MATH 1210 Assignment 4

March 21, 2014

Due: March 28, 2014, in class

Question 1. Consider the following system of linear equations:

$$\left\{\begin{array}{rcrcrcrcl} w & - & x & + & 2y & - & 3z & = & 0 \\ 3w & - & 3x & + & 8y & - & 5z & = & 0 \\ 2w & - & 2x & + & 5y & - & 4z & = & 0 \\ 3w & - & 3x & + & 7y & - & 7z & = & 0 \end{array}\right.$$

(a) Find the reduced row-echelon form of the augmented matrix of this system.

(b) Find all the basic solutions of this system.

(c) Use part (b) to find a solution in which $w = -4$ and $y = 4$.

Question 2. Suppose that $AX = B$ is the matrix representation of a system of linear equations.

(a) Suppose that Y is a solution of this system, and that Z_1 and Z_2 are solutions of the associated homogeneous system. Show that $Y - 17Z_1 + 16Z_2$ is also a solution of $AX = B$.

(b) Suppose that Y_1 and Y_2 are solutions of $AX = B$. Show that $2Y_1 - Y_2$ is also a solution of $AX = B$.

Question 3. Let $A = \begin{pmatrix} 1 & 2 & c \\ 3 & 4c & 12 \\ c & -1 & 2 \end{pmatrix}$ and let X be the column vector of variables x, y, and z. For which values of c, if any, does the system $AX = \mathbf{0}$ have non-trivial solutions?

Question 4. Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & -5 & 3 \\ -3 & 2 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

Evaluate the determinant of A in two ways:

(a) by expansion along column 3;

(b) by row reduction.

Question 5. For $n \geq 1$, let H_n be the $n \times n$ matrix with (i,j)-entry $\dfrac{1}{i+j-1}$. For instance,

$$H_1 = (1)\,, \qquad H_2 = \begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{pmatrix}, \text{ and } H_3 = \begin{pmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{pmatrix}.$$

This is a famous family of "bad examples" called the *Hilbert matrices*. The determinants of even the small sizes are awkward to calculate by hand, and the big sizes are good tests of the accuracy of computer numerical methods.

Showing all your work carefully, evaluate $|\,H_3\,|$ by any method.

The reason we didn't ask you for $|\,H_4\,|$ is that the answer is $\dfrac{1}{6048000}$.

Question 6. Suppose that A, B, and C are 5×5 matrices with $|\,A\,| = -2$ and $|\,B\,| = 3$.

(a) Find $\left|\,\frac{1}{6}B^3A^5\,\right|$.

(b) If $\left|\,A^3C^2B^T\,\right| = -96$, find $|\,C\,|$.

Question 7. Use Cramer's rule to find the value of z such that

$$\begin{cases} 2x + 3y & = & 3 \\ x - 2y - 5z & = & -2 \\ 4x - y + z & = & 1 \end{cases}$$

Question 8.

(a) Determine whether the following set of vectors is linearly dependent or linearly independent, by the method of determinants.

$$\{\,\langle\,2, 3, -1\,\rangle\,, \langle\,-1, 2, -10\,\rangle\,, \langle\,3, -1, 9\,\rangle\,\}$$

(b) For what values (if any) of a are the following vectors linearly independent?

$$\{\,\langle\,1, 2, 2\,\rangle\,, \langle\,1, a, 2\,\rangle\,, \langle\,1, a, 1\,\rangle\,\}$$

(c) Show that the following three vectors are linearly dependent, and write one as a linear combination of the other two.

$$\{\,\langle\,2, -1, 3, 4\,\rangle\,, \langle\,-3, 2, -5, -7\,\rangle\,, \langle\,1, 1, 0, -1\,\rangle\,\}$$

Question 9.

(a) Show that if $\mathbf{u}_1$ and $\mathbf{u_2}$ are a pair of linearly independent vectors, then $\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$ are also linearly independent.

(b) Write each of $\mathbf{u}_1$ and $\mathbf{u}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.