Math 1210: Techniques of Classical & Linear Algebra

Assignment 1 – Winter 2011 (A01 and A02)/ Solutions

1. Use mathematical induction to verify that, for every positive integer n, the quantity $n^3 + 6n^2 + 2n$ is divisible by 3.

   Solution: Let $P_n: \ f(n) = n^3 + 6n^2 + 2n$ is divisible by 3, $n \geq 1$.
   1) Let $n = 1$, $f(1) = 1 + 6 + 2 = 9 = 3(3)$. So, P_1 is true.
   2) Assume P_k , i.e. $3 \,|\, k^3 + 6k^2 + 2\,k = 3\,g(k)$. Where $g(k)$ is an integer.
   Prove P_{k+1} i.e. $3 \,|\, (k+1)^3 + 6(k+1)^2 + 2(k+1) = f(k+1)$.

   Proof.
   $$\begin{aligned} f(k+1) &= k^3 + 3k^2 + 3k + 1 + 6k^2 + 12k + 6 + 2k + 2 \\ &= k^3 + 6k^2 + 2k + 3k^2 + 15k + 9 \\ &= 3\,g(k) + 3(k^2 + 5k + 3) = 3[g(k) + k^2 + 5k + 5] \end{aligned}$$
   we see that, $3 \,|\, f(k+1)$ and $g(k+1) = g(k) + k^2 + 5k + 5$.
   3) By PMI, $3 \,|\, f(n)$, for every $n \geq 1$.

2. (a) For a general positive integer n, write out the sum $\sum_{i=1}^{2n}(i+1)$ explicitly in the form "(first term) + (second term) + (third term) $+ \cdots +$ (last term)" and describe precisely in words the "meaning" of this sum.
   Hint: You may find it useful to consider the special cases $n = 1, 2, 3, 4, 5$ before considering the case of the general positive integer n.

   Solution.
   $$2 + 3 + 4 + 5 + \cdots + 2n + 1$$
   is the sum of the integers from 2 to the odd number $(2n+1)$, which is always an even number of numbers.

   (b) Use mathematical induction to prove that for every positive integer n,
   $$\sum_{i=1}^{2n}(i+1) = n(2n+3).$$
   Solution: Let
   $$P_n: \ \sum_{i=1}^{2n}(i+1) = n(2n+3).$$
   1) P_1 is $2 + 3 = 1\,(2+3)$, which is true.
   2) <u>Induction step</u>: Assume $P_k: \ \sum_{i=1}^{2k}(i+1) = k(2k+3)$.
   To prove $P_{k+1}: \ \sum_{i=1}^{2k+2}(i+1) = (k+1)(2k+5)$.

Proof: RS of P_{k+1} is $2k^2+7k+5$.

LS of P_{k+1} is $\sum_{i=1}^{2k}(i+1)+(2k+1+1)+(2k+2+1)$

$= k(2k+3)+2k+2+2k+3$ (by the induction assumption)

$= 2k^2+7k+5 =$ RS.

3) By PMI, P_n is true for all $n \geq 1$.

(c) Use the result that $\sum_{j=1}^{n} j = \dfrac{n(n+1)}{2}$ for every positive integer n (equation 1.1 in section 1.1.1 of the notes) to verify the result of part (b).

Solution:

$$\sum_{i=1}^{2n}(i+1) = \sum_{i=1}^{2n} i + 2n(1) = \frac{2n(2n+1)}{2} + 2n = 2n^2+n+2n = n(2n+3)$$

as required.

3. (a) Rewrite in sigma notation, for general positive integer n, the sum

$$\frac{1}{1(4)} + \frac{1}{4(7)} + \cdots + \frac{1}{(3n-2)(3n+1)}.$$

Solution:

$$\sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)}$$

(b) Conjecture a simple formula for the sum appearing in part (a).

Solution: Let $S(n)$ be the sum of n terms.

$S(1) = \dfrac{1}{4}, \quad S(2) = \dfrac{1}{4} + \dfrac{1}{28} = \dfrac{7+1}{28} = \dfrac{8}{28} = \dfrac{2}{7}.$

$S(3) = \dfrac{1}{4} + \dfrac{1}{28} + \dfrac{1}{7(10)} = \dfrac{2}{7} + \dfrac{1}{70} = \dfrac{20+1}{70} = \dfrac{21}{70} = \dfrac{3}{10}.$

The obvious conjecture is

$$P_n: \quad \sum_{i=1}^{n} \frac{1}{(3i-2)(3i+1)} = \frac{n}{3n+1}.$$

(c) Use mathematical induction to prove true the conjecture you made in part (b).

Solution: We know P_1, P_2, P_3 (see part (b) above)

Induction step: Assume $P_k: \quad \displaystyle\sum_{i=1}^{k} \frac{1}{(3i-2)(3i+1)} = \frac{k}{3k+1},$

to prove $P_{k+1}: \quad \displaystyle\sum_{i=1}^{k+1} \frac{1}{(3i-2)(3i+1)} = \frac{k+1}{3k+4}.$

LS of P_{k+1} is $\displaystyle\sum_{i=1}^{k} \frac{1}{(3i-2)(3i+1)} + \frac{1}{(3k+1)(3k+4)}$ and by induction assumption

$$= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} = \frac{k\,(3k+4)+1}{(3k+1)(3k+4)} = \frac{3k^2+4k+1}{(3k+1)(3k+4)}$$

$$= \frac{(k+1)(3k+1)}{(3k+1)(3k+4)} = \frac{k+1}{3k+4} = \text{RS}.$$

By PMI, P_n is true for all $n \geq 1$.

4. Use mathematical induction to prove that $x-y$ is always a factor of $x^n - y^n$ for every positive integer n.
   Solution: Let P_n be $(x-y) \mid x^n - y^n$ for any positive integer n.
   For $n=1$, $(x-y)$ divides itself, P_1 is true.

   $x^2 - y^2 = (x+y)(x-y)$ shows P_2.
   Assume P_k : $(x-y) \mid x^k - y^k$, to show that P_{k+1} : $(x-y) \mid x^{k+1} - y^{k+1}$.

   $x^{k+1} - y^{k+1} = x^k\, x - y^k\, y - xy^k + xy^k = x\,(x^k - y^k) + y^k(x-y)$.
   The first term is divisible by $(x-y)$ by our assumption, and clearly the second term is divisible by $(x-y)$. So, P_{k+1} is true.
   By PMI, P_n is true for all $n \geq 1$.

5. Consider the sequence of real numbers $x_1, x_2, x_3, \ldots$ defined by the relations $x_1 = 1$ and $x_{n+1} = \sqrt{1+2x_n}$ for $n \geq 1$. Use mathematical induction to show that $x_{n+1} > x_n$ for all $n \geq 1$.
   Solution: $x_1 = 1,\ x_2 = \sqrt{1+2} = \sqrt{3},\ x_3 = \sqrt{1+2\sqrt{3}}$.

   Consider P_n : $x_{n+1} > x_n$, $n \geq 1$.
   P_1 is $x_2 > x_1$, true since $\sqrt{3} > 1$.
   Assume P_k : $x_{k+1} > x_k$, for some $k \geq 1$.
   Prove P_{k+1} : $x_{k+2} > x_{k+1}$.

   Proof: Since $x_{k+1} > x_k$, then

$$\begin{aligned} 2\,x_{k+1} &> 2\,x_k\,, \\ 1 + 2\,x_{k+1} &> 1 + 2\,x_k\,, \\ \sqrt{1+2\,x_{k+1}} &> \sqrt{1+2\,x_k}, \qquad \text{that is,} \\ x_{k+2} &> x_{k+1}. \end{aligned}$$

   Since all the numbers involved are positive that guarantees the square root can be taken.
   By PMI, $x_{n+1} > x_n$ for all $n \geq 1$.