MATH1210 Assignment #2

Due: 1:30 pm Friday 6 October 2006 T. G. Berry

NOTES:

(1) ***The assignment is due at the start of our class on Friday 6 October 2006.***

(2) ***Late assignments will NOT be accepted.***

(3) ***If your assignment is not accompanied by a Faculty of Science "Honesty Declaration", it will NOT be graded.***

1. In each of the following cases, evaluate the given complex number, writing your answer in (simplified) Cartesian form:

(a) $(1+i)-(7+2i)$ (b) i^{57}

(c) $i^2(-1+3i)$ (d) $\left(\dfrac{2-3i}{4+2i}\right)$

(e) $\overline{(7-2\bar{i})^2}$ (f) $\left(1-\sqrt{3}\,i\right)^4$

(g) $(4-2i)^6$ (h) $\left(2+\overline{(-1+i)}\right)^2$

(i) $\dfrac{(2+i)(4-6i)}{(3-i)i^3}$ (j) $\dfrac{2+i}{\left(1+\dfrac{1}{1-i}\right)}$

2. Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be any two complex numbers written in polar form.

(a) Prove that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

(where the latter equation must be interpreted in the manner described in the paragraph following equation (2.14b) on page 19 in the course notes).

(b) Let $z_1 = -\frac{1}{\sqrt{2}}(1+i)$ and $z_2 = \frac{-1+\sqrt{3}i}{2}$.

Evaluate $\frac{z_1}{z_2}$ directly, and confirm that $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$ (subject to the interpretation described in part (a)). HINT: You will need to use your calculator to verify the above result in this case, and will probably need to include a "correction" involving some multiple of 2π .

In addition, evaluate the principal values of $\arg\left(\frac{z_1}{z_2}\right)$, $\arg z_1$ and $\arg z_2$ and show that

$$p.v.\left(\arg\left(\frac{z_1}{z_2}\right)\right) \text{ is not equal to } p.v.(\arg(z_1)) - p.v.(\arg(z_2)) .$$

3. Let a and b be any real numbers and consider the complex number $a+ib$.

(a) Use the Binomial Theorem to evaluate $(a+ib)^n$ for $n = 2,3,4,5$, in each case simplifying your answer as far as possible, and expressing it in Cartesian form.

(b) Now let $a = \cos\theta$ and $b = \sin\theta$, so that $a+ib = e^{i\theta}$.

Use the theorem of de Moivre, written in the following form

$$\left(e^{i\theta}\right)^n = e^{i(n\theta)} ,$$

in order to derive the following compound-angle formulae for the sine and cosine functions:

$$\sin(2\theta) = 2\sin\theta\cos\theta ,$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta ,$$
$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta ,$$
$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta ,$$
$$\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta ,$$
$$\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta ,$$
$$\sin(5\theta) = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta ,$$
$$\cos(5\theta) = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta .$$

4. In each if the following cases, find all solutions of the given equation, writing your answers in both Cartesian and polar form, and graphically illustrating these solutions in the complex plane:

(a) $x^6 + 1 = 0$,

(b) $x^2 - i = 0$,

(c) $x^3 + 4\sqrt{2}(1+i) = 0$.

Comment: The above could be rephrased as (a) "find all sixth roots of -1 " , (b) "find all square roots of i ," and (c) "find all cube roots of $-4\sqrt{2}(1+i)$ " respectively.

5. Consider the function

$$w = i\,z \ ,$$

in which both z and w are complex numbers and $i = \sqrt{-1}$.

(a) Suppose that we write z and w in Cartesian form as $z = x + iy$ and $w = u + iv$, in which x , y , u and v are real numbers.

Show that the given complex function is equivalent to the two (real) equations

$$u = -y \quad \text{and} \quad v = x \ .$$

(b) Regard the latter two equations as defining a transformation of variables between the xy-plane and the uv-plane,. Illustrate graphically the effect of the transformation of part (a) on an arbitrary point (x,y) in the xy-plane.

(c) Use the above results to geometrically describe (in words) the effect of the function $w = i\,z$, if it is regarded as a transformation between two copies of the complex plane (with z representing any element in the first such complex plane and w representing the corresponding element in the second such complex plane).

(d) Use the exponential representation complex numbers in order to confirm the result of part (c). HINT: Let $z = re^{i\theta}$ and $w = Re^{i\phi}$ and determine the relationship between R and r, and between ϕ and θ .