MATH 2130 – Tutorial Problem Solutions, Thu Jan 18

Lines and planes

Example. Find the equation of the plane that is perpendicular to the plane $x - y + 2z + 3 = 0$ and that contains the line $\mathbf{r} = (2, 3, -1) + t(1, 0, -2)$.

Solution. A normal vector to the plane $x - y + 2z + 3 = 0$ is $(1, -1, 2)$. A normal vector to the plane we want must be perpendicular to $(1, -1, 2)$. Therefore $(1, -1, 2)$ is parallel to the desired plane.

From the equation of the line, another vector parallel to the desired plane is $(1, 0, -2)$. Therefore a normal vector is

$$(1, -1, 2) \times (1, 0, -2) = (2, 4, 1).$$

The equation of the plane has the form

$$2x + 4y + z + D = 0.$$

From the equation of the line, the plane must contain the point $(2, 3, -1)$. With this substitution, we find $D = -15$. Thus the equation of the plane is

$$2x + 4y + z - 15 = 0.$$

Example. Find the equation of the plane that contains the point $(0, 1, 0)$ and the line $\mathbf{r} = (1, -2, 1) + t(0, 2, 1)$, $t \in \mathbb{R}$.

Solution. The plane contains the points $(0, 1, 0)$ and $(1, -2, 1)$. Therefore one vector parallel to the plane is $(1, -2, 1) - (0, 1, 0) = (1, -3, 1)$. From the equation of the line, another vector parallel to the plane is $(0, 2, 1)$. Therefore a normal vector is

$$(1, -3, 1) \times (0, 2, 1) = (-5, -1, 2).$$

The equation of the plane takes the form

$$-5x - y + 2z + D = 0.$$

To find D, we substitute the point $(0, 1, 0)$, and get $D = 1$. Therefore the equation of the plane is

$$-5x - y + 2z + 1 = 0.$$

Example. Let ℓ be the line $\mathbf{r} = (-\frac{1}{2}, 0, 2) + t(0, 3, 2)$, $t \in \mathbb{R}$, and let m be the line $\mathbf{r} = (1, 0, 0) + s(1, 2, 0)$, $s \in \mathbb{R}$. Show that ℓ and m intersect. Then find the vector equation of the line n that passes through this point of intersection, making right angles with both ℓ and m.

Solution. A point on ℓ has the form $\left(-\frac{1}{2}, 3t, 2 + 2t\right)$ for some $t \in \mathbb{R}$, and a point on m has the form $(1 + s, 2s, 0)$ for some $s \in \mathbb{R}$. For the lines to intersect, there must be values of t and s such that

$$\left(-\frac{1}{2}, 3t, 2 + 2t\right) = (1 + s, 2s, 0)\,.$$

From the x-coordinates, we get $s = -\frac{3}{2}$. From the z-coordinates, we get $t = -1$. The point on ℓ corresponding to $t = -1$ is $\left(-\frac{1}{2}, -3, 0\right)$, and the point on m corresponding to $s = -\frac{3}{2}$ is $\left(-\frac{1}{2}, -3, 0\right)$. Since all three coordinates agree, this is the point of intersection of the two lines.

Now we are asked to find the vector equation of the line n through $\left(-\frac{1}{2}, -3, 0\right)$, perpendicular to ℓ and m. A vector parallel to ℓ is $(0, 3, 2)$, and a vector parallel to m is $(1, 2, 0)$. A vector perpendicular to them both is

$$(0, 3, 2) \times (1, 2, 0) = (-4, 2, -3).$$

Thus a vector equation for the desired line is

$$\mathbf{r} = \left(-\frac{1}{2}, -3, 0\right) + t(-4, 2, -3), \quad t \in \mathbb{R}.$$

Distances

Example. Find the equations of all planes that are perpendicular to the vector $\mathbf{v} = (1, 0, -1)$ and a distance 2 from the point $P = (1, 1, 2)$.

Solution. The equation of such a plane takes the form

$$x - z + D = 0.$$

To find the distance from a point to a plane, we need a unit normal vector to the plane, and an arbitrary point on the plane. A unit normal vector is

$$\widehat{\mathbf{v}} = \frac{1}{\sqrt{2}}(1, 0, -1).$$

Let $Q = (a, b, c)$ be a point on the plane. Then $a - c + D = 0$. The vector $\mathbf{PQ}$ is $(a-1, b-1, c-2)$, and the distance from P to the plane is

$$|\widehat{\mathbf{v}} \cdot \mathbf{PQ}| = \frac{1}{\sqrt{2}} |a - 1 - c + 2| = \frac{1}{\sqrt{2}} |a - c + 1|.$$

We set this distance equal to 2, and note that $a - c = -D$:

$$\frac{1}{\sqrt{2}}|1 - D| = 2.$$

Thus D must satisfy $|1-D| = 2\sqrt{2}$. There are two solutions, namely $D = 1+2\sqrt{2}$ and $D = 1-2\sqrt{2}$. Therefore the equations of the planes that satisfy the given conditions are

$$x - z + 1 + 2\sqrt{2} = 0, \quad x - z + 1 - 2\sqrt{2} = 0.$$

More generally, let $Ax + By + Cz + D = 0$ be a plane, and let $Q = (x, y, z)$ be a point on the plane. A normal vector for the plane is $\mathbf{v} = (A, B, C)$. Notice that $\mathbf{v} \cdot \mathbf{Q} = Ax + By + Cz = -D$.

Let $P = (p, q, r)$ be a point not on the plane. The distance between P and this plane is

$$\begin{aligned}
|\widehat{\mathbf{v}} \cdot \mathbf{PQ}| &= \frac{1}{\sqrt{A^2 + B^2 + C^2}} |\mathbf{v} \cdot (\mathbf{Q} - \mathbf{P})| \\
&= \frac{1}{\sqrt{A^2 + B^2 + C^2}} |-D - \mathbf{v} \cdot \mathbf{P}| \\
&= \frac{|Ap + Bq + Cr + D|}{\sqrt{A^2 + B^2 + C^2}}.
\end{aligned}$$

This is a shortcut for finding the distance from a point to a plane.