MATH 2130 – Tutorial Problem Solutions, Thu Jan 25

Sketching surfaces

Example (review). In the xy-plane, sketch the hyperbola given by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a, b > 0$. Include the intercepts of the hyperbola, and the equations of its asymptotes.

Solution. The hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

opens in the $\pm x$-directions. It has intercepts $(\pm a, 0)$.

To find the asymptotes, divide the equation by x^2:

$$\frac{1}{a^2} - \frac{1}{b^2}\frac{y^2}{x^2} = \frac{1}{x^2}.$$

The asymptotes represent the behavior of the curve as x^2 becomes very large. In this limit, $\frac{1}{x^2} \to 0$, so we get

$$\frac{1}{a^2} - \frac{1}{b^2}\left(\frac{y}{x}\right)^2 = 0,$$

which implies that

$$\frac{y}{x} = \pm\frac{b}{a}.$$

Thus the asymptotes for this hyperbola are $y = \pm\frac{b}{a}x$.

Example. Sketch the surface given by the equation $z = x^2 - 2y^2$.

Solution. At $z = 0$, the equation reduces to $x^2 = 2y^2$, which represents the two lines $x = \sqrt{2}y$ and $x = -\sqrt{2}y$.

Consider the slice at $z = t$, $t > 0$. The equation becomes

$$x^2 - 2y^2 = t,$$

which is a hyperbola, centre $(x = 0, y = 0)$, opening in the $\pm x$-directions, with intercepts $(x = \pm\sqrt{t}, y = 0)$ and asymptotes $y = \pm\frac{1}{\sqrt{2}}x$.

Now consider the slice at $z = -t$, $t > 0$. The equation becomes

$$2y^2 - x^2 = t,$$

which is a hyperbola, centre $(x = 0, y = 0)$, opening in the $\pm y$-directions, with intercepts at $(x = 0, y = \pm\sqrt{t/2})$ and asymptotes $y = \pm\frac{1}{\sqrt{2}}x$.

When I slice the surface this way, I have no idea how to draw it.

Alternatively: At $y = 0$, we get the curve $z = x^2$, which is a parabola in the xz-plane, opening in the $+z$-direction, global minimum $(x = 0, z = 0)$.

At $x = 0$, we get the curve $z = -2y^2$, which is a parabola in the yz-plane, opening in the $-z$-direction, global maximum $(y = 0, z = 0)$.

Set $x = t$, $t > 0$, and note that $x = t$ and $x = -t$ yield the same slice. In this plane, we get the curve $z = t^2 - 2y^2$, which is a parabola, opening in the $-z$-direction, global maximum $(y = 0, z = t^2)$. The global maxima of these parabolas follow the curve $z = x^2$, $y = 0$.

Compare Figure 11.28 in the textbook, p. 704.

Look for: slices of the surface that are hyperbolas. Then, if possible, avoid them!

Parametric representations of curves

Example. Find a parametric representation for the intersection of the surface $x^2 + (y+4)^2 + (z-2)^2 = 9$ with (a) the plane $z = 1$; (b) the plane $y + z = 2$.

Solution. The first surface is a sphere with centre $(0, -4, 2)$ and radius 3.

(a) The plane $z = 1$ is parallel to the xy-plane, through the point $(0, 0, 1)$. A sketch suggests that the curve of intersection is a circle.

Notice that all three variables are constrained. There is no clear choice of variable to set equal to t.

Set $z = 1$ in the equation of the sphere:

$$x^2 + (y+4)^2 + (-1)^2 = 9,$$

which implies that

$$x^2 + (y+4)^2 = 8.$$

This is the equation of a circle, radius $2\sqrt{2}$, centre $(x = 0, y = -4)$.

Let

$$x = 2\sqrt{2}\cos t, \quad y + 4 = 2\sqrt{2}\sin t, \quad 0 \le t \le 2\pi.$$

Then

$$x^2 + (y+4)^2 = 8\cos^2 t + 8\sin^2 t = 8,$$

as needed. Thus a parametric representation of the curve of intersection is

$$x = 2\sqrt{2}\cos t, \quad y = 2\sqrt{2}\sin t - 4, \quad z = 1, \quad 0 \le t \le 2\pi.$$

(b) The plane $y + z = 2$ has normal vector $(0, 1, 1)$ and passes through $(0, 1, 1)$. A sketch suggests that the intersection is an ellipse.

From the second equation, we get $z = 2 - y$. With this substitution, the first equation becomes

$$x^2 + (y+4)^2 + y^2 = 9.$$

We expand, then complete the square in y, and get

$$x^2 + 2(y+2)^2 = 1.$$

Let

$$x = \cos t, \quad \sqrt{2}(y+2) = \sin t, \quad 0 \le t \le 2\pi.$$

Then

$$y = \frac{1}{\sqrt{2}} \sin t - 2,$$

and

$$x^2 + 2(y+2)^2 = \cos^2 t + \sin^2 t = 1,$$

as needed. From the equation for z in terms of y, we get

$$z = 2 - y = 4 - \frac{1}{\sqrt{2}} \sin t.$$

Thus a parametrization of the intersection is

$$x = \cos t, \quad y = \frac{1}{\sqrt{2}} \sin t - 2, \quad z = 4 - \frac{1}{\sqrt{2}} \sin t, \quad 0 \leq t \leq 2\pi.$$

Look for: constraints of the form

$$A^2(x-a)^2 + B^2(y-b)^2 = r^2,$$

or similar expressions involving any two of the three variables. A solution is

$$A(x-a) = r\cos t, \quad B(y-b) = r\sin t, \quad 0 \leq t \leq 2\pi,$$

which rearranges to

$$x = \frac{r}{A}\cos t + a, \quad y = \frac{r}{B}\sin t + b, \quad 0 \leq t \leq 2\pi.$$

Then it only remains to solve for z in terms of x and y.

Example. Find a parametric representation for the intersection of the surfaces

$$x = \sqrt{y^2 + z^2} \quad \text{and} \quad x - 3z = 0$$

such that x increases when y is negative.

First we will find a parametric representation, then we will check whether the extra condition is satisfied.

The equation $x = \sqrt{y^2 + z^2}$ is a cone, opening in the $+x$-direction. The second equation is a plane through the origin with normal vector $(1, 0, -3)$. A sketch suggests that the curve of intersection consists of two rays.

The curve of intersection is restricted to $x \geq 0$. From the equation of the plane, we have $x = 3z$, so $z \geq 0$ also. There is no constraint on y.

Let $y = t$, $t \in \mathbb{R}$. Then $x = \sqrt{t^2 + z^2}$. We substitute $x = 3z$ in this equation, square both sides, and rearrange to get

$$z^2 = \frac{t^2}{8}.$$

Since z is nonnegative, $\sqrt{z^2} = z$. Since t can be any real value, $\sqrt{t^2} = |t|$. Thus, when we take the square root of the equation, we get

$$z = \frac{1}{2\sqrt{2}}|t|.$$

Lastly, $x = 3z = \frac{3}{2\sqrt{2}}|t|$. Thus a parametrization of this curve is

$$x = \frac{3}{2\sqrt{2}}|t|, \quad y = t, \quad z = \frac{1}{2\sqrt{2}}|t|, \quad t \in \mathbb{R}.$$

Now we check the extra condition. Notice that y is negative when $t < 0$. In this range of t,

$$x = \frac{3}{2\sqrt{2}}|t| = -\frac{3}{2\sqrt{t}}t,$$

which satisfies

$$\frac{dx}{dt} = -\frac{3}{2\sqrt{2}}.$$

Thus, in the parametrization we have constructed, x decreases when y is negative.

Since this is not what we were asked for, we must reverse direction. Let $s = -t$. Then

$$x = \frac{3}{2\sqrt{2}}|s|, \quad y = -s, \quad z = \frac{1}{2\sqrt{2}}|s|, \quad s \in \mathbb{R}.$$

This is a parametrization of the curve of intersection having the desired property.