EXERCISES FOR CHAPTER 4: Alternating Series

Determine which of the following series converge absolutely, converge conditionally or diverge.

1. $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$ $\sum_{1}^{\infty}\frac{\sin(\frac{\pi n}{2})}{n^2}$ $\sum_{1}^{\infty}\frac{\cos\pi n}{n^4}$

Solution

(a) $\sum_{1}^{\infty}\frac{1}{n^2}$ converges so series is absolutely convergent.

(b) $\sum_{1}^{\infty}\frac{1}{n^2}$ converges so series is absolutely convergent.

(c) $\sum_{1}^{\infty}\frac{1}{n^4}$ converges so series is absolutely convergent.

2. $\sum_{2}^{\infty}(-1)^{n+1}\frac{\ln n}{n}$ $\sum_{1}^{\infty}\frac{\sin(\frac{\pi n}{2})}{n}$ $\sum_{1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$

Solution

(a) Terms are decreasing since $\frac{d}{dn}\frac{\ln n}{n}=\frac{1-\ln n}{n^2}<0$ and $\lim_{n\to\infty}\frac{\ln n}{n}=0$ so series converges conditionally since $\sum_{2}^{\infty}\frac{\ln n}{n}$ diverges.

(b) Terms are clearly decreasing and $\lim_{n\to\infty}\frac{\ln n}{n}=0$ so series converges conditionally since $\sum_{1}^{\infty}\frac{1}{n}$ diverges.

(c) Terms are decreasing since $\frac{d}{dn}\frac{n}{n^2+1}=\frac{n^2+1-2n^2}{(n+1)^2}<0$ and $\lim_{n\to\infty}\frac{n}{n^2+1}=0$ so series converges conditionally since $\sum_{1}^{\infty}\frac{n}{n^2+1}$ diverges.

3. $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}+1}$ $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt[3]{n}}$

Solution

(a) The terms are clearly deceasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0$ so the series converges. The series of positive terms, $\sum_{1}^{\infty}\frac{1}{\sqrt{n}}$ diverges being a *p* series with $p=\frac{1}{2}$. Hence $\sum_{1}^{\infty}(-1)^{n+1}\frac{1}{\sqrt{n}}$ converges conditionally.

(b) The terms are clearly deceasing and $\lim_{n\to\infty}\frac{1}{\sqrt{n}+1}=0$ so the series converges. The series of positive terms, $\sum_1^\infty \frac{1}{\sqrt{n}+1}$ diverges (by comparison with $\sum_1^\infty \frac{1}{\sqrt{n}}$). Hence $\sum_1^\infty (-1)^{n+1}\frac{1}{\sqrt{n}+1}$ converges conditionally.

(c) The terms are clearly deceasing and $\lim_{n\to\infty}\frac{1}{\sqrt[3]{n}}=0$ so the series converges. The series of positive terms, $\sum_1^\infty \frac{1}{\sqrt[3]{n}}$ diverges being a p series with $p=\frac{1}{3}$. Hence $\sum_1^\infty (-1)^{n+1}\frac{1}{\sqrt[3]{n}}$ converges conditionally.

4. $\sum_1^\infty (-1)^{n+1}\frac{1+n}{2+n}$ $\quad\sum_2^\infty (-1)^{n+1}\frac{1}{n^2\ln n}$ $\quad\sum_1^\infty (-1)^{n+1}\frac{n^2}{n^3+1}$

Solution

(a) $\frac{d}{dn}\frac{1+n}{2+n}=\frac{n+2-(n+1)}{(2+n)^2}=\frac{1}{(2+n)^2}>0$ and $\lim_{n\to\infty}\frac{1+n}{2+n}=1$ so series diverges.

(b) The terms are clearly deceasing and $\lim_{n\to\infty}\frac{1}{n^2\ln n}=0$ so the series converges. The series of positive terms $\sum_2^\infty \frac{1}{n^2\ln n}$ converges since $\frac{1}{n^2\ln n}<\frac{1}{n^2}$ and $\sum_2^\infty \frac{1}{n^2}$ being a p series with $p=2$. Hence $\sum_2^\infty (-1)^{n+1}\frac{1}{n^2\ln n}$ converges absolutely.

(c) $\frac{d}{dn}\frac{n^2}{n^3+1}=\frac{2n(n^3+1)-(n^2)3n^2}{(n^3+1)^2}=\frac{2n-n^4}{(n^3+1)^2}<0$ and $\lim_{n\to\infty}\frac{n^2}{n^3+1}=0$ so the series converges. The series of positive terms $\sum_1^\infty \frac{n^2}{n^3+1}$ diverges using the limit comparison test with $\sum_1^\infty \frac{1}{n}$ and so the series $\sum_1^\infty (-1)^{n+1}\frac{n^2}{n^3+1}$ converges conditionally.

5. $\sum_1^\infty (-1)^{n+1}\frac{1+3^n}{1+2^n}$ $\quad\sum_2^\infty (-1)^{n+1}\frac{\ln n}{2^n}$ $\quad\sum_2^\infty (-1)^{n}\frac{n}{\ln n}$

Solution

(a) $\lim_{n\to\infty}\frac{1+3^n}{1+2^n}=\infty$ so the series diverges.

(b) $\frac{d}{dn}\frac{\ln n}{2^n}=\frac{\frac{2^n}{n}-\ln n2^n\ln 2}{2^{2n}}=\frac{1-n\ln n2^n\ln 2}{n2^n}<0$ so the terms are decreasing. Since $\lim_{n\to\infty}\frac{\ln n}{2^n}=\lim_{n\to\infty}\frac{\frac{1}{n}}{2^n\ln 2}=0$ the series converges. The series of positive terms converges (use the ratio test) and so the series converges absolutely.

(c) $\lim_{n\to\infty}\frac{n}{\ln n}=\lim_{n\to\infty}\frac{1}{\frac{1}{n}}=\infty$ so series diverges.

6. $\sum_{1}^{\infty}(-1)^{n+1}\sqrt[n]{2}$ $\quad\sum_{2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ $\quad\sum_{1}^{\infty}(-1)^{n+1}\frac{e^n+1}{e^n-1}$

Solution

(a) $\lim_{n\to\infty}\sqrt[n]{2}=1$ so the series diverges.

(b) The terms are clearly deceasing and $\lim_{n\to\infty}\frac{1}{n\ln n}=0$ so the series converges. The series of positive terms diverges by the integral test so the series $\sum_{2}^{\infty}(-1)^{n+1}\frac{1}{n\ln n}$ converges conditionally.

(c) $\lim_{n\to\infty}\frac{e^n+1}{e^n-1}=1$ so the series diverges.

7. $\sum_{1}^{\infty}(-1)^{n+1}\frac{\sqrt[n]{2}}{n!}$ $\quad\sum_{1}^{\infty}(-1)^{n+1}\frac{n!}{n^n}$ $\quad\sum_{1}^{\infty}(-1)^{n+1}\frac{(n+1)^n}{(2n)^n}$

Solution

(a) The terms are clearly decreasing and $\lim_{n\to\infty}\frac{\sqrt[n]{2}}{n!}=0$ so the series converges. The series of positive terms converges by the ratio test so the series converges absolutely.

(b) The series converges absolutely by applying the ratio test to the series with positive terms:

$$\rho=\lim_{n\to\infty}\frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}}=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n\frac{n+1}{n+1}=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)^n$$

$$=e^{-1}<1$$

(c) The ratio test on the series with positive terms gives

$$\rho=\lim_{n\to\infty}\frac{\frac{(n+2)^{n+1}}{(2n+2)^{n+1}}}{\frac{(n+1)^n}{(2n)^n}}=\lim_{n\to\infty}\left(\frac{n+2}{n+1}\right)^{n+1}\frac{(2n)^{n+1}}{(2n+2)^{n+1}}\frac{n+1}{2n}$$

$$=\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}\frac{2^{n+1}}{2^{n+1}}\left(1-\frac{1}{n+1}\right)^{n+1}\frac{n+1}{2n}=e\times e^{-1}\times\frac{1}{2}=\frac{1}{2}<1$$

and so the series converges absolutely.

8. $\sum_{1}^{\infty}(-1)^{n+1}n^2\left(\frac{2}{3}\right)^n$ $\quad\sum_{1}^{\infty}(-1)^{n+1}\frac{(n+1)!-n!}{(n+2)!}$ $\quad\sum_{1}^{\infty}(-1)^{n+1}\frac{(n!)^2}{(2n)!}$

Solution

(a) By the ratio test on the positive series $\sum_{1}^{\infty}n^2\left(\frac{2}{3}\right)^n$,

$$\rho = \lim_{n\to\infty} \frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n} = \frac{2}{3} < 1$$

and so the series converges absolutely.

(b) $\frac{(n+1)!-n!}{(n+2)!} = \frac{n!(n+1-1)}{(n+2)!} = \frac{n}{(n+2)(n+1)}$. The terms are decreasing and

$\lim_{n\to\infty} \frac{n}{(n+2)(n+1)} = 0$ so the series converges. The series of positive terms

$\sum_1^\infty \frac{n}{(n+2)(n+1)}$ diverges by the limit comparison test with the series $\sum_1^\infty \frac{1}{n}$

and so the series converges conditionally.

(c) By the ratio test on the series with positive terms,

$$\rho = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{\frac{((n+1)!)^2}{(2n+2)!}}{\frac{(n!)^2}{(2n)!}} = \lim_{n\to\infty} \frac{(2n)!}{(2n+2)!} \frac{((n+1)!)^2}{(n!)^2} = \lim_{n\to\infty} \frac{(n+1)^2}{(2n+2)(2n+1)}$$

$$= \lim_{n\to\infty} \frac{1}{2} \frac{n+1}{2n+1} = \frac{1}{4} < 1$$

and so the series converges absolutely.

9. Consider the conditionally convergent series $\sum_1^\infty \frac{(-1)^{n+1}}{n}$ which is known to converge with sum $\ln 2$. Find the sum of the first 10 and then 11 terms and verify that the alternating series sum theorem is satisfied.

Solution

$S_{10} = 0.645635, \quad S_{11} = 0,736544, \quad S_\infty = \ln 2 \approx 0.693147$, so $S_{10} < S_\infty < S_{11}$.

10. (a) Show that the series $\sum_0^\infty \frac{(-1)^n}{(2n)!}$ converges absolutely. (b) How many terms must be kept in this series so that the sum agrees with the exact sum of $\cos 1$ (in radians) to 4 decimal places?

Solution

(a) Use ratio test.

(b)

$$\frac{1}{(2N+2)!} < 5\times 10^{-5}$$

$$(2N+2)! > \frac{1}{5\times 10^{-5}} = 2\times 10^4$$

$$N \geq 3$$

With N=3 (the sum begins with zero so we need 4 terms),

$$\sum_0^3 \frac{(-1)^n}{(2n)!} = 1 - \frac{1}{2} + \frac{1}{24} - \frac{1}{720} = 0.5402777 \approx 0.5403 \text{ and } \cos(1) = 0.5403023 \approx 0.5403.$$

11. (a) Show that the series $\sum_1^\infty (-1)^{n+1} \frac{(\pi/2)^{2n-1}}{(2n-1)!}$ converges absolutely. (b) How many terms must be kept in this series so that the sum agrees with the exact sum which is 1, to 4 decimal places?

Solution

(a) Use ratio test.

(b)

$$\frac{(\pi/2)^{2N+1}}{(2N+1)!} < 5\times 10^{-5}$$

$$\frac{(2N+1)!}{(\pi/2)^{2N+1}} > \frac{1}{5\times 10^{-5}} = 2\times 10^4$$

$$\Rightarrow N \ge 5$$

With N=5, $S_5 = \sum_1^5 (-1)^{n+1} \frac{(\pi/2)^{2n-1}}{(2n-1)!} = \frac{\frac{\pi}{2}}{1} - \frac{\left(\frac{\pi}{2}\right)^3}{6} + \frac{\left(\frac{\pi}{2}\right)^5}{120} - \frac{\left(\frac{\pi}{2}\right)^7}{5040} + \frac{\left(\frac{\pi}{2}\right)^9}{362{,}880} = 1.000000.$

12. (a) Show that the series $\sum_0^\infty \frac{(-1)^n}{n!}$ converges absolutely. (b) How many terms must be kept in this series so that the sum agrees with the exact sum of e^{-1} to 5 decimal places?

Solution

$$\frac{1}{(N+1)!} < 5\times 10^{-6}$$

$$(N+1)! > \frac{1}{5\times 10^{-6}} = 2\times 10^5$$

$$\Rightarrow N+1 \ge 9$$

$$\Rightarrow N \ge 8$$

With N=8 (the sum begins with $n=0$ so we need 9 terms),

$$\sum_0^8 (-1)^n \frac{1}{n!} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \frac{1}{40320} = 0.367882 \approx 0.36788.$$

The exact answer is $e^{-1} \approx 0.367879 \approx 0.36788$ to five decimal places, in agreement with the answer above.

13. (a) Show that the series $\sum_1^\infty \frac{(-1)^{n+1}}{(2n-1)^3}$ converges absolutely. (b) Estimate the sum of this series by keeping 5 terms. (c) Confirm that your estimate is consistent with the alternating series sum theorem. (The exact sum is $\frac{\pi^3}{32}$.)

Solution

(a) The series of positive terms $\sum_{1}^{\infty}\frac{1}{(2n-1)^3}$ converges by the integral test. The terms are decreasing and $\int_1^{\infty}\frac{1}{(2n-1)^3}dk=-\lim_{\Lambda\to\infty}\frac{1}{4(2k-1)^2}\Big|_1^{\Lambda}=\frac{1}{4}$. (b) $\sum_{1}^{5}\frac{(-1)^{n+1}}{(2n-1)^3}=0.969419$.

(c) The next term neglected is $\frac{(-1)^7}{(2\times 3-1)^3}=-0.000751315$. The error is $\frac{\pi^3}{32}-0.969419\approx -0.000473107$ which is less than $\frac{(-1)^7}{(2\times 3-1)^3}$ in absolute value and has the same sign as the neglected term.

14. What is wrong with this argument?

Let $S=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$. We know that $S=\ln 2$. Then re-arranging the terms as

$$S=\left(1-\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{5}-\frac{1}{10}-\frac{1}{12}\right)+\cdots$$

and adding the first two terms in each bracket we get

$$\begin{aligned}S&=\left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{6}-\frac{1}{8}\right)+\left(\frac{1}{10}-\frac{1}{12}\right)+\cdots\\&=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots\right)\\&=\frac{1}{2}\ln 2\end{aligned}$$

Solution

The series is only conditionally convergent and so it is not permitted to rearrange the terms.

15. Rearrange the terms of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ in order to get a sum of zero.

Solution

$1>0$

$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}=-\frac{1}{24}<0$

$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}=\frac{7}{24}>0$

$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}=-\frac{43}{1680}<0$

$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}=\frac{293}{1680}>0$

Therefore the series with sum zero begins as

$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\cdots$.