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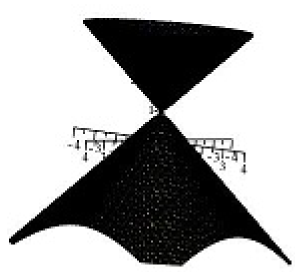
[4] 1. (a) *For the curve, $z^2 - 2z = x^2 + y^2 - 1$, identify the type of curve and give a sketch.*

Solution:
Completing the square yields

$$z^2 - 2z + 1 = x^2 + y^2 - 1 + 1 \Rightarrow (z-1)^2 = x^2 + y^2$$

which is a cone.



[4] (b) *Determine the projection of $x^2 = z^2 - 4$, $x^2 + y^2 - z^2 - 2z = 0$ in the yz-plane.*

Solution:
$x = \pm\sqrt{z^2 - 4}$ which inserting into the other equation yields

$$z^2 - 4 + y^2 - z^2 - 2z = 0 \Rightarrow y^2 - 2z - 4 = 0 \Rightarrow z = \frac{y^2 - 4}{2}$$

However since $z^2 - 4 \geq 0$, we also have the added restriction that $zge2$.) Hence the projection is

$$z = \frac{y^2 - 4}{2}, \; x = 0, \; z \geq 2.$$

Note that $z \leq -2$ also satisifes $z^2 - 4 \geq 0$, however since $z = (y^2 - 4)/2$, z cannot be less than -2.

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2. *Let l_1 be the line*

$$x - 5 = \frac{y+3}{-2} = \frac{z-4}{3}$$

and l_2 be the line

$$x = 7 + 2t,\ y = -5 - 3t,\ z = 8 + 5t.$$

[3] (a) *Show that the lines are intersecting and determine the point of intersection.*

Solution:
Inserting x, y and z from the second equation into the first one we can get

$$7 + 2t - 5 = \frac{-5 - 3t + 3}{-2} = \frac{8 + 5t - 4}{3}.$$

Hence the lines intersect if there exists some value of t which makes all of these equal to each other, and they don't intersect if there isn't any such t.
Equating the first two.

$$2t + 2 = \frac{-3t - 2}{-2} \Rightarrow -4t - 4 = -3t - 2 \Rightarrow -t = 2 \Rightarrow .$$

Inserting into the third part verifies that all 3 are -2. Hence the equation is satisifed.
$t = -2 \Rightarrow x = 3,\ y = 1,\ z = -2$ and thus the point of intersection is $P(3, 1, -2)$.

[3] (b) *Determine the cosine of the smallest angle between the lines.*

Solution:
A vector parallel to both lines are $\mathbf{u} = \langle 1, -2, 3\rangle$ and $\mathbf{v} = \langle 2, -3, 5\rangle$ respectively. The cosine of the angle between them is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|} = \frac{2 + 6 + 15}{\sqrt{14}\sqrt{38}} = \frac{23}{\sqrt{14}\sqrt{38}}.$$

[5] (c) Determine an equation of the plane containing both lines.

Solution:
Two vector parallel to the plane are the vectors parallel to both lines which are $\mathbf{u} = \langle 1, -2, 3\rangle$ and $\mathbf{v} = \langle 2, -3, 5\rangle$ respectively. Hence a vector normal to the plane is

$$\mathbf{u}\times\mathbf{v} = \langle -1, 1, 1\rangle.$$

Now we can take any point on the plane which could be any point on either line, say the $P(3, 1, -2)$ we found in part (a). Hence an equation of the plane is

$$-1(x - 3) + 1(y - 1) + 1(z + 2) = 0.$$

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3. *Let l_1 be the line with symmetric equations*

$$\frac{x-3}{2} = y+5 = \frac{z-1}{-2}$$

and l_2 be the line with parametric equations

$$x = 1-4t \quad y = 3-2t \quad x = 1+4t.$$

[2] (a) *Determine whether the lines are parallel, intersecting or skew.*

Solution:
A vector parallel to the first line is $\langle 2,\ 1,\ -2\rangle$ and a vector parallel to the second line is $\langle -4,\ -2,\ 4\rangle$ which are multiples of each other. Therefore the lines are parallel to each other.

[6] (b) Determine the shortest distance between the lines.

Solution:
Hence We need to take any point on line l_2, say $P(1,\ 3,\ 1)$ and find the distance to the line l_1. (or the other way around)
Let's find a point Q on l_1, say $Q(3, -5, 1)$ and therefore the distance is

$$\begin{aligned} d &= |\mathbf{PQ}| \sin\theta \\ &= \frac{|\mathbf{PQ}|\,|\mathbf{v}| \sin\theta}{|\mathbf{v}|} \\ &= \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|} \end{aligned}$$

where θ is the angle between $\mathbf{PQ}$ and $\mathbf{v}$.
$\mathbf{PQ} = \langle 2,\ -8,\ 0\rangle$ and $\mathbf{v} = \langle 2,\ 1,\ -2\rangle \Rightarrow |\mathbf{v}| = 3$. Hence the distance between the lines is

$$\begin{aligned} d &= \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|} \\ &= \frac{|\langle 2,\ -8,\ 0\rangle \times \langle 2,\ 1,\ -2\rangle|}{3} \\ &= \frac{|\langle 16,\ 4,\ 18\rangle|}{3} \\ &= \frac{\sqrt{256+16+324}}{3} \\ &= \frac{\sqrt{596}}{3} \end{aligned}$$

Note: There are other ways to do the question as well.

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[5] 4. *Let* $f(t) = t$ *and* $\mathbf{v}(t) = \left\langle t, \frac{1}{t^2+1}, e^t \right\rangle$. *Evaluate*

$$\int (f\mathbf{v})(t)\, dt$$

Solution:

$$\begin{aligned} I &= \int (f\mathbf{v})(t)\, dt \\ &= \int \left\langle t^2, \frac{t}{t^2+1}, te^t \right\rangle dt \end{aligned}$$

$$\int t^2\, dt = \frac{t^3}{3} + C_1$$

$$\int \frac{t}{t^2+1}\, dt = \int \frac{1}{2}\frac{1}{u}\, du = \frac{1}{2}\ln(t^2+1) + C_2$$

$$\int te^t\, dt = \int te^t - \int e^t\, dt = te^t - e^t + C_3.$$

Hence

$$\int (f\mathbf{v})(t)\, dt \left\langle \frac{t^3}{3}, \frac{1}{2}\ln(t^2+1), te^t - e^t \right\rangle + \mathbf{C}.$$

[6] 5. *Determine a parametric representation for the curves of intersection of* $z = \sqrt{9 - x^2 - y^2}$ *and* $x^2 - 2x + y^2 = 0$ *directed so that* x *decreases when* y *is positive. Justify your answer.*

Solution:

The second equation becomes $(x-1)^2 + y^2 = 1$ and therefore we can let $(x-1)^2 = \cos^2 t \Rightarrow x - 1 = \cos t \Rightarrow x = 1 + \cos t$ and $y^2 = \sin^2 t \Rightarrow y = \pm \sin t$

Since x decreases when y is positive, we have that $x'(t) < 0$ when $y > 0$ and therefore they must have opposite signs. $x'(t) = -\sin t$ and therefore $y = \sin t$.

$$\begin{aligned} z &= \sqrt{9 - x^2 - y^2} \\ &= \sqrt{9 - (1+\cos t)^2 - \sin^2 t} \\ &= \sqrt{9 - 1 - 2\cos t - \cos^2 t - \sin^2 t} \\ &= \sqrt{7 - 2\cos t} \end{aligned}$$

Hence

$$x = 1 + \cos t,\ y = \sin t,\ z = \sqrt{7 - 2\cos t}$$

Note: There are other way to justify the answer

6. *Let a curve C be defined by a position vector* $\mathbf{r}(t) = \langle t,\, t^{3/2},\, 4t^{3/2}\rangle$.

[4] (a) *Determine parametric equations for the tangent line to* $\mathbf{r}(t)$ *at the point* $(4, 8, 32)$.

Solution:
$\mathbf{r}'(t) = \langle 1,\, (3/2)t^{1/2},\, 6t^{1/2}\rangle$ and the point $(4, 8, 32)$ happens when $t = 4,\ t^{3/2} = 8,\ 32 = 4t^{3/2} \Rightarrow t = 4$.
Hence a vector parallel to the tangent line is
$\mathbf{r}'(4) = \langle 1,\, (3/2)(4)^{1/2},\, 6(4)^{1/2}\rangle = \langle 1,\, 3,\, 12\rangle$.
Therefore the equations of the tangent line are

$$x = 4 + s,\ y = 8 + 3s,\ z = 32 + 12s.$$

[2] (b) Determine the unit tangent vector to $\mathbf{r}(t)$ at the point $(4, 8, 32)$.

Solution: From part (a) the tangent vector at the point $(4, 8, 32)$ is $\langle 1,\, 3,\, 12\rangle$. Hence the unit tangent vector is

$$\hat{\mathbf{T}}(4) = \frac{\langle 1,\, 3,\, 12\rangle}{\sqrt{154}}.$$

[6] (c) *Determine the length of the curve C from the point* $(0, 0, 0)$ *to* $(4, 8, 32)$.

Solution:
The point $(0, 0, 0)$ occurs when $t = 0$ and $(4, 8, 32)$ occurs when $t = 4$ and therefore

$$\begin{aligned}
L &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt \\
&= \int_0^4 \sqrt{(1)^2 + ((3/2)t^{1/2})^2 + (6t^{1/2})^2}\, dt \\
&= \int_0^4 \sqrt{1 + \frac{9}{4}t + 36t}\, dt \\
&= \int_0^4 \sqrt{1 + \frac{153}{4}t}\, dt \\
&= \int_1^{154} \sqrt{u}\, \frac{4}{153}\, du \\
&= \frac{4}{153}\left(\frac{2}{3}u^{3/2}\Big|_1^{154}\right) \\
&= \frac{8}{459}\left((154)^{3/2} - 1\right)
\end{aligned}$$