MATH 1700 Problem Workshop 12 Solutions

1. (a) $r = y$. $\dfrac{dx}{dy} = 2y$. Hence the surface area is

$$
\begin{aligned}
SA &= \int_0^4 2\pi y\sqrt{1+(2y)^2}\,dy \\
&= 2\pi \int_0^4 y\sqrt{1+4y^2}\,dy \\
&= \frac{\pi}{4}\int_1^{65} \sqrt{w}\,dw \\
&= \frac{\pi}{4}\left(\frac{w^{3/2}}{3/2}\right|_1^{65} \\
&= \frac{\pi}{6}(65^{3/2}-1)
\end{aligned}
$$

(b) $r = y$. $\dfrac{dy}{dx} = \cos x$. Hence the surface area is

$$
\begin{aligned}
SA &= \int_0^{2\pi} 2\pi y\sqrt{1+(\cos x)^2}\,dx \\
&= 2\pi \int_0^{2\pi} (2+\sin x)\sqrt{1+\cos^2 x}\,dx
\end{aligned}
$$

(c) $r = x$. $\dfrac{dy}{dx} = \dfrac{1}{x}$. Hence the surface area is

$$
\begin{aligned}
SA &= \int_1^e 2\pi x\sqrt{1+\frac{1}{x^2}}\,dx \\
&= \int_1^e 2\pi\sqrt{x^2+1}\,dx
\end{aligned}
$$

Solving

$$\int \sqrt{x^2+1}\,dx$$

using $x = \tan\theta$

$$
\begin{aligned}
I &= \int \sqrt{x^2+1}\,dx \\
&= \int \sqrt{\tan^2\theta+1}\sec^2\theta\,d\theta \\
&= \int \sec^3\theta\,d\theta
\end{aligned}
$$

using $u = \sec\theta, \quad dv = \sec^2\theta\, d\theta$.

$$\begin{aligned} I &= \int \sec^3\theta\, d\theta \\ &= \sec\theta\tan\theta - \int sec\theta \tan^2\theta\, d\theta \\ &= \sec\theta\tan\theta - \int sec\theta(\sec^2\theta - 1)\, d\theta \\ &= \sec\theta\tan\theta - I + \int sec\theta\, d\theta \\ &= \sec\theta\tan\theta - I + \ln|sec\theta + \tan\theta| \end{aligned}$$

Hence

$$\begin{aligned} I &= \frac{1}{2}\big(\sec\theta\tan\theta + \ln|sec\theta + \tan\theta|\big) + C \\ &= \frac{1}{2}\big(x\sqrt{x^2+1} + \ln\left|\sqrt{x^2+1} + x\right|\big) + C \end{aligned}$$

Hence the surface area is

$$\begin{aligned} &2\pi\left(\frac{1}{2}\big(x\sqrt{x^2+1} + \ln\left|\sqrt{x^2+1} + x\right| \Big|_1^e\right) \\ &= \pi\Big(\big(e\sqrt{e^2+1} + \ln(\sqrt{e^2+1} + e)\big) - \big(\sqrt{1^2+1}1 + \ln\left|\sqrt{1^2+1} + 1\right|\big)\Big) \\ &= \pi\Big(\big(e\sqrt{e^2+1} + \ln(\sqrt{e^2+1} + e)\big) - \sqrt{2} - \ln(\sqrt{2}+1)\big)\Big) \end{aligned}$$

(d) $r = y$. $\dfrac{dy}{dx} = -e^{-x}$. Hence the surface area is

$$
\begin{aligned}
SA &= \int_0^\infty 2\pi e^{-x}\sqrt{1+(-e^{-x})^2}\,dx \\
&= 2\pi \int_0^\infty e^{-x}\sqrt{1+e^{-2x}}\,dx \\
&= \lim_{t\to\infty} 2\pi \int_0^t e^{-x}\sqrt{1+e^{-2x}}\,dx \\
&= \lim_{t\to\infty} 2\pi \int_1^{e^{-t}} -\sqrt{1+w^2}\,dw \text{ using } w=e^{-x} \\
&= \lim_{t\to\infty} -2\pi\left(\frac{1}{2}\big(w\sqrt{w^2+1}+\ln\left|\sqrt{w^2+1}+w\right|\,\Big|_1^{e^{-t}}\right) \\
&= \lim_{t\to\infty} -\pi\big(e^{-t}\sqrt{e^{-2t}+1}+\ln\left|\sqrt{e^{-2t}+1}+e^{-t}\right|\big)+\pi\big(1\sqrt{1^2+1}+\ln\left|\sqrt{1^2+1}+1\right|\big) \\
&= \pi\big(\sqrt{2}+\ln(\sqrt{2}+1)
\end{aligned}
$$

2. (a) $-\dfrac{\pi}{2} \le t \le \dfrac{\pi}{2}$

$$
\begin{aligned}
SA &= \int_{-\pi/2}^{\pi/2} 2\pi y\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt \\
&= \int_{-\pi/2}^{\pi/2} 2\pi a\sin^3 t\sqrt{(-a(\cos^2 t)^2\sin t)^2+(a(\sin^2 t)^2\cos t)^2}\,dt \\
&= \int_{-\pi/2}^{\pi/2} 2\pi a\sin^3 t\sqrt{a^2\cos^4 t\sin^2 t+a^2\sin^4 t\cos^2 t}\,dt \\
&= \int_{-\pi/2}^{\pi/2} 2\pi a\sin^3 t(a\sin t\cos t)\sqrt{\cos^2 t+\sin^2 t}\,dt \\
&= 2\pi a^2\int_{-\pi/2}^{\pi/2} \sin^4 t\cos t\,dt \\
&= 2\pi a^2\int_{-1}^{1} w^4\,dw \\
&= \frac{2\pi a^2 w^5}{5}\bigg|_{-1}^{1} \\
&= \frac{2\pi a^2(1)^5}{5}-\frac{2\pi a^2(-1)^5}{5} \\
&= \frac{4\pi a^2}{5}.
\end{aligned}
$$

(b)

$$\begin{aligned}
dL &= \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \\
&= \sqrt{4R^2 \sin^2\theta + 4R^2 \cos^2\theta} \\
&= 2R.
\end{aligned}$$

Hence

$$\begin{aligned}
SA &= \int_0^\pi 2\pi y \, dL \\
&= \int_0^\pi 2\pi r \sin\theta (2R) \, d\theta \\
&= 2\pi \int_0^\pi 2R \sin\theta \sin\theta (2R) \, d\theta \\
&= 8\pi R^2 \int_0^\pi \sin^2\theta \, d\theta \\
&= 4\pi R^2 \int_0^\pi (1 - \cos 2\theta) \, d\theta \\
&= 4\pi R^2 \left(\theta - \frac{\sin 2\theta}{2}\right|_0^\pi \\
&= 4\pi R^2 \left(\pi - \frac{\sin 2\pi}{2}\right) - \left(0 - \frac{\sin 2(0)}{2}\right) \\
&= 4\pi^2 R^2
\end{aligned}$$