UNIVERSITY OF MANITOBA

DATE: November 12, 2015

TERM TEST 2

EXAMINATION: Engineering Mathematical Analysis 2

TIME: 60 minutes

COURSE: MATH 2132

EXAMINER: M. Virgilio

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[7] 1. Evaluate the following limit using infinite series.

$$\lim_{x\to 0} \frac{\sqrt[5]{(1-x^3)^2} - x^3 - 1}{x^3}$$

(You are not allowed to use any other method).

Using Binomial series,

$$\sqrt[5]{(1-x^3)^2} = \left[(1-x^3)^2\right]^{1/5} = (1-x^3)^{2/5}$$

$$= 1 - \frac{2}{5}x^3 + \frac{\left(\frac{2}{5}\right)\left(-\frac{3}{5}\right)}{2!}(-x^3)^2 + \cdots$$

$$= 1 - \frac{2}{5}x^3 - \frac{3}{25}x^6 + \cdots$$

$$\lim_{x\to 0} \frac{\sqrt[5]{(1-x^3)^2} - x^3 - 1}{x^3} = \lim_{x\to 0} \frac{1}{x^3}\left[\left(1 - \frac{2}{5}x^3 - \frac{3}{25}x^6 + \cdots\right) - x^3 - 1\right]$$

$$= \lim_{x\to 0} \frac{1}{x^3}\left[-\frac{7}{5}x^3 - \frac{3}{25}x^6 + \cdots\right]$$

$$= \lim_{x\to 0} \left(-\frac{7}{5} - \frac{3}{25}x^3 + \cdots\right)$$

$$= -\frac{7}{5}.$$

[3] 2. Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{4^n}{n^2}$ converges or diverges. Justify your answer.

$$\lim_{n\to\infty} \frac{4^n}{n^2} = \lim_{x\to\infty} \frac{4^x}{x^2} \overset{H}{=} \lim_{x\to\infty} \frac{4^x \ln 4}{2x} \overset{H}{=} \lim_{x\to\infty} \frac{4^x(\ln 4)^2}{2} = \infty \neq 0$$

The series diverges by the alternating series test (or by the nth term test).

[16] 3. Use term-by-term differentiation to find the Taylor series of

$$f(x) = \frac{x}{(3x+7)^2}$$

about $x = -1$. Give the open interval of convergence. Express your final answer in sigma notation $\sum_n a_n(x+1)^n$.

$$\frac{1}{3x+7} = \frac{1}{3(x+1-1)+7} = \frac{1}{3(x+1)-3+7} = \frac{1}{3(x+1)+4}$$

$$= \frac{1}{4\left(1+\frac{3(x+1)}{4}\right)} = \frac{1}{4\left(1-\left(-\frac{3(x+1)}{4}\right)\right)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n \frac{3^n(x+1)^n}{4^n} \quad \text{for } -1 < \frac{3(x+1)}{4} < 1$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{3^n(x+1)^n}{4^{n+1}} \quad \text{for } -\frac{7}{3} < x < \frac{1}{3}$$

We have $\frac{d}{dx}\left(\frac{1}{3x+7}\right) = \frac{-3}{(3x+7)^2}$, so

$$\frac{1}{(3x+7)^2} = -\frac{1}{3}\sum_{n=1}^{\infty}(-1)^n \frac{n3^n(x+1)^{n-1}}{4^{n+1}} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n-1}}{4^{n+1}}.$$

Therefore, $\frac{x}{(3x+7)^2} = \frac{x+1-1}{(3x+7)^2} = \frac{x+1}{(3x+7)^2} - \frac{1}{(3x+7)^2}$

$$= (x+1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}n3^{n-1}(x+1)^{n-1}}{4^{n+1}} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n-1}}{4^{n+1}}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n}}{4^{n+1}} - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n-1}}{4^{n+1}}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n}}{4^{n+1}} - \sum_{n=0}^{\infty}(-1)^{n+2}\frac{(n+1)3^{n}(x+1)^{n}}{4^{n+2}}$$

$$= \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n3^{n-1}(x+1)^{n}}{4^{n+1}} - \left(\frac{1}{4^2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+2}(n+1)3^{n}(x+1)^{n}}{4^{n+2}}\right)$$

$$= -\frac{1}{4^2} + \sum_{n=1}^{\infty}\left[\frac{(-1)^{n+1}n3^{n-1}}{4^{n+1}} - \frac{(-1)^{n+2}(n+1)3^{n}}{4^{n+2}}\right](x+1)^n$$

for $-\frac{7}{3} < x < \frac{1}{3}$.

[12] 4. Find the sum and the open interval of convergence of the series

$$\sum_{n=1}^{\infty} n5^{2n-2}x^{2n+1}.$$

let $X = x^2$, then $a_n = n\,5^{2n-2}$ and $a_{n+1} = (n+1)\,5^{2n}$

$$R_X = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{n\,5^{2n-2}}{(n+1)\,5^{2n}}\right| = \frac{1}{5^2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{1}{25}$$

So $R_x = \sqrt{R_X} = \frac{1}{5}$. The series converges for $|x| < \frac{1}{5}$.

let $S(x) = \sum_{n=1}^{\infty} n\,5^{2n-2}x^{2n+1}$ on this interval, then

$$\frac{S(x)}{x^2} = \sum_{n=1}^{\infty} n\,5^{2n-2}x^{2n-1}.$$

$$\int \frac{S(x)}{x^2}\,dx = \sum_{n=1}^{\infty}\frac{\cancel{n}}{2\cancel{n}}\,5^{2n-2}x^{2n} + C = \sum_{n=1}^{\infty}\frac{5^{-2}}{2}\,5^{2n}x^{2n} + C$$

$$= \frac{1}{50}\sum_{n=1}^{\infty}(5^2x^2)^n + C$$

$$= \frac{1}{50}\,\frac{25x^2}{(1-25x^2)} + C$$

$$= \frac{x^2}{2(1-25x^2)} + C$$

If we differentiate, we have

$$\frac{S(x)}{x^2} = \frac{1}{2}\,\frac{2x(1-25x^2) + (50x)x^2}{(1-25x^2)^2} = \frac{x}{(1-25x^2)^2}$$

that is, $S(x) = \dfrac{x^3}{(1-25x^2)^2}$ for $-\frac{1}{5} < x < \frac{1}{5}$.

[12] 5. Find, in **explicit** form, a one parameter family of solutions for the differential equation

$$x^2\frac{dy}{dx} + x(x+2)y = e^x, \qquad x \neq 0.$$

The standard form is:

$$\frac{dy}{dx} + \frac{x(x+2)}{x^2}y = \frac{e^x}{x^2} \quad \text{or}$$

$$\frac{dy}{dx} + \frac{(x+2)}{x}y = \frac{e^x}{x^2} \quad \text{or} \quad \frac{dy}{dx} + \left(1+\frac{2}{x}\right)y = \frac{e^x}{x^2}.$$

An integrating factor is:

$$e^{\int\left(1+\frac{2}{x}\right)dx} = e^{(x+2\ln|x|)} = e^{(x+\ln x^2)}$$

$$= e^x e^{\ln x^2} = x^2e^x$$

Multiplying both sides of the standard form by x^2e^x yields

$$x^2e^x\frac{dy}{dx} + x^2e^x\left(1+\frac{2}{x}\right)y = x^2e^x\frac{e^x}{x^2}$$

$$\frac{d}{dx}\left[yx^2e^x\right] = e^{2x}$$

$$yx^2e^x = \frac{e^{2x}}{2} + C$$

Therefore, $y = \dfrac{e^{2x}}{2x^2e^x} + \dfrac{C}{x^2e^x}$ or

$$y = \frac{e^x}{2x^2} + \frac{Ce^{-x}}{x^2}, \quad x \neq 0.$$