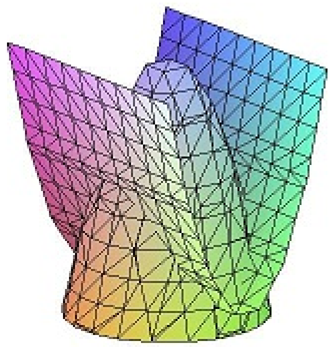
MATH 2130 Summer Evening 2013 Problem Workshop 7 Solutions

1. Even though $z = 9 - x^2 - y^2$ would work well in cylindrical coordinates, the parabolic cylinder $z = x^2$ would not. Hence we'll start in cartesian coordinates.

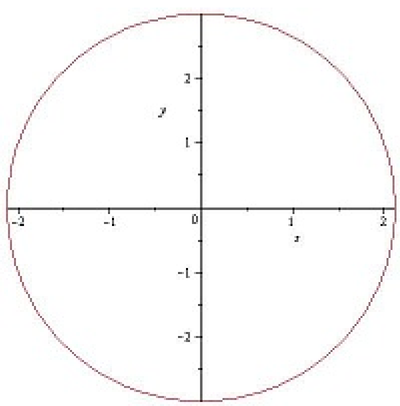
   The paraboloid $z = 9 - x^2 - y^2$ has a z-coordinate of 9 and opens down whereas $z = x^2$ opens up. Therefore the paraboloid would be on top and the parabolic cylinder is on the bottom. The sketch appears below.

   Hecne we have the bounds on z to be $x^2 \leq z \leq 9 - x^2 - y^2$. As for the bounds on x and y, we set the curves equal to each other to get

   $$9 - x^2 - y^2 = x^2 \Rightarrow 2x^2 + y^2 = 9 \Rightarrow \frac{x^2}{9/2} + \frac{y^2}{9} = 1$$

   Hence the region is an ellipse in the xy plane which appears below.

   Now unless you want to make a variable change so that we have a circle instead of an ellipse (which is beond the scope of our course) changing to polar will not be very helpful. Hence we will stay in cartesian.

   Now we can either use symmetry noting that both $9 - x^2 - y^2$ and x^2 are even functions of both x and y. Hence we only need to take the first quadrant and multiply by 4. Solving for y in terms of x in the ellipse would yield

$$y^2 = 9 - 2x^2 \Rightarrow y = \pm\sqrt{9-2x^2}.$$

As for bounds on x, we note that

$$2x^2 \leq 9 \Rightarrow x^2 \leq \frac{9}{2} \Rightarrow -\frac{3}{\sqrt{2}} \leq x \leq \frac{3}{\sqrt{2}}.$$

Hence if we use symmetry we get

$$4\int_0^{3/\sqrt{2}} \int_0^{\sqrt{9-2x^2}} \int_{x^2}^{9-x^2-y^2} dzdydx$$

and if we don't use symmetry we get

$$\int_{-3/\sqrt{2}}^{3/\sqrt{2}} \int_{-\sqrt{9-2x^2}}^{\sqrt{9-2x^2}} \int_{x^2}^{9-x^2-y^2} dzdydx$$

2. Setting up this volume can be tricky. Note that the function $z = xy$ is negative when $x < 0$ or $y < 0$ but not both, and positive otherwise. Hence when setting up the integral, we need different integrals for these regions. Now based on the cylinder $x^2 + y^2 = 1$, we are going to put this in cylindrical coordinates. Hence we have

$$0 \leq z \leq xy = r^2 \sin\theta\cos\theta \text{ when } 0 \leq \theta \leq \frac{\pi}{2}, \pi \leq \theta \leq \frac{3\pi}{2}$$

and

$$xy = r^2 \sin\theta\cos\theta \leq z \leq 0 \text{ when } \frac{\pi}{2} \leq \theta \leq \pi, \frac{3\pi}{2} \leq \theta \leq 2\pi.$$

Hence we get

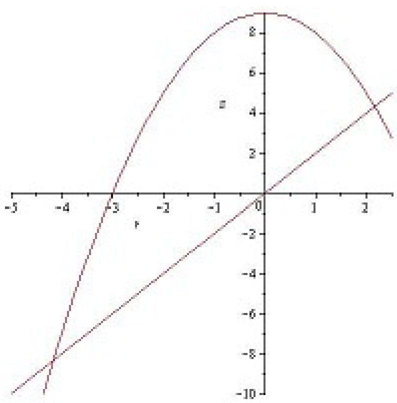
$$\begin{aligned} Vol = &\int_0^{\pi/2} \int_0^1 \int_0^{r^2\sin\theta\cos\theta} rdzdrd\theta + \int_{\pi/2}^{\pi} \int_0^1 \int_{r^2\sin\theta\cos\theta}^{0} rdzdrd\theta \\ &+ \int_{\pi}^{3\pi/2} \int_0^1 \int_0^{r^2\sin\theta\cos\theta} rdzdrd\theta + \int_{3\pi/2}^{2\pi} \int_0^1 \int_{r^2\sin\theta\cos\theta}^{0} \end{aligned}$$

Now we can evaluate all of these, or we can use symmetry to notice that they will all be the same. Either way we will get 4 times

$$\int_0^{\pi/2}\int_0^1\int_0^{r^2\sin\theta\cos\theta} r\,dz\,dr\,d\theta = \int_0^{\pi/2}\int_0^1 r^3\sin\theta\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/2} \frac{r^4}{4}\Big]_0^1 \sin\theta\cos\theta\,d\theta$$
$$= \int_0^{\pi/2} \frac{1}{4}\sin\theta\cos\theta\,d\theta$$
$$= \frac{1}{4}\frac{\sin^2\theta}{2}\Big]_0^{\pi/2}$$
$$= \frac{1}{4}\left(\frac{\sin^2(pi/2)}{2} - \frac{\sin^2(0)}{2}\right)$$
$$= \frac{1}{8}.$$

Therefore the total volume is $4(1/8) = 1/2$.

3. Both of these regions are better with cylindrical coordinates. The cone becomes $z = 2r$ and the paraboloid becomes $z = 9 - r^2$. Graphing the two functions (of z as a function of r) yields the picture



Hence the paraboloid is on top and the cone below (or you could graph it 3 dimensionally).

Therefore $2r \le z \le 9 - r^2$. Now finding where the cone and the parabolid meet gives

$$2r = 9 - r^2 \Rightarrow r^2 + 2r - 9 = 0 \Rightarrow r = \frac{-2 \pm \sqrt{4+36}}{2} = -1 + \sqrt{10}$$

using that r is positive. Hence the intersection is a circle

$$0 \le r \le \sqrt{10} - 1, \qquad 0 \le \theta \le 2\pi.$$

Therefore

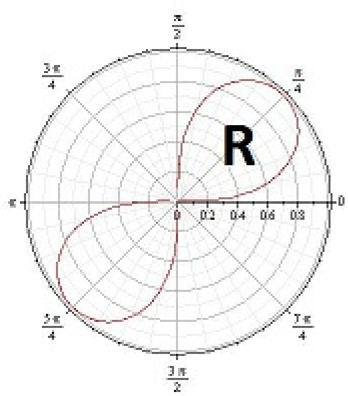
$$\begin{aligned}
Vol &= \int_0^{2\pi} \int_0^{\sqrt{10}-1} \int_{2r}^{9-r^2} r dr dz d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{10}-1} (9r - r^3 - 2r^2) dz d\theta \\
&= \int_0^{2\pi} \left(\frac{9}{2}r^2 - \frac{1}{4}r^4 - \frac{2}{3}r^3 \right]_0^{\sqrt{10}-1} d\theta \\
&= \int_0^{2\pi} \left(\frac{9}{2}(\sqrt{10}-1)^2 - \frac{1}{4}(\sqrt{10}-1)^4 - \frac{2}{3}(\sqrt{10}-1)^3 \right) d\theta \\
&= 2\pi \left(\frac{9}{2}(\sqrt{10}-1)^2 - \frac{1}{4}(\sqrt{10}-1)^4 - \frac{2}{3}(\sqrt{10}-1)^3 \right).
\end{aligned}$$

4. The sphere can work in either cylindrical or spherical coordinates. However $(x^2+y^2)^2 = 2xy$ is a function for which we use polar coordinates.

$$(r^2)^2 = 2(r\cos\theta)(r\sin\theta) \Rightarrow r^2 = 2\sin\theta\cos\theta = \sin(2\theta) \Rightarrow r = \sqrt{\sin(2\theta)}.$$

Note that $r < 1$, so the sphere hits the equation $r = \sqrt{\sin(2\theta)}$ before the xy-plane. Hence we are finding the volume where $0 \leq z \leq \sqrt{1 - x^2 - y^2} = \sqrt{1 - r^2}$.

As for the bounds on r, θ the region s just the region inside $r = \sqrt{\sin(2\theta)}$ which has the graph



Note that the region in the third quadrant matches the region in the first, the sphere is symmetric as well as the function x^2. Hence the integral over those pieces are equal and we can take 2 times the integral in the first octant. Using $x = r\cos\theta$, and the region

$$0 \leq r \leq \sqrt{\sin(2\theta)}, \qquad 0 \leq \theta \leq \pi/2,$$

we get the integral is

$$2 \int_0^{\pi/2} \int_0^{\sqrt{\sin(2\theta)}} \int_0^{\sqrt{1-r^2}} (r\cos\theta)^2 r dz dr d\theta.$$

5. In this case we want to use spherical coordinates since we are using spheres. Changing the equations to spherical we notice the small sphere has equation

$$x^2 + y^2 + z^2 = 9 \Rightarrow \mathcal{R}^2 = 9 \Rightarrow \mathcal{R} = 3.$$

Similarly for the other hemisphere we have $\mathcal{R} = 4$. Since both hemispheres have $y \geq 0$ and the region is bounded by the xz plane, we need to restrict the region to the values where $y \leq 0$. Hence we have the region is

$$3 \leq \mathcal{R} \leq 4, \qquad 0 \leq \phi \leq \pi, \qquad 0 \leq \theta \leq \pi.$$

Using that $x = r\cos\theta = \mathcal{R}\sin\phi\cos\theta$ and $dV = \mathcal{R}^2 \sin\phi d\mathcal{R}d\phi d\theta$, we get the integral is

$$\begin{aligned}
I &= \int_0^\pi \int_0^\pi \int_3^4 (\mathcal{R}\sin\phi\cos\theta)^2(\mathcal{R}^2\sin\phi d\mathcal{R}d\phi d\theta) \\
&= \int_0^\pi \int_0^\pi \int_3^4 \mathcal{R}^4 \sin^3\phi\cos^2\theta d\mathcal{R}d\phi d\theta \\
&= \int_0^\pi \int_0^\pi \left(\frac{1}{5}\mathcal{R}^5 \sin^3\phi\cos^2\theta\right]_3^4 d\phi d\theta \\
&= \int_0^\pi \int_0^\pi \frac{4^5 - 3^5}{5} \sin^3\phi\cos^2\theta d\phi d\theta
\end{aligned}$$

Using $\sin^2\phi = 1 - \cos^2\phi$, we can use the substitution $w = \cos\phi, dw = \sin\phi$ to get

$$\int \sin^3\phi d\phi = \int (1 - w^2)(-dw) = \frac{w^3}{3} - w + C = \frac{\cos^3\phi}{3} - \cos\phi + C.$$

So the integral becomes

$$\begin{aligned}
I &= \int_0^\pi \int_0^\pi \frac{4^5 - 3^5}{5} \sin^3\phi\cos^2\theta d\phi d\theta \\
&= \frac{781}{5}\int_0^\pi \left(\frac{\cos^3\phi}{3} - \cos\phi\right]_0^\pi \cos^2\theta d\theta \\
&= \frac{781}{5}\int_0^\pi \left[\left(\frac{-1}{3} - (-1)\right) - \left(\frac{1}{3} - (1)\right)\right] \cos^2\theta d\theta \\
&= \frac{781}{5}\int_0^\pi \frac{4}{3}\cos^2\theta d\theta
\end{aligned}$$

Now using $\cos^2\phi = \frac{1+\cos 2\theta}{2}$ we get that

$$\int \cos^2\theta d\theta = \int \frac{1 + \cos 2\theta}{2} d\theta = \frac{\theta}{2} + \frac{\sin 2\theta}{4} + C.$$

So the integral becomes

$$\begin{aligned}
I &= \frac{781}{5}\int_0^\pi \frac{4}{3}\cos^2\theta d\theta \\
&= \frac{3124}{15}\left(\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^\pi \\
&= \frac{3124}{15}\left[\left(\frac{\pi}{2}+\frac{\sin 2\pi}{4}\right)-\left(\frac{0}{2}+\frac{\sin 0}{4}\right)\right] \\
&= \frac{3124}{15}\left(\frac{\pi}{2}\right) \\
&= \frac{1562}{15}\pi.
\end{aligned}$$

6. The region is symmetric about both the x-axis and the y-axis, hence the x and y coordinate of the centroid is 0. Hence we are only concerned with the z-coordinate. First we need to find the volume. Since we are working with a cone and a sphere, we will use spherical coordinates.

   The cone becomes

$$z=\sqrt{x^2+y^2} \Rightarrow z=r \Rightarrow \mathcal{R}\sin\phi = \mathcal{R}\cos\phi \Rightarrow \tan\phi = 1 \Rightarrow \phi = \frac{\pi}{4}.$$

   The sphere becomes $\mathcal{R}^2=1 \Rightarrow \mathcal{R}=1$. Therefore we get

$$0\le \mathcal{R}\le 1, \qquad 0\le\phi\le\frac{\pi}{4}, \qquad 0\le\theta\le 2\pi.$$

   The volume is therefore

$$\begin{aligned}
Vol &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \mathcal{R}^2\sin\phi d\mathcal{R}d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\mathcal{R}^3\sin\phi\Big]_0^1 d\phi d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi d\phi d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\int_0^{\pi/4} -\cos\phi]_0^{\pi/4} d\theta \\
&= \frac{1}{3}\int_0^{2\pi}\left(1-\frac{1}{\sqrt{2}}\right)d\theta \\
&= \frac{2\pi}{3}\left(1-\frac{1}{\sqrt{2}}\right) \\
&= \frac{\pi}{3}(2-\sqrt{2}).
\end{aligned}$$

As for the centroid, we get that

$$Vol \cdot \overline{z} = M_{xy} = \iiint_V z dV$$

(We also get similar equations for $\overline{x}$ and $\overline{y}$, but by symmetry, $M_{xz} = M_{yz} = 0$, or it can be calculated.) Finding M_{xy} requires $z = \mathcal{R}\cos\phi$. Hence we get

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 (\mathcal{R}\cos\phi)\mathcal{R}^2 \sin\phi d\mathcal{R}d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \mathcal{R}^3 \sin\phi\cos\phi d\mathcal{R}d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{4}\mathcal{R}^4\sin\phi\cos\phi\Big]_0^1 d\phi d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4} \sin\phi\cos\phi d\phi d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4} \frac{\sin^2\phi}{2}\Big]_0^{\pi/4} d\theta \\
&= \frac{1}{4}\int_0^{2\pi}\int_0^{\pi/4} \left(\frac{\sin^2(\pi/4))}{2} - \frac{\sin^2(0))}{2}\right) d\theta \\
&= \frac{1}{4}\int_0^{2\pi} \frac{1}{4} d\theta \\
&= \frac{\pi}{8}.
\end{aligned}$$

Hence

$$\overline{z} = \frac{M_{xy}}{Vol} = \frac{\frac{\pi}{8}}{\frac{\pi}{3}(2-\sqrt{2})} = \frac{3}{8(2-\sqrt{2})}.$$