# MATH 1210 Problem Workshop 10 Solutions

1. (a) We have multiple options for the first determinant and none of them are nice. At the point the worksheet was handed out, the only option was row or column expansion. Other options include row/column reduction.

   Expanding along the first row

$$\begin{vmatrix} 2 & 1 & -3 & 0 \\ 4 & 2 & 1 & 5 \\ -3 & 3 & -2 & 2 \\ 4 & 5 & 2 & -4 \end{vmatrix} = 2\begin{vmatrix} 2 & 1 & 5 \\ 3 & -2 & 2 \\ 5 & 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 & 5 \\ -3 & -2 & 2 \\ 4 & 2 & -4 \end{vmatrix} - 3\begin{vmatrix} 4 & 2 & 5 \\ -3 & 3 & 2 \\ 4 & 5 & -4 \end{vmatrix} - 0$$

   The first 3 by 3 determinant is

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 5 \\ 3 & -2 & 2 \\ 5 & 2 & -4 \end{vmatrix} &= 2\begin{vmatrix} -2 & 2 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} + 5\begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} \\ &= 2(8-4) - 1(-12-10) + 5(6-(-10)) \\ &= 2(4) - 1(-22) + 5(16) \\ &= 8 + 22 + 80 \\ &= 110. \end{aligned}$$

   The second 3 by 3 determinant is

$$\begin{aligned} \begin{vmatrix} 4 & 1 & 5 \\ -3 & -2 & 2 \\ 4 & 2 & -4 \end{vmatrix} &= 4\begin{vmatrix} -2 & 2 \\ 2 & -4 \end{vmatrix} - 1\begin{vmatrix} -3 & 2 \\ 4 & -4 \end{vmatrix} + 5\begin{vmatrix} -3 & -2 \\ 4 & 2 \end{vmatrix} \\ &= 4(8-4) - 1(12-8) + 5(-6-(-8)) \\ &= 4(4) - 1(4) + 5(2) \\ &= 16 - 4 + 10 \\ &= 22 \end{aligned}$$

   The third 3 by 3 determinant is

$$\begin{aligned} \begin{vmatrix} 4 & 2 & 5 \\ -3 & 3 & 2 \\ 4 & 5 & -4 \end{vmatrix} &= 4\begin{vmatrix} 3 & 2 \\ 5 & -4 \end{vmatrix} - 2\begin{vmatrix} -3 & 2 \\ 4 & -4 \end{vmatrix} + 5\begin{vmatrix} -3 & 3 \\ 4 & 5 \end{vmatrix} \\ &= 4(-12-10) - 2(12-8) + 5(-15-12) \\ &= 4(-22) - 2(4) + 5(-27) \\ &= -88 - 8 - 135 \\ &= -231 \end{aligned}$$

Hence the 4 by 4 determinant is

$$2(110) - 1(22) - 3(-231) = 220 - 22 + 693 = 891.$$

(b) This matrix is very close to the Vandermonde determinant which comes up in solving for best fit polynomials. It's a lot cleaner to do this using row operations, however expanding along row 1 yields

$$\begin{aligned}
\begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} &= x\begin{vmatrix} y^2 & y^3 \\ z^2 & z^3 \end{vmatrix} - x^2\begin{vmatrix} y & y^3 \\ z & z^3 \end{vmatrix} + x^3\begin{vmatrix} y & y^2 \\ z & z^2 \end{vmatrix} \\
&= x(y^2z^3 - y^3z^2) - x^2(yz^3 - y^3z) + x^3(yz^2 - y^2z) \\
&= xy^2z^2(z-y) - x^2yz(z^2 - y^2) + x^3yz(z-y) \\
&= xyz\big(yz(z-y) - x(z^2-y^2) + x^2(z-y)\big) \\
&= xyz\big(yz(z-y) - x(z-y)(z+y) + x^2(z-y)\big) \\
&= xyz(z-y)\big(yz - x(y+z) + x^2\big) \\
&= xyz(z-y)(z-x)(y-x)
\end{aligned}$$

2. (a) There are many ways to do the row reduction to solve this. For example we can do

$\begin{vmatrix} 4 & 6 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ -3 & -2 & -1 & 1 \\ 2 & 5 & 7 & 2 \end{vmatrix}$ Performing $R_2 = \frac{1}{2}R_2$ yields

$2\begin{vmatrix} 4 & 6 & 2 & 2 \\ 1 & 2 & 3 & 4 \\ -3 & -2 & -1 & 1 \\ 2 & 5 & 7 & 2 \end{vmatrix}$ Performing $R_1 \leftrightarrow R_2$ yields

$-2\begin{vmatrix} 1 & 2 & 3 & 4 \\ 4 & 6 & 2 & 2 \\ -3 & -2 & -1 & 1 \\ 2 & 5 & 7 & 2 \end{vmatrix}$ Performing $R_2 \to R_2 - 4R_1$, $R_3 \to R_3 + 3R_1$ and $R_4 \to R_4 - 2R_1$ yields

$-2\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & -2 & -10 & -14 \\ 0 & 4 & 8 & 13 \\ 0 & 1 & 1 & -6 \end{vmatrix}$ Performing $R_2 \leftrightarrow R_4$ yields

$-(-2)\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -6 \\ 0 & 4 & 8 & 13 \\ 0 & -2 & -10 & -14 \end{vmatrix}$ Performing $R_3 \to R_3 - 4R_2$ and $R_4 \to R_4 + 2R_2$ yields

$$2\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 4 & 37 \\ 0 & 0 & -8 & -26 \end{vmatrix}$$ Performing $R_4 \to R_4 + 2R_3$ yields

$$2\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 4 & 37 \\ 0 & 0 & 0 & 48 \end{vmatrix} = 2(1)(1)(4)(48) = 384.$$

(b) $$\begin{vmatrix} a+b & a & a & a & a \\ a & a+b & a & a & a \\ a & a & a+b & a & a \\ a & a & a & a+b & a \\ a & a & a & a & a+b \end{vmatrix}$$ Performing the last four rows subtracting row 1 yields

$$\begin{vmatrix} a+b & a & a & a & a \\ -b & b & 0 & 0 & 0 \\ -b & 0 & b & 0 & 0 \\ -b & 0 & 0 & b & 0 \\ -b & 0 & 0 & 0 & b \end{vmatrix}$$ Factoring $b$ out of the last four rows yields

$$b^4\begin{vmatrix} a+b & a & a & a & a \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{vmatrix}$$ Performing row 1 minus $a$ times each of the bottom four rows yields

$$b^4\begin{vmatrix} 5a+b & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{vmatrix}$$ Since the matrix is now lower triangular, the determinant is $b^4(5a+b)(1)(1)(1)(1) = 5ab^4 + b^5$.

3. Let $A$ be the coefficient matrix

$$A = \begin{bmatrix} 1 & 2 & -3 & 6 \\ 2 & -1 & 0 & 1 \\ 0 & 5 & 1 & -2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Cramers rule says the system has a unique solution if $|A| \neq 0$. Since the system is homogeneous, the unique solution would be the trivial solution. Let's find $|A|$.

$\begin{vmatrix} 1 & 2 & -3 & 6 \\ 2 & -1 & 0 & 1 \\ 0 & 5 & 1 & -2 \\ 1 & 1 & 1 & 1 \end{vmatrix}$ Using $R_2 \to R_2 - 2R_1$ and $R_4 \to R_4 - R_1$ yields

$\begin{vmatrix} 1 & 2 & -3 & 6 \\ 0 & -5 & 6 & -11 \\ 0 & 5 & 1 & -2 \\ 0 & -1 & 4 & -5 \end{vmatrix}$ Using $R_4 \leftrightarrow R_2$ and then $R_2 \to -R_2$ yields

$(-1)(-1)\begin{vmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -4 & 5 \\ 0 & 5 & 1 & -2 \\ 0 & -5 & 6 & -11 \end{vmatrix}$ Using $R_3 \to R_3 - 5R_2$ and $R_4 \to R_4 + 5R_2$ yields

$\begin{vmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 21 & -27 \\ 0 & 0 & -14 & 14 \end{vmatrix}$ Using $R_3 \leftrightarrow R_4$ and then $R_3 \to \dfrac{1}{14}R_3$ yields

$14(-1)\begin{vmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 21 & -27 \end{vmatrix}$ Using $R_4 \to R_4 + 21R_3$ yields

$-14\begin{vmatrix} 1 & 2 & -3 & 6 \\ 0 & 1 & -4 & 5 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -6 \end{vmatrix} = -14(1)(1)(1)(-6) \neq 0.$

Hence $z = 0$ (and so is $x, y$ and $w$.)

4. Let $A$ be the coefficient matrix

$$A = \begin{bmatrix} 1 & 1 & -3 & 0 \\ 2 & 2 & 0 & -1 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{bmatrix}$$

Cramers rule says the system has a unique solution if $|A| \neq 0$. Let's find $|A|$. Being clever, we can turn it into triangular form.

$\begin{vmatrix} 1 & 1 & -3 & 0 \\ 2 & 2 & 0 & -1 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix}$ Using $R_2 \to R_2 - R_4$ yields

$$\begin{vmatrix} 1 & 1 & -3 & 0 \\ 2 & -1 & -2 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix} \text{ Using } R_2 \to R_2 + 2R_3 \text{ and } R_1 \to R_1 + 3R_3 \text{ yields}$$

$$\begin{vmatrix} 10 & -5 & 0 & 0 \\ 8 & -5 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix} \text{ Expanding along column 4 yields}$$

$$-1 \begin{vmatrix} 10 & -4 & 0 \\ 8 & -5 & 0 \\ 3 & -2 & 1 \end{vmatrix} \text{ Expanding along column 3 yields}$$

$$-1 \begin{vmatrix} 10 & -5 \\ 8 & -5 \end{vmatrix} = -1(10(-5) - (-5)(8)) = -1(-50 + 40) = 10.$$

To find $z$ we need to find

$$|A_3| = \begin{vmatrix} 1 & 2 & 5 & 0 \\ 2 & 2 & 0 & -1 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{vmatrix} \text{ Using } R_2 \to R_2 - R_4 \text{ yields}$$

$$\begin{vmatrix} 1 & 2 & 5 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix} \text{ Using } R_1 \to R_1 - 5R_3 \text{ yields}$$

$$\begin{vmatrix} -14 & 12 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix} \text{ Using } R_1 \to R_1 + 12R_2 \text{ yields}$$

$$\begin{vmatrix} 10 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 3 & -2 & 1 & 0 \\ 0 & 3 & 2 & -1 \end{vmatrix} = (10)(-1)(1)(-1) = 10.$$

Hence $z = \dfrac{|A_3|}{|A|} = \dfrac{10}{10} = 1.$