MATH 2130 Problem Workshop 3 Solutions

1. Note, especially for the limits which don't exist, there tends to be many paths that show different values. I'm including some here, but of course you may have used different ones.

   (a) First we try plugging in $(0,0)$, but we get the indeterminant form $0/0$. Then we try simplifying, but there doesn't appear to be anything, Next, see if there is a nice substitution, but again there isn't. Let's try some paths. Along $x = 0$,

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2 + 2y^2} = \lim_{y\to 0} \frac{0^2 - 0y + y^2}{0^2 + 2y^2} = \lim_{y\to 0} \frac{y^2}{2y^2} = \frac{1}{2}.$$

   Along $y = 0$,

   $$\lim_{(x,y)\to(0,0)} \frac{x^2 - xy + y^2}{x^2 + 2y^2} = \lim_{x\to 0} \frac{x^2 - x(0) + 0^2}{x^2 + 2(0)^2} = \lim_{y\to 0} \frac{x^2}{x^2} = 1.$$

   Therefore since we get two different value along two different paths, we get that there is no limit.

   (b) First we try plugging in $(0,0)$, but we get the indeterminant form $0/0$. Then we try simplifying, but there doesn't appear to be anything, Next, see if there is a nice substitution, but again there isn't. Let's try some paths. Along $x = 0, y = 0$, we get 0 both times. Trying along any line $y = mx$,

   $$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^3} = \lim_{x\to 0} \frac{mx^4}{x^6 + m^3x^3} = \lim_{x\to 0} \frac{mx}{x^3 + m^3} = 0$$

   unless $m = 0$ but then we had the original path $y = 0$ anyways. Trying along $y = mx^2$ yields

   $$\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6 + y^3} = \lim_{x\to 0} \frac{mx^5}{x^6 + m^3x^6} = \lim_{x\to 0} \frac{m}{x(1 + m^3)}$$

   which does not exist. Therefore the limit doesn't exist.

   (c) First we try plugging in $(2,-3)$, but we get the indeterminant form $0/0$. Then we try simplifying, but there doesn't appear to be anything, Next, see if there is a nice substitution, but again there isn't. Let's try some paths. Along $x = 2$, (note that since we are going to the point $(2,-3)$ we won't go along $x = 0$, we get

   $$\lim_{(x,y)\to(2,-3)} \frac{x^2 - 4x - y^2 - 6y - 5}{x^2 - 4x + y^2 + 6y + 13} = \lim_{y\to -3} \frac{-y^2 - 6y - 9}{y^2 + 6y + 9} = -1$$

   Along $y = -3$,

   $$\lim_{(x,y)\to(2,-3)} \frac{x^2 - 4x - y^2 - 6y - 5}{x^2 - 4x + y^2 + 6y + 13} = \lim_{x\to 2} \frac{x^2 - 4x + 4}{x^2 - 4x + 4} = 1.$$

   Therefore since we get two different value along two different paths, we get that there is no limit.

(d) First we try plugging in $(0,0)$, but we get the indeterminant form $0/0$. Then we try simplifying, but there doesn't appear to be anything, Next, see if there is a nice substitution. In this case if we let $u = 2x - 3y$, then we can turn the function into a single variable limit. As $(x,y) \to (3,2)$, $u \to 0$, so our limit becomes

$$\lim_{u\to 0} \frac{\sin u}{u} = 1.$$

(e) First we try plugging in $(0,0)$, but we get the indeterminant form $0/0$. Then we try simplifying, but there doesn't appear to be anything, Next, see if there is a nice substitution, but again there isn't. Trying some paths leads to getting 0 for a while, so we may suspect the limit exists and is equal to 0. Using the squeeze theorem requires us to find bounds on the function

$$0 \le \frac{x^4 y^4}{x^4 + y^4} \le \frac{x^4 y^4}{x^4} \le y^4$$

Since both 0 and y^4 go to 0 as $(x,y) \to (0,0)$, we know that the limit exists and is 0 by the squeeze theorem.

(f) Plugging in $(0,1)$ gives us a $1/0$ form for y/x, meaning that y/x goes to infinity or negative infinity (which it will do both depending on the path, see the next question). However $|y/x| \to \infty$ meaning that

$$\lim_{(x,y)\to(0,1)} \arctan\left|\frac{y}{x}\right| = \frac{\pi}{2}.$$

(g) This starts the same as the previous question, except that there is no absolute value to force $y/x \to \infty$. In fact even if we look along the path $y = 1$ we get that $y/x \to \infty$ as $x \to 0^+$ and $y/x \to -\infty$ as $x \to 0^-$. Therefore along different paths we get that

$$\arctan\left(\frac{y}{x}\right) \to \pi/2 \text{ or } \arctan\left(\frac{y}{x}\right) \to -\pi/2$$

making the limit not exist.

(h) Plugging in $(0,1)$ gives $\arctan\left(\frac{0}{1}\right) = 0$.

2. We must show the left hand side matched the right hand side when we use the given function. First we'll find the two partial deriatives and then plug them into the left hand side to see what happens.

$$\frac{\partial f}{\partial x} = 6x + y^2\left(-\sin\left(\frac{2x}{y}\right)\frac{2}{y}\right) = 6x - 2y\sin\left(\frac{2x}{y}\right)$$

and

$$\frac{\partial f}{\partial y} = 2y\cos\left(\frac{2x}{y}\right) + y^2\left(-\sin\left(\frac{2x}{y}\right)\left(-\frac{2x}{y^2}\right)\right) = 2y\cos\left(\frac{2x}{y}\right) + 2x\sin\left(\frac{2x}{y}\right)$$

Hence

$$
\begin{aligned}
x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} &= x\left(6x - 2y\sin\left(\frac{2x}{y}\right)\right) + y\left(2y\cos\left(\frac{2x}{y}\right) + 2x\sin\left(\frac{2x}{y}\right)\right) \\
&= 6x^2 - 2xy\sin\left(\frac{2x}{y}\right) + 2y^2\cos\left(\frac{2x}{y}\right) + 2xy\sin\left(\frac{2x}{y}\right) \\
&= 6x^2 + 2y^2\cos\left(\frac{2x}{y}\right) \\
&= 2f(x, y).
\end{aligned}
$$