Topology of the O(3) non-linear sigma model under the gradient flow

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Abstract

The O(3) non-linear sigma model (NLSM) is a prototypical field theory for QCD and ferromagnetism, featuring topological qualities. **Though the topological susceptibility χ_t should vanish in physical theories, lattice simulations of the NLSM find that χ_t diverges in the continuum limit. [1, 2]** We study the effect of the gradient flow on this quantity using a Markov chain Monte Carlo method, finding that a logarithmic divergence persists. This result supports a previous study and indicates that either the definition of topological charge is problematic or the NLSM has no well-defined continuum limit. We also introduce a θ-term and analyze the topological charge as a function of θ under the gradient flow.

Non-Linear σ Model

We specifically study the O(3) non-linear sigma model (NLSM) in 1+1 dimensions, defined by the Euclidean action

$$S_E = \frac{\beta}{2}\int d^2x\,\left[(\partial_t\vec{e})^2 + (\partial_x\vec{e})^2\right]$$

- $\vec{e}$ is 3-component vector constrained by $|\vec{e}| = 1$.
- β is the inverse coupling constant.

Topology of 1+1 O(3) NLSM

1. Since the Lagrangian must disappear as $x \to \infty$, the field $\vec{e}$ becomes uniform.
2. Therefore, we can envision 1+1 spacetime as a sphere.
3. The NLSM field $\vec{e}$ becomes mapping between two Riemann spheres ($S^2 \to S^2$).
4. Every configuration has an associated topological charge $Q \in \mathbb{Z}$ (see Fig. 1).

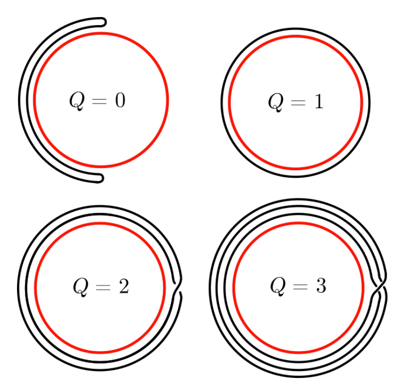
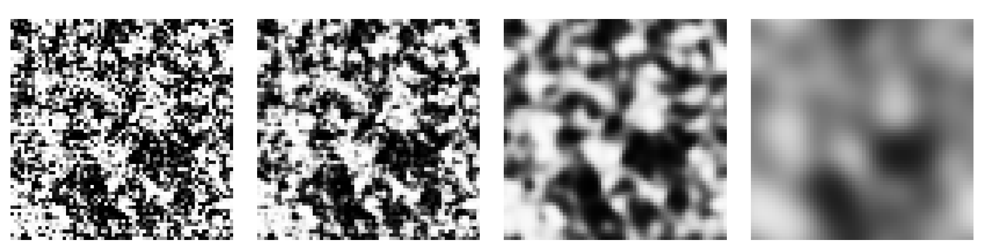


Figure 1: Visualization of homotopy group of $S^1 \to S^1$

The Gradient Flow

- Successful in removing χ_t divergence in QCD by reducing ultraviolet divergences.
- Introduces a new dimension, τ, called the "flow time," which pushes fields towards action minima



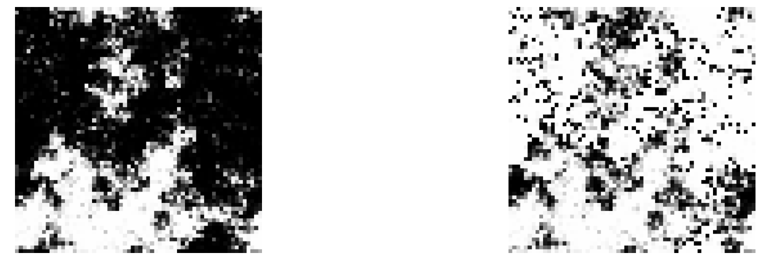
(a) $\tau = 0$ (b) $\tau = 0.001$ (c) $\tau = 0.01$ (d) $\tau = 0.1$

Figure 2: Visualization of gradient flow in ϕ^4 model

Computational Method

We use a Markov Chain Monte Carlo method to identify field configurations at the action minimum, produced using:

- Discretized spacetime (lattice regulation)
- Metropolis algorithm on each site, forming a "sweep" of the lattice.
- Wolff cluster algorithm every five Metropolis sweeps



(a) before cluster flip (b) after cluster flip

Figure 3: Visualization of Wolff cluster algorithm in ϕ^4 model

- Thermalize with 1,000 sweeps
- Sample every 50 sweeps to reduce autocorrelation

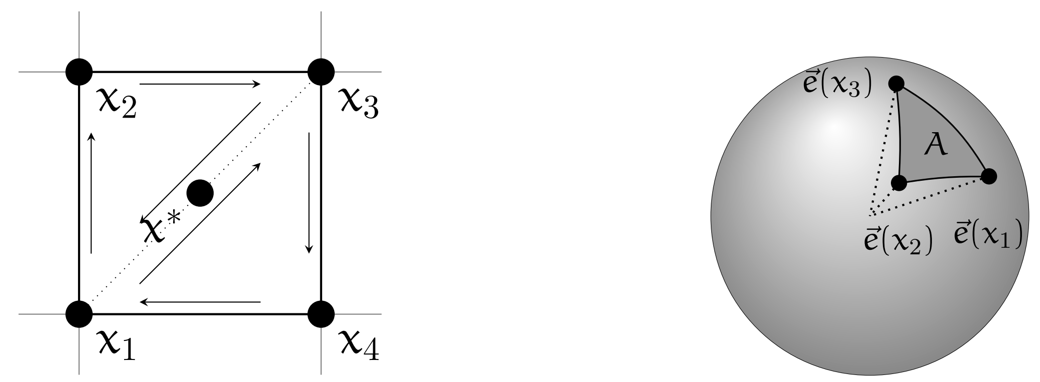
The ensemble has a topological susceptibility $\chi_t \equiv \left(\langle Q^2\rangle - \langle Q\rangle^2\right)/L^2$ which should disappear in the continuum limit.

Topological Charge on the Lattice

Following [2], we define topological charge density $q(x^*)$ for each plaquette x^* such that

$$Q = \sum_{x^*} q(x^*)$$

$$q(x^*) = \frac{1}{4\pi}\left[A\left(\vec{e}(x_1), \vec{e}(x_2), \vec{e}(x_3)\right) + A\left(\vec{e}(x_1), \vec{e}(x_3), \vec{e}(x_4)\right)\right].$$



(a) a plaquette broken up into two triangles (b) signed area of triangle in target space

Non-trivial theory

$$S[\vec{e}] \to S[\vec{e}] - i\theta Q[\vec{e}].$$

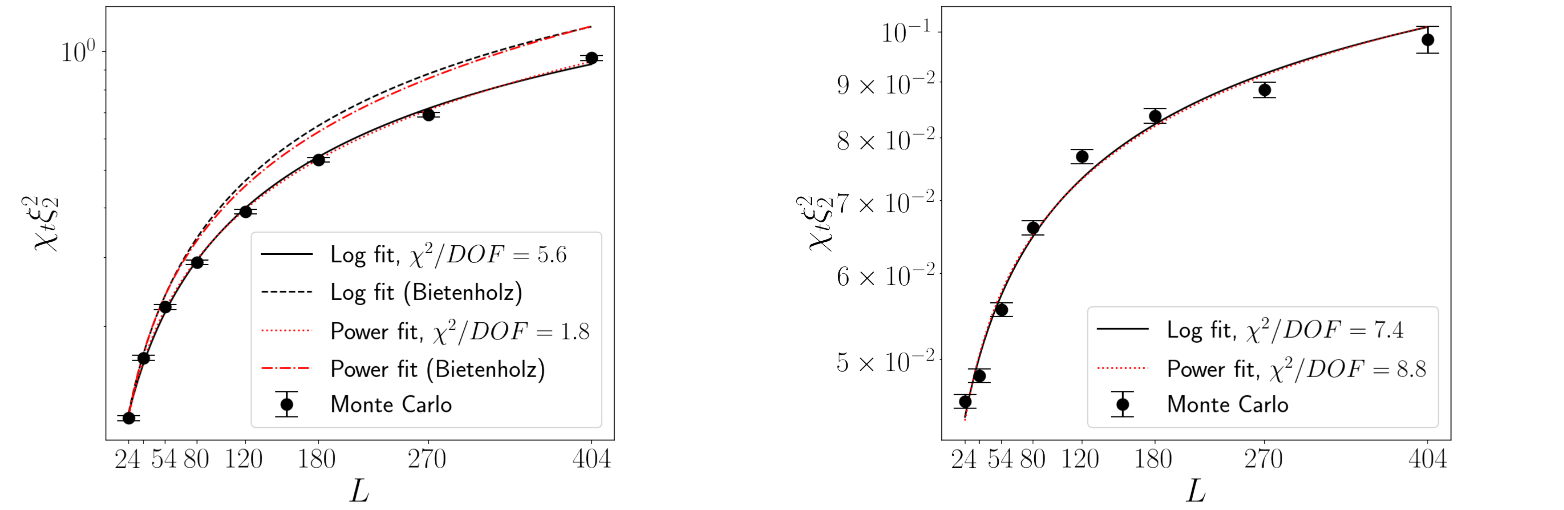
A nonzero θ implies nonzero $\langle Q\rangle$. Furthermore,

$$\chi_t \propto \left.\frac{d\,\mathrm{Im}\langle Q\rangle}{d\theta}\right|_{\theta=0}$$

[1] W. Bietenholz et al., Phys. Rev. D **98**, 114501 (2018).

[2] B. Berg and M. Lüscher, Nuclear Physics B **190**, 412 (1981).

χ_t Divergence



(a) $\tau = 0$ (b) $\tau = 5t_0$

Figure 5: $\chi_t\xi_2^2$ as a function of L. ξ_2 is the second moment of the correlation function and t_0 is a scale-independent unit of flow time. We fit the data with both a logarithmic and power fit. Simulation run with 10,000 measurements every 50 sweeps, 1,000 sweep thermalization. In the $\tau = 0$ case, we have compared our result with the curve fits found in [1].

Effect of θ-term on ⟨Q⟩

Using the path integral formulation, we show that

$$\langle Q\rangle_\theta = \int \mathcal{D}\vec{e}\; Q[\vec{e}]\, e^{-S[\vec{e}]+i\theta Q[\vec{e}]}$$
$$= \int \mathcal{D}\vec{e}\; \left(Q[\vec{e}]\, e^{i\theta Q[\vec{e}]}\right) e^{-S[\vec{e}]}$$
$$= \langle Q e^{i\theta Q}\rangle_{\theta=0}$$

Therefore, the trivial Monte Carlo method can calculate topological observables, shown in Fig. 6. The increasing slope at $\theta = 0$ as $L \to \infty$ indicates a nonzero suscpetibility in the continuum limit.

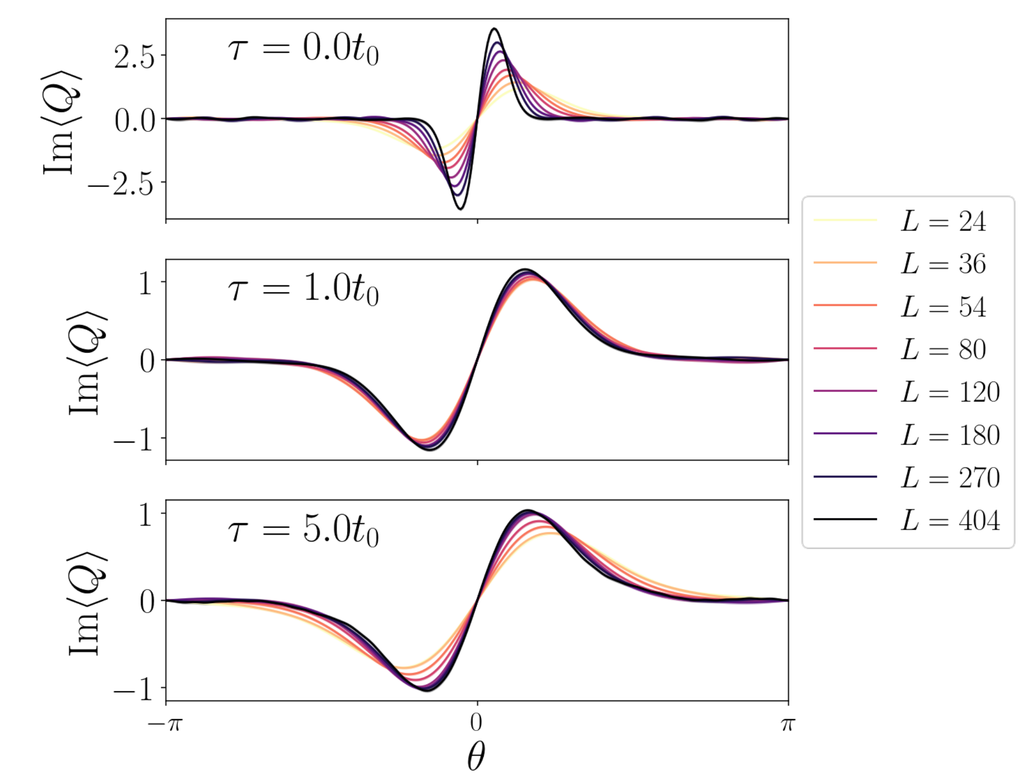


Figure 6: Nontrivial Im⟨Q⟩ as a function of θ. Simulation run with 10,000 measurements, every 50 sweeps, 1,000 sweep thermalization. Note the different scaling of the y-axis.

Conclusion

Berg & Lüscher [2] give three possible sources of divergence:

1. ultraviolet modes
2. the definition of Q on the lattice is problematic
3. the NLSM has no well-defined continuum limit

Our result undermines the first hypothesis and therefore supports the others.

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