MATH 1210 A01 Winter 2013 Problem Workshop 1 Solutions

1. Part A:

   When $n = 1$, the left hand side is 2. The right hand side is

   $$\frac{1}{2}(3(1)^2 + 1) = \frac{1}{2}(4) = 2.$$

   Therefore the formula is true for $n = 1$.

   Part B:

   Suppose the formula is true for $n = k$, that is

   $$2 + 5 + 8 + \cdots + (3k - 1) = \frac{3k^2 + k}{2}.$$

   We need to show that

   $$2 + 5 + 8 + \cdots + (3(k + 1) - 1) = \frac{3(k + 1)^2 + (k + 1)}{2} = \frac{3k^2 + 7k + 4}{2}.$$

   The left hand side is

   $$\begin{aligned} 2 + 5 + 8 + \cdots + (3k + 2) &= \big(2 + 5 + 8 + \cdots + (3k - 1)\big) + (3k + 2) \\ &= \frac{3k^2 + k}{2} + 3k + 2 \\ &= \frac{3k^2 + k}{2} + \frac{6k + 4}{2} \\ &= \frac{3k^2 + 7k + 4}{2} \end{aligned}$$

   which is equal to the right hand side. Hence the formula is true for $n = k + 1$. By the principle of mathematical induction the formula is true for all $n \geq 1$.

2. Let $p(n) = n^3 + 9n^2 + 26n + 24$. Part A:

   When $n = 1$, $p(n)$ is equal to $1^3 + 9(1^2) + 26(1) + 24 = 1 + 9 + 26 + 24 = 60$ which is a multiple of 6. Hence 6 divides $p(n)$ when $n = 1$.

   Part B:

   Suppose that 6 divides $p(n)$ for $n = k$, that is

   $$k^3 + 9k^2 + 26k + 24 = 6l$$

   for some integer l. 6 divides $p(n)$ for $n = k + 1$, that is

   $$(k + 1)^3 + 9(k + 1)^2 + 26(k + 1) + 24 = 6L$$

for some other integer L. The left hand side is

$$\begin{aligned}(k+1)^3+9(k+1)^2+26(k+1)+24 &= (k^3+3k^2+3k+1)+9(k^2+2k+1)+26(k+1)+24\\ &= k^3+12k^2+47k+60\\ &= (k^3+9k^2+26k+24)+(3k^2+21k+36)\\ &= 6l+3(k^2+7k+12)\\ &= 6l+3(k+3)(k+4)\end{aligned}$$

You could show separately by induction that $(k+3)(k+4)$ must be divisible by 2 however since $k+3$ and $k+4$ are consecutive integers, one of them must be even, and hence $(k+3)(k+4)$ is divisible by 2. Thus it can be written as $2K$ for some integer K.

Therefore

$$(k+1)^3+9(k+1)^2+26(k+1)+24 = 6l+3(k+3)(k+4) = 6l+3(2K) = 6(l+K)$$

and therefore $p(k+1)$ is divisible by 6. By the principle of mathematical induction $p(n)$ is divisible by 6 for all $n \geq 1$.

3. Part A:

When $n=1$, the left hand side is $1(3)=3$. The right hand side is

$$\frac{1}{4}\big((2(1)-1)3^{1+1}+3\big) = \frac{1}{4}(12) = 3.$$

Therefore the formula is true for $n=1$.

Part B:

Suppose the formula is true for $n=k$, that is

$$1(3)+2(3^2)+\cdots+k(3^k) = \frac{(2k-1)3^{k+1}+3}{4}.$$

We need to show it's true for $n=k+1$, that is

$$1(3)+2(3^2)+\cdots+(k+1)(3^{k+1}) = \frac{(2(k+1)-1)3^{k+1+1}+3}{4} = \frac{(2k+1)3^{k+2}+3}{4}.$$

The left hand side is

$$
\begin{aligned}
LHS &= 1(3) + 2(3^2) + \cdots + (k+1)(3^{k+1}) \\
&= 1(3) + 2(3^2) + \cdots + +k(3^k) + (k+1)(3^{k+1}) \\
&= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)(3^{k+1}) \\
&= \frac{(2k-1)3^{k+1}+3}{4} + \frac{(4k+4)(3^{k+1})}{4} \\
&= \frac{(6k+3)3^{k+1}+3}{4} \\
&= \frac{(2k+1)3^{k+2}+3}{4}
\end{aligned}
$$

which is equal to the right hand side. Hence the formula is true for $n = k+1$. By the principle of mathematical induction the formula is true for all $n \geq 1$.

4. Part A:

When $n = 1$, the left hand side is $1^2 + 2^2 + 3^2 = 14$. The right hand side is

$$\frac{1}{2}(1)(4)(7) = 14.$$

Therefore the formula is true for $n = 1$.

Part B:

Suppose the formula is true for $n = k$, that is

$$1^2 + 2^2 + 3^2 + \cdots + (3k)^2 = \frac{k(3k+1)(6k+1)}{2}.$$

We need to show that it's true for $n = k+1$, that is

$$1^2+2^2+3^2+\cdots+(3(k+1))^2 = \frac{(k+1)(3(k+1)+1)(6(k+1)+1)}{2} = \frac{(k+1)(3k+4)(6k+7)}{2}.$$

The left hand side is

$$
\begin{aligned}
LHS &= 1^2 + 2^2 + 3^2 + \cdots + (3(k+1))^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (3k)^2 + (3k+1)^2 + (3k+2)^2 + (3k+3)^2 \\
&= \frac{k(3k+1)(6k+1)}{2} + (3k+1)^2 + (3k+2)^2 + (3k+3)^2 \\
&= \frac{k(3k+1)(6k+1)}{2} + 27k^2 + 36k + 14 \\
&= \frac{18k^3 + 9k^2 + k}{2} + \frac{54k^2 + 72k + 28}{2} \\
&= \frac{18k^3 + 63k^2 + 73k + 28}{2} \\
&= \frac{(k+1)(3k+4)(6k+7)}{2}
\end{aligned}
$$

which is equal to the right hand side. Hence the formula is true for $n = k+1$. By the principle of mathematical induction the formula is true for all $n \geq 1$.

5. Part A:

   When $n = 1$, the left hand side is $3+5 = 8$. The right hand side is $3(1^2)+4(1)+1 = 8$. Therefore the formula is true for $n = 1$.

   Part B:

   Suppose the formula is true for $n = k$, that is

   $$(2k+1) + (2k+3) + (2k+5) + \cdots + (4k+1) = 3k^2 + 4k + 1.$$

   We need to show that

   $$
   \begin{aligned}
   (2(k+1)+1) + (2(k+1)+3) + \cdots + (4(k+1)+1) &= 3(k+1)^2 + 4(k+1) + 1 \\
   &= 3k^2 + 6k + 3 + 4k + 4 + 1 \\
   &= 3k^2 + 10k + 8.
   \end{aligned}
   $$

   The left hand side is

   $$
   \begin{aligned}
   LHS &= (2k+3) + (2k+5) + \cdots + (4k+5) \\
   &= (2k+3) + (2k+5) + \cdots + (4k+1) + (4k+3) + (4k+5).
   \end{aligned}
   $$

   However we don't have the $2k+1$ term at the beginning. Therefore we will add it in (and also subtract it)

$$
\begin{aligned}
LHS &= (2k+3)+(2k+5)+\cdots+(4k+1)+(4k+3)+(4k+5). \\
&= \big((2k+1)+(2k+3)+(2k+5)+\cdots+(4k+1)\big)+(4k+3)+(4k+5)-(2k+1) \\
&= \big((2k+1)+(2k+3)+(2k+5)+\cdots+(4k+1)\big)+6k+7 \\
&= \big(3k^2+4k+1\big)+6k+7 \\
&= 3k^2+10k+8.
\end{aligned}
$$

which is equal to the right hand side. Hence the formula is true for $n = k+1$. By the principle of mathematical induction the formula is true for all $n \geq 1$.

6. Part A:

   When $n = 1$, the left hand side is $a^2 - 1$. The right hand side is $(a-1)(a+1) = a^2 - 1$. Therefore the formula is true for $n = 1$.

   Part B:

   Suppose the formula is true for $n = k$, that is

$$a^{k+1} - 1 = (a-1)(a^k + \cdots + a + 1).$$

   We need to show that it's true for $n = k+1$, that is

$$a^{k+2} - 1 = (a-1)(a^{k+1} + a^k + \cdots + a + 1).$$

   The left hand side is

$$
\begin{aligned}
LHS &= a^{k+2} - 1 \\
&= a(a^{k+1}) - 1 \\
&= a(a^{k+1} - 1) + (a-1) \\
&= a(a-1)(a^k + \cdots + a + 1) + (a-1) \\
&= (a-1)(a^{k+1} + a^k + \cdots + a^2 + a) + (a-1)(1) \\
&= (a-1)(a^{k+1} + a^k + \cdots + a + 1)
\end{aligned}
$$

   which is equal to the right hand side. Hence the formula is true for $n = k+1$. By the principle of mathematical induction the formula is true for all $n \geq 1$.