, we get

$$\overline{\dim}_{\mathrm{B}} \pi_V (F_p \times F_p) < 1$$

for $\gamma_{2,1}$-almost all $V \in G(2,1)$. This is most striking when p is very close to 0 and (3.5.1) is close to 2. A direct calculation, which we omit, in fact reveals that for all $V \in G(2,1)$ apart from the horizontal and vertical projections,

$$\overline{\dim}_{\mathrm{B}} \pi_V (F_p \times F_p) = 1 - \left(\frac{p}{p+1}\right)^2.$$

An entertaining formula that we would not have come across if Corollary 3.5.5 had not lead us to it, see also [34, Proposition 5.1].

Furthermore, Theorem 3.4.1 together with Corollary 3.5.2 also have a surprising application to the box and Hausdorff dimensions of sets with continuity at $\theta = 0$. In the following, we use the notation

$$\underline{\dim}_{\mathrm{B}}^{n\alpha} E = \underline{\dim}_{1}^{n\alpha} E,$$

for $\alpha \in [0, 1]$, since our profiles extend the lower box-counting dimension profiles $\underline{\dim}_{\mathrm{B}}^{m}$ of Falconer [19] to non-integer values of m when $\theta = 1$ (and similarly for upper dimensions).

Corollary 3.5.6. *Let $E \subset \mathbb{R}^n$ be a bounded set such that $\underline{\dim}_{\,\theta} E$ is continuous at $\theta = 0$. If $\alpha > \frac{1}{n} \dim_{\mathrm{H}} E$, then*

$$\frac{1}{\alpha} \underline{\dim}_{\mathrm{B}}^{n\alpha} E < n.$$

On the other hand, if $\alpha \leq \frac{1}{n} \dim_{\mathrm{H}} E$, then

$$\frac{1}{\alpha} \underline{\dim}_{\mathrm{B}}^{n\alpha} E = n.$$

The analogous results hold for the upper box-counting dimension profiles.

Proof. Let $E \subset \mathbb{R}^n$ be such that $\underline{\dim}_{\,\theta} E$ is continuous at $\theta = 0$, and let $B_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ denote index-α fractional Brownian motion where

$$\alpha > \frac{\dim_{\mathrm{H}} E}{n}.$$

Hence, by [47, Corollary, pp. 267],

$$\dim_{\mathrm{H}} B_\alpha(E) = \frac{1}{\alpha} \dim_{\mathrm{H}} E < n \tag{3.5.2}$$

almost surely. Then, in order to reach a contradiction, suppose that $\frac{1}{\alpha} \underline{\dim}_{\mathrm{B}}^{n\alpha} E = n$. This implies $\underline{\dim}_{\mathrm{B}} B_\alpha(E) = n$ almost surely by Theorem 3.4.7. Then, by [8, Corollary 6.3],

$$\underline{\dim}_\theta B_\alpha(E) = n$$

almost surely for all $\theta \in (0, 1]$. By Corollary 3.5.2, $\underline{\dim}_\theta B_\alpha(E)$ is continuous at $\theta = 0$ which implies $\dim_{\mathrm{H}} B_\alpha(E) = n$, a contradiction to (3.5.2). The case for $\alpha \leq \frac{1}{n} \dim_{\mathrm{H}} E$ follows easily from [47, Corollary, pp. 267] and Theorem 3.4.7. $\square$

In particular, since $\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E$, the first part of Corollary 3.5.6 shows us that $\underline{\dim}^{n\alpha}_{\mathrm{B}} E$ is strictly less than the trivial upper bound of $n\alpha$ implied by Lemma 3.2.2 for

$$\alpha \in \left(\frac{\dim_{\mathrm{H}} E}{n}, \frac{\underline{\dim}_{\mathrm{B}} E}{n} \right),$$

and similarly for $\overline{\dim}^{n\alpha}_{\mathrm{B}} E$ and $\overline{\dim}_{\mathrm{B}} E$. Furthermore, Corollary 3.5.6 may immediately be translated into the context of fractional Brownian motion by Theorem 3.4.7.

Corollary 3.5.7. *Let $E \subset \mathbb{R}^n$ be a bounded set such that $\underline{\dim}_\theta E$ is continuous at $\theta = 0$ and $B_\alpha : \mathbb{R}^n \to \mathbb{R}^n$ denote index-α Brownian motion. If $\alpha > \frac{1}{n} \dim_{\mathrm{H}} E$, then*

$$\underline{\dim}_{\mathrm{B}} B_\alpha(E) < n.$$

almost surely. On the other hand, if $\alpha \leq \frac{1}{n} \dim_{\mathrm{H}} E$, then

$$\underline{\dim}_{\mathrm{B}} B_\alpha(E) = n.$$

almost surely. The analogous results hold for the upper box-counting dimension profiles.

It may be of interest to see how Corollary 3.5.7, which deals with box-counting dimension, differs from the related classical result of Kahane on the Hausforff dimensions of Brownian images [47, Corollary, pp. 267].

A further implication of Theorem 3.4.7 is that an inequality derived from a slight modification of the proof allows us to show that the dimension profiles are continuous for any Borel set $E \subset \mathbb{R}^n$. It is worth noting that this does not follow from Lemma 3.2.2, which describes how

$$\frac{\log C^{s,t}_{r,\theta}(E)}{-\log r} - s$$

changes in s and not how the root varies with t as $r \to 0$.

Corollary 3.5.8. *Let $E \subset \mathbb{R}^n$ be bounded and $\theta \in (0, 1]$. The functions $f, g : (0, n) \to [0, n]$ defined by*

$$f(t) = \underline{\dim}^t_\theta E$$

and

$$g(t) \to \overline{\dim}^t_\theta E$$

are continuous in t.

Proof. Let $0 < s < n$ and $\theta \in (0, 1]$. Fix $\alpha > 0$ such that $n\alpha = s$. Since E is bounded, there exists $B > 1$ such that

$$|x - y| < B$$

for all $x, y \in E$. Let $\varepsilon > 0$ be such that $n(\alpha + \varepsilon)/(1 - \varepsilon) < n$, and choose $C_\varepsilon \geq B^{\varepsilon(1+\alpha)/(1-\varepsilon)}$. Observe

$$\begin{aligned} C_\varepsilon &\geq |x - y|^{\varepsilon(1+\alpha)/(1-\varepsilon)} \\ &= \frac{|x - y|^{(\alpha+\varepsilon)/(1-\varepsilon)}}{|x - y|^\alpha} \end{aligned} \tag{3.5.3}$$

for all $x, y \in E$. Then, consider $B_\alpha : \mathbb{R}^n \to \mathbb{R}^n$. By (3.4.17) and (3.5.3),

$$\mathbb{P}(|B_\alpha(x) - B_\alpha(y)| \leq r) \leq 2^n \min\left\{1, \left(\frac{r^{1/\alpha}}{|x - y|}\right)^{n\alpha}\right\}$$

$$\leq 2^n C_\varepsilon^n \min\left\{1, \left(\frac{r^{(1-\varepsilon)/(\alpha+\varepsilon)}}{|x-y|}\right)^{n\frac{\alpha+\varepsilon}{1-\varepsilon}}\right\}$$
$$= 2^n C_\varepsilon^n \phi_{r^\gamma,\theta}^{n/\gamma,n/\gamma}(x-y)$$

for all $x, y \in E$ and $r > 0$, where $\gamma = (1-\varepsilon)/(\alpha+\varepsilon)$. Hence, from Theorem 3.4.1 and Theorem 3.4.4, we have

$$\frac{1-\varepsilon}{\alpha+\varepsilon}\underline{\dim}_\theta^{n(\alpha+\varepsilon)/(1-\varepsilon)} E \leq \underline{\dim}_\theta B_\alpha(E) \leq \frac{1}{\alpha-\varepsilon}\underline{\dim}_\theta^{n(\alpha-\varepsilon)} E$$

almost surely. The profiles are monotonically increasing, and so

$$\frac{1-\varepsilon}{\alpha+\varepsilon}\underline{\dim}_\theta^s E \leq \frac{1-\varepsilon}{\alpha+\varepsilon}\underline{\dim}_\theta^{n(\alpha+\varepsilon)/(1-\varepsilon)} E \leq \frac{1}{\alpha}\underline{\dim}_\theta^s E \leq \frac{1}{\alpha-\varepsilon}\underline{\dim}_\theta^{n(\alpha-\varepsilon)} E \leq \frac{1}{\alpha-\varepsilon}\underline{\dim}_\theta^s E$$

almost surely, since

$$\frac{n(\alpha+\varepsilon)}{1-\varepsilon} > s > n(\alpha-\varepsilon).$$

This holds for arbitrary sequences of sufficiently small positive ε tending to zero and so establishes continuity from above and below. The proof for $\overline{\dim}_\theta^s$ is similar. □

One final application concerns the Hausdorff dimension of the set of exceptional sets in the projection setting. The proof is based on an application of Theorem 3.4.4, which allows the proof of [19, Theorem 1.2 (ii), (iii)] to be generalised from box-counting dimension (the case where $\theta = 1$) to all intermediate dimensions.

Theorem 3.5.9. *Let $E \subset \mathbb{R}^n$ be compact, $m \in \{1, \ldots, n\}$ and $0 \leq \lambda \leq m$. Then*

$$\dim_{\mathrm{H}}\{V \in G(n,m) : \overline{\dim}_\theta \pi_V E < \overline{\dim}_\theta^\lambda E\} \leq m(n-m) - (m-\lambda). \tag{3.5.4}$$

The analagous results holds for $\underline{\dim}_\theta \pi_V E$ and $\underline{\dim}_\theta^\lambda E$.

Proof. First, define

$$A = \{V \in G(n,m) : \overline{\dim}_\theta \pi_V E < \overline{\dim}^\lambda_\theta E\}$$

and suppose, with the aim of deriving a contradiction, that

$$\dim_{\mathrm{H}} A > m(n-m) - (m-\lambda).$$

By Frostman's lemma, there exists a measure μ supported on a compact set $B \subseteq A$ and $c > 0$ such that

$$\mu(B_G(V,r)) \leq cr^{m(n-m)-(m-\lambda)}$$

for all $V \in G(n,m)$ and $r > 0$, where B_G is a ball defined via the natural metric of dimension $m(n-m)$ on $G(n,m)$. Hence, using [57, Inequality (5.12)] yields

$$\begin{aligned}
\mu(\{V \in G(n,m) : |\pi_V x - \pi_V y| < r\}) &\leq \left(\frac{r}{|x-y|}\right)^{m(n-m)-(m-\lambda)-m(n-m-1)} \\
&= \left(\frac{r}{|x-y|}\right)^{\lambda} \\
&\leq \phi^{\lambda,\lambda}_{r,\theta}(x-y).
\end{aligned}$$

Thus, the condition of Theorem 3.4.4 is satisfied with $\Omega = G(n,m)$, $\tau = \mu$ and $\gamma = m/\lambda$. Hence

$$\overline{\dim}_\theta \pi_V E \geq \overline{\dim}^\lambda_\theta E \tag{3.5.5}$$

for μ almost-all $V \in G(n,m)$. Since μ is supported on A, this is a contradiction, as it implies the existence of $V \in A$ satisfying (3.5.5). The proof for $\underline{\dim}_\theta$ follows similarly. $\square$

Recall that $\overline{\dim}^\lambda_\theta E$ and $\underline{\dim}^\lambda_\theta E$ decrease as λ decreases. Thus, Theorem 3.5.9 tells us that the there is a stricter upper bound on the dimension of the exceptional set the larger the drop in dimension from the expected value. We conclude by posing a slightly different question which is a mild strengthening of Theorem 3.5.9, an analogy of which was considered in [19, Theorem 1.3 (ii), (iii)].

Question 3.5.10. *Let $0 \leq \gamma \leq n - m$. What are the optimum upper bounds for*

$$\dim_{\mathrm{H}}\{V \in G(n,m) : \overline{\dim}_{\theta}\pi_V E < \overline{\dim}_{\theta}^{m+\gamma} E - \gamma\}$$

and

$$\dim_{\mathrm{H}}\{V \in G(n,m) : \underline{\dim}_{\theta}\pi_V E < \underline{\dim}_{\theta}^{m+\gamma} E - \gamma\}?$$

The method in [18] for box-counting dimensions relied on Fourier transforms and approximating the potential kernels by a Gaussian with a strictly positive Fourier transform. However, the natural family of kernels appropriate for working with intermediate dimension have a more complex shape, which complicates matters. A significantly different, but perhaps interesting, approach may be required.

Chapter 4

Elliptical polynomial spirals

4.1 Introduction

An infinitely wound spiral is a subset of the complex plane

$$S(\phi) = \{\phi(t) \exp(it) : 1 < t < \infty\}, \tag{4.1.1}$$

where $\phi : [1, \infty) \to (0, \infty)$, known as a *winding function*, is continuous, strictly decreasing and tends to zero as $t \to \infty$. Such forms arise throughout science and the natural world, from α-models of fluid turbulence and vortex formation to the structure of galaxies [27, 53, 60, 66, 67]. The self-similarity present within these spirals makes them natural candidates for fractal analysis, and one may wish to examine the fine local structure present at the origin [11, 31]. This may be quantified via a suitable notion of fractal dimension such as box-counting dimension [69].

The isotropic classical definition (4.1.1) may be too restrictive for the modelling of general natural or abstract phenomena. Most naturally occurring spirals are anisotropic, developing in systems with inherent asymmetry, such as elliptical whirlpools forming in a flowing body of water. Another simple example arises in Newtonian mechanics: suppose a weight attached to an elastic band is rotated about an axis parallel to the ground. At high velocities the centripetal force dominates gravity and the orbit is circular. However, if the system is allowed to decelerate, the weight will follow a spiral trajectory that will become increasingly elongated in the vertical direction as the relative contribution of gravitational force grows.

To account for these scenarios, flexibility may be introduced by controlling rate of contraction in each axis and introducing an additional functional parameter. Thus, for two winding functions $\phi, \psi : [1, \infty) \to (0, \infty)$, we define the associated *elliptical* spiral to be

$$S(\phi, \psi) = \{\phi(t) \cos t + i\psi(t) \sin t : 1 < t < \infty\}. \tag{4.1.2}$$

Our results concern the family of elliptical *polynomial* spirals $S_{p,q} = S(t^{-p}, t^{-q})$, where $0 < p \leq q$, although our arguments apply more generally. If $p = q$, then we write $S_{p,p} = S_p$ and (4.1.2) recovers the *generalised hyperbolic* spirals. Spirals such as these with polynomial winding functions typically arise in systems with an underlying dynamical process. On the other hand, spirals emerging from static settings are generally logarithmic with winding functions of the form $\exp(-ct)$ for $c > 0$ [31].

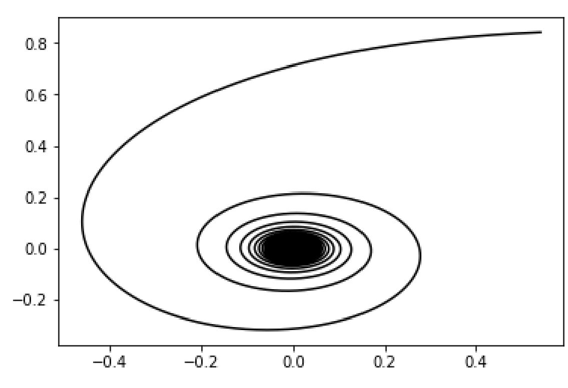

Figure 4.1: An elliptical polynomial spiral $S_{p,q}$ with $p = 0.7$ and $q = 0.75$.

This chapter serves two purposes. First, we offer a dimensional analysis of the family of elliptical polynomial spirals. This involves calculating the intermediate, box-counting and Assouad-type dimensions. Together, our results show the intermediate dimensions and the Assouad spectrum provide a continuous interpolation between the two extremes of the dimensional repertoire, as illustrated in Figure 4.2. One exciting outcome of this analysis was that Assouad spectrum of $S_{p,q}$ turned out to contain two points of non-differentiability, or phase transitions (see Theorem 4.2.7). The elliptical polynomial spirals are the first natural example to exhibit this behaviour, found before only as the product of delicate constructions.

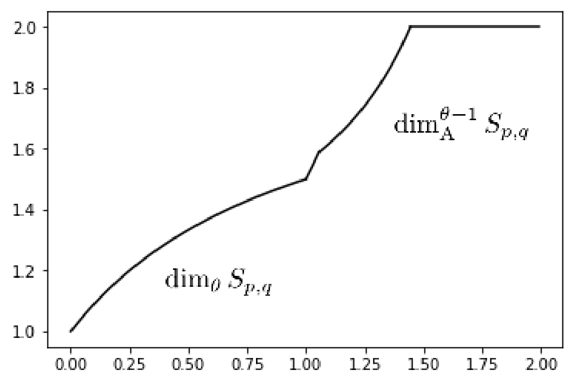

Figure 4.2: A plot of $\dim_\theta S_{p,q}$ against θ (x-axis) for $\theta \in [0,1]$ and $\dim_{\mathrm{A}}^{\theta-1} S_{p,q}$ against θ for $\theta \in [1,2]$. In this example, $p = 0.1$ and $q = 0.8$.

The second focus is to determine permissible α such that there may exist an α-Hölder function $f : S_{p,q} \to S_{r,s}$ that *deforms* one elliptical polynomial spiral into another. Recall a function $f : X \to Y$ is α-Hölder ($0 < \alpha \leq 1$) if there exists $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|^\alpha \qquad (x, y \in X).$$

Such maps may play a role within dynamical systems where spirals form and evolve over time. The Hölder exponent characterises the regularity of f by quantifying the degree of distortion at local scales. A number of related questions on regularity have been explored over the past few decades for different categories of spirals that arise from winding functions of various canonical forms. Katznelson, Nag and Sullivan show that the logarithmic spiral satisfies the bi-Lipschitz *winding problem* [48]. That is, it may be constructed as the image of a bi-Lipschitz homeomorphism on the unit interval. However, if ϕ decays sub-exponentially, i.e.

$$\frac{\log \phi(t)}{t} \to 0 \qquad (t \to \infty),$$

then no such bi-Lipschitz homeomorphism exists [26]. This led Fraser [31] to investigate Hölder solutions to the winding problem for generalised hyperbolic spirals.

Our methodology is based on the dimension profiles we saw in Chapter 3. Of course, if there is an α-Hölder map between $S_{p,q}$ and $S_{r,s}$ we immediately obtain

$$\alpha \leq \frac{\dim S_{p,q}}{\dim S_{r,s}}, \tag{4.1.3}$$

where dim denotes Hausdorff or box-counting dimension, since

$$\dim f(E) \leq \frac{1}{\alpha} \dim E$$

for $E \subset \mathbb{R}^n$ and α-Hölder $f : \mathbb{R}^n \to \mathbb{R}^n$. However, the upper dimension profiles (3.2.6) provide a strictly sharper bound on α by use of the formula

$$\alpha \leq \frac{\overline{\dim}_\theta^{2\alpha} S_{p,q}}{\dim_\theta S_{r,s}}, \tag{4.1.4}$$

derived from Falconer [18, Theorem 2.6] in the case $\theta = 1$ and Theorem 3.4.1 for $\theta \in [0, 1]$. Of course, we could analogously obtain an estimate using the lower dimension profiles (3.2.5), too.

While this approach seems promising at first sight, the definition of the profiles is potential-theoretic and rather challenging to compute in the case of $S_{p,q}$. This difficulty is circumvented by instead using the relationship to their fractional Brownian images given by Theorem 3.4.7. In fact, the method employed here may be used more generally to estimate the Hölder regularity of a function between any two sets for which the box or intermediate dimensions of the fractional Brownian images may be estimated from above.

In preparation for the main proofs, we conclude this introduction by setting notation and making a useful geometric observation that is applied frequently in our arguments.

Dimension concerns limiting processes for which fixed multiplicative constants are typically of little consequence. Therefore, we often write $x \lesssim y$ when it is clear there exists a uniform constant $c > 0$ not depending on x and y such that $x \leq cy$. Naturally, we analogously define $\gtrsim$, and write $x \approx y$ if $x \lesssim y$ and $x \gtrsim y$. In circumstances where c is not uniform but depends on certain parameters, say $t_1, t_2, \ldots$, we write $\lesssim_{t_1,t_2,\ldots}$, $\gtrsim_{t_1,t_2,\ldots}$ and $\approx_{t_1,t_2,\ldots}$ to make this clear.

A useful trick is to decompose $S_{p,q}$ into a countable disjoint union of *full turns*. In particular, we define

$$S_{p,q} := \bigcup_{k \geq 1} S_{p,q}^k, \tag{4.1.5}$$

where

$$S_{p,q}^k = \{t^{-p} \cos t + it^{-q} \sin t : 2\pi k \leq t < 2\pi(k+1)\}.$$

Note that, for arithmetic convenience, we have removed the part of $S_{p,q}$ corresponding to $1 < t < 2\pi$ in the definition (4.1.2) without meaningful loss of generality. The following geometric observation estimates the sum of the 1-dimensional Hausdorff measures, or length, over a collection of consecutive turns using standard number theoretic estimates.

Lemma 4.1.1. *Let $0 < p \le q$. For $k \ge 1$,*

$$\mathcal{H}^1(S_{p,q}^k) \approx_p k^{-p} \tag{4.1.6}$$

Moreover, for sufficiently large integers $N, M \in \mathbb{N}$ with $M < N$,

$$\sum_{k=M}^{N} \mathcal{H}^1(S_{p,q}^k) \approx_p \begin{cases} N^{1-p} - M^{1-p} & \text{if } p < 1 \\ \log N - \log M & \text{if } p = 1 \\ M^{1-p} - N^{1-p} & \text{if } p > 1 \end{cases} \cdot \tag{4.1.7}$$

Proof. By comparing $\mathcal{H}^1(S_{p,q}^k)$ with the perimeter of a square of sidelength $2(2k\pi)^{-p}$ centred on the origin we may deduce

$$(2k\pi)^{-p} \le \mathcal{H}^1(S_{p,q}^k) \le 8(2k\pi)^{-p},$$

from which (4.1.6) follows immediately. (4.1.7) may then be deduced in a standard way. Letting $\lfloor t \rfloor$ denote the integer part of $t \in \mathbb{R}$, observe that for $p \neq 1$,

$$\sum_{k=M}^{N} \mathcal{H}^1(S_{p,q}^k) \approx_p \sum_{k=M}^{N} k^{-p} = \sum_{k=M}^{N} \int_{k}^{k+1} \lfloor u \rfloor^{-p} \, du \approx_p \frac{1}{1-p}(N^{1-p} - M^{1-p}).$$

The case for $p = 1$ follows similarly. $\square$

4.2 Dimensions

For $0 < p \le q$, the Hausdorff and packing dimensions satisfy

$$\dim_{\mathrm{H}} S_{p,q} = \dim_{\mathrm{P}} S_{p,q} = 1,$$

due to the countable stability of these dimensions and the decomposition (4.1.5). We present the remaining dimensions of $S_{p,q}$, beginning with the intermediate dimensions.

It is convenient to start by proving an upper bound in the wider context of images of elliptical spirals under Hölder transformations. As we shall see, this becomes especially relevant in Section 4.3 when considering fractional Brownian images and dimension profiles, since index-α fractional Brownian motion is almost surely $(\alpha - \varepsilon)$-Hölder for all $\varepsilon > 0$.

Lemma 4.2.1. *Let $0 < p \leq q$, $\theta \in [0,1]$ and $f : S_{p,q} \to \mathbb{R}^2$ be α-Hölder $(0 < \alpha \leq 1)$. If $p < 1$, then*

$$\overline{\dim}_\theta f(S_{p,q}) \leq \begin{cases} 2 & 0 < \alpha \leq 1/2 \\ \frac{p+q+2\theta(1-p)}{\alpha(p+q)+\theta(1-p)} & 1/2 < \alpha \leq 1 \end{cases}.$$

Otherwise, if $p \geq 1$, then

$$\overline{\dim}_\theta f(S_{p,q}) \leq \begin{cases} 2 & 0 < \alpha \leq 1/2 \\ \frac{1}{\alpha} & 1/2 < \alpha \leq 1 \end{cases}.$$

Proof. Let $0 \leq s \leq 2$ and $0 < \delta < 1$. To aid readability when dealing with particularly complicated exponents, we write $t = -\log \delta$.

If $0 < \alpha \leq 1/2$, the bound is trivial. Thus, hereafter assume $1/2 < \alpha \leq 1$.

Choose $M \in \mathbb{N}$ to be the smallest integer satisfying

$$M \geq \exp\left(\frac{t(s - (1/\alpha) + \theta(2-s))}{1 - p + \alpha(p+q)}\right), \tag{4.2.1}$$

and note that by (4.1.6) from Lemma 4.1.1,

$$N_{\delta^{1/\alpha}}(S^k_{p,q}) \lesssim_p \frac{k^{-p}}{\delta^{1/\alpha}}. \tag{4.2.2}$$

Let the uniform constant associated with the Hölder property of f be $c > 0$. Then, for $k \leq M$, by considering the image of a cover satisfying (4.2.2) under f, we may obtain a cover of $f(S^k_{p,q})$ by

$$\approx_p \frac{k^{-p}}{\delta^{1/\alpha}}$$

balls of diameter $c2^{\alpha/2}\delta$. It the follows that there exists a constant $d_{c,p,\alpha}$, depending only on c, p and α, such that we may cover $f(S^k_{p,q})$ by

$$d_{c,p,\alpha}\frac{k^{-p}}{\delta^{1/\alpha}} \approx_{c,p,\alpha} \frac{k^{-p}}{\delta^{1/\alpha}}$$

balls of diameter δ. The remaining region will be covered by balls of diameter δ^θ. For $k > M$,

$$\begin{aligned}\bigcup_{k>M} f(S^k_{p,q}) &\subset f([-M^{-p}, M^{-p}] \times [-M^{-q}, M^{-q}]) \\ &\subseteq [-cM^{-p\alpha}, cM^{-p\alpha}] \times [-cM^{-q\alpha}, cM^{-q\alpha}],\end{aligned}$$

and such a rectangle may be covered by

$$\approx_c \frac{M^{-(p+q)\alpha}}{\delta^{2\theta}}$$

balls of diameter δ^θ. Summing over this cover, that we denote $\{U_i\}_i$, gives

$$\sum |U_i|^s \approx_{c,p,\alpha} \left(\frac{M^{-\alpha(p+q)}}{\delta^{2\theta}}\right)\delta^{\theta s} + \delta^s \sum_{k=1}^{M} \frac{k^{-p}}{\delta^{1/\alpha}}. \tag{4.2.3}$$

If $p \leq 1$, then (4.2.1) and (4.2.3) imply

$$\begin{aligned}\sum |U_i|^s &\approx_{c,p,\alpha} M^{-\alpha(p+q)}\delta^{\theta s-2\theta} + M^{1-p}\delta^{s-(1/\alpha)} \\ &\approx_{c,p,\alpha} 2\exp\left(-t\frac{s(\alpha(p+q)+\theta(1-p))-(p+q+2\theta(1-p))}{1-p+\alpha(p+q)}\right). \qquad (4.2.4)\end{aligned}$$

Hence, $\sum |U_i|^s \to 0$ as $\delta \to 0$ providing

$$s > \frac{p+q+2\theta(1-p)}{\alpha(p+q)+\theta(1-p)},$$

and so

$$\overline{\dim}_\theta f(S_{p,q}) \leq \frac{p+q+2\theta(1-p)}{\alpha(p+q)+\theta(1-p)}.$$

Note that if $p=1$ this bound equals $1/\alpha$, as required. On the other hand, if $p>1$, then (4.2.3) implies

$$\begin{aligned}\sum |U_i|^s &\approx_{c,p,\alpha} M^{-\alpha(p+q)}\delta^{\theta s-2\theta} + \delta^{s-(1/\alpha)} \\ &\approx_{c,p,\alpha} \exp\left(-t\frac{s(\alpha(p+q)+\theta(1-p))-(p+q+2\theta(1-p))}{1-p+\alpha(p+q)}\right) + \delta^{s-(1/\alpha)}.\end{aligned}$$

Clearly,

$$1-p+\alpha(p+q) \geq 1-p+\frac{1}{2}(p+p) = 1,$$

and so the left-hand term converges to 0 as $\delta \to 0$ if

$$s > \frac{p+q+2\theta-2p\theta}{\alpha(p+q)+\theta-p\theta},$$

while the right hand term requires $s > 1/\alpha$. Hence

$$\overline{\dim}_\theta f(S_{p,q}) \leq \max\left\{\frac{p+q+2\theta-2p\theta}{\alpha(p+q)+\theta-p\theta}, \frac{1}{\alpha}\right\} = \frac{1}{\alpha}.$$

□

The exact value of the intermediate dimensions may then be derived by applying Lemma 4.2.1 to the identity map, along with a lower bound that we obtain using the mass distribution principle for intermediate dimensions [21, Proposition 2.2]. Since this version of the mass distribution principle is less well-known, we include it below for convenience.

Proposition 4.2.2. [21, Proposition 2.2] *Let F be a Borel subset of $\mathbb{R}^n$ and let $0 \leq \theta \leq 1$ and $s \geq 0$. Suppose that there are numbers $a, c, \delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$ we can find a Borel measure μ_0 supported on F with $\mu_0(F) \geq a$, and*

$$\mu_\delta(U) \leq c|U|^s$$

for all Borel sets $U \subseteq \mathbb{R}^n$ with $\delta \leq |U| \leq \delta^\theta$. Then $\underline{\dim}_\theta F \geq s$. Moreover, if measures μ_δ with the above properties can be found only for a sequence of $\delta \to 0$, then the conclusion is weakened to $\overline{\dim}_\theta F \geq s$.

We also make use of ellipses to help bound the distance between consecutive turns of $S_{p,q}$ in the upper half plane. So, let us define

$$E_r = \{r^{-p} \cos t + ir^{-q} \sin t : 0 \leq t < 2\pi\} \tag{4.2.5}$$

for each $m \in \mathbb{N}$ and $r = m\pi$, which corresponds to the ellipse centred on the origin with major axis $2r^{-p}$ and minor axis $2r^{-q}$.

Theorem 4.2.3. *Let $\theta \in [0, 1]$ and $0 < p \leq q$. If $p < 1$, then*

$$\dim_\theta S_{p,q} = \frac{p + q + 2\theta(1 - p)}{p + q + \theta(1 - p)}.$$

Otherwise, if $p \geq 1$, then

$$\dim_\theta S_{p,q} = 1.$$

Proof. The upper bound follows from Lemma 4.2.1 applied to the identity mapping. If $p \geq 1$, the upper bound coincides with the trivial lower bound, and so it suffices to assume $0 < p < 1$. Let $0 < \delta < 1$, and define $M \in \mathbb{N}$ to be the smallest integer satisfying

$$M \geq \exp\left(\frac{t(s-1+\theta(2-s))}{1+q}\right),$$

recalling $t = -\log\delta$. Moreover, for the lower bound, it suffices by monotonicity of the intermediate dimensions to consider $S_{p,q}^{+} = S_{p,q} \cap U$, where U is the upper half-plane. The turns associated with $S_{p,q}^{+}$ are denoted $S_{p,q}^{+,k}$.

Next, define

$$s = \frac{p+q+2\theta(1-p)}{p+q+\theta(1-p)},$$

and construct a measure μ_δ supported on $S_{p,q}^{+}$ by

$$\mu_\delta = \delta^{s-1}\sum_{k=1}^{M} \mathcal{H}^1\big|_{S_{p,q}^{+,k}}, \tag{4.2.6}$$

where $\mathcal{H}^1\big|_{S_{p,q}^{+,k}}$ denotes the restriction of 1-dimensional Hausdorff measure to $S_{p,q}^{+,k}$.

It is easy to see that

$$\mu_\delta(S_{p,q}^{+}) = \delta^{s-1}\sum_{k=1}^{M}\mathcal{H}^1(S_{p,q}^{+,k}) \gtrsim_p \delta^{s-1}\sum_{k=1}^{M} k^{-p} \approx_p M^{1-p}\delta^{s-1} \approx_p 1,$$

with the final calculation similar to that which obtained (4.2.4).

Next, in order to apply the mass distribution principle for intermediate dimensions, we must estimate $\mu_\delta(U)$ for arbitrary Borel sets U satisfying $\delta \leq |U| \leq \delta^\theta$. This requires us to consider the spacing between consecutive turns of the spiral. Specifically, we wish to estimate the quantity

$$D_{p,q}(k-1,k) = \inf\{|x-y| : x \in S_{p,q}^{+,k-1}, y \in S_{p,q}^{+,k}\}.$$

It suffices to bound the distance between the pair of ellipses $E_{2k\pi-\pi}$ and $E_{2k\pi}$ that lie between $S_{p,q}^{+,k-1}$ and $S_{p,q}^{+,k}$ in the upper half plane, as illustrated by Figure 4.3.

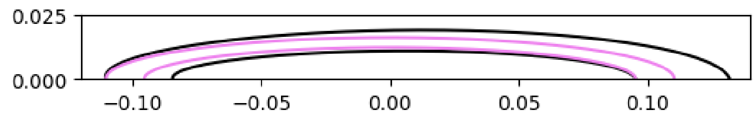

Figure 4.3: A plot illustrating the two ellipses $E_{2k\pi-\pi}$ and $E_{2k\pi}$ (violet) that lie between $S_{p,q}^{+,k-1}$ and $S_{p,q}^{+,k}$ (black). In this example, $p = 0.8$, $q = 1.5$ and $k = 3$.

To do this, we consider the transformation of $E_{2k\pi-\pi}$ and $E_{2k\pi}$ under the affine map $\phi((x,y)) = ((2k\pi)^{p-q}x, y)$. This gives

$$E'_{2k\pi-\pi} := \phi(E_{2k\pi-\pi}) = \left\{\left(\frac{2k\pi}{2k\pi-\pi}\right)^p (2k\pi)^{-q}\cos t + i(2k\pi-\pi)^{-q}\sin t : 0 \le t \le \pi\right\}$$

and

$$E'_{2k\pi} := \phi(E_{2k\pi}) = \left\{(2k\pi)^{-q}\cos t + i(2k\pi)^{-q}\sin t : 0 \le t \le \pi\right\}.$$

The distance between $E'_{2k\pi-\pi}$ and $E'_{2k\pi}$ on the horizontal axis is given by

$$(2k\pi)^{-q}\left(\frac{2k\pi}{2k\pi-\pi}\right)^p - (2k\pi)^{-q}.$$

Moreover, this is the minimal distance between $E'_{2k\pi-1}$ and $E'_{2k\pi}$, which by can seen by noting $p \le q$ implies

$$\left(\frac{2k\pi}{2k\pi-1}\right)^p - 1 \le \left(\frac{2k\pi}{2k\pi-1}\right)^q - 1,$$

and

$$\begin{aligned}
(2k\pi)^{-q}\left(\frac{2k\pi}{2k\pi-\pi}\right)^p-(2k\pi)^{-q} &= \left(\left(\frac{2k\pi}{2k\pi-\pi}\right)^p-1\right)(2k\pi)^{-q}\\
&\le \left(\left(\frac{2k\pi}{2k\pi-\pi}\right)^q-1\right)(2k\pi)^{-q}\\
&= (2k\pi-\pi)^{-q}-(2k\pi)^{-q}.
\end{aligned}$$

Then, by considering the appropriate Taylor expansions, observe that we may choose $k_0 > 0$ such that

$$\left(\frac{2k\pi}{2k\pi-\pi}\right)^p-1 \ge \frac{p}{2q}\left(\left(\frac{2k\pi}{2k\pi-\pi}\right)^q-1\right)$$

for $k > k_0$. Hence

$$\begin{aligned}
(2k\pi)^{-q}\left(\frac{2k\pi}{2k\pi-\pi}\right)^p-(2k\pi)^{-q} &\ge \frac{p}{2q}\left(\left(\frac{2k\pi}{2k\pi-\pi}\right)^q-1\right)(2k\pi)^{-q}\\
&= \frac{p}{2q}\left((2k\pi-\pi)^{-q}-(2k\pi)^{-q}\right).
\end{aligned}$$

Moreover, since ϕ^{-1} is expanding and increases distances, the minimum distance between $E_{2k\pi-\pi}$ and $E_{2k\pi}$ is also bounded below by

$$\frac{p}{2q}\left((2k\pi-\pi)^{-q}-(2k\pi)^{-q}\right)$$

for $k > k_0$. On the other hand, there must exist some $c > 0$ such that the minimum distance between $E_{2k\pi-\pi}$ and $E_{2k\pi}$ is bounded below by

$$c\left((2k\pi-\pi)^{-q}-(2k\pi)^{-q}\right).$$

for $2 \le k \le k_0$, since each distance is strictly positive and there are finitely many such k.

Hence, we conclude

$$D_{p,q}(k-1,k) \geq C_{p,q}\left((2k\pi-\pi)^{-q}-(2k\pi)^{-q}\right) \approx_{p,q} (2k\pi-\pi)^{-q}-(2k\pi)^{-q}, \qquad (4.2.7)$$

where $C_{p,q} = \min\{c, p/(2q)\}$. An application of the mean value theorem to $f(x) = x^{-q}$ then gives

$$f\left(2k\pi-\pi\right) - f\left(2k\pi\right) = (2k\pi-\pi)^{-q}-(2k\pi)^{-q} = -qc^{-q-1}(-1) \qquad (4.2.8)$$

for some $2k\pi - \pi \leq c \leq 2k\pi$. Together, (4.2.7) and (4.2.8) imply

$$\begin{aligned} D_{p,q}(k-1,k) &\gtrsim_{p,q} (2k\pi-\pi)^{-q}-(2k\pi)^{-q} \\ &\gtrsim_{p,q} \frac{1}{(2M\pi)^{1+q}} \\ &\approx_{p,q} \frac{1}{M^{1+q}} \end{aligned}$$

for $2 \leq k \leq M$. It follows that a set U satisfying $\delta \leq |U| \leq \delta^{\theta}$ may intersect at most $|U|M^{1+q}$ turns that contain mass, up to a constant depending only on p and q. Moreover, for each turn it intersects, U may cover a region of mass at most δ^{s-1} multiplied by the circumference of a ball of diameter U. Hence

$$\begin{aligned} \mu_{\delta}(U) &\lesssim_{p,q} (|U|\delta^{s-1})(|U|M^{1+q}) \\ &= |U|^2\delta^{s-1}\delta^{-s+1-\theta(2-s)} \\ &= |U|^2\delta^{\theta(s-2)} \\ &= |U|^2|U|^{s-2} \text{ (since } s<2 \text{ and } |U| \leq r^{\theta}) \\ &= |U|^s. \end{aligned}$$

The lower bound then follows from the mass distribution principle for intermediate dimensions, see Proposition 4.2.2. $\square$

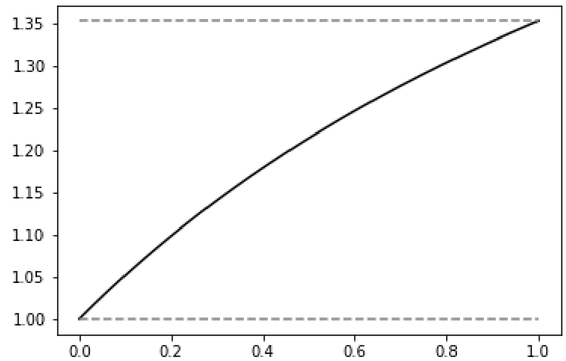

Figure 4.4: A plot of $\dim_\theta S_{p,q}$ against θ (x-axis) for $p = 0.4$ and $q = 0.7$, along with dashed horizontal lines that indicate $\dim_{\mathrm{H}} S_{p,q} = 1$ and $\dim_{\mathrm{B}} S_{p,q} = (2 + q - p)/(1 + q)$.

It is worth remarking that measures of a form similar to (4.2.6) could be useful for a wide range of sets E with a spiral structure. For example, we might consider the image of a spiral under a map f that distorts the local geometry while preserving the general form. If it were the case that $\dim_{\mathrm{H}} f(S_{p,q}^k) = t$ for all $k \in \mathbb{N}$, then measures of the form

$$\mu_\delta = \delta^{s-t} \sum_{k=1}^{M} \mathcal{H}^t\big|_{f(S_{p,q}^k)} \tag{4.2.9}$$

may be good candidates for use with Proposition 4.2.2.

By setting $\theta = 1$, Theorem 4.2.3 also offers the box-counting dimensions of elliptical polynomial spirals.

Corollary 4.2.4. *Let $\theta \in [0, 1]$ and $0 < p \leq q$. If $0 < p < 1$, then*

$$\dim_{\mathrm{B}} S_{p,q} = \frac{2 + q - p}{1 + q} = 1 + \frac{1 - p}{1 + q}.$$

Otherwise, if $p \geq 1$, then

$$\dim_{\mathrm{B}} S_{p,q} = 1.$$

In the special case $p = q$, Theorem 4.2.3 may be applied to determine the intermediate dimensions of generalised hyperbolic spirals, which have also been obtained independently by Tan [65].

Corollary 4.2.5. *Let $\theta \in [0, 1]$. If $0 < p < 1$, then*

$$\dim_\theta S_p = \frac{2p + 2\theta(1-p)}{2p + \theta(1-p)}.$$

Otherwise, if $p \geq 1$, then

$$\dim_\theta S_p = 1.$$

As we saw in previous chapters, a question of interest within the literature on intermediate dimensions has been the classification of sets that are continuous at $\theta = 0$ [8, 21]. Theorem 4.2.3 confirms that the elliptical polynomial spirals are within this class.

Corollary 4.2.6. *Let $0 < p \leq q$. The function $\theta \to \dim_\theta S_{p,q}$ is continuous on $[0, 1]$.*

Next, we move on into the realm of Assouad-type dimensions. As illustrated in Figure 4.5, the following theorem gives the value of $\dim_{\mathrm{A}}^\theta S_{p,q}$ for all $\theta \in (0, 1]$ and establishes the existence of two phase transitions, that is, points where the spectrum is non-differentiable. Moreover, these phase transitions are genuine in the sense that their left and right derivatives are necessarily distinct.

Theorem 4.2.7. *Let $0 < p \leq q$. If $0 < p < 1$, then*

$$\dim_{\mathrm{A}}^\theta S_{p,q} = \begin{cases} \frac{2+q-p}{(1+q)(1-\theta)} & \text{if } 0 \leq \theta < p/(1+q) \\ \frac{2+q-\theta(1+q)}{(1+q)(1-\theta)} & \text{if } p/(1+q) \leq \theta < q/(1+q) \\ 2 & \text{if } q/(1+q) \leq \theta < 1 \end{cases} \cdot$$

Otherwise, if $p \geq 1$, then

$$\dim_{\mathrm{A}}^{\theta} S_{p,q} = \begin{cases} \frac{p-\theta(p-1)}{p(1-\theta)} & \text{if } 0 \leq \theta < p/(1+q) \\ \frac{2+q-\theta(1+q)}{(1+q)(1-\theta)} & \text{if } p/(1+q) \leq \theta < q/(1+q) \\ 2 & \text{if } q/(1+q) \leq \theta < 1 \end{cases} .$$

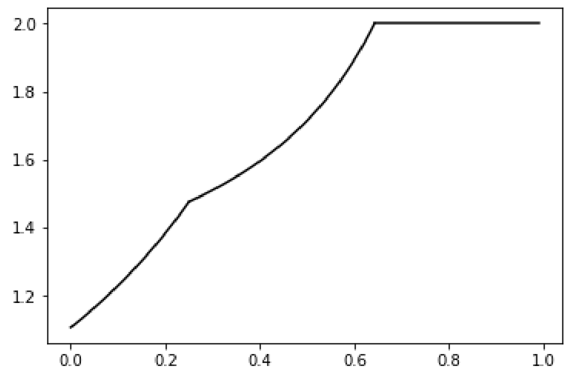

Figure 4.5: A plot of $\dim_{\mathrm{A}}^{\theta} S_{p,q}$ against θ (x-axis) for $p = 1.1$ and $q = 1.8$.

Proof. If $p = q$, then the result is [31, Theorem 4.4], so let $0 < p < q$. For each $0 < \delta < 1$, define $L_p, L_q \in \mathbb{N}$ to be the largest integers such that

$$\delta \leq \frac{1}{(\pi + 2\pi L_p)^p} - \frac{1}{(\pi + 2\pi(L_p + 1))^p} \tag{4.2.10}$$

and

$$\delta \leq \frac{1}{(\frac{3\pi}{2} + 2\pi L_q)^q} - \frac{1}{(\frac{3\pi}{2} + 2\pi(L_q + 1))^q}. \tag{4.2.11}$$

Geometrically, L_p and L_q are the maximal indices k, such that $S_{p,q}^k$ is separated on the horizontal and vertical axes by at least δ, respectively. In addition, define the integers l_p and l_q to be the minimal k such that $S_{p,q}^k$ intersects the ball $B(0, \delta^\theta)$ on the horizontal and vertical axes, respectively. In particular,

$$(\pi + 2\pi l_p)^{-p} \leq \delta^\theta < (\pi + 2\pi(l_p - 1))^{-p}$$

and

$$\left(\frac{3\pi}{2} + 2\pi l_q\right)^{-q} \leq \delta^\theta < \left(\frac{3\pi}{2} + 2\pi(l_q - 1)\right)^{-q}.$$

Figure 4.6 illustrates the geometric significance of the quantities L_p and l_p through an example. Of course, L_q and l_q may be understood similarly by considering the vertical, rather than horizontal, axis. Throughout, we use the fact that

$$S_{p,q} \cap B(0, \delta^\theta) \subseteq \bigcup_{k=l_q}^{\infty} S_{p,q}^k \cap B(0, \delta^\theta).$$

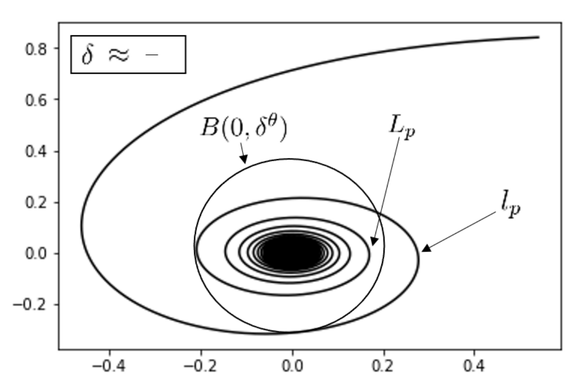

Figure 4.6: A plot illustrating the quantities l_p and L_p. Recall that l_p is the minimal k such that $S_{p,q}^k$ intersects $B(0, \delta^\theta)$ on the horizontal axis, and L_p is the maximal index k such that $S_{p,q}^k$ is separated from neighbouring turns on the horizontal axis by at least δ. The size of δ is indicated pictorially, to scale, by a line segment in the legend.

The ordering of L_p, L_q, l_p and l_q depends on θ, and gives rise to phase transitions within the spectrum. To determine the order based on a value of θ, first note that

$$l_t \approx_t \delta^{-\theta/t} \tag{4.2.12}$$

for $t \in \{p, q\}$. Then, analogously to (4.2.8), for $t \in \{p, q\}$, it follows from an application of the mean value theorem applied to $f(x) = x^{-t}$ that

$$\frac{t}{(k+1)^{1+t}} \leq \frac{1}{k^t} - \frac{1}{(k+1)^t} \leq \frac{t}{k^{1+t}}.$$

This, along with the fact L_p and L_q are the maximal integers satisfying (4.2.10) and (4.2.11), respectively, implies

$$L_t \approx_t \delta^{-\frac{1}{1+t}}. \tag{4.2.13}$$

It is immediate that $l_p \gtrsim_{p,q} l_q$ and $L_p \gtrsim_{p,q} L_q$ for all $\theta \in [0,1)$ since $p < q$, but we must divide into cases to learn more. By continuity of the Assouad spectrum [39, Corollary 3.5] and [39, Corollary 3.6], it suffices to consider θ in the ranges $0 \leq \theta < p/(1+q)$ and $p/(1+q) < \theta < q/(1+q)$. Throughout, we use the estimate

$$N_\delta(S_{p,q} \cap B(z, \delta^\theta)) \lesssim_{p,q} N_\delta(S_{p,q} \cap B(0, \delta^\theta))$$

for all $z \in \mathbb{C}$. This in intuitively clear, since the origin is the densest part of the set $S_{p,q}$. [39] provides further details on this reduction in the case of S_p and similar arguments would apply here.

Case 1: suppose $\frac{p}{1+q} < \theta < \frac{q}{1+q}$.

In order to simplify some geometric estimates, it is convenient to adopt an equivalent definition of the Assouad spectrum in this case. Specifically, we consider minimal coverings of the set $D(0, \delta^\theta) \cap S_{p,q}$, where $D(0, \delta^\theta)$ is a square centred on the origin of sidelength $2\delta^\theta$ and orientated with the co-ordinate axes. By (4.2.12) and (4.2.13), for sufficiently small $\delta > 0$,

$$l_p^{-p} < L_q^{-p} < l_q^{-p}.$$

For $l_q \leq k \leq L_q$, the set $S^k_{p,q} \cap D(0,\delta^\theta)$ contains at least one arc A such that

$$\mathcal{H}^1(A) \approx \delta^\theta,$$

and so

$$N_\delta(A) \approx \frac{\delta^\theta}{\delta}.$$

Turns in the range $l_q \leq k \leq L_q$ are separated by at least δ on the vertical and horizontal axes, and thus any square of sidelength δ may intersect at most two of the corresponding arcs.

It follows that, recalling (4.2.12) and (4.2.13),

$$\begin{aligned} N_\delta(S_{p,q} \cap D(0,\delta^\theta)) &\gtrsim \sum_{k=l_q}^{L_q} \delta^{\theta-1} \\ &\approx_{p,q} \delta^{\theta-1}\left(\delta^{-\frac{1}{1+q}} - \delta^{-\frac{\theta}{q}}\right) \\ &\gtrsim_{p,q} \left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-\theta(1+q)}{(1+q)(1-\theta)}}. \end{aligned} \tag{4.2.14}$$

Hence

$$\dim_{\mathrm{A}}^\theta S_{p,q} \geq \frac{2+q-\theta(1+q)}{(1+q)(1-\theta)}.$$

On the other hand, observe

$$\bigcup_{k=L_q}^{\infty} S^k_{p,q} \cap D(0,\delta^\theta) \subseteq [-\delta^\theta, \delta^\theta] \times [-(2\pi L_q)^{-q}, (2\pi L_q)^{-q}],$$

and such a rectangle may be covered by

$$\approx_q \frac{\delta^\theta L_q^{-q}}{\delta^2}$$

squares of sidelength δ. The remaining portion may be covered in a similar manner as in (4.2.14), and we conclude

$$\begin{aligned}
N_\delta(S_{p,q} \cap B(0,\delta^\theta)) &\lesssim_q \frac{\delta^\theta L_q^{-q}}{\delta^2} + \sum_{k=l_q}^{L_q} \delta^{\theta-1} \\
&\approx_{p,q} \left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-\theta(1+q)}{(1+q)(1-\theta)}} + \left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-\theta(1+q)}{(1+q)(1-\theta)}} \\
&= 2\left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-\theta(1+q)}{(1+q)(1-\theta)}}.
\end{aligned}$$

Case 2: suppose $0 \leq \theta < \frac{p}{1+q}$.

By (4.2.12) and (4.2.13), for sufficiently small $\delta > 0$,

$$L_p^{-p} < L_q^{-p} < l_p^{-p} < l_q^{-p},$$

with the gaps between the four integers L_p, L_q, l_p and l_q arbitrarily large. Then, for $k = l_p + 1, \ldots, L_q$, we have

$$S_{p,q}^k \subset B(0,\delta^\theta),$$

while the turns in this region are separated by at least δ on the horizontal and vertical axes. Therefore they should be covered individually by at least

$$\frac{\mathcal{H}^1(S_{p,q}^k)}{\delta} \approx_p \frac{k^{-p}}{\delta}$$

squares of sidelength δ.

Hence

$$N_\delta(S_{p,q} \cap B(0,\delta^\theta)) \gtrsim_p \sum_{k=l_p}^{L_q} \frac{k^{-p}}{\delta}. \tag{4.2.15}$$

This sum may be estimated using Lemma 4.1.1. If $p < 1$, then

$$\begin{aligned}
N_\delta(S_{p,q} \cap B(0, \delta^\theta)) &\gtrsim_p \frac{L_q^{1-p} - l_p^{1-p}}{\delta} \\
&\approx_{p,q} \delta^{\frac{p-1}{1+q} - 1} \\
&= \left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-p}{(1+q)(1-\theta)}}.
\end{aligned}$$

On the other hand, if $p = 1$, then

$$\begin{aligned}
N_\delta(S_{p,q} \cap B(0, \delta^\theta)) &\gtrsim_p \frac{\log(L_q) - \log(l_p)}{\delta} \\
&\approx_{p,q} \delta^{-1} |\log(\delta)| \\
&\geq \left(\frac{\delta^\theta}{\delta}\right)^{\frac{1}{(1-\theta)}}.
\end{aligned} \tag{4.2.16}$$

Finally, if $p > 1$, then

$$\begin{aligned}
N_\delta(S_{p,q} \cap B(0, \delta^\theta)) &\gtrsim_p \frac{l_p^{1-p} - L_q^{1-p}}{\delta} \\
&\approx_{p,q} \delta^{\frac{(p-1)\theta}{p} - 1} \\
&= \left(\frac{\delta^\theta}{\delta}\right)^{\frac{p-\theta(p-1)}{p(1-\theta)}}.
\end{aligned}$$

In each case we obtain the desired lower bound.

For the upper bound, we consider a cover of three parts. First, cover turns indexed by $k \geq L_q$ by covering the rectangle

$$[-(2\pi L_q)^{-p}, (2\pi L_q)^{-p}] \times [-(2\pi L_q)^{-q}, (2\pi L_q)^{-q}]$$

by

$$\approx_{p,q} \frac{L_q^{-p} L_q^{-q}}{\delta^2}$$

squares of sidelength δ. The remaining two portions may then be covered as in (4.2.14) and (4.2.15). Hence

$$N_\delta(S_{p,q} \cap B(0,\delta^\theta)) \lesssim_{p,q} \frac{L_q^{-p} L_q^{-q}}{\delta^2} + \sum_{k=l_p}^{L_q} \frac{k^{-p}}{\delta} + \sum_{k=l_q}^{l_p} \delta^{\theta-1}.$$

We now apply Lemma 4.1.1 in each case. If $p < 1$, then

$$\begin{aligned} N_\delta(S_{p,q} \cap B(0,\delta^\theta)) &\lesssim_{p,q} \delta^{\frac{p}{1+q}+\frac{q}{1+q}-2} + \delta^{-1}(L_q^{1-p} - l_p^{1-p}) + \delta^{\theta-1}(l_p - l_q) \\ &\lesssim_{p,q} \left(\frac{\delta^\theta}{\delta}\right)^{\frac{2+q-p}{(1-\theta)(1+q)}}. \end{aligned}$$

On the other hand, if $p = 1$, then

$$\begin{aligned} N_\delta(S_{p,q} \cap B(0,\delta^\theta)) &\lesssim_{p,q} \delta^{\frac{p}{1+q}+\frac{q}{1+q}-2} + \delta^{-1}(\log L_q - \log l_p) + \delta^{\theta-1}(l_p - l_q) \\ &\lesssim_{p,q} \left(\frac{\delta^\theta}{\delta}\right)^{\frac{1}{1-\theta}}. \end{aligned}$$

Finally, if $p > 1$, then

$$\begin{aligned} N_\delta(S_{p,q} \cap B(0,\delta^\theta)) &\lesssim_{p,q} \delta^{\frac{p}{1+q}+\frac{q}{1+q}-2} + \delta^{-1}(l_p^{1-p} - L_q^{1-p}) + \delta^{\theta-1}(l_p - l_q) \\ &\lesssim_{p,q} \left(\frac{\delta^\theta}{\delta}\right)^{\frac{p-(p-1)\theta}{(1-\theta)p}}, \end{aligned}$$

which completes the proof. □

The reader familiar with [31] may be surprised to see that the first phase transition occurs at $p/(1+q)$, rather than $p/(1+p)$. Indeed, this shows an unexpected and subtle interaction between the parameters. Theorem 4.2.7 also shows that elliptical polynomial spirals have maximal Assouad dimension.

Corollary 4.2.8. *For all* $0 < p \le q$, $\dim_{\mathrm{A}} S_{p,q} = 2$.

Lastly, the relationship between elliptical polynomial spirals and concentric ellipses indicated by the proof of Theorem 4.2.3 is worthy of further comment. Let us define

$$C_{p,q} = \bigcup_{n\in\mathbb{N}} E_{2\pi n},$$

where $E_{2\pi n}$ denotes the ellipse given by (4.2.5). See Figure 4.7 for a visual representation of $C_{p,q}$. It is not surprising that $C_{p,q}$ is dimensionally equivalent to $S_{p,q}$ and our arguments apply equally well to such sets, since it is not too hard to show that the covering number of $S^k_{p,q}$ is equal to that of $E_{2\pi k}$ up to multiplicative constants depending only on p and q.

Corollary 4.2.9. *Theorem 4.2.3 and Theorem 4.2.7 hold with $S_{p,q}$ replaced by $C_{p,q}$.*

Proof. This follows immediately upon observing that $S_{p,q} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$ is bi-Lipschitz equivalent to $C_{p,q} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}$. $\square$

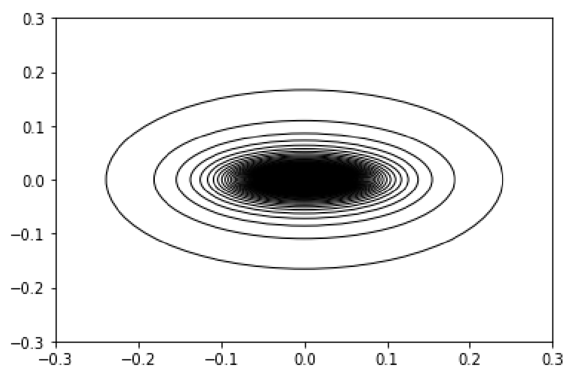

Figure 4.7: A family of concentric ellipses $C_{p,q}$ dimensionally equivalent to $S_{p,q}$, where $p = 0.4$ and $q = 0.6$.

4.3 Regularity of spiral deformations

In this section we shall see how dimension theoretic information may be applied to examine the regularity of Hölder mappings that deform one elliptical polynomial spiral into another. The behaviour of dimension under Hölder mappings has been widely studied, and offers insight into permissible α for which there may exist an α-Hölder map transforming a set X onto a set Y. For example, Corollary 4.2.4 allows us to glean such information from the box-counting dimensions of $S_{p,q}$ and $S_{r,s}$.

Proposition 4.3.1. *Let $0 < p \leq q$ and $0 < r \leq s$ with $r \leq 1$. Suppose $f : S_{p,q} \to S_{r,s}$ is α-Hölder. If $p \leq 1$, then*

$$\alpha \leq \frac{(2+q-p)(1+s)}{(2+s-r)(1+q)}.$$

Otherwise, if $p > 1$, then

$$\alpha \leq \frac{1+s}{2+s-r}.$$

Proof. Let $p \leq 1$. By the standard properties of box-counting dimensions, see [17, Chapter 2],

$$\frac{2+s-r}{1+s} = \dim_{\mathrm{B}} f(S_{p,q}) \leq \frac{1}{\alpha} \dim_{\mathrm{B}} S_{p,q} = \frac{1}{\alpha} \frac{2+q-p}{1+q},$$

from which the first result follows. The case for $p > 1$ is similar. □

Proposition 4.3.1 provides a non-trivial bound on α when $\dim_{\mathrm{B}} S_{r,s} > \dim_{\mathrm{B}} S_{p,q}$. However, it is possible to do better using dimension profiles. In the following lemma, we bound the 2α-profiles of $S_{p,q}$ by a quantity strictly less than the dimension for $\theta > 0$, $p < 1$ and $1/2 < \alpha < 1$. This is depicted in Figure 4.8.

Lemma 4.3.2. *Let $0 < p \leq q$ and $\theta \in [0,1]$. If $p \leq 1$, then*

$$\overline{\dim}_{\theta}^{2\alpha} S_{p,q} \leq \begin{cases} 2\alpha & 0 < \alpha \leq 1/2 \\ \frac{\alpha(p+q+2\theta(1-p))}{\alpha(p+q)+\theta(1-p)} & 1/2 < \alpha < 1 \end{cases}.$$

Proof. Recall that index-α fractional Brownian motion is almost surely $(\alpha - \varepsilon)$-Hölder for all $\varepsilon > 0$ [47].

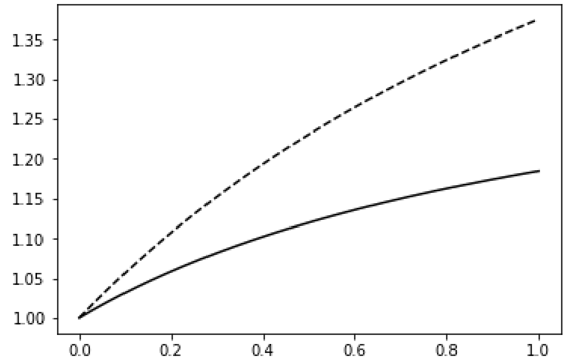

Figure 4.8: A plot of the upper bound of $\overline{\dim}_\theta^{2\alpha} S_{p,q}$ (solid) and $\dim_\theta S_{p,q}$ (dashed) against θ (x-axis) for $\alpha = 0.7$, $p = 0.4$ and $q = 0.6$.

Hence, for each $\varepsilon > 0$, Lemma 4.2.1 tells us that

$$\overline{\dim}_\theta B_\alpha(S_{p,q}) \leq \begin{cases} 2 & 0 < \alpha \leq 1/2 \\ \frac{p+q+2\theta(1-p)}{(\alpha-\varepsilon)(p+q)+\theta(1-p)} & 1/2 < \alpha < 1 \end{cases}$$

almost surely. Then, letting $\varepsilon \to 0$, by Theorem 3.4.7 we have

$$\overline{\dim}_\theta^{2\alpha} S_{p,q} = \alpha \overline{\dim}_\theta B_\alpha(S_{p,q}) \leq \begin{cases} 2\alpha & 0 < \alpha \leq 1/2 \\ \frac{\alpha(p+q+2\theta(1-p))}{\alpha(p+q)+\theta(1-p)} & 1/2 < \alpha < 1 \end{cases}$$

almost surely. This concludes the proof, since $\overline{\dim}_\theta^{2\alpha} S_{p,q}$ has no random component. $\square$

It is clear from Lemma 4.3.2 that we may produce a bound strictly superior to that from Theorem 4.3.1 for all parameter configurations with $p < 1$ using dimension profiles. This improvement is illustrated in Figure 4.9. For larger p, the two approaches are equivalent.

Theorem 4.3.3. *Let $0 < p \leq q$ and $0 < r \leq s$. If $p \leq 1$, $r \leq 1$ and $f : S_{p,q} \to S_{r,s}$ is α-Hölder, then*

$$\alpha \leq \frac{p+q+r+s-pr+qs}{(2+s-r)(p+q)}.$$

Proof. The target bound is strictly greater than $1/2$, and so we may assume without loss of generality that $\alpha > 1/2$. The discrepancy between the profile and the dimension is maximised when $\theta = 1$. Thus, set $\theta = 1$, and observe from Theorem 3.4.1, Lemma 4.3.2 and Corollary 4.2.4 that

$$\dim_1 S_{r,s} = \frac{2+s-r}{1+s} \leq \frac{1}{\alpha} \overline{\dim}_1^{2\alpha} S_{p,q} \leq \frac{p+q+2(1-p)}{\alpha(p+q)+(1-p)},$$

from which the result follows on re-expressing the inequality in terms of α. $\square$

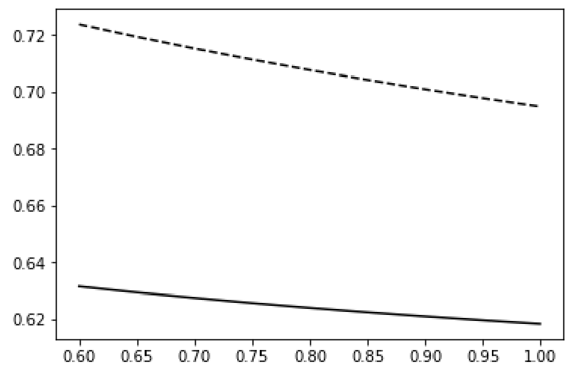

Figure 4.9: Bounds on the Hölder exponent of $f : S_{p,q} \to S_{r,s}$ against the value of q (x-axis) when $p = 0.6, r = 0.2$ and $s = 0.1$. The bounds derived from the dimension profiles (Theorem 4.3.3) and the box-counting dimension (Proposition 4.3.1) correspond to the solid and dashed lines, respectively.

Recall that if $p = q$, then $S_{p,p} = S_p$ is a generalised hyperbolic spiral. In this case, Theorem 4.3.3 offers an appealing upper bound on α.

Corollary 4.3.4. *Let $p > q$ and $f : S_p \to S_q$ be α-Hölder. If $p \leq 1$, then*

$$\alpha \leq \frac{p+q}{2p}.$$

Proof. Apply Theorem 4.3.3 to $f : S_{p,p} \to S_{q,q}$. $\square$

In [31], it was seen that the Assouad spectrum provided the most information on Hölder exponents in the context of the winding problem (mapping a line segment to a spiral). However, it is easily verified that the same tool, [39, Theorem 4.11], provides only trivial information in our setting (mapping a spiral to a spiral). Conversely, in the context of the winding problem, dimension profiles provide no new information. Thus, it is interesting to see that the regimes are inverted in the context of spiral deformation, with the Assouad spectrum providing the least information and the dimension profiles the most.

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