MATH 2130 – Tutorial Problem Solutions, Thu Mar 8

Example. Find the longest and the shortest distance between the origin and the curve given by the intersection of the surfaces $y^2 + 2z^2 = 2$ and $x^2 - y^2 = 4$. (Hint: if you simplify the problem enough, no calculus is required.)

Solution. Let $D(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. This is the distance between the point (x, y, z) and the origin. We need to find the absolute maximum and absolute minimum of D over the given curve.

Since $x^2 = 4 + y^2$ everywhere on $\mathcal{C}$, we have

$$D = \sqrt{4 + 2y^2 + z^2}.$$

Further, since $y^2 = 2 - 2z^2$ on $\mathcal{C}$, we have

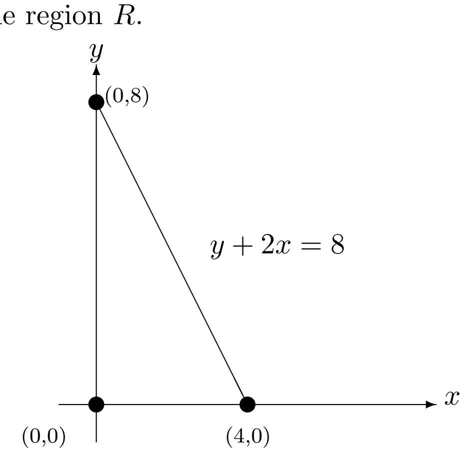
$$D = \sqrt{4 + 2(2 - 2z^2) + z^2} = \sqrt{8 - 3z^2}.$$

Since $z^2 \geq 0$ for all z, the largest value that D could possibly have is $\sqrt{8} = 2\sqrt{2}$. This occurs if there is a point on $\mathcal{C}$ such that $z = 0$. Notice that the point $(\sqrt{6}, \sqrt{2}, 0)$ lies on $\mathcal{C}$. Therefore z attains the value 0 on $\mathcal{C}$, so the absolute maximum of D over $\mathcal{C}$ is $2\sqrt{2}$.

The expression $\sqrt{8 - 3z^2}$ is minimized when z^2 attains its largest possible value on $\mathcal{C}$. Any value of z corresponding to a point on $\mathcal{C}$ must satisfy $y^2 + 2z^2 = 2$. The value of z^2 is largest when $y = 0$, in which case $z^2 = 1$. Therefore z^2 can be no larger than 1 on $\mathcal{C}$. Note that the point $(2, 0, 1)$ lies on $\mathcal{C}$, so z^2 does attain the value 1. Thus the absolute minimum of D over $\mathcal{C}$ is $\sqrt{8 - 3} = \sqrt{5}$.

Example. Let $f(x, y) = xy - x - y$. Find the absolute maximum and absolute minimum of f over the region R bounded by the positive x- and y- axes and the line $2x + y = 8$.

Solution. First, we sketch the region R.



It is a triangle with vertices $(0, 0)$, $(4, 0)$ and $(0, 8)$.

The gradient of f is

$$\nabla f = (y - 1, x - 1).$$

The only critical point of f is $(1, 1)$, which is within R. The value of f at $(1, 1)$ is

$$f(1, 1) = 1 - 1 - 1 = -1.$$

On the boundary line $x = 0$, the function becomes

$$f(0, y) = -y, \quad 0 \leq y \leq 8.$$

This function has no critical points. Its values on the endpoints are

$$f(0, 0) = 0, \quad f(0, 8) = -8.$$

On the boundary line $y = 0$, the function becomes

$$f(x, 0) = -x, \quad 0 \leq x \leq 4.$$

This function also has no critical points. We already have its value on the endpoint $x = 0$. At the other endpoint, we get

$$f(4, 0) = -4.$$

Finally, on the boundary line $y = 8 - 2x$, the function becomes

$$f(x, 8 - 2x) = x(8 - 2x) - x - (8 - 2x) = -2x^2 + 9x - 8, \quad 0 \leq x \leq 4.$$

The derivative of this function is

$$\frac{df}{dx} = -4x + 9,$$

so f has a critical point at $x = \frac{9}{4}$, which is within the specified interval for x. The corresponding y value is $y = 8 - \frac{9}{2} = \frac{7}{2}$, and the function value is

$$f\left(\frac{9}{4}, \frac{7}{2}\right) = \frac{63}{8} - \frac{9}{4} - \frac{7}{2} = \frac{17}{8}.$$

Comparing all of the values of f at the critical point and on the boundary, we find that the absolute maximum of f over R is $\frac{17}{8}$, and the absolute minimum is -8.

Example. Let $f(x, y, z) = x^2 + \frac{1}{2}y + \frac{1}{4}z$. Find the maximum and minimum values of f, subject to the constraint $x^2 + y^2 + z^2 = 1$.

Solution. Let $G(x, y, z) = x^2 + y^2 + z^2 - 1$. Then the constraint equation is $G(x, y, z) = 0$. Let

$$L(x, y, z, \lambda) = x^2 + \frac{1}{2}y + \frac{1}{4}z + \lambda(x^2 + y^2 + z^2 - 1).$$

We find the critical points of L:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2x + 2x\lambda = 0, & \frac{\partial L}{\partial y} &= \frac{1}{2} + 2y\lambda = 0, \\ \frac{\partial L}{\partial z} &= \frac{1}{4} + 2z\lambda = 0, & \frac{\partial L}{\partial \lambda} &= x^2 + y^2 + z^2 - 1 = 0. \end{aligned}$$

We obtain the constraint equation, and the system

$$\begin{aligned} x(1 + \lambda) &= 0, \quad (1) \\ y\lambda &= -\frac{1}{4}, \quad (2) \\ z\lambda &= -\frac{1}{8} \quad (3). \end{aligned}$$

The solutions to (1) are $x = 0$ or $\lambda = -1$. Proceed by cases.

Suppose $x = 0$. Then $y = -\frac{1}{4\lambda}$, $z = -\frac{1}{8\lambda}$, and $y^2 + z^2 = 1$. Substitute the expressions for y and z into the constraint equation:

$$\frac{1}{16\lambda^2} + \frac{1}{64\lambda^2} = 1,$$

which implies that

$$\lambda = \pm\frac{\sqrt{5}}{8}.$$

The solutions for (x, y, z) are $\left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $\left(0, -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$. We test the function at these points:

$$f\left(0, \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) = \frac{\sqrt{5}}{4},$$
$$f\left(0, -\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right) = -\frac{\sqrt{5}}{4}.$$

Now suppose $\lambda = -1$. We get $y = \frac{1}{4}$, $z = \frac{1}{8}$, and $x^2 + y^2 + z^2 = 1$. Using the values of y and z in the constraint equation:

$$x^2 = 1 - \frac{5}{64} = \frac{59}{64}.$$

The solutions for (x, y, z) are $\left(\frac{\sqrt{59}}{64}, \frac{1}{4}, \frac{1}{8}\right)$ and $\left(-\frac{\sqrt{59}}{64}, \frac{1}{4}, \frac{1}{8}\right)$. Evaluate f at these points:

$$f\left(\pm\frac{\sqrt{59}}{64}, \frac{1}{4}, \frac{1}{8}\right) = \frac{59}{64} + \frac{1}{4} + \frac{1}{8} = \frac{83}{64}.$$

Comparing all of the values of f, we find that the maximum value is $\frac{83}{64}$ and the minimum value is $-\frac{\sqrt{5}}{4}$.

Example. Let $f(x, y, z) = x - 2y - 3z$. Find the maximum and minimum values of f, subject to the constraints $x^2 + y^2 + z^2 = 1$ and $x + y + z = 0$.

Solution. The constraint functions are $x^2 + y^2 + z^2 - 1 = 0$ and $x + y + z = 0$. Let

$$L(x, y, z, \lambda, \mu) = x - 2y - 3z + \lambda(x^2 + y^2 + z^2 - 1) + \mu(x + y + z).$$

We find the critical points of L:

$$\frac{\partial L}{\partial x} = 1 + 2x\lambda + \mu = 0, \qquad \frac{\partial L}{\partial y} = -2 + 2y\lambda + \mu = 0,$$
$$\frac{\partial L}{\partial z} = -3 + 2z\lambda + \mu = 0, \qquad \frac{\partial L}{\partial \lambda} = x^2 + y^2 + z^2 - 1 = 0,$$
$$\frac{\partial L}{\partial \mu} = x + y + z = 0.$$

We obtain the two constraint equations, and the system

$$\begin{aligned} 1 + 2x\lambda + \mu &= 0, \quad (1) \\ -2 + 2y\lambda + \mu &= 0, \quad (2) \\ -3 + 2z\lambda + \mu &= 0, \quad (3). \end{aligned}$$

Take (1) − (3) and (2) − (3):

$$\begin{aligned} 4 + 2\lambda(x - z) &= 0, \quad (1)' \\ 1 + 2\lambda(y - z) &= 0, \quad (2)'. \end{aligned}$$

From $(1)'$, we get $x = z - \frac{2}{\lambda}$, and from $(2)'$, we get $y = z - \frac{1}{2\lambda}$. Make these substitutions in the constraint $x + y + z = 0$:

$$z - \frac{2}{\lambda} + z - \frac{1}{2\lambda} + z = 0,$$

which implies that

$$z = \frac{5}{6\lambda},$$

and so

$$x = z - \frac{2}{\lambda} = -\frac{7}{6\lambda}, \quad y = z - \frac{1}{2\lambda} = \frac{1}{3\lambda}.$$

Now we make all of these substitutions in the constraint $x^2 + y^2 + z^2 = 1$:

$$\frac{1}{\lambda^2}\left[\left(-\frac{7}{6}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{5}{6}\right)^2\right] = 1,$$

which reduces to $\lambda^2 = \frac{78}{36}$. The solutions are $\lambda = \pm\frac{\sqrt{78}}{6}$. The corresponding (x, y, z) values are $\left(-\frac{7}{\sqrt{78}}, \frac{2}{\sqrt{78}}, \frac{5}{\sqrt{78}}\right)$ and $\left(\frac{7}{\sqrt{78}}, -\frac{2}{\sqrt{78}}, -\frac{5}{\sqrt{78}}\right)$ We test f at these points:

$$\begin{aligned} f\left(-\frac{7}{\sqrt{78}}, \frac{2}{\sqrt{78}}, \frac{5}{\sqrt{78}}\right) &= -\frac{26}{\sqrt{78}}, \\ f\left(\frac{7}{\sqrt{78}}, -\frac{2}{\sqrt{78}}, -\frac{5}{\sqrt{78}}\right) &= \frac{26}{\sqrt{78}}. \end{aligned}$$

These are the only critical points. The maximum value of f is $\frac{26}{\sqrt{78}}$, and the minimum value is $-\frac{26}{\sqrt{78}}$.