MATH 2130 – Tutorial Problem Solutions, Thu Mar 1

Critical Points and Extreme Values

Example. Let $f(x,y) = y^4 + x^2y + x^2$. Find all critical points of f, and classify them as relative minima, relative maxima, saddle points, or none of these.

Solution. We calculate the first partial derivatives:

$$f_x = 2xy + 2x = 2x(y+1),$$

and

$$f_y = 4y^3 + x^2.$$

Notice that both partial derivatives are defined everywhere. Therefore ∇f exists everywhere, and the critical points are those points where $\nabla f = \mathbf{0}$. We find

$$f_x = 0 \quad \text{when} \quad x = 0 \quad \text{or} \quad y = -1,$$

and

$$f_y = 0 \quad \text{when} \quad x^2 = -4y^3.$$

Proceed by cases on the conditions such that $f_x = 0$:

- Suppose $x = 0$. Then $x^2 = -4y^3$ implies that $y = 0$. One critical point is $(0,0)$.
- Suppose $y = -1$. Then $x^2 = -4y^3 = 4$, which is satisfied when $x = \pm 2$. Two more critical points are $(2,-1)$ and $(-2,-1)$.

Thus we have three critical points: $(0,0)$, $(2,-1)$ and $(-2,-1)$.

To classify the critical points, we calculate

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 2y+2 & 2x \\ 2x & 12y^2 \end{pmatrix}.$$

At $(2,-1)$, the determinant of the matrix of second partial derivatives is

$$\Delta = \begin{vmatrix} 0 & 4 \\ 4 & 12 \end{vmatrix} = -16 < 0.$$

Therefore $(2,-1)$ is a saddle point. At $(-2,-1)$, the determinant is

$$\Delta = \begin{vmatrix} 0 & -4 \\ -4 & 12 \end{vmatrix} = -16 < 0.$$

Therefore $(-2,-1)$ is also a saddle point. Finally, at $(0,0)$, the determinant is

$$\Delta = \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0,$$

and this test gives us no information.

We return to f, and write

$$f(x,y) = y^4 + x^2(y+1).$$

Clearly $f(0,0) = 0$. If we can find a path through $(0,0)$ such that $f > 0$ on one side of the path and $f < 0$ on the other, then $(0,0)$ is a saddle point. On the other hand, if we can argue that there is a neighborhood of $(0,0)$ such that $f \geq 0$ everywhere on the neighborhood, then $(0,0)$ is a relative minimum for f.

Notice that y^4 and x^2 are both nonnegative for all (x,y). The only contribution to f that could possibly be negative is the factor $(y+1)$. But $y+1 \geq 0$ as long as $y \geq -1$. Therefore any disk centered on origin that is restricted to the region where $y \geq -1$ will have $f(x,y) \geq 0$ everywhere.

Example. Let $f(x,y) = x - |y-2|$. Find all critical points of f, and classify them as relative minima, relative maxima, saddle points, or none of these.

Solution. Observe that $\frac{\partial f}{\partial y}$ does not exist whenever $y = 2$. Therefore all points of the form $(x,2)$, $x \in \mathbb{R}$, are critical points for f.

Assume that $y \neq 2$. Then we can rewrite f as

$$f = \begin{cases} x - y + 2, & y > 2, \\ x + y - 2, & y < 2. \end{cases}$$

The gradient is

$$\nabla f = \begin{cases} (1,-1), & y > 2, \\ (1,1), & y < 2. \end{cases}$$

There are no points where $\nabla f = \mathbf{0}$, so the only critical points are the points $(x,2)$, $x \in \mathbb{R}$.

Let $x_0 \in \mathbb{R}$ be fixed. Then $f(x_0,2) = x_0$. If we can find a path through $(x_0,2)$ such that $f(x,y) > x_0$ on one side and $f(x,y) < x_0$ on the other, then the point $(x_0,2)$ cannot be a relative maximum or minimum. It also cannot be a saddle point, since $\nabla f \neq \mathbf{0}$ there, so we will conclude that it fits none of our categories.

Fix $y = 2$, and consider the path $(x,2)$. On this path, $f(x,2) = x$. Clearly $f(x,2) > x_0$ when $x > x_0$ and $f(x,2) < x_0$ when $x < x_0$. Therefore $(x_0,2)$ cannot be a local max or min, and it is not a saddle point.

Since this construction works for every $x_0 \in \mathbb{R}$, we find that all of the critical points of this function are uncategorized.