Math 2130 - Engineering Mathematical Analysis 1

Tutorial 10 - Questions for §13.8 to 13.12.

13.8.1. Evaluate the triple integral of the function $f(x, y, z) = x$ over the volume bounded by the surfaces

$$2x + 3y + z = 6, \quad x = 0, \quad y = 0, \quad z = 0.$$

13.9.1. Find the volume in the first octant bounded by the surfaces

$$4x + 4y + z = 16, \quad z = 0, \quad y = x/2, \quad y = 2x.$$

13.9.2. Set up, but do **NOT** evaluate, a triple iterated integral for the volume in the first octant bounded by the surfaces

$$z = 2x + y, \quad 9x^2 + 4y^2 = 1, \quad x = 0, \quad y = 0, \quad z = 0.$$

13.9.3. Set up, but do **NOT** evaluate, a triple iterated integral for the volume bounded by the surfaces

$$z = 9 - x^2 - y^2, \quad z = x^2.$$

13.11.1. Find the volume bounded by the surfaces

$$z = xy, \quad x^2 + y^2 = 1, \quad z = 0.$$

Note: By volume we mean the *geometric* volume.

13.11.2. Consider the solid bounded by the surfaces $z = 2\sqrt{x^2 + y^2}$ and $z = 9 - x^2 - y^2$. Set up triple iterated integrals in each of Cartesian and cylindrical coordinates for the volume of this solid. Calculate the value of this volume, and obtain a numerical answer, but do not simplify it.

The next question covers material from sections 13.9, 10, 11, and 12.

13.12.1. Consider the solid bounded by the upper hemisphere $z = \sqrt{16 - x^2 - y^2}$ and the cone $z = \sqrt{x^2 + y^2}$. The density of this solid is $\rho = 1$. Set up triple iterated integrals in each of Cartesian, cylindrical and spherical coordinates for the volume, the mass and the centre of mass (as needed). Calculate the mass and centre of mass using both cylindrical and spherical coordinates and approximate your answer to two decimal places. Show that the centre of mass lies just below the level where the hemisphere and cone meet.

Answers:

13.8.1. $9/2$. **13.9.1.** $128/9$. **13.9.2.** $\displaystyle\int_0^{1/3}\int_0^{(1/2)\sqrt{1-9x^2}}\int_0^{2x+y} dz\,dy\,dx.$

13.9.3. $\displaystyle 4\int_0^{3/\sqrt{2}}\int_0^{\sqrt{9-2x^2}}\int_{x^2}^{9-x^2-y^2} dz\,dy\,dx.$

13.11.1. $1/2$.

13.11.2. Volume $\displaystyle = 4\int_0^{\sqrt{10}-1}\int_0^{\sqrt{(\sqrt{10}-1)^2-x^2}}\int_{2\sqrt{x^2+y^2}}^{9-x^2-y^2} dz\,dy\,dx;$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\sqrt{10}-1}\int_{2r}^{9-r^2} r\;dz\,dr\,d\theta$$
$$= 2\pi\left[\frac{9(\sqrt{10}-1)^2}{2} - \frac{(\sqrt{10}-1)^4}{4} - \frac{2(\sqrt{10}-1)^3}{3}\right].$$

13.12.1. Since $\rho = 1$, the mass equals the volume.

$$\text{Volume} = 4\int_0^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\,dy\,dx;$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r\;dz\,dr\,d\theta;$$

$$\text{Volume} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 R^2 sin(\phi)\;dR\,d\phi\,d\theta.$$

Volume = Mass (M) $= 64\,(2-\sqrt{2})\,\dfrac{\pi}{3}$.

By symmetry, $\bar{x} = \bar{y} = 0$.

$$M\bar{z} = 4\int_0^{2\sqrt{2}}\int_0^{\sqrt{8-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} z\,dz\,dy\,dx = \int_0^{2\pi}\int_0^{2\sqrt{2}}\int_r^{\sqrt{16-r^2}} r\;z\,dz\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^4 R^3 sin(\phi)cos(\phi)\;dR\,d\phi\,d\theta$$
$$= 32\,\pi.$$

Therefore, $\bar{z} = \dfrac{3}{2\,(2-\sqrt{2})} \approx 2.56$.

The hemisphere and cone meet at $z = 2\,\sqrt{2} \approx 2.83$.
