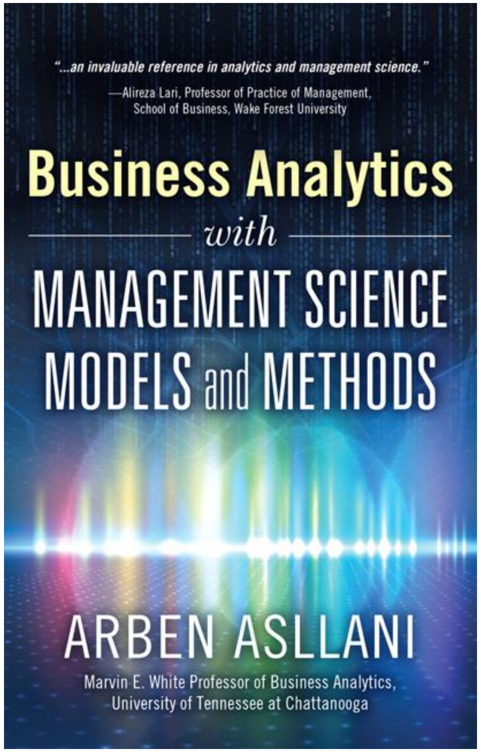
Business Analytics Prescriptive Models



Based on

Business Analytics
With
Management Science
Models and Methods
by
Arben Asllani

Chapter 2

Introduction to Linear Programming

Business Analytics with Management Science Models and Methods

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Chapter Outline

- Chapter Objectives
- Chevron Optimizes Processing of Crude Oil
- Introduction
- LP Formulation
- Solving LP Models: A Graphical Approach
- Possible Outcome Solutions to LP Model
- Solving LP Problems with Solver
- Wrap up!

Chapter Objectives

- Discuss the importance of linear programming as a business analytics tool in business organizations
- Explore the components of linear programming models and relate them to business goals and constraints
- Provide a step by step formulation methodology for linear programming models
- Solve linear programming models graphically
- Solve linear programming models with Excel's Solver Add in
- Understand the constructs of linear programming modeling, its formulation and solutions
- Discuss challenges of using linear programming models in everyday decision making

Chevron Optimizes Processing of Crude Oil

- *World's leading integrated energy companies*
- *Production systems are limited by thousands of operational constraints:*
- *Designed Petro, an in-house LP software tool*
  - *Used by 25--30 people each day*
  - *400 to 500 variables*
  - *Build models within a few seconds*
  - *Execute those cases in fractions of seconds*
- *$600 million per year from operating the business*
- *$400 million per year by enabling better decisions in capital allocation*
- *$10 billion over the past 30 years*

Introduction to LP

- One of the most commonly used techniques
  - Objective function
  - Set of constraints
- Mathematical approach: the assumptions of linearity
  - proportionality
  - additivity
- The assumption of certainty and divisibility

Introduction to LP

Algebraically, linear equations must have simple variables, such as “x" or "y", but not x^2, or xy. Assume that a production line produces two products: *tables* and *chairs*. If the production cost for one table is $50 and the line produces x units of them, then the total production cost for tables is 50x. That is a *linear* relationship. If the production cost for one chair is $30 and the line produces y units of them, then total production costs for chairs is 30y.

This is another linear function. The total production cost for both tables and chairs can then be expressed as an additive linear function: 50x + 30y. If the production line were to increase its capacity by ten percent, the production cost would increase by ten percent as well. That is *proportionality* in a linear relationship.

Introduction to LP

Similarly, linear equations can be constructed to represent constraints. Suppose that the same production line is constrained by a limited number of labor hours available every week. If the production uses five hours of labor for a table and four hours of labor for a chair, then the total labor hours used for tables is $5x$ and the total labor hours for chairs is $4y$. The total use of labor can then be expressed as a linear function: $5x + 4y$. If the production line were to produce more tables and chairs, then the labor hours used would increase *proportionally* and *additively*.

LP Formulation: Example 1

- Rolls Bakery produces two products: DRC, SRC
  - Total of 150 machine hours avaliable, at labor cost $10/h
  - Each is produced in lots of 1000 cases
  - Demand: minimum 3000 cases of DRCs and 4000 of SRCs

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Product	Wholesale Price per Case	Wholesale Price per Lot	Processing time (in hours) per Lot	Cost of Raw Materials per Lot	**Net Profit per Lot**	Demand for Cases	**Demand for Production Lots**
DRC	$0.75	$750	10	$250	**?**	3000	**?**
SRC	$0.65	$650	15	$200	**?**	4000	**?**

- How many lots should run every week to maximize profit?

LP Formulation: Example 1

- Rolls Bakery produces two products: DRC, SRC
  - net profit for a lot of DRCs is: $750 - ($250 + 10h*$10/h) = $400
  - net profit for a lot of SRCs is: $650 - ($200 + 15h*$10/h) = $300
  - demand for DRCs: 3000/1000 = 3 lots
  - demand for SRCs: 4000/1000 = 4 lots

	(1)	(2)	(3)		(5)	(6)	(7)
Product	Wholesale Price per Case	Wholesale Price per Lot	Processing time (in hours) per Lot	Cost of Raw Materials per Lot	**Net Profit per Lot**	Demand for Cases	**Demand for Production Lots**
DRC	$0.75	$750	10	$250	$400	3000	**3**
SRC	$0.65	$650	15	$200	$300	4000	**4**

LP Formulation Steps

- Define decision variables
  - Look for the decision variables in the problem description
  - Include a time framework in the definition
- Formulate an objective function
  - Define whether the goal is to maximize or minimize
  - Identify contribution coefficients
  - Create the equation for the objective function
- Identify a set of constraints
  - Identify the right-hand side of the constraint
  - Express the left- hand side in the form of an equation
  - Select the directions of the constraint ($=$, $\le$, $\ge$) But not ($<$, $>$)!
- Identify a set of non-negativity constraints
  - All decision variables must be non-negative

LP Overall Formulation: Ex.1

x_1 – number of DRC lots to be produced during the week
x_2 – number of SRC lots to be produced during the week

Max $Z = 400x_1 + 300x_2$

Subject to:

- $10x_1 + 15x_2 \leq 150$ (Constraint 1)
- $x_1 \geq 3$ (Constraint 2)
- $x_2 \geq 4$ (Constraint 3)
- $x_1, x_2 \geq 0$ (Non-negativity Constraints)

LP Formulation: Example 2

- Political Communication: specializes in political campaigns
  - To reach at least 1,000,000 voters every week
  - One TV advertisement reaches 25,000 viewers
  - One radio advertisement reaches 12,500 listeners
  - TV spot is $1,200
  - Radio spot is $400
  - Minimum 30 TV spots per week
  - No more than 60 radio spots per week
- How many TV and radio spots must be purchased every week to minimize campaign cost?

LP Formulation: Example 2

- Political Communication: specializes in political campaigns
- Follow the formulation steps:
  - Step 1: Define decision variables
  - Step 2: Formulate the objective function
  - Step 3: Identify Constraints
  - Step 4: Identify non-negativity constraints

LP Overall Formulation: Ex. 2

x_1 – number of TV spots to be purchased during the week
x_2 – number of radio spots to be purchased during the week

Min $Z = 1200x_1 + 400x_2$

Subject to:

- $25000x_1 + 12500x_2 \geq 1000000$ (Voter reach constraint)
- $x_1 \geq 30$ (Weekly TV spots constraint)
- $x_2 \leq 60$ (Weekly radio spots constraint)
- $x_1, x_2 \geq 0$ (Non-negativity constraints)

Graphical Approach

1. Graph the area of feasible solutions
2. Graph the objective function
3. Find the coordinates for the optimal point
4. Find the value of the objective function at the optimal solution

Rolls Bakery Production Run

Max $Z = 400x_1 + 300x_2$

Subject to:

- $10x_1 + 15x_2 \leq 150$ (Constraint 1)
- $x_1 \geq 3$ (Constraint 2)
- $x_2 \geq 4$ (Constraint 3)
- $x_1, x_2 \geq 0$ (Non-negativity Constraints)

Rolls Bakery Production Run

Constraints

(1) $10x_1 + 15x_2 = 150 \Longrightarrow \begin{cases} x_1 = 0 \Longrightarrow 15x_2 = 150 \Longrightarrow x_2 = 10 \Rightarrow \begin{cases} x_1 = 0 \\ x_2 = 10 \end{cases} \\ x_2 = 0 \Longrightarrow 10x_1 = 150 \Longrightarrow x_1 = 15 \Rightarrow \begin{cases} x_1 = 15 \\ x_2 = 0 \end{cases} \end{cases}$

(2) $x_1 = 3$

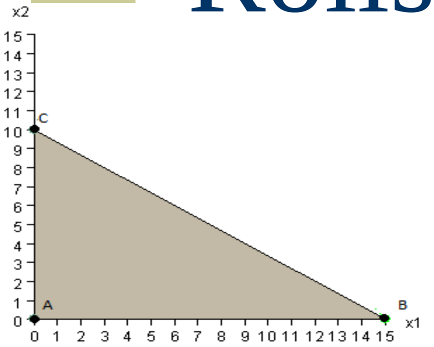
(3) $x_2 = 4$

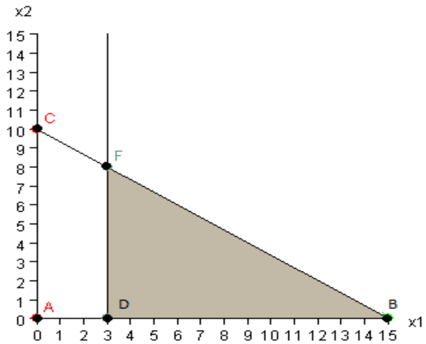
Objective function

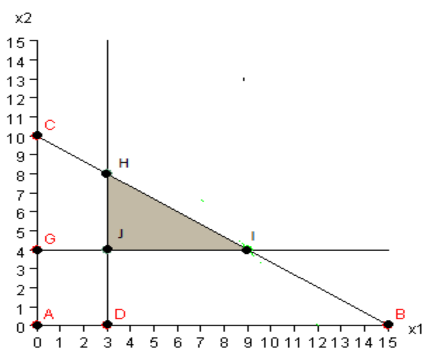
$$\begin{cases} x_1 = 0 \\ x_2 = 0 \end{cases} \Longrightarrow Z = 400 \cdot 0 + 300 \cdot 0 = 0$$

$$x_1 = -3 \Longrightarrow\ Z = 400 \cdot (-3) + 300 \cdot x_2 = 0 \Rightarrow 300x_2 = 1200 \Longrightarrow x_2 = 4 \Rightarrow \begin{cases} x_1 = -3 \\ x_2 = 4 \end{cases}$$

Rolls Bakery Production Run

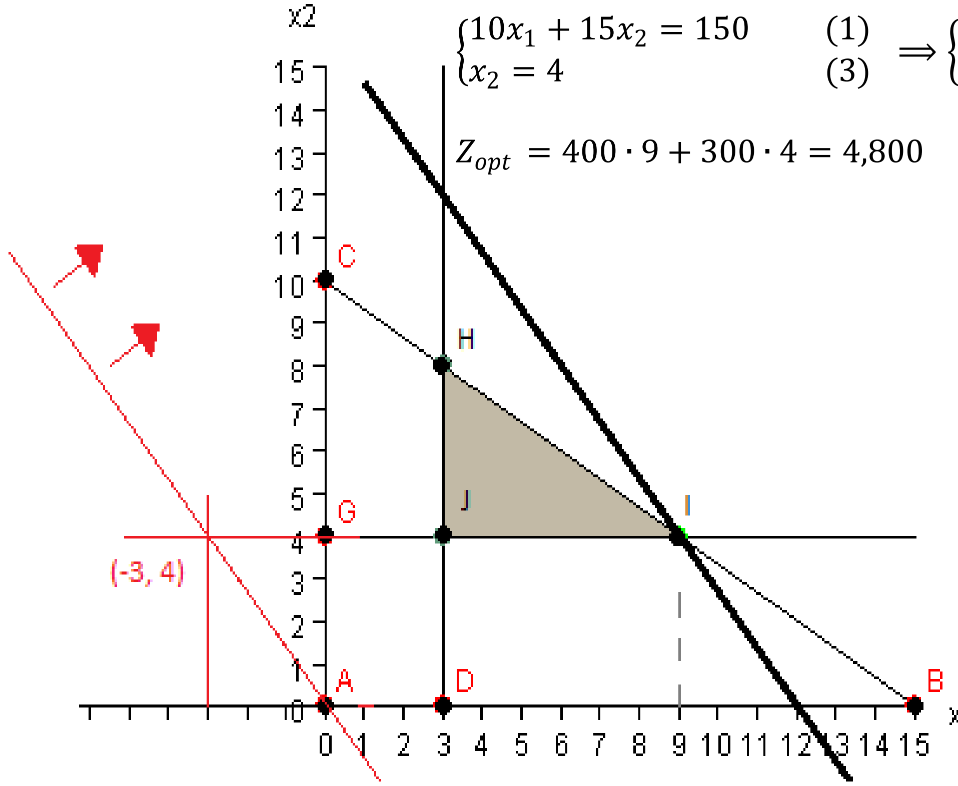






$$\begin{cases}10x_1 + 15x_2 = 150 & (1)\\ x_2 = 4 & (3)\end{cases} \Longrightarrow \begin{cases}x_1 = 9\\ x_2 = 4\end{cases}$$

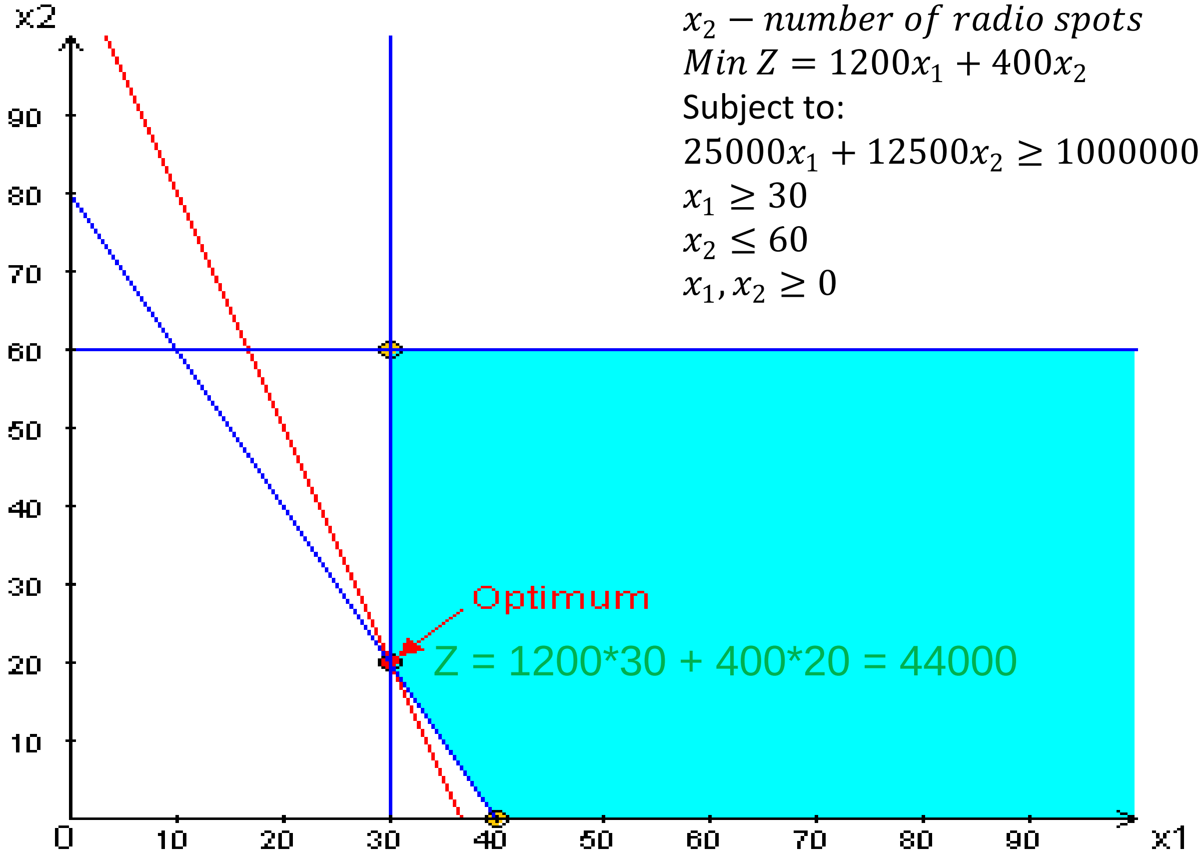
$$Z_{opt} = 400 \cdot 9 + 300 \cdot 4 = 4{,}800$$



Optimal decision: 9 lots of DRCs and 4 lots of SRCs, what gives $4,800 net profit

PoliCom Campaign

$x_1 - number\ of\ TV\ spots$
$x_2 - number\ of\ radio\ spots$
$Min\ Z = 1200x_1 + 400x_2$
Subject to:
$25000x_1 + 12500x_2 \geq 1000000$
$x_1 \geq 30$
$x_2 \leq 60$
$x_1, x_2 \geq 0$



Optimal values for the number of TV spots and Radio spots are: $x_1 = 30$ and $x_2 = 20$.

PoliCom Campaign

Step 1: Graph the area of feasible solutions

The non-negativity constraints require that both decision variables remain positive, so the area of feasible solution always remains in the upper-right side of the Cartesian axes where all values for x_1 and x_2 are positive as shown above.

The first constraint $25000x_1 + 12500x_2 \leq 1000000$ can be graphed as follows:

Assign $x_1 = 0$; This results in $12500x_2 = 1000000$ which makes $x_2 = 80$

Assign $x_2 = 0$; This results in $25000x_1 = 1000000$ which makes $x_1 = 40$

Thus, constraint 1 will go through two points: (0, 80) and (40, 0). Since the left-hand side (LHS) of the equation for Constraint 1 is greater than or equal to the right-hand side (RHS) value, then the area of feasible solution will be on the upper right side of the line. Constraint 2 $(x_1 \geq 30)$ is represented by the vertical line which goes through point $(x_1 = 30, x_2 = 0)$. Constraint 3 $(x_2 \leq 60)$ is represented by the horizontal line which goes through point $(x_1 = 0, x_2 = 60)$. The shaded area shows the feasible solution. Every point in this area satisfies all three constraints. The objective function is used to find the point where the value of objective function (cost of campaign) is minimized.

PoliCom Campaign

Step 2: Graph the objective function

Assigning a value of zero to the objective function (Z=0) leads to the following equation:

$1200x_1 + 400x_2 = 0$

One point in this line is the origin: $x_1 = 0 \; then \; 400x_2 = 0$ and $x_2 = 0$

Another point can be identified by simply assigning a value to any of the decision variables.

For example if $x_2 = 30$ then $1200x_1 + 400 \cdot 30 = 0$ which leads to:

$1200x_1 + 12000 = 0$, thus $x_1 = -10$

As a result, the objective function line will connect the following two points:

(0, 0) and (-10, 30)

A more practical way to create a starting line for the objective function is to equate the value of Z to the product of the contribution coefficients or to a common factor of them. For example, if Z = 48000 then:

$x_1 = 0$ then $400x_2 = 48000$ and $x_2 = 120$

$x_2 = 0$ then $1200x_1 = 48000$ which leads to $x_1 = 40$

The objective function line will now connect the following two points: (0, 120) and (40, 0).

The set of lines representing the objective functions at different values for Z are parallel to each other. The line representing the objective function is known as the iso-profit (or in this case iso-cost) line, since all points in the line indicate a combination of coordinates, which result in an equal level of profit or cost.

PoliCom Campaign

Step 3: Find the coordinates for the optimal point

The goal is to move the line parallel to itself until it reaches the lowest point of the feasible area. In the example under consideration, the optimal solution occurs at one of the corners of the feasible region. As shown, the last point of contact (when moving downward) of the objective function with the area of feasible solution is (30, 20). That means that the optimal values for the number of TV spots and Radio spots are: $x_1 = 30$ and $x_2 = 20$.

Step 4: Find the value of the objective function at the optimal point

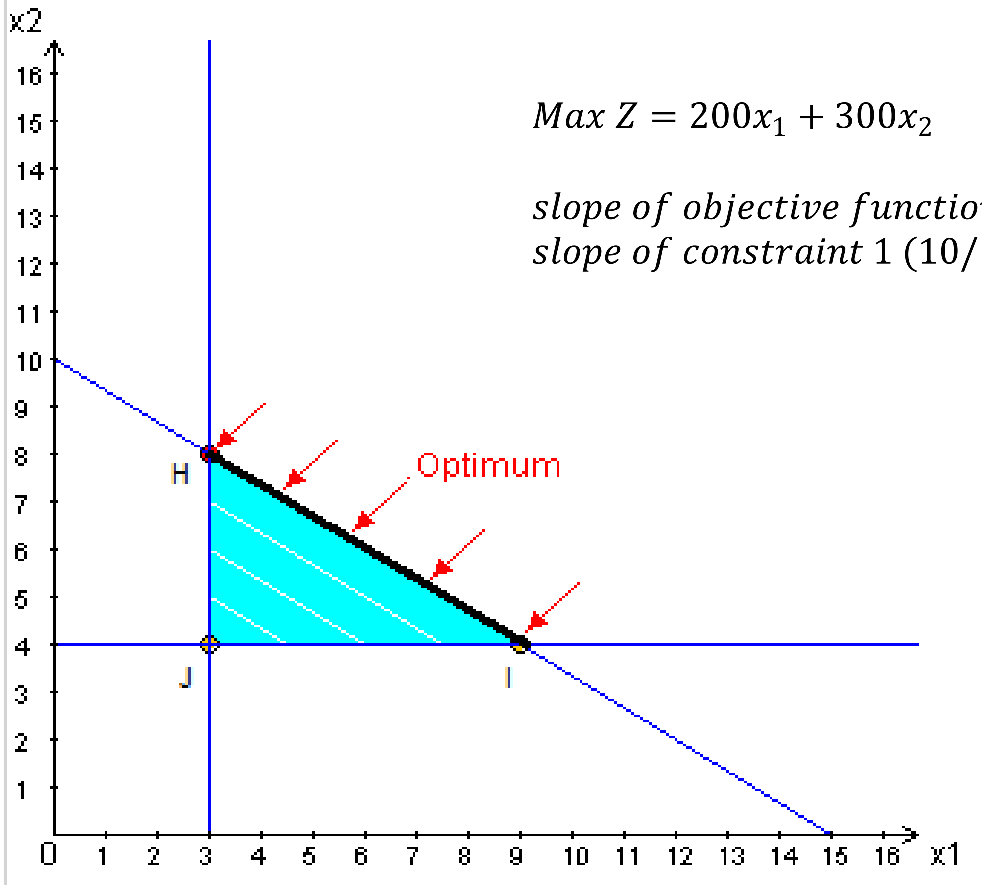
The values of x_1 and x_2 found in the previous step are substituted into the objective function to find the value of the objective function at the optimal solution. The minimum value of the objective function is: $Z = 1200 \cdot 30 + 400 \cdot 20 = 44000$. As shown graphically, in order to minimize the advertisement cost, PoliCom needs to run 30 TV spots and 20 radio spots on a weekly basis. This will ensure that that the campaign reaches at least one million users and limits the number of radio spots to 60 per week and assures that the company offers no fewer than 30 TV spots.

Possible Outcome Solutions to LP Models

1. Multiple Solutions

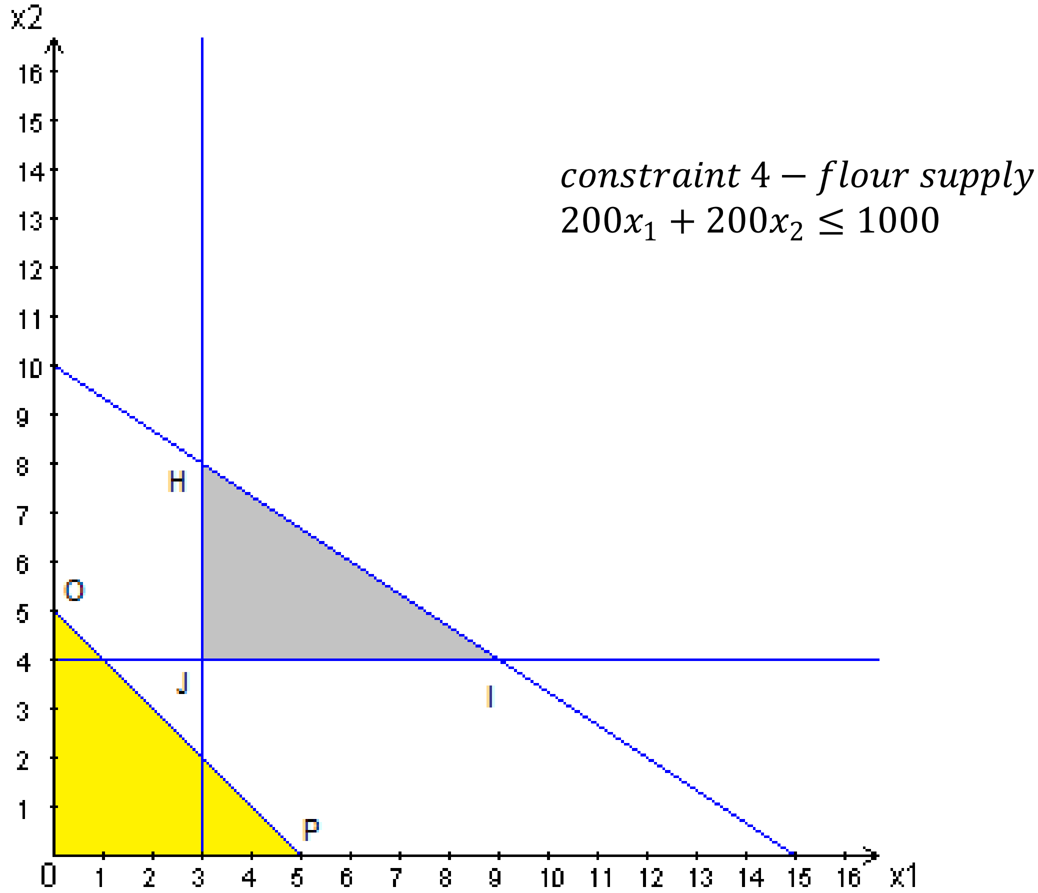
$$Max\ Z = 200x_1 + 300x_2$$

$slope\ of\ objective\ function\ (200/300)$
$slope\ of\ constraint\ 1\ (10/15)$



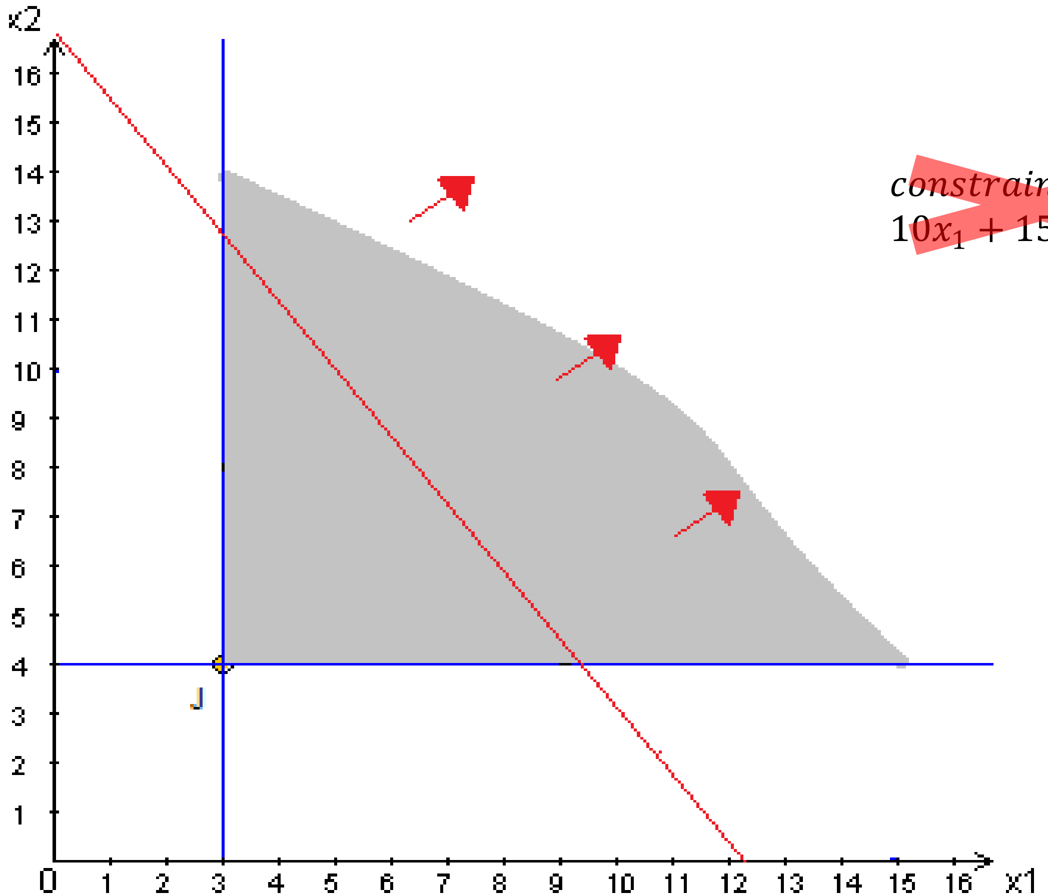
Possible Outcome Solutions to LP Models

2. No Solutions



Possible Outcome Solutions to LP Models

3. Unbounded Solutions



~~*constraint* 1~~
$\cancel{10x_1 + 15x_2 \leq 150}$

Solving Rolls Bakery Problem with Solver

1. Excel Template

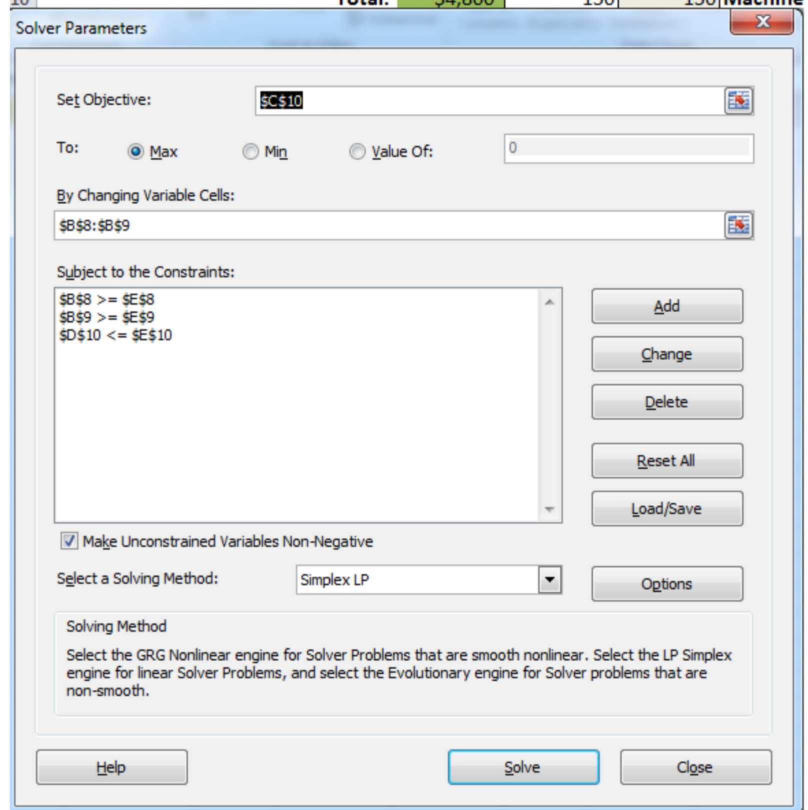
	A	B	C	D	E	F	G	H
1	Rolls Bakery Production Information							
2	Products	Wholesale Price per Case	Wholesale Price per Lot	Processing Time (in hours) per Lot	Cost of Raw Materials per	Net Profit per Lot	Demand for Cases	Demand for Production Lots
3	Dinner Roll Case (DRC)	$0.75	$750	10	$250	$400	3000	3
4	Sandwich Roll Case (SRC)	$0.65	$650	15	$200	$300	4000	4
5								
6	What-If Analysis							
7	Product	Number of Lots per Week	Net Profit Per Lot	Total Machine Time Used (in hours)	Right hand-side values			
8	Dinner Roll Case Lots	1	$400	10	3	Minimum DRC (Constraint 2)		
9	Sandwich Roll Case Lots	1	$300	15	4	Minimum SRC (Constraint 3)		
10	Total:		$700	25	150	Machine Hours (Constraint 1)		

Ctrl + ~ shows formulas

	A	B	C	D	E	F	G	H
1	Rolls Bakery Production Information							
2	Products	Wholesale Price per Case	Wholesale Price per Lot	Processing Time (in hours) per Lot	Cost of Raw Materials per	Net Profit per Lot	Demand for Cases	Demand for Production Lots
3	Dinner Roll Case (DRC)	0.75	=B3*1000	10	250	=C3-D3*10-E3	3000	=G3/1000
4	Sandwich Roll Case (SRC)	0.65	=B4*1000	15	200	=C4-D4*10-E4	4000	=G4/1000
5								
6	What-If Analysis							
7	Product	Number of Lots per Week	Net Profit Per Lot	Total Machine Time Used (in hours)	Right hand-side values			
8	Dinner Roll Case Lots	1	=B8*F3	=B8*D3	=H3	Minimum DRC (Constraint 2)		
9	Sandwich Roll Case Lots	1	=B9*F4	=B9*D4	=H4	Minimum SRC (Constraint 3)		
10	Total:		=C8+C9	=D8+D9	150	Machine Hours (Constraint 1)		

Solving Rolls Bakery Problem with Solver

	A	B	C	D	E	F	G	H
1	**Rolls Bakery Production Information**							
2	Products	Wholesale Price per Case	Wholesale Price per Lot	Processing Time (in hours) per Lot	Cost of Raw Materials per Lot	Net Profit per Lot	Demand for Cases	Demand for Production Lots
3	Dinner Roll Case (DRC)	$0.75	$750	10	$250	$400	3000	3
4	Sandwich Roll Case (SRC)	$0.65	$650	15	$200	$300	4000	4
5								
6	**What-If Analysis**							
7	Product	Number of Lots per Week	Net Profit Per Lot	Total Machine Time Used	Right hand-side values			
8	Dinner Roll Case Lots	9	$3,600	90	3	**Minimum DRC (Constraint 2)**		
9	Sandwich Roll Case Lots	4	$1,200	60	4	**Minimum SRC (Constraint 3)**		
10	**Total:**		$4,800	150	150	**Machine Hours (Constraint 1)**		



2. Apply Solver

Solving Rolls Bakery Problem with Solver

3. Interpret Solver Solution
   - Objective Cell
   - Variable Cells
   - The Constraints

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$C$10	Total: Net Profit Per Lot	$700	$4,800

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$8	Dinner Roll Case Lots Number of Lots per Week	1	9	Contin
$B$9	Sandwich Roll Case Lots Number of Lots per Week	1	4	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$10	Total: Total Machine Time Used (in hours)	150	$D$10<=$E$10	Binding	0
$B$8	Dinner Roll Case Lots Number of Lots per Week	9	$B$8>=$E$8	Not Binding	6
$B$9	Sandwich Roll Case Lots Number of Lots per Week	4	$B$9>=$E$9	Binding	0

Solving PoliCom Problem with Solver

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$E$10	Total: Cost	$1,600	$44,000

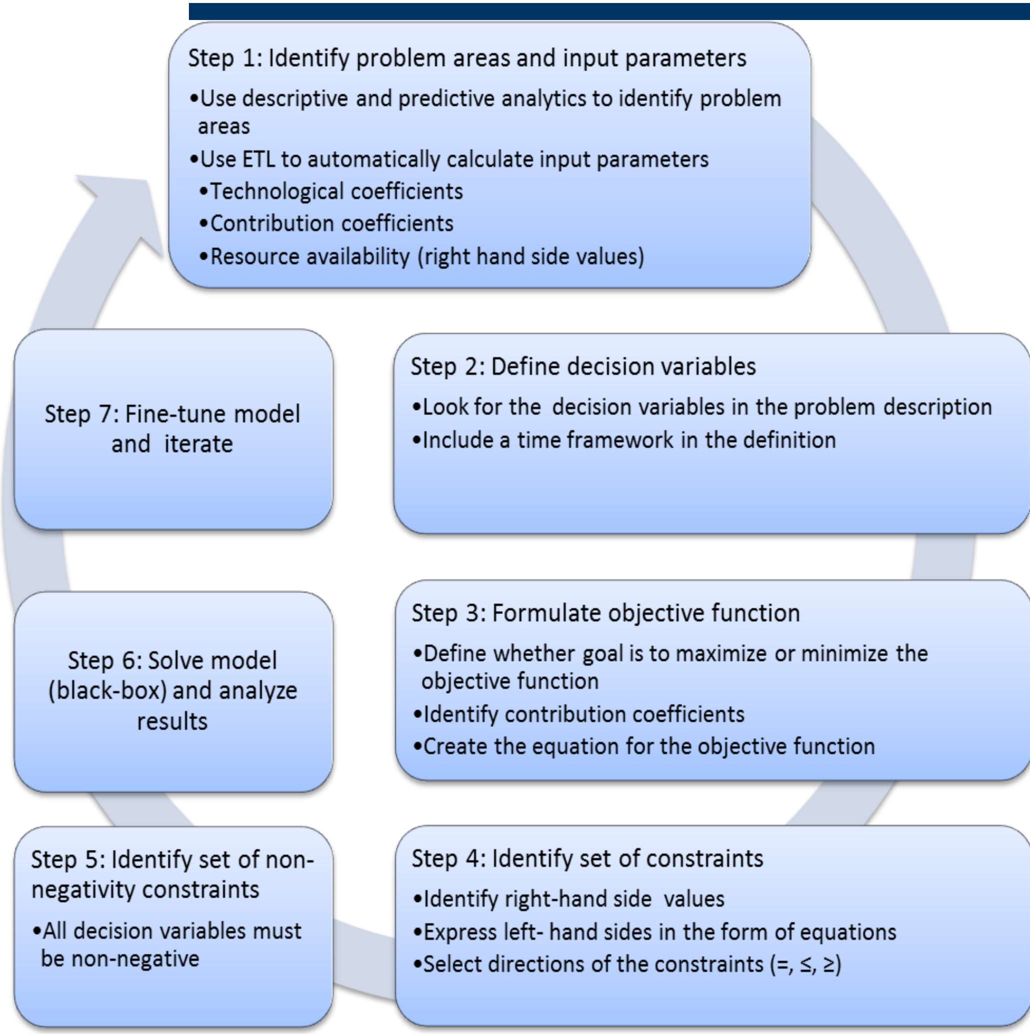
Variable Cells

Cell	Name	Original Value	Final Value	Integer
$C$8	TV Number of Media Ads per Week	1	30	Contin
$C$9	Radio Number of Media Ads per Week	1	20	Contin

Constraints

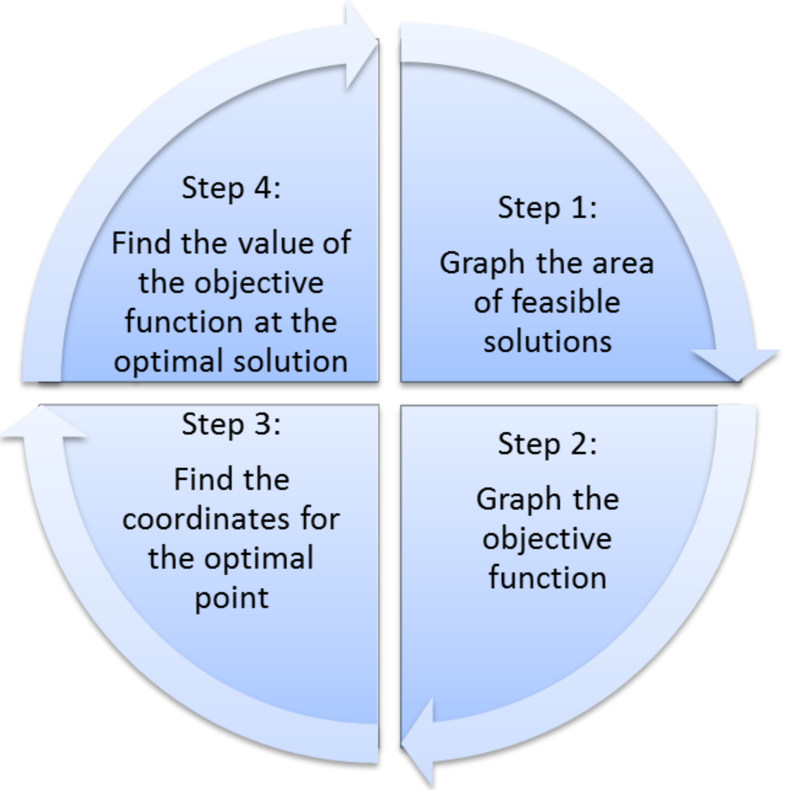
Cell	Name	Cell Value	Formula	Status	Slack
$D$10	Total: Voter Reach	1000000	$D$10>=$F$10	Binding	0
$C$8	TV Number of Media Ads per Week	30	$C$8>=$F$8	Binding	0
$C$9	Radio Number of Media Ads per Week	20	$C$9<=$F$9	Not Binding	40

Exploring Big Data with LP Models



Wrap Up

- 1. Solve LP Models graphically



Wrap Up

- 2. Solve LP Models with Solver

Stage 1: Create a template

- Build a “what-if” template for a trial solution
- Use a formula to calculate the profit/cost
- Use formula to calculate the left-hand side value

Stage 2: Apply solver

- Activate solver dialog box
- Set the cell holding profit/cost as the objective
- Select max or min for the objective
- Add constraints accordingly
- Select non-negativity for the decision variables
- Choose simplex LP method
- Click solve

Stage 3: Interpret solver solution

- Interpret the results for the objective cell
- Interpret the results for the variable cells
- Interpret the results for the constraints

End of lecture 2

Thank You