MATH 1210 Assignment 1 Winter 2014 **Due date: January 31**

1. Use mathematical induction on positive integer n to prove each of the following:

(a) $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \dfrac{1}{2}\, n(6n^2 - 3n - 1)$, **for** $n \geq 1$ **;**

(b) $3 + 7 + 11 + \ldots + (8n-1) = 2n(4n+1)$, **for** $n \geq 1$ **;**

(c) $2^{4n} - 3^{2n}$ **is divisible by** 7 **for** $n \geq 1$ **.**

Solution:

(a) Let $P(n)$ be the statement $1^2 + 4^2 + 7^2 + \cdots + (3n-2)^2 = \dfrac{1}{2}\, n(6n^2 - 3n - 1)$.
If $n = 1$, then $P(1)$ is true beacuse $1^2 = 1$ **and** $\frac{1}{2}(1)[6(1)^2 - 3(1) - 1] = \frac{1}{2}(2) = 1$.
We assume that for $n = k$, $P(k)$ is valid that is

$$1^2 + 4^2 + 7^2 + \cdots + (3k-2)^2 = \frac{1}{2}\, k(6k^2 - 3k - 1) \qquad (*)$$

We need to prove that for $n = k+1$, $P(k+1)$ is valid that is we must verify that

$$1^2 + 4^2 + 7^2 + \cdots + (3k+1)^2 = \frac{1}{2}\,(k+1)\Big(6(k+1)^2 - 3(k+1) - 1\Big).$$

But

$$\begin{aligned} 1^2 + 4^2 + 7^2 + \cdots + (3k+1)^2 &= [1^2 + 4^2 + 7^2 + \cdots + (3k-2)^2] + (3k+1)^2 \\ &= \frac{1}{2}\, k(6k^2 - 3k - 1) + (3k+1)^2 \quad \textbf{by } (*) \\ &= \frac{1}{2}[\, k(6k^2 - 3k - 1) + 2(3k+1)^2] \\ &= \frac{1}{2}[\, 8k^3 + 15k^2 + 11k + 2]\,. \end{aligned}$$

Also

$$\frac{1}{2}\,(k+1)\Big(6(k+1)^2 - 3(k+1) - 1\Big) \;=\; \frac{1}{2}\,(k+1)\Big(6k^2 + 9k + 2\Big) \;=\; \frac{1}{2}[\, 8k^3 + 15k^2 + 11k + 2]\,.$$

Hence $1^2 + 4^2 + 7^2 + \cdots + (3k+1)^2 = \dfrac{1}{2}\,(k+1)\Big(6(k+1)^2 - 3(k+1) - 1\Big)$.
Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 1$.

(b) Let $P(n)$ be the statement $3 + 7 + 11 + \ldots + (8n-1) = 2n(4n+1)$.
If $n = 1$, then $P(1)$ is true beacuse $3 + 7 = 10$ **and** $2(1)(4(1)+1) = 2(5) = 10$. **We assume that for $n = k$, $P(k)$ is valid that is**

$$3 + 7 + 11 + \ldots + (8k-1) = 2k(4k+1) \qquad (*)$$

We need to prove that for $n = k+1$, $P(k+1)$ is valid that is we must verify that

$$3 + 7 + 11 + \ldots + (8k+7) = 2(k+1)\big(4(k+1)+1\big)$$

But

$$\begin{aligned} 3+7+11+\ldots+(8k+7) &= [3+7+11+\ldots+(8k-1)] + (8k+3) + (8k+7) \\ &= 2k(4k+1) + (8k+3) + (8k+7) \quad \textbf{by } (*) \\ &= 8k^2+18k+10 \\ &= 2(k^2+9k+5) \\ &= 2(k+1)(4k+5) \\ &= 2(k+1)\big(4(k+1)+1\big). \end{aligned}$$

Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 1$.

(c) Let $P(n)$ be the statement " $2^{4n} - 3^{2n}$ is divisible by 7 ".
If $n = 1$, then $P(1)$ is true beacuse $2^4 - 3^2 = 7$ which is divisible by 7. We assume that for $n = k$, $P(k)$ is valid that is $2^{4k} - 3^{2k}$ is divisible by 7. We need to prove that for $n = k+1$, $P(k+1)$ is valid that is we must prove that $2^{4(k+1)} - 3^{2(k+1)}$ is divisible by 7. But

$$\begin{aligned} 2^{4(k+1)} - 3^{2(k+1)} = 2^{4k+4} - 3^{2k+2} &= 2^{4k+4} - 2^4\,3^{2k} + 2^4\,3^{2k} - 3^{2k+2} \\ &= 2^4\,(2^{4k} - 3^{2k}) + 3^{2k}(2^4 - 3^2) \\ &= 16\,(2^{4k} - 3^{2k}) + 7\,(3^{2k}). \end{aligned}$$

Now $7\,(3^{2k})$ is divisible by 7 and by the induction assumption $2^{4k} - 3^{2k}$ is also divisible by 7; so $16\,(2^{4k}-3^{2k}) + 7\,(3^{2k})$ is divisible by 7 whcih means $2^{4(k+1)} - 3^{2(k+1)}$ is divisible by 7.
Therefore by the principle of mathematical induction $2^{4n} - 3^{2n}$ is is divisible by 7 for all $n \geq 1$.

2. Consider the sum $(5)^2 + (11)^2 + (17)^2 + \cdots + (18n-1)^2$:

(a) Write the sum in sigma notation.

Solution: Since $5 = 6(1) - 1$, $11 = 6(2) - 1$ and $(18n-1) = [6(3n) - 1]$ so

$$(5)^2 + (11)^2 + (17)^2 + \cdots + (18n-1)^2 = \sum_{j=1}^{3n} (6j-1)^2.$$

(b) Use identities $\displaystyle\sum_{k=1}^{m} k = \frac{1}{2}\,[m(m+1)]$ and $\displaystyle\sum_{k=1}^{m} k^2 = \frac{1}{6}\,[m(m+1)(2m+1)]$ to prove that

$$(5)^2 + (11)^2 + (17)^2 + \cdots + (18n-1)^2 = 3n\,(108n^2 + 36n + 1).$$

Solution:

$$(5)^2 + (11)^2 + (17)^2 + \cdots + (18n-1)^2 = \sum_{j=1}^{3n} (6j-1)^2$$

$$= \sum_{j=1}^{3n} (36j^2 - 12j + 1)$$

$$= 36 \sum_{j=1}^{3n} j^2 - 12 \sum_{j=1}^{3n} j + \sum_{j=1}^{3n} 1$$

$$= 36\,[\frac{1}{6}(3n)(3n+1)(6n+1)] - 12[\frac{1}{2}(3n)(3n+1)] + 3n$$

$$= 3n\,[6(3n+1)(6n+1) - 6(3n+1) + 1]$$

$$= 3n\,(108n^2 + 36n + 1)\,.$$

3. Prove that $\sum_{\ell=1}^{n} \ell(\ell+2) = \frac{1}{6}\,[n(n+1)(2n+7)]$ **by each of the following two methods:**

(a) **By mathematical induction on positive integer** $n \geq 1$ **.**

(b) **By using the identities mentioned in part (b) of question** 2 **.**

Solution:

(a) Let $P(n)$ **be the statement** $\sum_{\ell=1}^{n} \ell(\ell+2) = \frac{1}{6}\,[n(n+1)(2n+7)]$**.**

If $n = 1$**, then** $P(1)$ **is true beacuse** $\sum_{\ell=1}^{1} \ell(\ell+2) = 1(1+2) = 3$ **and also** $\frac{1}{6}[1(1+1)(2+7)] = \frac{1}{6}(18) = 3$**.**

We assume that for $n = k$**,** $P(k)$ **is valid that is**

$$\sum_{\ell=1}^{k} \ell(\ell+2) \;=\; \frac{1}{6}[k(k+1)(2k+7)] \qquad (*)$$

We need to prove that for $n = k+1$**,** $P(k+1)$ **is valid that is we must prove that**

$$\sum_{\ell=1}^{k+1} \ell(\ell+2) \;=\; \frac{1}{6}\big[(k+1)(k+2)(2k+9)\big]\,.$$

But

$$\begin{aligned}
\sum_{\ell=1}^{k+1} \ell(\ell+2) &= \sum_{\ell=1}^{k} \ell(\ell+2) \;+\; (k+1)(k+3) \\
&= \frac{1}{6}[k(k+1)(2k+7)] \;+\; (k+1)(k+3) \quad \textbf{by } (*) \\
&= \frac{1}{6}\big[k(k+1)(2k+7) \;+\; 6(k+1)(k+3)\big] \\
&= \frac{1}{6}\Big[(k+1)\big(k(2k+7) \;+\; 6(k+3)\big)\Big] \\
&= \frac{1}{6}\big[(k+1)(2k^2+13k+18)\big] \\
&= \frac{1}{6}\big[(k+1)(k+2)(2k+9)\big]\,.
\end{aligned}$$

Therefore by the principle of mathematical induction $P(n)$ is valid for all $n \geq 1$.

(b) Using the given formulas we have

$$\begin{aligned}
\sum_{\ell=1}^{n} \ell(\ell+2) = \sum_{\ell=1}^{n} \ell^2 \;+\; 2\sum_{\ell=1}^{n} \ell &= \frac{1}{6}[n(n+1)(2n+1)] \;+\; 2\left[\frac{1}{2}\big(n(n+1)\big)\right] \\
&= \frac{1}{6}\big[n(n+1)(2n+1) \;+\; 6n(n+1)\big] \\
&= \frac{1}{6}\Big[(n+1)\big(n(2n+1) \;+\; 6n\big)\Big] \\
&= \frac{1}{6}\Big[(n+1)\big(2n^2 \;+\; 7n\big)\Big] \\
&= \frac{1}{6}[n(n+1)(2n+7)]\,.
\end{aligned}$$

4. For each of the following sums, rewrite the sum such that it starts from the given number. Keep your answer in sigma notation but simplify it.

(a) $\displaystyle\sum_{j=6}^{25} \big[(3j-15)^3 + j(j-10) + 25\big]$ **starting with** $j = 1$**.**

Solution: First we notice that $\displaystyle\sum_{j=6}^{25} \big[(3j-15)^3 + j(j-10) + 25\big] = \sum_{j=6}^{25} \big[(3j-15)^3 + (j-5)^2\big]$**.**

Now if we replace j by $j+5$, then we get

$$\begin{aligned}\sum_{j=6}^{25}\left[(3j-15)^3+j(j-10)+25\right] &= \sum_{j=6}^{25}\left[(3j-15)^3+(j-5)^2\right]\\ &= \sum_{j=1}^{20}\left[\big(3(j+5)-15\big)^3 + (j+5-5)^2\right]\\ &= \sum_{j=1}^{20}\left[(3j)^3 + j^2\right]\\ &= 27\sum_{j=1}^{20} j^3 + \sum_{j=1}^{20} j^2 .\end{aligned}$$

(b) $\displaystyle\sum_{k=-3}^{n-3}\left[(6+2k)^2+\frac{k+3}{k(k+4)}\right]$ **starting with** $k=0$**.**

Solution: If we replace k **by** $k-3$ **, then we get**

$$\begin{aligned}\sum_{k=-3}^{n-3}\left[(6+2k)^2+\frac{k+3}{k(k+4)}\right] &= \sum_{k=0}^{n}\left[\big(6+2(k-3)\big)^2+\frac{k-3+3}{(k-3)(k-3+4)}\right]\\ &= \sum_{k=0}^{n}\left[(2k)^2+\frac{k}{(k-3)(k+1)}\right]\\ &= 4\sum_{k=0}^{n} k^2 + \sum_{k=0}^{n}\frac{k}{k^2-2k-3} .\end{aligned}$$

5. Find all 4^{th} **roots of** -17 **in Cartesian form. Simplify as much as possible.**

Solution: Either simply notice that $-17 = 17(-i) = 17\,e^{\pi i}$ **, or say** $-17 = -17+0i$ **so** $r=\sqrt{(-17)^2+0^2}=17$ **and** $\tan\theta = \frac{0}{-17}$ **so** $\theta=\pi$**.**

Now all 4^{th} **roots of** $-17 = 17\,e^{\pi i}$ **are of form** $z_k = \sqrt[4]{17}\,e^{\dfrac{\pi+2k\pi}{4}i}$ **where** $k=0,1,2,3$**.**

If $k=0$ **then**

$$z_0 = \sqrt[4]{17}\,e^{\frac{\pi}{4}i} = \sqrt[4]{17}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) + i\,\sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\sqrt[4]{68} + i\,\frac{1}{2}\sqrt[4]{68}.$$

If $k=1$ **then**

$$z_1 = \sqrt[4]{17}\,e^{\frac{3\pi}{4}i} = \sqrt[4]{17}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt[4]{17}\left(\frac{-\sqrt{2}}{2}\right) + i\,\sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}\sqrt[4]{68} + i\,\frac{1}{2}\sqrt[4]{68}.$$

If $k=2$ **then**

$$z_2 = \sqrt[4]{17}\,e^{\frac{5\pi}{4}i} = \sqrt[4]{17}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = \sqrt[4]{17}\left(\frac{-\sqrt{2}}{2}\right) - i\,\sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) = -\frac{1}{2}\sqrt[4]{68} - i\,\frac{1}{2}\sqrt[4]{68}.$$

If $k=3$ **then**

$$z_3 = \sqrt[4]{17}\,e^{\frac{7\pi}{4}i} = \sqrt[4]{17}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) - i\,\sqrt[4]{17}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{2}\sqrt[4]{68} - i\,\frac{1}{2}\sqrt[4]{68}.$$

6. Let $z = \frac{1}{2}i - \frac{\sqrt{3}}{2}$, evaluate $z^{123} + 2i\,\overline{z} + \dfrac{-3+\sqrt{3}i}{1+\sqrt{3}i}$. Simplify as much as possible.

Solution: Since $r = \sqrt{(\frac{-\sqrt{3}}{2})^2 + \frac{1}{2})^2} = \sqrt{\frac{3}{4}+\frac{1}{4}} = 1$ **and** $\tan\theta = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \frac{1}{-\sqrt{3}} = -\frac{\sqrt{3}}{3}$ **so** $\theta = \frac{5\pi}{6}$ **that is** $z = 1e^{\frac{5\pi}{6}\,i}$**. Therefore**

$$z^{123} = 1^{123}\left(e^{\frac{5\pi}{6}\,i}\right)^{123} = e^{\frac{5(123)\pi}{6}\,i} = e^{\frac{205\pi}{2}\,i} = e^{\left(51(2\pi)+\frac{\pi}{2}\right)\,i} = e^{\frac{\pi}{2}\,i} = \cos\frac{\pi}{2} + i\,\sin\frac{\pi}{2} = 0+i = i\,.$$

Also $2i\,\overline{z} = 2i\,(\overline{\frac{-\sqrt{3}}{2} + \frac{1}{2}\,i}\,) = 2i\,(\frac{-\sqrt{3}}{2} - \frac{1}{2}\,i\,) = -\sqrt{3}\,i - i^2 = 1 - \sqrt{3}\,i$ **.**
On the other hand

$$\frac{-3+\sqrt{3}i}{1+\sqrt{3}i} = \frac{-3+\sqrt{3}i}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{-3+3\sqrt{3}\,i+\sqrt{3}\,i+3}{1+3} = \frac{4\sqrt{3}\,i}{4} = \sqrt{3}\,i\,.$$

Hence $z^{123} + 2i\,\overline{z} + \dfrac{-3+\sqrt{3}i}{1+\sqrt{3}i} = i + (1 - \sqrt{3}\,i) + \sqrt{3}\,i = 1 + i$**.**

7. For each of the following statements, if it is true prove it, and if it is false give a counter example.

(a) $\overline{z} = \dfrac{|z|^2}{z}, \quad (z \neq 0)$**;**

(b) $\arg(z) = \arg(\overline{z})$**;**

(c) $z\,\overline{(z + z|z|)} = |z|^2\,(1 + |z|)$**;**

(d) $\dfrac{e^{i\,\theta^2}\,(e^{i\,\theta})^2}{e^{i^3}} = \cos(\theta+1)^2 + i\sin(\theta+1)^2$**.**

Solution:

(a) It is true. Let $z = x + yi$ **then**

$$\frac{|z|^2}{z} = \frac{x^2+y^2}{x+yi} = \frac{x^2+y^2}{x+yi}\times\frac{x-yi}{x-yi} = \frac{(x^2+y^2)(x-yi)}{x^2-y^2i^2} = \frac{(x^2+y^2)(x-yi)}{x^2+y^2} = x-yi = \overline{z}\,.$$

(b) It is false. For example if $z = i$ **then** $\overline{z} = -i$ **and** $\arg(z) = \frac{\pi}{2}$ **while** $\arg(\overline{z}) = \pi$**.**

(c) It is true. From part (a) **we get** $z\overline{z} = |z|^2$**; also we know that in general** $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$ **and** $\overline{(kz)} = k\,\overline{z}$**. Therefore**

$$z\,\overline{(z + z|z|)} = z\left(\overline{z} + \overline{z|z|}\,\right) = z\left(\overline{z} + |z|\,\overline{z}\,\right) = z\overline{z}\,(1 + |z|) = |z|^2\,(1 + |z|)\,.$$

(d) It is true, because

$$\frac{e^{i\,\theta^2}\,(e^{i\,\theta})^2}{e^{i^3}}$$
$$= e^{i\,\theta^2}\,e^{i\,(2\theta)}\,e^{-i^3} = e^{i\,\theta^2}\,e^{i\,(2\theta)}\,e^{i} = e^{(\theta^2+2\theta+1)\,i} = e^{(\theta+1)^2\,i} = \cos(\theta+1)^2 + i\sin(\theta+1)^2\,.$$