MAT2130: Engineering Mathematical Analysis 1

Midterm 1 Practice Problems – Selected Solutions

1. Let $A = (A_x, A_y, A_z)$, $B = (B_x, B_y, B_z)$ and $C = (C_x, C_y, C_z)$. Then the midpoint of AB is $P = \left(\frac{A_x+B_x}{2}, \frac{A_y+B_y}{2}, \frac{A_z+B_z}{2}\right)$, the midpoint of BC is $Q = \left(\frac{B_x+C_x}{2}, \frac{B_y+C_y}{2}, \frac{B_z+C_z}{2}\right)$, and the midpoint of AC is $R = \left(\frac{A_x+C_x}{2}, \frac{A_y+C_y}{2}, \frac{A_z+C_z}{2}\right)$. Lastly, the midpoint of PR is $X = \left(\frac{2A_x+B_x+C_x}{4}, \frac{2A_y+B_y+C_y}{4}, \frac{2A_z+B_z+C_z}{4}\right)$.

   We are asked to show that A, Q and X are collinear. Observe that

$$X - A = \left(\frac{B_x + C_x - 2A_x}{4}, \frac{B_y + C_y - 2A_y}{4}, \frac{B_z + C_z - 2A_z}{4 \in}\right),$$

while

$$Q - A = \left(\frac{B_x + C_x - 2A_x}{2}, \frac{B_y + C_y - 2A_y}{2}, \frac{B_z + C_z - 2A_z}{2}\right).$$

Therefore $Q - A = 2(X - A)$, which shows that these three points all lie on the same line.

2. Let $A = (0, 2, 1)$ and $B = (2\sqrt{3}, 0, 1)$. Then

$$\|AB\|^2 = (2\sqrt{3})^2 + 2^2 + 0 = 16.$$

We are asked for a point C in the yz-plane such that ABC is an equilateral triangle. Let $C = (0, C_y, C_z)$. We require that $\|AC\|^2 = \|BC\|^2 = 16$. From these conditions, we get

$$\|AC\|^2 = (C_y - 2)^2 + (C_z - 1)^2 = 16, \tag{1}$$

and

$$\|BC\|^2 = (2\sqrt{3})^2 + C_y^2 + (C_z - 1)^2 = 16. \tag{2}$$

Subtract (2) from (1):

$$-4C_y + 4 - 12 = 0,$$

which implies that $C_y = -2$. With this substitution in (1), we find that $C_z = 1$. Therefore the desired point is $C = (0, -2, 1)$.

3. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Let the angle between them be θ. If $\mathbf{u}$ and $\mathbf{v}$ are in the same direction, then $\widehat{\mathbf{u}} + \widehat{\mathbf{v}}$ is also in that direction, and the angle between all of these vectors is 0. If $\mathbf{u}$ and $\mathbf{v}$ are in opposite directions, then $\widehat{\mathbf{u}} + \widehat{\mathbf{v}} = \mathbf{0}$, which does not have a well defined angle with either $\mathbf{u}$ or $\mathbf{v}$.

   Assume that $0 < \theta < \pi$. Let $\mathbf{w} = \widehat{\mathbf{u}} + \widehat{\mathbf{v}} \neq \mathbf{0}$. Let the angle between $\mathbf{u}$ and $\mathbf{w}$ be α, and let the angle between $\mathbf{v}$ and $\mathbf{w}$ be β. Then

$$\mathbf{u} \cdot \mathbf{w} = |\mathbf{u}||\mathbf{w}| \cos \alpha,$$

and

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}||\mathbf{w}| \cos \beta.$$

Observe that

$$\begin{aligned}\mathbf{u}\cdot\mathbf{w} =&\mathbf{u}\cdot(\widehat{\mathbf{u}}+\widehat{\mathbf{v}})\\ =&\mathbf{u}\cdot\widehat{\mathbf{u}}+\mathbf{u}\cdot\widehat{\mathbf{v}}\\ =&|\mathbf{u}|+|\mathbf{u}|\cos\theta.\end{aligned}$$

Therefore

$$|\mathbf{u}|+|\mathbf{u}|\cos\theta = |\mathbf{u}||\mathbf{w}|\cos\alpha,$$

which implies that

$$\cos\alpha = \frac{1+\cos\theta}{|\mathbf{w}|}.$$

Similarly,

$$\begin{aligned}\mathbf{v}\cdot\mathbf{w} =&\mathbf{v}\cdot(\widehat{\mathbf{u}}+\widehat{\mathbf{v}})\\ =&\mathbf{v}\cdot\widehat{\mathbf{u}}+\mathbf{v}\cdot\widehat{\mathbf{v}}\\ =&|\mathbf{v}|\cos\theta+|\mathbf{v}|.\end{aligned}$$

Therefore

$$|\mathbf{v}|\cos\theta+|\mathbf{v}| = |\mathbf{v}||\mathbf{w}|\cos\beta,$$

which implies that

$$\cos\beta = \frac{1+\cos\theta}{|\mathbf{w}|} = \cos\alpha.$$

We conclude that $\alpha=\beta$, as desired.

4. (a) Let $\mathbf{v}=(2,-1,1)$ and $\mathbf{u}=(1,1,1)$. Then the component of $\mathbf{v}$ in the direction of $\mathbf{u}$ is

$$\begin{aligned}\mathbf{v}\cdot\widehat{\mathbf{u}} =&(2,-1,1)\cdot\frac{1}{\sqrt{3}}(1,1,1)\\ =&\frac{2}{\sqrt{3}}.\end{aligned}$$

(b) Let $\mathbf{w}=(w_x,w_y,w_z)$. We require that $\mathbf{w}$ be a vector of length 12 such that its component in the direction of $\mathbf{v}$ is $\sqrt{6}$ and its component in the direction of $\mathbf{u}$ is $2\sqrt{3}$. From the constraint on the length, we get

$$w_x^2+w_y^2+w_z^2=144.$$

The component of $\mathbf{w}$ in the direction of $\mathbf{v}$ is

$$\begin{aligned}\mathbf{w}\cdot\widehat{\mathbf{v}} =&(w_x,w_y,w_z)\frac{1}{\sqrt{6}}(2,-1,1)\\ =&\frac{2w_x-w_y+w_z}{\sqrt{6}}.\end{aligned}$$

When we set this equal to $\sqrt{6}$, we get

$$2w_x-w_y+w_z=6. \tag{3}$$

Further, the component of $\mathbf{w}$ in the direction of $\mathbf{u}$ is

$$\begin{aligned}\mathbf{w}\cdot\widehat{\mathbf{v}} =&(w_x, w_y, w_z)\frac{1}{\sqrt{3}}(1,1,1)\\ =&\frac{w_x+w_y+w_z}{\sqrt{3}}.\end{aligned}$$

When we set this equal to $2\sqrt{3}$, we get

$$w_x + w_y + w_z = 6. \tag{4}$$

We subtract (3) from (4) to find that

$$w_x = 2w_y.$$

With this substitution in (4), we get

$$w_z = 6 - 3w_y.$$

Lastly, from the condition on the length, we get

$$4w_y^2 + w_y^2 + (6-3w_y)^2 = 144.$$

This is a quadratic equation in w_y which has two roots. These give us the two possible vectors $\mathbf{w}$ satisfying all of the constraints.

5. Let the position of the mass be $X = (a, b, -c)$, where $a, b, c > 0$. We are told that the cables are attached to points $A = (0,0,0)$, $B = (2\sqrt{3}, 0, 0)$ and $C = (\sqrt{3}, 3, 0)$, and they each have length d. Then $|\mathbf{XA}| = |\mathbf{XB}| = |\mathbf{XC}| = d$, which implies that

$$a^2 + b^2 + c^2 = d^2, \tag{5}$$

$$(2\sqrt{3}-a)^2 + b^2 + c^2 = d^2, \tag{6}$$

$$(\sqrt{3}-a)^2 + (b-3)^2 + c^2 = d^2. \tag{7}$$

From (5) and (6), we get $a = \sqrt{3}$. With this substitution in (6) and (7), we find $b = 1$. Lastly, it follows that $c = \sqrt{d^2-4}$.

Let the tensions in cables XA, XB and XC be $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$, respectively. Unit vectors in these directions are

$$\begin{aligned}\widehat{\mathbf{T}}_1 &= \frac{1}{d}\mathbf{XA} = \left(-\frac{\sqrt{3}}{d}, -\frac{1}{d}, \frac{\sqrt{d^2-4}}{d}\right),\\ \widehat{\mathbf{T}}_2 &= \frac{1}{d}\mathbf{XB} = \left(\frac{\sqrt{3}}{d}, -\frac{1}{d}, \frac{\sqrt{d^2-4}}{d}\right),\\ \widehat{\mathbf{T}}_3 &= \frac{1}{d}\mathbf{XC} = \left(0, \frac{2}{d}, \frac{\sqrt{d^2-4}}{d}\right).\end{aligned}$$

We write

$$\mathbf{T}_1 = T_1\widehat{\mathbf{T}}_1, \quad \mathbf{T}_2 = T_2\widehat{\mathbf{T}}_2, \quad \mathbf{T}_3 = T_3\widehat{\mathbf{T}}_3.$$

The force of gravity is

$$\mathbf{F}_g = -Mg\widehat{\mathbf{k}} = (0, 0, -Mg).$$

We set

$$\mathbf{T}_1 + \mathbf{T}_2 + \mathbf{T}_3 + \mathbf{F}_g = \mathbf{0}.$$

From each of the three components, we get the equations

$$-\frac{\sqrt{3}}{d}T_1 + \frac{\sqrt{3}}{d}T_2 = 0, \tag{8}$$

$$-\frac{1}{d}T_1 - \frac{1}{d}T_2 + \frac{2}{d}T_3 = 0, \tag{9}$$

$$\frac{\sqrt{d^2-4}}{d}T_1 + \frac{\sqrt{d^2-4}}{d}T_2 + \frac{\sqrt{d^2-4}}{d}T_3 - Mg = 0. \tag{10}$$

From (8), it follows that $T_1 = T_2$. With this substitution in (9), we get $T_2 = T_3$. Thus all three tensions are equal in magnitude. Finally, (10) reduces to

$$\frac{3\sqrt{d^2-4}}{d}T_1 - Mg = 0,$$

which implies that

$$T_1 = \frac{Mgd}{3\sqrt{d^2-4}}.$$

6. (c) We are asked to find the line ℓ that lies in the plane $2x - y - z - 1 = 0$ and intersects the line $\frac{8-x}{3} = y = \frac{z+7}{4}$ at right angles. First, we rewrite the line

$$\frac{8-x}{3} = y = \frac{z+7}{4}$$

in vector form by setting each fraction equal to t. The result is

$$\mathbf{r}(t) = (8-3t, t, 4t-7), \quad t \in \mathbb{R}.$$

This line passes through the point $(8, 0, -7)$ and is parallel to the vector $(-3, 1, 4)$

The line ℓ lies in the plane $2x - y - z - 1 = 0$, which implies that it is perpendicular to the plane's normal vector, $(2, -1, -1)$. It is also perpendicular to the vector $(-3, 1, 4)$. Therefore a vector parallel to the desired line is

$$(-3, 1, 4) \times (2, -1, -1) = (-1+4, 8-3, 3-2) = (3, 5, 1).$$

Now we need to find a point on ℓ. This line lies in the plane $2x - y - z - 1 = 0$ and also intersects the line $\mathbf{r}(t) = (8-3t, t, 4t-7)$. Therefore we need to find the intersection between the plane and the line. Substitute the components of the line into the equation of the plane:

$$2(8-3t) - t - (4t-7) - = 0,$$

which is satisfied when $t = 2$. Thus the line intersects the plane at the point $(2, 2, 1)$.

Hence ℓ passes through $(2, 2, 1)$ and is parallel to $(3, 5, 1)$, so its vector representation is

$$\mathbf{r}(t) = (2+3t, 2+5t, 1+t), \quad t \in \mathbb{R}.$$

The corresponding parametric representation is

$$x = 2+3t, \quad y = 2+5t, \quad z = 1+t, \quad t \in \mathbb{R}.$$

When we solve for t in each equation, we obtain the symmetric equations

$$\frac{x-2}{3} = \frac{y-2}{5} = z - 1.$$

8. (c) We are asked for the plane containing the line $x = y - 1$, $z = -1$, perpendicular to the plane $3x - 4y + z + 7 = 0$. Let the equation of the plane we want be $Ax + By + Cz + D = 0$. Then its normal vector is (A, B, C).

Note that the parametric representation of the line we are given is

$$x = t, \quad y = t + 1, \quad z = -1, \quad t \in \mathbb{R},$$

which is parallel to the vector $(1, 1, 0)$ and passes through $(0, 1, -1)$. Therefore (A, B, C) is perpendicular to $(1, 1, 0)$.

A normal vector to the plane we are given is $(3, -4, 1)$. Since the plane we want is perpendicular to this plane, (A, B, C) is also perpendicular to $(3, -4, 1)$. Thus

$$(A, B, C) = (1, 1, 0) \times (3, -4, 1) = (1, -1, -4 - 3) = (1, -1, -7).$$

Thus the equation of the plane has the form $x - y - 7z + D = 0$. It remains to find D. A point on the plane is $(0, 1, -1)$. Substituting this point into the equation of the plane yields $D = -6$. Hence the equation of the desired plane is

$$x - y - 7z - 6 = 0.$$

9. (c) We want to find the distance between the line ℓ given by $x = 1$, $\frac{y-1}{3} = \frac{4-z}{2}$ and the line m given by $x - 1 = \frac{1-y}{2}$, $z = 0$. First, we convert the lines to their vector representations:

$$\begin{aligned} \ell: \quad \mathbf{r}(t) &= (1, 1 + 3t, 4 - 2t), \quad t \in \mathbb{R}, \\ m: \quad \mathbf{r}(t) &= (1 + t, 1 - 2t, 0), \quad t \in \mathbb{R}. \end{aligned}$$

A point on ℓ is $R = (1, 1, 4)$, and a point on m is $S = (1, 1, 0)$. The vector from R to S is $\mathbf{RS} = (0, 0, -4)$. A vector parallel to ℓ is $(0, 3, -2)$, and a vector parallel to m is $(1, -2, 0)$. Therefore a vector perpendicular to both ℓ and m is

$$\mathbf{v} = (1, -2, 0) \times (0, 3, -2) = (4, 2, 3).$$

To find the distance between ℓ and m, we need the absolute value of the component of $\mathbf{RS}$ in the direction of $(4, 2, 3)$. This is

$$|\mathbf{RS} \cdot \hat{\mathbf{v}}| = \left| (0, 0, -4) \cdot \frac{1}{\sqrt{16 + 4 + 9}} (4, 2, 3) \right| = \frac{12}{\sqrt{29}}.$$

11. (a) An elliptic cylinder centered on the y-axis.

(b) Complete the square to put this in the form of a sphere.

(c) A cone, centered on the line $y = 2$, $z = 0$, opening in the positive x-axis.

(d) Rearrange in the form $x^2 + y^2 + z^2 = 4$, noting that $z \geq 0$. This is a hemisphere.

(e) Take the curve $y = -\frac{2}{x}$ in the xy-plane, and translate it in the $\pm z$-directions.

12. (a) $(x-1)^2 + y^2 + (z+1)^2 = 25$.

(b) $x^2 + (z-5)^2 = 4$.

(c) $x = 1 - \sqrt{y^2 + z^2}$

(d) $x = \sqrt{1 - (y-1)^2 - z^2}$

13. (c) We want to find a parametric representation for the intersection of $x = y + z^2$ and $x + 2y = 0$. We make the substitution $x = -2y$ and find $-3y = z^2$. This implies $y \geq 0$, and so $x \leq 0$. There are no constraints on z. Therefore we let $z = t$, $t \in \mathbb{R}$. Then $y = -\frac{1}{3}z^2 = -\frac{1}{3}t^2$, and $x = -2y = \frac{2}{3}t^2$. Thus a parametric representation of this curve is

$$x(t) = \frac{2}{3}t^2, \quad y(t) = -\frac{1}{3}t^2, \quad z(t) = t, \quad t \in \mathbb{R}.$$

(e) We want a parametric representation of the intersection of $y = \sqrt{1 - (x-1)^2 - z^2}$ and $x = 1$. When we substitute $x = 1$ in the first equation, we get $y = \sqrt{1 - z^2}$, which is a semicircle with center $(y = 0, z = 0)$, restricted to the region with $y \geq 0$.

In particular, we want the parametrization to start in the region where $z > 0$ and end in the region where $z < 0$. Let $z = \cos t$, where $0 \leq t \leq \pi$. Then $y = \sqrt{\sin^2 t} = \sin t$, since $\sin t \geq 0$ everywhere on this interval. Thus a parametric representation is

$$x(t) = 1, \quad y(t) = \sin t, \quad z(t) = \cos t, \quad 0 \leq t \leq \pi.$$

15. (a) We are given $\mathbf{r}(t) = 3\cos t\widehat{\mathbf{i}} + 2\sin t\widehat{\mathbf{j}} - \frac{t^2}{\pi^2}\widehat{\mathbf{k}}$. The derivative is

$$\mathbf{r}'(t) = -3\sin t\widehat{\mathbf{i}} + 2\cos t\widehat{\mathbf{j}} - \frac{2t}{\pi^2}\widehat{\mathbf{k}}.$$

This is a vector tangent to the curve, pointing in the direction of increasing t.

In particular, we need the value of t such that

$$\mathbf{r}(t) = \left(3\cos t, 2\sin t, -\frac{t^2}{\pi^2}\right) = \left(-\frac{3}{\sqrt{2}}, \sqrt{2}, -\frac{25}{16}\right).$$

From the z-component, we find that the possible values of t are $\pm\frac{5\pi}{4}$. The one that satisfies $2\sin t = \sqrt{2}$ is $t = -\frac{5\pi}{4}$, which also satisfies $3\cos t = -\frac{3}{\sqrt{2}}$.

At $t = -\frac{5\pi}{4}$, a tangent vector in the correct direction is

$$\mathbf{T} = \mathbf{r}'(-5\pi/4) = \left(-\frac{3}{\sqrt{2}}, -\sqrt{2}, \frac{5}{2\pi}\right).$$

(d) We are given the curve $\mathbf{r}(t) = (3t^3 - 2t)\widehat{\mathbf{i}} + e^{2t}\widehat{\mathbf{j}} + (t^2+1)^{3/2}\widehat{\mathbf{k}}$. A tangent vector pointing in the direction of decreasing t is

$$\mathbf{T} = -\mathbf{r}'(t) = -(9t^2 - 2)\widehat{\mathbf{i}} - 2e^{2t}\widehat{\mathbf{j}} - 3t(t^2+1)^{1/2}\widehat{\mathbf{k}}.$$

In particular, at $t = 0$, this vector is

$$\mathbf{T} = (2, -2, 0).$$

The corresponding unit vector is

$$\widehat{\mathbf{T}} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right).$$

16. (c) Consider the curve

$$\mathbf{r}(t) = \left(t\cos t, t\sin t, \frac{2\sqrt{2}}{3}t^{3/2}\right)$$

over the interval $0 \leq t \leq 2$. A tangent vector is

$$\mathbf{r}'(t) = \left(\cos t - t\sin t, \sin t + t\cos t, \sqrt{2}t^{1/2}\right).$$

This vector satisfies

$$\begin{aligned}|\mathbf{r}'(t)|^2 =&(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 2t\\ =&\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + 2t\\ =&1 + t^2 + 2t\\ =&(1+t)^2.\end{aligned}$$

Therefore

$$|\mathbf{r}'(t)| = 1 + t$$

for $0 \leq t \leq 2$. The length of the curve is then

$$\begin{aligned}L &= \int_0^2 |\mathbf{r}'(t)|\,dt\\ &= \int_0^2 (1+t)\,dt\\ &= t + \frac{1}{2}t^2\Big|_{t=0}^2\\ &= 4.\end{aligned}$$