6 1. Find the open interval of convergence for the series

$$\sum_{n=3}^{\infty} \frac{(n+1)!}{n^n}(x+1)^{4n+3}.$$

$$\sum_{n=3}^{\infty} \frac{(n+1)!}{n^n}(x+1)^{4n+3} = (x+1)^3 \sum_{n=3}^{\infty} \frac{(n+1)!}{n^n}(x+1)^{4n}$$

We set $y = (x+1)^4$, and consider the series

$$\sum_{n=3}^{\infty} \frac{(n+1)!}{n^n} y^n$$

Its radius of convergence is

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{n^n}}{\frac{(n+2)!}{(n+1)^{n+1}}} \right| = \lim_{n\to\infty} \left[\frac{(n+1)!}{n^n} \frac{(n+1)^{n+1}}{(n+2)(n+1)!} \right]$$

$$= \lim_{n\to\infty} \left[\left(\frac{n+1}{n+2}\right)\left(\frac{n+1}{n}\right)^n \right] = e.$$

Hence, $R_x = e^{1/4}$, and the open interval of convergence is

$$|x+1| < e^{1/4}$$

$$-e^{1/4} < x+1 < e^{1/4}$$

$$-1-e^{1/4} < x < -1+e^{1/4}$$

14 **2.** Find the Taylor series for the function

$$f(x) = \sqrt{1+3x},$$

about $x = 2$. You must use a method that guarantees that the series converges to the function. Write the series in sigma notation, simplified as much as possible. What is the radius of convergence of the series?

$\boxed{x-2}$

$$f(x) = \sqrt{7+3(x-2)} = \sqrt{7}\left[1+\frac{3(x-2)}{7}\right]^{1/2}$$ 2

$$= \sqrt{7}\left\{1 + \left(\tfrac{1}{2}\right)\frac{3}{7}(x-2) + \frac{\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)}{2!}\frac{3^2(x-2)^2}{7^2} + \frac{\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)}{3!}\frac{3^3(x-2)^3}{7^3}+\cdots\right.$$

$$= \sqrt{7}\left\{1 + \frac{3}{2\cdot 7}(x-2) - \frac{3^2}{2^2 2!\cdot 7^2}(x-2)^2 + \frac{3\cdot 3^3}{2^3 3!\,7^3}(x-2)^3 - \frac{(3\cdot 5)3^4}{2^4 4!\,7^4}(x-2)^4+\cdots\right.$$

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$$= \sqrt{7}\left\{1 + \frac{3(x-2)}{2\cdot 7} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3^n[1\cdot 3\cdot 5\cdots(2n-3)](x-2)^n}{2^n n!\cdot 7^n}\right\}$$

$$= \sqrt{7}\left\{1 + \frac{3}{2\cdot 7}(x-2) + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3^n[1\cdot 2\cdot 3\cdot 4\cdots(2n-2)]}{2^n n![2\cdot 4\cdots(2n-2)]\cdot 7^n}(x-2)^n\right\}$$

$$= \sqrt{7}\left\{1 + \frac{3(x-2)}{2\cdot 7} + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}3^n(2n-2)!}{2^{2n-1}n!\,(n-1)!\,7^n}(x-2)^n\right\}$$

$$= \sqrt{7} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}3^n(2n-2)!}{2^{2n-1}7^{n-1/2}n!\,(n-1)!}(x-2)^n$$ 3

Since the series is valid for

$$\left|\frac{3(x-2)}{7}\right| < 1 \Rightarrow |x-2| < \frac{7}{3},$$ 1

the radius of convergence is $7/3$. 1

10 3. Find the sum of the series

$$\sum_{n=1}^{\infty}(n+1)2^n x^{n-1}.$$

~~Fthrerd~~ $\sum_{n=1}^{\infty}(n+1)2^n x^{n-1} = \sum_{n=0}^{\infty}(n+2)2^{n+1}x^n$

The radius of convergence of the series is

$$R = \lim_{n\to\infty}\left|\frac{(n+2)2^{n+1}}{(n+3)2^{n+2}}\right| = \frac{1}{2}$$

If we set $S(x) = \sum_{n=1}^{\infty}(n+1)2^n x^{n-1}$, then

$$x\,S(x) = \sum_{n=1}^{\infty}(n+1)2^n x^n.$$

We may integrate the series term-by-term,

$$\int x\,S(x)\,dx = \sum_{n=1}^{\infty}2^n x^{n+1} + C$$

$$= \frac{2x^2}{1-2x} + C.$$

Differentiation ~~now~~ gives

$$x\,S(x) = \frac{(1-2x)(4x)-2x^2(-2)}{(1-2x)^2} = \frac{4x-4x^2}{(1-2x)^2}$$

Division by x now gives

$$S(x) = \frac{4-4x}{(1-2x)^2}, \text{ provided } x \neq 0.$$

Since the sum of the series when $x=0$ is

$$\sum_{n=1}^{\infty}(n+1)2^n(0)^{n-1} = 4,$$

and the above formula also gives $S(0)=4$, we may write that

$$\sum_{n=1}^{\infty}(n+1)2^n x^{n-1} = \frac{4(1-x)}{(1-2x)^2}, \quad -\frac{1}{2} < x < \frac{1}{2}.$$

12 **4.** Solve the initial-value problem

$$y'' = yy', \quad y(0) = 1, \quad y'(0) = 1/2.$$

If we set $v = \frac{dy}{dx}$ and $\frac{d^2y}{dx^2} = v\frac{dv}{dy}$, 1

$$v\frac{dv}{dy} = vy \quad 1$$

$$dv = y\,dy \quad 1 \quad , \text{provided } v \neq 0.$$

A 1-parameter family of solutions of this equation is defined implicitly by

$$v = \frac{y^2}{2} + C. \quad 2$$

Since $v = y'$ is equal to $\frac{1}{2}$ when $y = 1$,

$$\frac{1}{2} = \frac{1}{2} + C \implies C = 0. \quad 1$$

$$\therefore v = \frac{y^2}{2} \quad 1$$

$$\frac{dy}{dx} = \frac{y^2}{2}$$

$$\frac{dy}{y^2} = \frac{dx}{2} \quad 1 \quad \text{provided } y \neq 0.$$

A 1-parameter family of solutions of this equation is defined implicitly by

$$-\frac{1}{y} = \frac{x}{2} + D \quad 1$$

Since $y(0) = 1$, $-1 = D$ 1

$$\therefore -\frac{1}{y} = \frac{x}{2} - 1 = \frac{x-2}{2} \quad 1$$

$$\therefore y = \frac{2}{2-x}. \quad 1$$

6 5. Find the form of a particular solution of the differential equation

$$D(D^2-1)(D^2+4)y = 3x^2e^{-x} + 10x + 5\cos x$$

as predicted by the method of undetermined coefficients. Do **NOT** solve for the coefficients.

The auxiliary equation is

$$m(m^2-1)(m^2+4)=0$$

Roots are $m = 0, \pm 1, \pm 2i$.

$$y_h(x) = C_1 + C_2e^x + C_3e^{-x} + C_4\cos 2x + C_5\sin 2x$$

$$y_p(x) = Ax^3e^{-x} + Bx^2e^{-x} + Cxe^{-x} + Dx^2 + Ex + F\cos x + G\sin x$$

12 6. A 100 gram mass is suspended from a spring with constant 25/2 newtons per metre. It is set into vertical motion by pulling it 5 centimetres below its equilibrium position and giving it velocity 2 metres per second upward. During its subsequent motion, damping is equal to 3 times velocity.

(a) Find the position of the mass as a function of time.

(b) Is the motion underdamped, overdamped, or critically damped?

(c) Determine whether the mass ever passes through its equilibrium position. If it does, find the time(s) when this occurs.

(a) The initial value problem for displacements is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + \frac{25}{2}x = 0, \quad x(0) = -\frac{1}{20}, \quad x'(0) = 2.$$

The auxiliary equation is

$$0 = \frac{1}{10}m^2 + 3m + \frac{25}{2} = \frac{1}{10}(m^2 + 30m + 125)$$

$$= \frac{1}{10}(m+5)(m+25)$$

Roots are $m = -5, -25$. Thus,

$$x(t) = Ae^{-5t} + Be^{-25t}.$$

The initial conditions give

$$-\frac{1}{20} = A + B, \quad 2 = -5A - 25B.$$

Solutions are $A = \frac{3}{80}$ and $B = -\frac{7}{80}$.

Thus, $x(t) = \frac{3}{80}e^{-5t} - \frac{7}{80}e^{-25t}$ m.

(b) Motion is overdamped.

(c) The mass passes through equilibrium if

$$0 = \frac{3}{80}e^{-5t} - \frac{7}{80}e^{-25t}$$

$$3e^{-5t} = 7e^{-25t}$$

$$e^{20t} = \frac{7}{3}$$

$$20t = \ln(7/3)$$

$$t = \frac{1}{20}\ln(7/3) \text{ s.}$$

THE UNIVERSITY OF MANITOBA

DATE: December 16, 2013 (Afternoon) FINAL EXAMINATION

DEPARTMENT & COURSE NO: MATH2132 TIME: 3 hours

EXAMINATION: Engineering Mathematical Analysis 2 EXAMINER: D. Trim

8 7. Find the Laplace transform of the function

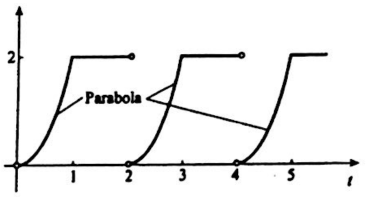
$$f(t) = e^{4t} \sin 3t\, h(t-2).$$

$$F(s) = e^{-2s} \mathcal{L}\{e^{4(t+2)} \sin 3(t+2)\}$$
$$= e^{-2s} e^{8} \mathcal{L}\{e^{4t}(\sin 3t \cos 6 + \cos 3t \sin 6)\}$$
$$= e^{8-2s}\left[\frac{\cos 6\,(3)}{(s-4)^2+9} + \frac{(s-4)\sin 6}{(s-4)^2+9}\right]$$

or

$$F(s) = \mathcal{L}\{\sin 3t\, h(t-2)\}_{|s-4}$$
$$= e^{-2(s-4)} \mathcal{L}\{\sin 3(t+2)\}_{|s-4}$$
$$= e^{8-2s} \mathcal{L}\{\sin 3t \cos 6 + \cos 3t \sin 6\}_{|s-4}$$
$$= e^{8-2s}\left[\frac{3\cos 6}{(s-4)^2+9} + \frac{(s-4)\sin 6}{(s-4)^2+9}\right]$$

10 8. Find the Laplace transform of the function in the figure below. You need not simplify your result.



$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(t) e^{-st}\,dt$$
$$= \frac{1}{1-e^{-2s}} \mathcal{L}\{2t^2[h(t)-h(t-1)] + 2[h(t-1)-h(t-2)]\}$$
$$= \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - 2e^{-s}\mathcal{L}\{(t+1)^2\} + \frac{2}{s}e^{-s} - \frac{2}{s}e^{-2s}\right]$$
$$= \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - 2e^{-s}\mathcal{L}\{t^2+2t+1\} + \frac{2}{s}e^{-s} - \frac{2}{s}e^{-2s}\right]$$
$$= \frac{1}{1-e^{-2s}}\left[\frac{4}{s^3} - 2e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \cancel{\frac{1}{s}}\right) + \cancel{\frac{2}{s}e^{-s}} - \frac{2}{s}e^{-2s}\right]$$

10 **9.** Find the inverse Laplace transform for the function

$$\frac{(3s^2+s-6)e^{-2s}}{s^3+3s^2}.$$

Since $\dfrac{3s^2+s-6}{s^3+3s^2} = \dfrac{3s^2+s-6}{s^2(s+3)} = \dfrac{1}{s} - \dfrac{2}{s^2} + \dfrac{2}{s+3}$,

$$\mathcal{L}^{-1}\left\{\frac{3s^2+s-6}{s^3+3s^2}\right\} = 1 - 2t + 2e^{-3t}$$

Thus, $\mathcal{L}^{-1}\left\{\dfrac{(3s^2+s-6)e^{-2s}}{s^3+3s^2}\right\} = \left[1-2(t-2)+2e^{-3(t-2)}\right]h(t-2)$

$$= (5-2t+2e^{6-3t})h(t-2)$$

12 **10.** Solve the initial-value problem

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 6y = 4\delta(t-3), \quad y(0) = 2, \quad y'(0) = 1.$$

When we take Laplace transforms,

$$[s^2Y - 2s - 1] + 4[sY - 2] + 6Y = 4e^{-3s}.$$

$$Y(s) = \frac{2s + 9 + 4e^{-3s}}{s^2+4s+6}$$

$$= \frac{2s+9}{s^2+4s+6} + \frac{4e^{-3s}}{s^2+4s+6}$$

$$= \frac{2(s+2)+5}{(s+2)^2+2} + \frac{4e^{-3(s+2)+6}}{(s+2)^2+2}$$

$$y(t) = e^{-2t}\mathcal{L}^{-1}\left\{\frac{2s+5}{s^2+2}\right\} + 4e^{6-2t}\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s^2+2}\right\}$$

$$= e^{-2t}\left\{2\cos\sqrt{2}t + \frac{5}{\sqrt{2}}\sin\sqrt{2}t\right\}$$

$$+ \frac{4e^{6-2t}}{\sqrt{2}}\sin\sqrt{2}(t-3)h(t-3)$$