[7] 1. Find the radius of convergence and the open interval of convergence for the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1} \sqrt{(2n)!}}{4^{2n}(n+2)!}(x+2)^{2n}.$$

Let $X = (x+2)^2$, then the series becomes $\sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n+1}\sqrt{(2n)!}}{4^{2n}(n+2)!} X^n$

Let $a_n = \frac{(-1)^n 3^{2n+1}\sqrt{(2n)!}}{4^{2n}(n+2)!}$, then $a_{n+1} = \frac{(-1)^{n+1} 3^{2n+3}\sqrt{(2n+2)!}}{4^{2n+2}(n+3)!}$

$$R_X = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^n 3^{2n+1}\sqrt{(2n)!}}{4^{2n}(n+2)!}\,\frac{4^{2n}4^2(n+2)!(n+3)}{(-1)^{n+1}3^{2n+1}3^2\sqrt{(2n)!(2n+1)(2n+2)}}\right|$$

$$= \lim_{n\to\infty}\frac{4^2(n+3)}{3^2\sqrt{(2n+1)(2n+2)}} = \lim_{n\to\infty}\frac{16n}{9\sqrt{4n^2}} = \lim_{n\to\infty}\frac{16n}{18n} = \frac{8}{9}$$

Since $|X| < R_X$, then $|X| < \frac{8}{9}$ and $|x+2|^2 < \frac{8}{9}$

It follows that $|x+2| < \frac{\sqrt{8}}{3} = R_x$ which is the radius of convergence. The open interval is

$$-\frac{\sqrt{8}}{3} < x+2 < \frac{\sqrt{8}}{3}$$

$$-\frac{\sqrt{8}}{3} - 2 < x < \frac{\sqrt{8}}{3} - 2$$

$$\frac{-\sqrt{8}-6}{3} < x < \frac{\sqrt{8}-6}{3}$$

$$\frac{-2\sqrt{2}-6}{3} < x < \frac{2\sqrt{2}-6}{3}$$

[10] 2. Let $f(x) = \dfrac{2x+2}{-15x^2+2x+1}$; given the partial decomposition

$$\frac{2x+2}{-15x^2+2x+1} = \frac{1}{1-3x} + \frac{1}{1+5x},$$

find the Taylor series of $f(x)$ about $x = -1$. Give the open interval of convergence. Express your final answer in sigma notation $\sum_n a_n(x+1)^n$.

$$\cdot)\ \frac{1}{1-3x} = \frac{1}{1-3(x+1-1)} = \frac{1}{4-3(x+1)} = \frac{1}{4\left(1-\frac{3(x+1)}{4}\right)}$$

$$= \frac{1}{4}\sum_{n=0}^{\infty} \frac{3^n}{4^n}(x+1)^n = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}}(x+1)^n$$

for $\left|\frac{3(x+1)}{4}\right| < 1$ or $-\frac{7}{3} < x < \frac{1}{3}$

$$\cdot)\ \frac{1}{1+5x} = \frac{1}{1+5(x+1-1)} = \frac{1}{-4+5(x+1)} = -\frac{1}{4}\frac{1}{\left(1-\frac{5(x+1)}{4}\right)}$$

$$= -\frac{1}{4}\sum_{n=0}^{\infty} \frac{5^n}{4^n}(x+1)^n = \sum_{n=0}^{\infty} -\frac{5^n}{4^{n+1}}(x+1)^n$$

for $\left|\frac{5(x+1)}{4}\right| < 1$

or $-\frac{9}{5} < x < -\frac{1}{5}$

$$f(x) = \sum_{n=0}^{\infty} \frac{3^n}{4^{n+1}}(x+1)^n + \sum_{n=0}^{\infty} -\frac{5^n}{4^{n+1}}(x+1)^n$$

$$= \sum_{n=0}^{\infty}\left[\frac{3^n - 5^n}{4^{n+1}}\right](x+1)^n$$

The open interval of convergence is the intersection between $\left(-\frac{7}{3}, \frac{1}{3}\right)$ and $\left(-\frac{9}{5}, -\frac{1}{5}\right)$ which is $\left(-\frac{9}{5}, -\frac{1}{5}\right)$.

[10] 3. Find the MacLaurin series for the function

$$f(x) = x^2 \ln(1-3x).$$

Give the open interval of convergence. Express your final answer in sigma notation $\sum_n a_n x^n$.

Consider $g(x) = \ln(1-3x)$, $g'(x) = \dfrac{-3}{1-3x}$.

then $-3 \cdot \dfrac{1}{1-3x} = -3 \sum_{n=0}^{\infty} 3^n x^n = -\sum_{n=0}^{\infty} 3^{n+1} x^n$

for $|3x| < 1$ or $-\frac{1}{3} < x < \frac{1}{3}$.

$$\ln|1-3x| = \int \frac{-3}{1-3x}\,dx = \int -\sum_{n=0}^{\infty} 3^{n+1} x^n$$

$$= -\sum_{n=0}^{\infty} \frac{3^{n+1}}{n+1} x^{n+1} + C$$

let $x=0$, we have $\ln 1 = C = 0$ and therefore

$$\ln|1-3x| = -\sum_{n=0}^{\infty} \frac{3^{n+1} x^{n+1}}{n+1}.$$

Since $1-3x > 0$ when $-\frac{1}{3} < x < \frac{1}{3}$, we can write

$$\ln(1-3x) = -\sum_{n=0}^{\infty} \frac{3^{n+1} x^{n+1}}{n+1} \text{ and}$$

$$x^2 \ln(1-3x) = -\sum_{n=0}^{\infty} \frac{3^{n+1} x^{n+3}}{n+1}$$

$$= -\sum_{n=3}^{\infty} \frac{3^{n-2}}{n-2} x^n \quad \text{for } -\frac{1}{3} < x < \frac{1}{3}.$$

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[7] 4. Find the value of x for which the **fourth** term of the binomial expansion of $f(x) = \dfrac{1}{(1+x)^2}$ is equal to $-\dfrac{27}{2}$. Show your work.

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = 1+(-2)x + \frac{(-2)(-3)}{2!}x^2 + \frac{(-2)(-3)(-4)}{3!}x^3$$

$$+ \frac{(-2)(-3)(-4)(-5)}{4!}x^4 + \cdots$$

$$= 1-2x+\frac{3!}{2!}x^2 - \frac{4!}{3!}x^3 + \frac{5!}{4!}x^4 + \cdots$$

$$= 1-2x+3x^2-4x^3+5x^4+\cdots$$

The fourth term is $-4x^3$. So $-4x^3 = -\dfrac{27}{2}$ when $x^3 = \dfrac{27}{8}$ or $x = \dfrac{3}{2}$.

[9] 5. Find a one parameter family of solutions for the differential equation

$$\frac{dy}{dx} = \frac{xy + 4y - 2x - 8}{xy - y + 3x - 3}.$$

$$\frac{dy}{dx} = \frac{y(x+4) - 2(x+4)}{y(x-1) + 3(x-1)} = \frac{(x+4)(y-2)}{(x-1)(y+3)} \Leftrightarrow$$

$$\frac{y+3}{y-2}\,dy = \frac{x+4}{x-1}\,dx$$

$$\frac{y-2+5}{y-2}\,dy = \frac{x-1+5}{x-1}\,dx$$

$$\left(1 + \frac{5}{y-2}\right)dy = \left(1 + \frac{5}{x-1}\right)dx$$

Integrating both sides, we have

$$y + 5\ln|y-2| = x + 5\ln|x-1| + C.$$

[9] 6. Find, in **explicit** form, a one parameter family of solutions for the differential equation

$$x^2\frac{dy}{dx} + x(x+2)y = e^x, \qquad x > 0.$$

The standard form is

$$\frac{dy}{dx} + \left(1+\frac{2}{x}\right)y = \frac{e^x}{x^2}.$$ Thus, $P(x) = 1+\frac{2}{x}$ so an

integrating factor is

$$e^{\int P(x)dx} = e^{\int\left(1+\frac{2}{x}\right)dx} = e^{(x+2\ln|x|)} = e^{(x+\ln x^2)}$$
$$= x^2e^x.$$

Multiplying the standard form by x^2e^x gives us

$$x^2e^x\frac{dy}{dx} + x^2e^x\left(1+\frac{2}{x}\right)y = e^{2x}$$
$$\frac{d}{dx}\left[x^2e^xy\right] = e^{2x}$$

By integrating both sides,

$$x^2e^xy = \int e^{2x}\,dx$$
$$x^2e^xy = \frac{1}{2}e^{2x} + C$$
$$y = \frac{e^{2x}}{2x^2e^x} + \frac{C}{x^2e^x}$$

or $$y = \frac{e^x}{2x^2} + \frac{Ce^{-x}}{x^2}$$

[10] 7. Find the solution of the initial-value problem

$$2y'' = 3y^2, \qquad y(0) = 1, \quad y'(0) = 1.$$

let $v = y'$ so that $y'' = v\frac{dv}{dy}$. The equation becomes $2v\frac{dv}{dy} = 3y^2$. Separating variables we obtain

$$2v\,dv = 3y^2\,dy \Rightarrow v^2 = y^3 + c_1$$

When $x=0$, $y=1$ and $y' = v = 1$ so $1 = 1 + c_1$ and $c_1 = 0$. Then

$$v^2 = y^3 \Rightarrow \left(\frac{dy}{dx}\right)^2 = y^3 \Rightarrow \frac{dy}{dx} = y^{3/2} \Rightarrow$$

$$y^{-3/2}dy = dx \Rightarrow -2y^{-1/2} = x + c_2 \Rightarrow y = \frac{4}{(x+c_2)^2}$$

When $x=0$, $y=1$, so $1 = \frac{4}{c_2^2} \Rightarrow c_2 = \pm 2$. Thus

$$y = \frac{4}{(x+2)^2} \text{ and } y = \frac{4}{(x-2)^2}.$$

Note however that when $y = \frac{4}{(x+2)^2}$, $y' = -\frac{8}{(x+2)^3}$ and $y'(0) = -1 \neq 1$. Thus, the solution of the IVP is $y = \frac{4}{(x-2)^2}$.

[8] 8. Given that $(m-1)^2(m+3)^3(m^2+4)^2 = 0$ is the auxiliary equation associated with the linear differential equation

$$\phi(D)y = 2x + \sin 2x + xe^x + x^2e^{-3x},$$

what is the **form** of the particular solution $y_p(x)$?
DO NOT EVALUATE THE COEFFICIENTS IN $y_p(x)$.

The roots are $m = 1$ (multiplicity 2), $m = -3$ (multiplicity 3), $m = \pm 2i$ (multiplicity 2). So,

$$y_h(x) = (C_1 + C_2x)e^x + (C_3 + C_4x + C_5x^2)e^{-3x} + (C_6 + C_7x)\cos 2x + (C_8 + C_9x)\sin 2x$$

Assume

$$y_p(x) = Ax + B + \underbrace{C\cos 2x + E\sin 2x}_{\text{multiply by } x^2} + \underbrace{(Fx+G)e^x}_{\text{multiply by } x^2} + \underbrace{(Hx^2+Ix+J)e^{-3x}}_{\text{multiply by } x^3}$$

Finally,

$$y_p(x) = Ax + B + Cx^2\cos 2x + Ex^2\sin 2x + (Fx^3 + Gx^2)e^x + (Hx^5 + Ix^4 + Jx^3)e^{-3x}.$$

[10] 9. Find the Laplace transform of the function

$$f(t) = \begin{cases} t^2 & \text{if } 0 \le t < 3 \\ e^{-3t} & \text{if } 3 \le t < 5 \\ (t-5)^3 & \text{if } t \ge 5. \end{cases}$$

$$f(t) = t^2 - t^2 u(t-3) + e^{-3t}u(t-3) - e^{-3t}u(t-5) + (t-5)^3 u(t-5)$$

$\cdot)\ \mathcal{L}\{t^2\} = \dfrac{2}{s^3}$

$$\cdot)\ \mathcal{L}\{t^2 u(t-3)\} = e^{-3s}\mathcal{L}\{(t+3)^2\}$$
$$= e^{-3s}\left(\mathcal{L}\{t^2\} + 6\mathcal{L}\{t\} + 9\mathcal{L}\{1\}\right)$$
$$= e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right)$$

$$\cdot)\ \mathcal{L}\{e^{-3t}u(t-3)\} = e^{-3s}\mathcal{L}\{e^{-3(t+3)}\} = e^{-3s}e^{-9}\mathcal{L}\{e^{-3t}\}$$
$$= \frac{e^{-(3s+9)}}{s+3}$$

$$\cdot)\ \mathcal{L}\{e^{-3t}u(t-5)\} = e^{-5s}\mathcal{L}\{e^{-3(t+5)}\} = e^{-5s}e^{-15}\mathcal{L}\{e^{-3t}\}$$
$$= \frac{e^{-(5s+15)}}{s+3}$$

$$\cdot)\ \mathcal{L}\{(t-5)^3 u(t-5)\} = e^{-5s}F(s) \text{ with } F(s) = \mathcal{L}\{t^3\} = \frac{3!}{s^4}$$
$$= e^{-5s}\frac{3!}{s^4} = \frac{6e^{-5s}}{s^4}$$

$$\mathcal{L}\{f(t)\} = \frac{2}{s^3} - e^{-3s}\left(\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right) + \frac{e^{-(3s+9)} - e^{-(5s+15)}}{s+3}$$
$$+ \frac{6e^{-5s}}{s^4}.$$

[10] 10. Find $\mathscr{L}^{-1}\left\{\dfrac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\}$.

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3s+5}{s^2-2s+5} - \frac{1}{s+1}$$

By completing the square, $s^2-2s+5 = (s-1)^2+4$

$$\frac{2s^2+10s}{(s^2-2s+5)(s+1)} = \frac{3s+5}{(s-1)^2+4} - \frac{1}{s+1}$$

$$= \frac{3(s-1)+5+3}{(s-1)^2+4} - \frac{1}{s+1}$$

$$= \frac{3(s-1)+8}{(s-1)^2+4} - \frac{1}{s+1}$$

$$= \frac{3(s-1)}{(s-1)^2+2^2} + \frac{8}{(s-1)^2+2^2} - \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\left\{\frac{2s^2+10s}{(s^2-2s+5)(s+1)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{s-1}{(s-1)^2+2^2}\right\} + 4\mathcal{L}^{-1}\left\{\frac{2}{(s-1)^2+2^2}\right\}$$

$$-\mathcal{L}\left\{\frac{1}{s+1}\right\}$$

$$= 3e^{t}\cos 2t + 4e^{t}\sin 2t - e^{-t}.$$

[10] 11. Use Laplace transforms to solve the initial-value problem

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} = 1 + \delta(t-2), \qquad y(0) = 0, \;\; y'(0) = 1.$$

$$s^2 Y(s) - s y(0) - y'(0) - 2[sY(s) - y(0)] = \mathcal{L}\{1\} + \mathcal{L}\{\delta(t-2)\}$$

$$s^2 Y(s) - 1 - 2s Y(s) = \frac{1}{s} + e^{-2s}$$

$$(s^2 - 2s) Y(s) = \frac{1}{s} + 1 + e^{-2s}$$

$$Y(s) = \frac{1}{s(s^2-2s)} + \frac{1}{s^2-2s} + \frac{e^{-2s}}{s^2-2s}$$

$$= \frac{s+1}{s^2(s-2)} + \frac{e^{-2s}}{s(s-2)}$$

Using partial fraction decomposition,

$$Y(s) = \frac{3}{4}\frac{1}{s-2} - \frac{3}{4}\frac{1}{s} - \frac{1}{2}\frac{1}{s^2} + \left[\frac{1}{2}\frac{1}{s-2} - \frac{1}{2}\frac{1}{s}\right]e^{-2s}$$

Therefore,

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \frac{3}{4}e^{2t} - \frac{3}{4} - \frac{1}{2}t + \left[\frac{1}{2}e^{2(t-2)} - \frac{1}{2}\right]u(t-2)$$