PART-1

Black Body Radiation, Stefan's Law, Wien's Law, Rayleigh-Jeans Law and Plank's Law, Wave Particle Duality, Matter Waves.

CONCEPT OUTLINE : PART-1

Wave Particle Duality :
According to Einstein, the energy of light is concentrated in small bundles called photon. Hence, light behaves as a wave on one hand and as a particle on the other hand. This nature of light is known as dual nature, while this property of light is known as wave particle duality.

de-Broglie Wave or Matter Waves :
de-Broglie wavelength is given by

$$\lambda = \frac{h}{p}$$

de-Broglie wavelength in terms of temperature is given by

$$\lambda = \frac{h}{p} = \frac{h}{\sqrt{3mkT}}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **Explain black body radiation and discuss energy distribution in the spectrum of a black body radiation.**

Answer

A. Black Body :

1. A body which absorbs completely all the radiations incident upon it, reflecting none and transmitting none, is called a black body.
2. Absorptivity of a black body is unity for all wavelengths.
3. It appears black whatever the wavelength of incident radiation is.
4. When a black body is heated to a suitable high temperature it emits total radiation which is known as black body radiations.
5. From the energy point of view, black body radiation is equivalent to the radiation of an infinitely large number of non-interacting harmonic oscillations, the so called radiation oscillations.

6. No actual body is a perfect black body, it is only an ideal conception.
7. Lamp black is the nearest approach to black body which absorbs nearly 99 % of the incident radiation.
8. Platinum black is another example of a black body.

B. Energy Distribution :

1. Results of the studies of black body radiation spectra are shown in Fig. 3.1.1 in which variation of intensity with wavelength for various temperatures are shown.

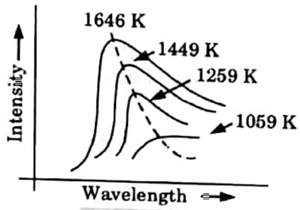

Fig. 3.1.1.

2. The energy distribution in the radiation spectrum of black body is not uniform. As the temperature of the body rises, the intensity of radiation for each wavelength increases.
3. At a given temperature, the intensity of radiation increases with increase in wavelength and becomes maximum at a particular wavelength. With further increase in wavelength, the intensity of radiation decreases.
4. The point of maximum energy shifts towards the shorter wavelengths as the temperature increases.
5. For a given temperature the total energy of radiation is represented by the area between the curve and the horizontal axis and the area increases with increase of temperature, being directly proportional to the fourth power of absolute temperatures.

Que 3.2. **Discuss Stefan's law.**

Answer

1. The total amount of heat (E) radiated by a perfectly black body per second per unit area is directly proportional to the fourth power of its absolute temperature (T), *i.e.*,

$$E \propto T^4 \quad \text{or} \quad E = \sigma T^4$$

where, σ = Universal constant and is called Stefan's constant.

2. This law is also called as Stefan's fourth power law.
3. If a black body at absolute temperature T is surrounded by another black body at absolute temperature T_0 then the net amount of heat (E) lost by the former per second per cm^2 is given by

$$E = \sigma(T^4 - T_0^4)$$

4. This law is also known as Stefan-Boltzmann law.

Que 3.3. **Explain Wien's laws of energy distribution.**

Answer

A. Fifth Power Law :

1. The total amount of energy emitted by a black body per unit volume at an absolute temperature T and contained in the spectral region included within the wavelength λ and $\lambda + d\lambda$ is given as,

$$u_\lambda\, d\lambda = \frac{A}{\lambda^5} f(\lambda T)\, d\lambda \quad \text{...(3.3.1)}$$

where, A = Constant, and

$f(\lambda T)$ = A function of the product λT and is given as

$f(\lambda T) = e^{-ch/\lambda kT} = e^{-a/\lambda T}$

$$\therefore \quad u_\lambda\, d\lambda = A\lambda^{-5} e^{-a/\lambda T}\, d\lambda \quad \text{...(3.3.2)}$$

2. For $\lambda = \infty$, $u_\lambda = 0$ and for $\lambda = 0$, $u_\lambda = 0$.
3. Thus eq. (3.3.2) shows that no energy is emitted by a wave of infinite wavelength as well as by a wave of zero wavelength.
4. For $T = \infty$, eq. (3.3.2) reduces to : $u_\lambda\, d\lambda = A\lambda^{-5}$, which is finite quantity and is in open contradiction with the Stefan's fourth power law (σT^4).
5. Wien's law of energy distribution, however, explains the energy distribution at short wavelengths at higher temperature and fails for long wavelengths.

B. Displacement Law :

1. As the temperature of the body is raised the maximum energy tends to be associated with shorter wavelength, *i.e.*,

$$\lambda_m T = \text{constant} \quad \text{...(3.3.3)}$$

where, λ_m = wavelength at which the energy is maximum , and

T = absolute temperature.

2. Thus, if radiation of a particular wavelength at a certain temperature is adiabatically altered to another wavelengths then temperature changes in the inverse ratio.

Que 3.4. **Discuss Rayleigh-Jean's Law.**

Answer

1. The total amount of energy emitted by a black body per unit volume at an absolute temperature T in the wavelength range λ and $\lambda + d\lambda$ is given as,

$$u_\lambda\, d\lambda = \frac{8\pi kT}{\lambda^4}\, d\lambda$$

where, k = Boltzmann's constant = 1.381×10^{-23} J/K.

2. The energy radiated in a given wavelength range λ and $\lambda + d\lambda$ increases rapidly as λ decreases and approaches infinity for very short wavelengths which however can't be true.
3. This law, thus, explains the energy distribution at longer wavelengths at all temperatures and fails totally for shorter wavelengths.
4. The energy distribution curves of black body show a peak while going towards the ultra-violet wavelength (shorter λ) and then fall while Rayleigh-Jean's law indicate continuous rise only.
5. This is the failure of classical physics and this failure was often called as the "Ultra-violet catastrophe" of classical physics.

Que 3.5. **Derive Planck's radiation law and show how this law successfully explained observed spectrum of black body radiation.**

Answer

A. Planck's Radiation Law :

1. According to Planck's quantum hypothesis the exchange of energy by radiations with matters do not take place continuously but discontinuously and discretely as an integral multiple of an elementary quantum of energy represented by the relation

$$E = h\nu$$

where, ν = frequency of radiations, and
h = Planck's constant.

2. Thus, the resonators can oscillate only with integral energy values $h\nu$, $2h\nu$, $3h\nu$, ... , $nh\nu$ or in general $E_n = n\,h\nu$ (n = 1, 2, 3,...).
3. Hence, emission and absorption of energy by the particles of a radiating body interchanging energy with the radiation oscillation occur discretely, not in a continuous sequence.
4. In relation $E_n = n\,h\nu$, n is called a quantum number and the energies of the radiators are said to be quantised and allowed energy states are called quantum states.
5. On the basis of his assumptions Planck derived a relation for energy density (u_ν) of resonators emitting radiation of frequency lying between ν and $\nu + d\nu$ which is given as follows :

$$u_\nu\, d\nu = \frac{8\pi h\nu^3}{c^3}\frac{d\nu}{e^{h\nu/kT} - 1} \quad ...(3.5.1)$$

or

$$u_\lambda\, d\lambda = \frac{8\pi\, hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT} - 1} \quad ...(3.5.2)$$

6. The eq. (3.5.1) and eq. (3.5.2) are known as Planck's radiation law.

B. Experimental Verification of Planck's Radiation Law : According to Planck's radiation law expression for energy density is given as

$$u_\lambda\, d\lambda = \frac{8\pi\, hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT} - 1}$$

a. Wien's Law from Planck's Radiation Law :

1. For shorter wavelengths λT will be small and hence $e^{hc/\lambda kT} >> 1$
2. Hence, for small values of λT Planck's formula reduces to

$$u_\lambda\, d\lambda = \frac{8\pi\, hc}{\lambda^5}\frac{d\lambda}{e^{hc/\lambda kT}} = 8\pi\, hc\, \lambda^{-5} e^{-hc/\lambda kT} d\lambda$$

or

$$u_\lambda\, d\lambda = A\lambda^{-5} e^{-a/\lambda T} d\lambda \quad ...(3.5.3)$$

where, A = Constant (= $8\pi\, hc$), and
a = Constant (= hc/k).

3. Eq. (3.5.3) is Wien's law.
4. This result shows that at shorter wavelengths Planck's law approaches Wien's law and hence at shorter wavelengths Planck's law and Wien's law agrees (Fig. 3.5.1).

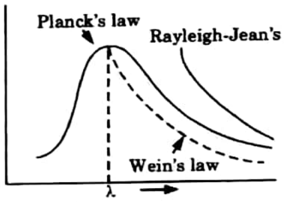

Fig. 3.5.1.

b. Rayleigh-Jean's Law :

1. For longer wavelengths $e^{hc/\lambda kT}$ is small and can be expanded as follows :

$$e^{hc/\lambda kT} = 1 + \frac{hc}{\lambda\, kT} \approx \frac{hc}{\lambda\, kT}$$

2. Hence, for longer wavelengths Planck's formula reduces to

$$u_\lambda d\lambda = \frac{8\pi\, hc}{\lambda^5}\frac{\lambda kT}{hc}\, d\lambda$$

or

$$u_\lambda d\lambda = \frac{8\pi\, kT}{\lambda^4}\, d\lambda \quad ...(3.5.4)$$

3. Eq. (3.5.4) shows that for longer wavelengths Planck's law approaches to Rayleigh-Jean's law and thus at longer wavelengths Planck's law and Rayleigh-Jean's law agree (Fig. 3.5.1).
4. Thus, it is concluded that the Planck's radiation law successfully explained the entire shape of the curves giving the energy distribution in black body radiation.

Que 3.6. **Discuss the wave particle duality.**

Answer

1. According to the Planck's theory of thermal radiation; Einstein's explanation of photoelectric effect; emission and absorption of radiation by substance; black body radiation etc., the electromagnetic radiation consist of discrete indivisible packets of energy ($h\nu$) called photons which manifest particle character of radiation. On the other hand, macroscopic optical phenomena like interference, diffraction and polarisation reveal and firmly confirm the wave character of electromagnetic radiation. Therefore, we conclude that the electromagnetic radiation has dual character, in certain situation it exhibits wave properties and in other it acts like a particle.
2. The particle and wave properties of radiation can never be observed simultaneously. To study the path of a beam of monochromatic radiation, we use the wave theory, while to calculate the amount of energy transactions of the same beam, we have to recourse to the photon or particle theory.
3. It has been found impossible to separate the particle and wave aspects of electromagnetic radiation.

Que 3.7. **What are de-Broglie's waves or matter waves ?**

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Answer

1. When a material particle moves in a medium, a group of waves is associated with it due to which it shows the wave particle duality. These waves are known as matter waves or de-Broglie waves.
2. According to de-Broglie's concept, each material particle in motion behaves as waves, having wavelength 'λ' associated with moving particle of momentum p.

$$\lambda = \frac{h}{p} \Rightarrow \lambda \propto \frac{1}{p}$$

$$\text{Wave nature} \propto \frac{1}{\text{Particle nature}}$$

Que 3.8. **Deduce expression for wavelength of de-Broglie wave.**

Answer

1. Let a photon having energy,

$$E = h\nu = \frac{hc}{\lambda} \quad \text{...(3.8.1)}$$

2. If a photon possesses mass, it is converted into energy.
3. Now according to Einstein's law,

$$E = mc^2 \quad \text{...(3.8.2)}$$

4. From eq. (3.8.1) and eq. (3.8.2),

$$mc^2 = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{mc^2}$$

$$\lambda = \frac{h}{mc} \Rightarrow \lambda = \frac{h}{p} \qquad [\because \; mc = p]$$

5. In place of photon, we take material particle having mass 'm' moving with velocity 'v'. The momentum,

$$p = mv$$

6. The wavelength of wave associated with particle is,

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

This is de-Broglie's wavelength.

7. If E_k is kinetic energy of material particle of mass 'm' moving with velocity 'v' then,

$$E_k = \frac{1}{2} mv^2$$

$$E_k = \frac{m^2 v^2}{2m}$$

or $$E_k = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

or $$p = \sqrt{2mE_k}$$

8. The de-Broglie's wavelength, $\lambda = \dfrac{h}{\sqrt{2mE_k}}$
9. According to kinetic theory of gases, the average kinetic energy (E_k) of the material particle is given as

$$E_k = \frac{3}{2} KT$$

10. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2m \times \frac{3KT}{2}}} = \frac{h}{\sqrt{3mKT}}$$

where, $K = 1.38 \times 10^{-23}$ J/K

T = temperature (K).

11. Suppose material particle is accelerated by potential difference of V volt then,

$$E_k = qV$$

where, q = charge of particle.

12. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2mqV}}$$

Que 3.9. **Derive an expression for de-Broglie wavelength of helium atom having energy at temperature T K.**

Answer

1. According to kinetic theory of gases, the average kinetic energy (E_k) of the material particle is given as

$$E_k = \frac{3}{2}KT$$

2. The de-Broglie's wavelength,

$$\lambda = \frac{h}{\sqrt{2mE_k}} = \frac{h}{\sqrt{2m \times \frac{3KT}{2}}} = \frac{h}{\sqrt{3mKT}}$$

where, $K = 1.38 \times 10^{-23}$ J/K and

T = temperature (K).

Que 3.10. **The kinetic energy of an electron is 4.55×10^{-25} J. Calculate the velocity, momentum and wavelength of the electron.**

Answer

Given : $E_k = 4.55 \times 10^{-25}$ J

To Find : i. Velocity
ii. Momentum
iii. Wavelength of electron.

1. If m_o is the rest mass of electron, v is the velocity of the electron, then its kinetic energy (E_k) is given by

$$E_k = \frac{1}{2}m_o v^2$$

$$v = \sqrt{\frac{2E_k}{m_o}} = \sqrt{\frac{2 \times 4.55 \times 10^{-25}}{9.1 \times 10^{-31}}} = 1 \times 10^3 \text{ m/s}$$

[$\because$ $m_o = 9.1 \times 10^{-31}$ kg]

2. Momentum of electron,

$$p = m_o v = 9.1 \times 10^{-31} \times 10^3 = 9.1 \times 10^{-28} \text{ kg m/s}$$

3. Wavelength of electron,

$$\lambda = h/p = (6.63 \times 10^{-34})/(9.1 \times 10^{-28}) = 7.29 \times 10^{-7} \text{ m}$$

Que 3.11. **Find the de-Broglie wavelength of neutron of energy 12.8 MeV (given that $h = 6.625 \times 10^{-34}$ J-s, mass of neutron (m_n) = 1.675×10^{-27} kg and 1 eV = 1.6×10^{-19} Joule).**

Answer

Given : E_k = 12.8 MeV, $h = 6.625 \times 10^{-34}$ J-s, $m_n = 1.675 \times 10^{-27}$ kg, 1 eV = 1.6×10^{-19} J.

To Find : de-Broglie wavelength.

1. Rest mass energy of neutron is given as

$$m_o c^2 = 1.675 \times 10^{-27} \times (3 \times 10^8)^2$$
$$= 1.5075 \times 10^{-10} \text{ J}$$
$$= \frac{1.507 \times 10^{-10}}{1.6 \times 10^{-19}} = 942.18 \text{ MeV}$$

2. The given energy 12.8 MeV is very less compared to the rest mass energy of neutron, therefore relativistic consideration in this case is not applicable.

3. Now de-Broglie wavelength of the neutron is given as

$$\lambda = \frac{h}{\sqrt{2mE_k}}$$

$$E_k = 12.8 \times 10^6 \times (1.6 \times 10^{-19}) \text{ J}$$

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 1.675 \times 10^{-27} \times 12.8 \times 10^6 \times 1.6 \times 10^{-19}}}$$

$$= \frac{6.625 \times 10^{-34}}{8.28 \times 10^{-20}}$$

$$= 8 \times 10^{-15} \text{ m}$$

$$= 8 \times 10^{-5} \text{ Å}$$

Que 3.12. **Calculate the de-Broglie's wavelength associated with a proton moving with a velocity equal to $\frac{1}{20}$ th of light velocity.**

Answer

Given : v = (1/20) c = $\frac{1}{20} \times 3 \times 10^8$ m/s = 1.5×10^7 m/s

To Find : de-Broglie wavelength

1. Formula for de-Broglie's wavelength :

$$\lambda = \frac{h}{mv}$$

$$\lambda = \frac{6.63 \times 10^{-34}}{1.67 \times 10^{-27} \times 1.5 \times 10^{7}} = 2.646 \times 10^{-14} \text{ m}$$

[$\because$ $m = 1.67 \times 10^{-27}$ kg and $h = 6.63 \times 10^{-34}$ J-s]

$= 2.646 \times 10^{-4}$ Å

PART-2

Time Dependent and Time Independent Schrodinger's Wave Equation, Born Interpretation of Wave Function, Solution to Stationary State, Schrodinger Wave Equation for One-Dimensional Particle in a Box, Compton Effect.

CONCEPT OUTLINE : PART-2

Wave Function and its Significance :
The wave function ψ is described as mathematical function whose variation builds up matter waves. $|\psi|^2$ defines the probability density of finding the particle within the given confined limits. ψ is defined as probability amplitude and $|\psi|^2$ is defined as probability density.

Schrodinger's Wave Equation :
This wave equation is a fundamental equation in quantum mechanics and describes the variation of wave function ψ in space and time.

Compton Effect : The phenomenon in which the wavelength of the incident X-rays increases and hence the energy decreases due to scattering from an atom is known as Compton effect.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.13. **What is Schrodinger wave equation ? Derive time independent and time dependent Schrodinger wave equations.**

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Answer

1. Schrodinger's equation, which is the fundamental equation of quantum mechanics, is a wave equation in the variable ψ.

A. Time Independent Schrodinger Wave Equation :

1. Consider a system of stationary wave to be associated with particle and the position coordinate of the particle (x, y, z) and ψ is the periodic displacement at any instant time 't'.
2. The general wave equation in 3-D in differential form is given as

$$\nabla^2\psi = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2} \quad ...(3.13.1)$$

where, v = velocity of wave, and

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \text{Laplacian operator.}$$

3. The wave function may be written as

$$\psi = \psi_o e^{-i\omega t} \quad ...(3.13.2)$$

4. Differentiate eq. (3.13.2) w.r.t. time, we get,

$$\frac{\partial\psi}{\partial t} = -i\,\omega\,\psi_o e^{-i\omega t} \quad ...(3.13.3)$$

5. Again differentiating eq. (3.13.3), we get

$$\frac{\partial^2\psi}{\partial t^2} = i^2\omega^2\psi_o e^{-i\omega t}$$

$$\frac{\partial^2\psi}{\partial t^2} = -\omega^2\psi \quad ...(3.13.4)$$

6. Putting these value in eq. (3.13.1),

$$\nabla^2\psi = \frac{-\omega^2}{v^2}\psi \quad ...(3.13.5)$$

7. But $\quad \omega = 2\pi v = \dfrac{2\pi v}{\lambda} \Rightarrow \dfrac{\omega}{v} = \dfrac{2\pi}{\lambda} \quad ...(3.13.6)$
8. From eq. (3.13.6) and eq. (3.13.5), we get

$$\nabla^2\psi = -\frac{4\pi^2}{\lambda^2}\psi \quad ...(3.13.7)$$

9. From de-Broglie's wavelength, $\lambda = \dfrac{h}{mv}$

then, $$\nabla^2\psi = \frac{-4\pi^2 m^2 v^2}{h^2}\psi \quad ...(3.13.8)$$

10. If E and V are the total energy and potential energy of a particle and E_k is kinetic energy, then,

$$E_k = E - V \quad \text{or} \quad \frac{1}{2}mv^2 = E - V$$

$$m^2v^2 = 2m(E - V) \quad ...(3.13.9)$$

11. From eq. (3.13.8) and eq. (3.13.9), we get

$$\nabla^2\psi = \frac{-4\pi^2 2m[E-V]\psi}{h^2}$$

$$\therefore \quad \nabla^2\psi + \frac{2m[E-V]\psi}{\hbar^2} = 0 \qquad \left[\text{where, } \hbar = \frac{h}{2\pi}\right] \qquad ...(3.13.10)$$

This is required time-independent Schrodinger wave equation.

12. For free particle, $V = 0$

$$\therefore \quad \nabla^2\psi + \frac{2m}{\hbar^2}E\psi = 0$$

B. Time Dependent Schrodinger Wave Equation :

1. We know that wave function is $\psi = \psi_o e^{-i\omega t}$
2. On differentiating w.r.t. time, we get,

$$\frac{\partial\psi}{\partial t} = -i\omega\,\psi_o e^{-i\omega t}$$

or
$$\frac{\partial\psi}{\partial t} = -i\,(2\pi\nu)\,\psi \qquad ...(3.13.11)$$

3. But
$$E = h\nu \Rightarrow \nu = \frac{E}{h}$$

4. So, eq. (3.13.11) becomes,

$$\frac{\partial\psi}{\partial t} = -i2\pi\left(\frac{E}{h}\right)\psi$$

$$\frac{\partial\psi}{\partial t} = -\frac{i}{\hbar}E\psi \qquad \left[\because \hbar = \frac{h}{2\pi}\right]$$

or
$$E\psi = -\frac{\hbar}{i}\frac{\partial\psi}{\partial t}$$

or
$$E\psi = i\hbar\frac{\partial\psi}{\partial t} \qquad ...(3.13.12)$$

5. Now time independent Schrodinger wave equation is,

$$\nabla^2\psi + \frac{2m}{\hbar^2}(E-V)\psi = 0$$

$$\nabla^2\psi + \frac{2m}{\hbar^2}[E\psi - V\psi] = 0$$

6. Using eq. (3.13.12), we get,

$$\nabla^2\psi + \frac{2m}{\hbar^2}\left[i\hbar\frac{\partial\psi}{\partial t} - V\psi\right] = 0$$

$$\nabla^2\psi - \frac{2m}{\hbar^2}V\psi = -\frac{2m}{\hbar^2}i\hbar\frac{\partial\psi}{\partial t}$$

$$\left(\nabla^2 - \frac{2m}{\hbar^2}V\right)\psi = -\frac{2m}{\hbar^2}i\hbar\frac{\partial\psi}{\partial t}$$

or
$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial\psi}{\partial t}$$

This is required time dependent Schrodinger wave equation.

7.
$$-\frac{\hbar^2}{2m}\nabla^2 + V = H \rightarrow \text{is known as Hamiltonian operator.}$$

and,
$$i\hbar\frac{\partial\psi}{\partial t} = E\psi \rightarrow \text{energy operator.}$$

Then,
$$H\psi = E\psi$$

Que 3.14. **Discuss the Born's interpretation of wave function.**

Answer

1. Max Born interpreted the relation between the wave function $\psi(r, t)$ and the location of the particle by drawing an analogy between the intensity of light or photon beam and the intensity of electron beam.
2. Consider a beam of light (EM-wave) incident normally on a screen. The magnitude of electric field vector $\vec{E}$ of the beam is given by

$$E = E_0 \sin(kx - \omega t)$$

where, E_0 = Amplitude of the electric field.

3. For an EM-wave, the intensity I at a point due to a monochromatic beam of frequency ν is given by

$$I = c\varepsilon_0 < E^2 > \qquad ...(3.14.1)$$

Here, $<E^2>$ = Average of the square of the instantaneous magnitudes of the electric field vector of the wave over a complete cycle,
c = Velocity of light in free space, and
ε_0 = Electric permittivity of free space.

4. The intensity may also be interpreted as the number N of photons each of energy $h\nu$ crossing unit area in unit time at the point under consideration normal to the direction of the photons.
Thus,
$$I = N\,h\nu \qquad ...(3.14.2)$$

5. Comparing eq. (3.14.1) and eq. (3.14.2), we get,

$$N\,h\nu = c\varepsilon_0 < E^2 >$$

or
$$N = \frac{c\varepsilon_0}{h\nu} < E^2 >$$

or
$$N \propto < E^2 > \qquad ...(3.14.3)$$

This relation is valid only when a large number of photons are involved, *i.e.*, the beam has the large intensity.

6. If we consider the scattering of only a single photon by a crystal or the passage of only a single photon through a narrow slit, then it is impossible to observe the usual pattern of intensity variation or diffraction.
7. In this situation, we can only say that the probability of photon striking the screen is highest at places where the wave theory predicts a maximum and lowest at places where the wave theory predicts a minimum.
8. Eq. (3.14.3) shows that $< E^2 >$ is a measure of the probability of photon crossing unit area per second at the point under consideration. Hence, in one dimension, $<E^2>$ is a measure of the probability per unit length of finding the photon at the position x at time t.

Que 3.15. **The wave function of a particle confined to a box of length L is**

$$\psi(x) = \sqrt{\frac{2}{L}} \sin\frac{\pi x}{L} \qquad 0 < x < L$$

and $\psi(x) = 0$ everywhere else.
Calculate probability of finding the particle in region $0 < x < \frac{L}{2}$.

Answer

1. The probability of finding the particle in interval dx at distance x is

$$p(x)dx = |\psi|^2 dx = \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)dx$$

2. The probability in region $0 < x < \frac{L}{2}$ is

$$P = \int_0^{L/2} p(x)dx = \int_0^{L/2} \frac{2}{L}\sin^2\left(\frac{\pi x}{L}\right)dx$$

$$= \frac{1}{L}\int_0^{L/2}\left(1 - \cos\frac{2\pi x}{L}\right)dx$$

$$= \frac{1}{L}\cdot\frac{L}{2} = \frac{1}{2}$$

Que 3.16. **Discuss the stationary state solutions in brief.**

Answer

1. A state of the system in which probability distribution function $\psi\psi^*$ is independent of time is called stationary state of the system.
   where, ψ = wave function, and
   ψ^* = complex conjugate of wave function.
2. Let the probability distribution function $\psi\psi^*$ for a system in the state is given by the wave function

$$\psi(x, y, z, t) = \sum_{n=1}^{\infty} a_n f_n(x, y, z)\, e^{-iE_n t/\hbar} \qquad ...(3.16.1)$$

The complex conjugate of eq. (3.16.1) is,

$$\psi^*(x, y, z, t) = \sum_{m=1}^{\infty} a_m^* f_m^*(x, y, z)\, e^{iE_m t/\hbar} \qquad ...(3.16.2)$$

The product of ψ and ψ^* or probability distribution function $\psi\psi^*$ is given by

$$\psi\psi^* = \left[\sum_{n=1}^{\infty} a_n f_n(x, y, z)\, e^{-iE_n t/\hbar}\right]\left[\sum_{m=1}^{\infty} a_m^* f_m^*(x, y, z)\, e^{iE_m t/\hbar}\right]$$

$$= \sum a_n a_n^* f_n(x, y, z) f_n^*(x, y, z)$$

$$+ \sum_m \sum_n a_n a_m^* f_n(x, y, z) f_m^*(x, y, z)\, e^{i(E_m - E_n)t/\hbar} \qquad ...(3.16.3)$$

3. The probability distribution function that is, $\psi\psi^*$ will be independent of time only when a_n's are zero for all values except for one value of E_n. In such cases the wave function contains only a single term and expressed as

$$\psi_n(x, y, z, t) = f_n(x, y, z)\; e^{-iE_n t/\hbar} \qquad ...(3.16.4)$$

4. Since $\psi\psi^* = f_n f_n^*$ is independent of time, the solution represented by eq. (3.16.4) is stationary state solution.

Que 3.17. **Write the Schrodinger wave equation for the particle in a box and solve it to obtain the eigen value and eigen function.**

AKTU 2013-14, 2017-18

Answer

1. Let a particle is confined in one-dimensional box of length 'L'. The particle is free *i.e.*, no external force, so potential energy inside box is zero ($V = 0$).

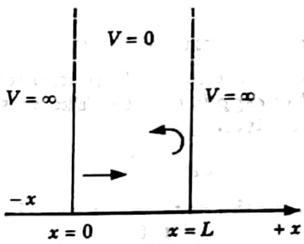

Fig. 3.17.1.

$V = 0$ for $0 < x < L$
$V = \infty$ for $x < 0$ and $x > L$

i.e., outside the box the potential energy is infinite.

2. The particle cannot exist outside the box.

$$\psi = 0 \text{ for } x = L \text{ and } x = 0$$

3. Schrodinger time independent equation for free particle ($V = 0$),

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{2m}{\hbar^2} E\psi = 0 \quad \text{...(3.17.1)}$$

or
$$\frac{\partial^2 \psi}{\partial x^2} + k^2 \psi = 0 \quad \left[\text{where, } k^2 = \frac{2mE}{\hbar^2}\right] \quad \text{...(3.17.2)}$$

4. Solution of eq. (3.17.2) is

$$\psi(x) = A \sin kx + B \cos kx \quad \text{...(3.17.3)}$$

5. Using boundary condition, $\psi = 0$ at $x = 0$

$$0 = A \sin 0 + B \cos 0$$

or
$$B = 0$$

and
$$\psi = 0 \text{ at } x = L$$

$$0 = A \sin kL + B \cos kL$$

or
$$A \sin kL = 0 \quad \text{or} \quad \sin kL = \sin n\pi \quad [\because B = 0]$$

$$kL = n\pi, \quad n = 1, 2, 3, \ldots \text{ But } n \neq 0$$

$$k = \frac{n\pi}{L}$$

6. Now eq. (3.17.3) becomes,

$$\psi_n(x) = A \sin \frac{n\pi x}{L} \quad \text{[Eigen function]}$$

and
$$\frac{n^2\pi^2}{L^2} = \frac{2mE}{\hbar^2} \Rightarrow E = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

7. Then,
$$E_n = \frac{n^2h^2}{8mL^2} \quad \text{[Eigen energy]} \quad \left[\hbar = \frac{h}{2\pi}\right]$$

8. For $n = 1$, $E_1 = \dfrac{h^2}{8mL^2}$, it is ground state energy of particle.

Que 3.18. **A particle is in motion along a line between $x = 0$ and $x = L$ with zero potential energy. At points for which $x < 0$ and $x > L$, the potential energy is infinite. The wave function for the particle in n^{th} state is given by :**

$$\psi_n = A \sin \frac{n\pi x}{L}$$

Find the expression for the normalized wave function.

OR

Derive normalization wave function.

Answer

1. The eigen function is,

$$\psi_n(x) = A \sin \frac{n\pi x}{L} \quad \text{...(3.18.1)}$$

2. Now applying normalisation condition to find constant A,

$$\int_0^L |\psi_n(x)|^2 \, dx = 1$$

$$\int_0^L A^2 \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\frac{A^2}{2} \int_0^L \left(1 - \cos \frac{2n\pi}{L} x\right) dx = 1$$

$$\frac{A^2}{2} \left[x - \frac{\sin \frac{2n\pi x}{L}}{\frac{2n\pi}{L}}\right]_0^L = 1$$

$$\frac{A^2}{2} L = 1$$

$$A = \sqrt{\frac{2}{L}}$$

3. So, eq. (3.18.1) becomes, $\psi_n(x) = \sqrt{\dfrac{2}{L}} \sin\left(\dfrac{n\pi x}{L}\right)$.

This is normalization function.

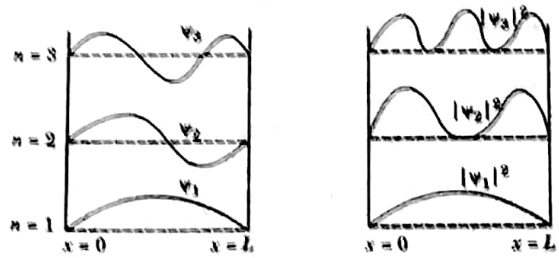

Fig. 3.18.1.

Que 3.19. **An electron is bound in one dimensional potential box which has width 2.5×10^{-10} m. Assuming the height of the box to be infinite, calculate the lowest two permitted energy values of the electron.**

AKTU 2014-15, Marks 05

Answer

Given : $L = 2.5 \times 10^{-10}$ m, $n = 1, 2$.

To Find : Lowest two permitted energy values of electron.

1. We know that,

$$E_n = \frac{n^2 h^2}{8mL^2}$$

$$E_n = \frac{n^2 (6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (2.5 \times 10^{-10})^2}$$

$[\because h = 6.63 \times 10^{-34}$ J-s, $m = 9.1 \times 10^{-31}$ kg$]$

$$= 9.66 \times 10^{-19} n^2 \text{ J}$$

$$= \frac{9.66 \times 10^{-19}}{1.6 \times 10^{-19}} \text{ eV}$$

$$= 6.037 n^2 \text{ eV}$$

2. For $n = 1$, $E_1 = 6.037$ eV
$n = 2$, $E_2 = 24.15$ eV

Que 3.20. **Compute the energy difference between the ground state and first excited state of an electron in one-dimensional box of length 10^{-8} m.**

Answer

Given : $L = 10^{-8}$m.
To Find : Energy difference between ground state and first excited state.

1. We know that eigen energy, $E_n = \frac{n^2h^2}{8mL^2}$

$$E_n = n^2 \frac{(6.63 \times 10^{-34})^2}{8 \times 9.1 \times 10^{-31} \times (10^{-8})^2} = 0.60 \times 10^{-21} n^2 \text{ J}$$

$$= \frac{0.6 \times 10^{-21} n^2}{1.6 \times 10^{-19}} \text{ eV}$$

$$E_n = 3.75 \times 10^{-3} n^2 \text{ eV}$$

2. For ground state ($n = 1$), $E_1 = 3.75 \times 10^{-3}$ eV
3. First excited state ($n = 2$), $E_2 = 0.015$ eV
4. Difference between first excited and ground state,

$$E_2 - E_1 = (15 - 3.75)\, 10^{-3} \text{ eV} = 11.25 \text{ meV}$$

Que 3.21. **A particle confined to move along x-axis has the wave function $\psi = ax$ between $x = 0$ and $x = 1.0$, and $\psi = 0$ elsewhere. Find the probability that the particle can be found between $x = 0.35$ to $x = 0.45$. Also, find the expectation value $<x>$ of particle's position.**

Answer

Given : $\psi = ax$.
To Find : i. Probability of particle can be found between $x = 0.35$ to $x = 0.45$.
ii. Expectation value.

1. The probability of finding the particle between x_1 and x_2 when it is in n^{th} state is,

$$P = \int_{x_1}^{x_2} |\psi_n|^2 dx$$

2. Here, $x_1 = 0.35$ and $x_2 = 0.45$
Therefore, $P = \int_{0.35}^{0.45} (ax)^2 dx = a^2 \int_{0.35}^{0.45} x^2 dx$

$$P = \frac{a^2}{3}\left[x^3\right]_{0.35}^{0.45} = \frac{a^2}{3}\,[(0.45)^3 - (0.35)^3]$$

$$= \frac{a^2}{3}\,[0.091125 - 0.042875] = 0.0161\, a^2$$

3. The expectation value of the position of particle is given by

$$<x> = \int_{-\infty}^{\infty} x|\psi_n(x)|^2 dx$$

4. Since, the particle is confined in a box having its limit $x = 0$ to $x = 1$ then,

$$<x> = \int_0^1 x.(ax)^2 dx = a^2 \int_0^1 x^3 dx$$

$$<x> = \frac{a^2}{4} = 0.25\, a^2$$

Que 3.22. **Determine the probabilities of finding a particle trapped in a box of length L in the region from $0.45\,L$ to $0.55\,L$ for the ground state.**

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Answer

Given : $x_1 = 0.45L$, $x_2 = 0.55L$
To Find : Probabilities of finding a particle trapped in a box.

1. The eigen function of particle trapped in a box of length L is

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

2. Probability, $P = \int_{x_1}^{x_2} |\psi_n(x)|^2 dx = \frac{2}{L}\int_{x_1}^{x_2} \sin^2 \frac{n\pi x}{L} dx$

$$P = \frac{2}{L}\int_{x_1}^{x_2} \frac{1}{2}\left(1 - \cos\frac{2\pi n x}{L}\right)dx = \frac{1}{L}\left[x - \frac{L}{2\pi n}\sin\frac{2n\pi x}{L}\right]_{x_1}^{x_2}$$

3. Since, $x_1 = 0.45\,L$ and $x_2 = 0.55\,L$, for ground state, $n = 1$

$$P = \frac{1}{L}\left[x - \frac{L}{2\pi}\sin\frac{2\pi x}{L}\right]_{0.45L}^{0.55L}$$

$$= \frac{1}{L}\left[\left(0.55L - \frac{L}{2\pi}\sin 1.1\pi\right) - \left(0.45L - \frac{L}{2\pi}\sin 0.9\pi\right)\right]$$

$$= \left[\left(0.55 - \frac{1}{2\pi}\sin 198^\circ\right) - \left(0.45 - \frac{1}{2\pi}\sin 162^\circ\right)\right]$$

$$P = 0.198362 = 19.8\ \%$$

Que 3.23. **Discuss Compton effect and derive an expression for Compton shift.**

OR

Derive an expression for Compton shift showing dependency on angle of scattering.

Answer

1. When a monochromatic beam of high frequency radiation is scattered by a substance, the scattered radiation contain two components-one having a lower frequency or greater wavelength and the other having the same frequency or wavelength.
2. The radiation of unchanged frequency in the scattered beam is known as 'unmodified radiation' while the radiation of lower frequency or slightly higher wavelength is called as 'modified radiation'.
3. This phenomenon is known as 'Compton effect'.
4. Let a photon of energy $h\nu$ collides with an electron at rest.
5. During the collision it gives a fraction of energy to the free electron. The electron gains kinetic energy and recoil as shown in Fig. 3.23.1.

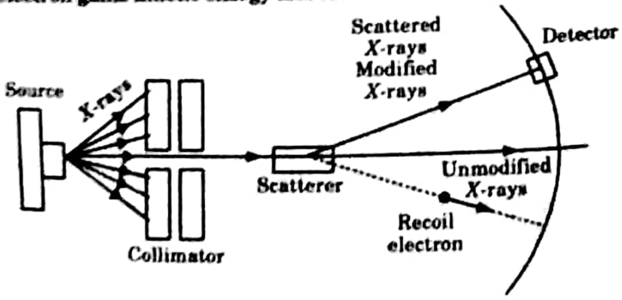

Fig. 3.23.1. Compton effect.

(a) Geometry of Compton scattering (b) Components of momentum before and after collision

Fig. 3.23.2.

a. Before collision
   i. Energy of incident photon = $h\nu$
   ii. Momentum of incident photon = $\frac{h\nu}{c}$
   iii. Rest energy of electron = m_0c^2
   iv. Momentum of rest electron = 0
b. After collision :
   i. Energy of scattered photon = $h\nu'$
   ii. Momentum of scattered photon = $\frac{h\nu'}{c}$
   iii. Energy of electron = mc^2
   iv. Momentum of recoil electron = mv

6. According to the principle of conservation of energy,

$$h\nu + m_0c^2 = h\nu' + mc^2 \quad ...(3.23.1)$$

7. Again using the principle of conservation of momentum along and perpendicular to the direction of incident, we get,
   Momentum before collision = Momentum after collision

$$\frac{h\nu}{c} + 0 = \frac{h\nu'}{c}\cos\theta + mv\cos\phi \quad ...(3.23.2)$$

$$0 + 0 = \frac{h\nu'}{c}\sin\theta - mv\sin\phi \quad ...(3.23.3)$$

8. From eq. (3.23.2), we get,

$$mvc\cos\phi = h\nu - h\nu'\cos\theta \quad ...(3.23.4)$$

9. From eq. (3.23.3), we get,

$$mvc\sin\phi = h\nu'\sin\theta \quad ...(3.23.5)$$

10. Squaring eq. (3.23.4) and eq. (3.23.5) and then adding, we get,

$$m^2v^2c^2 = (h\nu - h\nu'\cos\theta)^2 + (h\nu'\sin\theta)^2$$
$$= h^2\nu^2 - 2h^2\nu\nu'\cos\theta + h^2\nu'^2\cos^2\theta + h^2\nu'^2\sin^2\theta$$
$$= h^2[\nu^2 + \nu'^2 - 2\nu\nu'\cos\theta] \quad ...(3.23.6)$$

11. From eq. (3.23.1), we get,

$$mc^2 = h(\nu - \nu') + m_0c^2$$

Squaring, $m^2c^4 = h^2(v^2 - 2vv' + v'^2) + 2h(v - v')\,m_0c^2 + m_0^2c^4$ ...(3.23.7)

12. Subtracting eq. (3.23.6) from eq. (3.23.7), we have,

$$m^2c^4 - m^2v^2c^2 = -2h^2vv'(1-\cos\theta) + 2h(v-v')\,m_0c^2 + m_0^2c^4$$

or $$m^2c^2(c^2 - v^2) = -2h^2vv'(1-\cos\theta) + 2h(v-v')\,m_0c^2 + m_0^2c^4$$

or $$\frac{m_0^2c^2}{1-\frac{v^2}{c^2}}(c^2 - v^2) = -2h^2vv'(1-\cos\theta) + 2h(v-v')\,m_0c^2 + m_0^2c^4$$

$$\left[\because\ m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}}\right]$$

or $$m_0^2c^4 = -2h^2vv'(1-\cos\theta) + 2h(v-v')\,m_0c^2 + m_0^2c^4$$

$$\therefore\ 2h(v-v')\,m_0c^2 = 2h^2vv'(1-\cos\theta) \quad ...(3.23.8)$$

or $$\frac{v-v'}{vv'} = \frac{h}{m_0c^2}(1-\cos\theta)$$

$$\frac{1}{v'} - \frac{1}{v} = \frac{h}{m_0c^2}(1-\cos\theta) \quad ...(3.23.9)$$

13. Eq. (3.23.9) shows that $v' < v$ as m_0, c, h are the constants with positive values and the maximum value of $\cos\theta = 1$. This shows that the scattered frequency is always smaller than the incident frequency.

14. From eq. (3.23.9), we have,

$$\frac{c}{v'} - \frac{c}{v} = \frac{h}{m_0c}(1-\cos\theta)$$

or $$\lambda' - \lambda = \frac{h}{m_0c}(1-\cos\theta)$$

or $$\Delta\lambda = \frac{2h}{m_0c}\sin^2\frac{\theta}{2} \quad ...(3.23.10)$$

15. From eq. (3.23.10), it is noted that Compton shift depends on angle of scattering.

Que 3.24. **Explain the experimental verification of Compton effect.**

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Answer

1. The apparatus used by Compton for experimental verification of Compton effect is shown in Fig. 3.24.1.

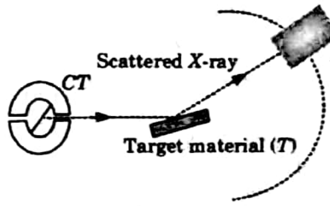

Fig. 3.24.1. Experimental verification of Compton effect.

2. Monochromatic X-rays of wavelength λ from a Coolidge tube CT are allowed to fall on a target material T such as a small block of carbon.
3. The scattered X-rays of wavelength λ' are received by a Bragg's spectrometer BS, which can move along the arc of a circle.
4. The wavelength of the scattered X-rays is measured for different values of the scattering angle.

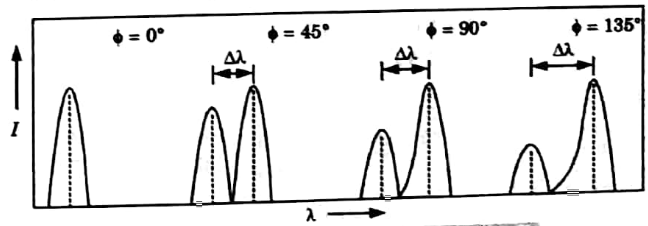

Fig. 3.24.2. Intensity of scattered X-rays.

5. The plots of intensity of scattered X-rays against their wavelength are shown in Fig. 3.24.2 for $\phi = 0°, 45°, 90°$ and $135°$.
6. These plots show the two peaks for non-zero values of ϕ, which is the indication of the presence of two distinct lines in the scattered X-ray radiation.
7. One of these lines is known as the unmodified line which has the same wavelength as the incident radiation and the other line is known as the modified line which has the comparatively longer wavelength. The Compton shift $\Delta\lambda$ is found to vary with the angle of scattering.

Que 3.25. **Why Compton shift is not observed with visible light ?**

Answer

1. The energy of a visible light photon say of wavelength λ = 6000 Å ($= 6 \times 10^{-7}$ m) is given by

$$E = hv = \frac{hc}{\lambda}$$

$$= \frac{(6.6 \times 10^{-34} \times 3 \times 10^{8})}{(6 \times 10^{-7})} \text{ J}$$

$$= \frac{(6.6 \times 10^{-34} \times 3 \times 10^{8})}{(1.6 \times 10^{-19} \times 6 \times 10^{-7})} \text{ eV}$$

$$= 2.06 \text{ eV} \approx 2 \text{ eV}$$

Whereas the energy of X-ray photon, say of wavelength $\lambda = 1$ Å $= 10^{-10}$ m will be more than 1000 times the above value.

2. The binding energy of the electron in the atoms is of the order of the 10 eV. For example, the binding energy of the electron in the hydrogen atom is,

$$E_b = \frac{2\pi^2 k^2 Z^2 m_0 e^4}{h^2}$$

Where, $k = 1/4\pi\varepsilon_0 = 9 \times 10^9$ N-m²/C²,

$m_0 = 9.1 \times 10^{-31}$ kg,

$h = 6.6 \times 10^{-34}$ J-s,

$e = 1.6 \times 10^{-19}$ C,

$Z = 1$.

Thus, $$E_b = \frac{2\pi^2 (9 \times 10^9)^2 (9.1 \times 10^{-31})(1.6 \times 10^{-19})^4}{(6.6 \times 10^{-34})^2} \text{ J}$$

$$= \frac{2\pi^2 (81 \times 10^{18})(9.1 \times 10^{-31})(1.6 \times 10^{-19})^4}{(6.6 \times 10^{-34})^2 (1.6 \times 10^{-19})} \text{ eV}$$

$$= 13.68 \text{ eV}$$

Hence, this electron can be treated as free when X-rays are incident but this electron cannot be treated as free for visible light. So, the Compton effect cannot be observed for visible light.

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