MATH 2130 – Midterm 2 Solutions

1. Let $f(x,y,z) = ze^{x^2-y^2}$, and let $\mathcal{C}$ be the curve with vector representation

$$\mathbf{r}(t) = (t^2)\,\widehat{\mathbf{i}} + (t^2-2t-2)\,\widehat{\mathbf{j}} + 2\cos(t+1)\,\widehat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

Find the rate of change of f in the direction of $\mathcal{C}$ at the point $(1,1,2)$.

Solution. We have

$$\begin{aligned}\nabla f =& \left(2xze^{x^2-y^2}, -2yze^{x^2-y^2}, e^{x^2-y^2}\right),\\ \nabla f|_{(1,1,2)} =&(4,-4,1).\end{aligned}$$

Set

$$\mathbf{r}(t) = (t^2, t^2-2t-2, 2\cos(t+1)) = (1,1,2).$$

The only solution is $t=-1$. Then

$$\begin{aligned}\mathbf{r}'(t) =&(2t, 2t-2, -2\sin(t+1)),\\ \mathbf{r}'(-1) =&(-2,-4,0).\end{aligned}$$

The corresponding unit vector is

$$\widehat{\mathbf{T}} = \frac{1}{\sqrt{5}}(-1,-2,0).$$

($-\widehat{\mathbf{T}}$ is also acceptable.) The rate of change of f in the direction of $\mathcal{C}$ is then

$$\nabla f|_{(1,1,2)} \cdot \widehat{\mathbf{T}} = (4,-4,1)\cdot\frac{1}{\sqrt{5}}(-1,-2,0) = \frac{4}{\sqrt{5}}.$$

2. Evaluate each of the following limits, or show that it does not exist.

(a) $$\lim_{(x,y)\to(2,1)} \frac{x^2-xy-2y^2}{\sqrt{4x+y}-\sqrt{3x+3y}}$$

Solution.

$$\begin{aligned}\lim_{(x,y)\to(2,1)} \frac{x^2-xy-2y^2}{\sqrt{4x+y}-\sqrt{3x+3y}} &= \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x+y)(\sqrt{4x+y}+\sqrt{3x+3y})}{(4x+y)-(3x+3y)}\\ &= \lim_{(x,y)\to(2,1)} \frac{(x-2y)(x+y)(\sqrt{4x+y}+\sqrt{3x+3y})}{x-2y}\\ &= \lim_{(x,y)\to(2,1)} (x+y)(\sqrt{4x+y}+\sqrt{3x+3y})\\ &=(2+1)(\sqrt{9}+\sqrt{9}) = 18.\end{aligned}$$

(b) $$\lim_{(x,y)\to(1,-3)} \frac{x^2-y^2-2x-6y-8}{4x^2+y^2-8x+6y+13}$$

Solution.

$$\lim_{(x,y)\to(1,-3)} \frac{x^2-y^2-2x-6y-8}{4x^2+y^2-8x+6y+13} = \lim_{(x,y)\to(1,-3)} \frac{(x-1)^2-(y+3)^2}{4(x-1)^2+(y+3)^2}.$$

There are no obvious cancellations, so we conjecture that this limit does not exist.

Try the path $x = 1$, $y \to -3$:

$$\lim_{y\to-3} \frac{-(y+3)^2}{(y+3)^2} = -1.$$

Now try the path $x \to 1$, $y = -3$:

$$\lim_{x\to1} \frac{(x-1)^2}{4(x-1)^2} = \frac{1}{4}.$$

Since the two paths yield different values, the two-dimensional limit does not exist.

3. Let $u = s + t^2 + te^s$, where s and t are defined as functions of x and y by

$$\begin{aligned} F(s,t,x,y) = xs - yt =& 1, \\ G(s,t,x,y) = xt + ys =& 2. \end{aligned}$$

Find $\dfrac{\partial u}{\partial x}$.

Solution. We are given u as a function of s and t, where s and t are functions of x and y. By the chain rule,

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}.$$

From the given expression for u, we find

$$\frac{\partial u}{\partial s} = 1 + te^s, \quad \frac{\partial u}{\partial t} = 2t + e^s.$$

We use implicit differentiation to find $\frac{\partial s}{\partial x}$ and $\frac{\partial t}{\partial x}$.

$$\frac{\partial s}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,t)}}{\frac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}} = -\frac{\begin{vmatrix} s & -y \\ t & x \end{vmatrix}}{\begin{vmatrix} x & -y \\ y & x \end{vmatrix}} = -\frac{sx + ty}{x^2 + y^2},$$

$$\frac{\partial t}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(s,x)}}{\frac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{x^2 + y^2} = -\frac{\begin{vmatrix} x & s \\ y & t \end{vmatrix}}{x^2 + y^2} = -\frac{xt - ys}{x^2 + y^2}.$$

Thus

$$\frac{\partial u}{\partial x} = (1 + te^s)\left(-\frac{sx + ty}{x^2 + y^2}\right) + (2t + e^s)\left(-\frac{xt - ys}{x^2 + y^2}\right).$$

4. Let S_1 be the surface $y = 2x^2 - xz$, and let S_2 be the surface $x^2 + 2yz + yz^2 = 4$.

(a) Find the equation of the tangent plane to S_1 at the point $(1, 1, 1)$.

Solution. Let $F(x, y, z) = 2x^2 - xz - y$. Then S_1 is given by $F(x, y, z) = 0$. We find

$$\begin{aligned} \nabla F =& (4x - z, -1, -x), \\ \nabla F|_{(1,1,1)} =& (3, -1, -1). \end{aligned}$$

A normal for the tangent plane is $(3, -1, -1)$, and a point on it is $(1, 1, 1)$. Thus an equation for the tangent plane is

$$3(x-1) - (y-1) - (z-1) = 0.$$

(b) Let $\mathcal{C}$ be the curve formed by the intersection of S_1 and S_2. Find a vector representation for the tangent line to $\mathcal{C}$ at $(1, 1, 1)$.

Solution. Let $G(x, y, z) = x^2 + 2yz + yz^2$. Then S_2 is given by $G(x, y, z) = 4$. We have

$$\begin{aligned} \nabla G =& (2x, 2z + z^2, 2y + 2yz), \\ \nabla G|_{(1,1,1)} =& (2, 3, 4). \end{aligned}$$

The gradient vectors $\nabla F|_{(1,1,1)} = (3, -1, -1)$ and $\nabla G|_{(1,1,1)} = (2, 3, 4)$ are both perpendicular to the curve of intersection $\mathcal{C}$. Therefore a tangent vector to $\mathcal{C}$ is

$$\nabla F|_{(1,1,1)} \times \nabla G|_{(1,1,1)} = (3, -1, -1) \times (2, 3, 4) = (-4 + 3, -2 - 12, 9 + 2) = (-1, -14, 11).$$

A vector representation for the tangent line to $\mathcal{C}$ at $(1, 1, 1)$ is

$$\mathbf{r}(t) = (1, 1, 1) + t(-1, -14, 11), \quad t \in \mathbb{R}.$$

5. Let $f(x, y) = 5x^2 - 4y + 10xy + 2y^3$. Find all critical points of f. Choose **one** critical point and determine if it is a relative maximum, relative minimum, saddle point, or none of these.

We calculate

$$\nabla f = (f_x, f_y) = (10x + 10y, -4 + 10x + 6y^2).$$

The gradient exists everywhere. To find the critical points, we set $\nabla f = \mathbf{0}$:

$$f_x = 10x + 10y = 0 \iff x = -y,$$

$$f_y = -4 + 10x + 6y^2 = - \iff 3y^2 + 5x - 2 = 0.$$

Substitute $x = -y$ in the second equation:

$$3y^2 - 5y - 2 = 0 \iff (3y + 1)(y - 2) = 0.$$

There are two solutions, $y = 2$ and $y = -\frac{1}{3}$. These yield the two critical points $(-2, 2)$ and $\left(\frac{1}{3}, -\frac{1}{3}\right)$.

To classify a critical point, we calculate

$$f_{xx} = 10, \quad f_{xy} = 10, \quad f_{yy} = 12y.$$

At $(2, -2)$, we get

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 10 & 10 \\ 10 & 24 \end{vmatrix} = 140 > 0,$$

and $f_{xx} = 10 > 0$. Therefore $(2, -2)$ is a relative minimum. At $\left(\frac{1}{3}, -\frac{1}{3}\right)$, we get

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 10 & 10 \\ 10 & -4 \end{vmatrix} = -140 < 0,$$

so $\left(\frac{1}{3}, -\frac{1}{3}\right)$ is a saddle point.