Isotropic Second and Fourth-Order Tensors

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1 Introduction

We seek to show that the most general form of an isotropic 2^{nd}-order tensor is:

$$T_{ij} = \alpha \delta_{ij} \tag{1}$$

and that the most general form for an isotropic 4^{th}-order tensor, T_{ijkl} is given by:

$$T_{ijkl} = \alpha \delta_{ij} \delta_{kl} + \beta \delta_{ik} \delta_{jl} + \gamma \delta_{il} \delta_{jk}, \tag{2}$$

for constants α, β, γ where δ_{ij} is the Kronecker delta function defined by:

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}. \tag{3}$$

This document follows identically to the proof provided by Hodge (1961), [1] but attempts to expand on some details. As a preliminary, consider the Euler representation of a rotation tensor which corresponds to a rotation of angle ϕ about the vector $\mathbf{r}$:

$$\mathbf{R}(\phi, \mathbf{r}) = (\mathbf{I} - \mathbf{r} \otimes \mathbf{r}) \cos \phi - (\boldsymbol{\epsilon} \mathbf{r}) \sin \phi + \mathbf{r} \otimes \mathbf{r} \tag{4}$$

where $\boldsymbol{\epsilon}$ is the alternating tensor defined by (given an orthonormal basis $\{\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k\}$):

$$\boldsymbol{\epsilon} := \epsilon_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k, \tag{5}$$

where it is also noted that ϵ_{ijk} is the Levi-Civita symbol defined by:

$$\epsilon_{ijk} := \begin{cases} +1 & \text{if } (i,j,k) \text{ is } (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{if } (i,j,k) \text{ is } (3,2,1), (1,3,2), (2,1,3) \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

Consider now a general infinitesimal rotation, $\mathbf{Q}$, that is a rotation about an arbitrary axis $\mathbf{r}$ by an infinitesimal amount $d\nu$. Using the small angle approximations, $\sin x = x$ and $\cos x = 1$, with reference to Eq. 4, we may write the infinitesimal rotation in the Euler representation as:

$$\begin{aligned} \mathbf{Q} \equiv \mathbf{R}(d\nu, \mathbf{r}) &= \mathbf{I} - \mathbf{r} \otimes \mathbf{r} - d\nu \boldsymbol{\epsilon} \mathbf{r} + \mathbf{r} \otimes \mathbf{r} && (7) \\ &= \mathbf{I} - d\nu \boldsymbol{\epsilon} \mathbf{r} && (8) \\ &= \mathbf{I} + \boldsymbol{\epsilon} d\boldsymbol{\omega} && (9) \end{aligned}$$

for $d\boldsymbol{\omega} := -d\nu \mathbf{r}$. Thus, $Q_{ij} = \delta_{ij} + \epsilon_{ijk} d\omega_k$. Additionally, we will make use of the $\epsilon - \delta$ identity (not proven here) which states:

$$\epsilon_{ijk} \epsilon_{rsk} = \delta_{ir} \delta_{js} - \delta_{is} \delta_{jr}. \tag{10}$$

2 Proof: Isotropic Second-Order Tensors

We will now prove the most general form of a 2^{nd}-order isotropic tensor. By definition, an isotropic tensor maintains the same components with respect to a basis before and after the application of *any* coordinate transformation. This means that after transforming the basis vectors of a 2^{nd}-order tensor, its components will remain unchanged. Mathematically, we have:

$$\mathbf{T} = \mathbf{R}^T\mathbf{T}\mathbf{R} \leftrightarrow T_{ij} = R_{mi}R_{nj}T_{mn}. \tag{11}$$

The same holds true using our infinitesimal rotation $\mathbf{Q}$ instead of $\mathbf{R}$:

$$\mathbf{T} = \mathbf{Q}^T\mathbf{T}\mathbf{Q} \leftrightarrow T_{ij} = Q_{mi}Q_{nj}T_{mn}. \tag{12}$$

Now, substituting Eq. 9 into Eq. 12, we find:

$$\begin{aligned}
T_{ij} &= (\delta_{mi} + \epsilon_{mik}d\omega_k)(\delta_{nj} + \epsilon_{njl}d\omega_l)T_{mn} && (13)\\
&= (\delta_{mi}\delta_{nj} + \delta_{nj}\epsilon_{mik}d\omega_k + \delta_{mi}\epsilon_{njl}d\omega_l + \epsilon_{mik}\epsilon_{njl}d\omega_k d\omega_l)\,T_{mn} && (14)\\
&= T_{ij} + \epsilon_{mik}d\omega_k T_{mj} + \epsilon_{njl}d\omega_l T_{in} && (15)\\
&= T_{ij} + \epsilon_{mik}d\omega_k T_{mj} + \epsilon_{mjk}d\omega_k T_{im} && (16)
\end{aligned}$$

where we have ignored the second-order $d\omega$ term. The dummy indices were changed in the final step. Thus,

$$(\epsilon_{mik}T_{mj} + \epsilon_{mjk}T_{im})\,d\omega_k = 0. \tag{17}$$

Since this must hold true for any arbitrary rotation, we have that:

$$\epsilon_{mik}T_{mj} + \epsilon_{mjk}T_{im} = 0. \tag{18}$$

Multiplying Eq. 18 by ϵ_{tik} and using the $\epsilon - \delta$ identity, we find:

$$\begin{aligned}
0 &= \epsilon_{tik}\epsilon_{mik}T_{mj} + \epsilon_{tik}\epsilon_{mjk}T_{im} && (19)\\
&= 2\delta_{tm}T_{mj} + (\delta_{tm}\delta_{ij} - \delta_{tj}\delta_{im})\,T_{im} && (20)\\
&= 2T_{tj} + T_{jt} - \delta_{tj}T_{ii} && (21)
\end{aligned}$$

where we have used an additional identity (not proven here) $\epsilon_{imn}\epsilon_{jmn} = 2\delta_{ij}$. Then,

$$2T_{tj} + T_{jt} = T_{ii}\delta_{tj} \tag{22}$$

where $T_{ii} = \mathrm{tr}\,\mathbf{T}$. Taking the transpose of the previous equation,

$$2T_{jt} + T_{tj} = T_{ii}\delta_{jt} = T_{ii}\delta_{tj} \tag{23}$$

by the symmetry of the identity tensor. Setting the left-hand side of Eq. 22 equal to the left-hand side of Eq. 23, we find that:

$$2T_{tj} + T_{jt} = 2T_{jt} + T_{tj} \rightarrow T_{tj} = T_{jt}. \tag{24}$$

Hence, 2^{nd}-order isotropic tensors are necessarily symmetric. Returning to Eq. 22 and substituting Eq. 24, we find that:

$$T_{ij} = \frac{1}{3}T_{mm}\delta_{ij} = \frac{1}{3}\,(T_{11} + T_{22} + T_{33})\,\delta_{ij} \tag{25}$$

Setting $i = j = 1$,

$$3T_{11} = T_{11} + T_{22} + T_{33} \tag{26}$$

$$2T_{11} = T_{22} + T_{33} \tag{27}$$

Now setting $i = j = 2$ in Eq. 25, we have:

$$3T_{22} = T_{11} + T_{22} + T_{33}, \tag{28}$$

so

$$T_{22} = \frac{T_{11} + T_{33}}{2}. \tag{29}$$

Substituting Eq. 29 into Eq. 27, we have

$$T_{11} = T_{33}. \tag{30}$$

Further, substituting Eq. 30 into Eq. 29 , we come to the conclusion that

$$T_{11} = T_{22} = T_{33} \tag{31}$$

which is obvious looking at Eq. 25. Thus, a 2^{nd}-order isotropic tensor only has one independent component which makes up the diagonal of its matrix representation, so in general a 2^{nd}-order isotropic tensor is proportional to the identity tensor. Summarizing our results:

$$\mathbf{T} = \mathbf{T}^T = \lambda\,\mathbf{I} \tag{32}$$

where λ is a scalar constant.

3 Proof: Isotropic Fourth-Order Tensors

We start by following a procedure very similar to that of the 2^{nd}-order case. A 4^{th}-order tensor is isotropic if after the change of basis the components remain the same:

$$T_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}T_{pqrs}. \tag{33}$$

Applying 9 to Eq. 33, we have:

$$\begin{aligned}
T_{ijkl} &= \left(\delta_{ip} + \epsilon_{ipw}d\omega_w\right)\left(\delta_{jq} + \epsilon_{jqx}d\omega_x\right)\left(\delta_{kr} + \epsilon_{kry}d\omega_y\right)\left(\delta_{ls} + \epsilon_{lsz}d\omega_z\right)T_{pqrs} &(34)\\
&= \left(\delta_{ip}\delta_{jq}\delta_{kr}\delta_{ls} + \delta_{jq}\delta_{kr}\delta_{ls}\epsilon_{ipw}d\omega_w + \delta_{ip}\delta_{kr}\delta_{ls}\epsilon_{jqx}d\omega_x + \delta_{ip}\delta_{jq}\delta_{ls}\epsilon_{kry}d\omega_y + \delta_{ip}\delta_{jq}\delta_{kr}\epsilon_{lsz}d\omega_z\right)T_{pqrs} + \mathcal{O}(d\omega^2) &(35)\\
&= T_{ijkl} + T_{pjkl}\epsilon_{ipw}d\omega_w + T_{iqkl}\epsilon_{jqx}d\omega_x + T_{ijrl}\epsilon_{kry}d\omega_y + T_{ijks}\epsilon_{lsz}d\omega_z. &(36)
\end{aligned}$$

Cancelling out the T_{ijkl} terms,

$$\begin{aligned}
0 &= T_{pjkl}\epsilon_{ipw}d\omega_w + T_{iqkl}\epsilon_{jqx}d\omega_x + T_{ijrl}\epsilon_{kry}d\omega_y + T_{ijks}\epsilon_{lsz}d\omega_z &(37)\\
&= \left(T_{pjkl}\epsilon_{ipw} + T_{iqkl}\epsilon_{jqw} + T_{ijrl}\epsilon_{krw} + T_{ijks}\epsilon_{lsw}\right)d\omega_w &(38)\\
&= \left(T_{pjkl}\epsilon_{ipw} + T_{ipkl}\epsilon_{jpw} + T_{ijpl}\epsilon_{kpw} + T_{ijkp}\epsilon_{lpw}\right)d\omega_w. &(39)
\end{aligned}$$

Again, since this holds for an arbitrary rotation, we must have:

$$T_{pjkl}\epsilon_{ipw} + T_{ipkl}\epsilon_{jpw} + T_{ijpl}\epsilon_{kpw} + T_{ijkp}\epsilon_{lpw} = 0. \tag{40}$$

Multiplying Eq. 40 by ϵ_{tsw}:

$$\epsilon_{tsw}\left(T_{pjkl}\epsilon_{ipw} + T_{ipkl}\epsilon_{jpw} + T_{ijpl}\epsilon_{kpw} + T_{ijkp}\epsilon_{lpw}\right) = 0. \tag{41}$$

Breaking up each term and making use of the $\epsilon - \delta$ identity (Eq. 10):

$$\begin{aligned}
\epsilon_{tsw}\epsilon_{ipw}T_{pjkl} &= \left(\delta_{ti}\delta_{sp} - \delta_{tp}\delta_{si}\right)T_{pjkl} = \delta_{ti}T_{sjkl} - \delta_{si}T_{tjkl} &(42)\\
\epsilon_{tsw}\epsilon_{jpw}T_{ipkl} &= \left(\delta_{tj}\delta_{sp} - \delta_{tp}\delta_{sj}\right)T_{ipkl} = \delta_{tj}T_{iskl} - \delta_{sj}T_{itkl} &(43)\\
\epsilon_{tsw}\epsilon_{kpw}T_{ijpl} &= \left(\delta_{tk}\delta_{sp} - \delta_{tp}\delta_{sk}\right)T_{ijpl} = \delta_{tk}T_{ijsl} - \delta_{sk}T_{ijtl} &(44)\\
\epsilon_{tsw}\epsilon_{lpw}T_{ijkp} &= \left(\delta_{tl}\delta_{sp} - \delta_{tp}\delta_{sl}\right)T_{ijkp} = \delta_{tl}T_{ijks} - \delta_{sl}T_{ijkt}. &(45)
\end{aligned}$$

Then, adding the terms together and setting equal to zero per Eq. 41:

$$\delta_{ti}T_{sjkl} - \delta_{si}T_{tjkl} + \delta_{tj}T_{iskl} - \delta_{sj}T_{itkl} + \delta_{tk}T_{ijsl} - \delta_{sk}T_{ijtl} + \delta_{tl}T_{ijks} - \delta_{sl}T_{ijkt} = 0. \tag{46}$$

Setting $s = i$ in Eq. 46, we find:

$$\begin{aligned} 0 &= \delta_{ti}T_{ijkl} - \delta_{ii}T_{tjkl} + \delta_{tj}T_{iikl} - \delta_{ij}T_{itkl} + \delta_{tk}T_{ijil} - \delta_{ik}T_{ijtl} + \delta_{tl}T_{ijki} - \delta_{il}T_{ijkt} & (47)\\ &= T_{tjkl} - 3T_{tjkl} + \delta_{tj}T_{iikl} - T_{jtkl} + \delta_{tk}T_{ijil} - T_{kjtl} + \delta_{tl}T_{ijki} - T_{ljkt}, & (48)\end{aligned}$$

which results in the equation (re-indexing first $i \leftrightarrow m$, then $t \leftrightarrow i$):

$$2T_{ijkl} + T_{jikl} + T_{kjil} + T_{ljki} = \delta_{ij}T_{mmkl} + \delta_{ik}T_{mjml} + \delta_{il}T_{mjkm}. \tag{49}$$

At this point, it is important to note that terms with a contracted index like T_{mmkl} are 2^{nd}-order isotropic tensors owing to the fact that we are assuming T_{ijkl} is isotropic. Using our result for 2^{nd}-order isotropic tensors, these terms are then proportional to the identity $T_{mmkl} = \lambda\delta_{kl}$, $T_{mjml} = \mu\delta_{jl}$, etc. For further detail of this assertion, see the appendix. Then Eq. 49 simplifies to:

$$2T_{ijkl} + T_{jikl} + T_{kjil} + T_{ljki} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \tag{50}$$

Repeating the same procedure but setting $s = j$, $s = k$, and $s = l$ in Eq. 46, we obtain the following three equations, respectively:

$$\begin{aligned} 2T_{ijkl} + T_{jikl} + T_{ikjl} + T_{ilkj} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} & (51)\\ 2T_{ijkl} + T_{kjil} + T_{ikjl} + T_{ijlk} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} & (52)\\ 2T_{ijkl} + T_{ljki} + T_{ilkj} + T_{ijlk} &= \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. & (53)\end{aligned}$$

Summarizing the four equations obtained by setting $s = i$, $s = j$, $s = k$, and $s = l$ in Eq. 46:

$$\begin{aligned} \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} &= 2T_{ijkl} + T_{jikl} + T_{kjil} + T_{ljki} & (54)\\ &= 2T_{ijkl} + T_{jikl} + T_{ikjl} + T_{ilkj} & (55)\\ &= 2T_{ijkl} + T_{kjil} + T_{ikjl} + T_{ijlk} & (56)\\ &= 2T_{ijkl} + T_{ljki} + T_{ilkj} + T_{ijlk}. & (57)\end{aligned}$$

Next, adding together Eq. 54 and Eq. 55 and subtracting from this the sum of Eq. 56 and Eq. 57, we find the relation $T_{jikl} = T_{ijlk}$. Repeating this for the other two possible pairs we get the additional relations $T_{kjil} = T_{ilkj}$ and $T_{ikjl} = T_{ljki}$. Substituting these three relations into Eqs. 54, 55, 56, and 57, we find that they are all identical and we are left with one equation:

$$2T_{ijkl} + T_{ijlk} + T_{ilkj} + T_{ikjl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \tag{58}$$

Through a cyclic permutation of the indices j, k, l and leaving i the same, two other forms of Eq. 58 read:

$$\begin{aligned} 2T_{iljk} + T_{ilkj} + T_{ikjl} + T_{ijlk} &= \lambda\delta_{il}\delta_{jk} + \mu\delta_{ij}\delta_{lk} + \nu\delta_{ik}\delta_{lj}, & (59)\\ 2T_{iklj} + T_{ikjl} + T_{ijlk} + T_{ilkj} &= \lambda\delta_{ik}\delta_{lj} + \mu\delta_{il}\delta_{kj} + \nu\delta_{ij}\delta_{kl}. & (60)\end{aligned}$$

On their own, Eqs. 58, 59, and 60 are identical, but by summing them and exploiting the properties of the Kronecker delta function allows us to proceed to our desired result. Adding these three equations gives:

$$3(T_{ijlk} + T_{ilkj} + T_{ikjl}) + 2\left(T_{ijkl} + T_{iljk} + T_{iklj}\right) = \lambda\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{lj}\right) + \mu\left(\delta_{ik}\delta_{jl} + \delta_{ij}\delta_{lk} + \delta_{il}\delta_{kj}\right) + \nu\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{lj} + \delta_{ij}\delta_{kl}\right) \tag{61}$$

Again, invoking the symmetry of the Kronecker delta function, ie. $\delta_{ij} = \delta_{ji}$, Eq. 61 reduces to:

$$3\left(T_{ijlk} + T_{ilkj} + T_{ikjl}\right) + 2\left(T_{ijkl} + T_{iljk} + T_{iklj}\right) = (\lambda + \mu + \nu)\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right). \tag{62}$$

Now, consider interchanging the indices $k \leftrightarrow l$ in Eq. 62 to get:

$$3\left(T_{ijkl} + T_{iklj} + T_{iljk}\right) + 2\left(T_{ijlk} + T_{ikjl} + T_{ilkj}\right) = (\lambda + \mu + \nu)\left(\delta_{ij}\delta_{lk} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{63}$$

Again, through symmetry of the $\delta_{kl} = \delta_{lk}$ term, we may set Eq. 62 equal to Eq. 63 to find:

$$T_{ijkl} + T_{iklj} + T_{iljk} = T_{ijlk} + T_{ikjl} + T_{ilkj}. \tag{64}$$

Then by substituting Eq. 64 into Eq. 62 we have:

$$T_{ijkl} + T_{iklj} + T_{iljk} = T_{ijlk} + T_{ikjl} + T_{ilkj} = \frac{1}{5}\left(\lambda + \mu + \nu\right)\left(\delta_{ij}\delta_{lk} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{65}$$

Finally, substituting Eq. 65 into Eq. 58, we obtain:

$$2T_{ijkl} = \tfrac{4}{5}\lambda\delta_{ij}\delta_{lk} - \tfrac{1}{5}\lambda\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) + \tfrac{4}{5}\mu\delta_{ik}\delta_{jl} - \tfrac{1}{5}\mu\left(\delta_{ij}\delta_{lk} + \delta_{il}\delta_{jk}\right) + \tfrac{4}{5}\nu\delta_{il}\delta_{jk} - \tfrac{1}{5}\nu\left(\delta_{ij}\delta_{lk} + \delta_{ik}\delta_{jl}\right), \tag{66}$$

and by defining:

$$\alpha \coloneqq \frac{4\lambda - \mu - \nu}{10} \tag{67}$$
$$\beta \coloneqq \frac{4\mu - \lambda - \nu}{10} \tag{68}$$
$$\gamma \coloneqq \frac{4\nu - \lambda - \mu}{10} \tag{69}$$

we arrive at our desired result, that is, the general form of a fourth-order isotropic tensor is:

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}. \tag{70}$$

4 Corollary: Isotropic Fourth-Order Tensors with Minor Symmetry

Many times, such as in elasticity, we may end up with a minor symmetry so that $T_{ijkl} = T_{ijlk}$ or $T_{ijkl} = T_{jikl}$. Let's see what effect this has on the general form of an isotropic fourth-order tensor. Consider the first case $T_{ijkl} = T_{ijlk}$, then by exchanging indices $k \leftrightarrow l$ in Eq. 70 we have:

$$\alpha\delta_{ij}\delta_{kl} + \beta\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk} = \alpha\delta_{ij}\delta_{lk} + \beta\delta_{il}\delta_{jk} + \gamma\delta_{ik}\delta_{jl}. \tag{71}$$

This provides:

$$\delta_{ik}\delta_{jl} = \delta_{il}\delta_{jk}. \tag{72}$$

Then, substituting this back into Eq. 70 and defining $\eta \coloneqq \beta + \gamma$, we have:

$$T_{ijkl} = \alpha\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl}. \tag{73}$$

That is to say, in linear elasticity, for example, a material that is isotropic may be represented by just two material parameters, here denoted as α and η, known as the Lamé moduli $\mu \coloneqq \frac{1}{2}\alpha$ and $\lambda \coloneqq \eta$.

5 Appendix: Isotropic Fourth-Order Tensor with Repeated Index Is an Isotropic Second-Order Tensor

In the proof of the fourth-order isotropic tensor, we used the fact that tensors with a repeated index like T_{mmkl} are proportional to the identity tensor, ie. $T_{mmkl} = \lambda\delta_{kl}$. Here we expand on this fact. Consider an isotropic fourth-order tensor with a repeated index T_{mmkl}. Then, using the definition of isotropy Eq. 33, we have:

$$T_{mmkl} = Q_{mp}Q_{mq}Q_{kr}Q_{ls}T_{pqrs}. \tag{74}$$

The defining property of an orthogonal matrix $\mathbf{Q}$ is that $\mathbf{Q}^T\mathbf{Q} = \mathbf{Q}\mathbf{Q}^T = \mathbf{I}$, thus, in indicial notation:

$$Q_{mp}Q_{mq} = \delta_{pq}. \tag{75}$$

Using Eq. 75 in Eq. 76, we have:

$$T_{mmkl} = Q_{kr}Q_{ls}T_{qqrs}. \tag{76}$$

We continue by using the same trick of substituting our infinitesimal rotation $Q_{ij} = \delta_{ij} + \epsilon_{ijk}d\omega_k$:

$$T_{mmkl} = (\delta_{kr} + \epsilon_{kri}d\omega_i)(\delta_{ls} + \epsilon_{lsj}d\omega_j)\,T_{qqrs}. \tag{77}$$

Again, ignoring higher order terms of $d\boldsymbol{\omega}$ and re-indexing some terms, we have:

$$T_{mmkl} = T_{mmkl} + (\delta_{kr}\epsilon_{lsj}\omega_j + \delta_{ls}\epsilon_{krj}\omega_j)\,T_{qqrs}. \tag{78}$$

Then, we have

$$(\epsilon_{lsj}T_{qqks} + \epsilon_{ksj}T_{qqsl})\,\omega_j = 0 \tag{79}$$

which holds for all ω_j; thus,

$$\epsilon_{lsj}T_{qqks} + \epsilon_{ksj}T_{qqsl} = 0. \tag{80}$$

Multiplying the previous expression by ϵ_{tjl} and applying the ϵ-δ identity:

$$\begin{aligned} 0 &= 2\delta_{st}T_{qqks} + \delta_{kl}\delta_{st}T_{qqsl} - \delta_{kt}\delta_{sl}T_{qqsl} && (81)\\ &= 2T_{qqkt} + T_{qqtk} - \delta_{kt}T_{qqss}. && (82) \end{aligned}$$

At this point, with a little re-indexing, we have:

$$2T_{mmkl} + T_{mmlk} = T_{mmnn}\delta_{kl}. \tag{83}$$

Exchanging the k and l indices and using the symmetry of the identity tensor, $\delta_{kl} = \delta_{lk}$, we get the equation:

$$2T_{mmlk} + T_{mmkl} = T_{mmnn}\delta_{kl}, \tag{84}$$

and setting Eq. 83 equal to Eq. 84 we find:

$$T_{mmkl} = T_{mmlk}. \tag{85}$$

These tensors are symmetric in their non-repeated indices. Using Eq. 85 in Eq. 83, we get:

$$3T_{mmkl} = T_{mmnn}\delta_{kl}. \tag{86}$$

Now, let $k = l = 1$:

$$3T_{mm11} = T_{mm11} + T_{mm22} + T_{mm33}. \tag{87}$$

Next, letting $k = l = 2$,

$$3T_{mm22} = T_{mm11} + T_{mm22} + T_{mm33}. \tag{88}$$

it is clear that $T_{mm11} = T_{mm22} = T_{mm33} = \lambda$ where the constant λ depends on the location of the mm components of T. That is to say, if the mm components fall in a different location ie. T_{mjml}, then we could follow the same steps but would arrive at a different constant, say μ. In summary:

$$\begin{aligned} T_{mmkl} &= \lambda\delta_{kl} && (89)\\ T_{mjml} &= \mu\delta_{jl} && (90)\\ T_{mjkm} &= \nu\delta_{jk} && (91) \end{aligned}$$

References

[1] Philip G. Hodge Jr. “On Isotropic Cartesian Tensors”. In: *The American Mathematical Monthly* 68.8 (1961), pp. 793–795. DOI: `https://doi.org/10.2307/2311997`.