MATH 1210 Problem Workshop 12 Solutions

1. For E, we find the cofactors to be

$$c_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = 2 - 3 = -1$$

$$c_{12} = -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} = -(2 - 6) = 4$$

$$c_{13} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = 1 - 2 = -1$$

$$c_{21} = -\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = -(2 - 2) = 0$$

$$c_{22} = \begin{vmatrix} 0 & 2 \\ 2 & 2 \end{vmatrix} = 0 - 4 = -4$$

$$c_{23} = -\begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - (0 - 2) = 2$$

$$c_{31} = \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 3 - 2 = 1$$

$$c_{32} = -\begin{vmatrix} 0 & 2 \\ 1 & 3 \end{vmatrix} = -(0 - 2) = 2$$

$$c_{33} = \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = (0 - 1) = -1.$$

Hence

$$C = \begin{bmatrix} -1 & 4 & -1 \\ 0 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \Rightarrow adj(E) = C^T = \begin{bmatrix} -1 & 0 & 1 \\ 4 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

Since $E^{-1} = \dfrac{1}{|E|} adj(E)$ and

$$\begin{aligned} det(E) &= \begin{vmatrix} 0 & 1 & 2 \\ 1 & 1 & 3 \\ 2 & 1 & 2 \end{vmatrix} \\ &= 0\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ &= 0 - 1(2 - 6) + 2(1 - 2) \\ &= 0 - (-4) + 2(-1) \\ &= 4 - 2 \\ &= 2 \end{aligned}$$

Hence

$$E^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 0 & 1 \\ 4 & -4 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

2. (a) We use that $|AB| = |A|\,|B|$ and $AA^{-1} = I$ to get that

$$1 = |I| = \left|AA^{-1}\right| = |A|\left|A^{-1}\right| \Rightarrow \left|A^{-1}\right| = \frac{1}{|A|}.$$

(b) The row operation says if you multiply a row by c then the determinant is multiplied by c. Since cA multiplies all n rows by c, the determinant is multiplied by c^n. Hence

$$|cA| = c^n\,|A|\,.$$

(c) Since the determinant of a matrix is a constant, and $Aadj(A) = |A|\,I$ we use part (b) to get

$$|A|\,|adj(A)| = ||A|\,I| = |A|^n\,|I| = |A|^n\,(1).$$

Provided $|A| \neq 0$,

$$|adj(A)| = \frac{|A|^n}{|A|} = |A|^{n-1}\,.$$

In fact it can be shown that even if $|A| = 0$, that $adj(A)$ is not invertible. (Hint: Show $adj(adj(A)) = |A|^{n-2}\,A = 0_{n\times n}$) and hence $|adj(A)| = 0$. Therefore the question is true for all matrices A.

3.

$$\begin{aligned} \left|5AB^{-2}\right| &= 5^3\left|AB^{-2}\right| \\ &= 125\,|A|\left|B^{-1}\right|\left|B^{-1}\right| \\ &= \frac{125\,|A|}{|B|\,|B|} \\ &= \frac{125(2)}{(3)(3)} \\ &= \frac{250}{9}. \end{aligned}$$

4. The coefficient matrix is

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 1 & 1 & 2 \\ 3 & 4 & 0 \end{bmatrix}.$$

Hence finding our inverse require row reducing

$$\left[\begin{array}{ccc|ccc} 2 & 3 & -4 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \text{ Using } R_1 \leftrightarrow R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 2 & 3 & -4 & 1 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 1 \end{array}\right] \text{ Using } R_2 \leftrightarrow R_2 - 2R_1 \text{ and } R_3 \leftrightarrow R_3 - 3R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & 0 \\ 0 & 1 & -6 & 0 & -3 & 1 \end{array}\right] \text{ Using } R_2 \leftrightarrow R_2 - 2R_1 \text{ and } R_3 \leftrightarrow R_3 - 3R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 10 & -1 & 3 & 0 \\ 0 & 1 & -8 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] \text{ Using } R_1 \leftrightarrow R_1 - R_2 \text{ and } R_3 \leftrightarrow R_3 - R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 10 & -1 & 3 & 0 \\ 0 & 1 & -8 & 1 & -2 & 0 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] \text{ Using } R_1 \leftrightarrow R_1 - 5R_3 \text{ and } R_2 \leftrightarrow R_2 + 4R_3 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 8 & -5 \\ 0 & 1 & 0 & -3 & -6 & 4 \\ 0 & 0 & 2 & -1 & -1 & 1 \end{array}\right] \text{ Using } R_3 \leftrightarrow \frac{1}{2}R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 8 & -5 \\ 0 & 1 & 0 & -3 & -6 & 4 \\ 0 & 0 & 1 & -1/2 & -1/2 & 1/2 \end{array}\right].$$

Hence

$$A^{-1} = \begin{bmatrix} 4 & 8 & -5 \\ -3 & -6 & 4 \\ -1/2 & -1/2 & 1/2 \end{bmatrix}$$

Solving for x, y, z requires

$$X = A^{-1}B = \begin{bmatrix} 4 & 8 & -5 \\ -3 & -6 & 4 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 12 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

Therefore $x = 1, y = 2$ and $z = -1$.

5. Using the same coefficient matrix as the previous question and therefore the same inverse, we get

$$X = A^{-1}B = \begin{bmatrix} 4 & 8 & -5 \\ -3 & -6 & 4 \\ -1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4a + 8b - 5c \\ -3a - 6b + 4c \\ -1/2a - 1/2b + 1/2c \end{bmatrix}.$$

Therefore $x = 4a + 8b - 5c, \quad y = -3a - 6b + 4c, \quad z = \dfrac{-a - b + c}{2}$

6. (a) From the first definition, we can see it's linear because for any vectors $\mathbf{u}$ and $\mathbf{v}$ and constant c

$$T(\mathbf{u}+\mathbf{v}) = k(\mathbf{u}+\mathbf{v}) = k\mathbf{u}+k\mathbf{v} = T(\mathbf{u})+T(\mathbf{v})$$

and

$$T(c\mathbf{u}) = k(c\mathbf{u}) = c(k\mathbf{u}) = cT(\mathbf{u}).$$

From the second definition we get that

$$T(\mathbf{v}) = T(\langle v_1, v_2, \ldots, v_n\rangle) = \langle kv_1, kv_2, \ldots, kv_n\rangle.$$

Since $kv_1, kv_2, \ldots, kv_n$ are all linear combinations of $v_1, v_2, \ldots, v_n$ we have that T is linear.

(b) If $\mathbf{c}$ is not-zero, then T is not linear since it violates the first condition when $\mathbf{u} = \langle 1,0,0\rangle$ and $\mathbf{v} = \langle 0,1,0\rangle$.

$$T(\mathbf{u}+\mathbf{v}) = T(\langle 1,1,0\rangle) = \langle 1,1,0\rangle + \mathbf{c}$$

and

$$T(\mathbf{u})+T(\mathbf{v}) = T(\langle 1,0,0\rangle)+T(\langle 0,1,0\rangle) = \langle 1,0,0\rangle+\mathbf{c}+\langle 0,1,0\rangle+\mathbf{c} = \langle 1,1,0\rangle+2\mathbf{c}$$

These are only equal if $\mathbf{c}$ is the zero vector, so since it isn't the zero vector, $T(\mathbf{u}+\mathbf{v}) \neq T(\mathbf{u})+T(\mathbf{v})$ and hence T is not linear.

Using the second definition they are not linear since $\mathbf{c}$ is not the zero vector. This implies one of the components (for example the first component) and hence the first component of $T(\mathbf{v})$ is $v_1 + c_1$ which is not a linear combination of the v_i.

(c) T is not linear since it violates the first condition when $\mathbf{u} = \langle 1,0,0\rangle$ and $\mathbf{v} = \langle 0,1,0\rangle$.

$$T(\mathbf{u}+\mathbf{v}) = T(\langle 1,1,0\rangle) = \frac{\langle 1,1,0\rangle}{\sqrt{2}}$$

and

$$T(\mathbf{u})+T(\mathbf{v}) = T(\langle 1,0,0\rangle)+T(\langle 0,1,0\rangle) = \langle 1,0,0\rangle+\langle 0,1,0\rangle+\mathbf{c} = \langle 1,1,0\rangle \neq T(\mathbf{u}+\mathbf{v}).$$

Using the second definition, T is not linear since

$$T(\mathbf{v}) = \frac{\langle v_1, v_2, v_3\rangle}{\sqrt{v_1^2+v_2^2+v_3^2}}$$

which is not a linear combination of v_1, v_2 and v_3.