MATH 1210 Assignment 3

March 07, 2014

Solutions

Question 1. Consider the two points $P(1,2,3)$ and $Q(2,3,1)$, and let $\mathbf{v} = \vec{OQ}$ be the vector with its tail at the origin and with its head at Q.

(a) Find an equation of the line passing through P and Q.

Solution: A direction vector from P to Q is given by $\overline{PQ} = \langle 2,3,1 \rangle - \langle 1,2,3 \rangle = \langle 1,1,-2 \rangle$. Thus an equation of the line through these two points is $\langle x,y,z \rangle = \langle 1,2,3 \rangle + t \langle 1,1,-2 \rangle$, or

$$\begin{cases} x &= 1+t \\ y &= 2+t \\ z &= 3-2t \end{cases}$$

Comment: It is equally correct to compute $\overline{QP}$ and/or to use Q as the given point on the line rather than P.

(b) Find an equation of the line passing through P in the direction of $\mathbf{v}$.

Solution: So $\mathbf{v}$ is just the vector with the coordinates of the point Q: $\mathbf{v} = \langle 2,3,1 \rangle$. So an equation of the line is $\langle x,y,z \rangle = \langle 1,2,3 \rangle + t \langle 2,3,1 \rangle$, or

$$\begin{cases} x &= 1+2t \\ y &= 2+3t \\ z &= 3+t \end{cases}$$

(c) Find an equation of the plane through the origin containing P and Q.

Solution: In other words, we are being asked to find the equation of a plane containing 3 points not all on the same straight line. Vectors between any two of these points lie in the plane, in particular (since the origin is one of the three points in this question) the vectors $\langle 1,2,3 \rangle$ and $\langle 2,3,1 \rangle$ lie in the plane. A vector perpendicular to both of them (and hence, a normal vector to the plane) is given by $\langle 1,2,3 \rangle \times \langle 2,3,1 \rangle = \hat{\mathbf{i}}(2-9) - \hat{\mathbf{j}}(1-6) + \hat{\mathbf{k}}(3-4) = \langle -7,5,-1 \rangle$. An equation of the plane is then any of the following:

$$\begin{aligned} \mathbf{x} &= \langle -7,5,-1 \rangle (\mathbf{x} - \langle 1,2,3 \rangle) = 0 \\ -7(x-1) + 5(y-2) - (z-3) &= 0 \\ -7x + 5y - z + 20 &= 0 \\ -7x + 5y - z &= -20 \\ 7x - 5y + z &= 20 \end{aligned}$$

(d) Find an equation of the plane containing P and normal to $\mathbf{v}$.

Solution:

$$\begin{aligned}
\mathbf{x} &= \langle 2,3,1 \rangle (\mathbf{x} - \langle 1,2,3 \rangle) = 0 \\
2(x-1) + 3(y-2) + (z-3) &= 0 \\
2x + 3y + z - 11 &= 0 \\
2x + 3y + z &= 11
\end{aligned}$$

Question 2. Let ℓ_1 be the line with equation $\langle x,y,z \rangle = \langle 1,4,2 \rangle + s \langle 2,0,1 \rangle$ and let ℓ_2 be the line with equation $\langle x,y,z \rangle = \langle 2,-3,1 \rangle + t \langle 1,1,3 \rangle$.

(a) Show that the two lines do not intersect.

Solution: If they did intersect we would be able to find values for the parameters s and t which produce the same point:

$$\begin{aligned}
x &= 1+2s &= 2+t \\
y &= 4 &= -3+t \\
z &= 2+s &= 1+3t
\end{aligned}$$

From the second of these equations, we see that $t = 7$. Substituting this into the first equation we find that $s = 4$, but substituting it into the third equation we find that $s = 20$, contradictory statements. Therefore the two lines do not intersect.

(b) Find a vector perpendicular to both lines.

Solution: We get direction vectors for the lines directly from their equatioons: $\langle 2,0,1 \rangle$ and $\langle 1,1,3 \rangle$. A vector perpendicular to both of them is $\langle 2,0,1 \rangle \times \langle 1,1,3 \rangle = \hat{\mathbf{i}}(0-1) - \hat{\mathbf{j}}(6-1) + \hat{\mathbf{k}}(2-0) = \langle -1,-5,2 \rangle$.

(c) Find equations of the two parallel planes Π_1 and Π_2 such that Π_1 contains ℓ_1 and Π_2 contains ℓ_2.

Solution: We have a normal vector from the previous part, and the equations for the two lines gives us points on the lines. So an equation of Π_1 is $\langle -1,-5,2 \rangle (\mathbf{x} - \langle 1,4,2 \rangle) = 0$ and an equation of Π_2 is $\langle -1,-5,2 \rangle (\mathbf{x} - \langle 2,-3,1 \rangle) = 0$, in other words

$$\begin{aligned}
-x - 5y + 2z &= -17 && [\Pi_1] \\
-x - 5y + 2z &= 15 && [\Pi_2]
\end{aligned}$$

Question 3. Solve the following systems of linear equations by Gaussian elimination.

$$\begin{cases} 2x + 2y + \quad = 3 \\ x \qquad + z = 4 \\ \quad 3y + 3z = 9 \end{cases} \qquad \begin{cases} 2X + 2Y \qquad = 3 \\ X \qquad + Z = 4 \\ \quad 3Y - 3Z = 9 \end{cases}$$

Solution: In each case we start with the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 2 & 2 & 0 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 3 & 3 & 9 \end{array}\right) \qquad R_2 \leftrightarrow R_1 \qquad \left(\begin{array}{ccc|c} 2 & 2 & 0 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 3 & -3 & 9 \end{array}\right)$$

Some of you may have taken a shortcut by interchanging Rows 1 and 2 mentally as you wrote down the augmented matrices of these systems. This is perfectly acceptable, but you should tell us what you have done as part of your solution.

Some of you may have been even more perceptive and will have realized that the two systems can be solved by essentially the same steps, at least up to the last step. The row operations we display here apply to BOTH augmented matrices.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & 2 & 0 & 3 \\ 0 & 3 & 3 & 9 \end{array}\right) \qquad R_2 \to R_2 - 2R_1 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & 2 & 0 & 3 \\ 0 & 3 & -3 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & -2 & -5 \\ 0 & 3 & 3 & 9 \end{array}\right) \qquad R_2 \to \tfrac{1}{2}R_2 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & -2 & -5 \\ 0 & 3 & -3 & 9 \end{array}\right)$$

This is the "strict" method according to the rules of Gaussian elimination. You could avoid fractions by the two steps $R_2 \to R_2 - R_3$, $R_2 \to -R_2$.

Another way of accomplishing the same thing is by the two steps $R_3 \to \frac{1}{3}R_3$, $R_2 \leftrightarrow R_3$. These are both acceptable forms of the solution.

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 3 & 3 & 9 \end{array}\right) \qquad R_3 \to R_3 - 3R_2 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 3 & -3 & 9 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 6 & \frac{33}{2} \end{array}\right) \qquad R_3 \to \tfrac{1}{6}R_3 \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 0 & \frac{33}{2} \end{array}\right)$$

Of course, at the end, the row operation is applied only to the left hand augmented matrix!

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 1 & \frac{11}{4} \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 0 & \frac{11}{4} \end{array}\right)$$

So the first system has a unique solution: $z = \frac{11}{4}$, $y = -\frac{5}{2} + \frac{11}{4} = \frac{1}{4}$, and $x = 4 - \frac{11}{4} = \frac{5}{4}$.

On the other hand, the second system is inconsistent (has no solutions).

Question 4. Solve the following system of linear equations by Gaussian elimination.

$$\begin{cases} \quad w - 2x - 2y + \ 7z &= \ -1 \\ -2w + 4x + 3y - 12z &= \ -1 \\ \quad w - 2x \qquad\quad + \ 3z &= \quad 5 \end{cases}$$

Solution: We start by setting up an augmented matrix for the system:

$$\left(\begin{array}{rrrr|r} 1 & -2 & -2 & 7 & -1 \\ -2 & 4 & 3 & -12 & -1 \\ 1 & -2 & 0 & 3 & 5 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left(\begin{array}{rrrr|r} 1 & -2 & -2 & 7 & -1 \\ 0 & 0 & -1 & 2 & -3 \\ 0 & 0 & 2 & -4 & 6 \end{array}\right) \qquad \begin{array}{l} \text{First: } R_2 \to -R_2 \\ \text{Followed by: } R_3 \to R_3 - 2R_2 \end{array}$$

$$\left(\begin{array}{rrrr|r} 1 & -2 & -2 & 7 & -1 \\ 0 & 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This matrix is in row-echelon form with leading 1s in the columns corresponding to w and y. So we take $x = s$ and $z = t$ as parameters, and use the two equations represented by the non-zero rows of this matrix to write w and y in terms of the parameters:

$$\begin{cases} w &= 5 + 2s - 3t \\ x &= \qquad s \\ y &= 3 \qquad + 2t \\ z &= \qquad\qquad t \end{cases}$$

Question 5. Solve the following system of linear equations by Gauss-Jordan elimination:

$$\begin{cases} 4x + 6y - \ z &= \quad 9 \\ 2x + 3y &= \quad 1 \\ 2x + 2y - 3z &= -9 \end{cases}$$

Solution: We start by setting up the augmented matrix of this system. We anticipate some computational difficulties by interchanging the first and second equations as we write them down.

$$\left(\begin{array}{rrr|r} 2 & 3 & 0 & 1 \\ 4 & 6 & -1 & 9 \\ 2 & 2 & -3 & -9 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

We don't see any urgent need now to make a leading 1 at the beginning of row 1, so we will avoid the fractions as long as possible!

$$\left(\begin{array}{rrr|r} 2 & 3 & 0 & 1 \\ 0 & 0 & -1 & 7 \\ 0 & -1 & -3 & -10 \end{array}\right) \qquad \begin{array}{c} R_2 \to -R_2 \\ R_3 \to -R_3 \\ \hline R_2 \leftrightarrow R_3 \end{array}$$

Note that FIRST we multiply each of the last two rows by -1, THEN we interchange the resulting rows.

$$\left(\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 1 & 3 & 10 \\ 0 & 0 & 1 & -7 \end{array}\right) \qquad R_2 \to R_2 - 3R_3$$

$$\left(\begin{array}{ccc|c} 2 & 3 & 0 & 1 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & -7 \end{array}\right) \qquad R_1 \to R_1 - 3R_2$$

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & -92 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & -7 \end{array}\right) \qquad R_1 \to \tfrac{1}{2}R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & -46 \\ 0 & 1 & 0 & 31 \\ 0 & 0 & 1 & -7 \end{array}\right)$$

This system of equations has the unique solution $x = -46,\ y = 31,\ z = -7$.

Question 6. Solve the following system of linear equations by Gauss-Jordan elimination.

$$\left\{\begin{array}{rcr} x_1 + 2x_2 + 2x_3 - 5x_4 \phantom{{}+2x_5} & = & -1 \\ 2x_1 + 4x_2 + 5x_3 - 12x_4 + x_5 & = & 1 \\ -3x_1 - 6x_2 - 9x_3 + 21x_4 - 2x_5 & = & -1 \end{array}\right.$$

Solution: First we set up an augmented matrix for the system:

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 2 & -5 & 0 & -1 \\ 2 & 4 & 5 & -12 & 1 & 1 \\ -3 & -6 & -9 & 21 & -2 & -1 \end{array}\right) \qquad \begin{array}{l} R_2 \longrightarrow R_2 - 2R_1 \\ R_3 \longrightarrow R_3 + 3R_1 \end{array}$$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 2 & -5 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & -3 & 6 & -2 & -4 \end{array}\right) \qquad R_3 \longrightarrow R_3 + 3R_2$$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 2 & -5 & 0 & -1 \\ 0 & 0 & 1 & -2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right) \qquad R_2 \to R_2 - R_3$$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 2 & -5 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right) \qquad R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{rrrrr|r} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & -2 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{array}\right)$$

There are leading ones corresponding to x_1 , x_3 , and x_5 , so we take the other two variables as parameters and read off the general solution from the reduced row-echelon form:

$$\begin{cases} x_1 &= & 3 - 2s + t \\ x_2 &= & s \\ x_3 &= & -2 + 2t \\ x_4 &= & t \\ x_5 &= & 5 \end{cases}$$

Question 7. Consider the following matrix A, representing the augmented matrix of a system of linear equations, where c is some unknown constant.

$$A = \left[\begin{array}{rrr|r} 1 & 2 & -1 & c \\ -2 & -3 & 2-c & -3c \\ 3 & 6+c & -c-3 & c \end{array}\right]$$

(a) By a few simple row operations, you can find the first two rows of a row-echelon form of this matrix, and reduce the third row to the point where it has a (quadratic) polynomial in c as its leading entry. Leave it in this form, and use this information to answer the remaining parts of this question:

Solution: We begin by setting up an augmented matrix for the system:

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & c \\ -2 & -3 & 2-c & -3c \\ 3 & 6+c & -c-3 & c \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & c \\ 0 & 1 & -c & -c \\ 0 & c & -c & -2c \end{array}\right) \qquad R_3 \to R_3 - cR_2$$

It is an *error* at this stage to simplify Row 3 by multiplying by $\frac{1}{c}$. It might be the case that $c = 0$, in which case this operation is undefined. If you wish to use this method, you would then have to break the remaining part of your work into two cases, $c = 0$ and $c \neq 0$. There is nothing wrong with doing it this way, but it is more work than the solution that we give here.

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & c \\ 0 & 1 & -c & -c \\ 0 & 0 & c^2-c & c^2-2c \end{array}\right)$$

It helps a lot in the remaining parts to notice that the last two entries in the last row factor as $c(c-1)$ and $c(c-2)$.

(b) Find all values of c, if any, such that the system has no solutions.

Solution: There is no solution when the last row looks like $(0\ 0\ 0 \mid d)$, where d is some number different from 0. To get zeroes for the coefficients, we need $c = 0$ or $c = 1$; but $c = 0$ makes the constant column 0 as well.

So there is no solution when $c = 1$.

(c) Find all values of c, if any, such that the system has a unique solution.

Solution: There is a uniqe solution as long as we can get a leading 1 in the third row/third column, that is, when there are no parameter variables. We can find a leading 1 in this position as long as $c^2 - c \neq 0$, that is, when $c \neq 0$ AND $c \neq 1$.

(d) Find all the values of c, if any, such that the system has infinitely many solutions.

Solution: There will be infinitely many solutions whenever the system is consistent, but not every variable corresponds to a leading 1, that is, in this case, the third row being entirely 0 will be enough.

We already observed in part (b) that this happens when $c = 0$.

Note that the value $c = 2$ plays no special role in the answers to this problem.

Question 8. Students in three sections of a course take a midterm exam. An average percentage score is computed for each of the sections. There are 100 students in Section A01, 180 students in Section A02, and 120 students in Section A03. You are given the following pieces of information:

- The average of the three averages is 72%;
- The overall average of the three sections together is 74%;
- The average for section A02 is 20% more than the combined average for sections A01 and A03 together.

Set up a suitable system of linear equations and solve it in order to find the average percentage score for each section.

Solution: You can choose whatever names you like for the variables. We will call the average score for Section A01 a_1, the average score for Section A02 a_2, and the average score for Section A03 a_3.

Remember that the Average Score is the Total Score divided by the Number of Students. So we can recover the Total Score in each section by multiplying the average by the number of students in the section: $100a_1$, $180a_2$, and $120a_3$ respectively.

We observe that there are $100 + 180 + 120 = 400$ students altogether.

We get the following three equations, corresponding to the three pieces of information in order:

$$\begin{cases} \dfrac{a_1+a_2+a_3}{3} &= 72 \\ \dfrac{100a_1+180a_2+120a_3}{400} &= 74 \\ a_2 &= 20+\dfrac{100a_1+120a_3}{220} \end{cases}$$

Comment: Unfortunately, this question was not stated precisely. The third equation that is written down here reflects the intended interpretation of the third bulleted statement: namely that, since all the marks are being reported as percentages, percent should be regarded as a unit of measurement, and therefore the average in A02 is 20 percentage points more than the combined average of A01 and A03.

However it is also completely reasonable to interpret the third statement as saying that the average for section A02 is 1.2 times the combined average for the other two sections. This would yield the following equation:

$$a_2 = 1.2\left(\frac{100a_1+120a_3}{220}\right)$$

This leads to a much more difficult system of linear equations to solve than what we had intended.

We reorganize this system into standard form and clear fractions from the denominators before writing down an augmented matrix for the system. It looks like a good idea to start out by doing some 'housecleaning'!

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 216 \\ 100 & 180 & 120 & 74\times 400 \\ -100 & 220 & -120 & 20\times 220 \end{array}\right) \qquad \begin{array}{l} R_2 \to \frac{1}{20}R_2 \\ R_3 \to \frac{1}{20}R_3 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 216 \\ 5 & 9 & 6 & 74\times 20 = 1480 \\ -5 & 11 & -6 & 220 \end{array}\right) \qquad \begin{array}{l} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 + 5R_1 \end{array}$$

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 216 \\ 0 & 4 & 1 & 400 \\ 0 & 16 & -1 & 1300 \end{array}\right) \qquad R_3 \to R_3 - 4R_2$$

Better than dealing with fractions in row 2, we can finish row 3 right away.

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 216 \\ 0 & 4 & 1 & 400 \\ 0 & 0 & -5 & -300 \end{array}\right) \qquad R_3 \to \frac{-1}{5}R_3$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 216 \\ 0 & 4 & 1 & 500 \\ 0 & 0 & 1 & 60 \end{array}\right) \qquad \begin{array}{l} R_1 \to R_1 - R_3 \\ R_2 \to R_2 - R_3 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 156 \\ 0 & 4 & 0 & 340 \\ 0 & 0 & 1 & 60 \end{array}\right) \qquad \begin{array}{l} \text{First: } R_2 \to \frac{1}{4}R_2 \\ \text{Second: } R_1 \to R_1 - R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 71 \\ 0 & 1 & 0 & 85 \\ 0 & 0 & 1 & 60 \end{array}\right)$$

Therefore the average in section A01 was 71%, the average in section A02 was 85%, and the average in section A03 was 60%.