MATH2132 Test1 February 7, 2008 60 minutes

Student Name - Student Number -

Values

4 1. Find the limit of the sequence of functions $\{f_n(x)\}$ on the interval $0 \le x \le 5$, if it exists. Justify your answer.

$$f_n(x) = \frac{2n^2x + nx}{n^2+1}$$

$$\lim_{n\to\infty} \frac{2n^2x+nx}{n^2+1} = \lim_{n\to\infty} \frac{2x + \frac{x}{n}}{1+\frac{1}{n^2}} = 2x$$

5 2. Find the Taylor series about $x = -2$ for the function $f(x) = e^{2x+1}$. Include its interval of convergence.

$$e^{2x+1} = e^{2(x+2)-3} = e^{-3}e^{2(x+2)} = e^{-3}\sum_{n=0}^{\infty} \frac{[2(x+2)]^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n}{e^3 n!}(x+2)^n$$

This is valid for $-\infty < 2(x+2) < \infty$ or $-\infty < x < \infty$.

9 **3.** Find the open interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{n^3}(x+1)^{3n+1}.$$

Express your answer in the form $a < x < b$ for appropriate values of a and b.

$$\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{n^3}(x+1)^{3n+1} = (x+1)\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{n^3}(x+1)^{3n}$$

We set $y = (x+1)^3$, and consider the series $\sum_{n=3}^{\infty} \frac{(-1)^n 2^n}{n^3} y^n$. Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n 2^n}{n^3}}{\frac{(-1)^{n+1} 2^{n+1}}{(n+1)^3}} \right| = \frac{1}{2}.$$

It follows that $R_x = 1/2^{1/3}$. The open interval of convergence is

$$\begin{aligned} |x+1| &< \frac{1}{2^{1/3}} \\ -\frac{1}{2^{1/3}} < x+1 &< \frac{1}{2^{1/3}} \\ -1 - \frac{1}{2^{1/3}} < x &< -1 + \frac{1}{2^{1/3}} \end{aligned}$$

10 **4.** Find the Maclaurin series for the function $f(x) = \dfrac{x}{(2+x)^2}$. What is the interval of convergence of the series?

Method 1: $\dfrac{1}{2+x} = \dfrac{1}{2(1+x/2)} = \dfrac{1}{2}\sum_{n=0}^{\infty}\left(-\dfrac{x}{2}\right)^n = \sum_{n=0}^{\infty}\dfrac{(-1)^n}{2^{n+1}}x^n$ which is valid for $\left|\dfrac{x}{2}\right| < 1$ or $|x| < 2$. Since the radius of convergence of this series is positive, we may differentiate with respect to x to get

$$-\frac{1}{(2+x)^2} = \sum_{n=0}^{\infty}\frac{(-1)^n n}{2^{n+1}}x^{n-1}.$$

If we multiply by $-x$,

$$\frac{x}{(2+x)^2} = \sum_{n=0}^{\infty}\frac{(-1)^{n+1} n}{2^{n+1}}x^n.$$

The open interval of convergence is $-2 < x < 2$. Since differentiation of a series never picks up an end point, this is also the interval of convergence.

Method 2: Using the binomial expansion,

$$\begin{aligned}
\frac{x}{(2+x)^2} &= \frac{x}{4}\left(1+\frac{x}{2}\right)^{-2} \\
&= \frac{x}{4}\left[1+(-2)\left(\frac{x}{2}\right)+\frac{(-2)(-3)}{2!}\left(\frac{x}{2}\right)^2+\frac{(-2)(-3)(-4)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right] \\
&= \frac{x}{4}\left[1-\frac{2x}{2}+\frac{3x^2}{2^2}-\frac{4x^3}{2^3}+\cdots\right] \\
&= \frac{x}{4}-\frac{2x^2}{2^3}+\frac{3x^3}{2^4}-\frac{4x^4}{2^5}+\cdots \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{2^{n+1}}x^n.
\end{aligned}$$

The open interval of convergence is given by $|x/2| < 1$ or $-2 < x < 2$. At $x = 2$, the series becomes $\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}n}{2}$, which diverges. At $x = -2$, the series becomes $\sum_{n=1}^{\infty}-\dfrac{n}{2}$, which also diverges. The interval of convergence is $-2 < x < 2$.

12 5. Find the Maclaurin series for the function $f(x) = \dfrac{1}{\sqrt[3]{8+3x}}$. Find the radius of convergence of the series.

$$\begin{aligned}
\frac{1}{(8+3x)^{1/3}} &= \frac{1}{2}\left(1+\frac{3x}{8}\right)^{-1/3} \\
&= \frac{1}{2}\left[1+(-1/3)\left(\frac{3x}{8}\right)+\frac{(-1/3)(-4/3)}{2!}\left(\frac{3x}{8}\right)^2+\frac{(-1/3)(-4/3)(-7/3)}{3!}\left(\frac{3x}{8}\right)^3+\cdots\right] \\
&= \frac{1}{2}\left[1-\frac{1}{8}x+\frac{1\cdot 4}{2!\,8^2}x^2-\frac{1\cdot 4\cdot 7}{3!\,8^3}x^3+\cdots\right] \\
&= \frac{1}{2}-\frac{1}{2^4}x+\frac{1\cdot 4}{2!\,2^7}x^2-\frac{1\cdot 4\cdot 7}{3!\,2^{10}}x^3+\cdots \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 4\cdot 7\cdots(3n-2)]}{n!\,2^{3n+1}}x^n
\end{aligned}$$

The open interval of convergence is given by $\left|\dfrac{3x}{8}\right| < 1$ or $|x| < \dfrac{8}{3}$. The radius of convergence is therefore $8/3$.

[5] 1. Find the limit of the sequence of functions $\{f_n(x)\}$ on the interval $1 \le x \le 7$, if it exists. Justify your answer.

$$f_n(x) = \frac{n^4x^3 + n^3x^2 + n^4x + 3n}{n^3x^3 + n^4x + 4}$$

Solution:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^4x^3 + n^3x^2 + n^4x + 3n}{n^3x^3 + n^4x + 4}$$

$$= \lim_{n\to\infty} \frac{\frac{n^4x^3}{n^4} + \frac{n^3x^2}{n^4} + \frac{n^4x}{n^4} + \frac{3n}{n^4}}{\frac{n^3x^3}{n^4} + \frac{n^4x}{n^4} + \frac{4}{n^4}}$$

$$= \lim_{n\to\infty} \frac{x^3 + \frac{x^2}{n} + x + \frac{3}{n^3}}{\frac{x^3}{n} + x + \frac{4}{n^4}}$$

$$= \frac{x^3 + x}{x} = x^2 + 1$$

[6] 2. Find the radius of convergence and the open interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(2n+1)!\,(x-3)^{2n}}{4^{3n}\,n!\,(n-1)!}$$

Solution:

We let $y = (x-3)^2$, and do the following to find the radius of convergence of y.

$$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{\frac{(2n+1)!}{4^{3n}n!(n-1)!}}{\frac{(2n+3)!}{4^{3n+3}(n+1)!n!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(2n+1)!}{4^{3n}n!(n-1)!}\cdot\frac{4^{3n+3}(n+1)!n!}{(2n+3)!}\right|$$

$$= \lim_{n\to\infty}\left|\frac{4^3(n+1)(n)}{(2n+3)(2n+2)}\right|$$

$$= \lim_{n\to\infty}\left|\frac{4^3n^2+4^3n}{4n^2+10n+6}\right|$$

$$= 4^2 = 16$$

Hence we know that $|y| < 16$
or $|(x-3)^2| < 16$,
so $|(x-3)| < 4$.

This gives $-4 < x-3 < 4$
or $-1 < x < 7$.

The radius of convergence (of x) is 4, the open interval of convergence is $-1 < x < 7$.

[5] 3. (a) Find the first 4 terms of the Taylor series of $f(x) = e^{3x}$ about $x = 2$.

Solution: (Note, there are a few correct ways to handle this question)

$f(x) = e^{3x}$ $\quad f(2) = e^6$

$f'(x) = 3e^{3x}$ $\quad f'(2) = 3e^6$

$f''(x) = 9e^{3x}$ $\quad f''(2) = 9e^6$

$f'''(x) = 27e^{3x}$ $\quad f'''(2) = 27e^6$

$\vdots$

$f^{(n)}(x) = 3^n e^{3x}$

Using the formula for the Taylor series as follows :

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)(x-c)^2}{2!} + \frac{f'''(c)(x-c)^3}{3!} + \cdots$$

We get

$$f(x) = e^6 + 3e^6(x-2) + \frac{9e^6(x-2)^2}{2!} + \frac{27e^6(x-c)^3}{3!} + \cdots$$

[2] (b) What is $R_n(2, x)$ (The n$^{\text{th}}$ Taylor Remainder)?

Solution:

$$R_n(2,x) = \frac{3^{n+1}e^{3z_0}(x-2)^{n+1}}{(n+1)!}$$

where z_0 is between 2 and x.

[5] 4. Find the sum of the power series

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(2n)!\, 3^{2n+2}}\, x^{2n+1}$$

Solution:

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, 2^{2n}}{(2n)!\, 3^{2n+2}}\, x^{2n+1}$$

$$= \frac{1}{3^2}\, x \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{2}{3}x\right)^{2n}$$

$$= \frac{x}{9} \cos\left(\frac{2}{3}x\right)$$

[10] 5. Find the Taylor series of $f(x) = \dfrac{3x+4}{2x^2+7x+3}$ about $x = 1$. Express in sigma notation, and include the open interval of convergence.

Hint : Find A and B so that $\dfrac{3x+4}{2x^2+7x+3} = \dfrac{A}{2x+1} + \dfrac{B}{x+3}$.

Solution:

To solve the partial fraction, we need to solve:
$A(x-3) + B(2x+1) = 3x+4$ or $Ax + 2Bx + 3A + B = 3x+4$

The system $\begin{array}{rcrl} A & + & 2B = & 3 \\ 3A & + & B = & 4 \end{array}$ has solutions $A = 1$ $B = 1$.

So $f(x) = \dfrac{1}{2x+1} + \dfrac{1}{x+3}$

Since $\dfrac{1}{2x+1} = \dfrac{1}{3+2(x-1)} = \dfrac{\frac{1}{3}}{1+\frac{2}{3}(x-1)} = \dfrac{\frac{1}{3}}{1-(-\frac{2}{3}(x-1))}$

When $|-\frac{2}{3}(x-1)| < 1$ (so $|x-1| < \frac{3}{2}$), we have

$$\frac{1}{2x+1} = \sum_{n=0}^{\infty} \frac{1}{3}\left(-\frac{2}{3}(x-1)\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{3^{n+1}}(x-1)^n$$

Since $\dfrac{1}{x+3} = \dfrac{1}{4+(x-1)} = \dfrac{\frac{1}{4}}{1+\frac{1}{4}(x-1)} = \dfrac{\frac{1}{4}}{1-(-\frac{1}{4}(x-1))}$

When $|-\frac{1}{4}(x-1)| < 1$ (so $|x-1| < 4$), we have

$$\frac{1}{x+3} = \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{1}{4}(x-1)\right)^n = \sum_{n=0}^{\infty}(-1)^n \frac{1}{4^{n+1}}(x-1)^n$$

So

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{2^n}{3^{n+1}}(x-1)^n + \sum_{n=0}^{\infty}(-1)^n \frac{1}{4^{n+1}}(x-1)^n$$

$$= \sum_{n=0}^{\infty}(-1)^n \left[\frac{2^n}{3^{n+1}} + \frac{1}{4^{n+1}}\right](x-1)^n$$

on the interval $-\frac{1}{2} < x < \frac{5}{2}$

[7] 6. Use the binomial theorem to express $f(x) = \dfrac{1}{\sqrt{1-x}}$ as a power series. Express in sigma notation, and include the open interval of convergence.

Solution:

We write $f(x) = (1-x)^{\frac{1}{2}}$, so we find $\begin{pmatrix} k \\ n \end{pmatrix}$ for $k = -\frac{1}{2}$ as follows:

$$\begin{pmatrix} -\frac{1}{2} \\ n \end{pmatrix} = \frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})(\frac{-7}{2})\cdots(\frac{-1}{2}-n+1)}{n!}$$

$$= \frac{(\frac{-1}{2})(\frac{-3}{2})(\frac{-5}{2})(\frac{-7}{2})\cdots(\frac{-2n+1}{2})}{n!}$$

$$= \frac{(-1)^n(\frac{1}{2})^n \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{n!}$$

So when $|-x| < 1$ we have

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n\, n!}(-x)^n$$

$$= 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-1)}{2^n\, n!} x^n$$

on the open interval $-1 < x < 1$.

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Student Name - Student Number -

Values

5 1. Find the limit of the sequence of functions

$$\left\{\frac{n^2x^3+3nx}{2n^2x+1}\mathrm{Tan}^{-1}\left(\frac{nx}{n+3}\right)\right\}$$

on the interval $0 \le x \le 3$, if it exists. Justify your answer.

$$f(x) = \lim_{n\to\infty}\frac{n^2x^3+3nx}{2n^2x+1}\mathrm{Tan}^{-1}\left(\frac{nx}{n+3}\right) = \lim_{n\to\infty}\frac{x^3+\dfrac{3x}{n}}{2x+\dfrac{1}{n^2}}\mathrm{Tan}^{-1}\left(\frac{x}{1+\dfrac{3}{n}}\right)$$

$$= \frac{x^3}{2x}\mathrm{Tan}^{-1}x, \quad \text{provided } x \neq 0$$

$$= \frac{x^2}{2}\mathrm{Tan}^{-1}x.$$

Since each function in the sequence has value 0 at $x=0$, the limit of the sequence at $x=0$ is 0. Since $(x^2/2)\mathrm{Tan}^{-1}x$ has value 0 at $x=0$, we can write that

$$f(x) = \frac{x^2}{2}\mathrm{Tan}^{-1}x, \quad 0 \le x \le 3.$$

8 2. Determine whether the following series converge or diverge. Justify you answers. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^2}{1+2n^2}$ (b) $\displaystyle\sum_{n=3}^{\infty}\frac{2^n}{3^{n+1}}$

(a) Since $\displaystyle\lim_{n\to\infty}\frac{n^2}{2n^2+1} = \frac{1}{2}$, it follows that $\displaystyle\lim_{n\to\infty}\frac{(-1)^n n^2}{2n^2+1}$ does not exist. The series therefore diverges by the n^{th}-term test.

(b) This is a geometric series with common ratio 2/3. It therefore converges and has sum

$$\frac{2^3/3^4}{1-2/3} = \frac{8}{27}.$$

3. (a) Find the first four Taylor polynomials $P_0(x)$, $P_1(x)$, $P_2(x)$, and $P_3(x)$ about $x = 0$ for the function $\cos 3x$.

(b) Use Taylor's remainder formula to verify that the Maclaurin series for $\cos 3x$ converges to $\cos 3x$ for all x.

(a) With $f(x) = \cos 3x$,

$$\begin{aligned}
f(0) &= 1 \\
f'(0) &= -3\sin 3x_{|x=0} = 0, \\
f''(0) &= -9\cos 3x_{|x=0} = -9, \\
f'''(0) &= 27\sin 3x_{|x=0} = 0.
\end{aligned}$$

The first four Taylor polynomials are therefore

$$\begin{aligned}
P_0(x) &= f(0) = 1, \\
P_1(x) &= f(0) + f'(0)x = 1, \\
P_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 = 1 - \frac{9x^2}{2}, \\
P_3(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 = 1 - \frac{9x^2}{2}.
\end{aligned}$$

(b) Taylor's remainder formula gives

$$f(x) = 1 - \frac{9x^2}{2} + \cdots + \text{term in } x^n + R_n(0,x),$$

where $R_n(0,x) = \frac{d^{n+1}\cos 3x}{dx^{n+1}}_{|x=z_n} \frac{x^{n+1}}{(n+1)!}$. Since

$$\frac{d^{n+1}\cos 3x}{dx^{n+1}} = 3^{n+1}\left(\text{one of } \pm\sin 3x \text{ and } \pm\cos 3x\right),$$

it follows that

$$|R_n(0,x)| \leq \frac{3^{n+1}|x|^{n+1}}{(n+1)!} = \frac{|3x|^{n+1}}{(n+1)!}.$$

This approaches zero as $n \to \infty$ for all x. The Maclaurin series therefore converges to $\cos 3x$ for all x.

8 **4.** Find the interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{3^n} x^{2n+1}.$$

We set $y = x^2$, and write $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{3^n} x^{2n+1} = x \sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{3^n} y^n$. The radius of convergence of this series is

$$R_y = \lim_{n\to\infty} \left| \frac{\dfrac{(-1)^{n+1}n^2}{3^n}}{\dfrac{(-1)^{n+2}(n+1)^2}{3^{n+1}}} \right| = 3.$$

The radius of convergence of the x-series is therefore $R_x = \sqrt{3}$. The open interval of convergence is $-\sqrt{3} < x < \sqrt{3}$. At $x = \sqrt{3}$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{3^n} (\sqrt{3})^{2n+1} = \sqrt{3} \sum_{n=3}^{\infty} (-1)^{n+1}n^2.$$

Since $\lim_{n\to\infty} (-1)^{n+1}n^2$ does not exist, this series diverges (by the n^{th}-term test). At $x = -\sqrt{3}$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(-1)^{n+1}n^2}{3^n} (-\sqrt{3})^{2n+1} = -\sqrt{3} \sum_{n=3}^{\infty} (-1)^{n+1}n^2.$$

This is the negative of the series at $x = \sqrt{3}$, and it therefore diverges. The interval of convergence is $-\sqrt{3} < x < \sqrt{3}$.

7 **5.** Find the open interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{2^{n+1}}{n^3 + 100n^2} (x+2)^n.$$

The radius of convergence is

$$R = \lim_{n\to\infty} \left| \frac{\dfrac{2^{n+1}}{n^3 + 100n^2}}{\dfrac{2^{n+2}}{(n+1)^3 + 100(n+1)^2}} \right| = \frac{1}{2}.$$

The open interval of convergence is therefore

$$|x+2| < \frac{1}{2} \implies -\frac{1}{2} < x + 2 < \frac{1}{2} \implies -\frac{5}{2} < x < -\frac{3}{2}.$$

MATH2132 Test1 February 1, 2011 60 minutes

Student Name - Student Number -

Values

2 **1.** The limit of the sequence $\left\{\dfrac{(-1)^n n^2 + 3n}{2n^2+5}\right\}$ is

(a) 1/2 (b) $\pm 1/2$ (c) ∞ (d) $-\infty$ (e) None of these

Answer e

2 **2.** The limit of the sequence $\left\{\dfrac{2n^2+3}{5-3n^2}\mathrm{Sin}^{-1}\left(\dfrac{n+2}{2n-3}\right)\right\}$ is

(a) -1 (b) $\pi/10$ (c) $-\pi/9$ (d) $\pi/6$ (e) None of these

Answer c

2 **3.** The sum of the series $\displaystyle\sum_{n=1}^{\infty}\left(-\frac{3}{4}\right)^{n+1}$ is

(a) 9/28 (b) 9/4 (c) $-3/7$ (d) -3 (e) None of these

Answer a

2 **4.** The sum of the series $\displaystyle\sum_{n=1}^{\infty} n\left(\frac{7}{4}\right)^{n}$ is

(a) $-7/3$ (b) 7/3 (c) ∞ (d) $-\infty$ (e) None of these

Answer c

2 **5.** The limit of the sequence of functions $\left\{\left(1+\dfrac{x}{2n}\right)^n\right\}$ on the interval $0 \le x < 1$ is

(a) 1 (b) $e^{x/2}$ (c) $x/2$ (d) Does not exist (e) None of these

Answer b

10 **6.** Prove that the Maclaurin series for e^{3x} converges to e^{3x} for all x.

Taylor's remainder formula gives

$$e^{3x} = 1 + 3x + \frac{3^2}{2!}x^2 + \cdots + \frac{3^n}{n!}x^n + R_n(0,x),$$

where

$$R_n(0,x) = \frac{d^{n+1}(e^{3x})}{dx^{n+1}}_{|x=z_n}\frac{x^{n+1}}{(n+1)!} = 3^{n+1}e^{3z_n}\frac{x^{n+1}}{(n+1)!}.$$

When $x > 0$, we know that $0 < z_n < x$, and therefore

$$|R_n(0,x)| < 3^{n+1}e^{3x}\frac{|x|^{n+1}}{(n+1)!} = e^{3x}\frac{|3x|^{n+1}}{(n+1)!},$$

and this approaches zero as $n \to \infty$. When $x < 0$, we know that $x < z_n < 0$, and therefore

$$|R_n(0,x)| < 3^{n+1}e^{0}\frac{|x|^{n+1}}{(n+1)!} = \frac{|3x|^{n+1}}{(n+1)!},$$

and this approaches zero as $n \to \infty$. Since remainders approach zero for all x, the Maclaurin series for e^{3x} converges to e^{3x} for all x.

8 **7.** What is the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{n+1}{n4^n}x^n?$$

Justify all results.

The radius of convergence of the series is

$$R = \lim_{n\to\infty}\left|\frac{\dfrac{n+1}{n4^n}}{\dfrac{n+2}{(n+1)4^{n+1}}}\right| = 4.$$

The open interval of convergence is $-4 < x < 4$. At $x = 4$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n}$. Since $\displaystyle\lim_{n\to\infty}\frac{n+1}{n} = 1$, the series diverges by the n^{th}-term test. At $x = -4$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n+1}{n}$. Since $\displaystyle\lim_{n\to\infty}(-1)^n\frac{n+1}{n}$, does not exist, the series diverges by the n^{th}-term test. The interval of convergence is therefore $-4 < x < 4$.

12 **8.** Find the Taylor series about $x = 4$ for the function

$$f(x) = \frac{1}{(x-2)^2}.$$

Express your answer in sigma notation simplified as much as possible. You must use a technique that guarantees that the Taylor series converges to the function. What is the radius of convergence of the series?

In the box is $x - 4$.

$$\frac{1}{x-2} = \frac{1}{(x-4)+2} = \frac{1}{2\left[1+\left(\dfrac{x-4}{2}\right)\right]} = \frac{1}{2}\sum_{n=0}^{\infty}\left[-\left(\frac{x-4}{2}\right)\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(x-4)^n,$$

valid for $\left|-\left(\frac{x-4}{2}\right)\right| < 1 \implies |x-4| < 2$. Since the radius of convergence is $R = 2 > 0$, we may differentiate the series term-by-term,

$$-\frac{1}{(x-2)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{2^{n+1}}(x-4)^{n-1}.$$

Thus,

$$\frac{1}{(x-2)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{2^{n+1}}(x-4)^{n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{2^{n+2}}(x-4)^n.$$

Since differentiation preserves radii of convergence, the radius of convergence is $R = 2$.

Summer 2011

MATH2132 Test1 Solutions

Values

10 **1.** Find limits for the following sequences, if they exist.

(a) $\left\{\left(\frac{n+1}{n}\right)^n\left(\frac{n^2}{2n^2+1}\right)\right\}$ (b) $\left\{\frac{2^n+\mathrm{Cot}^{-1}n}{3(2^n+4)}\right\}$

(a) Since $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n=e$, and $\lim_{n\to\infty}\frac{n^2}{2n^2+1}=\frac{1}{2}$,

$$\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\left(\frac{n^2}{2n^2+1}\right)=\frac{e}{2}.$$

(b) Since $\lim_{n\to\infty}\mathrm{Cot}^{-1}n=0$, it follows that $\lim_{n\to\infty}\frac{2^n+\mathrm{Cot}^{-1}n}{3(2^n+4)}=\lim_{n\to\infty}\frac{1+\frac{1}{2^n}\mathrm{Cot}^{-1}n}{3\left(1+\frac{4}{2^n}\right)}=\frac{1}{3}$.

6 **2.** Find the limit for the following sequence of functions on the interval $-1<x\leq 100$, if it exists. Show your reasoning or calculations.

$$\left\{\frac{n^2x^2+5x+n}{3n^2-x^{15}}+x\right\}$$

$$\lim_{n\to\infty}\left(\frac{n^2x^2+5x+n}{3n^2-x^{15}}+x\right)=\lim_{n\to\infty}\left(\frac{x^2+\frac{5x}{n^2}+\frac{1}{n}}{3-\frac{x^{15}}{n^2}}+x\right)=\frac{x^2}{3}+x$$

8 **3.** Determine whether the following series converge or diverge. Justify your conclusions.

(a) $\sum_{n=1}^{\infty}\frac{n^2}{3n^2+2n+5}$ (b) $\sum_{n=2}^{\infty}\frac{e^n}{3^{2n}}$

(a) Since $\lim_{n\to\infty}\frac{n^2}{3n^2+2n+5}=\frac{1}{3}\neq 0$, the series diverges by the n^{th}-term test.

(b) This is a geometric series with common ration $e/9<1$, and the series therefore converges.

6 4. Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{5^n}(x+1)^n$. Include its interval of convergence.

This is a geometric series with sum

$$\sum_{n=1}^{\infty} - \left[-\frac{(x+1)}{5}\right]^n = -\frac{-(x+1)/5}{1+(x-1)/5} = \frac{x+1}{x+6}.$$

The interval of convergence is

$$\left|-\frac{(x+1)}{5}\right| < 1 \implies |x+1| < 5 \implies -5 < x+1 < 5 \implies -6 < x < 4.$$

10 5. Find the interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{1}{n4^n}(x-2)^{2n}?$$

Justify all results.

If we set $y = (x-2)^2$, the series becomes $\sum_{n=3}^{\infty} \frac{1}{n4^n} y^n$. Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{1}{n4^n}}{\frac{1}{(n+1)4^{n+1}}} \right| = 4.$$

Hence, $R_x = 2$. The open interval of convergence is $|x-2| < 2$, from which $0 < x < 4$. At both ends $x = 0$ and $x = 4$, the series becomes $\sum_{n=3}^{\infty} \frac{1}{n}$, the harmonic series without its first two terms. The series diverges. The interval of convergence is therefore $0 < x < 4$.

MATH2132 Test1 October 6, 2011 60 minutes

Student Name - Student Number -

Values

12 1. Determine whether the following series converge or diverge. Justify your answers. If a series converges, find its sum.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{2^{2n+3}}{5^{n+1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n-4}{10n+5}$

(a) If we write the series in the form $\displaystyle\sum_{n=2}^{\infty} \frac{8}{5}\left(\frac{4}{5}\right)^n$, we see that it is geometric with common ratio 4/5. The series therefore converges with sum

$$\frac{2^7/125}{1-4/5} = \frac{128}{25}.$$

(b) Since $\displaystyle\lim_{n\to\infty} \frac{n-4}{10n+5} = \frac{1}{10} \neq 0$, the series diverges by the n^{th}-term test.

13 2. Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{n+1} x^{2n+2}.$$

We set $y = x^2$ and write the series in the form $\displaystyle x^2 \sum_{n=1}^{\infty} \frac{2^n}{n+1} y^n$. Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{2^n}{n+1}}{\frac{2^{n+1}}{n+2}} \right| = \frac{1}{2}.$$

The radius of convergence of the original series is therefore $R_x = 1/\sqrt{2}$. Its open interval of convergence is $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$. At the end points $x = \pm 1/\sqrt{2}$, the series becomes

$$\sum_{n=1}^{\infty} \frac{1}{2(n+1)} = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n+1}.$$

Since this is the harmonic series (less the first term), the series diverges. The interval of convergence is $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.

11 **3.** Use Taylor's remainder formula to verify that the Maclaurin series for e^{-2x} converges to e^{-2x} for $x \le 0$.

Taylor remainders are

$$R_n(0,x) = \frac{d^{n+1}(e^{-2x})}{dx^{n+1}}\bigg|_{x=z_n} \frac{x^{n+1}}{(n+1)!} = (-2)^{n+1} e^{-2z_n} \frac{x^{n+1}}{(n+1)!}.$$

Since $x \le 0$, it follows that $x < z_n < 0$, and therefore $e^{-2z_n} < e^{-2x}$. Consequently,

$$|R_n(0,x)| < e^{-2x} \frac{|2x|^{n+1}}{(n+1)!} \quad \longrightarrow \quad 0 \quad \text{as} \quad n \to \infty.$$

6 **4.** Is it possible for the Maclaurin series for a function $f(x)$ to converge at $x = 5$, but not at $x = 4$? Explain.

No. If the Maclaurin series converges at $x = 5$, then its radius of convergence R must be greater than or equal to 5. It follows that the series converges for $-R < 5 \le x \le 5 < R$, and therefore converges at $x = 4$.

8 **5.** Determine whether the sequence of functions

$$\left\{ \frac{n^2x^2 + 3n^2x + n}{2n^2x + 5nx + 4} \right\}$$

has a limit as $n \to \infty$. If the sequence has a limit, find it; if the sequence does not have a limit, indicate why not. Do this on the following intervals:

(a) $x \ge 1$ (b) $-1 < x < 1$

(a) When $x \ge 1$,

$$\lim_{n\to\infty} \frac{n^2x^2 + 3n^2x + n}{2n^2x + 5nx + 4} = \lim_{n\to\infty} \frac{x^2 + 3x + \frac{1}{n}}{2x + \frac{5x}{n} + \frac{4}{n^2}} = \frac{x^2 + 3x}{2x} = \frac{x + 3}{2}.$$

(b) When $-1 < x < 1$,

$$\lim_{n\to\infty} \frac{n^2x^2 + 3n^2x + n}{2n^2x + 5nx + 4} = \lim_{n\to\infty} \frac{x^2 + 3x + \frac{1}{n}}{2x + \frac{5x}{n} + \frac{4}{n^2}} = \frac{x^2 + 3x}{2x},$$

but not at $x = 0$. When $x = 0$, the sequence of functions becomes the sequence of constants $\left\{\frac{n}{4}\right\}$, which diverges. Thus, there is no limit function on the interval $-1 < x < 1$.