MATH 1210 A01 Summer 2013 Problem Workshop 13 Solutions

1. If $\mathbf{v} = T\mathbf{u}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ defined by

$$\begin{aligned} v_1 &= 3u_1 - 2u_2 \\ v_2 &= 4u_1 + 3u_2 + u_3 \\ v_3 &= -u_1 + 2u_2 + 3u_3 : \end{aligned}$$

(a)

$$T\langle 2,-1,3\rangle = \langle 3(2)-2(-1), 4(2)+3(-1)+(3), -2+2(-1)+3(3)\rangle = \langle 8,8,5\rangle$$

(b) We are solving for u_1, u_2 and u_3 such that

$$\begin{aligned} 1 &= 3u_1 - 2u_2 \\ 1 &= 4u_1 + 3u_2 + u_3 \\ -1 &= -u_1 + 2u_2 + 3u_3 : \end{aligned}$$

Putting this into an augmented matrix yields

$\left[\begin{array}{ccc|c} 3 & -2 & 0 & 1 \\ 4 & 3 & 1 & 1 \\ -1 & 2 & 3 & -1 \end{array}\right]$ Using $R_3 \to -R_3$ and $R_1 \leftrightarrow R_3$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 4 & 3 & 1 & 1 \\ 3 & -2 & 0 & 1 \end{array}\right]$ Using $R_2 \to R_2 - 4R_1$ and $R_3 \to R_3 - 3R_1$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 11 & 13 & -3 \\ 0 & 4 & 9 & -2 \end{array}\right]$ Using $R_2 \to R_2 - 3R_3$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & -1 & -14 & 3 \\ 0 & 4 & 9 & -2 \end{array}\right]$ Using $R_2 \to -R_2$ yields

$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 1 \\ 0 & 1 & 14 & -3 \\ 0 & 4 & 9 & -2 \end{array}\right]$ Using $R_1 \to R_1 + 2R_2$ and $R_3 \to R_3 - 4R_2$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 25 & -5 \\ 0 & 1 & 14 & -3 \\ 0 & 0 & -47 & 10 \end{array}\right]$ Using $R_3 \to -\frac{1}{47}R_3$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 25 & -5 \\ 0 & 1 & 14 & -3 \\ 0 & 0 & 1 & -10/47 \end{array}\right]$ Using $R_1 \to R_1 - 25R_3$ and $R_1 \to R_1 - 14R_3$ yields

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 15/47 \\ 0 & 1 & 0 & -1/47 \\ 0 & 0 & 1 & -10/47 \end{array}\right].$$

Hence $\mathbf{u} = \dfrac{1}{47}\langle 15, -1, -10\rangle$.

(c) We are solving for u_1, u_2 and u_3 such that

$$\begin{aligned} 2u_1 &= 3u_1 - 2u_2 \\ 2u_2 &= 4u_1 + 3u_2 + u_3 \\ 2u_3 &= -u_1 + 2u_2 + 3u_3 \end{aligned}$$

or equivalently

$$\begin{aligned} 0 &= u_1 - 2u_2 \\ 0 &= 4u_1 + u_2 + u_3 \\ 0 &= -u_1 + 2u_2 + u_3 \end{aligned}$$

which is homogeneous system. Since the coeffeicient matrix has determinant

$$\begin{aligned} |A| &= \begin{vmatrix} 1 & -2 & 0 \\ 4 & 1 & 1 \\ -1 & 2 & 1 \end{vmatrix} \\ &= 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 4 & 1 \\ -1 & 1 \end{vmatrix} \\ &= 1(1-2) + 2(4-(-1)) \\ &= -1 + 10 \\ &= 9 \neq 0. \end{aligned}$$

Since $|A| \neq 0$. The only solution is the trivial solution $u_1 = u_2 = u_3 = 0$. Hence the only solution is $\langle 0, 0, 0\rangle$.

2. The eigenvalues are the solution to $6\lambda^4 + 11\lambda^3 - 4\lambda^2 + 11\lambda - 10 = 0$. Form the rational root theorem, we know that any rational solution must be one of

$$\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{2}, \pm\frac{5}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm\frac{1}{6}, \pm\frac{5}{6}$$

of which the bounds theorem could disqualify $\pm 5, \pm 10, \pm 10/3$.

Testing values leads to $\lambda = \dfrac{2}{3}$ as a solution and hence $3\lambda - 2$ is a factor. Divison yields

$$(3\lambda - 2)(2\lambda^3 + 5\lambda^2 + 2\lambda + 5) = 0.$$

Using the rational root thereom (and Descartes rule of signs) on the remaining cubic yields the possibilities of

$$-1, -5, -\frac{1}{2}, -\frac{5}{2}$$

of which $\lambda = -5/2$ is a solution. Divison leads to

$$(3\lambda - 2)(2\lambda + 5)(\lambda^2 + 1) = 0.$$

Hence the eignevalues are

$$\frac{2}{3}, -\frac{5}{2}, \pm i.$$

3. Since $Ix = x$, the eigenvalue is $\lambda = 1$ and every (non-zero) vector is an eigenvector corresponding to 1.

4. (a) First we find the characteristic equation. $|A - \lambda I| = 0$.

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{bmatrix}.$$

Hence

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 5-\lambda & 4 & 2 \\ 4 & 5-\lambda & 2 \\ 2 & 2 & 2-\lambda \end{vmatrix} \\ &= (5-\lambda)\begin{vmatrix} 5-\lambda & 2 \\ 2 & 2-\lambda \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 2 & 2-\lambda \end{vmatrix} + 2\begin{vmatrix} 4 & 5-\lambda \\ 2 & 2 \end{vmatrix} \\ &= (5-\lambda)\big((5-\lambda)(2-\lambda) - 4\big) - 4\big(4(2-\lambda) - 4\big) + 2\big(8 - 2(5-\lambda)\big) \\ &= (5-\lambda)\big(\lambda^2 - 7\lambda + 6\big) - 4\big(4 - 4\lambda\big) + 2\big(-2 + 2\lambda\big) \\ &= -\lambda^3 + 12\lambda^2 - 41\lambda + 30 - 16 + 16\lambda - 4 + 4\lambda \\ &= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 \end{aligned}$$

Testing $\lambda = 1$ yields a solution, hence $\lambda - 1$ is a factor and we get

$$\begin{aligned} 0 &= -\lambda^3 + 12\lambda^2 - 21\lambda + 10 \\ &= -(\lambda - 1)(\lambda^2 - 11\lambda + 10) \\ &= -(\lambda - 1)(\lambda - 1)(\lambda - 10). \end{aligned}$$

Therefore the eigenvalues are $1, 1, 10$. Finding the eigenvectors requires solving for non-zero vectors where $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
For $\lambda = 1$

$$A - \lambda I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 4 & 4 & 2 & 0 \\ 4 & 4 & 2 & 0 \\ 2 & 2 & 1 & 0 \end{array}\right] \text{ row reduces to}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ which has solutions}$$

$$x = -y - \frac{z}{2}, \quad y, z \text{ are arbitrary.}$$

Hence the eigenvalues are of the form.

$$y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} z \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

For $\lambda = 10$

$$A - \lambda I = \begin{bmatrix} -5 & 4 & 2 \\ 4 & -5 & 2 \\ 2 & 2 & -8 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{array}\right] \text{ row reduces to}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ which has solutions}$$

$$x = 2z, \quad y = 2z, \quad , z \text{ arbitrary.}$$

Hence the eigenvalues are of the form.

$$z \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

(b) First we find the characteristic equation. $|A - \lambda I| = 0$.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{bmatrix}.$$

Hence

$$\begin{aligned}
|A-\lambda I| &= \begin{vmatrix} 1-\lambda & 2 & -1 \\ 1 & -\lambda & 1 \\ 4 & -4 & 5-\lambda \end{vmatrix} \\
&= (1-\lambda)\begin{vmatrix} -\lambda & 1 \\ -4 & 5-\lambda \end{vmatrix} - 2\begin{vmatrix} 1 & 1 \\ 4 & 5-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & -\lambda \\ 4 & -4 \end{vmatrix} \\
&= (1-\lambda)\big((-\lambda)(5-\lambda)-(-4)\big) - 2\big(1(5-\lambda)-4\big) - 1\big(-4-4(-\lambda)\big) \\
&= (1-\lambda)\big(\lambda^2-5\lambda+4\big) - 2\big(1-\lambda\big) - 1\big(-4+4\lambda\big) \\
&= -\lambda^3+6\lambda^2-9\lambda+4-2+2\lambda+4-4\lambda \\
&= -\lambda^3+6\lambda^2-11\lambda+6
\end{aligned}$$

Testing $\lambda = 1$ yields a solution, hence $\lambda - 1$ is a factor and we get

$$\begin{aligned}
0 &= -\lambda^3+6\lambda^2-11\lambda+6 \\
&= -(\lambda-1)(\lambda^2-5\lambda+6) \\
&= -(\lambda-1)(\lambda-2)(\lambda-3).
\end{aligned}$$

Therefore the eigenvalues are $1, 2, 3$. Finding the eigenvectors requires solving for non-zero vectors where $(A - \lambda I)\mathbf{x} = \mathbf{0}$.
For $\lambda = 1$

$$A-\lambda I = \begin{bmatrix} 0 & 2 & -1 \\ 1 & -1 & 1 \\ 4 & -4 & 4 \end{bmatrix}$$

$\left[\begin{array}{ccc|c} 0 & 2 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 4 & -4 & 4 & 0 \end{array}\right]$ row reduces to
$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ which has solutions

$$x = -\frac{z}{2}, \quad y = \frac{z}{2}, z \text{ arbitrary.}$$

Hence the eigenvalues are of the form.

$$\frac{1}{2}z\begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

For $\lambda = 2$

$$A-\lambda I = \begin{bmatrix} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 4 & -4 & 3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 2 & -1 & 0 \\ 1 & -2 & 1 & 0 \\ 4 & -4 & 3 & 0 \end{array}\right] \text{ row reduces to}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ which has solutions}$$

$$x = -\frac{z}{2}, \quad y = \frac{z}{4}, z \text{ arbitrary.}$$

Hence the eigenvalues are of the form.

$$\frac{1}{4}z\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}.$$

For $\lambda = 3$

$$A - \lambda I = \begin{bmatrix} -2 & 2 & -1 \\ 1 & -3 & 1 \\ 4 & -4 & 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 2 & -1 & 0 \\ 1 & -3 & 1 & 0 \\ 4 & -4 & 2 & 0 \end{array}\right] \text{ row reduces to}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/4 & 0 \\ 0 & 1 & -1/4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \text{ which has solutions}$$

$$x = -\frac{z}{4}, \quad y = \frac{z}{4}, z \text{ arbitrary.}$$

Hence the eigenvalues are of the form.

$$\frac{1}{4}z\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}.$$

5. If $\lambda = 0$ is an eigenvalue, then $A\mathbf{x} = 0\mathbf{x} = \mathbf{0}$ has non zero solutions. If A was invertible, then the only solution to $A\mathbf{x} = \mathbf{0}$ would be $\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$. Hence A cannot be invertible.