UNIVERSITY OF MANITOBA

DATE: November 5, 2014 TERM TEST 2

EXAMINATION: Engineering Mathematical Analysis 1 TIME: 70 minutes

COURSE: MATH 2130 EXAMINER: various

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1. *Evaluate the limit, if it exists. Justify your answer.*

[3] (a) $\displaystyle\lim_{(x,y)\to(0,0)} \left(\frac{x^2-y^2}{x^2+y^2}\right)^2$.

[3] (b) $\displaystyle\lim_{(x,y)\to(1,-4)} \frac{(x-1)^3-2(y+4)^2}{3(x-1)^2+(y+4)^2}$.

Solution:

(a) Trying along the path $y = mx$ for any number m yields

$$\lim_{x\to 0}\left(\frac{x^2-m^2x^2}{x^2+m^2x^2}\right)^2 = \lim_{x\to 0}\left(\frac{1-m^2}{1+m^2}\right)^2 = \left(\frac{1-m^2}{1+m^2}\right)^2$$

Since the limit depends on the path, the limit does not exist.

(b) Trying along the path $y = -4$

$$\lim_{x\to 1}\frac{(x-1)^3-2(0)^2}{3(x-1)^2+(0)^2} = \lim_{x\to 1}\frac{x-1}{3} = 0$$

Trying along the path $x = 1$

$$\lim_{y\to -4}\frac{(0)^3-2(y+4)^2}{3(0)^2+(y+4)^2} = \lim_{y\to -4} -2 = -2$$

Since the limit depends on the path, the limit does not exist.

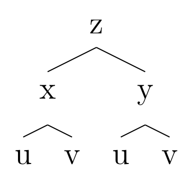
2. *Let* $z = \cos(xy^2)$ *where* $x = 3u^2 + v^3, \quad y = 2uv^2$.

[5] (a) *Use the chain rule to determine* $\left.\dfrac{\partial z}{\partial u}\right)_v$.

[5] (b) *Use the chain rule to determine* $\left.\dfrac{\partial^2 z}{\partial u^2}\right)_v$. *Do not simplify your answer.*

Solution:

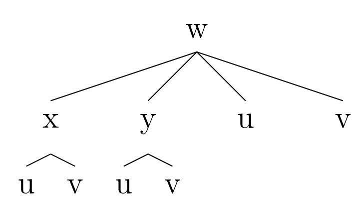
(a) The tree is



Hence

$$\begin{aligned}\frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ &= -y^2 \sin(xy^2)(6u) - 2xy\sin(xy^2)(2v^2) \\ &= -6uy^2\sin(xy^2) - 4xyv^2\sin(xy^2)\end{aligned}$$

(b) Let $w = \dfrac{\partial z}{\partial u} = -6uy^2\sin(xy^2) - 4xyv^2\sin(xy^2)$. Then $\left.\dfrac{\partial^2 z}{\partial u^2}\right)_v = \left.\dfrac{\partial w}{\partial u}\right)_v$ which has a tree



Hence

$$\begin{aligned}\left.\frac{\partial^2 z}{\partial u^2}\right)_v &= \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \left.\frac{\partial w}{\partial u}\right)_{x,y,u} \\ &= \Big(-6uy^2(y^2)\cos(xy^2) - (4yv^2\sin(xy^2) + 4xyv^2(y^2)\cos(xy^2))\Big)(6u) \\ &\quad + \Big(-(6u(2y)\sin(xy^2) + 6uy^2(2xy)\cos(xy^2)) \\ &\qquad - (4xv^2\sin(xy^2) + 4xyv^2(2xy)\cos(xy^2))\Big)(2v^2) - 6y^2\sin(xy^2)\end{aligned}$$

[10] 3. *The equations*

$$x^2y^3\cos v + 2u^2\sin w = \sqrt{2}$$
$$y^3 - \cos v + u\cos^2 v + y^2 = 0$$
$$yu + x\cos v + \sin w = \frac{\sqrt{2}}{2} - 1$$

define u, v, and w as functions of x and y.

Compute $\left.\frac{\partial v}{\partial y}\right)_x$ *when* $x = 1, y = -1, u = 1, v = \pi/2, w = \pi/4$.

Solution:

Let

$$F = x^2y^3\cos v + 2u^2\sin w - \sqrt{2}$$
$$G = y^3 - \cos v + u\cos^2 v + y^2$$
$$H = yu + x\cos v + \sin w - \frac{\sqrt{2}}{2} + 1$$

$$\left.\frac{\partial v}{\partial y}\right)_x = -\frac{\dfrac{\partial(F,G,H)}{\partial(u,y,w)}}{\dfrac{\partial(F,G,H)}{\partial(u,v,w)}} = -\frac{\begin{vmatrix} F_u & F_y & F_w \\ G_u & G_y & G_w \\ H_u & H_y & H_w \end{vmatrix}}{\begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}}$$

The denominator is

The determinant in the denominator is

$$\begin{vmatrix} 4u\sin w & -x^2y^3\sin v & 2u^2\cos w \\ \cos^2 v & \sin v - 2u\cos v\sin v & 0 \\ y & -x\sin v & \cos w \end{vmatrix}$$
$$= \begin{vmatrix} 2\sqrt{2} & 1 & \sqrt{2} \\ 0 & 1 & 0 \\ -1 & -1 & \frac{1}{\sqrt{2}} \end{vmatrix}$$
$$= 2 + \sqrt{2}$$

at $x = 1, y = -1, u = 1, v = \pi/2, w = \pi/4$.

The determinant in the numerator is

$$\begin{vmatrix} 4u\sin w & 3x^2y^2\cos v & 2u^2\cos w \\ \cos^2 v & 3y^2 + 2y & 0 \\ y & u & \cos w \end{vmatrix}$$
$$= \begin{vmatrix} 2\sqrt{2} & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ -1 & 1 & \frac{1}{\sqrt{2}} \end{vmatrix}$$
$$= 2 + \sqrt{2}$$

at $x = 1, y = -1, u = 1, v = \pi/2, w = \pi/4$.

Thus the derivative is

$$\left.\frac{\partial v}{\partial y}\right)_x = -\frac{2+\sqrt{2}}{2+\sqrt{2}} = -1.$$

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4. *The temperature at a point (x, y, z) is given by*

$$T(x, y, z) = 2e^{-x^2-2y^2-3z^2}$$

where T is measured in degrees Celsius and x, y, z in meters. Further let P be the point $P(2, -1, 2)$.

[4] (a) *Compute the directional derivative of the function T at the point P in the direction toward the point $Q(2, -3, 3)$. Include units in your answer.*

[2] (b) *In which direction does the temperature increase fastest at P ?*

[2] (c) *Compute the maximum rate of increase of T at P. Include units in your answer.*

Solution:

(a) $\mathbf{v} = \mathbf{PQ} = \langle 0, -2, 1\rangle$ which has unit vector $\hat{\mathbf{v}} = \frac{1}{\sqrt{5}}\langle 0, -2, 1\rangle$.

$\nabla T = \langle -4xe^{-x^2-2y^2-3z^2},\ -8ye^{-x^2-2y^2-3z^2},\ -12ze^{-x^2-2y^2-3z^2}\rangle$ and therefore

$\nabla T(2, -1, 2) = \langle -8e^{-18},\ 8e^{-18},\ -24e^{-18}\rangle$ Hence the directional dervia-tive is

$$D_{\mathbf{v}}T = \langle -8e^{-18},\ 8e^{-18},\ -24e^{-18}\rangle \cdot \frac{1}{\sqrt{5}}\langle 0, -2, 1\rangle = -8\sqrt{5}e^{-18}\ {}^\circ C/m.$$

(b) The direction which the rate of change increases the fastest is in the direction of gradient $\langle -8e^{-18},\ 8ye^{-18},\ -24e^{-18}\rangle$ or as a unit vector $\frac{\langle -1,\ 1,\ -3\rangle}{\sqrt{11}}$

(c) The maximum rate of increase is the length of the gradient which is

$$\left|\langle -8e^{-18},\ 8ye^{-18},\ -24e^{-18}\rangle\right| = 8e^{-18}\sqrt{(-1)^2 + (1)^2 + (-3)^2} = 8\sqrt{11}e^{-18}\ {}^\circ C/m$$

[6] 5. *Determine parametric equations for the tangent line to the curve*

$$x^2z + 3y^2x - 3z^2y = -11, \qquad x^2 + 2xy + z^2 + 3y^2 = 10$$

at $P_0(1, 1, -2)$.

Solution:

The vector parallel to the tangent line must be perpendicular to the normal to both surfaces at the point P_0. Therefore

$$\mathbf{v} = \nabla F_1(1, 1, -2) \times \nabla F_1(1, 1, -2)$$

where

$$F_1 = x^2z + 3y^2x - 3z^2y + 11$$
$$F_2 = x^2 + 2xy + z^2 + 3y^2 - 10$$

$$\nabla F_1(1, 1, -2) = \langle 2xz + 3y^2,\ 6xy - 3z^2,\ x^2 - 6yz\rangle_{(1,1,-2)} = \langle -1,\ -6,\ 13\rangle.$$

$$\nabla F_2(1, 1, -2) = \langle 2x + 2y,\ 2x + 6y,\ 2z\rangle_{(1,1,-2)} = \langle 4,\ 8,\ -4\rangle.$$

Hence

$$\mathbf{v} = \langle -1,\ -6,\ 13\rangle \times \langle 4,\ 8,\ -4\rangle = \langle -80,\ 48,\ 16\rangle$$

(or you can use the parallel vector $\langle -5,\ 3,\ 1\rangle$.)

Hence the equations of the tangent line are

$$x = 1 - 5t$$
$$y = 1 + 3t$$
$$z = -2 + t$$

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[10] 6. *Determine all critical points of $f(x, y) = -x^3 + 4xy - 2y^2 + 1$, and classify each point as yielding a relative maximum, relative minimum, a saddle point, or none of these.*

Solution:

$f_x = -3x^2 + 4y = 0$ and $f_y = 4x - 4y = 0$ imply that $y = x \Rightarrow -3x^2 + 4x = 0 \Rightarrow x = 0$, $x = 4/3$.

Hence the critical points are $(0, 0)$ and $(4/3, 4/3)$.

$f_{xx} = -6x, \quad f_{xy} = 4, \quad f_{yy} = -4.$

For $(0, 0)$ we have that

$$A = 0,\ B = 4,\ C = -4 \Rightarrow B^2 - AC = 16 > 0$$

and therefore $(0, 0)$ yields a saddle point.

For $(4/3,\ 4/3)$ we have that

$$A = -8,\ B = 4,\ C = -4 \Rightarrow B^2 - AC = -16 > 0 \text{ and } A < 0$$

and therefore $(4/3,\ 4/3)$ yields a relative maximum.