Unit 6 Assignment Solutions

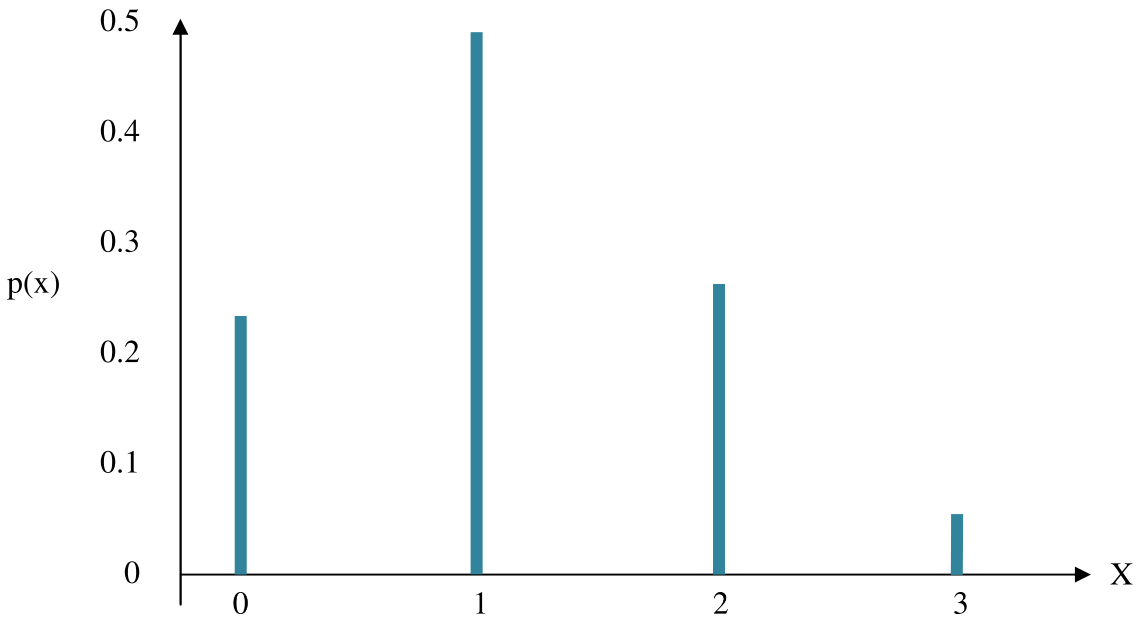
1. (a) The outcomes in the sample space and the probabilities of each outcome are shown below. Note that probabilities are calculated by multiplying the probabilities for individual games, as the outcomes of the games are clearly independent.

Sample Space S	Probability
CEO	(0.6)(0.2)(0.3) = 0.036
CEN	(0.6)(0.2)(0.7) = 0.084
CPO	(0.6)(0.8)(0.3) = 0.144
CPN	(0.6)(0.8)(0.7) = 0.336
MEO	(0.4)(0.2)(0.3) = 0.024
MEN	(0.4)(0.2)(0.7) = 0.056
MPO	(0.4)(0.8)(0.3) = 0.096
MPN	(0.4)(0.8)(0.7) = 0.224

(b) P(X = 0) = P(MPN) = 0.224.
P(X = 1) = P(CPN) + P(MEN) + P(MPO) = 0.336 + 0.056 + 0.096 = 0.488.
P(X = 2) = P(CEN) + P(CPO) + P(MEO) = 0.084 + 0.144 + 0.024 = 0.252.
P(X = 3) = P(CEO) = 0.036.

The p.m.f. of X and its graph are shown below:

x	**0**	**1**	**2**	**3**
P(X = x)	0.224	0.488	0.252	0.036



(c) The c.d.f. of X is as follows:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.224, & 0 \le x < 1 \\ 0.712 & 1 \le x < 2 \\ 0.964, & 2 \le x < 3 \\ 1, & x \ge 3 \end{cases}$$

(d) $E(X) = \sum_{i=0}^{3} p_i x_i = 0.224(0) + 0.488(1) + 0.252(2) + 0.036(3) = 1.1.$

$$E(X^2) = \sum_{i=0}^{3} p_i x_i^2 = 0.224(0^2) + 0.488(1^2) + 0.252(2^2) + 0.036(3^2) = 1.82.$$

$$Var(X) = E(X^2) - [E(X)]^2 = 1.82 - (1.1)^2 = 0.61.$$

2. (a) The sample space for the experiment is shown below:

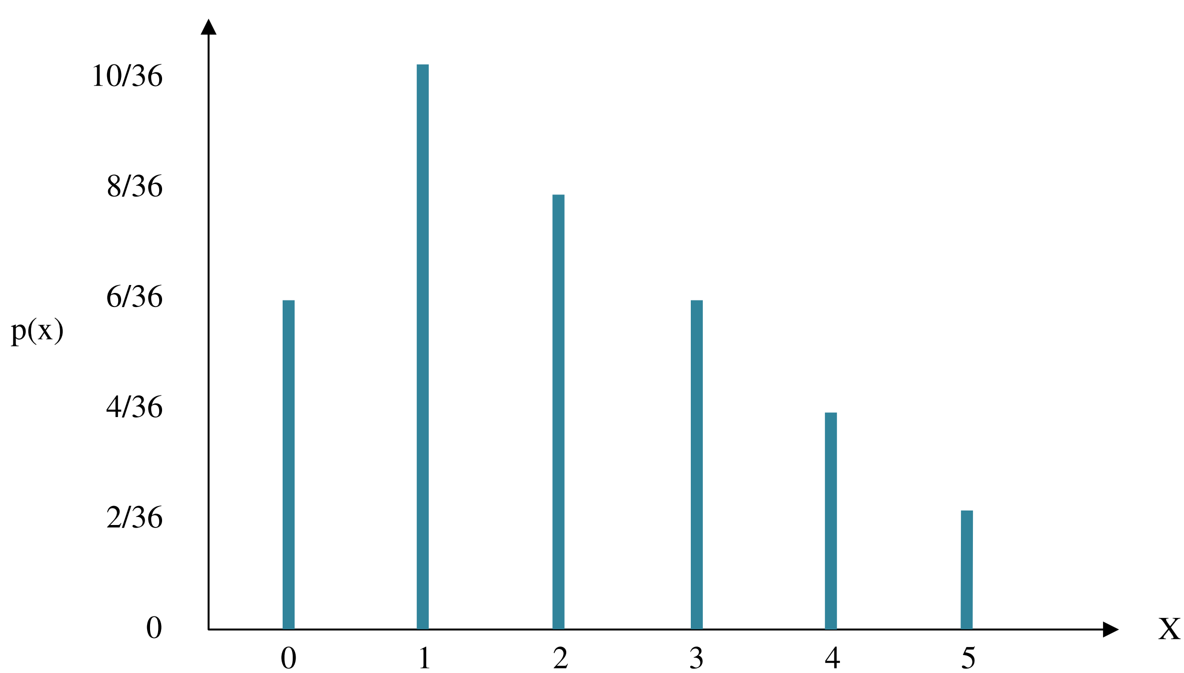
S = {11, 12, 13, 14, 15, 16,
21, 22, 23, 24, 25, 26,
31, 32, 33, 34, 35, 36,
41, 42, 43, 44, 45, 46,
51, 52, 53, 54, 55, 56,
61, 62, 63, 64, 65, 66}

The absolute differences of each of these outcomes are shown below:

0, 1, 2, 3, 4, 5,
1, 0, 1, 2, 3, 4,
2, 1, 0, 1, 2, 3,
3, 2, 1, 0, 1, 2,
4, 3, 2, 1, 0, 1,
5, 4, 3, 2, 1, 0

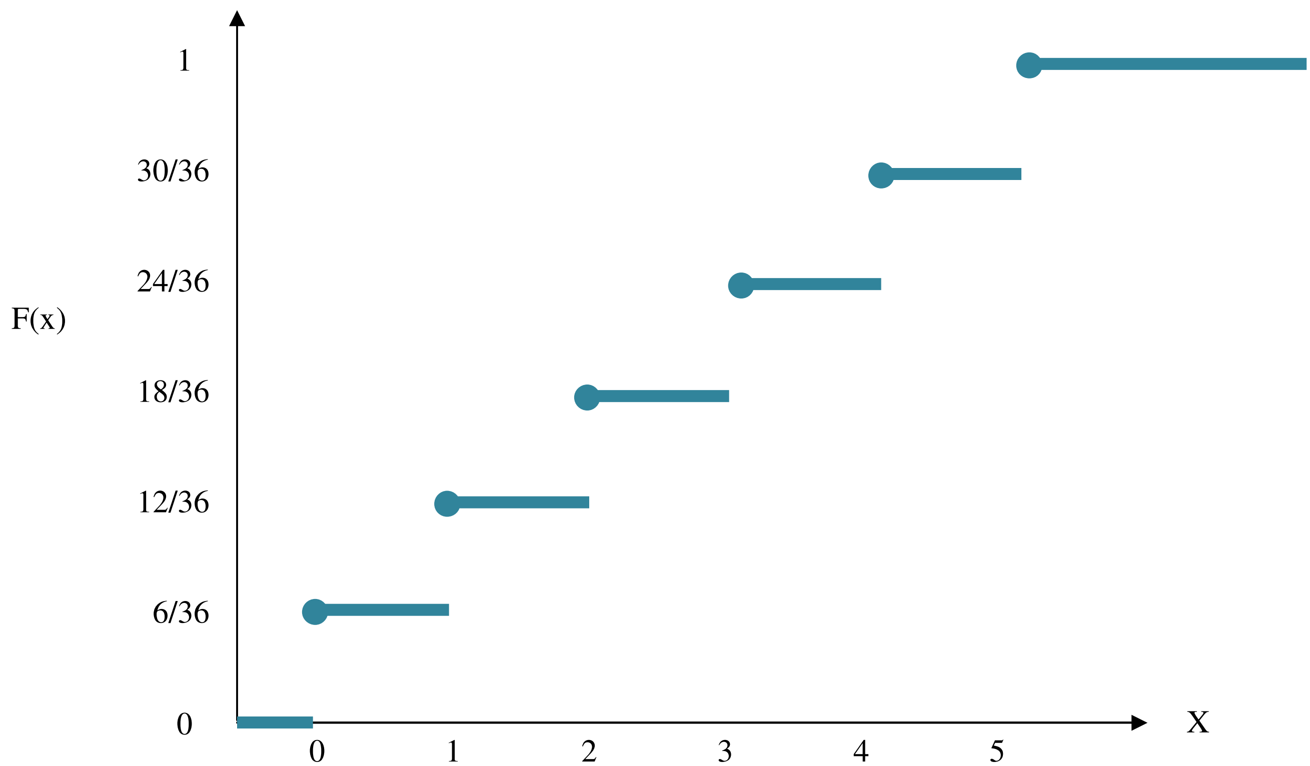
The probability mass function of X and its graph are shown below:

x_i	0	1	2	3	4	5
$P(X = x_i)$	6/36 = 0.1667	10/36 = 0.2778	8/36 = 0.2222	6/36 = 0.1667	4/36 = 0.1111	2/36 = 0.0556



(b) The c.d.f. of X and its graph are shown below:

$$F(x) = \begin{cases} 0, & x < 0 \\ 6/36 = 0.1667, & 0 \le x < 1 \\ 16/36 = 0.4444 & 1 \le x < 2 \\ 24/36 = 0.6667, & 2 \le x < 3 \\ 30/36 = 0.8333, & 3 \le x < 4 \\ 34/36 = 0.9444, & 4 \le x < 5 \\ 1, & x \ge 3 \end{cases}$$



(c) The expected value of X is

$$E(X) = \sum_x x_i P(X = x_i) = (0)(6/36) + (1)(10/36) + (2)(8/36) + (3)(6/36) + (4)(4/36) + (5)(2/36) = \frac{70}{36} = 1.944$$

The expected value of X^2 is

$$E(X^2) = \sum_x x_i P(X = x_i) = (0)^2(6/36) + (1)^2(10/36) + (2)^2(8/36) + (3)^2(6/36) + (4)^2(4/36) + (5)^2(2/36)$$

$$= \frac{210}{36} = 5.833$$

And so the variance of X is

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 5.833 - (1.944)^2 = 2.054$$

3. (a) In order for f(x) to be a valid density curve, the area under the curve and above the x-axis must be equal to one.

$$\int_1^{\infty} \frac{c}{x^3}\,dx = c\left[-\frac{1}{2x^2}\right]_1^{\infty} = \frac{c}{2} = 1 \quad \Rightarrow \quad c = 2.$$

(b) $F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_1^{x} \frac{2}{y^3}\,dy = \left[-\frac{1}{y^2}\right]_1^{x} = 1 - \frac{1}{x^2}$ for $x \geq 1$.

We also know that $F(x) = 0$ for $x < 1$.

(c) Using the p.d.f., $P(X > 1.5) = \int_{1.5}^{\infty} f(x)\,dx = \int_{1.5}^{\infty} \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_{1.5}^{\infty} = \frac{1}{(1.5)^2} = \frac{1}{2.25} = 0.4444.$

Using the c.d.f., $P(X > 1.5) = 1 - P(X \leq 1.5) = 1 - F(1.5) = 1 - \left[1 - \frac{1}{(1.5)^2}\right] = \frac{1}{(1.5)^2} = \frac{1}{2.25} = 0.4444.$

(d) The median of the distribution of X is the value x such that $P(X \leq x) = 0.5$, i.e. $F(x) = 0.5$.

$$F(x) = 0.5 \quad \Rightarrow \quad 1 - \frac{1}{x^2} = 0.5 \quad \Rightarrow \quad x^2 = 2 \quad \Rightarrow x = \sqrt{2} = 1.4142.$$

(e) $E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_1^{\infty} \frac{2}{x^2}\,dx = \left[-\frac{2}{x}\right]_1^{\infty} = 2.$

4. (a) $f(x) = \frac{d}{dx}F(x) = \frac{d}{dx}\left(\frac{x^2 - 196}{29}\right) = \frac{2}{29}x$ for $14 \le x \le 15$, and 0 otherwise.

(b) Using the p.d.f.,

$$P(17.2 \le X \le 17.5) = \int_{14.3}^{14.6} \frac{2}{29}x = \frac{1}{29}\left[x^2\right]_{14.3}^{14.6} = \frac{1}{29}\left[(14.6)^2 - (14.3)^2\right] = \frac{1}{29}(8.67) = 0.2990.$$

Using the c.d.f.,

$$P(14.3 \le X \le 14.6) = F(14.6) - F(14.3) = \left[\frac{(14.6)^2 - 289}{29} - \frac{(14.3)^2 - 289}{29}\right] = \frac{8.67}{29} = 0.2990.$$

(c) The median of the distribution of X is the value x such that $P(X \le x) = 0.5$, i.e. $F(x) = 0.5$.

$$F(x) = \frac{x^2 - 196}{29} = 0.5 \quad \Rightarrow \quad x = \sqrt{29(0.5) + 196} = 14.5086.$$

(d) $$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_{14}^{15} \frac{2}{29}x^2\, dx = \frac{2}{87}\left[x^3\right]_{14}^{15} = \frac{2}{87}\left[(15)^3 - (14)^3\right] = \frac{1262}{87} = 14.5057.$$

(e) $$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{14}^{15} \frac{2}{29}x^3\, dx = \frac{1}{58}\left[x^4\right]_{14}^{15} = \frac{1}{58}\left[(15)^4 - (14)^4\right] = \frac{12209}{58} = 210.5.$$

And so $\mathrm{Var}(X) = E(X^2) - [E(X)]^2 = 210.5 - (14.5057)^2 = 0.0847.$

(f) Mass = Density * Volume, so

$$\text{Mass} < 100 \quad \Leftrightarrow \quad \text{Density} * \text{Volume} < 100 \quad \Leftrightarrow \quad 0.00785\left(\frac{4}{3}\pi x^3\right) < 100$$

$$\Leftrightarrow \quad x < \left(\frac{3(100)}{0.00785(4\pi)}\right)^{\frac{1}{3}} \quad \Leftrightarrow \quad x < (3041.18)^{1/3} = 14.488.$$

So $$P(\text{Mass} < 100) = P(X < 14.488) = F(14.488) = \frac{(14.488)^2 - 196}{29} = 0.4794.$$

5. (a) $E(Y) = E[0.5W + 0.5X] = 0.5E(W) + 0.5E(X) = 0.5(10) + 0.5(15) = 5 + 7.5 = 12.5.$

$$\text{Var}(Y) = \text{Var}[0.5W + 0.5X] = (0.5)^2\,\text{Var}(W) + (0.5)^2\,\text{Var}(X) = 0.25(6) + 0.25(8) = 1.5 + 2 = 3.5.$$

(b) $E(Y) = E[7W - 3X] = 7E(W) - 3E(X) = 7(10) - 3(15) = 70 - 45 = 25.$

$$\text{Var}(Y) = \text{Var}[7W - 3X] = (7)^2\,\text{Var}(W) + (3)^2\,\text{Var}(X) = 49(6) + 9(8) = 144 + 28 = 366.$$

(c) $\text{Var}(Y) = \text{Var}(\delta W + (1-\delta)X) = \text{Var}(\delta W) + \text{Var}((1-\delta)X) + \delta^2\text{Var}(W) + (1-\delta)^2\,\text{Var}(X)$

$$= 6\delta^2 + 8(1-\delta)^2 = 14\delta^2 - 16\delta + 8.$$

Minimizing the standard deviation is equivalent to minimizing the variance. To do this, we take the derivative of the variance with respect to δ, set it to zero, and solve for δ:

$$\frac{d}{d\delta}\text{Var}(Y) = \frac{d}{d\delta}14\delta^2 - 16\delta + 8 = 28\delta - 16 = 0 \;\Rightarrow\; \delta = \frac{16}{28} = 0.5714.$$

Note that the second derivative evaluated at $\delta = 0.5714$ is positive, and so this must be a minimum.