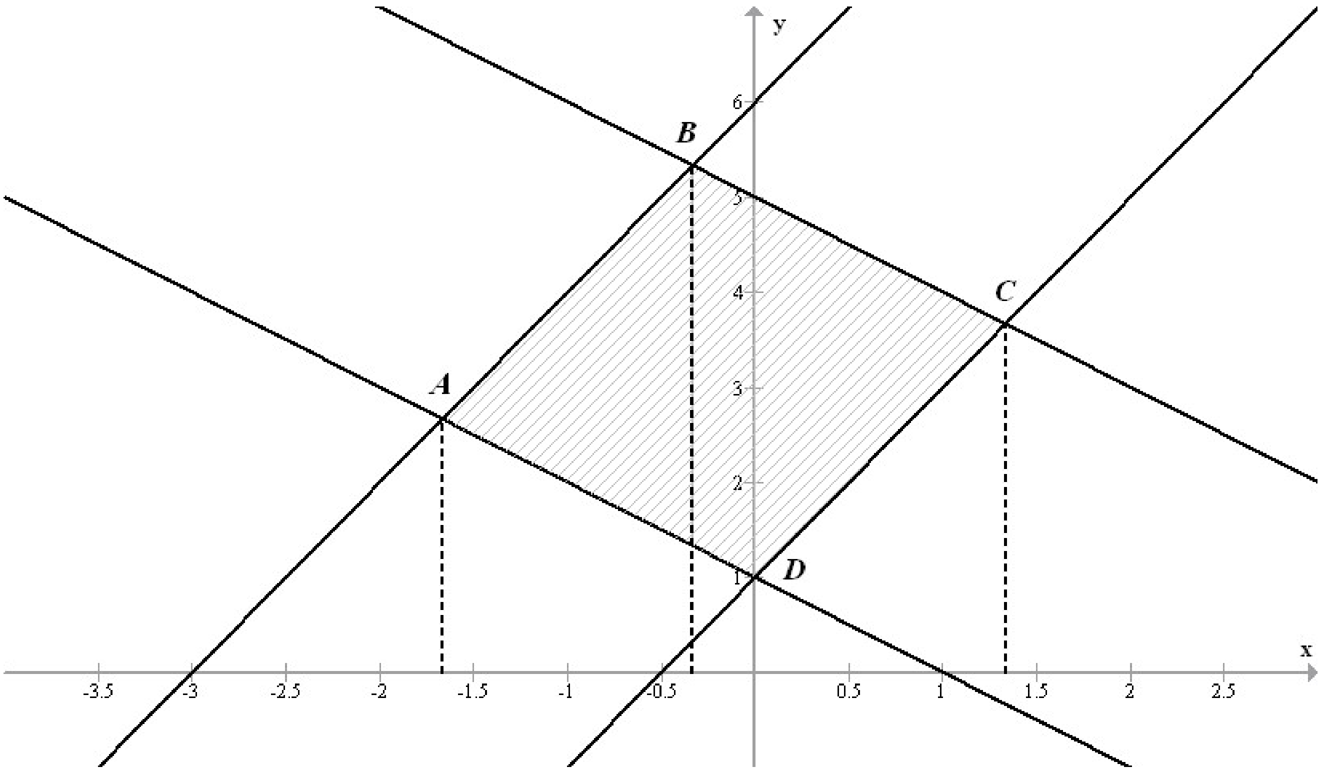
Math 1710. Homework Problems V (January 13, 2012)

Find the area of the region bounded by the curves:

12. $x + y = 1$, $x + y = 5$, $y = 2x + 1$, $y = 2x + 6$.

Solution: The graphs of all four functions are lines:

$$y = -x + 1,\ y = -x + 5,\ y = 2x + 1,\ y = 2x + 6$$



We find the coordinates of the points of intersection:

$$A \quad : \quad \begin{cases} y = 2x + 6, \\ y = -x + 1 \end{cases} \quad \Rightarrow 2x + 6 = -x + 1 \Rightarrow x_A = -5/3$$

$$B \quad : \quad \begin{cases} y = 2x + 6, \\ y = -x + 5 \end{cases} \quad \Rightarrow 2x + 6 = -x + 5 \Rightarrow x_B = -1/3$$

$$C \quad : \quad \begin{cases} y = 2x + 1, \\ y = -x + 1 \end{cases} \quad \Rightarrow 2x + 1 = -x + 1 \Rightarrow x_C = 0$$

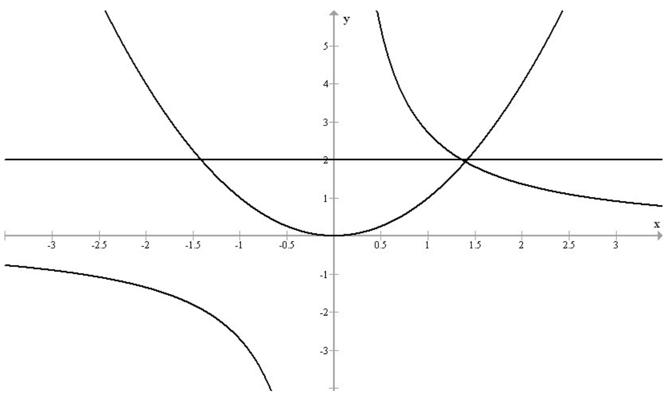
$$D \quad : \quad \begin{cases} y = 2x + 1, \\ y = -x + 5 \end{cases} \quad \Rightarrow 2x + 1 = -x + 5 \Rightarrow x_D = -4/3.$$

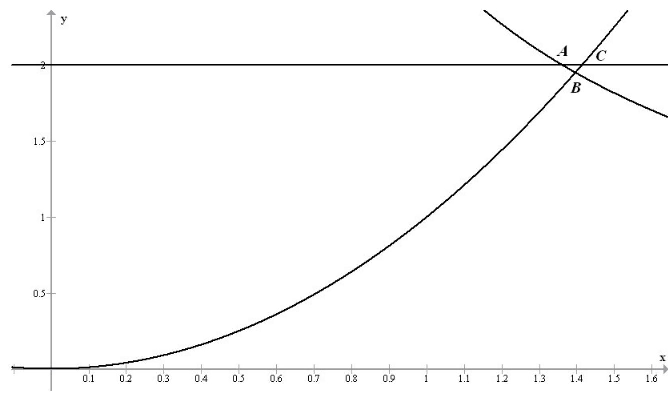
Therefore,

$$\begin{aligned}
Area &= \int_{-5/3}^{-1/3} [(2x+6)-(-x+1)]\,dx + \int_{-1/3}^{0} [(-x+5)-(-x+1)]\,dx + \int_{0}^{4/3} [(-x+5)-(2x+1)]\,dx \\
&= \int_{-5/3}^{-1/3} (3x+5)\,dx + \int_{-1/3}^{0} 4\,dx + \int_{0}^{4/3} (-3x+4)\,dx = \frac{8}{3}+\frac{4}{3}+\frac{8}{3} = \boxed{\frac{20}{3}}
\end{aligned}$$

14. $xy = e$, $y = x^2$, $y = 2$.

Solution: $xy = e$ is a hyperbola, $y = x^2$ is a vertical parabola, $y = 2$ is a horizontal line (the region is very small):





We use integration along the y-axis. Hence we only need to find the y-coordinate of point B. We find the coordinates of the points of intersection:

$$B: \quad \begin{cases} y = x^2, \\ xy = e \end{cases} \quad \Rightarrow x^3 = e \Rightarrow x_B = \sqrt[3]{e} \Rightarrow y_B = x_B^2 = e^{2/3}$$

Note that the points A, B, C are exactly in the order shown on the picture since $e^{2/3} < 2 \Leftrightarrow e^2 < 2^3 = 8$ (can be easily checked by computations)

Therefore, we write x as a function of y:

$$x = \frac{e}{y}, \quad x = \sqrt{y},$$

and find the area of the given region

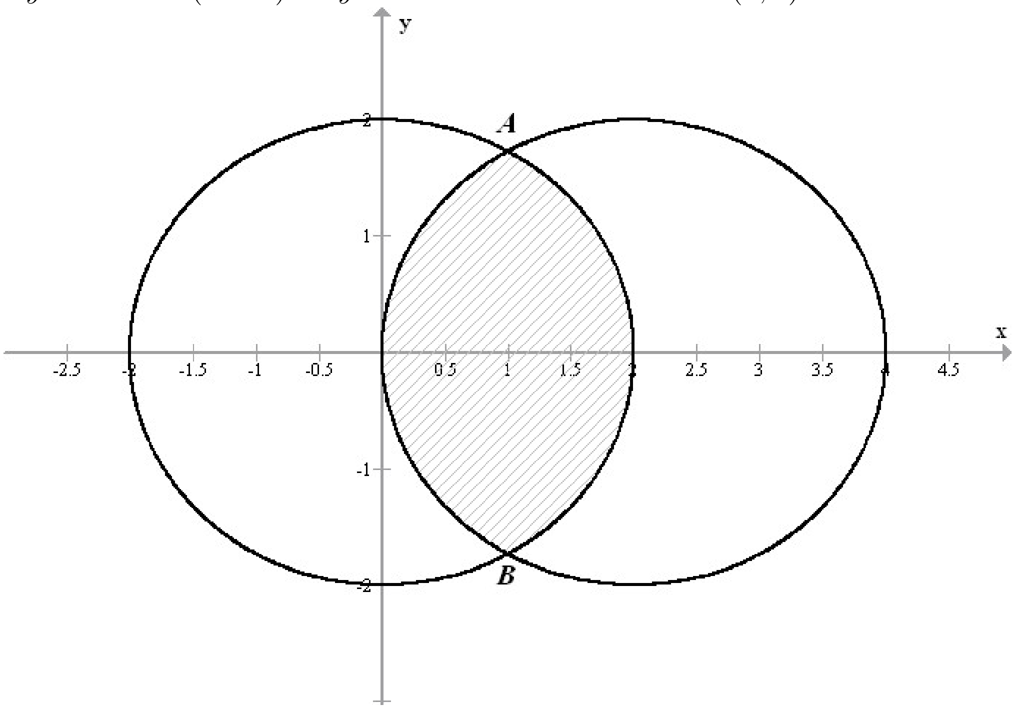
$$Area = \int_{e^{2/3}}^{2} \left[\sqrt{y} - \frac{e}{y}\right] dy = \left[\frac{2y^{3/2}}{3} - e\ln|y|\right]\Big|_{e^{2/3}}^{2} = \left(\frac{4\sqrt{2}}{3} - e\ln 2\right) - \left(\frac{2e}{3} - \frac{2e}{3}\right) = \boxed{\frac{4\sqrt{2}}{3} - e\ln 2}$$

20. $x^2 + y^2 = 4$, $x^2 + y^2 = 4x$ (set up the integral only).

Solution: Both curves represent a circle:

$x^2 + y^2 = 4$ – a circle centered at $(0, 0)$ with radius $r = 2$;

$x^2 + y^2 = 4x \Leftrightarrow (x-2)^2 + y^2 = 4$ – a circle centered at $(2, 0)$ with radius $r = 2$.



We find the points of intersection:

$$\begin{cases} x^2 + y^2 = 4, \\ x^2 + y^2 = 4x \end{cases} \Rightarrow 4x = 4 \Rightarrow x = 1 \Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}.$$

If we choose to use integration along the y-axis, we will have only one integral to calculate. However, we need to express x in terms of y:

$$x^2 + y^2 = 4 \Leftrightarrow x^2 = 4 - y^2 \Rightarrow x = \sqrt{4 - y^2},$$
$$(x-2)^2 + y^2 = 4 \Leftrightarrow (x-2)^2 = 4 - y^2 \Rightarrow x - 2 = -\sqrt{4 - y^2} \Rightarrow x = 2 - \sqrt{4 - y^2}$$

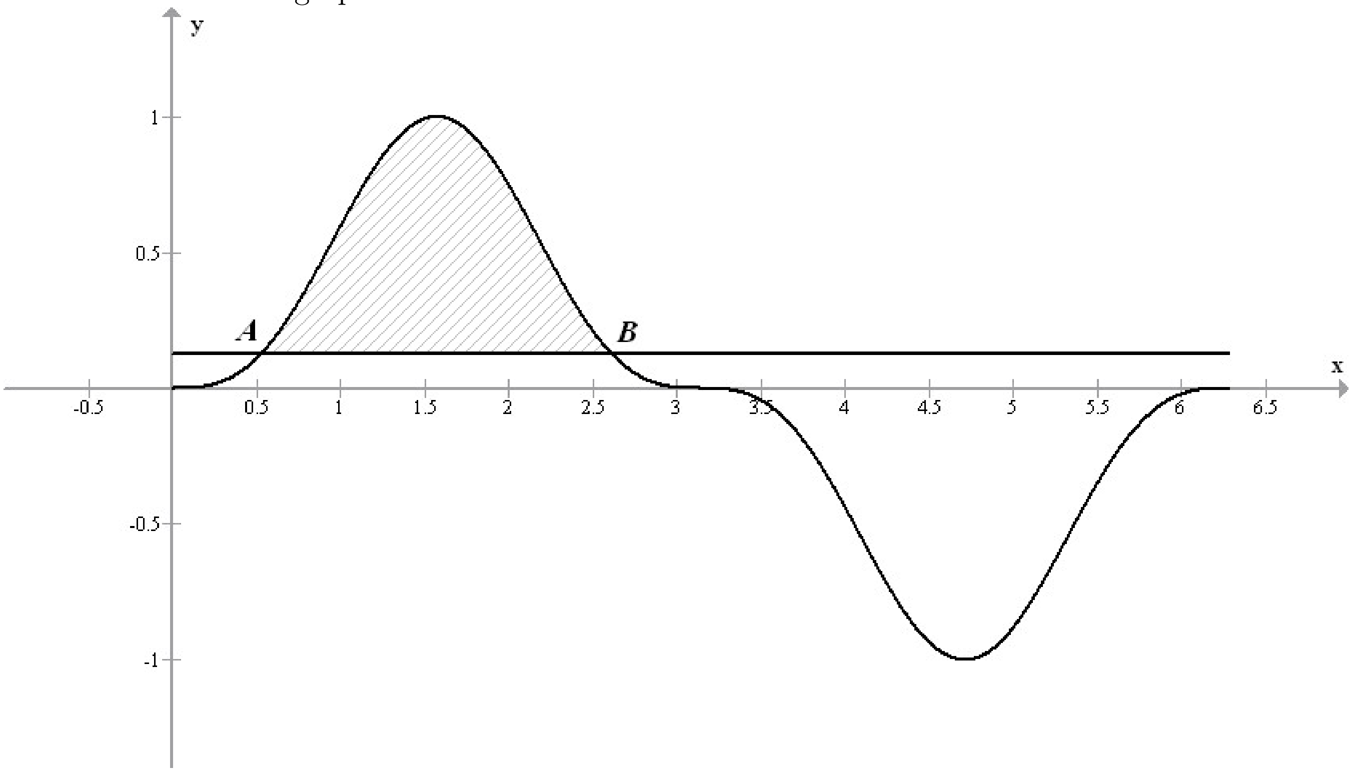
Note that we take positive sign in front of the square root in the first equation since $x \geq 0$ in the region, and we take negative sign in front of the square root in the second equation since $(x - 2) \leq 0$ is negative in the region.

Hence,

$$Area = \boxed{\int_{-\sqrt{3}}^{\sqrt{3}} \left[(2 - \sqrt{4 - y^2}) - (\sqrt{4 - y^2})\right] \, dy}$$

29. $y = \sin^3 x$, $y = \dfrac{1}{8}$, $0 \le x \le 2\pi$.

Solution: We sketch the graph:



We find the points of intersection first:

$$\begin{cases} y = \sin^3 x, \\ y = 1/8 \end{cases} \Rightarrow \sin^3 x = 1/8 \Rightarrow \sin x = 1/2 \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$$

Therefore,

$$Area = \int_{\pi/6}^{5\pi/6} \left[\sin^3 x - \frac{1}{8}\right] dx = \int_{\pi/6}^{5\pi/6} (1 - \cos^2 x) \sin x \, dx - \frac{1}{8}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$$

Make substitution $u = \cos x$, $du = -\sin x dx$, $\quad x = \pi/6 \to u = \dfrac{\sqrt{3}}{2}$, $x = 5\pi/6 \to u = -\dfrac{\sqrt{3}}{2}$

$$= \int_{\sqrt{3}/2}^{-\sqrt{3}/2} (1 - u^2)(-du) - \frac{\pi}{12} = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (1 - u^2) \, du - \frac{\pi}{12} = \left(u - \frac{u^3}{3}\right)\Bigg|_{-\sqrt{3}/2}^{\sqrt{3}/2} - \frac{\pi}{12}$$

$$= \sqrt{3} - \frac{\sqrt{3}}{4} - \frac{\pi}{12} = \boxed{\frac{3\sqrt{3}}{4} - \frac{\pi}{12}}$$