[5] 1. Find the equation of the tangent plane to the surface $x^2 + 2yz - 1 = 0$ at the point $(1, 0, 3)$.

Let $f(x,y,z) = x^2+2yz-1$. A normal to surface $f=0$ is $\nabla f = [2x, 2z, 2y]$, $\nabla f(1,0,3) = [2,6,0]$.

$\therefore$ equation of tangent plane is $(x-1)\cdot 2 + (y-0)\cdot 6 + (z-3)\cdot 0 = 0$

or $2x + 6y - 2 = 0$.

or $x + 3y - 1 = 0$.

[7] 2. Given a smooth surface $f(x, y, z) = 0$. (a) Find a formula for $\left.\dfrac{\partial z}{\partial x}\right)_y$. (b) Evaluate $\left.\dfrac{\partial z}{\partial x}\right)_y \left.\dfrac{\partial x}{\partial y}\right)_z \left.\dfrac{\partial y}{\partial z}\right)_x$.

Using implicit differentiation,

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \Rightarrow \left.\frac{\partial z}{\partial x}\right)_y = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

$$\therefore \left.\frac{\partial z}{\partial x}\right)_y \left.\frac{\partial x}{\partial y}\right)_z \left.\frac{\partial y}{\partial z}\right)_x = -\frac{\cancel{\frac{\partial f}{\partial x}}}{\cancel{\frac{\partial f}{\partial z}}} \cdot (-1)\frac{\cancel{\frac{\partial f}{\partial y}}}{\cancel{\frac{\partial f}{\partial x}}} \cdot (-1)\frac{\cancel{\frac{\partial f}{\partial z}}}{\cancel{\frac{\partial f}{\partial y}}}$$

$$= -1$$

[7] 3. Find all unit vector(s) $\mathbf{v}$ so that the rate of change of the function $f(x,y) = xy^2 + x^3$ at the point $(1,-1)$ in direction $\mathbf{v}$ is zero.

$\nabla f = (y^2+3x^2, 2xy)$, $\quad \nabla f(1,-1) = (4,-2)$

find $v = (v_1, v_2)$ so that $v_1^2 + v_2^2 = 1$ and

$0 = v \cdot \nabla f(1,-1) = 4v_1 - 2v_2 \Rightarrow v_2 = 2v_1$

$\therefore v_1^2 + 4v_1^2 = 1 \Rightarrow v_1 = \pm \frac{1}{\sqrt{5}}$

$\therefore$ possible directions are $\left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ and $\left(-\frac{1}{\sqrt{5}}, -\frac{2}{\sqrt{5}}\right)$.

[7] 4. Find and classify the critical point(s) of $f(x,y) = x^3 + xy - x + 2y$. Justify your answer.

$0 = f_x = 3x^2 + y - 1$

$0 = f_y = x + 2 \quad \Rightarrow \quad x = -2$

From first equation, $y = 1 - 3x^2 = -11$.

$\therefore$ one critical pt $(-2,-11)$.

$f_{xx} = 6x$, $\quad f_{yy} = 0$, $\quad f_{xy} = 1$.

(A01) $\quad D = f_{xx}f_{yy} - f_{xy}^2 = 0 - 1^2 = -1 < 0$.

$\therefore (-2,-11)$ is a saddle pt. by the 2nd deriv. test.

(A02) $\quad B^2 - AC = f_{xy}^2 - f_{xx}f_{yy} = 1^2 - 0 = 1 > 0$.

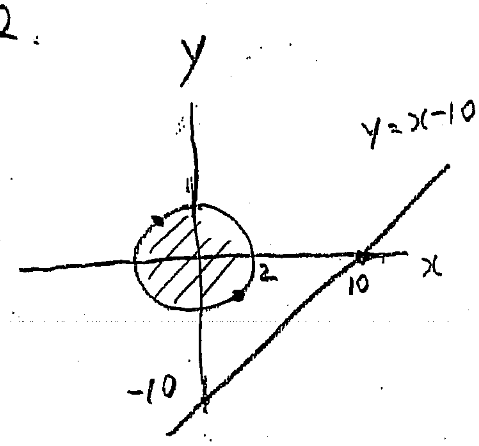
$\therefore (-2,-11)$ is a saddle pt. by the 2nd deriv. test.

[11] 5. Find the maximum and minimum values of $f(x, y) = (x - y - 10)^2$ in the region $\{(x, y),\ x^2 + y^2 \le 4\}$. Give also the coordinates of all the points where the values are attained.

For critical points,

$0 = f_x = 2(x-y-10)$, $0 = f_y = 2(x-y-10)(-1)$

$\Rightarrow$ critical pts along line $y = x-10$, which clearly does not intersect circle (or disk) of radius 2.



$\therefore$ absolute extrema occur along circle of radius 2.

Let $x = 2\cos\theta$, $y = 2\sin\theta$, $0 \le \theta \le 2\pi$

$$g(\theta) = f(2\cos\theta, 2\sin\theta) = (2\cos\theta - 2\sin\theta - 10)^2, \quad \theta \in [0, 2\pi]$$

$$g'(\theta) = 2\underbrace{(2\cos\theta - 2\sin\theta - 10)}_{<0}(-2\sin\theta - 2\cos\theta)$$

For critical pts of g, $g'(\theta) = 0$. $\therefore \sin\theta + \cos\theta = 0$

$\therefore \sin\theta = -\cos\theta$; $\therefore \tan\theta = -1$.

$\Rightarrow \theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

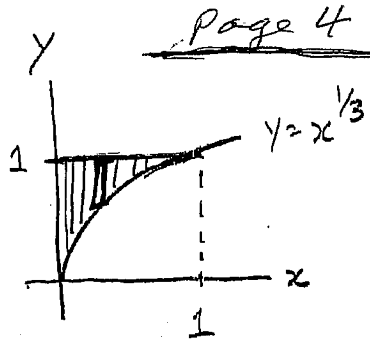
$$g\left(\tfrac{3\pi}{4}\right) = \left(-\tfrac{2}{\sqrt{2}} - \tfrac{2}{\sqrt{2}} - 10\right)^2 = (2\sqrt{2}+10)^2, \quad g\left(\tfrac{7\pi}{4}\right) = \left(\tfrac{2}{\sqrt{2}} + \tfrac{2}{\sqrt{2}} - 10\right)^2 = (2\sqrt{2}-10)^2$$

$\therefore$ abs max is $(2\sqrt{2}+10)^2$, occurs at $(-\sqrt{2}, \sqrt{2})$

min is $(10-2\sqrt{2})^2$, occurs at $(\sqrt{2}, -\sqrt{2})$

Note that $g(0) = g(2\pi) = 8^2$ which is strictly between above extrema.

[7] 6. Evaluate the integral $\int_0^1 \int_{x^{1/3}}^1 e^{y^4}\,dy\,dx$.



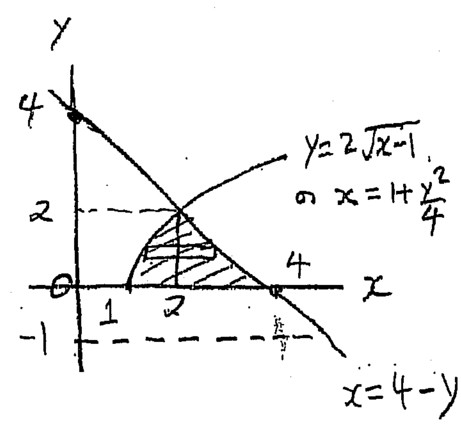
$$= \int_0^1 \int_0^{y^3} e^{y^4}\,dx\,dy$$

(interchange order of integration)

$$= \int_0^1 y^3 e^{y^4}\,dy$$

$$= \frac{1}{4} e^{y^4}\Big|_0^1 = \frac{1}{4}(e-1).$$

[6] 7. Set up, but DO NOT EVALUATE, a double iterated integral for the volume of the solid of revolution obtained by rotating the region bounded by $x + y = 4$, $y = 2\sqrt{x-1}$ and $y = 0$ about the line $y = -1$.



First find intersection of parabola and line:

$$y = 4 - x = 2\sqrt{x-1} \qquad (1)$$

Square: $16 - 8x + x^2 = 4(x-1)$

$$x^2 - 12x + 20 = 0$$

$$(x-2)(x-10) = 0 \Rightarrow x = 2, 10$$

$x = 2 \Rightarrow y = 2$. $\quad x = 10$ does not satisfy (1)

$$\therefore \text{volume} = 2\pi \int_0^2 \int_{1+\frac{y^2}{4}}^{4-y} (y+1)\,dx\,dy.$$

Alternative: NOT as good an answer; it requires two integrals:

$$\text{Volume} = 2\pi\left[\int_1^2 \int_0^{2\sqrt{x-1}} (y+1)\,dy\,dx + \int_2^4 \int_0^{4-x} (y+1)\,dy\,dx\right].$$