MATH 2132 Problem Workshop 6

1. Compute the Laplace transforms of the following functions. For part (a) use the definition. For the others, you can use the table

   (a) $f(t) = e^{-2t}\cos 4t$

Solution:

$$\begin{aligned} F(s) &= \int_0^\infty e^{-st}e^{-2t}\cos 4t\,dt \\ &= \int_0^\infty e^{(-s-2)t}\cos 4t\,dt \end{aligned}$$

Let $I = \int e^{(-s-2)t}\cos 4t\,dt$.
Using integration by parts with $dv = e^{(-s-2)t}\,dt$ we get

$$I = \frac{1}{-s-2}\cos 4te^{(-s-2)t} - \frac{4}{-s-2}\int e^{(-s-2)t}\sin 4t\,dt$$

Using integration by parts again with $dv = e^{(-s-2)t}\,dt$ we get

$$\begin{aligned} I &= \frac{1}{-s-2}\cos 4te^{(-s-2)t} + \frac{4}{-s-2}\left(\frac{1}{-s-2}\sin 4te^{(-s-2)t} - \frac{4}{-s-2}\int e^{(-s-2)t}\cos 4t\,dt\right) \\ &= \frac{e^{(-s-2)t}}{-s-2}\cos 4t + \frac{4e^{(-s-2)t}}{(-s-2)^2}\sin 4t - \frac{16}{(-s-2)^2}I \\ &\Rightarrow \left(1+\frac{16}{(s+2)^2}\right)I = \frac{e^{(-s-2)t}}{-s-2}\cos 4t - \frac{4e^{(-s-2)t}}{(-s-2)^2}\sin 4t \\ \Rightarrow I &= \frac{1}{1+\dfrac{16}{(s+2)^2}}\left(\frac{e^{(-s-2)t}}{-s-2}\cos 4t + \frac{4e^{(-s-2)t}}{(-s-2)^2}\sin 4t\right) \\ &= \frac{1}{(s+2)^2+16}\Big(-(s+2)\cos 4t + 4\sin 4t\Big)e^{(-s-2)t} \end{aligned}$$

Thus

$$\begin{aligned} F(s) &= \lim_{z\to\infty}\frac{1}{(s+2)^2+16}\Big(-(s+2)\cos 4t + 4\sin 4t\Big)e^{(-s-2)t}\Big|_0^z \\ &= \frac{1}{(s+2)^2+16}\lim_{z\to\infty}\left(\Big(\Big(-(s+2)\cos 4z + 4\sin 4z\Big)e^{(-s-2)z}\Big) - (-(s+2))\right) \\ &= \frac{1}{(s+2)^2+16}(s+2) \end{aligned}$$

Where $\lim_{z\to\infty}\Big(-(s+2)\cos 4z+4\sin 4z\Big)e^{(-s-2)z}=0$ as in absolute value

$$\Big|-(s+2)\cos 4z+4\sin 4z\Big|e^{(-s-2)z}\le (s+6)e^{(-s-2)z}\to 0.$$

Hence $F(s)=\dfrac{s+2}{(s+2)^2+16}$.

(b) $f(t)=e^{-2t}\cos 4t h(t-3)$

Solution:
We use that $\mathcal{L}(h(t-a)f(t))=e^{-as}\mathcal{L}(f(t+a))$ to get

$$\begin{aligned}\mathcal{L}(e^{-2t}\cos 4t h(t-3)) &= e^{-3s}\mathcal{L}(e^{-2(t+3)}\cos 4(t+3))\\ &= e^{-3s}\mathcal{L}(e^{-2t-6}\cos 4(t+3))\\ &= e^{-3s-6}\mathcal{L}(e^{-2t}\cos(4t+12))\end{aligned}$$

Since $\cos(4t+12)=\cos(12)\cos 4t-\sin(12)\sin 4t$,

$$\begin{aligned}\mathcal{L}(e^{-2t}\cos 4t h(t-3)) &= e^{-3s-6}\mathcal{L}(e^{-2t}\cos(4t+12))\\ &= e^{-3s-6}\big(\cos(12)\mathcal{L}(e^{-2t}\cos 4t)-\sin(12)\mathcal{L}(e^{-2t}\sin 4t)\big)\\ &= e^{-3s-6}\left(\frac{\cos(12)(s+2)}{(s+2)^2+16}-\frac{4\sin(12)}{(s+2)^2+16}\right)\end{aligned}$$

(c) $f(t)=\begin{cases} 2t-5 & 0\le t<4\\ t^2 & 4\le t<8\\ 1 & t\ge 8\end{cases}$

Solution:
Option 1: Definition

$$\mathcal{L}(f(t))=\int_0^4(2t-5)e^{-st}\,dt+\int_4^8 t^2e^{-st}\,dt+\int_8^\infty e^{-st}\,dt.$$

$$\begin{aligned}
\int_0^4 (2t-5)e^{-st}\,dt &= \frac{1}{-s}(2t-5)e^{-st}\bigg|_0^4 + \frac{1}{s}\int_0^4 2e^{-st}\,dt \\
&= \frac{1}{-s}(2t-5)e^{-st} - \frac{2e^{-st}}{s^2}\bigg|_0^4 \\
&= -\frac{3}{s}e^{-4s} - \frac{2}{s^2}e^{-4s} - \frac{5}{s} + \frac{2}{s^2}
\end{aligned}$$

$$\begin{aligned}
\int_4^8 t^2e^{-st}\,dt &= \frac{1}{-s}t^2e^{-st}\bigg|_4^8 + \frac{1}{s}\int_4^8 2te^{-st}\,dt \\
&= -\frac{1}{s}t^2e^{-st} - \frac{2t}{s^2}e^{-st}\bigg|_4^8 + \frac{1}{s^2}\int_4^8 2e^{-st}\,dt \\
&= -\frac{1}{s}t^2e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\bigg|_4^8 \\
&= e^{-8s}\left(-\frac{64}{s} - \frac{16}{s^2} - \frac{2}{s^3}\right) + e^{-4s}\left(\frac{16}{s} + \frac{8}{s^2} + \frac{2}{s^3}\right)
\end{aligned}$$

$$\begin{aligned}
\int_8^\infty e^{-st}\,dt &= \lim_{z\to\infty} \frac{1}{-s}e^{-st}|_8^z \\
&= \lim_{z\to\infty}\left(\frac{1}{-s}e^{-sz} + \frac{1}{s}e^{-8s}\right) \\
&= \frac{1}{s}e^{-8s}
\end{aligned}$$

Thus the Laplace transform is

$$\frac{2}{s^2} - \frac{5}{s} + e^{-4s}\left(\frac{13}{s} + \frac{6}{s^2} + \frac{2}{s^3}\right) + e^{-8s}\left(-\frac{63}{s} - \frac{16}{s^2} - \frac{2}{s^3}\right)$$

Solution 2: Re-write as step functions

$$\left\{\begin{array}{cc} 2t-5 & 0\le t<4 \\ 0 & 4\le t \end{array}\right. + \left\{\begin{array}{cc} 0 & 0\le t<4 \\ t^2 & 4\le t<8 \\ 0 & t\ge 8 \end{array}\right. + \left\{\begin{array}{cc} 0 & 0\le t<8 \\ 1 & t\ge 8 \end{array}\right.$$

The first is

$$\begin{aligned}\mathcal{L}(2t-5-(2t-5)h(t-4)) &= \frac{2}{s^2}-\frac{5}{s}-e^{-4s}\mathcal{L}(2(t+4)+5)\\ &= \frac{2}{s^2}-\frac{5}{s}-e^{-4s}\mathcal{L}(2t+3)\\ &= \frac{2}{s^2}-\frac{5}{s}-e^{-4s}\left(\frac{2}{s^2}+\frac{3}{s}\right).\end{aligned}$$

The second is

$$\begin{aligned}\mathcal{L}(t^2h(t-4)-t^2h(t-8)) &= e^{-4s}\mathcal{L}((t+4)^2)-e^{-8s}\mathcal{L}((t+8)^2)\\ &= e^{-4s}\mathcal{L}(t^2+8t+16)-e^{-4s}\mathcal{L}(t^2+16t+64)\\ &= e^{-4s}\left(\frac{16}{s}+\frac{8}{s^2}+\frac{2}{s^3}\right)-e^{-8s}\left(\frac{64}{s}+\frac{16}{s^2}+\frac{2}{s^3}\right).\end{aligned}$$

The last is

$$\begin{aligned}\mathcal{L}(h(t-8)) &= e^{-8s}\mathcal{L}(1)\\ &= \frac{e^{-8s}}{s}.\end{aligned}$$

Thus the Laplace transform is

$$\frac{2}{s^2}-\frac{5}{s}+e^{-4s}\left(\frac{13}{s}+\frac{6}{s^2}+\frac{2}{s^3}\right)+e^{-8s}\left(-\frac{63}{s}-\frac{16}{s^2}-\frac{2}{s^3}\right)$$

(d) $f(t)=t^2-2t+3, \qquad 0\le t<2 \qquad f(t+2)=f(t)$

Solution:
The Laplace transform of a periodic function $f(t)$ with period p is

$$\mathcal{L}(f(t))=\frac{1}{1-e^{-ps}}\int_0^p e^{-st}f(t)\,dt$$

First lets find

$$\int (t^2-2t+3)e^{-st}\,dt.$$

Using integration by parts twice with $dv=e^{-st}\,dt$

$$
\begin{aligned}
I &= \int (t^2 - 2t + 3)e^{-st}\,dt \\
&= -\frac{1}{s}(t^2 - 2t + 3)e^{-st} + \frac{1}{s}\int (2t-2)e^{-st}\,dt \\
&= -\frac{1}{s}(t^2 - 2t + 3)e^{-st} + \frac{1}{s}\left(-\frac{1}{s}(2t-2)e^{-st} + \frac{2}{s}\int e^{-st}\,dt\right) \\
&= -\frac{1}{s}(t^2 - 2t + 3)e^{-st} - \frac{1}{s^2}(2t-2)e^{-st} + \frac{2}{s^2}\int e^{-st}\,dt \\
&= -\frac{1}{s}(t^2 - 2t + 3)e^{-st} - \frac{1}{s^2}(2t-2)e^{-st} - \frac{2}{s^3}e^{-st} + C
\end{aligned}
$$

Since the period of the function is 2 we have

$$
\begin{aligned}
\int_0^2 (t^2 - 2t + 3)e^{-st}\,dt &= -\frac{1}{s}(t^2 - 2t + 3)e^{-st} - \frac{1}{s^2}(2t-2)e^{-st} - \frac{2}{s^3}e^{-st}\Bigg|_0^2 \\
&= \left(-\frac{1}{s}(3)e^{-2s} - \frac{1}{s^2}(2)e^{-2s} - \frac{2}{s^3}e^{-2s}\right) - \left(-\frac{1}{s}(3) - \frac{1}{s^2}(-2) - \frac{2}{s^3}\right) \\
&= e^{-2s}\left(-\frac{3}{s} - \frac{2}{s^2} - \frac{2}{s^3}\right) + \left(\frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3}\right).
\end{aligned}
$$

Therefore the Laplace transform is

$$
\frac{1}{1-e^{-2s}}\left(e^{-2s}\left(-\frac{3}{s} - \frac{2}{s^2} - \frac{2}{s^3}\right) + \left(\frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3}\right)\right).
$$

2. Compute the inverse Laplace Transform for the following functions.

(a) $F(s) = \dfrac{s^2+3}{s^3+2s^2+s}$

Solution:
This can be re-written as
$F(s) = \dfrac{s^2+3}{s(s+1))^2}$. Using partial fractions

$$
\frac{s^2+3}{s(s+1))^2} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{A(s+1)^2 + Bs(s+1) + Cs}{s(s+1))^2}.
$$

Therefore

$$s^2 + 3 = s^2(A + B) + s(2A + B + C) + A$$

Hence $A = 3$, $A + B = 1 \Rightarrow B = -2$ and $2A + B + C = 0 \Rightarrow C = -4$.
Thus we are finding

$$\mathcal{L}^{-1}\left(\frac{3}{s} - \frac{2}{s+1} - \frac{4}{(s+1)^2}\right) = 3 - 2e^{-t} - 4te^{-t}.$$

(b) $F(s) = \dfrac{e^{-s}(1 + e^{-2s})}{s^2 - s}$

Solution:
This can be rearranged to be

$$\frac{e^{-s}}{s^2 - s} + \frac{e^{-3s}}{s^2 - s}$$

Recall that $\mathcal{L}(h(t-a)f(t)) = e^{-as}\mathcal{L}(f(t+a))$.
For $\dfrac{e^{-s}}{s^2 - s}$, we have $a = 1$. We then find the inverse Laplace transform of $\dfrac{1}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$ which is $e^t - 1$.
Thus $f(t+1) = e^t - 1 \Rightarrow f(t) = e^{t-1} - 1$. So $\mathcal{L}^{-1}\left(\dfrac{e^{-s}}{s^2 - s}\right) = h(t-1)\big(e^{t-1} - 1\big)$.
For $\dfrac{e^{-3s}}{s^2 - s}$, we have $a = 3$. We then find the inverse Laplace transform of $\dfrac{1}{s(s-1)} = \dfrac{1}{s-1} - \dfrac{1}{s}$ which is $e^t - 1$.
Thus $f(t+3) = e^t - 1 \Rightarrow f(t) = e^{t-3} - 1$. So $\mathcal{L}^{-1}\left(\dfrac{e^{-3s}}{s^2 - s}\right) = h(t-3)\big(e^{t-3} - 1\big)$.
Therefore

$$\mathcal{L}^{-1}\left(\frac{e^{-s}}{s^2 - s} + \frac{e^{-3s}}{s^2 - s}\right) = h(t-1)\big(e^{t-1} - 1\big) + h(t-3)\big(e^{t-3} - 1\big).$$

As a piecewise function this would be

$$\begin{cases} 0 & t < 1 \\ e^{t-1} - 1 & 1 \le t < 3 \\ e^{t-1} + e^{t-3} - 2 & 3 \le t \end{cases}$$

(c) $F(s) = \dfrac{1}{e^{2s}(s^3 + 2s^2 + 6s)}$

Solution:
This is similar to the last question with heavyside step function $h(t-2)$. Hence we must compute

$$\mathcal{L}^{-1}\left(\frac{1}{s^3 + 2s^2 + 6s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s(s^2 + 2s + 6)}\right)$$

Now

$$\frac{1}{s(s^2 + 2s + 6)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 2s + 6} = \frac{A(s^2 + 2s + 6) + (Bs + C)s}{s(s^2 + 2s + 6)}.$$

Therefore $1 = s^2(A + B) + s(2A + C) + 6A$ which implies

$$A = \frac{1}{6}.\ A + B = 0 \Rightarrow B = -\frac{1}{6}.\ 2A + C = 0 \Rightarrow C = -\frac{1}{3}.$$

Hence we are finding

$$\frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s} + \frac{-s-2}{s^2 + 2s + 6}\right)$$

Completing the square on the last term yields

$$\frac{-s-2}{(s+1)^2 + 5} = -\frac{s+1}{(s+1)^2 + 5} - \frac{1}{(s+1)^2 + 5}$$

Hence

$$\frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s} + -\frac{s+1}{(s+1)^2 + 5} - \frac{1}{(s+1)^2 + 5}\right) = \frac{1}{6}\left(1 - e^{-t}\cos\sqrt{5}t - \frac{1}{\sqrt{5}}e^{-t}\sin\sqrt{5}t\right)$$

Since this must be $f(t+2)$ we have

$$f(t) = \frac{1}{6}\left(1 - e^{-(t-2)}\cos\sqrt{5}(t-2) - \frac{1}{\sqrt{5}}e^{-(t-2)}\sin\sqrt{5}(t-2)\right)$$

Hence the inverse Laplace transform is

$$\frac{1}{6}h(t-2)\left(1 - e^{-(t-2)}\cos\sqrt{5}(t-2) - \frac{1}{\sqrt{5}}e^{-(t-2)}\sin\sqrt{5}(t-2)\right)$$

or

$$\begin{cases} 0 & t < 2 \\ \dfrac{1}{6}\left(1 - e^{-(t-2)}\cos\sqrt{5}(t-2) - \dfrac{1}{\sqrt{5}}e^{-(t-2)}\sin\sqrt{5}(t-2)\right) & 2 \leq t \end{cases}$$

3. Is it possible for $F(s) = \dfrac{s(s^2+3s-6)}{4s^3-3s+10}$ to be the Laplace transform for a piecewise continuous function of exponential order.

Solution:

If the function has exponential order, then the Laplace transform exists as defined below.

$F(s) = \displaystyle\int_0^\infty e^{-st} f(t)\, dt$

If the integral exists, then the limit

$$\lim_{s\to\infty} F(s) = \int_0^\infty (0) f(t)\, dt = 0.$$

However our given function does not go to 0. Therefore it cannot be a Laplace transform.