UNIVERSITY OF MANITOBA

DATE: March 7, 2016 | TERM TEST 2

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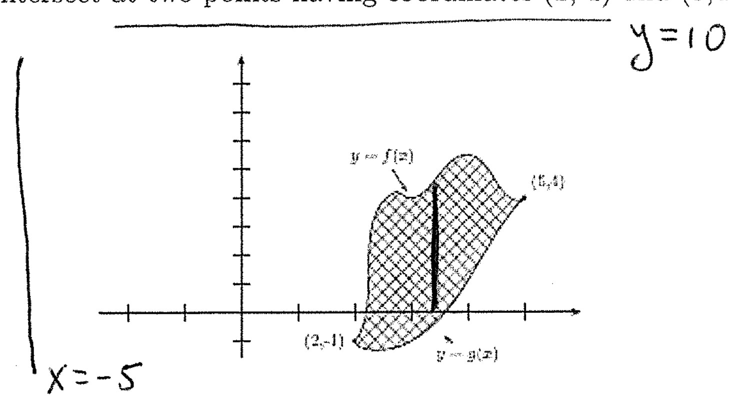
EXAMINATION: Applied Calculus 2 | TIME: 60 minutes

COURSE: MATH 1710 | EXAMINER: Harland, Kopotun

Use the following region for questions 1 and 2

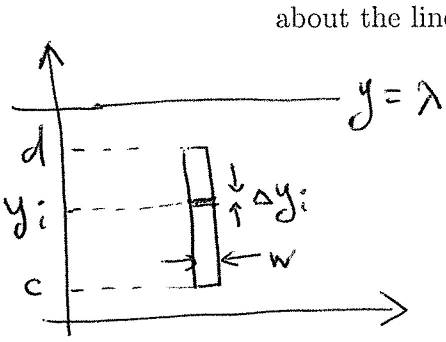
A thin plate with constant mass per unit area ρ occupies the region R shown in the figure below. Note that R lies between the curves $y = f(x)$ and $y = g(x)$, which intersect at two points having coordinates (2,-1) and (5,4).



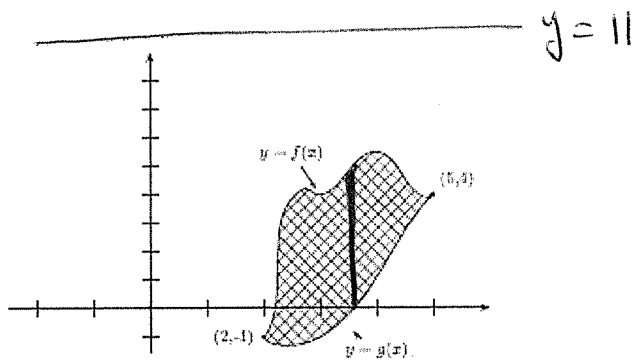
[5] 1. (a) Set up an integral for the first moment of this plate about the line $x = -5$.

$$M_i = \text{mass} \times \text{dist} = \rho \times \text{area} \times (x_i - (-5)) =$$
$$= \rho \times [f(x_i) - g(x_i)] \times \Delta x_i \times (x_i + 5)$$
$$M = \lim_{\|P\| \to 0} \sum_{i=1}^{n} M_i = \int_2^5 \rho (f(x) - g(x))(x+5)\, dx$$

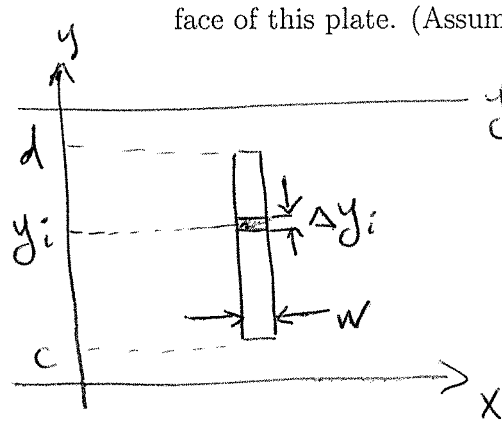
[5] (b) Set up an integral for the moment of inertia (second moment) of this plate about the line $y = 10$.



$$I_{\text{rect. about } y=\lambda} = \int_c^d \rho w (\lambda - y)^2\, dy =$$
$$= \rho w \left[-\frac{(\lambda - y)^3}{3} \Big|_c^d \right] = \frac{1}{3} \rho w \left[(\lambda - c)^3 - (\lambda - d)^3 \right]$$
$$I_i = \frac{1}{3} \rho \Delta x_i \left[(10 - g(x_i))^3 - (10 - f(x_i))^3 \right]$$
$$I = \lim_{\|P\| \to 0} \sum_{i=1}^{n} I_i = \int_2^5 \frac{1}{3} \rho \left([10 - g(x)]^3 - [10 - f(x)]^3 \right) dx$$



[6] 2. Suppose that this plate is immersed vertically into water so that the surface of water is at $y = 11$. Set up an integral for the total force exerted by water on one face of this plate. (Assume that x and y coordinates are measured in meters.)



$$F_i = \underbrace{\rho g(\lambda - y_i)}_{\text{pressure}} \times \underbrace{(w \Delta y_i)}_{\text{area}}$$

$$F = \int_c^d \rho g(\lambda - y) w \, dy =$$

$$= \rho g w \left[-\frac{(\lambda - y)^2}{2} \Big|_c^d \right] =$$

$$= \frac{1}{2} \rho g w \left[(\lambda - c)^2 - (\lambda - d)^2 \right]$$

$$\therefore \; F_{\text{vert. strip at } x_i} = \frac{1}{2} \rho g \Delta x_i \left([11 - g(x_i)]^2 - [11 - f(x_i)]^2 \right)$$

$$F_{\text{total}} = \lim_{\|P\| \to 0} \sum_{i=1}^{n} F_{\text{vert. strip at } x_i} =$$

$$= \int_2^5 \frac{1}{2} \rho g \left([11 - g(x)]^2 - [11 - f(x)]^2 \right) dx \text{ with}$$

$\rho = 1000 \, \text{kg/m}^3$ and $g \approx 9.81 \, \text{m/s}^2$.

3. Simplify (if possible) either by constructing an appropriate diagram and labelling all relevant information, or by using trigonometric identities.

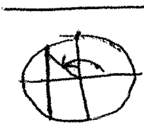
[3] (a) $\operatorname{Cos}^{-1}\left(\sin\left(-\frac{\pi}{6}\right)\right)$

Method 1: $\operatorname{Sin}^{-1}x + \operatorname{Cos}^{-1}x = \frac{\pi}{2}$, $-1 \le x \le 1$

$\therefore \operatorname{Cos}^{-1}(\sin(-\frac{\pi}{6})) = \frac{\pi}{2} - \operatorname{Sin}^{-1}(\sin(-\frac{\pi}{6})) = \frac{\pi}{2} - (-\frac{\pi}{6}) = \frac{2}{3}\pi$

(Note: $\operatorname{Sin}^{-1}(\sin x) = x$ if $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$).

Method 2: $\sin(-\frac{\pi}{6}) = -\frac{1}{2}$ and $\cos^{-1}(-\frac{1}{2}) = \alpha \iff \cos\alpha = -\frac{1}{2}$ and $0 \le \alpha \le \pi$

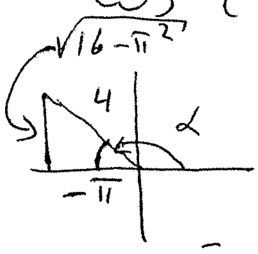


$\therefore \alpha = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$ $\quad\therefore$ Answer: $\frac{2\pi}{3}$.

[3] (b) $\cot\left(\operatorname{Cos}^{-1}\left(-\frac{\pi}{4}\right)\right) = \cot\alpha$, where

Method 1

$\operatorname{Cos}^{-1}(-\frac{\pi}{4}) = \alpha \iff \cos\alpha = -\frac{\pi}{4}$ and $0 \le \alpha \le \pi$.



$\therefore \cot\alpha = -\frac{\pi}{\sqrt{16-\pi^2}}$

Method 2: $\cot(\cos^{-1}(-\frac{\pi}{4})) = \frac{\cos(\cos^{-1}(-\frac{\pi}{4}))}{\sin(\cos^{-1}(-\frac{\pi}{4}))} =$

$= \frac{-\frac{\pi}{4}}{\sqrt{1-(-\frac{\pi}{4})^2}} = -\frac{\pi}{\sqrt{16-\pi^2}}$ (Note: $\cos(\cos^{-1}x) = x$, $-1 \le x \le 1$

$\sin(\cos^{-1}(-\frac{\pi}{4})) = \sqrt{1-\cos^2(\cos^{-1}(-\frac{\pi}{4}))}$

since $\sin\alpha \ge 0$ if $0 \le \alpha \le \pi$).

$\therefore$ Answer: $-\frac{\pi}{\sqrt{16-\pi^2}}$.

[3] (c) $\tan\left(\operatorname{Sin}^{-1}\left(\frac{\pi}{2}\right)\right)$

$\operatorname{Sin}^{-1}(\frac{\pi}{2}) = \alpha \iff \sin\alpha = \frac{\pi}{2}$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$

This is impossible since $|\sin x| \le 1$ for all $x \in \mathbb{R}$.

Alternatively, notice that $\operatorname{Sin}^{-1}(\frac{\pi}{2})$ is not defined since $\frac{\pi}{2} \notin \operatorname{Dom}(\operatorname{Sin}^{-1}x) = [-1, 1]$.

4. For the function $f(x) = \ln(\mathrm{Sin}^{-1}(x^2))$.

[3] (a) Determine the domain of $f(x)$.

$\mathrm{Sin}^{-1} x^2 = \alpha \iff \sin\alpha = x^2$ and $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, i.e., $|x^2| \le 1$

(or use the fact that the domain of $\mathrm{Sin}^{-1} x$ is $[-1, 1]$).

$\therefore$ x has to be such that $-1 \le x \le 1$ but we need more!

For f to be defined, we also need $\mathrm{Sin}^{-1}(x^2) > 0$, i.e.,

x should be s.t. $\mathrm{Sin}^{-1} x^2 = \alpha > 0 \iff \sin\alpha = x^2$ and $0 < \alpha \le \frac{\pi}{2}$,

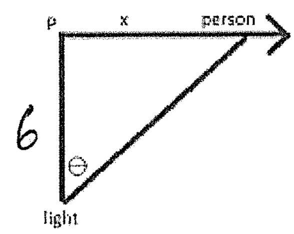
i.e., $0 < x^2 \le 1$. $\therefore$ x should be s.t. $|x| \le 1$ and $x \ne 0$, i.e.,

Domain of f is $[-1, 0) \cup (0, 1]$.

[4] (b) Compute the derivative of $f(x)$.

$$f'(x) = \frac{1}{\mathrm{Sin}^{-1}(x^2)} \cdot \left[\mathrm{Sin}^{-1} x^2\right]' = \frac{1}{\mathrm{Sin}^{-1}(x^2)} \cdot \frac{1}{\sqrt{1-(x^2)^2}} \cdot 2x$$

5. A person walks along a straight path at a speed of 2 m/s. A searchlight is located on the ground 6 metres from the path and is kept focused on the person. See the picture



[2] (a) Express θ as a function of x.

$\tan\theta = \frac{x}{6} \iff \theta = \tan^{-1}\frac{x}{6}$ (clearly, θ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. In fact, $0 \le \theta < \frac{\pi}{2}$).

[6] (b) Let P be the point on the path closest to the light. (see picture) How quickly is the search light rotating when the person is 5 metres from point P, walking away? Express your answer in radians per second.

We need to find $\frac{d\theta}{dt}(t_0)$ at the time t_0 when $x(t_0) = 5$m.

$$\frac{d\theta}{dt} = \frac{d}{dt}\left[\tan^{-1}\frac{x}{6}\right] = \frac{1}{1+\left(\frac{x}{6}\right)^2}\cdot\frac{1}{6}\cdot\frac{dx}{dt} =$$

$$= \frac{6}{x^2+36}\cdot\frac{dx}{dt}$$

$$\frac{d\theta}{dt}(t_0) = \frac{6}{[x(t_0)]^2+36}\cdot\frac{dx}{dt}(t_0) = \frac{6}{25+36}\cdot 2 = \frac{12}{61}\ \text{rad/s}$$