December 2009 Final Exam

1. Prove using induction that $\sum_{r=1}^{n}(-1)^r = \dfrac{(-1)^n - 1}{2}$ for all integers $n \geq 1$.

2. Consider the polynomial $p(x) = 7x^{27} - 34x^8 + 15x^2 - x + 10$.

   (a) List all the possible rational roots of $p(x)$.

   (b) Determine the number of positive and negative roots.

3. Let $p(x) = x^{2009} - 3x^{1492} + 2$. Find the remainder when $p(x)$ is divided by $x + i$.

4. Let $x_1 = 1 + i$ and $z_2 = -1 + i\sqrt{3}$.

   (a) Express z_1 and z_2 in exponential form.

   (b) Find the modulus and the prinicipal value of the argument of $z = z_1^2 z_2$.

5. Given $L_1 : \langle x, y, z\rangle = \langle 7, 1, 4\rangle + s\langle 1, 1, 3\rangle$ and $L_2 : \langle x, y, z\rangle = \langle 3, 2, 0\rangle + t\langle 2, -3, -2\rangle$. Find the point of interesection of L_1 and L_2.

6. Find $\sin\theta$ where θ is the angle between $\mathbf{u} = \langle 3, 0, -2\rangle$ and $\mathbf{v} = \langle 1, 1, -1\rangle$.

7. Given $L_1 : \dfrac{x-1}{3} = \dfrac{y-2}{-2} = \dfrac{z-5}{4}$ and $\Pi : x+y+z = 3$ Find the point of interesection of L_1 and Π.

8. Let $A = \begin{pmatrix} 1 & 0 \\ i & -i \end{pmatrix}$ Find (a) A^2 and (b) A^{-1}.

9. Find the value of k for which the following system is consistent:

$$3x - 2y = k, \quad 2x + y = 6, \quad -x + 3y = 4.$$

10. Find a such that the following vectors are linearly dependent:

$$\langle -1, 2, a\rangle, \quad \langle 2, 7, 4\rangle, \quad \langle 3, 5, -2a\rangle$$

11. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 3 \end{pmatrix}$ Find:

    (a) $|A|$

    (b) the element in the 2nd row and 3rd column of A^{-1}.

12. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -3 & 4 & 2 \end{pmatrix}$ Find A^{-1}.

13. Let $A = \begin{pmatrix} 5 & 0 & -3 \\ -4 & 1 & 3 \\ 8 & 0 & -4 \end{pmatrix}$.

   (a) Find the characteristic equation of A.

   (b) Given that $\lambda = 1$ is one of the eigenvalues, find the other two.

14. Consider the transformation from $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T : v_1' = 2v_1 - v_2 + v_3, v_1 - 4v_3.$$

   (a) Write down the matrix corresponding to T.

   (b) Find the image of the vector $\langle 1, -6, 3 \rangle$ under T.

   (c) Find a non-zero vector $\mathbf{v}$ such that $T(\mathbf{v}) = \mathbf{0}$.

15. Consider the symmetric matrix $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$. It is given that 3 and 0 are eigenvalues of A.

   (a) Find an eigenvector $\mathbf{u}$ corresponding to the eigenvalue 3.

   (b) Find an eigenvector $\mathbf{v}$ corresponding to the eigenvalue 0.

   (c) Find the angle between $\mathbf{u}$ and $\mathbf{v}$.

   (d) Find the remaining eigenvalue of A.