Math 2130 Multivariable Limits Collection from old Tests(*with Solution*)
(Fall2011-Summer2015

[4] 5. Evaluate $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^2+y^2}{x^2+y^2}$.

approach along x-axis ($y=0$): $\lim\limits_{x\to 0} \dfrac{2x^2}{x^2} = 2$

approach along y-axis ($x=0$): $\lim\limits_{y\to 0} \dfrac{y^2}{y^2} = 1$.

$\boxed{\text{D.N.E.}}$

6 **5.** Show that the following limit does not exist,

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}.$$

If we approach $(0,0)$ along the cubic curves $x = ay^3$, then

$$\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6} = \lim_{y\to 0} \frac{ay^6}{a^2y^6+y^6} = \frac{a}{a^2+1}.$$

Since this result depends on a, it follows that the original limit does not exist.

Values

6 **1.** Determine whether the limit

$$\lim_{(x,y)\to(0,1)} \frac{x^2+y^2-2y+1}{3x^2-4y^2+8y-4}$$

exists. If the limit exists, find its value; if the limit does not exist, give reasons for its nonexistence.

If we approach $(0,1)$ along straight lines $y = mx+1$, then

$$\lim_{(x,y)\to(0,1)} \frac{x^2+y^2-2y+1}{3x^2-4y^2+8y-4} = \lim_{x\to 0} \frac{x^2+(y-1)^2}{3x^2-4(y-1)^2} = \lim_{x\to 0} \frac{x^2+m^2x^2}{3x^2-4m^2x^2}$$
$$= \lim_{x\to 0} \frac{1+m^2}{3-4m^2} = \frac{1+m^2}{3-4m^2}.$$

Because this depends on m, the function has different limits for different modes of approach. Therefore the given limit does not exist.

[4] 5. (a) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ does not exist.

Solution:
We could go along paths $x = 0$, $y = 0$ and $y = mx$ and we notice that the limit is 0 each time. Trying along the path $y = ax^2$ yields

$$\lim_{x\to 0} \frac{ax^4}{x^4 + a^2x^4} = \lim_{x\to 0} \frac{a}{1+a^2} = \frac{a}{1+a^2}.$$

Hence the limit depends on the path and thus the limit does not exist.

[4] (b) Show that $\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2} = 0$.

Solution:
Because of the even exponents and the absolute value we know the function is positive and

$$0 \le \frac{x^2y^2}{x^4+y^2} \le \frac{x^2y^2}{y^2} = x^2.$$

Since

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} x^2 = 0$$

we have that

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^4+y^2} = 0$$

by the squeeze theorem.

[4] 1. (a) *Show* $\lim_{(x,y)\to(0,0)} \frac{x^2y}{x^4+y^2}$ *does not exist.*

We could go along paths $x = 0$, $y = 0$ and $y = mx$ and we notice that the limit is 0 each time. Trying along the path $y = ax^2$ yields

$$\lim_{x\to 0} = \frac{ax^4}{x^4 + a^2x^4} = \lim_{x\to 0} = \frac{a}{1+a^2} = \frac{a}{1+a^2}.$$

Hence the limit depends on the path and thus the limit does not exist.

[3] (b) *Show* $\lim_{(x,y)\to(0,0)} \frac{x^2y^4}{x^4+y^4} = 0.$

Because of the even exponents we know the function is positive and

$$0 \leq \frac{x^2y^4}{x^4+y^4} \leq \frac{x^2y^4}{y^4} = x^2.$$

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by the squeeze theorem.

[4] 1. (a) Show $\lim_{(x,y)\to(0,0)} \frac{x^4 y}{x^6 + y^3}$ does not exist.

Solution:
We could go along paths $x = 0$, $y = 0$ and $y = mx$ and we notice that the limit is 0 each time. Trying along the path $y = ax^2$ yields

$$\lim_{x\to 0} \frac{ax^6}{x^6 + a^3x^6} = \lim_{x\to 0} \frac{a}{1+a^3} = \frac{a}{1+a^3}.$$

Hence the limit depends on the path and thus the limit does not exist.

[3] (b) Show $\lim_{(x,y)\to(0,0)} \frac{|x|\, y^2}{x^2 + y^2} = 0$.

Solution:

Because of the even exponents and the absolute value we know the function is positive and

$$0 \le \frac{|x|\, y^2}{x^2 + y^2} \le \frac{|x|\, y^2}{y^2} = |x|\,.$$

Since

$$\lim_{(x,y)\to(0,0)} 0 = \lim_{(x,y)\to(0,0)} |x| = 0$$

we have that

$$\lim_{(x,y)\to(0,0)} \frac{|x|\, y^2}{x^2 + y^2} = 0$$

by the squeeze theorem.

6. *Find the following limits, or show why they do not exist.*

[5] (a) $\lim_{(x,y)\to(0,0)} \frac{-3xy}{2x^2+y^2}$

Along the path $x=0$, we get

$$\lim_{y\to 0} \frac{-3(0)y}{2(0)^2+y^2} = \lim_{y\to 0} \frac{0}{y^2} = 0$$

Along the path $y=0$, we get

$$\lim_{x\to 0} \frac{-3x(0)}{2x^2+(0)^2} = \lim_{y\to 0} \frac{0}{2x^2} = 0$$

Along the path $y=mx$, we get

$$\lim_{x\to 0} \frac{-3x(mx)}{2x^2+(mx)^2} = \lim_{y\to 0} \frac{-3mx^2}{2x^2+m^2x^2} = \lim_{y\to 0} \frac{-3m}{2+m^2} = -\frac{3m}{2+m^2}$$

Since the limit is not always the same since it depends on m, the limit does not exist.

[4] (b) $\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{2x^3+2xy^2-3x^2-3y^2}$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{2x^3+2xy^2-3x^2-3y^2} &= \lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{(x^2+y^2)(2x-3)} \\ &= \lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{2x-3}\end{aligned}$$

Using a substitution $u=x^2+y^2$ for the first limit yields

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{2x^3+2xy^2-3x^2-3y^2} &= \lim_{(x,y)\to(0,0)} \frac{e^{x^2+y^2}-1}{x^2+y^2} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{2x-3} \\ &= \lim_{u\to 0} \frac{e^u-1}{u} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{2x-3} \\ &= \lim_{u\to 0} \frac{e^u}{1} \cdot \lim_{(x,y)\to(0,0)} \frac{1}{2x-3} \\ &= \frac{e^0}{1} \cdot \frac{1}{2(0)-3} \\ &= -\frac{1}{3}.\end{aligned}$$

9 **4.** Determine whether the following limit exists,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - x^2y + 2y^4}{2x^4 - 5x^2y + 4y^4}.$$

If the limit does not exist, explain why not.

If we approach $(0,0)$ along the parabolas $y = ax^2$, the limit becomes

$$\lim_{x\to 0} \frac{x^4 - x^2(ax^2) + 2(ax^2)^4}{2x^4 - 5x^2(ax^2) + 4(ax^2)^4} = \lim_{x\to 0} \frac{1 - a + 2a^4x^4}{2 - 5a + 4a^4x^4} = \frac{1-a}{2-5a}.$$

Since this value depends on which parabola is used, it follows that the original limit does not exist.

4. Find the following limits, or show why they do not exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2}$ [4]

(b) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^3+xy^2+2x^2+2y^2}$ [3]

4. (a) This is a 0/0 indeterminant form was can't be simplified or substituted. Trying along $x = 0$ yields

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2} = \lim_{y\to 0} \frac{0}{3y^2} = 0$$

. Trying along $y = 0$ yields

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2} = \lim_{x\to 0} \frac{0}{x^2} = 0$$

. Trying along $y = mx$ yields

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+3y^2} = \lim_{x\to 0} \frac{2mx^2}{(1+m^2)x^2} = \frac{2}{1+m^2}.$$

Since the value fo the limit depends on the path, we conclude that the limit does not exist.

(b) This is a 0/0 indeterminant form, however we can do some simplification.

$$\begin{aligned}\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^3+xy^2+2x^2+2y^2} &= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)(x+2)} \\ &= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} \lim_{(x,y)\to(0,0)} \frac{1}{x+2} \\ &= (1)(1/2) = 1/2.\end{aligned}$$