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FINAL EXAMINATION

[9] 1. Find the Maclaurin series of

$$f(x) = \frac{x^3}{\sqrt{1+x^3}}.$$

$\gg$ ans) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n-2)!}{2^{2n-2}[(n-1)!]^2} x^{3n}$

Express your answer in sigma notation and simplify as much as possible. Find its open interval of convergence.

[6] 2. Let a and b be two numbers such that $0 < b < a$. Find the values of a and b if the radii of convergence of the series $\sum_{n=1}^{\infty} \frac{(a+b)^n}{4^n} x^n$ and $\sum_{n=1}^{\infty} (a-b)^n n\, 3^n x^n$ are 1 and $\frac{1}{6}$ respectively.

$\gg$ ans) $a=3,\ b=1$

[7] 3. Evaluate the following limit using infinite series.

$$\lim_{x\to\infty} \left[x \cos\left(\frac{1}{x}\right) - x^2\left(e^{\frac{1}{x}} - 1\right)\right]$$

$\gg$ ans) $-\frac{1}{2}$

[8] 4. Solve the differential equation

$$x^3 \frac{dy}{dx} = -1 + x^3 + 2x^2 y.$$

$\gg$ ans) $Cx^2 - x + \frac{1}{4x^2}$ or $\frac{1}{4x^2}\left(Dx^4 - 4x^2 + 1\right)$

[8] 5. Find an explicit two-parameter family of solutions for $\frac{1}{3} y'' = 2x \sqrt[3]{(y')^2}$.

$\gg$ ans) $\frac{x^7}{7} + \frac{3D}{5}x^5 + D^2 x^3 + E$

[8] 6. Chemical reactors are of third order when the amount $x(t)$ of substance being formed satisfies a differential equation of the form

$$\frac{dx}{dt} = k(a-x)(b-x)(c-x).$$

Solve this differential equation and find the exact amount of the substance when $a = b = c = 1$ and $k = \frac{1}{20}$ and $x(0) = 0$.

$\gg$ ans) $1 - \sqrt{\frac{10}{t+10}}$

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[8] 7. Given that $m^3(m+4)^2(m^2+1)^3 = 0$ is the auxiliary equation associated with the linear differential equation

$$\phi(D)y = 1 + \cos 3x + x^3e^{-4x},$$

what is the form of a particular solution $y_p(x)$?

DO NOT EVALUATE THE COEFFICIENTS IN $y_p(x)$.

$\rightarrowtail$ ans) $Ax^3 + B\cos 3x + C\sin 3x + (Dx^5 + Ex^4 + Fx^3 + Gx^2)e^{-4x}$

[10] 8. Consider the differential equation $y'' - y' = 2xe^x + 3e^x$.

(a) Given that $y_p = Ax^2e^x + Bxe^x$ is a particular solution, find the values of A and B.

$\rightarrowtail$ ans) $A=1,\ B=1$

(b) Find a general solution for $y'' - y' = 2xe^x + 3e^x$.

$\rightarrowtail$ ans) $c_1 + c_2e^x + x^2e^x + xe^x$

[8] 9. Find Laplace transform of $f(t)$ using only definition of Laplace transform, where

$$f(t) = t\,\mathscr{U}(t-1).$$

No mark will be given for any other method.

$\rightarrowtail$ ans) $\dfrac{e^{-s}}{s^2}(s+1)$

[9] 10. Find $\mathscr{L}\{f(t)g(t)\}$ where $f(t) = \delta(t-4) + e^{2t}\sin(t-3)$ and $g(t) = \mathscr{U}(t-3)$.

$\rightarrowtail$ ans) $e^{-4s} + \dfrac{e^{6-3s}}{s^2-4s+5}$

[8] 11. Find $\mathscr{L}^{-1}\left\{e^{-4s}\left(\dfrac{s^2+2s-1}{s^4-s^2}\right)\right\}$.

$\rightarrowtail$ ans) $\mathcal{U}(t-4)\left(2-t+2e^{t-4}\right)$

[11] 12. Use Laplace transforms to solve the initial-value problem

$$y'' + 4y' + 4y = e^{-t}(\sin t + \cos t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

$\rightarrowtail$ ans) $-\tfrac{1}{2}e^{-2t} + \tfrac{1}{2}e^{-t}(\sin t - \cos t)$