PROCEEDINGS OF SCIENCE

Topology of the $O(3)$ non-linear sigma model under the gradient flow

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The $O(3)$ non-linear sigma model (NLSM) is a prototypical field theory for QCD and ferromagnetism, featuring topological qualities. Though the topological susceptibility χ_t should vanish in physical theories, lattice simulations of the NLSM find that χ_t diverges in the continuum limit. We study the effect of the gradient flow on this quantity using a Markov Chain Monte Carlo method, finding that a logarithmic divergence persists. This result supports a previous study and indicates that either the definition of topological charge is problematic or the NLSM has no well-defined continuum limit. We also introduce a θ-term and analyze the topological charge as a function of θ under the gradient flow.

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1. Introduction

Spin models provide a framework for understanding the physics of strongly-coupled systems, from solid state and condensed matter systems to nuclear and particle physics. The non-linear sigma model (NLSM), in particular, has provided a rich arena in which to study nonperturbative effects. In solid-state systems, this model describes Heisenberg ferromagnets [1] and in nuclear physics, it acts as a prototype for quantum chromodynamics (QCD), the gauge theory of the strong nuclear force. In general, the NLSM shares key features with non-Abelian gauge theories such as QCD, including a mass gap and asymptotic freedom [2], and has proved a useful model for exploring the effect of these properties in a simpler system.

We consider the $O(3)$ NLSM in 1+1 dimensions (one dimension of space, one dimension of time). This theory exhibits topological properties, such as *instantons*, or classical field solutions at local minima of the action in Euclidean space. These topologically protected solutions cannot evolve into the vacuum state via local fluctuations. This property has become critically important to quantum field theories in cosmology and high energy physics [3]. CJM: Can we provide a more explicit example(s) here? Statement is very vague at the moment. Additionally, topological stability may become a key tool for fault-tolerant quantum computers [4]. In these devices, topology protects the delicate quantum states necessary for information processing.

2. The Non-Linear Sigma Model

We study the $O(3)$ NLSM in two dimensions CJM: [In Euclidean space it does not really make sense to distinguish time and space, unless we start in Minkowski spacetime and then Wick rotate.], defined by the Euclidean action

$$S_E = \frac{\beta}{2} \int d^2x \left[(\partial_t \vec{e}\,)^2 + (\partial_x \vec{e}\,)^2 \right],$$

where $\vec{e}$ is 3-component real vector constrained by $|\vec{e}\,| = 1$ and β is the inverse coupling constant.

Following [5], we define the topological charge density, $q(x^*)$, for each plaquette x^* such that the total topological charge is

$$Q = \sum_{x^*} q(x^*), \tag{1}$$

where

$$q(x^*) = \frac{1}{4\pi}\left[A\Big(\vec{e}(x_1), \vec{e}(x_2), \vec{e}(x_3)\Big) + A\Big(\vec{e}(x_1), \vec{e}(x_3), \vec{e}(x_4)\Big)\right]. \tag{2}$$

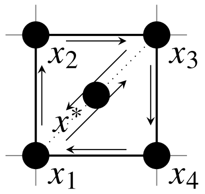
Here A is the signed area of the triangle in target space, which we represent in the left-hand figure of Fig. 1.

CJM: Need text here to explain the next equations

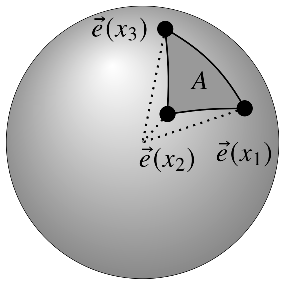
$$S[\vec{e}\,] \to S[\vec{e}\,] - i\theta Q[\vec{e}\,].$$

A nonzero θ implies nonzero $\langle Q \rangle$. Furthermore,

$$\chi_t \propto \left. \frac{d\,\mathrm{Im}\langle Q \rangle}{d\theta} \right|_{\theta=0} \tag{3}$$



(a) Visualization of a plaquette x^*. The dotted line separates the plaquette into two signed areas, used to define the topological charge density $q(x^*)$. Arrows represent the order of signed area.



(b) The signed area of a triangle in target space.

Figure 1

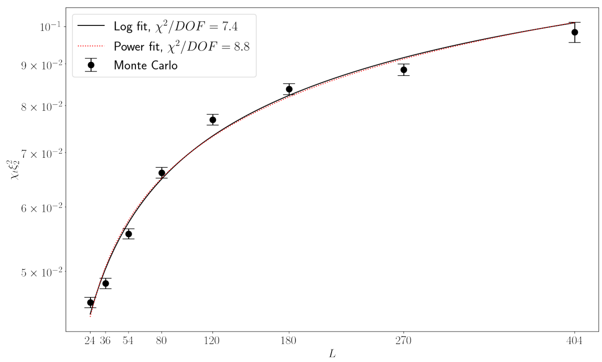
3. Computational Methods

CJM: Don't need to mention phi-4 model, or python, except to note that the phi-4 model served as a check of your code. We implement a numerical Monte Carlo method to simulate the lattice in two dimensions and verify our program with the well-studied ϕ^4 scalar field theory. We then generalize our model to a vector field to simulate the NLSM and implement a numerical solution to the gradient flow. We use the data produced by this program to study the topological charge and susceptibility under the gradient flow.

We first implement the ϕ^4 model in Python. Afterwards, we transition to the NLSM, using C++ for increased efficiency . To compile the C++ simulation, we use the `gcc` compiler with the highest level of optimization.

4. Results

(a) $\tau = 0$



(b) $\tau = 5t_0$

Figure 2: $\chi_t \xi_2^2$ as a function of L. ξ_2 is the second moment of the correlation function and t_0 is a scale-independent unit of flow time. We fit the data with both a logarithmic and power fit. Simulation run with 10,000 measurements every 50 sweeps, 1,000 sweep thermalization. In the $\tau = 0$ case, we have compared our result with the curve fits found in [6].

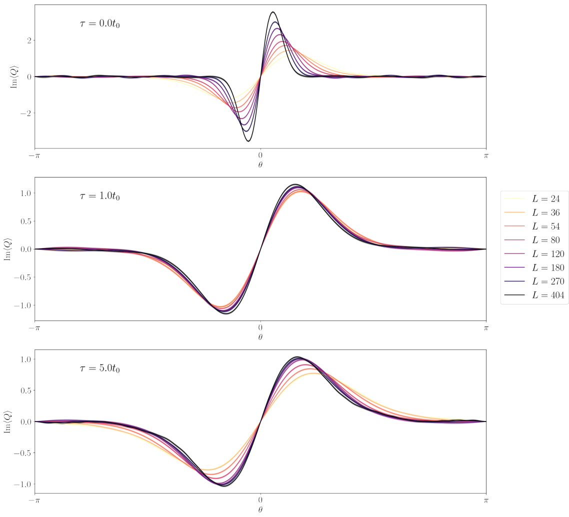


Figure 3: Nontrivial $\mathrm{Im}\langle Q\rangle$ as a function of θ. Simulation run with 10,000 measurements, every 50 sweeps, 1,000 sweep thermalization. Note the different scaling of the y-axis.

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