Discrete Mathematics: Lecture 2

translation, precedence, truth table, tautology, contradiction, contingency, satisfiable, rule of substitution, logically equivalent, rule of replacement

Jiahua Jiang*

*School of Information Science and Technology, ShanghaiTech University

Summary

Proposition: a <u>declarative</u> sentence that is <u>either true or false</u>.

- simple, compound, propositional constant/variable

Logical Connectives: $\neg$ (unary), $\wedge, \vee, \rightarrow, \leftrightarrow$ (binary)

- Truth table

Well-Formed Formulas (WFFs): formulas

- propositional constants (T, F) and propositional variables are WFFs
- $\neg A, (A \wedge B), (A \vee B), (A \leftarrow B), (A \leftrightarrow B)$
- Finite

Precedence of Logical Operators

Precedence (priority): $\neg, \land, \lor, \rightarrow, \leftrightarrow$

- formulas inside () are computed firstly
- different connectives: $\neg, \land, \lor, \rightarrow, \leftrightarrow$ (Decreasing Precedence)
- same connectives: from left to the right

- Example 1: $\neg p \land q$: $(\neg p) \land q$
- Example 2:$\neg(p \land q)$: First (.), then $\neg$.
- Example 3: $p \lor q \land r$: $p \lor (q \land r)$
- Example 4:

$$\underline{\underline{\underline{(p \rightarrow q)} \land \underline{(q \rightarrow r)}} \leftrightarrow \underline{(p \rightarrow r)}}$$

From Natural Language to WFFs

The Method of Translation:

- Introduce symbols to represent simple propositions
- Connect the symbols with logical connectives to obtain WFFs

Example:

- "It is not the case that snow is black."
  - p: "Snow is black"
  - Translation: $\neg p$
  - **Remark:** it is better to choose the simple proposition to be affirmative sentence.
- "π and e are both irrational"
  - p : "π is irrational"; q : "e. is irrational"
  - Translation: $p \wedge q$
- "If π is irrational, then 2π is irrational"
  - $p : \pi$ is irrational; $q : 2\pi$ is irrationals
  - Translation: $p \rightarrow q$

Example:

- "$e^{\pi} > \pi^{e}$ if and only if $\pi > e \ln \pi$"
    - $p : e^{\pi} > \pi^{e}$; $q : \pi > e \ln \pi$
    - Translation: $p \leftrightarrow q$
- **" $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational."** (ambiguity in natural language)
    - $p =$ "$(\sqrt{2})^{\sqrt{2}}$ is rational "; $q =$ "$(\sqrt{2})^{\sqrt{2}}$ is irrational"
    - Explanation 1: $(\sqrt{2})^{\sqrt{2}}$ cannot be neither rational nor irrational.
    - Emphasis: $(\sqrt{2})^{\sqrt{2}}$ is a real number, only two possibility
    - Translation 1: $p \vee q$ (by default, this is the translation of "or")

    - Explanation 2: $(\sqrt{2})^{\sqrt{2}}$ cannot be both rational or irrational
    - It is obvious that $(\sqrt{2})^{\sqrt{2}}$ is real number. Emphasis: not both
    - Translation 2: $(p \wedge \neg q) \vee (\neg p \wedge q)$ (not both)
- The specific translations remove the ambiguity.

Example:

- We agree that $p \vee q$ is the correct translation of " $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational."

- We agree that $(p \wedge \neg q) \vee (\neg p \wedge q)$ is the translation of " $(\sqrt{2})^{\sqrt{2}}$ is rational or irrational, but not both."

Example

- You are eligible to be President of the U.S.A. only if you are at least 35 years old, were born in the U.S.A, or at the time of your birth both of your parents were citizens, and you have lived at least 14 years in the country.
  - e: "You are eligible to be President of the U.S.A.,"
  - a: "You are at least 35 years old,"
  - b: "You were born in the U.S.A,"
  - p: "At the time of your birth, both of your parents were citizens,"
  - r: "You have lived at least 14 years in the U.S.A."
    - Translation: $e \rightarrow (a \wedge (b \vee p) \wedge r)$

Example

- A comes to the party if and only if B doesn't come, but, if B comes, then C doesn't come and D comes.
- A sufficient condition for A coming to the party is that, if B does not come, then at least one of C and D must come.
  - a: "A comes to the party."
  - b: "B comes to the party."
  - c: "C comes to the party."
  - d: "D comes to the party."
    - Translation 1: $(a \leftrightarrow \neg b) \wedge \big(b \rightarrow (\neg c \wedge d)\big)$
    - Translation 2: $(\neg b \rightarrow (c \vee d)) \rightarrow a$

System Specifications

System Specifications: Determine if there is a system that satisfy all of the following requirements.

1. The diagnostic message is stored in the buffer or it is retransmitted.
2. The diagnostic message is not stored in the buffer.
3. If the diagnostic message is stored in the buffer, then it's retransmitted.
   - $s =$ " The diagnostic message is stored in the buffer."
   - $r =$ " The diagnostic message is retransmitted."
   - 1: $s \vee r$; 2: $\neg s$; 3: $s \rightarrow r$
     Find an assignment of s and r to make all 1, 2, 3 true.
   - There is system that satisfies 1, 2, 3. ($s =$ **F**, $r =$ **T**)
4. Add one more requirement "The diagnostic message is not retransmitted"
   - 1: $s \vee r$; 2: $\neg s$; 3: $s \rightarrow r$; 4: $\neg r$
     There is no system that satisfies 1, 2, 3 and 4.

Truth Table

Definition: Let F be a WFF of $p_1, \ldots, p_n$, n propositional variables

- A truth assignment for F is a map $\alpha : \{p_1, \ldots, p_n\} \to \{\boldsymbol{T}, \boldsymbol{F}\}$.
- There are 2^n different truth assignments.

p_1	p_2	$\cdots$	p_n	F
T	**T**	$\cdots$	**T**	.
T	**T**	$\cdots$	**F**	.
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
F	**F**	$\cdots$	**F**	.

Example: Truth tables of $A = p \vee \neg p$, $B = p \wedge \neg p$, $C = p \to \neg p$

p	$\neg p$	A
T	**F**	
F	**T**	

p	$\neg p$	B
T	**F**	
F	**T**	

p	$\neg p$	C
T	**F**	
F	**T**	

Truth Table

Definition: Let F be a WFF of $p_1, \ldots, p_n$, n propositional variables

- A truth assignment for F is a map $\alpha : \{p_1, \ldots, p_n\} \rightarrow \{\boldsymbol{T}, \boldsymbol{F}\}$.
- There are 2^n different truth assignments.

Example: Truth tables of $A = p \vee \neg p$, $B = p \wedge \neg p$, $C = p \rightarrow \neg p$

p	$\neg p$	A
T	**F**	**T**
F	**T**	**T**

p	$\neg p$	B
T	**F**	**F**
F	**T**	**F**

p	$\neg p$	C
T	**F**	**F**
F	**T**	**T**

Types of WFFs

Tautology(always true): a WFF whose truth value is $\boldsymbol{T}$ for all truth assignment

- $p \vee \neg p$ is tautology

Contradiction(always false): a WFF whose truth value is $\boldsymbol{F}$ for all truth assignment

- $p \wedge \neg p$ is contradiction

Contingency(possible to be true and possible to be false): neither tautology nor contradiction

- $p \rightarrow \neg p$ is a contingency

Satisfiable: a WFF is satisfiable if it is true for at least one truth assignment

Rules of Substitution(the way to get Tautology): Let B be a formula obtained from a tautology A by substituting a propositional variable in A with an arbitrary formula. Then B must be a tautology.

- $p \vee \neg p$ is a tautology, then $(q \wedge r) \vee \neg(q \wedge r)$ is a tautology as well.

Summary

Translation: from natural language to formulas.

- introduce symbols; construct formulas

Precedence: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$

- Truth Table

Types of formulas:

- Tautology; contradiction; contingency; satisfiable; unsatisfiable
- Rule of substitution

Logically Equivalent ($A \leftrightarrow B$ is tautology)

Definition: Let A and B be WFFs in propositional variables $p_1, \ldots, p_n$.

- A and B are **logically equivalent** if they always have the same truth value for every truth assignment of $p_1, \ldots, p_n$.
- Notation: $A \equiv B$
- Same true table

Theorem: $A \equiv B$ iff $A \leftrightarrow B$ is a tautology.

- $A \equiv B$
- iff for any truth assignment, A, B take the same truth values
- iff for any truth assignment, $A \leftrightarrow B$ is true
- iff $A \leftrightarrow B$ is a tautology

Logically Equivalent ($A \leftrightarrow B$ is tautology)

Theorem: $A \equiv A$; if $A \equiv B$, then $B \equiv A$; if $A \equiv B$, $B \equiv C$, then $A \equiv C$

Theorem: Let $A^{-1}(\mathbf{T})$ be the set of truth assignments such that A is true. Then $A \equiv B$ if and only if $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$.

- $A \equiv B$ if and only if $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$.

Theorem: How to prove $A \equiv B$?

Proving $A \equiv B$

EXAMPLE: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ **//distributive law**

- Idea: Show that A, B have the same truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	**T**	**T**					
T	**T**	**F**					
T	**F**	**T**					
T	**F**	**F**					
F	**T**	**T**					
F	**T**	**F**					
F	**F**	**T**					
F	**F**	**F**					

Proving $A \equiv B$

EXAMPLE: $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ **//distributive law**

- Idea: Show that A, B have the same truth table.

P	Q	R	$Q \vee R$	$P \wedge (Q \vee R)$	$P \wedge Q$	$P \wedge R$	$(P \wedge Q) \vee (P \wedge R)$
T	**T**	**T**	**T**	**T**	**T**	**T**	**T**
T	**T**	**F**	**T**	**T**	**T**	**F**	**T**
T	**F**	**T**	**T**	**T**	**F**	**T**	**T**
T	**F**	**F**	**F**	**F**	**F**	**F**	**F**
F	**T**	**T**	**T**	**F**	**F**	**F**	**F**
F	**T**	**F**	**T**	**F**	**F**	**F**	**F**
F	**F**	**T**	**T**	**F**	**F**	**F**	**F**
F	**F**	**F**	**F**	**F**	**F**	**F**	**F**

Remark: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ can be shown similarly.

Proving $A \equiv B$

Commutative law: $P \wedge Q \equiv Q \wedge P$

- Idea: Show that $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$
- $A = P \wedge Q$; $B = Q \wedge P$
  - $A = \mathbf{T}$ if and only if $(P, Q) = (\mathbf{T}, \mathbf{T})$
    - $A^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
  - $B = \mathbf{T}$ if and only if $(P, Q) = (\mathbf{T}, \mathbf{T})$
    - $B^{-1}(\mathbf{T}) = \{(\mathbf{T}, \mathbf{T})\}$
- $A^{-1}(\mathbf{T}) = B^{-1}(\mathbf{T})$
- $A \equiv B$

Remark: $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ can be shown similarly. (**Associate law**)

Proving $A \equiv B$

Commutative law: $P \vee Q \equiv Q \vee P$

- Idea: Show that $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$
- $A = P \vee Q$; $B = Q \vee P$
    - $A = \mathbf{F}$ if and only if $(P, Q) = (\mathbf{F}, \mathbf{F})$
        - $A^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
    - $B = \mathbf{F}$ if and only if $(P, Q) = (\mathbf{F}, \mathbf{F})$
        - $B^{-1}(\mathbf{F}) = \{(\mathbf{F}, \mathbf{F})\}$
- $A^{-1}(\mathbf{F}) = B^{-1}(\mathbf{F})$
- $A \equiv B$

Remark: $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$ can be shown similarly. (**Associate law**)

Logical Equivalences

Name	Logical Equivalences	NO.
Double Negation Law 双重否定律	$\neg(\neg P) \equiv P$	1
Identity Laws 同一律	$P \wedge \mathbf{T} \equiv P$	2
	$P \vee \mathbf{F} \equiv P$	3
Idempotent Laws 等幂律	$P \vee P \equiv P$	4
	$P \wedge P \equiv P$	5
Domination Laws 零律	$P \vee \mathbf{T} \equiv \mathbf{T}$	6
	$P \wedge \mathbf{F} \equiv \mathbf{F}$	7
Negation Laws 补余律	$P \vee \neg P \equiv \mathbf{T}$	8
	$P \wedge \neg P \equiv \mathbf{F}$	9

Logical Equivalences

Name	Logical Equivalences	NO.
Commutative Laws 交换律	$P \vee Q \equiv Q \vee P$	10
	$P \wedge Q \equiv Q \wedge P$	11
Associative Laws 结合律	$P \vee (Q \vee R) \equiv (P \vee Q) \vee R$	12
	$P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$	13
Distributive Laws 分配律	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$	14
	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	15
De Morgan's Laws 摩根律	$\neg(P \wedge Q) \equiv (\neg P) \vee (\neg Q)$	16
	$\neg(P \vee Q) \equiv (\neg P) \wedge (\neg Q)$	17
Absorption Laws 吸收律	$P \vee (P \wedge Q) \equiv P$	18
	$P \wedge (P \vee Q) \equiv P$	19

Logical Equivalences

Name	Logical Equivalences	NO.
Laws Involving Implication $\rightarrow$	$P \rightarrow Q \equiv \neg P \vee Q$	20
	$P \rightarrow Q \equiv \neg Q \rightarrow \neg P$	21
	$(P \rightarrow R) \wedge (Q \rightarrow R) \equiv (P \vee Q) \rightarrow R$	22
	$P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R$	23
	$P \rightarrow (Q \rightarrow R) \equiv Q \rightarrow (P \rightarrow R)$	24
Laws Involving Bi-Implication $\leftrightarrow$	$P \leftrightarrow Q \equiv (P \rightarrow Q) \wedge (Q \rightarrow P)$	25
	$P \leftrightarrow Q \equiv (\neg P \vee Q) \wedge (P \vee \neg Q)$	26
	$P \leftrightarrow Q \equiv (P \wedge Q) \vee (\neg P \wedge \neg Q)$	27
	$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$	28