MATH 2132 Problem Workshop 2

1. Determine whether the sequence of constants converge diverge. Justify your answer. Find the sum of any convergent series

   (a) $\sum_{n=1}^{\infty} \frac{n^2+3n+1}{3n^2-4}$

   Solution:

   $$\lim_{n\ to\infty} \frac{n^2+3n+1}{3n^2-4} = \lim_{n\ to\infty} \frac{1+3/n+1/n^2}{3-4/n^2} = \frac{1}{3} \neq 0.$$

   Since the terms do not converge to 0, the series diverges.

   (b) $\sum_{n=2}^{\infty} \left(-\frac{7}{3}\right)^{n+1}$

   Solution:
   This is a geometric series with common ratio $r = -\frac{7}{3}$. Since $|r| \geq 1$, the series diverges.

   (c) $\sum_{n=2}^{\infty} \frac{3^{n+3}}{4^{2n-5}}$

   Solution: Changing the series to start at 0 leads to $m = n-2$, $n = m+2$ so the series becomes

   $$\sum_{m=0}^{\infty} \frac{3^{m+5}}{4^{2m-1}} = \sum_{m=0}^{\infty} \frac{3^5}{4^{-1}} \left(\frac{3}{16}\right)^m.$$

   First note that this is geometric series with common ratio $r = \frac{3}{16}$. Since $|r| < 1$ the series converges
   Using $\sum_{n=1}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$. We get the sum is

   $$\frac{3^5/4^{-1}}{1-(3/16)} = \frac{243 \cdot 4}{13/16} = \frac{15552}{13}.$$

   (d) $\sum_{n=3}^{\infty} \left(1+\frac{1}{n}\right)^n$

Solution:

$$\lim_{n\ to\infty} \left(1+\frac{1}{n}\right)^n = e \neq 0.$$

Since the terms do not converge to 0, the series diverges.

(e) $\sum\limits_{n=1}^{\infty} \frac{2^n+3^n}{4^{3n}}$

Solution:
$\sum\limits_{n=1}^{\infty} \frac{2^n+3^n}{4^{3n}} = \sum\limits_{n=1}^{\infty} \left(\left(\frac{1}{32}\right)^n + \left(\frac{3}{64}\right)^n\right).$

These are both geometric series with common ratios $\frac{1}{32}$ and $\frac{3}{64}$ which are both less than 1 in absolute value. Hence they both converge and thus the series can be written as

$$\begin{aligned}
\sum_{n=1}^{\infty}\left(\frac{1}{32}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{64}\right)^n &= \sum_{n=1}^{\infty}\frac{1}{32}\left(\frac{1}{32}\right)^{n-1} + \sum_{n=1}^{\infty}\frac{3}{64}\left(\frac{3}{64}\right)^{n-1} \\
&= \frac{1/32}{1-1/32} + \frac{3/64}{1-3/64} \\
&= \frac{1/32}{31/32} + \frac{3/64}{61/64} \\
&= \frac{1}{31} + \frac{3}{61} \\
&= \frac{154}{1891}.
\end{aligned}$$

(f) $\sum\limits_{n=1}^{\infty} (-e)^{-n}$

Solution: $\sum\limits_{n=1}^{\infty} (-e)^{-n} = \sum\limits_{n=1}^{\infty} \left(-\frac{1}{e}\right)^n$ so this is a geometric series with common ratio less than 1 in absolute value. Hence the series converges.

$$\sum_{n=1}^{\infty}\Big(-\frac{1}{e})^n = \sum_{n=1}^{\infty}-\frac{1}{e}\Big(-\frac{1}{e})^{n-1}$$
$$= \frac{-1/e}{1+1/e}$$
$$= -\frac{1}{e+1}.$$

(g) $\sum\limits_{n=100}^{\infty}\frac{1}{n}\tan^{-1} n$

Solution:
We know the harmonic series $\sum\limits_{n=100}^{\infty}\frac{1}{n}$ diverges to ∞. Since the individual terms of $\frac{1}{n}\tan^{-1} n$ are larger than $1/n$, the series just gets larger. Hence the series $\sum\limits_{n=100}^{\infty}\frac{1}{n}\tan^{-1} n$ diverges to ∞.

(h) $\sum\limits_{n=1}^{\infty}\frac{1}{n^2+n}$ Hint: Find the sequence of partial sums

Solution: We can try to take the sequence of partial sums as in the hint:

$$S_1 = \frac{1}{2}$$
$$S_2 = \frac{1}{2}+\frac{1}{6} = \frac{2}{3}$$
$$S_3 = \frac{2}{3}+\frac{1}{12} = \frac{3}{4}$$
$$S_4 = \frac{3}{4}+\frac{1}{20} = \frac{4}{5}.$$

We seem to have enough evidence to make the guess that $S_n = 1-\frac{1}{n+1}$ for $n \geq 1$ which we can prove by the principle of mathematical induction.
Let $P(n)$ be the statement $S_n = 1-\frac{1}{n+1}$
Let $n = 1$. The left hand side is $S_1 = \frac{1}{2}$. The right hand side is $1-\frac{1}{2} = \frac{1}{2}$.
Therefore $P(1)$ is true.
Suppose $P(k)$ is true for some integer $k \geq 1$. That is $S_k = 1-\frac{1}{k+1}$. We want to show $P(k+1)$ is true. That is $S_{k+1} = 1-\frac{1}{k+2}$.

$$
\begin{aligned}
LHS &= S_{k+1} \\
&= S_k + c_{k+1} \\
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)^2 + (k+1)} \\
&= 1 - \frac{k+2}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{k+1}{(k+1)(k+2)} \\
&= 1 - 1 - \frac{1}{k+2} = RHS
\end{aligned}
$$

Thus $P(k+1)$ is true.
Hence by the principle of mathematical induction, $S_n = 1 - \frac{1}{n+1}$ for all $n \geq 1$.
Since $\lim_{n\to\infty} S_n = 1$ the series converges to 1.
Note that there is an alternative to the guess and check way we created the formula for S_n. We could have used partial fractions to show that the terms

$$\frac{1}{n(n+1)} = \frac{1}{n} = \frac{1}{n+1}$$

Then

$$S_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

2. (a) Find the first five Taylor polynomials for the function $f(x) = \cos 2x$ about $x = 0$.

Solution:

$$f(x) = \cos 2x,\ f'(x) = -2\sin 2x,\ f''(x) = -4\cos 2x,\ f'''(x) = 8\sin 2x,\ f''''(x) = 16\cos 2x.$$

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -4,\ f'''(0) = 0,\ f''''(0) = 16.$$

Therefore

$$
\begin{aligned}
P_0(x) &= 1 \\
P_1(x) &= P_0(x) + \frac{0}{1!}x = 1 \\
P_2(x) &= P_1(x) + \frac{-4}{2!}x^2 = 1 - 2x^2 \\
P_3(x) &= P_2(x) + \frac{0}{3!}x^3 = 1 - 2x^2 \\
P_4(x) &= P_3(x) + \frac{16}{4!}x^4 = 1 - 2x^2 + \frac{2}{3}x^4
\end{aligned}
$$

(b) Show that the Maclaurin series for $\cos 2x$ converges to $\cos 2x$ for all x using the remainder formula.

Solution:
We must show that for all x, the remainder term goes to 0.
From the above we can see that $f^{(n)}(x)$ is $\pm 2^n$ multiplied by either $\cos 2x$ or $\sin 2x$ which is at most 1 in absolute value.
Recall that the remainder term for a Taylor or Maclaurin series is

$$R_n = \frac{f^{(n+1)}(z_n)x^{n+1}}{(n+1)!}$$

where x_n is some number between 0 and x. Since the term $f^{(n+1)}(z_n)$ is at most 2^{n+1} in absolute value, the remainder is at most

$$\frac{(2\,|x|)^{n+1}}{(n+1)!}$$

in absolute value which goes to 0 for all x.

3. Find the Taylor series about $x = 1$ for the function $f(x) = \dfrac{1}{(x-2)^2}$. Express your answer in sigma notation, simplified as much as possible.

Solution: We need to find a pattern for the derivatives.

$$
\begin{aligned}
f(x) &= (x-2)^{-2} \\
f'(x) &= -2(x-2)^{-3} \\
f''(x) &= -2(-3)(x-2)^{-4} \\
f'''(x) &= -2(-3)(-4)(x-2)^{-5} \\
f''''(x) &= -2(-3)(-4)(-5)(x-2)^{-6}
\end{aligned}
$$

Hence we can see that $f^{(n)}(x) = (-1)^n(n+1)!(x-2)^{-(n+2)}$.

Thus

$$f^{(n)}(1) = (-1)^n(n+1)!(1-2)^{-(n+2)} = (n+1)!.$$

Hence the series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n = \sum_{n=0}^{\infty} \frac{(n+1)!}{n!}(x-1)^n = \sum_{n=0}^{\infty}(n+1)(x-1)^n.$$

4. Find the Maclaurin series for the function $f(x) = \dfrac{1}{(8+3x)^{1/3}}$. Express your answer in sigma notation, simplified as much as possible.

Solution: We need to find a pattern for the derivatives.

$$
\begin{aligned}
f(x) &= (8+3x)^{-1/3} \\
f'(x) &= 3(-1/3)(8+3x)^{-4/3} \\
f''(x) &= 3(-1/3)(3)(-4/3)(8+3x)^{-7/3} \\
f'''(x) &= 3(-1/3)(3)(-4/3)(3)(-7/3)(8+3x)^{-10/3} \\
f''''(x) &= 3(-1/3)(3)(-4/3)(3)(-7/3)(3)(-10/3)(8+3x)^{-13/3}
\end{aligned}
$$

Hence we can see that

$$
\begin{aligned}
f^{(n)}(x) &= 3^n(-1)^n\frac{(1\cdot 4\cdot 7\cdots(3n-2))}{3^n}(8+3x)^{-(3n+1)/3} \\
&= (-1)^n(1\cdot 4\cdot 7\cdots(3n-2))(8+3x)^{-(3n+1)/3}.
\end{aligned}
$$

Thus

$$f^{(n)}(0) = (-1)^n(1\cdot 4\cdot 7\cdots(3n-2))(8)^{-(3n+1)/3} = (-1)^n(1\cdot 4\cdot 7\cdots(3n-2))2^{-(3n+1)}.$$

Hence the series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n (1 \cdot 4 \cdot 7 \cdots (3n-2))}{n! 2^{3n+1}} x^n.$$

5. Find the open interval of convergence for the power series.

(a) $\sum_{n=3}^{\infty} \frac{2^n}{n3^{n+1}} x^n$

Solution:

$$\begin{aligned} R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\ &= \lim_{n\to\infty} \left| \frac{\frac{2^n}{n3^{n+1}}}{\frac{2^{n+1}}{(n+1)3^{n+2}}} \right| \\ &= \lim_{n\to\infty} \frac{3(n+1)}{2n} \\ &= \frac{3}{2}. \end{aligned}$$

Hence the open interval of convergence is $(-3/2, 3/2)$

(b) $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{n!} x^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{(-1)^n 3^n}{n!}}{\frac{(-1)^{n+1}3^{n+1}}{(n+1)!}}\right| \\
&= \lim_{n\to\infty}\frac{(n+1)}{3} \\
&= \infty.
\end{aligned}
$$

Hence the open interval of convergence is $(-\infty, \infty)$

(c) $\sum_{n=0}^{\infty} \frac{(n+5)^4}{3^n} x^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{(n+5)^4}{3^n}}{\frac{(n+6)^4}{3^{n+1}}}\right| \\
&= \lim_{n\to\infty}\frac{3(n+5)^4}{(n+5)^6} \\
&= 3.
\end{aligned}
$$

Hence the open interval of convergence is $(-3, 3)$

(d) $\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 5\cdot 8\cdots(3n+2)}(2x)^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \lim_{n\to\infty}\left| 2^n \frac{\dfrac{1\cdot 3\cdot 5\cdots(2n+1)}{2\cdot 5\cdot 8\cdots(3n+2)}}{2^{n+1}\dfrac{1\cdot 3\cdot 5\cdots(2n+1)(2n+3)}{2\cdot 5\cdot 8\cdots(3n+2)(3n+5)}}\right| \\
&= \lim_{n\to\infty}\frac{3n+5}{2(2n+3)} \\
&= \frac{3}{4}.
\end{aligned}
$$

Hence the open interval of convergence is $(-3/4,\, 3/4)$

6. Find the sum of the series $\sum_{n=1}^{\infty}\frac{(-1)^n}{2^{2n}}(x-1)^n$. What is its interval of convergence.

Solution: The series can be rearranged to be

$\sum_{n=1}^{\infty}\frac{-(x-1)}{4}\left(\frac{-1}{4}(x-1)\right)^{n-1}$ which is geometric with common ratio $\frac{-1}{4}(x-1)$

Hence the sum is

$$
\frac{-(x-1)/4}{1+(x-1)/4} = \frac{1-x}{3+x}
$$

when $\left|\frac{-1}{4}(x-1)\right| < 1 \Rightarrow -3 < x < 5.$