Topology of the $O(3)$ non-linear sigma model under the gradient flow

Stuart Thomas

Christopher Monahan

May 3, 2021

Quantum Field Theory

$$\langle \hat{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \, \hat{O}[\phi] \, e^{iS[\phi]} \qquad ()$$

ϕ^4 model

$$S_E[\phi] = \int d^2 x_E \left[\frac{1}{2} \left(\partial_t \phi \right)^2 + \frac{1}{2} \left(\partial_x \phi \right)^2 + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right]. \quad ()$$

Non-Linear σ Model

$$S_E = \frac{\beta}{2} \int d^2x \; \left[(\partial_t \vec{e}\,)^2 + (\partial_x \vec{e}\,)^2 \right] \qquad ()$$

- a nonperturbative model in Quantum Field Theory which provides a good toy model for Quantum Chromodynamics
- characterized by the action

  $$S = \frac{1}{2g} \int d^2x \, \partial_\mu \vec{\phi}(x) \cdot \partial^\mu \vec{\phi}(x)$$

  where the $\vec{\phi}(x)$ is a 3-component vector with norm 1 and g is a coupling constant.
- Merits
  - mass gap
  - asymptotic freedom
  - $O(2)$ renormalizability
  - in condensed matter, models Heisenberg ferromagnets

Monte Carlo and the Markov Chain

Topological Charge

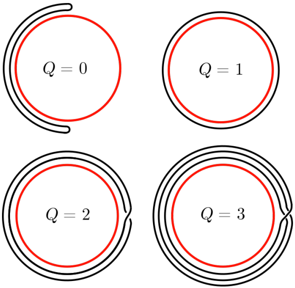


Figure: Homotopy group

Topological Action

$$S[\vec{e}] \to S[\vec{e}] - i\theta Q[\vec{e}]. \qquad ()$$

Topology Visualization

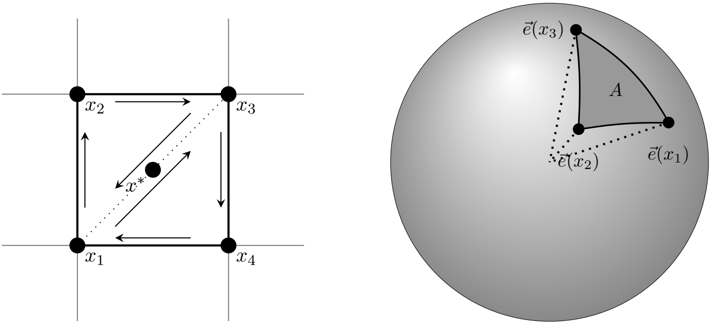
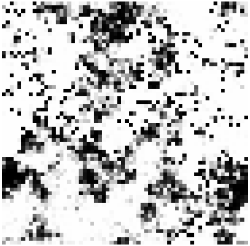


Figure: some plots

Wolff Cluster Algorithm

(a) before cluster flip



(b) after cluster flip

Figure: An example of the Wolff cluster algorithm in the ϕ^4 model. White represents positive values of ϕ while black represents negative. $\lambda = 0.5$, $m_0^2 = -0.9$.

Lattice Quantum Chromodynamics

- Method to solve non-perturbative theories
- Discretizes spacetime to numerically calculate fields.
- Uses path integral formulation of Quantum Field Theory to solve for correlation functions.

Problem

As lattice spacing approaches zero, some operators do not remain finite.

The Gradient Flow

- A form of smearing
- Introduces a new dimension, τ, called the "flow time," through which the field evolves as follows:

$$\frac{\partial \rho(\tau, x)}{\partial \tau} = \partial^2 \rho(\tau, x) \qquad ()$$

  with the boundary condition $\rho(0, x) = \phi(x)$.
- Suppresses infinities while keeping observables constant.

Goals

- Computationally implement the simpler ϕ^4 theory to verify code.
- Transition to $O(3)$ nonlinear σ model.
- Calculate twist-2 operators and vacuum expectation value with gradient flow applied.
- Other future paths:
  - Consider topological charge
  - Explore Wilson coefficients

Broad Motivation

To understand Deep Inelastic Scattering (DIS)

- Twist-2 operators $\rightarrow$ Mellin moments $\rightarrow$ Parton Distribution Functions $\rightarrow$ DIS Cross Sections

Computational Methods

- Use a Markov Chain Monte Carlo method to simulate lattice in 2D and 3D
  - Metropolis algorithm to generate random changes
  - Wolff cluster algorithm to remove metastable states
  - Implement checkerboard algorithm to parallelize Metropolis algorithm
- **By sampling the lattice at set intervals, we can calculate statical estimates of arbitrary operators.**

Markov Chain Monte Carlo

1. Start with random lattice (known as “hot start”)
2. Repeatedly transition from old configuration μ to new configuration ν with probability $P(\mu \to \nu)$.
3. After a thermalization period, record observables at certain intervals.
4. Take the statistical mean

Metropolis Algorithm

For each site, propose a new value and accept change with probability

$$P(\mu \to \nu) = \begin{cases} e^{-(S_\nu - S_\mu)} & S_\nu > S_\mu \\ 1 & \mathrm{S}_\nu \leq \mathrm{S}_\mu \end{cases}$$

A **sweep** consists of perfoming this on every site.

Wolff Cluster Algorithm

Problem: Markov Chain gets “stuck” in metastable states, i.e. local minima of the action.
Solution: Identify cluster of similar-values points and flip the sign.

1. Recursively grow a cluster with the probability of adding a new site
$$P_{add} = 1 - e^{-2\phi_a \cdot \phi_b}.$$
2. Flip sign of sites

ϕ^4 model

- Given by the Euclidean action

$$S_E = \int d^D x \left[\partial^\mu \phi \partial_\mu \phi + \frac{1}{2} m_0^2 \phi^2 + \frac{\lambda}{4} \phi^4 \right] \qquad ()$$

- Simplest interacting field
- In 1+1 dimensions, exhibits spontaneous symmetry breaking at critical m_0^2.

We measure…

- Magnetization: $\langle |\bar{\phi}| \rangle$
- Susceptibility: $\chi = \langle \bar{\phi}^2 \rangle - \langle \bar{\phi} \rangle^2$
- Binder cumulant: $U = 1 - \frac{\langle \bar{\phi}^4 \rangle}{3 \langle \bar{\phi}^2 \rangle^2}$

Computational Implementation

1. Begin with simple Python ϕ^4 model with Cython for efficiency
2. Transition to C++ code, generalized to nonlinear σ model
3. Compare results of Python and C++ code.

Parallelization

Message Passing Interface: shared RAM between processes.

Spontaneous Symmetry Breaking in ϕ^4 model

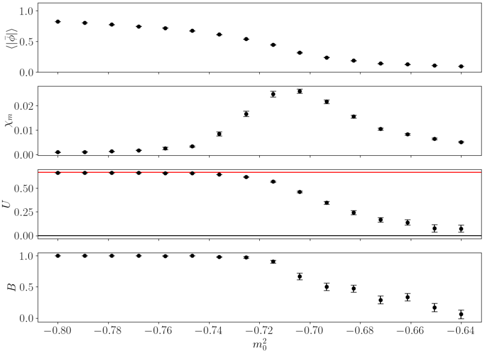
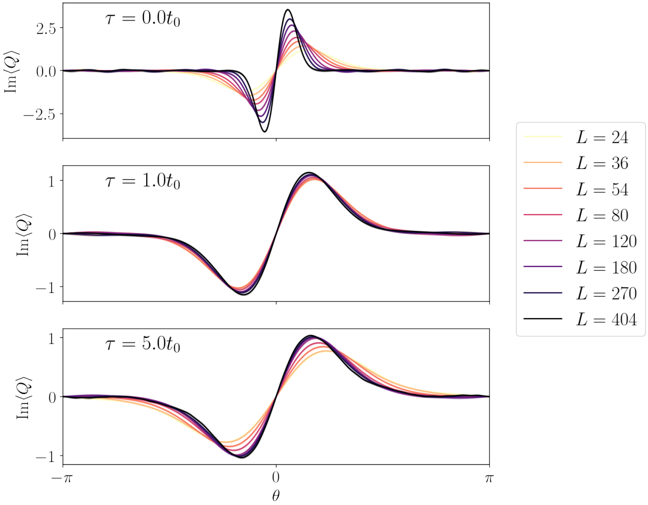


Figure: Average $\bar{\phi}$, Binder cumulant and susceptibility for various m_0^2 values. $\lambda = 0.5$, $L = 64$, 1000 measurements, 1000 sweep thermalization with measurements taken every 100 sweeps.

Effect of θ-term on $\langle Q \rangle$



Future Work

- Fully implement nonlinear σ model.
- Calculate twist-2 operators and vacuum expectation value
- Expand to 3 dimensions
- Profile and optimize C++ code
- Mathematical analysis of results