FAI HW4

Problem 1.

$$\varphi'(s) = \theta(s) + s \cdot \theta'(s) = \frac{1}{1+\exp(-s)} + s \cdot \frac{\exp(-s)}{(1+\exp(-s))^2}$$

$$= \frac{1}{1+\exp(-s)}\left[\frac{1+(1+s)\exp(-s)}{1+\exp(-s)}\right] = \frac{1+(1+s)\exp(-s)}{1+2\exp(-s)+\exp(-2s)}$$

Problem 2.

A) $v_1 = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/6 \\ 1/3 \end{pmatrix}$ $v_3 = \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/2 \\ 1/6 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 1/6 \\ 1/2 \end{pmatrix}$

$v_4 = \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/6 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 1/4 \\ 1/3 \end{pmatrix}$ $v_5 = \begin{pmatrix} 0 & 1 & 1/2 \\ 0 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 5/12 \\ 1/4 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 5/12 \\ 1/6 \\ 5/12 \end{pmatrix}$

B) To solve this, we have to solve $Pv^* = v^* \Rightarrow (P-I)v^* = 0 \Rightarrow \begin{bmatrix} -1 & 1 & 1/2 \\ 0 & -1 & 1/2 \\ 1 & 0 & -1 \end{bmatrix} v^* = 0$

Using Gaussian elimination,

$$\begin{bmatrix} -1 & 1 & 1/2 \\ 0 & -1 & 1/2 \\ 1 & 0 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1/2 \\ 0 & 1 & -1/2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow v^* = \begin{pmatrix} v_3 \\ \frac{1}{2}v_3 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} t$$

$$\Rightarrow v^* = \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \Rightarrow \frac{v^*}{\|v\|} = \frac{1}{\sqrt{1+\frac{1}{4}}}\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} = \frac{2}{\sqrt{5}}\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} \#$$

Problem 3.

A) 1st Iter: $S_1 = \{(1,2)\}$, $S_2 = \{(3,4), (7,0), (10,2)\} \Rightarrow \mu_1 = (1,2)$, $\mu_2 = (20/3, 2)$

2nd Iter: $S_1 = \{(1,2), (3,4)\}$, $S_2 = \{(7,0), (10,2)\} \Rightarrow \mu_1 = (2,3)$, $\mu_2 = (17/2, 1)$

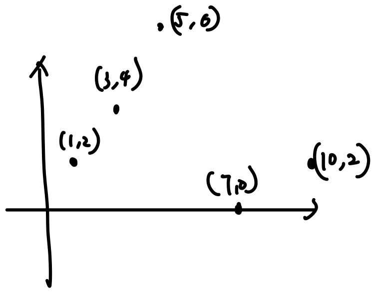
3rd Iter: $S_1 = \{(1,2), (3,4)\}$, $S_2 = \{(7,0), (10,2)\}$

$\Rightarrow$ Thus the algo converges.

B) 1st Iter: $S_1 = \{(1,2), (3,4)\}$, $S_2 = \{(7,0), (10,2)\}$ $\Rightarrow$ $\mu_1 = (2,3)$, $\mu_2 = (17/2, 1)$

$\Rightarrow$ It is obvious that the algo converges immediately. $\Rightarrow$ The result is the same.

C)



We can see that the global minimum is achieved with

$S_1 = \{(1,2), (3,4), (5,6)\}$, $S_2 = \{(7,0), (10,2)\}$.

Suppose we initialize with $\mu_1 = (1,2)$, $\mu_2 = (3,4)$

$\Rightarrow S_1 = \{(1,2)\}$, $S_2 = \{(3,4), (5,6), (7,0), (10,2)\}$ $\Rightarrow$ $\mu_1 = (1,2)$, $\mu_2 = (25/4, 3)$

$\Rightarrow S_1 = \{(1,2), (3,4)\}$, $S_2 = \{(5,6), (7,0), (10,2)\}$ $\Rightarrow$ $\mu_1 = (2,3)$, $\mu_2 = (8, 8/3)$

$\Rightarrow S_1 = \{(1,2), (3,4)\}$, $S_2 = \{$

hw4

June 2, 2025

1 Hw4 Report

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
from hw4 import *
from src.pca import PCA

%load_ext autoreload
%autoreload 2
```

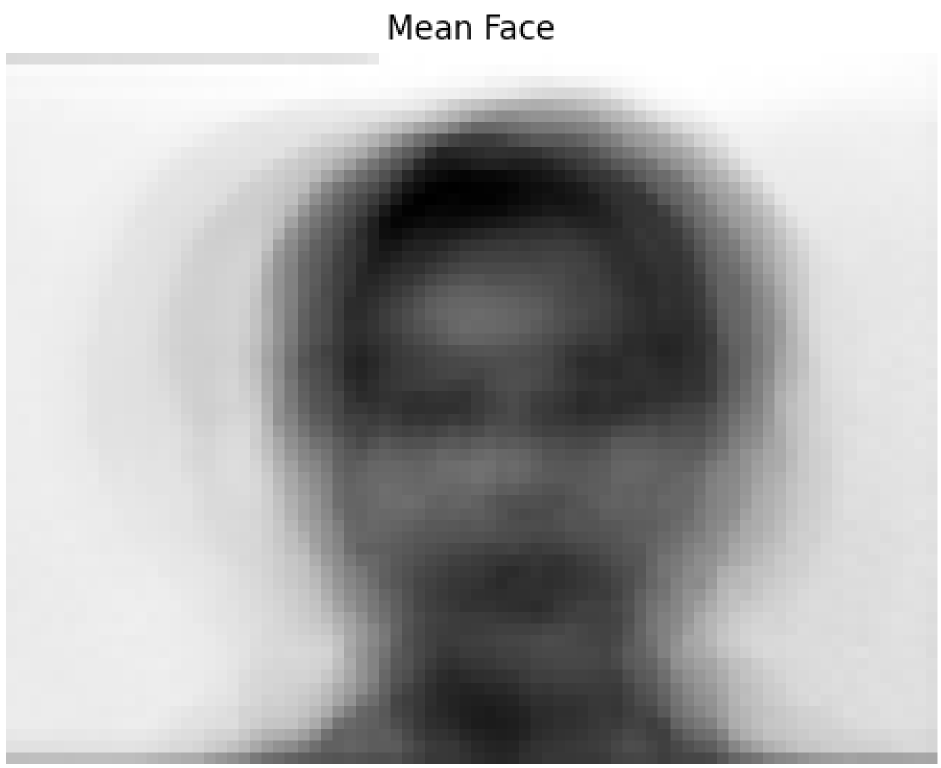
[2]:
```
# load date
X_train, y_train = load_data("train")
X_val, y_val = load_data("val")
```

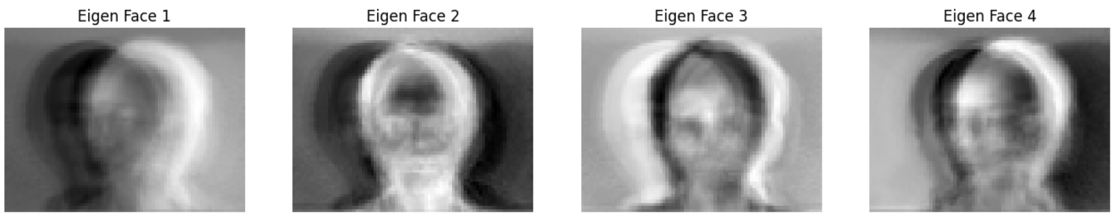
[3]:
```
# Train pca
pca = PCA(n_components=40)
pca.fit(X_train)
pca.components
```

[3]:
```
array([[-0.00251804, -0.00202934, -0.0019007 , …,  0.01042043,
         0.00970501,  0.00952482],
       [-0.00581508, -0.00549877, -0.00501899, …, -0.01489392,
        -0.01421267, -0.0147486 ],
       [-0.00328324, -0.00348955, -0.00360748, …,  0.00068049,
         0.00076611,  0.00179045],
       …,
       [ 0.01437973,  0.01673194,  0.01529846, …, -0.01356296,
        -0.00914527, -0.00976794],
       [-0.01058609, -0.01044993, -0.01005245, …, -0.00516778,
        -0.01495942, -0.02190599],
       [ 0.00892173,  0.01136742,  0.01095737, …, -0.00188989,
        -0.00220494, -0.00326373]], shape=(40, 4880))
```

1.1 (a)

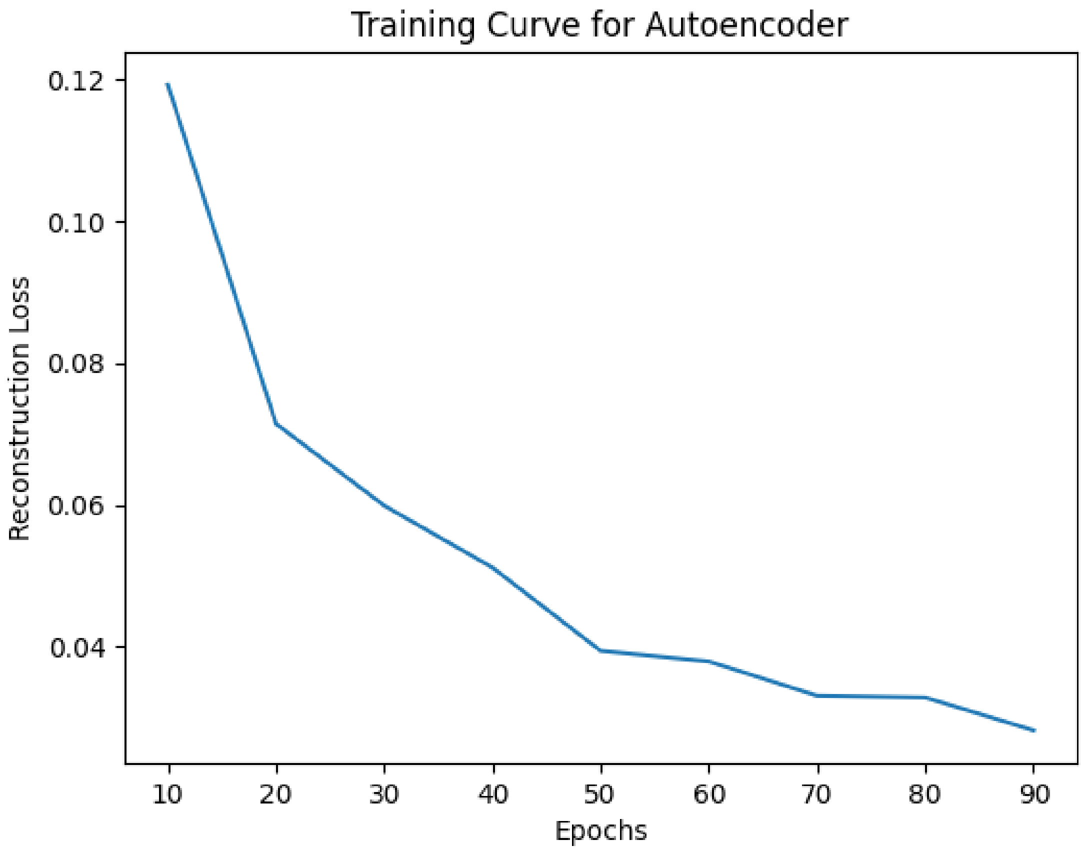
[4]:
```
# (a) plot mean and eigen face

plt.imshow(pca.mean.reshape(61, 80), cmap='gray')
plt.title('Mean Face')
plt.axis('off')
plt.show()

fig, axes = plt.subplots(1, 4, figsize=(16, 4))

for i in range(4):
    axes[i].imshow(pca.components[i].reshape(61, 80), cmap='gray')
    axes[i].set_title(f'Eigen Face {i+1}')
    axes[i].axis('off')

plt.show()
```



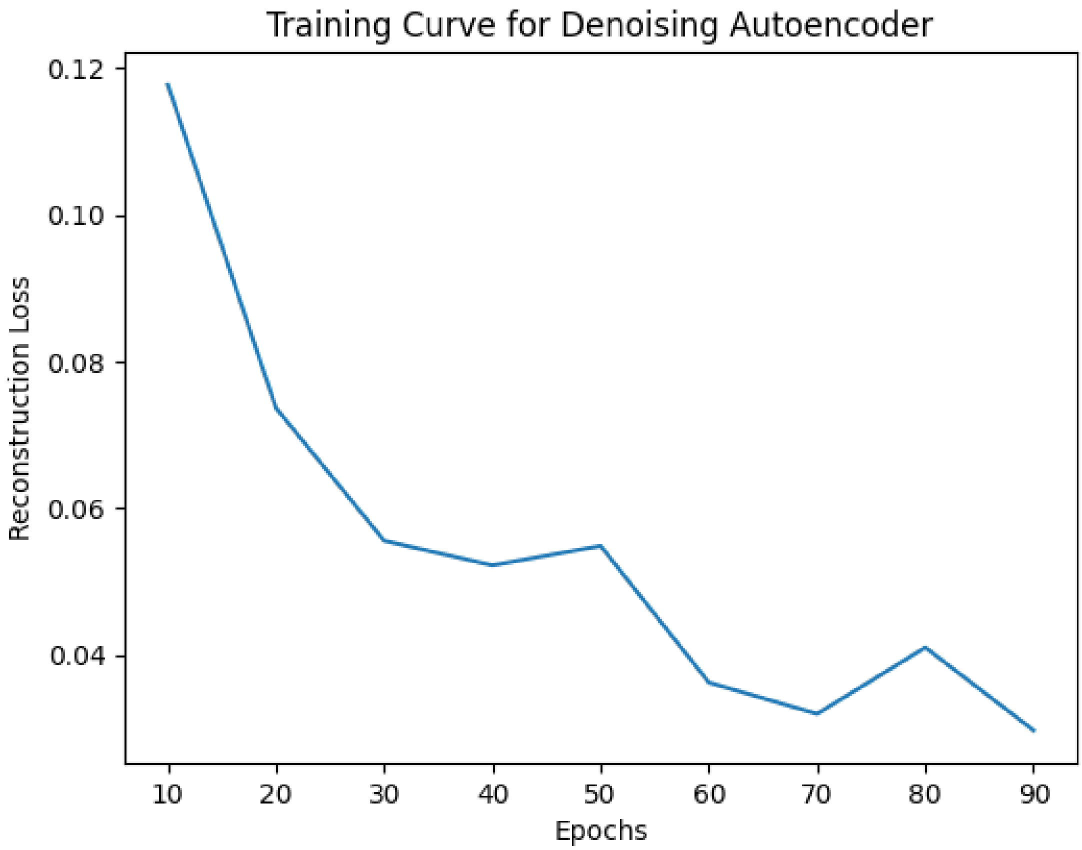


1.1.1 (b) Training curve for Autoencoder and DenoisingAutoencoder

[5]:
```
# (b) the training curve for the autoencoder

err_list = []
for epochs in range(10, 100, 10):
    autoencoder = Autoencoder(input_dim=4880, encoding_dim=488)
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list.append(reconstruction_loss(X_train, autoencoder.
 ↪reconstruct(X_train)))

err = pd.Series(err_list, index=range(10, 100, 10))

plt.plot(err)
plt.title('Training Curve for Autoencoder')
plt.xlabel('Epochs')
plt.ylabel('Reconstruction Loss')
plt.show()
```



[6]:
```
# the training curve for the denoising autoencoder

err_list = []
for epochs in range(10, 100, 10):
    autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488)
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list.append(reconstruction_loss(X_train, autoencoder.
 ↪reconstruct(X_train)))

err = pd.Series(err_list, index=range(10, 100, 10))
plt.plot(err)
plt.title('Training Curve for Denoising Autoencoder')
plt.xlabel('Epochs')
plt.ylabel('Reconstruction Loss')
plt.xticks(range(10, 100, 10))
plt.show()
```

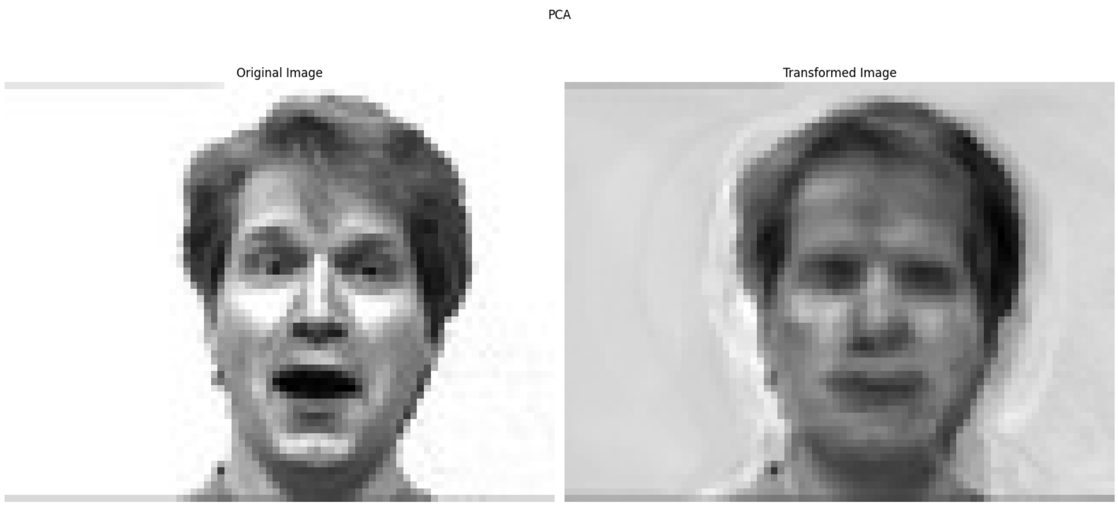


1.1.2 (c) Faces

This part plot the reconstructed faces by three different encoder: PCA, Autoencoder and DenoisingAutoencoder.

[7]:
```
# plot original image

img = read_image()


def plot_image(img, model = pca, title = "PCA"):

    img_transformed = model.transform(img)
    img_reconstructed = model.reconstruct(img)

    # plot the original and reconstructed images
    fig, axes = plt.subplots(1, 2, figsize=(16, 8))
    axes[0].imshow(np.real(img).reshape(61, 80), cmap='gray')
    axes[0].axis('off')
    axes[0].set_title('Original Image')
```

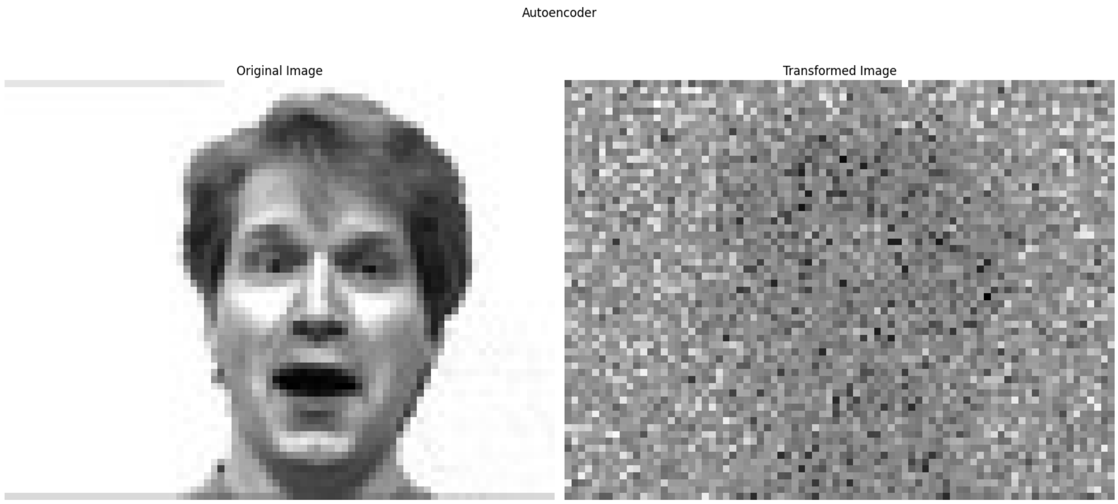
```
    axes[1].imshow(np.real(img_reconstructed).reshape(61, 80), cmap='gray')
    axes[1].axis('off')
    axes[1].set_title('Transformed Image')
    plt.suptitle(title)
    plt.tight_layout()
    plt.show()
    print(f"Reconstruction Loss: {reconstruction_loss(img, img_reconstructed)}")

plot_image(img, pca)
```




```
Reconstruction Loss: 0.010710469688056324
```

[73]:
```
# train autoencoder

encoder = Autoencoder(input_dim=4880, encoding_dim=488)
encoder.fit(X_train, epochs=100, batch_size=32)
plot_image(img, encoder, title = "Autoencoder")
```

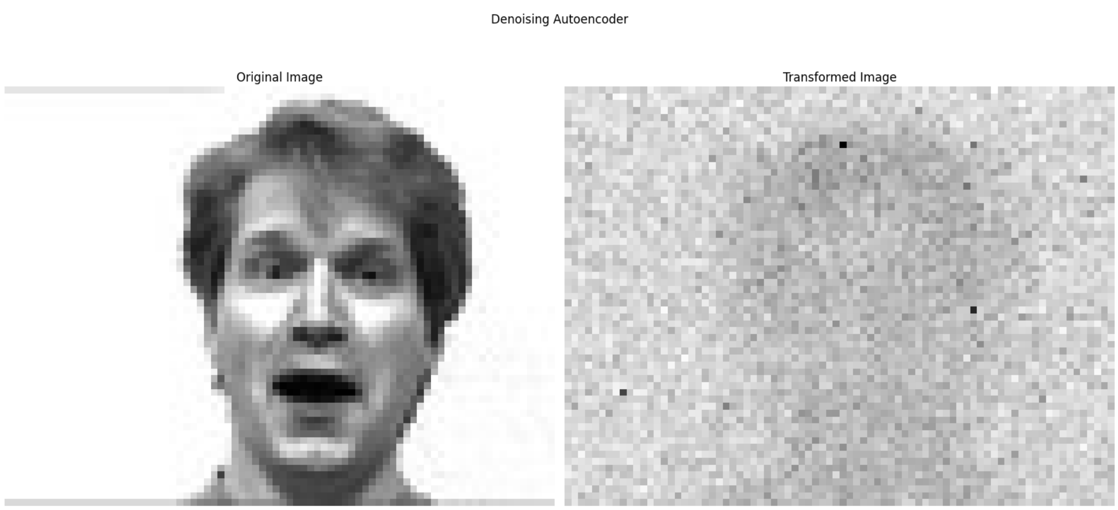
```
  0%|          | 0/100 [00:00<?, ?it/s]
```




```
Reconstruction Loss: 0.10303349307408938
```

[72]:
```
encoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488)
encoder.fit(X_train, epochs=100, batch_size=32)
plot_image(img, encoder, title = "Denoising Autoencoder")
```

```
  0%|          | 0/100 [00:00<?, ?it/s]
```




```
Reconstruction Loss: 0.07361398801586323
```

1.1.3 (d) Compare model architectures

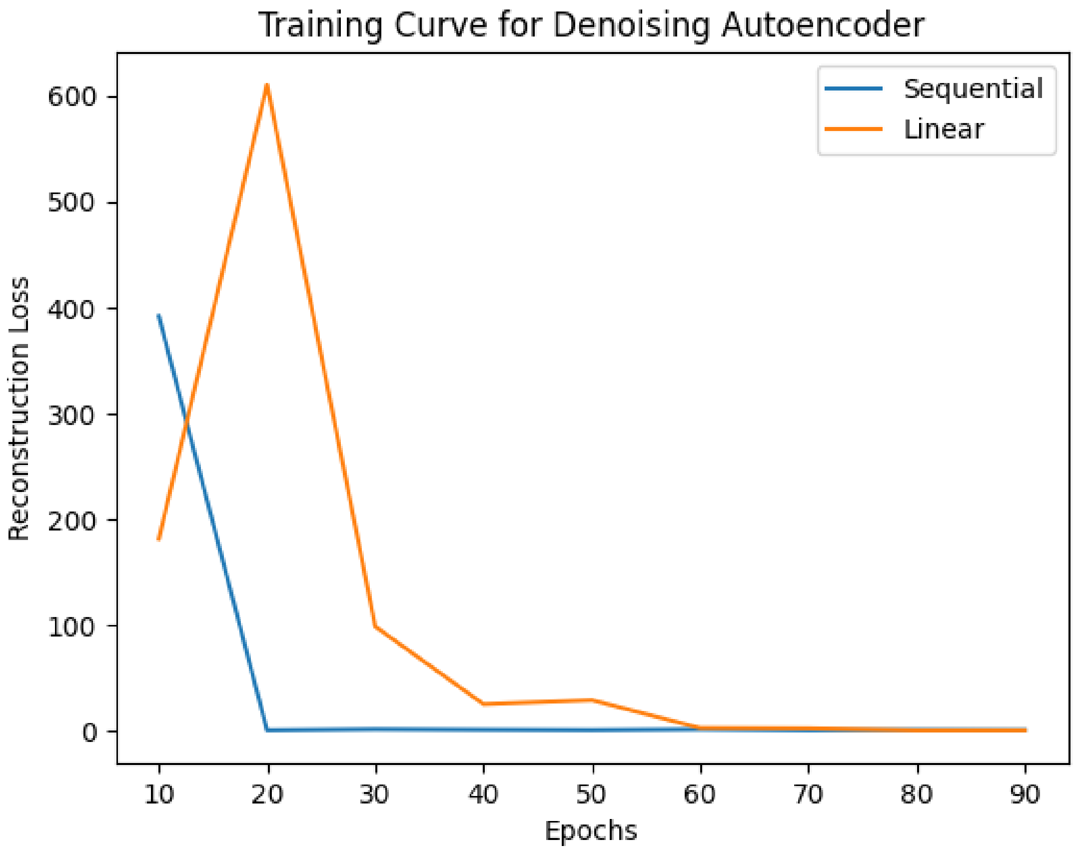
We try two model architectures: The first one is the default architecture provided with the sample code, and the second one are linear architecture with one layer of linear encoder and one layer of linear decoder.

As see in the training curve, as number of epochs increases, the default (sequential) model converges much faster then the linear model. However, as epochs approach 100, the reconstruction error surpasses that of default model.

We also plot the reconstructed image for the face vector using the same parameter as the one plotted above. We can see that the linear model can reconstruct a much clearer image than the default model.

[ ]:
```
# (d) compare two archetecture:

# sequential architecture
err_list_sequential = []
for epochs in range(10, 100, 10):
    autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488,␣
  ↪architecture="sequential")
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list_sequential.append(reconstruction_loss(X_train, autoencoder.
  ↪reconstruct(X_train)))

# linear architecture

err_list_linear = []
for epochs in range(10, 100, 10):
    autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488,␣
  ↪architecture="linear")
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list_linear.append(reconstruction_loss(X_train, autoencoder.
  ↪reconstruct(X_train)))
```

[55]:
```
err = pd.DataFrame({"Sequential": err_list_sequential, "Linear":␣
  ↪err_list_linear}, index=range(10, 100, 10))
plt.plot(err)
plt.title('Training Curve for Denoising Autoencoder')
plt.xlabel('Epochs')
plt.ylabel('Reconstruction Loss')
plt.xticks(range(10, 100, 10))
plt.legend(["Sequential", "Linear"])
plt.show()
```



[74]:
```
autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488,␣
↪architecture="linear")
autoencoder.fit(X_train, epochs=100, batch_size=32)
plot_image(img, autoencoder, title = "Denoising Autoencoder")
```

```
  0%|          | 0/100 [00:00<?, ?it/s]
```


```
Reconstruction Loss: 0.054513760697187066
```

1.1.4 (e) Optimizer Setting

Useing the same model architecture (default), we test two differnt optimizer: Adam and SGD. The below code plot the learning curve. The learning rates are both set to lr = 0.01

We can see that the Adam optimizer overall has about the same reconstruction loss as SGD. As shown in below output, as epochs grow, the reconstruction error sees no improvement for SGD, but improve significantly for Adam. Thus we conclude that SGD converges much faster then Adam.

[45]:
```
# (e) compare two optimizer:

# Adam
err_list_adam = []
for epochs in range(10, 100, 10):
    autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488)
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list_adam.append(reconstruction_loss(X_train, autoencoder.
 ↪reconstruct(X_train)))
```

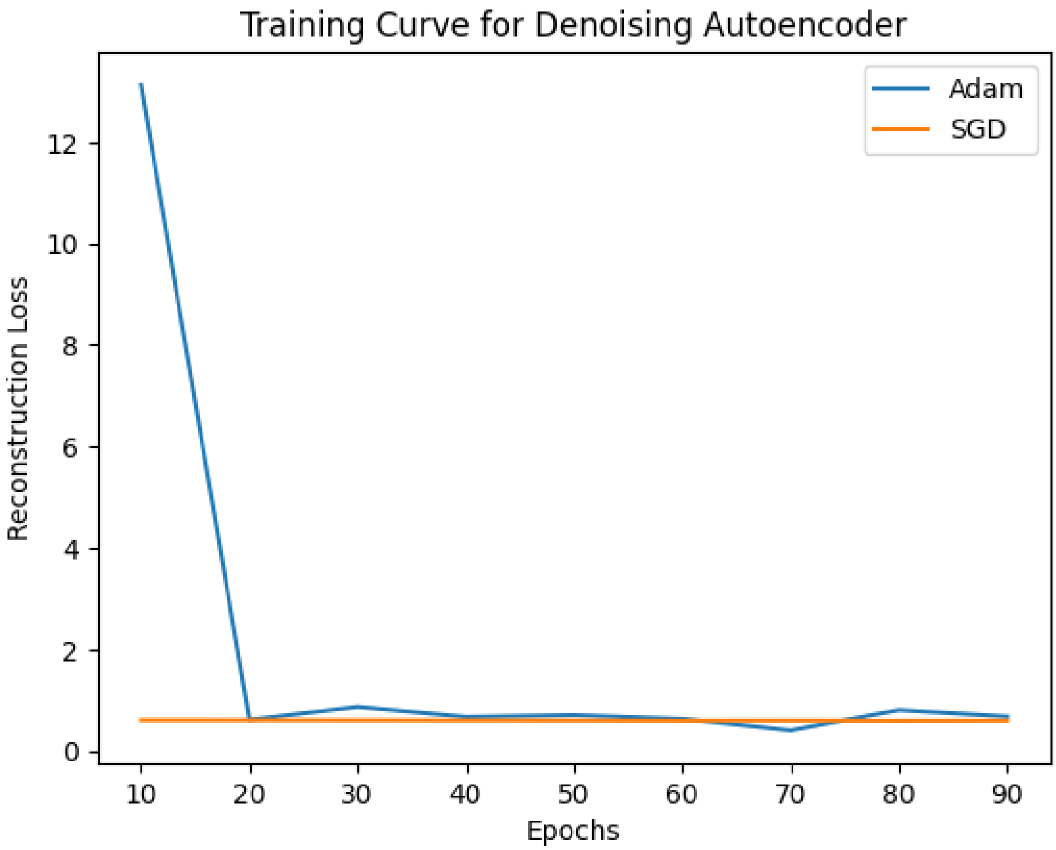
```
0%|          | 0/10 [00:00<?, ?it/s]

0%|          | 0/20 [00:00<?, ?it/s]

0%|          | 0/30 [00:00<?, ?it/s]

0%|          | 0/40 [00:00<?, ?it/s]

0%|          | 0/50 [00:00<?, ?it/s]

0%|          | 0/60 [00:00<?, ?it/s]

0%|          | 0/70 [00:00<?, ?it/s]
```

```
  0%|          | 0/80 [00:00<?, ?it/s]

  0%|          | 0/90 [00:00<?, ?it/s]
```

[46]:
```
# SGD
err_list_sgd = []

for epochs in range(10, 100, 10):
    autoencoder = DenoisingAutoencoder(input_dim=4880, encoding_dim=488,␣
 ↪optimizer_setting="SGD")
    autoencoder.fit(X_train, epochs=epochs, batch_size=135)
    err_list_sgd.append(reconstruction_loss(X_train, autoencoder.
 ↪reconstruct(X_train)))
```

```
  0%|          | 0/10 [00:00<?, ?it/s]

  0%|          | 0/20 [00:00<?, ?it/s]

  0%|          | 0/30 [00:00<?, ?it/s]

  0%|          | 0/40 [00:00<?, ?it/s]

  0%|          | 0/50 [00:00<?, ?it/s]

  0%|          | 0/60 [00:00<?, ?it/s]

  0%|          | 0/70 [00:00<?, ?it/s]

  0%|          | 0/80 [00:00<?, ?it/s]

  0%|          | 0/90 [00:00<?, ?it/s]
```

[48]:
```
err = pd.DataFrame({"Adam": err_list_adam, "SGD": err_list_sgd},␣
 ↪index=range(10, 100, 10))
plt.plot(err)
plt.title('Training Curve for Denoising Autoencoder')
plt.xlabel('Epochs')
plt.ylabel('Reconstruction Loss')
plt.xticks(range(10, 100, 10))
plt.legend(["Adam", "SGD"])
plt.show()
```


Training Curve for Denoising Autoencoder
Adam
SGD
Reconstruction Loss
Epochs
0
2
4
6
8
10
12
10
20
30
40
50
60
70
80
90