Practice Midterm 2B Answers

Part A

1. D
2. B
3. E
4. D
5. A
6. B
7. A
8. D
9. A
10. A
11. B
12. A
13. C
14. B
15. A

Part B

1. (a) The complete list of outcomes in the sample space is shown below, together with their respective probabilities.

Outcome	Probability
00	(0.2)(0.2) = 0.04
01	(0.2)(0.5) = 0.10
02	(0.2)(0.3) = 0.06
10	(0.5)(0.2) = 0.10
11	(0.5)(0.5) = 0.25
12	(0.5)(0.3) = 0.15
20	(0.3)(0.2) = 0.06
21	(0.3)(0.5) = 0.15
22	(0.3)(0.3) = 0.09

(b) $P(X = 0) = P(00) + P(01) + P(02) + P(10) + P(20) = 0.04 + 0.10 + 0.06 + 0.10 + 0.06 = 0.36$
$P(X = 1) = P(11) = 0.25$
$P(X = 2) = P(12) + P(21) = 0.15 + 0.15 = 0.30$
$P(X = 4) = P(22) = 0.09$

The p.m.f. of X is shown in tabular form below:

x	0	1	2	4
P(X = x)	0.36	0.25	0.30	0.09

(c)
$$F(x) = \begin{cases} 0, & x < 0 \\ 0.36, & 0 \le x < 1 \\ 0.61 & 1 \le x < 2 \\ 0.91 & 2 \le x < 4 \\ 1, & x > 4 \end{cases}$$

2. (a) $$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{25}^{x} \frac{1}{10\sqrt{y}}\,dy = \left[\frac{\sqrt{y}}{5}\right]_{25}^{x} = \frac{\sqrt{x}}{5} - 1 \text{ for } 25 \le x \le 100.$$

We also know that $F(x) = 0$ for $x < 25$ and $F(x) = 1$ for $x > 100$.

(b) The third quartile of the distribution of X is the value x such that

$$F(x) = 0.75 \quad \Rightarrow \quad \frac{\sqrt{x}}{5} - 1 = 0.75 \quad \Rightarrow \quad \sqrt{x} = 5(0.75 + 1) = 8.75 \quad \Rightarrow \quad x = (8.75)^2 = 76.56$$

(c) We can calculate this probability in either of two ways:

(i) Using the p.d.f.:

$$P(36 < X < 81) = \int_{36}^{81} f(x)\,dx = \int_{36}^{81} \frac{1}{10\sqrt{x}}\,dx = \left[\frac{\sqrt{x}}{5}\right]_{36}^{81} = \frac{\sqrt{81} - \sqrt{36}}{5} = \frac{9-6}{5} = \frac{3}{5} = 0.6$$

(ii) Using the c.d.f.:

$$P(36 < X < 81) = F(81) - F(36) = \left(\frac{\sqrt{81}}{5} - 1\right) - \left(\frac{\sqrt{36}}{5} - 1\right) = 1.8 - 1.2 = 0.6$$

(d) $$E(X) = \int_{-\infty}^{\infty} x\,f(x)\,dx = \int_{25}^{100} \frac{\sqrt{x}}{10}\,dx = \left[\frac{x^{3/2}}{15}\right]_{25}^{100} = \frac{(100)^{3/2}}{15} - \frac{(25)^{3/2}}{15} = \frac{1000 - 125}{15} = \frac{875}{15} = 58.33$$

3. (a) $$P(X > s + t \mid X > s) = \frac{P(X > s \ \cap \ X > s + t)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)} = \frac{\sum_{x=s+t+1}^{\infty}(1-p)^{x-1}p}{\sum_{x=s+1}^{\infty}(1-p)^{x-1}p}$$

For any integer k > 0,

$$P(X > k) = \sum_{x=k+1}^{\infty}(1-p)^{x-1}p$$

This is an infinite geometric series with first term $a = (1 - p)^k p$ and multiplicative term $r = 1 - p$. Therefore,

$$\sum_{x=k+1}^{\infty}(1-p)^{x-1}p = \frac{(1-p)^k p}{1-(1-p)} = \frac{(1-p)^k p}{p} = (1-p)^k$$

It follows that

$$P(X > s + t \mid X > s) = \frac{\sum_{x=s+t+1}^{\infty}(1-p)^{x-1}p}{\sum_{x=s+1}^{\infty}(1-p)^{x-1}p} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t = P(X > t)$$

(b) $$P(X > s + t \mid X > s) = \frac{P(X > s \;\cap\; X > s+t)}{P(X > s)} = \frac{P(X > s+t)}{P(X > s)} = \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{s}^{\infty} \lambda e^{-\lambda x}\, dx}$$

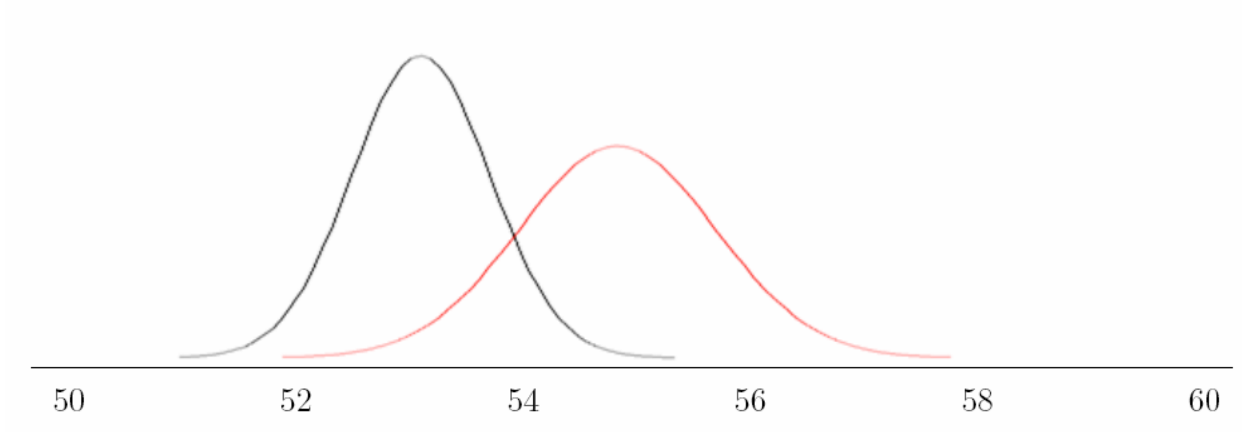
For any integer $k > 0$,

$$P(X > k) = \int_{k}^{\infty} \lambda e^{-\lambda x}\, dx = \left[-e^{-\lambda k}\right]_{k}^{\infty} = e^{-\lambda k}$$

It follows that

$$P(X > s + t \mid X > s) = \frac{\int_{s+t}^{\infty} \lambda e^{-\lambda x}\, dx}{\int_{s}^{\infty} \lambda e^{-\lambda x}\, dx} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda s}} = e^{-\lambda t} = P(X > t)$$

4. (a) The density curves for X_F (in black) and X_B (in red) are shown below:



Note the following:

- Since the standard deviation for X_B is greater than that for X_F , the red curve is wider than the black curve.
- Since the area under both curves must be equal to one, and since the red curve is wider, the height of the black curve must be greater.
- Both curves extend out from the mean by about three standard deviations in either direction until they get very close to the x-axis, although they never touch the x-axis.

(b) $P(52.5 < X_F < 54.0) = P\left(\frac{52.5-53}{0.7} < Z < \frac{54-53}{0.7}\right) = P(-0.71 < Z < 1.43)$
$= P(Z < 1.43) - P(Z < -0.71) = 0.9236 - 0.2389 = 0.6847$

(c) $P(X_B < X_F) = P(X_B - X_F < 0)$. We find the distribution of $X_B - X_F$ as follows: Since X_B and X_F are independent,

$$X_B - X_F \sim N\left(\mu_B - \mu_F, \sqrt{\sigma_B^2 + \sigma_F^2}\right)$$

$$\Rightarrow \quad X_B - X_F \sim N\left(55-53, \sqrt{(1.0)^2 + (0.7)^2}\right)$$

$$\Rightarrow \quad X_B - X_F \sim N(2, 1.22)$$

And so the probability the swimmer has a faster time in the butterfly race is

$$P(X_B - X_F < 0) = P\left(Z < \frac{0-2}{1.22}\right) = P(Z < -1.64) = 0.0505$$

(d) To determine which time was best relative to his previous times, we calculate the swimmer's z-score for his two races:

$$z_F = \frac{52.3 - 53}{0.7} = -1.00 \qquad z_B = \frac{53 - 55}{1.0} = -2.00$$

Since the swimmer's z-score is lower for the butterfly race, he did better in the butterfly relative to his previous times.

(To see why this is true, we calculate that the proportion of freestyle races the swimmer finishes in under 52.3 seconds is $P(Z < -1.00) = 0.1587$ and the proportion of freestyle races the swimmer finishes in under 53 seconds is $P(Z < -2.00) = 0.0228$.)