Solutions to Even-numbered Exercises

CHAPTER 16

EXERCISES 16.1

2. $\mathcal{L}\{t+e^t\} = \mathcal{L}\{t\} + \mathcal{L}\{e^t\} = \dfrac{1}{s^2} + \dfrac{1}{s-1}$ **4.** $\mathcal{L}\{e^{-2t}+2e^t\} = \mathcal{L}\{e^{-2t}\} + 2\mathcal{L}\{e^t\} = \dfrac{1}{s+2} + \dfrac{2}{s-1}$

6. $\mathcal{L}\{\cos 2t - 3\sin 4t\} = \mathcal{L}\{\cos 2t\} - 3\mathcal{L}\{\sin 4t\} = \dfrac{s}{s^2+4} - \dfrac{12}{s^2+16}$

8. $\mathcal{L}^{-1}\left\{\dfrac{2}{s} - \dfrac{3}{s^4}\right\} = 2\mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\dfrac{1}{s^4}\right\} = 2(1) - 3\left(\dfrac{t^3}{3!}\right) = 2 - \dfrac{t^3}{2}$

10. $\mathcal{L}^{-1}\left\{\dfrac{3}{s-1}\right\} = 3\mathcal{L}^{-1}\left\{\dfrac{1}{s-1}\right\} = 3e^t$

12.
$$\mathcal{L}^{-1}\left\{\frac{2s}{s^2+2} - \frac{5}{s^2+9}\right\} = 2\mathcal{L}^{-1}\left\{\frac{s}{s^2+2}\right\} - 5\mathcal{L}^{-1}\left\{\frac{1}{s^2+9}\right\} = 2\cos\sqrt{2}t - 5\left(\frac{1}{3}\sin 3t\right)$$
$$= 2\cos\sqrt{2}t - \frac{5}{3}\sin 3t$$

14.
$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^4 e^{-st}\,dt + \int_4^\infty 2e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_0^4 + 2\left\{\frac{e^{-st}}{-s}\right\}_4^\infty$$
$$= -\frac{e^{-4s}}{s} + \frac{1}{s} + \frac{2e^{-4s}}{s} = \frac{1+e^{-4s}}{s}, \quad \text{provided } s>0$$

16.
$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 t^2e^{-st}\,dt = \left\{-\frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_0^1$$
$$= \frac{2}{s^3} - \frac{e^{-s}}{s^3}(s^2+2s+2), \quad \text{provided } s>0$$

18.
$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_1^\infty (t-1)^2e^{-st}\,dt = \left\{-\frac{(t-1)^2}{s}e^{-st} - \frac{2(t-1)}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_1^\infty$$
$$= \frac{2}{s^3}e^{-s}, \quad \text{provided } s>0$$

20.
$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 te^{-st}\,dt + \int_1^2 (2-t)e^{-st}\,dt$$
$$= \left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_0^1 + \left\{\frac{t-2}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right\}_1^2 = \frac{1-2e^{-s}+e^{-2s}}{s^2}, \quad \text{provided } s>0$$

22.
$$F(s) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^1 (1+t^2)e^{-st}\,dt + \int_1^\infty 2te^{-st}\,dt$$
$$= \left\{\frac{e^{-st}}{-s} - \frac{t^2}{s}e^{-st} - \frac{2t}{s^2}e^{-st} - \frac{2}{s^3}e^{-st}\right\}_0^1 + 2\left\{-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right\}_1^\infty$$
$$= \frac{1}{s} + \frac{2(1-e^{-s})}{s^3}, \quad \text{provided } s>0$$

24. $F(s) = \displaystyle\int_0^\infty e^{-st}f(t)\,dt = \int_a^b e^{-st}\,dt = \left\{\frac{e^{-st}}{-s}\right\}_a^b = \frac{e^{-as}-e^{-bs}}{s}$, provided $s>0$

26. If we set $u=\sqrt{t}$, or, $t=u^2$, then $F(s) = \displaystyle\int_0^\infty \frac{1}{\sqrt{t}}e^{-st}\,dt = \int_0^\infty \frac{1}{u}e^{-su^2}(2u\,du) = 2\int_0^\infty e^{-su^2}\,du$. We now set $v=\sqrt{s}u$, in which case
$$F(s) = 2\int_0^\infty e^{-v^2}\left(\frac{dv}{\sqrt{s}}\right) = \frac{2}{\sqrt{s}}\int_0^\infty e^{-v^2}\,dv = \frac{2}{\sqrt{s}}\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\frac{\pi}{s}}.$$

EXERCISES 16.2

2. Since $f(t) = [h(t) - h(t-4)] + 2h(t-4) = 1 + h(t-4)$,

$$F(s) = \mathcal{L}\{1 + h(t-4)\} = \frac{1}{s} + \frac{e^{-4s}}{s} = \frac{1+e^{-4s}}{s}.$$

4. Since $f(t) = t^2[h(t) - h(t-1)] = t^2 - t^2 h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{t^2 - t^2 h(t-1)\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{(t+1)^2\} = \frac{2}{s^3} - e^{-s}\mathcal{L}\{t^2 + 2t + 1\} \\ &= \frac{2}{s^3} - e^{-s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) = \frac{2}{s^3} - \frac{e^{-s}(s^2+2s+2)}{s^3}. \end{aligned}$$

6. Since $f(t) = (t-1)^2 h(t-1)$,

$$F(s) = \mathcal{L}\{(t-1)^2 h(t-1)\} = e^{-s}\mathcal{L}\{(t+1-1)^2\} = e^{-s}\mathcal{L}\{t^2\} = \frac{2e^{-s}}{s^3}.$$

8. Since $f(t) = t[h(t) - h(t-1)] + (2-t)[h(t-1) - h(t-2)] = t + (2-2t)h(t-1) + (t-2)h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{t + (2-2t)h(t-1) + (t-2)h(t-2)\} = \frac{1}{s^2} + e^{-s}\mathcal{L}\{2 - 2(t+1)\} + e^{-2s}\mathcal{L}\{(t+2) - 2\} \\ &= \frac{1}{s^2} + e^{-s}\mathcal{L}\{-2t\} + e^{-2s}\mathcal{L}\{t\} = \frac{1}{s^2} - \frac{2}{s^2}e^{-s} + \frac{1}{s^2}e^{-2s} = \frac{1 - 2e^{-s} + e^{-2s}}{s^2}. \end{aligned}$$

10. Since $f(t) = (1+t^2)[h(t) - h(t-1)] + 2t\,h(t-1) = 1 + t^2 + (2t - 1 - t^2)h(t-1)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1 + t^2 + (2t-1-t^2)h(t-1)\} = \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{2(t+1) - 1 - (t+1)^2\} \\ &= \frac{1}{s} + \frac{2}{s^3} + e^{-s}\mathcal{L}\{-t^2\} = \frac{1}{s} + \frac{2}{s^3} - \frac{2}{s^3}e^{-s} = \frac{1}{s} + \frac{2(1-e^{-s})}{s^3}. \end{aligned}$$

12. Since $f(t) = h(t-a) - h(t-b)$, $F(s) = \mathcal{L}\{h(t-a) - h(t-b)\} = \dfrac{e^{-as}}{s} - \dfrac{e^{-bs}}{s} = \dfrac{e^{-as} - e^{-bs}}{s}$.

14. Since $f(t) = 2[h(t) - h(t-1)] + [h(t-1) - h(t-2)] + (t-2)h(t-2) = 2 - h(t-1) + (t-3)h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{2 - h(t-1) + (t-3)h(t-2)\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{(t+2) - 3\} = \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\mathcal{L}\{t - 1\} \\ &= \frac{2}{s} - \frac{e^{-s}}{s} + e^{-2s}\left(\frac{1}{s^2} - \frac{1}{s}\right) = \frac{2 - e^{-s}}{s} + \frac{(1-s)e^{-2s}}{s^2}. \end{aligned}$$

16. Since $f(t) = (1-t)[h(t) - h(t-1)] + (t-1)^2[h(t-1) - h(t-2)] = 1 - t + (t^2 - t)h(t-1) - (t-1)^2 h(t-2)$,

$$\begin{aligned} F(s) &= \mathcal{L}\{1 - t + (t^2 - t)h(t-1) - (t-1)^2 h(t-2)\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{(t+1)^2 - (t+1)\} - e^{-2s}\mathcal{L}\{(t+2-1)^2\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\mathcal{L}\{t^2 + t\} - e^{-2s}\mathcal{L}\{t^2 + 2t + 1\} \\ &= \frac{1}{s} - \frac{1}{s^2} + e^{-s}\left(\frac{2}{s^3} + \frac{1}{s^2}\right) - e^{-2s}\left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right) \\ &= \frac{s-1}{s^2} + \frac{(s+2)e^{-s}}{s^3} - \frac{(s^2 + 2s + 2)e^{-2s}}{s^3}. \end{aligned}$$

18. Since $f(t) = \sin t\, h(t - 2\pi)$,

$$F(s) = \mathcal{L}\{\sin t\, h(t-2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin(t + 2\pi)\} = e^{-2\pi s}\mathcal{L}\{\sin t\} = \frac{e^{-2\pi s}}{s^2+1}.$$

20. Since $f(t) = 2e^{-t}[h(t) - h(t-\ln 2)] + h(t-\ln 2) = 2e^{-t} + (1-2e^{-t})h(t-\ln 2)$,

$$F(s) = \mathcal{L}\{2e^{-t} + (1-2e^{-t})h(t-\ln 2)\} = \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1 - 2e^{-(t+\ln 2)}\}$$
$$= \frac{2}{s+1} + e^{-s\ln 2}\mathcal{L}\{1-e^{-t}\} = \frac{2}{s+1} + e^{-s\ln 2}\left(\frac{1}{s} - \frac{1}{s+1}\right) = \frac{2}{s+1} + \frac{e^{-s\ln 2}}{s(s+1)}.$$

22. $\mathcal{L}\{t^2e^{3t}\} = \mathcal{L}\{t^2\}_{|s-3} = \left(\dfrac{2}{s^3}\right)_{|s-3} = \dfrac{2}{(s-3)^3}$

24. $\mathcal{L}\{5e^{at} - 5e^{-at}\} = \dfrac{5}{s-a} - \dfrac{5}{s+a} = \dfrac{10a}{s^2-a^2}$

26. $\mathcal{L}\{2e^{-3t}\sin 3t + 4e^{3t}\cos 3t\} = 2\mathcal{L}\{\sin 3t\}_{|s+3} + 4\mathcal{L}\{\cos 3t\}_{|s-3}$

$$= 2\left(\frac{3}{s^2+9}\right)_{|s+3} + 4\left(\frac{s}{s^2+9}\right)_{|s-3} = \frac{6}{(s+3)^2+9} + \frac{4(s-3)}{(s-3)^2+9}$$

28. $\mathcal{L}\{\sin 3(t-4)h(t-4)\} = e^{-4s}\mathcal{L}\{\sin 3(t+4-4)\} = e^{-4s}\mathcal{L}\{\sin 3t\} = \dfrac{3e^{-4s}}{s^2+9}$

30. $\mathcal{L}\{(t+5)h(t-3)\} = e^{-3s}\mathcal{L}\{(t+3+5\} = e^{-3s}\mathcal{L}\{t+8\} = e^{-3s}\left(\dfrac{1}{s^2} + \dfrac{8}{s}\right) = \dfrac{(8s+1)e^{-3s}}{s^2}$

32. $\mathcal{L}\{\cos t\, h(t-\pi)\} = e^{-\pi s}\mathcal{L}\{\cos(t+\pi)\} = e^{-\pi s}\mathcal{L}\{-\cos t\} = \dfrac{-se^{-\pi s}}{s^2+1}$

34. $\mathcal{L}\{e^t h(t-4)\} = e^{-4s}\mathcal{L}\{e^{t+4}\} = e^4e^{-4s}\mathcal{L}\{e^t\} = \dfrac{e^{4-4s}}{s-1}$

36. $\mathcal{L}\{e^t\cos 2t\, h(t-1)\} = e^{-s}\mathcal{L}\{e^{t+1}\cos 2(t+1)\} = e^{-s}e\mathcal{L}\{\cos 2(t+1)\}_{|s-1}$

$$= e^{1-s}\mathcal{L}\{\cos 2\cos 2t - \sin 2\sin 2t\}_{|s-1} = e^{1-s}\left(\frac{s\cos 2}{s^2+4} - \frac{2\sin 2}{s^2+4}\right)_{|s-1}$$
$$= e^{1-s}\left[\frac{(s-1)\cos 2}{(s-1)^2+4} - \frac{2\sin 2}{(s-1)^2+4}\right] = \frac{e^{1-s}[s\cos 2 - (\cos 2 + 2\sin 2)]}{s^2-2s+5}$$

38. $F(s) = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{[h(t)-h(t-a)] - [h(t-a)-h(t-2a)]\} = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{1 - 2h(t-a) + h(t-2a)\}$

$$= \frac{1}{1-e^{-2as}}\left(\frac{1}{s} - \frac{2e^{-as}}{s} + \frac{e^{-2as}}{s}\right) = \frac{(1-e^{-as})^2}{s(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s(1+e^{-as})}$$

40. $F(s) = \dfrac{1}{1-e^{-2as}}\mathcal{L}\{t[h(t)-h(t-a)] + (2a-t)[h(t-a)-h(t-2a)]\}$

$$= \frac{1}{1-e^{-2as}}\mathcal{L}\{t + (2a-2t)h(t-a) + (t-2a)h(t-2a)\}$$
$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} + e^{-as}\mathcal{L}\{2a - 2(t+a)\} + e^{-2as}\mathcal{L}\{t+2a-2a\}\right]$$
$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2e^{-as}}{s^2} + \frac{e^{-2as}}{s^2}\right] = \frac{(1-e^{-as})^2}{s^2(1+e^{-as})(1-e^{-as})} = \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

42. $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2-2s+5}\right\} = \mathcal{L}^{-1}\left\{\dfrac{1}{(s-1)^2+4}\right\} = e^t\mathcal{L}^{-1}\left\{\dfrac{1}{s^2+4}\right\} = \dfrac{1}{2}e^t\sin 2t$

44. Since $\mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$, $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s^2}\right\} = (t-2)h(t-2)$.

46. Since $\mathcal{L}^{-1}\left\{\dfrac{s}{s^2+2}\right\} = \cos\sqrt{2}t$, $\mathcal{L}^{-1}\left\{\dfrac{se^{-5s}}{s^2+2}\right\} = \cos\sqrt{2}(t-5)\, h(t-5)$.

48. $$\mathcal{L}^{-1}\left\{\frac{1}{4s^2-6s-5}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s^2-3s/2-5/4}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s-3/4)^2-29/16}\right\}$$
$$=\frac{1}{4}e^{3t/4}\mathcal{L}^{-1}\left\{\frac{1}{s^2-29/16}\right\}=\frac{1}{4}e^{3t/4}\mathcal{L}^{-1}\left\{\frac{-2/\sqrt{29}}{s+\sqrt{29}/4}+\frac{2/\sqrt{29}}{s-\sqrt{29}/4}\right\}$$
$$=\frac{1}{2\sqrt{29}}e^{3t/4}(-e^{-\sqrt{29}t/4}+e^{\sqrt{29}t/4})=\frac{\sqrt{29}}{58}[e^{(3+\sqrt{29})t/4}-e^{(3-\sqrt{29})t/4}]$$

50. $$\mathcal{L}^{-1}\left\{\frac{4s+1}{(s^2+s)(4s^2-1)}\right\}=\frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4s+1}{s(s+1)(s+1/2)(s-1/2)}\right\}$$
$$=\frac{1}{4}\mathcal{L}^{-1}\left\{-\frac{4}{s}+\frac{4}{s+1}-\frac{4}{s+1/2}+\frac{4}{s-1/2}\right\}=-1+e^{-t}-e^{-t/2}+e^{t/2}$$

52. Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+3s+2}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{s+1}-\frac{1}{s+2}\right\}=e^{-t}-e^{-2t}$,
$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s^2+3s+2}\right\}=[e^{-(t-2)}-e^{-2(t-2)}]h(t-2)=[e^{2-t}-e^{2(2-t)}]h(t-2).$$

54. $$\mathcal{L}^{-1}\left\{\frac{5s-2}{3s^2+4s+8}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{5s-2}{s^2+4s/3+8/3}\right\}=\frac{1}{3}\mathcal{L}^{-1}\left\{\frac{5(s+2/3)-16/3}{(s+2/3)^2+20/9}\right\}$$
$$=\frac{1}{3}e^{-2t/3}\mathcal{L}^{-1}\left\{\frac{5s-16/3}{s^2+20/9}\right\}=\frac{1}{3}e^{-2t/3}\left(5\cos\frac{2\sqrt{5}t}{3}-\frac{8}{\sqrt{5}}\sin\frac{2\sqrt{5}t}{3}\right)$$

56. $$\mathcal{L}^{-1}\left\{\frac{s}{(s+1)^5}\right\}=\mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^5}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^4}-\frac{1}{(s+1)^5}\right\}$$
$$=e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^4}-\frac{1}{s^5}\right\}=e^{-t}\left(\frac{t^3}{3!}-\frac{t^4}{4!}\right)=\frac{t^3(4-t)e^{-t}}{24}$$

58. $$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2-4)^2}\right\}=\mathcal{L}^{-1}\left\{\frac{1/8}{s-2}+\frac{1/4}{(s-2)^2}-\frac{1/8}{s+2}+\frac{1/4}{(s+2)^2}\right\}$$
$$=\frac{1}{8}e^{2t}\mathcal{L}^{-1}\left\{\frac{1}{s}+\frac{2}{s^2}\right\}+\frac{1}{8}e^{-2t}\mathcal{L}^{-1}\left\{\frac{2}{s^2}-\frac{1}{s}\right\}=\frac{1}{8}e^{2t}(1+2t)+\frac{1}{8}e^{-2t}(2t-1)$$

60. $$F(s)=\frac{1}{1-e^{-4s}}\mathcal{L}\Bigg\{\frac{t^2}{4}[h(t)-h(t-1)]-\frac{1}{4}(t^2-4t+2)[h(t-1)-h(t-3)]$$
$$+\frac{1}{4}(t-4)^2[h(t-3)-h(t-4)]\Bigg\}$$
$$=\frac{1}{1-e^{-4s}}\mathcal{L}\left\{\frac{t^2}{4}-\frac{1}{2}(t^2-2t+1)h(t-1)+\frac{1}{2}(t^2-6t+9)h(t-3)-\frac{1}{4}(t-4)^2h(t-4)\right\}$$
$$=\frac{1}{4(1-e^{-4s})}\Bigg[\frac{2}{s^3}-2e^{-s}\mathcal{L}\{(t+1)^2-2(t+1)+1\}+2e^{-3s}\mathcal{L}\{(t+3)^2-6(t+3)+9\}$$
$$-e^{-4s}\mathcal{L}\{(t+4-4)^2\}\Bigg]$$
$$=\frac{1}{4(1-e^{-4s})}\left[\frac{2}{s^3}-2e^{-s}\mathcal{L}\{t^2\}+2e^{-3s}\mathcal{L}\{t^2\}-e^{-4s}\mathcal{L}\{t^2\}\right]$$
$$=\frac{1}{4(1-e^{-4s})}\left[\frac{2}{s^3}+\frac{2}{s^3}(-2e^{-s}+2e^{-3s}-e^{-4s})\right]=\frac{1}{2s^3(1-e^{-4s})}(1-2e^{-s}+2e^{-3s}-e^{-4s})$$
$$=\frac{(1-e^{-s})^2(1-e^{-2s})}{2s^3(1+e^{-2s})(1-e^{-2s})}=\frac{(1-e^{-s})^2}{2s^3(1+e^{-2s})}$$

EXERCISES 16.3

2. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 2[sY - 1] - Y = \frac{1}{s-1}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{(s-1)(s^2+2s-1)} + \frac{s+4}{s^2+2s-1}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s^2+2s-1)} + \frac{s+4}{s^2+2s-1}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s-1} + \frac{s/2+5/2}{s^2+2s-1}\right\} \\
&= \frac{1}{2}\left[e^t + \mathcal{L}^{-1}\left\{\frac{s+5}{s^2+2s-1}\right\}\right] = \frac{1}{2}\left[e^t + \mathcal{L}^{-1}\left\{\frac{(s+1)+4}{(s+1)^2-2}\right\}\right] \\
&= \frac{1}{2}\left[e^t + e^{-t}\mathcal{L}^{-1}\left\{\frac{s+4}{s^2-2}\right\}\right] = \frac{1}{2}e^t + \frac{1}{2}e^{-t}\mathcal{L}^{-1}\left\{\frac{-\sqrt{2}+1/2}{s+\sqrt{2}} + \frac{\sqrt{2}+1/2}{s-\sqrt{2}}\right\} \\
&= \frac{1}{2}e^t + \frac{1}{2}e^{-t}\left[\left(\frac{1}{2}-\sqrt{2}\right)e^{-\sqrt{2}t} + \left(\frac{1}{2}+\sqrt{2}\right)e^{\sqrt{2}t}\right] \\
&= \frac{1}{2}e^t + \left(\frac{1}{4}-\frac{\sqrt{2}}{2}\right)e^{-(1+\sqrt{2})t} + \left(\frac{1}{4}+\frac{\sqrt{2}}{2}\right)e^{(-1+\sqrt{2})t}.
\end{aligned}$$

4. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(0) - 1] + 2[sY] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2+2s+1} + \frac{1}{s^2(s^2+2s+1)} = \frac{1}{(s+1)^2} + \frac{1}{s^2(s+1)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2} + \frac{1}{s^2(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{(s+1)^2} - \frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1}\right\} \\
&= -2 + t + 2e^{-t} + e^{-t}\mathcal{L}^{-1}\left\{\frac{2}{s^2}\right\} = t - 2 + 2e^{-t} + 2te^{-t}.
\end{aligned}$$

6. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) + 2] + Y = \frac{1}{s^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-2}{s^2+1} + \frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} + \frac{s-3}{s^2+1}\right\} = t + \cos t - 3\sin t.$$

8. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - 2s - 1] + 6[sY - 2] + Y = \frac{3}{s^2+9}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s+13}{s^2+6s+1} + \frac{3}{(s^2+9)(s^2+6s+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s+13}{s^2+6s+1} + \frac{3}{(s^2+9)(s^2+6s+1)}\right\} \\
&= \mathcal{L}^{-1}\left\{-\frac{9s/194+12/194}{s^2+9} + \frac{397s/194+2588/194}{s^2+6s+1}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{1}{194}\mathcal{L}^{-1}\left\{\frac{397(s+3)+1397}{(s+3)^2-8}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{194}\mathcal{L}^{-1}\left\{\frac{397s+1397}{s^2-8}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{194}\mathcal{L}^{-1}\left\{\frac{(794\sqrt{2}-1397)/(4\sqrt{2})}{s+2\sqrt{2}} + \frac{(794\sqrt{2}+1397)/(4\sqrt{2})}{s-2\sqrt{2}}\right\} \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{e^{-3t}}{776\sqrt{2}}\left[(794\sqrt{2}-1397)e^{-2\sqrt{2}t} + (1397+794\sqrt{2})e^{2\sqrt{2}t}\right] \\
&= -\frac{9}{194}\cos 3t - \frac{2}{97}\sin 3t + \frac{1}{776\sqrt{2}}\left[(794\sqrt{2}-1397)e^{-(3+2\sqrt{2})t} + (1397+794\sqrt{2})e^{(-3+2\sqrt{2})t}\right].
\end{aligned}$$

10. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(-1) - 2] - 4[sY - (-1)] + 5Y = \frac{1}{(s+3)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{-s+6}{s^2-4s+5} + \frac{1}{(s+3)^2(s^2-4s+5)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{5/338}{s+3} + \frac{1/26}{(s+3)^2} + \frac{-343s/338+2050/338}{s^2-4s+5}\right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{1}{338}\mathcal{L}^{-1}\left\{\frac{-343(s-2)+1364}{(s-2)^2+1}\right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}\mathcal{L}^{-1}\left\{\frac{-343s+1364}{s^2+1}\right\} \\
&= \frac{5}{338}e^{-3t} + \frac{t}{26}e^{-3t} + \frac{e^{2t}}{338}(-343\cos t + 1364\sin t).
\end{aligned}$$

12. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] - 4Y = \mathcal{L}\{\cos^2 t\} = \mathcal{L}\left\{\frac{1+\cos 2t}{2}\right\} = \frac{1}{2s} + \frac{s}{2(s^2+4)}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{2s(s^2+2s-4)} + \frac{s}{2(s^2+2s-4)(s^2+4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2s(s^2+2s-4)} + \frac{s}{2(s^2+2s-4)(s^2+4)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{-1/8}{s} + \frac{-s/20+1/20}{s^2+4} + \frac{7s/40+12/40}{s^2+2s-4}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{7(s+1)+5}{(s+1)^2-5}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{40}\mathcal{L}^{-1}\left\{\frac{7s+5}{s^2-5}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{40}\mathcal{L}^{-1}\left\{\frac{(7\sqrt{5}-5)/(2\sqrt{5})}{s+\sqrt{5}} + \frac{(7\sqrt{5}+5)/(2\sqrt{5})}{s-\sqrt{5}}\right\}\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{e^{-t}}{80\sqrt{5}}[(7\sqrt{5}-5)e^{-\sqrt{5}t} + (7\sqrt{5}+5)e^{\sqrt{5}t}]\\
&= -\frac{1}{8} - \frac{1}{20}\cos 2t + \frac{1}{40}\sin 2t + \frac{1}{80}[(7-\sqrt{5})e^{-(1+\sqrt{5})t} + (7+\sqrt{5})e^{(-1+\sqrt{5})t}].
\end{aligned}$$

14. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 4[sY] - 2Y = \frac{4}{s^2+16}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{(s^2+16)(s^2+4s-2)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{4}{(s^2+16)(s^2+4s-2)}\right\} = \mathcal{L}^{-1}\left\{\frac{-4s/145-18/145}{s^2+16} + \frac{4s/145+34/145}{s^2+4s-2}\right\}\\
&= \frac{1}{145}\left[-4\cos 4t - \frac{9}{2}\sin 4t + \mathcal{L}^{-1}\left\{\frac{4(s+2)+26}{(s+2)^2-6}\right\}\right]\\
&= \frac{1}{290}\left[-8\cos 4t - 9\sin 4t + 2e^{-2t}\mathcal{L}^{-1}\left\{\frac{4s+26}{s^2-6}\right\}\right]\\
&= \frac{1}{290}\left[-8\cos 4t - 9\sin 4t + 4e^{-2t}\mathcal{L}^{-1}\left\{\frac{(2\sqrt{6}-13)/(2\sqrt{6})}{s+\sqrt{6}} + \frac{(2\sqrt{6}+13)/(2\sqrt{6})}{s-\sqrt{6}}\right\}\right]\\
&= \frac{1}{290}\left\{-8\cos 4t - 9\sin 4t + 4e^{-2t}\left[\left(1-\frac{13}{2\sqrt{6}}\right)e^{-\sqrt{6}t} + \left(1+\frac{13}{2\sqrt{6}}\right)e^{\sqrt{6}t}\right]\right\}\\
&= -\frac{1}{290}(8\cos 4t + 9\sin 4t) + \frac{\sqrt{6}}{870}\left[(2\sqrt{6}-13)e^{-(2+\sqrt{6})t} + (2\sqrt{6}+13)e^{(-2+\sqrt{6})t}\right].
\end{aligned}$$

16. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y] + 2[sY] + Y = \mathcal{L}\{t[h(t) - h(t-1)]\} = \frac{1}{s^2} - e^{-s}\mathcal{L}\{t+1\} = \frac{1}{s^2} - e^{-s}\left(\frac{1}{s^2} + \frac{1}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{1}{s^2(s^2+2s+1)} - \frac{e^{-s}(s+1)}{s^2(s^2+2s+1)} = \frac{1}{s^2(s+1)^2} + \frac{e^{-s}}{s^2(s+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{1}{s^2(s+1)^2} + \frac{e^{-s}}{s^2(s+1)}\right\} \\ &= \mathcal{L}^{-1}\left\{-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{1}{(s+1)^2} - e^{-s}\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)\right\} \\ &= -2 + t + 2e^{-t} + te^{-t} + [1 - (t-1) - e^{-(t-1)}]h(t-1). \end{aligned}$$

18. We set $y'(0) = A$. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - A] + 3[sY - 1] - 4Y = \frac{2}{s+4}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+A+3}{s^2+3s-4} + \frac{2}{(s+4)(s^2+3s-4)}.$$

The inverse transform of this function is the solution of the boundary-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+A+3}{s^2+3s-4} + \frac{2}{(s+4)(s^2+3s-4)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+A+3}{(s+4)(s-1)} + \frac{2}{(s+4)^2(s-1)}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{5A/25 + 22/25}{s-1} + \frac{(3-5A)/25}{s+4} - \frac{2/5}{(s+4)^2}\right\} \\ &= \left(\frac{A}{5} + \frac{22}{25}\right)e^t + \left(\frac{3}{25} - \frac{A}{5}\right)e^{-4t} - \frac{2t}{5}e^{-4t}. \end{aligned}$$

Since $y(1) = 1$,

$$1 = \left(\frac{A}{5} + \frac{22}{5}\right)e + \left(\frac{3}{25} - \frac{A}{5}\right)e^{-4} - \frac{2}{5}e^{-4} \quad \Longrightarrow \quad A = \frac{25e^4 - 22e^5 + 7}{5(e^5 - 1)}.$$

Thus,

$$\begin{aligned} y(t) &= \left[\frac{25e^4 - 22e^5 + 7}{25(e^5 - 1)} + \frac{22}{25}\right]e^t + \left[\frac{3}{25} - \frac{25e^4 - 22e^5 + 7}{25(e^5 - 1)}\right]e^{-4t} - \frac{2t}{5}e^{-4t} \\ &= \left(\frac{5e^4 - 3}{5e^5 - 5}\right)e^t + \left(\frac{5e^5 - 5e^4 - 2}{5e^5 - 5}\right)e^{-4t} - \frac{2t}{5}e^{-4t}. \end{aligned}$$

20. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1)] - 4[sY - 1] + 3Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s-4}{s^2-4s+3} + \frac{F(s)}{s^2-4s+3} = \frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = \mathcal{L}^{-1}\left\{\frac{s-4}{(s-1)(s-3)} + \frac{F(s)}{(s-1)(s-3)}\right\} = \mathcal{L}^{-1}\left\{\frac{3/2}{s-1} + \frac{-1/2}{s-3} + \left(\frac{-1/2}{s-1} + \frac{1/2}{s-3}\right)F(s)\right\}$$
$$= \frac{3}{2}e^t - \frac{1}{2}e^{3t} + \frac{1}{2}\int_0^t [-e^{t-u} + e^{3(t-u)}]f(u)\,du.$$

22. We set $y(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] + 16Y = F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As+B}{s^2+16} + \frac{F(s)}{s^2+16}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t) = A\cos 4t + \frac{B}{4}\sin 4t + \frac{1}{4}\int_0^t \sin 4(t-u)f(u)\,du$$
$$= A\cos 4t + C\sin 4t + \frac{1}{4}\int_0^t \sin 4(t-u)f(u)\,du.$$

24. Since $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and $\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$,

$$f(t) = \int_0^t e^{-u}\,du = \{-e^{-u}\}_0^t = 1 - e^{-t}.$$

26. Since $\mathcal{L}^{-1}\left\{\frac{1}{s+4}\right\} = e^{-4t}$ and

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-2}\right\} = \mathcal{L}^{-1}\left\{\frac{1/2}{s+\sqrt{2}} + \frac{1/2}{s-\sqrt{2}}\right\} = \frac{1}{2}(e^{-\sqrt{2}t} + e^{\sqrt{2}t}),$$

it follows that

$$f(t) = \frac{1}{2}\int_0^t (e^{-\sqrt{2}u} + e^{\sqrt{2}u})e^{-4(t-u)}\,du = \frac{1}{2}\int_0^t \left[e^{-4t+(4-\sqrt{2})u} + e^{-4t+(4+\sqrt{2})u}\right]du$$
$$= \frac{1}{2}\left\{\frac{e^{-4t+(4-\sqrt{2})u}}{4-\sqrt{2}} + \frac{e^{-4t+(4+\sqrt{2})u}}{4+\sqrt{2}}\right\}_0^t = \frac{1}{2}\left[\frac{e^{-\sqrt{2}t} - e^{-4t}}{4-\sqrt{2}} + \frac{e^{\sqrt{2}t} - e^{-4t}}{4+\sqrt{2}}\right]$$
$$= \frac{1}{2}\left[\left(\frac{4+\sqrt{2}}{14}\right)e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{14}\right)e^{\sqrt{2}t} + \left(-\frac{4+\sqrt{2}}{14} - \frac{4-\sqrt{2}}{14}\right)e^{-4t}\right]$$
$$= \left(\frac{4+\sqrt{2}}{28}\right)e^{-\sqrt{2}t} + \left(\frac{4-\sqrt{2}}{28}\right)e^{\sqrt{2}t} - \frac{2}{7}e^{-4t}.$$

28. We set $y'(0) = A$ and $y'(0) = B$. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - As - B] - 2[sY - A] + 4Y = \frac{2}{s^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{As + B - 2A}{s^2 - 2s + 4} + \frac{2}{s^3(s^2 - 2s + 4)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+4}+\frac{2}{s^3(s^2-2s+4)}\right\} = \mathcal{L}^{-1}\left\{\frac{As+B-2A}{s^2-2s+4}+\frac{1/4}{s^2}+\frac{1/2}{s^3}-\frac{1/4}{s^2-2s+4}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{As+C}{(s-1)^2+3}+\frac{1/4}{s^2}+\frac{1/2}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{A(s-1)+D}{(s-1)^2+3}+\frac{1/4}{s^2}+\frac{1/2}{s^3}\right\} \\ &= e^t\mathcal{L}^{-1}\left\{\frac{As+D}{s^2+3}\right\}+\frac{t}{4}+\frac{t^2}{4} = e^t\left(A\cos\sqrt{3}t+\frac{D}{\sqrt{3}}\sin\sqrt{3}t\right)+\frac{t}{4}+\frac{t^2}{4} \\ &= e^t(A\cos\sqrt{3}t+E\sin\sqrt{3}t)+\frac{t}{4}+\frac{t^2}{4}. \end{aligned}$$

30. We set $y'(0)=A$ and $y'(0)=B$. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y-As-B]+Y=F(s).$$

We solve this for the transform $Y(s)$,

$$Y(s)=\frac{As+B}{s^2+1}+\frac{F(s)}{s^2+1}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(t)=\mathcal{L}^{-1}\left\{\frac{As+B}{s^2+1}+\frac{F(s)}{s^2+1)}\right\}=A\cos t+B\sin t+\int_0^t f(u)\sin(t-u)\,du.$$

32. We set $y'(0)=A$ and $y'(0)=B$. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y-As-B]+4[sY-A]+Y=\frac{1}{s^2}+\frac{2}{s}.$$

We solve this for the transform $Y(s)$,

$$Y(s)=\frac{As+B+4A}{s^2+4s+1}+\frac{2s+1}{s^2(s^2+4s+1)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B+4A}{s^2+4s+1}+\frac{2s+1}{s^2(s^2+4s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{As+B+4A}{s^2+4s+1}-\frac{2}{s}+\frac{1}{s^2}+\frac{2s+7}{s^2+4s+1}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{Cs+D}{s^2+4s+1}-\frac{2}{s}+\frac{1}{s^2}\right\} = \mathcal{L}^{-1}\left\{\frac{C(s+2)+D-2C}{(s+2)^2-3}\right\}-2+t \\ &= t-2+e^{-2t}\mathcal{L}^{-1}\left\{\frac{Cs+E}{s^2-3}\right\} = t-2+e^{-2t}\mathcal{L}^{-1}\left\{\frac{F}{s+\sqrt{3}}+\frac{G}{s-\sqrt{3}}\right\} \\ &= t-2+e^{-2t}(Fe^{-\sqrt{3}t}+Ge^{\sqrt{3}t}) = t-2+Fe^{-(2+\sqrt{3})t}+Ge^{(-2+\sqrt{3})t}. \end{aligned}$$

34. We set $y'(0)=A$ and $y'(0)=B$. Assuming that the solution of $y''+9y=te^{ti}$ satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y-As-B]+9Y=\frac{1}{(s-i)^2}.$$

We solve this for the transform $Y(s)$,

$$Y(s)=\frac{As+B}{s^2+9}+\frac{1}{(s-i)^2(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{As+B}{s^2+9} + \frac{1}{(s-i)^2(s^2+9)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{As+B}{s^2+9} - \frac{i/32}{s-i} + \frac{1/8}{(s-i)^2} + \frac{-is/32 - 5/32}{s^2+9}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{Cs+D}{s^2+9} - \frac{i/32}{s-i} + \frac{1/8}{(s-i)^2}\right\} = C\cos 3t + \frac{D}{3}\sin 3t - \frac{i}{32}e^{ti} + \frac{t}{8}e^{ti}.
\end{aligned}$$

If we take imaginary parts, we get $y(t) = C\cos 3t + E\sin 3t - \dfrac{1}{32}\cos t + \dfrac{t}{8}\sin t$.

36. Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^3Y - s^2(1) + 2] - 3[s^2Y - s(1)] + 3[sY - 1] - Y = \frac{2}{(s-1)^3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s^2 - 3s + 1}{s^3 - 3s^2 + 3s - 1} + \frac{2}{(s-1)^3(s^3 - 3s^2 + 3s - 1)} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-1} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}\right\} \\
&= e^t\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s^2} - \frac{1}{s^3} + \frac{2}{s^6}\right\} = e^t\left(1 - t - \frac{t^2}{2} + \frac{t^5}{60}\right).
\end{aligned}$$

38. The initial-value problem is

$$\frac{1}{5}\frac{d^2x}{dt^2} + 10x = 0 \quad \Longrightarrow \quad x'' + 50x = 0, \qquad x(0) = -0.03, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2X + 0.03s] + 50X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = -\frac{0.03s}{s^2 + 50}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{-\frac{0.03s}{s^2 + 50}\right\} = -0.03\cos 5\sqrt{2}t \text{ m}.$$

40. The initial-value problem is

$$\frac{1}{5}\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 10x = 4\sin 10t \quad \Longrightarrow \quad 2x'' + 15x' + 100x = 40\sin 10t, \qquad x(0) = 0, \quad x'(0) = 0.$$

Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2[s^2X] + 15[sX] + 100X = \frac{400}{s^2 + 100}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{400}{(s^2+100)(2s^2+15s+100)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{400}{(s^2+100)(2s^2+15s+100)}\right\} = \mathcal{L}^{-1}\left\{\frac{-12s/65-80/65}{s^2+100} + \frac{24s/65+340/65}{2s^2+15s+100}\right\} \\
&= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{1}{65}\mathcal{L}^{-1}\left\{\frac{12s+170}{s^2+15s/2+50}\right\} \\
&= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{1}{65}\mathcal{L}^{-1}\left\{\frac{12(s+15/4)+125}{(s+15/4)^2+575/16}\right\} \\
&= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{e^{-15t/4}}{65}\mathcal{L}^{-1}\left\{\frac{12s+125}{s^2+575/16}\right\} \\
&= -\frac{1}{65}(12\cos 10t + 8\sin 10t) + \frac{e^{-15t/4}}{65}\left(12\cos\frac{5\sqrt{23}t}{4} + \frac{100}{\sqrt{23}}\sin\frac{5\sqrt{23}t}{4}\right) \text{ m.}
\end{aligned}$$

42. The initial-value problem is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{1}{20}\frac{dx}{dt} + 5x = 0 \implies 2x'' + x' + 100x = 0, \qquad x(0) = -\frac{1}{20}, \quad x'(0) = 2.$$

Assuming that the solution satisfies the conditions of Corollary 16.5.1, we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$2\left[s^2X + \frac{s}{20} - 2\right] + \left[sX + \frac{1}{20}\right] + 100X = 0.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{-s/10+79/20}{2s^2+s+100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{-s/10+79/20}{2s^2+s+100}\right\} = \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{-2s+79}{s^2+s/2+50}\right\} \\
&= \frac{1}{40}\mathcal{L}^{-1}\left\{\frac{-2(s+1/4)+159/2}{(s+1/4)^2+799/16}\right\} = \frac{e^{-t/4}}{40}\mathcal{L}^{-1}\left\{\frac{-2s+159/2}{s^2+799/16}\right\} \\
&= \frac{e^{-t/4}}{40}\left[-2\cos\frac{\sqrt{799}t}{4} + \frac{159}{2}\left(\frac{4}{\sqrt{799}}\right)\sin\frac{\sqrt{799}t}{4}\right] \\
&= \frac{e^{-t/4}}{20}\left(\frac{159}{\sqrt{799}}\sin\frac{\sqrt{799}t}{4} - \cos\frac{\sqrt{799}t}{4}\right) \text{ m.}
\end{aligned}$$

EXERCISES 16.4

2. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$[s^2Y - s(1) - 2] + 9Y = \mathcal{L}\{2[h(t) - h(t-4)]\} = 2\left(\frac{1}{s} - \frac{e^{-4s}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + \frac{2(1-e^{-4s})}{s(s^2+9)}\right\} = \mathcal{L}^{-1}\left\{\frac{s+2}{s^2+9} + 2\left(\frac{1/9}{s} + \frac{-s/9}{s^2+9}\right)(1-e^{-4s})\right\} \\ &= \cos 3t + \frac{2}{3}\sin 3t + \frac{2}{9}(1-\cos 3t) - \frac{2}{9}[1-\cos 3(t-4)]h(t-4). \\ &= \frac{2}{9} + \frac{7}{9}\cos 3t + \frac{2}{3}\sin 3t - \frac{2}{9}[1-\cos 3(t-4)]h(t-4). \end{aligned}$$

4. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(-1)] + 4[sY+1] + 4Y &= \mathcal{L}\{(2-t)[h(t)-h(t-2)] + (t-2)h(t-2)\} \\ &= \mathcal{L}\{2 - t + 2(t-2)h(t-2)\} = \frac{2}{s} - \frac{1}{s^2} + 2e^{-2s}\mathcal{L}\{t\} \\ &= \frac{2}{s} - \frac{1}{s^2} + 2\frac{e^{-2s}}{s^2}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = -\frac{s+4}{s^2+4s+4} + \frac{2}{s(s^2+4s+4)} - \frac{1-2e^{-2s}}{s^2(s^2+4s+4)} = -\frac{s+4}{(s+2)^2} + \frac{2}{s(s+2)^2} - \frac{1-2e^{-2s}}{s^2(s+2)^2}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{-\frac{s+4}{(s+2)^2} + \frac{2}{s(s+2)^2} - \frac{1-2e^{-2s}}{s^2(s+2)^2}\right\} \\ &= \mathcal{L}^{-1}\left\{\frac{1/2}{s} - \frac{3/2}{s+2} - \frac{3}{(s+2)^2} - \left[\frac{-1/4}{s} + \frac{1/4}{s^2} + \frac{1/4}{s+2} + \frac{1/4}{(s+2)^2}\right](1-2e^{-2s})\right\} \\ &= \frac{1}{2} - \frac{3}{2}e^{-2t} - 3te^{-2t} + \frac{1}{4}(1 - t - e^{-2t} - te^{-2t}) \\ &\quad + \frac{1}{2}[-1 + (t-2) + e^{-2(t-2)} + (t-2)e^{-2(t-2)}]h(t-2) \\ &= \frac{3}{4} - \frac{t}{4} - \frac{7}{4}e^{-2t} - \frac{13t}{4}e^{-2t} + \frac{1}{2}[-3 + t - e^{2(2-t)} + te^{2(2-t)}]h(t-2). \end{aligned}$$

6. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned} [s^2Y - s(1) - 2] + 4[sY-1] + 3Y &= \mathcal{L}\{\sin t[h(t) - h(t-\pi)]\} = \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{\sin(t+\pi)\} \\ &= \frac{1}{s^2+1} - e^{-\pi s}\mathcal{L}\{-\sin t\} = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}. \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{s+6}{s^2+4s+3} + \frac{1+e^{-\pi s}}{(s^2+1)(s^2+4s+3)} = \frac{s+6}{(s+1)(s+3)} + \frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{s+6}{(s+1)(s+3)} + \frac{1+e^{-\pi s}}{(s^2+1)(s+1)(s+3)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{5/2}{s+1} - \frac{3/2}{s+3} + \left(\frac{1/4}{s+1} - \frac{1/20}{s+3} + \frac{-s/5+1/10}{s^2+1}\right)(1+e^{-\pi s})\right\} \\
&= \frac{5}{2}e^{-t} - \frac{3}{2}e^{-3t} + \frac{1}{20}\left[5e^{-t} - e^{-3t} - 4\cos t + 2\sin t\right] \\
&\quad + \frac{1}{20}\left[5e^{-(t-\pi)} - e^{-3(t-\pi)} - 4\cos(t-\pi) + 2\sin(t-\pi)\right] h(t-\pi) \\
&= \frac{11}{4}e^{-t} - \frac{31}{20}e^{-3t} - \frac{1}{5}\cos t + \frac{1}{10}\sin t + \frac{1}{20}\left[5e^{\pi-t} - e^{3(\pi-t)} + 4\cos t - 2\sin t\right] h(t-\pi).
\end{aligned}$$

8. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y] + 2[sY] + 5Y &= \mathcal{L}\{4[h(t)-h(t-1)] - 4[h(t-1)-h(t-2)]\} = \mathcal{L}\{4 - 8h(t-1) + 4h(t-2)\} \\
&= \frac{4}{s} - \frac{8e^{-s}}{s} + \frac{4e^{-2s}}{s}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{4}{s(s^2+2s+5)}(1 - 2e^{-s} + e^{-2s}).$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{4}{s(s^2+2s+5)}(1-2e^{-s}+e^{-2s})\right\} = 4\mathcal{L}^{-1}\left\{\left(\frac{1/5}{s} - \frac{s/5+2/5}{s^2+2s+5}\right)(1-2e^{-s}+e^{-2s})\right\} \\
&= \frac{4}{5}\mathcal{L}^{-1}\left\{\left[\frac{1}{s} - \frac{(s+1)+1}{(s+1)^2+4}\right](1-2e^{-s}+e^{-2s})\right\} \\
&= \frac{4}{5}\left[1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)\right] - \frac{8}{5}\left\{1 - e^{-(t-1)}\left[\cos 2(t-1) + \frac{1}{2}\sin 2(t-1)\right]\right\} h(t-1) \\
&\quad + \frac{4}{5}\left\{1 - e^{-(t-2)}\left[\cos 2(t-2) + \frac{1}{2}\sin 2(t-2)\right]\right\} h(t-2) \\
&= \frac{2}{5}\left[2 - e^{-t}(2\cos 2t + \sin 2t)\right] + \frac{4}{5}\left\{-2 + e^{1-t}\left[2\cos 2(t-1) + \sin 2(t-1)\right]\right\} h(t-1) \\
&\quad + \frac{2}{5}\left\{2 - e^{2-t}\left[2\cos 2(t-2) + \sin 2(t-2)\right]\right\} h(t-2).
\end{aligned}$$

10. When we take Laplace transforms of both sides of the differential equation and use the initial conditions,

$$\begin{aligned}
[s^2Y - s(2)] + 16Y &= \frac{1}{1-e^{-2s}}\mathcal{L}\{t[h(t)-h(t-1)] + (2-t)[h(t-1)-h(t-2)]\} \\
&= \frac{1}{1-e^{-2s}}\mathcal{L}\left\{t + (2-2t)h(t-1) + (t-2)h(t-2)\right\} \\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} + e^{-s}\mathcal{L}\{2-2(t+1)\} + e^{-2s}\mathcal{L}\{t\}\right] \\
&= \frac{1}{1-e^{-2s}}\left[\frac{1}{s^2} - \frac{2e^{-s}}{s^2} + \frac{e^{-2s}}{s^2}\right] = \frac{(1-e^{-s})^2}{s^2(1-e^{-s})(1+e^{-s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})}.
\end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{2s}{s^2+16} + \frac{1-e^{-s}}{s^2(s^2+16)(1+e^{-s})}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1-e^{-s}}{s^2(s^2+16)(1+e^{-s})}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \left(\frac{1/16}{s^2} - \frac{1/16}{s^2+16}\right)(1-e^{-s})\sum_{n=0}^{\infty}(-1)^n e^{-ns}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{2s}{s^2+16} + \frac{1}{16}\left(\frac{1}{s^2} - \frac{1}{s^2+16}\right)\left[\sum_{n=0}^{\infty}(-1)^n e^{-ns} + \sum_{n=0}^{\infty}(-1)^{n+1}e^{-(n+1)s}\right]\right\} \\
&= 2\cos 4t + \frac{1}{16}\sum_{n=0}^{\infty}(-1)^n\left[(t-n) - \frac{1}{4}\sin 4(t-n)\right]h(t-n) \\
&\qquad + \frac{1}{16}\sum_{n=0}^{\infty}(-1)^{n+1}\left[(t-n-1) - \frac{1}{4}\sin 4(t-n-1)\right]h(t-n-1) \\
&= 2\cos 4t + \frac{1}{64}\sum_{n=0}^{\infty}(-1)^n\left[4(t-n) - \sin 4(t-n)\right]h(t-n) \\
&\qquad + \frac{1}{64}\sum_{n=0}^{\infty}(-1)^{n+1}\left[4(t-n-1) - \sin 4(t-n-1)\right]h(t-n-1).
\end{aligned}$$

12. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10 - 2}{s^2+400} + \frac{1000e^{-4s}}{s(s^2+400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10-2}{s^2+400} + \frac{1000e^{-4s}}{s(s^2+400)}\right\} \\
&= \mathcal{L}^{-1}\left\{\frac{s/10-2}{s^2+400} + \frac{5}{2}\left(\frac{1}{s} - \frac{s}{s^2+400}\right)e^{-4s}\right\} \\
&= \frac{1}{10}\cos 20t - \frac{1}{10}\sin 20t + \frac{5}{2}[1 - \cos 20(t-4)]h(t-4) \text{ m.}
\end{aligned}$$

14. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 50\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 50\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10+3}{s^2+50s+400} + \frac{1000e^{-4s}}{s(s^2+50s+400)} = \frac{s/10+3}{(s+10)(s+40)} + \frac{1000e^{-4s}}{s(s+10)(s+40)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10+3}{(s+10)(s+40)} + \frac{1000e^{-4s}}{s(s+10)(s+40)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1/15}{s+10} + \frac{1/30}{s+40} + 1000\left(\frac{1/400}{s} - \frac{1/300}{s+10} + \frac{1/1200}{s+40}\right) - e^{-4s}\right\}\\
&= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \left[\frac{5}{2} - \frac{10}{3}e^{-10(t-4)} + \frac{5}{6}e^{-40(t-4)}\right]h(t-4)\\
&= \frac{1}{15}e^{-10t} + \frac{1}{30}e^{-40t} + \frac{5}{6}\left[3 - 4e^{10(4-t)} + e^{40(4-t)}\right]h(t-4)\ \text{m}.
\end{aligned}$$

16. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{1}{10}\frac{d^2x}{dt^2} + \frac{dx}{dt} + 40x = 100h(t-4), \qquad x(0) = \frac{1}{10}, \quad x'(0) = -2.$$

We write the differential equation in the form

$$\frac{d^2x}{dt^2} + 10\frac{dx}{dt} + 400x = 1000h(t-4),$$

and take Laplace transforms,

$$\left[s^2X - \frac{s}{10} + 2\right] + 10\left[sX - \frac{1}{10}\right] + 400X = \frac{1000e^{-4s}}{s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{s/10-1}{s^2+10s+400} + \frac{1000e^{-4s}}{s(s^2+10s+400)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned}
x(t) &= \mathcal{L}^{-1}\left\{\frac{s/10-1}{s^2+10s+400} + \frac{1000e^{-4s}}{s(s^2+10s+400)}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left(\frac{s-10}{s^2+10s+400}\right) + 1000\left(\frac{1/400}{s} - \frac{s/400+1/40}{s^2+10s+400}\right)e^{-4s}\right\}\\
&= \mathcal{L}^{-1}\left\{\frac{1}{10}\left[\frac{(s+5)-15}{(s+5)^2+375}\right] + \frac{5}{2}\left[\frac{1}{s} - \frac{(s+5)+5}{(s+5)^2+375}\right]e^{-4s}\right\}\\
&= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right)\\
&\qquad + \frac{5}{2}\left\{1 - e^{-5(t-4)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4)\\
&= \frac{e^{-5t}}{10}\left(\cos 5\sqrt{15}t - \frac{\sqrt{15}}{5}\sin 5\sqrt{15}t\right)\\
&\qquad + \frac{5}{2}\left\{1 - e^{5(4-t)}\left[\cos 5\sqrt{15}(t-4) + \frac{1}{\sqrt{15}}\sin 5\sqrt{15}(t-4)\right]\right\}h(t-4)\ \text{m}.
\end{aligned}$$

18. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 80\frac{dx}{dt} + 512x = \delta(t), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X] + 80[sX] + 512X = 1.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{1}{2s^2 + 80s + 512} = \frac{1}{2(s^2 + 40s + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} x(t) &= \mathcal{L}^{-1}\left\{\frac{1}{2(s^2+40s+256)}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{1}{(s+20)^2 - 144}\right\} \\ &= \frac{e^{-20t}}{2}\mathcal{L}^{-1}\left\{\frac{1}{s^2-144}\right\} = \frac{e^{-20t}}{2}\mathcal{L}^{-1}\left\{\frac{-1/24}{s+12} + \frac{1/24}{s-12}\right\} \\ &= \frac{e^{-20t}}{48}\left(-e^{-12t} + e^{12t}\right) = \frac{1}{48}\left(e^{-8t} - e^{-32t}\right) \text{ m.} \end{aligned}$$

20. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \qquad x(0) = x_0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$2[s^2X - x_0 s] + 512X = e^{-t_0 s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0 s}{2s^2 + 512} + \frac{e^{-t_0 s}}{2s^2 + 512} = \frac{x_0 s}{s^2 + 256} + \frac{e^{-t_0 s}}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0 s}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}\right\} = x_0 \cos 16t + \frac{1}{32}\sin 16(t - t_0)\, h(t - t_0) \text{ m.}$$

22. The initial-value problem for displacement of the mass from its equilibrium position is

$$2\frac{d^2x}{dt^2} + 512x = \delta(t - t_0), \qquad x(0) = x_0, \quad x'(0) = v_0.$$

When we take Laplace transforms,

$$2[s^2X - x_0 s - v_0] + 512X = e^{-t_0 s}.$$

We solve this for the transform $X(s)$,

$$X(s) = \frac{2x_0 s + 2v_0}{2s^2 + 512} + \frac{e^{-t_0 s}}{2s^2 + 512} = \frac{x_0 s + v_0}{s^2 + 256} + \frac{e^{-t_0 s}}{2(s^2 + 256)}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\frac{x_0 s + v_0}{s^2+256} + \frac{e^{-t_0 s}}{2(s^2+256)}\right\} = x_0 \cos 16t + \frac{v_0}{16}\sin 16t + \frac{1}{32}\sin 16(t - t_0)\, h(t - t_0) \text{ m.}$$

24. The initial-value problem for displacement of the mass from its equilibrium position is

$$\frac{d^2x}{dt^2} + 100x = \sum_{n=0}^{\infty} \delta(t-n), \qquad x(0) = 0, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$[s^2X] + 100X = \sum_{n=0}^{\infty} e^{-ns}.$$

We solve this for the transform $X(s)$,

$$X(s) = \sum_{n=0}^{\infty} \frac{e^{-ns}}{s^2+100}.$$

The inverse transform of this function is the solution of the initial-value problem

$$x(t) = \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty} \frac{e^{-ns}}{s^2+100}\right\} = \frac{1}{10}\sum_{n=0}^{\infty} \sin 10(t-n)\, h(t-n) \text{ m}.$$

EXERCISES 16.5

2. (a) The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{mg}{EIL}[h(x) - h(x-L)], \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

If we set $y''(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4Y - As - B = -\frac{mg}{EIL}\left(\frac{1}{s} - \frac{e^{-Ls}}{s}\right) = -\frac{mg}{EIL}\left(\frac{1-e^{-Ls}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} - \frac{mg}{EIL}\left(\frac{1-e^{-Ls}}{s^5}\right).$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} - \frac{mgx^4}{24EIL} + \frac{mg(x-L)^4}{24EIL}h(x-L).$$

Since the last term contributes nothing to the solution, we drop it from further consideration. The boundary conditions at $x = L$ require

$$0 = y''(L) = A + BL - \frac{mgL}{2EI}, \qquad 0 = y'''(L) = B - \frac{mg}{EI}.$$

The solution of these equations is $A = -\dfrac{mgL}{2EI}$ and $B = \dfrac{mg}{EI}$. Thus,

$$y(x) = -\frac{mgLx^2}{4EI} + \frac{mgx^3}{6EI} - \frac{mgx^4}{24EIL} = -\frac{mg}{24EIL}(x^4 - 4Lx^3 + 6L^2x^2).$$

(b) The deflection at $x = L$ is

$$y(L) = -\frac{mg}{24EIL}(L^4 - 4L^4 + 6L^4) = -\frac{mgL^3}{8EI}.$$

4. (a) The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{mg}{EIL}[h(x) - h(x-L)] - \frac{3Mg}{EIL}[h(x) - h(x-L/3)], \quad y(0) = y'(0) = 0, \; y(L) = y''(L) = 0.$$

If we set $y''(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$\begin{aligned} s^4Y - As - B &= -\frac{mg}{EIL}\left(\frac{1}{s} - \frac{e^{-Ls}}{s}\right) - \frac{3Mg}{EIL}\left(\frac{1}{s} - \frac{e^{-Ls/3}}{s}\right) \\ &= -\frac{mg}{EIL}\left(\frac{1-e^{-Ls}}{s}\right) - \frac{3Mg}{EIL}\left(\frac{1-e^{-Ls/3}}{s}\right). \end{aligned}$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} - \frac{mg}{EIL}\left(\frac{1-e^{-Ls}}{s^5}\right) - \frac{3Mg}{EIL}\left(\frac{1-e^{-Ls/3}}{s^5}\right).$$

The inverse transform of this function is the solution of the initial-value problem

$$\begin{aligned} y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} - \frac{mgx^4}{24EIL} + \frac{mg(x-L)^4}{24EIL}h(x-L) \\ - \frac{Mgx^4}{8EIL} + \frac{Mg}{8EIL}(x-L/3)^4h(x-L/3). \end{aligned}$$

Since the fourth term term contributes nothing to the solution, we drop it from further consideration. The boundary conditions at $x = L$ require

$$0 = y(L) = \frac{AL^2}{2} + \frac{BL^3}{6} - \frac{mgL^3}{24EI} - \frac{MgL^3}{8EI} + \frac{Mg}{8EIL}\left(\frac{2L}{3}\right)^4,$$
$$0 = y''(L) = A + BL - \frac{mgL}{2EI} - \frac{3MgL}{2EI} + \frac{3Mg}{2EIL}\left(\frac{2L}{3}\right)^2.$$

The solution of these equations is $A = -\dfrac{mgL}{8EI} - \dfrac{25MgL}{216EI}$ and $B = \dfrac{5mg}{8EI} + \dfrac{205Mg}{216EI}$.
Thus,

$$\begin{aligned} y(x) &= \left(-\frac{mgL}{16EI} - \frac{25MgL}{432EI}\right)x^2 + \left(\frac{5mg}{48EI} + \frac{205Mg}{1296EI}\right)x^3 - \frac{mgx^4}{24EIL} \\ &\quad - \frac{Mgx^4}{8EIL} + \frac{Mg}{8EIL}(x - L/3)^4 h(x - L/3) \\ &= -\frac{gL(27m + 25M)}{432EI}x^2 + \frac{g(135m + 205M)}{1296EI}x^3 - \frac{mgx^4}{24EIL} - \frac{Mgx^4}{8EIL} \\ &\quad + \frac{Mg}{8EIL}(x - L/3)^4 h(x - L/3). \end{aligned}$$

6. (a) The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{P}{EI}\delta(x - L/3), \qquad y(0) = y'(0) = 0, \quad y(L) = y'(L) = 0.$$

If we set $y''(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4 Y - As - B = -\frac{P}{EI}e^{-Ls/3}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} - \frac{Pe^{-Ls/3}}{EIs^4}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} - \frac{P}{6EI}(x - L/3)^3 h(x - L/3).$$

The boundary conditions at $x = L$ require

$$0 = y(L) = \frac{AL^2}{2} + \frac{BL^3}{6} - \frac{P}{6EI}\left(\frac{2L}{3}\right)^3, \quad 0 = y'(L) = AL + \frac{BL^2}{2} - \frac{P}{2EI}\left(\frac{2L}{3}\right)^2.$$

The solution of these equations is $A = -\dfrac{4PL}{27EI}$ and $B = \dfrac{20P}{27EI}$.
Thus,

$$y(x) = -\frac{2PLx^2}{27EI} + \frac{10Px^3}{81EI} - \frac{P}{6EI}(x - L/3)^3 h(x - L/3).$$

(b) Since maximum deflection should be to the right of $x = L/3$, we set

$$0 = y'(x) = -\frac{4PLx}{27EI} + \frac{10Px^2}{27EI} - \frac{P}{2EI}(x - L/3)^2.$$

The solutions are $x = 3L/7$ and $x = L$. Maximum deflection is therefore at $x = 3L/7$.
(c) Theory indicates that with a delta function nonhomogeneity, $y(x)$, $y'(x)$, and $y''(x)$ should be continuous at $x = L/3$, but not $y'''(x)$. Let us show this. There is no question that the terms without the

Heaviside function have continuous derivatives of all orders. Consider then, the Heaviside term, less the leading constant, $f(x) = (x - L/3)^3 h(x - L/3)$. Clearly,

$$\lim_{x \to L/3^+} f(x) = \lim_{x \to L/3^-} f(x).$$

Since $f'(x) = 3(x - L/3)^2 h(x - L/3)$ and $f''(x) = 6(x - L/3)h(x - L/3)$, we also see that

$$\lim_{x \to L/3^+} f'(x) = \lim_{x \to L/3^-} f'(x) \quad \text{and} \quad \lim_{x \to L/3^+} f''(x) = \lim_{x \to L/3^-} f''(x).$$

On the other hand, since $f'''(x) = 6h(x - L/3)$,

$$\lim_{x \to L/3^+} f'''(x) = 6 \qquad \text{and} \qquad \lim_{x \to L/3^-} f'''(x) = 0.$$

8. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{P}{EI}\delta(x - L/2), \qquad y(0) = y'(0) = 0, \quad y''(L) = y'''(L) = 0.$$

If we set $y''(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4 Y - As - B = -\frac{P}{EI}e^{-Ls/2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^3} + \frac{B}{s^4} - \frac{Pe^{-Ls/2}}{EIs^4}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = \frac{Ax^2}{2} + \frac{Bx^3}{6} - \frac{P}{6EI}(x - L/2)^3 h(x - L/2).$$

The boundary conditions at $x = L$ require

$$0 = y''(L) = A + BL - \frac{P}{EI}\left(\frac{L}{2}\right), \quad 0 = y'''(L) = B - \frac{P}{EI}.$$

The solution of these equations is $A = -\dfrac{PL}{2EI}$ and $B = \dfrac{P}{EI}$.
Thus,

$$y(x) = -\frac{PLx^2}{4EI} + \frac{Px^3}{6EI} - \frac{P}{6EI}(x - L/2)^3 h(x - L/2).$$

10. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{P}{EI}\delta(x - L/2), \qquad y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0.$$

If we set $y'(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4 Y - As^2 - B = -\frac{P}{EI}e^{-Ls/2}.$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^2} + \frac{B}{s^4} - \frac{Pe^{-Ls/2}}{EIs^4}.$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = Ax + \frac{Bx^3}{6} - \frac{P}{6EI}(x - L/2)^3 h(x - L/2).$$

The boundary conditions at $x = L$ require

$$0 = y(L) = AL + \frac{BL^3}{6} - \frac{P}{6EI}\left(\frac{L}{2}\right)^3, \quad 0 = y''(L) = BL - \frac{P}{EI}\left(\frac{L}{2}\right).$$

The solution of these equations is $A = -\dfrac{PL^2}{16EI}$ and $B = \dfrac{P}{2EI}$.
Thus,

$$y(x) = -\frac{PL^2x}{16EI} + \frac{Px^3}{12EI} - \frac{P}{6EI}(x - L/2)^3 h(x - L/2).$$

12. In this situation, the beam will rotate until it is vertical, hanging from the pin at $x = 0$. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{mg}{EIL}[h(x) - h(x - L)], \qquad y(0) = y''(0) = 0, \quad y''(L) = y'''(L) = 0.$$

If we set $y'(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4Y - As^2 - B = -\frac{mg}{EIL}\left(\frac{1}{s} - \frac{e^{-Ls}}{s}\right) = -\frac{mg}{EIL}\left(\frac{1 - e^{-Ls}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^2} + \frac{B}{s^4} - \frac{mg}{EIL}\left(\frac{1 - e^{-Ls}}{s^5}\right).$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = Ax + \frac{Bx^3}{6} - \frac{mgx^4}{24EIL} + \frac{mg(x - L)^4}{24EIL}h(x - L).$$

Since the last term contributes nothing to the solution, we drop it from further consideration. The boundary conditions at $x = L$ require

$$0 = y''(L) = BL - \frac{mgL}{2EI}, \qquad 0 = y'''(L) = B - \frac{mg}{EI}.$$

These equations give conflicting values for B. Hence, the boundary-value problem does not give the physical solution.

14. The boundary-value problem for deflections of the beam is

$$\frac{d^4y}{dx^4} = -\frac{mg}{EIL}[h(x) - h(x - L)], \qquad y(0) = y''(0) = 0, \quad y(L) = y''(L) = 0.$$

If we set $y'(0) = A$ and $y'''(0) = B$, and take Laplace transforms, we obtain

$$s^4Y - As^2 - B = -\frac{mg}{EIL}\left(\frac{1}{s} - \frac{e^{-Ls}}{s}\right) = -\frac{mg}{EIL}\left(\frac{1 - e^{-Ls}}{s}\right).$$

We solve this for the transform $Y(s)$,

$$Y(s) = \frac{A}{s^2} + \frac{B}{s^4} - \frac{mg}{EIL}\left(\frac{1 - e^{-Ls}}{s^5}\right).$$

The inverse transform of this function is the solution of the initial-value problem

$$y(x) = Ax + \frac{Bx^3}{6} - \frac{mgx^4}{24EIL} + \frac{mg(x - L)^4}{24EIL}h(x - L).$$

Since the last term contributes nothing to the solution, we drop it from further consideration. The boundary conditions at $x = L$ require

$$0 = y(L) = AL + \frac{BL^3}{6} - \frac{mgL^3}{24EI}, \qquad 0 = y''(L) = BL - \frac{mgL}{2EI}.$$

The solution of these equations is $A = -\dfrac{mgL^2}{24EI}$ and $B = \dfrac{mg}{2EI}$. Thus,

$$y(x) = -\frac{mgL^2x}{24EI} + \frac{mgx^3}{12EI} - \frac{mgx^4}{24EIL} = -\frac{mg}{24EIL}(x^4 - 2Lx^3 + L^3x).$$

Maximum deflection occurs at $x = L/2$,

$$y(L/2) = -\frac{mg}{24EIL}\left[\left(\frac{L}{2}\right)^4 - 2L\left(\frac{L}{2}\right)^3 + L^3\left(\frac{L}{2}\right)\right] = -\frac{5mgL^3}{384EI}.$$

For this to be less than $L/100$,

$$\frac{5mgL^3}{384EI} < \frac{L}{100} \qquad \Longrightarrow \qquad m < \frac{384EIL}{500gL^3} = \frac{96EI}{125gL^2}.$$