[6] 1. Evaluate $\lim_{x \to 2} \frac{x^3 - 3x^2 + 4}{x^2 - 4x + 4}$, if it exists. Show all calculations.

[6] 2. Find all intervals on which the function $f(x) = \frac{x+3}{x^2+3}$ is increasing.

$$f'(x) = \frac{(x+3)'(x^2+3) - (x+3)(x^2+3)'}{(x^2+3)^2} = \frac{x^2+3 - 2x(x+3)}{(x^2+3)^2} =$$

$$= \frac{x^2+3-2x^2-6x}{(x^2+3)^2} = -\frac{x^2+6x-3}{(x^2+3)^2}$$

$x^2+6x-3=0 \Leftrightarrow x = \frac{-6 \pm \sqrt{36+4\cdot 3}}{2} = \frac{-6 \pm \sqrt{48}}{2} = -3 \pm 2\sqrt{3}$,

i.e. $x_1 = -3-2\sqrt{3}$ and $x_2 = -3+2\sqrt{3}$ are the critical pts of $f(x)$.

$$f'(x) = -\frac{(x-x_1)(x-x_2)}{(x^2+3)^2}$$

		$-3-2\sqrt{3}$		$-3+2\sqrt{3}$	
$f'(x)$	$-$		$+$		$-$
f	$\searrow$		$\nearrow$		$\searrow$

$\therefore$ f is increasing on $[-3-2\sqrt{3}, -3+2\sqrt{3}]$.

[6] 3. Let $f(x) = x^5 - x^6$. Determine the intervals on which $f(x)$ is concave up or concave down, and find all inflection points (if any).

$f'(x) = 5x^4 - 6x^5$, $f''(x) = 20x^3 - 30x^4 = 10x^3(2-3x)$

$f''(x) = 0 \iff x = 0$ or $x = \frac{2}{3}$

		0		$\frac{2}{3}$	
$f''(x)$	$-$		$+$		$-$
f	$\frown$		$\smile$		$\frown$

f is concave up on $[0, \frac{2}{3}]$ and concave down on $(-\infty, 0]$ and $[\frac{2}{3}, +\infty)$.

$f(0) = 0$, $f(\frac{2}{3}) = (\frac{2}{3})^5 - (\frac{2}{3})^6 = (\frac{2}{3})^5(1 - \frac{2}{3}) = \frac{2^5}{3^6} = \frac{32}{729}$

Hence, f has inflection points at $(0,0)$ and $(\frac{2}{3}, \frac{32}{729})$.

[6] 4. Find the maximum value of the function $f(x) = \dfrac{\ln x}{x^3}$ on the interval $1 \le x \le 2$.

f is continuous (and continuously differentiable) on $[1,2]$. Hence, it achieves its maximum on this interval (which is achieved either at the endpoints or the critical pts).

$$f'(x) = \frac{(\ln x)' x^3 - \ln x \cdot (x^3)'}{x^6} = \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3\ln x}{x^4}$$

$f'(x) = 0 \iff 1 - 3\ln x = 0 \iff \ln x = \frac{1}{3} \iff x = e^{\frac{1}{3}}$

Hence, $x = e^{\frac{1}{3}}$ is the only critical pt of f.

Now, we could simply compare $f(0)$, $f(2)$ and $f(e^{\frac{1}{3}})$ (note that $1 < e^{\frac{1}{3}} < 2$ and so $e^{\frac{1}{3}}$ is in $(1,2)$) but this is messy without a calculator. Therefore, we use the first derivative (and the fact that $\ln x$ is increasing).

	1		$e^{\frac{1}{3}}$		2
f'		$+$		$-$	
f		$\nearrow$		$\searrow$	

Since $f \uparrow$ on $[1, e^{\frac{1}{3}}]$ and $f \downarrow$ on $[e^{\frac{1}{3}}, 2]$, f achieves its absolute maximum at $x = e^{\frac{1}{3}}$.

$$f(e^{\frac{1}{3}}) = \frac{\ln e^{\frac{1}{3}}}{(e^{\frac{1}{3}})^3} = \frac{\frac{1}{3}}{e} = \frac{1}{3e}.$$

$\therefore$ Max value of f on $[1,2]$ is $\frac{1}{3e}$.

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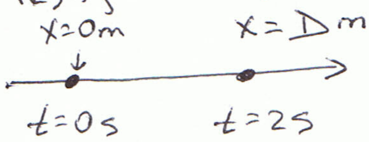
[6] 5. From a standing start, a cheetah reaches a velocity of 72 km/hr in 2 seconds. If the acceleration of the cheetah is constant, and it runs in a straight line, what distance does it cover in the 2 seconds? You are not permitted to use any physics formulas. You must derive your own.

First, note that $72 \ km/h = 72 \cdot \frac{1000}{3600} \ m/s = 20 \ m/s$



Let x be the distance (in m) from the starting point at time t s.

$\frac{d^2x}{dt^2} = a = \text{const} \to$ acceleration

$\frac{dx}{dt} = v(t) = \int a\,dt = at + C.$

Now, $v(0) = 0$ and $v(2) = 20$, i.e. $C = 0$ and $20 = a \cdot 2$
$\Leftrightarrow a = 10 \ (m/s^2)$

Hence, $\frac{dx}{dt} = v(t) = 10t$

$x(t) = \int 10t\,dt = 5t^2 + C_1$ and $x(0) = 0$, i.e. $C_1 = 0$

Hence, $x(t) = 5t^2$ and so $x(2) = 5 \cdot 2^2 = 20$ m.

$\therefore$ Cheetah covers 20 m in 2 seconds.

[5] 6. A particle travels along the curve $y = x^3 - 2x^2 - x + 5$ where x and y are in metres in such a way that its x-coordinate is increasing at a constant rate of 2 metres per second. Find all points on the curve where its y-coordinate is decreasing at 4 metres per second.

$\frac{dx}{dt} = 2 \ m/s$

$\frac{dy}{dt} = (3x^2 - 4x - 1)\frac{dx}{dt} = 2(3x^2 - 4x - 1) \ (m/s)$

We need to find all pts on the curve s.t. $\frac{dy}{dt} = -4 \ (m/s)$

$\therefore 2(3x^2 - 4x - 1) = -4 \Leftrightarrow 3x^2 - 4x - 1 = -2 \Leftrightarrow 3x^2 - 4x + 1 = 0$

$\Leftrightarrow x = \frac{4 \pm \sqrt{16 - 4 \cdot 3}}{2 \cdot 3} = \frac{4 \pm 2}{2 \cdot 3} = \frac{2 \pm 1}{3}$, i.e. $x_1 = \frac{1}{3}$ and $x_2 = 1$

When $x = 1$, $y = 1 - 2 - 1 + 5 = 3$

$x = \frac{1}{3}$, $y = \frac{1}{27} - \frac{2}{9} - \frac{1}{3} + 5 = \frac{1 - 6 - 9 + 5 \cdot 27}{27} = \frac{121}{27} = 4\frac{13}{27}$

Answer: two points $(1, 3)$ and $(\frac{1}{3}, 4\frac{13}{27})$.