Discrete Mathematics: Lecture 3

tautological implication, predicate, individual, function of individuals, universal quantifier, existential quantifier

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Go over: truth table and prove

$P \to Q \ \equiv \neg P \vee Q$

P	Q	$\neg P$	$P \to Q$	$\neg P \vee Q$
T	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**T**
F	**F**	**T**	**T**	**T**

Rule of Replacement: Replacing a sub-formula in a formula F with a logically equivalent sub-formula gives a formula logically equivalent to the formula F

Example: prove $P \to Q \equiv \neg Q \to \neg P$

$$P \to Q \equiv \neg P \vee Q \equiv Q \vee \neg P \equiv \neg(\neg Q) \vee \neg P \equiv \neg Q \to P \tag{1}$$

Tautological Implications

Definition: Let $\mathbf{A}$ and $\mathbf{B}$ be WFFs in propositional variables $p_1, \dots, p_n$.

- **A tautologically implies B** if every truth assignment that causes $\mathbf{A}$ to be true causes $\mathbf{B}$ to be true.
- Notation: $\mathbf{A} \Rightarrow \mathbf{B}$, called a **tautological implication**
- $\mathbf{A}^{-1}(\mathbf{T}) \subseteq \mathbf{B}^{-1}(\mathbf{T})$
- $\mathbf{A}^{-1}(\mathbf{F}) \supseteq \mathbf{B}^{-1}(\mathbf{F})$

Theroem: $\mathbf{A} \Rightarrow \mathbf{B}$ iff $\mathbf{A} \rightarrow \mathbf{B}$ is a tautology.

- $\mathbf{A} \Rightarrow \mathbf{B}$ iff $\mathbf{A}^{-1}(\mathbf{T}) \subseteq \mathbf{B}^{-1}(\mathbf{T})$ iff $\mathbf{A} \rightarrow \mathbf{B}$ is a tautology

Theroem: $\mathbf{A} \Rightarrow \mathbf{B}$ iff $A \wedge \neg \mathbf{B}$ is a contradiction.

- $\mathbf{A} \rightarrow \mathbf{B} \equiv \neg \mathbf{A} \vee \mathbf{B} \equiv \neg(\mathbf{A} \wedge \neg \mathbf{B})$

Proving $\mathbf{A} \Rightarrow \mathbf{B}$:

1. $\mathbf{A}^{-1}(\mathbf{T}) \subseteq \mathbf{B}^{-1}(\mathbf{T})$;
2. $\mathbf{A} \rightarrow \mathbf{B}$ is a tautology;
3. $\mathbf{A} \wedge \neg \mathbf{B}$ is a contradiction

Proving $\mathbf{A} \Rightarrow \mathbf{B}$

Example: show the tautological implication $p \wedge (p \to q) \Rightarrow q$

Proof: Let $\mathbf{A} \equiv p \wedge (p \to q)$; $\mathbf{B} \equiv q$. Need to show that $\mathbf{A} \Rightarrow \mathbf{B}$

1. To show that $\mathbf{A}^{-1}(\mathbf{T}) \subseteq \mathbf{B}^{-1}(\mathbf{T})$:
   $A^{-1}(\mathbf{T}) = \{(\mathbf{T},\mathbf{T})\}; B^{-1}(\mathbf{T}) = \{(\mathbf{T},\mathbf{T}),(\mathbf{F},\mathbf{T})\}: A^{-1}(\mathbf{T}) \subseteq B^{-1}(\mathbf{T})$.

p	q	$p \to q$	A	B
T	**T**	**T**	<u>**T**</u>	<u>**T**</u>
T	**F**	**F**	**F**	**F**
F	**T**	**T**	**F**	**T**
F	**F**	**T**	**F**	**F**

2. To show that $\mathbf{A} \to \mathbf{B}$ is tautology
3. To show that $\mathbf{A} \wedge \neg\mathbf{B}$ is contradiction

Tautological Implications

- ✓ $(P) \wedge (Q) \Rightarrow P \wedge Q$
- $P \wedge Q \Rightarrow P$
- $P \Rightarrow P \vee Q$
- $\neg P \Rightarrow P \to Q$
  - $\neg(P \to Q) \Rightarrow P$
- $Q \Rightarrow P \to Q$
  - $\neg(P \to Q) \Rightarrow \neg Q$
- $\neg P \wedge (P \vee Q) \Rightarrow Q$
- ✓ $P \wedge (P \to Q) \Rightarrow Q$
- $\neg Q \wedge (P \to Q) \Rightarrow \neg P$

- ✓ $(P \to Q) \wedge (Q \to R) \Rightarrow (P \to R)$
- $(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow (P \leftrightarrow R)$
- $(P \vee Q) \wedge (\neg P \vee R) \Rightarrow Q \vee R$
- $(Q \to R) \Rightarrow ((P \vee Q) \to (P \vee R))$
- $(Q \to R) \Rightarrow ((P \to Q) \to (P \to R))$
- $(P \to R) \wedge (Q \to R) \wedge (P \vee Q) \Rightarrow R$
- $(P \to R) \wedge (Q \to S) \wedge (P \vee Q) \Rightarrow R \vee S$
- $(P \to R) \wedge (Q \to S) \wedge (\neg R \vee \neg S) \Rightarrow \neg P \vee \neg Q$

EXAMPLE: What is the underlying tautological implication in the following proof?

- If 1/3 is a rational number, then 1/3 is a real number.
- 1/3 is a rational number.
- 1/3 is a real number.
  - $q \to r$ =“If 1/3 is a rational number, then 1/3 is a real number.
  - q =“1/3 is a rational number”
  - r =“1/3 is a real number”
    - What is the underlying tautological implication?
      - $(q \to r) \land q \Rightarrow r$
        - YES. This is a tautological implication.

Limitation of Propositional Logic

EXAMPLE: What is the underlying tautological implication in the following proof?

- All rational numbers are real numbers
- 1/3 is a rational number
- 1/3 is a real number
  - p =“All rational numbers are real numbers”
  - q =“1/3 is a rational number”
  - r =“1/3 is a real number”
    - What is the underlying tautological implication?
      - $p \land q \Rightarrow r$?
        - NO. $p \land q \rightarrow r$ is not a tautology.
          - Why is this a proof?
            - We need **predicate logic**.

Predicate and Individual

Predicate (informal): describe the property of the subject term (in a sentence)

Individual (informal): the object you are considering

- "$\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$ is an integer"
- "e^{π} is greater than π^{e}"
  - **Individual Constant个体常项:** $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}$, e^{π}, π^{e}
  - **Individual Variable个体变项:** x, y, z
  - **Domain of individuals个体域:** the set of all individuals you are considering

Predicate(formal): a function from a domain of individuals to $\{\mathbf{T}, \mathbf{F}\}$

- **n-ary predicate$n$元谓词:** a predicate on n individuals
  - I: "is an integer" // a unary predicate
  - G: "is greater than" //a binary predicate
- **Predicate constant谓词常项:** a concrete predicate // I, G
- **Predicate variable谓词变项:** a symbol that represents any predicate

From Predicates to Propositions

Propositional function: $P(x_1, \ldots, x_n)$, where P is an n-ary predicate

- $P(x, y)$:"x is greater than y"
- $P(x, y)$ gives a proposition when we assign values to x, y
  - $P(e^{\pi}, \pi^{e})$ is a proposition (a true proposition)
- $P(x, y)$ is not a proposition

Example: p : "Alice's father is a doctor"; q : "Bob's father is a doctor"

- Individuals: Alice's father, Bob's father; Predicate D: "is a doctor"
- $p = D(\text{Alice's father}), q = D(\text{Bob's father})$

Function of Individuals: a map on the domain of individuals

- $f(x) = x$'s father
- $p = D\big(f(\text{Alice})\big); q = D(f(\text{Bob}))$

More examples: white board

Universal Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **universal quantification**全称量化 of $P(x)$ is "$P(x)$ for all x in the domain".

- notation: $\forall x\, P(x)$; read as "for all x $P(x)$" or "for every x $P(x)$"
  - "$\forall$" is called the **universal quantifier**全称量词
  - "$\forall x\, P(x)$" is true iff $P(x)$ is true for every x in the domain
  - "$\forall x\, P(x)$" is false iff there is an x_0 in the domain such that $P(x_0)$ is false
    - **Counterexample**反例: an x_0 such that $P(x_0)$ is false

Example: $P(n)$: "$n^2 + n + 41$ is a prime"

- When domain = natural numbers, "$\forall n P(n)$" is "for every natural number n, $n^2 + n + 41$ is a prime"
- When domain is $D = \{0,1, \ldots, 39\}$, "$\forall n P(n)$" is "for every $n \in D$, $n^2 + n + 41$ is a prime"

Remark: If the domain is empty, then " $\forall x\ P(x)$" is true for any P, because there are no elements x in the domain for which $P(x)$ is false.

Existential Quantifier

DEFINITION: Let $P(x)$ be a propositional function. The **existential quantification**存在量化 of $P(x)$ is "there is an x in the domain such that $P(x)$"

- notation: $\exists x\, P(x)$; read as "for some x $P(x)$" or "there is an x s. t. $P(x)$"
  - "$\exists$" is called the **existential quantifier**存在量词
  - "$\exists x\, P(x)$" is true iff there is an x in the domain such that $P(x)$ is true
  - "$\exists x\, P(x)$" is false iff $P(x)$ is false for every x in the domain

Example: $P(x)$: "$x^2 - x + 1 = 0$"

- When $D = \mathbb{N}$, $\exists x\, P(x)$ is **F**
- When $D = \mathbb{R}$, $\exists x\, P(x)$ is **F**
- When $D = \mathbb{C}$, $\exists x\, P(x)$ is **T**

Remark: If the domain is empty, then " $\exists x\, P(x)$" is false for any P, because there doesn't exist any x in the domain making $P(x)$ to be true.
Remark: If the domain of individuals is not stated, the individual can be anything.