MAT2130: Engineering Mathematical Analysis 1

Midterm 1 Practice Problems

1. Let A, B and C be three distinct, noncollinear points in 3D space. Let P, Q and R be the midpoints of AB, BC and AC respectively. Show that A, Q and the midpoint of PR are all collinear.

2. Let $A = (0, 2, 1)$ and $B = (2\sqrt{3}, 0, 1)$. Find a point C in the yz-plane such that ABC is an equilateral triangle.

3. Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors. Show that the vector $\widehat{\mathbf{u}} + \widehat{\mathbf{v}}$ makes equal angles with $\mathbf{u}$ and $\mathbf{v}$.

4. Let $\mathbf{v} = (2, -1, 1)$ and $\mathbf{u} = (1, 1, 1)$.

   (a) Find the component of $\mathbf{v}$ in the direction of $\mathbf{u}$.

   (b) Find a vector $\mathbf{w}$ of length 12 such that the component of $\mathbf{w}$ in the direction of $\mathbf{v}$ is $\sqrt{6}$ and the component of $\mathbf{w}$ in the direction of $\mathbf{u}$ is $2\sqrt{3}$.

5. Define a coordinate system such that the ceiling lies in the xy-plane. There are three hooks in the ceiling at the points $(0, 0, 0)$, $(2\sqrt{3}, 0, 0)$ and $(\sqrt{3}, 3, 0)$. A mass M hangs from these hooks by three cables of equal length d, where $d > 2$. The force of gravity on the mass is $|\mathbf{F}_g| = Mg$, in the negative z-direction. Find the tension in each cable as a function of M and d.

6. Find parametric, vector and symmetric equations (if applicable) for the following lines.

   (a) Through the points $(3, 4, -5)$ and $(0, 2, 3)$.

   (b) The intersection of the planes $x + 3y = 0$ and $2x - y + z = 2$.

   (c) Lying in the plane $2x - y - z - 1 = 0$, intersecting the line $\frac{8-x}{3} = y = \frac{z+7}{4}$ at right angles.

   (d) Through the intersection of the lines $(2, 4, 6) + t(1, 1, 1)$ and $(-4, 3, 6) + t(4, -1, -2)$, perpendicular to both.

7. Let $A = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$ be distinct points in 3D space.

   (a) Find the equation of the plane perpendicular to $\mathbf{AB}$ and passing through the midpoint of the line segment AB.

   (b) Let $P = (x, y, z)$ be any point on the plane from part (a). Show that P is equidistant from A and B.

8. Find equations for the following planes.

   (a) Through the point $(1, 0, 1)$, perpendicular to the vector $(-1, 1, -1)$.

   (b) Containing the points $(7, 5, 12)$ and $(1, -1, -6)$ and parallel to the vector $(0, 1, 2)$.

   (c) Containing the line $x = y - 1$, $z = -1$ and perpendicular to the plane $3x - 4y + z + 7 = 0$.

   (d) Containing the lines $(3, 10, -6) + t(1, 3, -2)$ and $(-10, -1, 6) + t(5, 1, -3)$.

   (e) Containing the lines $\frac{x-1}{2} = \frac{y}{3} = z - 1$ and $x - 2 = \frac{2y-1}{3} = 2z - 3$.

9. Find the following distances.

   (a) Between the point $(-1, 2, 3)$ and the line $\mathbf{r} = (0, 1, 0) + t(0, 1, 1)$.
   (b) Between the point $(1, 1, 0)$ and the point $y - 2z = 0$.
   (c) Between the lines $x = 1$, $\frac{y-1}{3} = \frac{4-z}{2}$ and $x - 1 = \frac{1-y}{2}$, $z = 0$.
   (d) Between the line $(0, 2, 0) + t(1, 0, -5)$ and the plane $5x + 2y + z - 2 = 0$.

10. Let ℓ be the line $(3, 2, 3)+t(1, -1, 0)$, $t \in \mathbb{R}$, and let P be the point $(5, 0, 0)$. Find the distance between P and an arbitrary point Q on ℓ as a function of the parameter t. Use calculus to minimize this function and find the point R on ℓ that is closest to P. Verify that $\mathbf{PR}$ is perpendicular to ℓ.

11. Sketch the surface in 3D space described by each equation. Clearly label important features (e.g. cross sections, asymptotes, intercepts...) and show your work.

   (a) $x^2 + 3z^2 = 27$
   (b) $x^2 + y^2 + z^2 - 2x + 8y - 4z = 12$
   (c) $x = \sqrt{(y - 2)^2 + z^2}$
   (d) $z = \sqrt{4 - x^2 - y^2}$
   (e) $xy = -2$

12. Sketch each surface in 3D space, and write down an equation in the variables (x, y, z) that describes it.

   (a) A sphere with radius 5 and center $(1, 0, -1)$.
   (b) A cylinder whose cross section is a circle with radius 2, centered on the line $x = 0$, $z = 5$.
   (c) A cone opening in the negative x-direction, centered on the x-axis, with its vertex at $(1, 0, 0)$. Its cross sections in the xz- and yz-planes consist of lines of slope ± 1.
   (d) A hemisphere, center $(0, 1, 0)$, radius 1, restricted to the region where $x \geq 0$.

13. Find a parametric representation for each curve in 3D space, subject to the given conditions (if any).

   (a) The intersection of $x + y = 1$, $y^2 + z^2 = 4$.
   (b) The intersection of $x = y + z^2$, $x = y$.
   (c) The intersection of $x = y + z^2$, $x + 2y = 0$.
   (d) The intersection of $x + y = 1$, $z = x^2 + y^2 + 4$, such that z decreases when y is negative.
   (e) The intersection of $y = \sqrt{1 - (x - 1)^2 - z^2}$ and $x = 1$, such that the curve starts with $z > 0$ and ends with $z < 0$.

14. Perform the indicated calculation.

   (a) $\frac{d\mathbf{v}}{dt}$, where $\mathbf{v}(t) = \cos t\widehat{\mathbf{i}} - \cos t\widehat{\mathbf{j}} + 3t^2\widehat{\mathbf{k}}$.
   (b) $\frac{d}{dt}(\mathbf{u} \cdot \mathbf{v})$, where $\mathbf{u}(t) = \frac{1}{\sqrt{t}}\widehat{\mathbf{i}} + t^{5/2}\widehat{\mathbf{j}} + te^{2t}\widehat{\mathbf{k}}$ and $\mathbf{v}(t) = (t^2 - 5)\widehat{\mathbf{i}} + \ln(t^2 + 2)\widehat{\mathbf{j}} + \sin t\widehat{\mathbf{k}}$.
   (c) $\frac{d}{dt}(\mathbf{u} \times \mathbf{v})$, where $\mathbf{u}(t) = 4\cos t\widehat{\mathbf{i}} + 2\sin t\widehat{\mathbf{k}}$ and $\mathbf{v}(t) = e^{t^2}\widehat{\mathbf{i}} + \frac{t^2}{2}\widehat{\mathbf{j}}$.
   (d) $\int \mathbf{v}(t)\, dt$, where $\mathbf{v}(t) = (4t^3 + 6t^2)\widehat{\mathbf{j}} - \frac{1}{t^2}\widehat{\mathbf{k}}$.
   (e) $\int \mathbf{v}(t)\, dt$, where $\mathbf{v}(t) = \frac{1}{2t}\widehat{\mathbf{i}} + \cos\frac{t}{2}\widehat{\mathbf{j}} + e^{3t}\widehat{\mathbf{k}}$.

15. Find a tangent vector $\mathbf{T}$ to each curve, subject to the given constraints. There may be more than one possible answer.

   (a) $\mathbf{T}$ is tangent to the curve $\mathbf{r}(t) = 3\cos t\widehat{\mathbf{i}} + 2\sin t\widehat{\mathbf{j}} - \frac{t^2}{\pi^2}\widehat{\mathbf{k}}$ at the point $\left(-\frac{3}{\sqrt{2}}, \sqrt{2}, -\frac{25}{16}\right)$, pointing in the direction of increasing t.

   (b) $\mathbf{T}$ is a vector of length 2, tangent to the curve $\mathbf{r}(t) = 3t^2\widehat{\mathbf{i}} + (2t-7)\widehat{\mathbf{j}} + e^t\widehat{\mathbf{k}}$ at $t = 0$.

   (c) $\mathbf{T}$ is tangent to the curve $t\sin 2\pi t\widehat{\mathbf{i}} - t\cos 2\pi t\widehat{\mathbf{j}} + t\ln t\widehat{\mathbf{k}}$ at $t = \frac{1}{2}$, pointing in the direction of decreasing t.

   (d) $\widehat{\mathbf{T}}$ is a unit vector tangent to the curve $\mathbf{r}(t) = (3t^3 - 2t)\widehat{\mathbf{i}} + e^{2t}\widehat{\mathbf{j}} + (t^2+1)^{3/2}\widehat{\mathbf{k}}$ at $t = 0$, pointing in the direction of increasing t.

   (e) $\mathbf{T}$ is tangent to the curve $\mathbf{r}(t) = t^{1/2}\widehat{\mathbf{i}} - t^2\widehat{\mathbf{j}} + (t+2)\widehat{\mathbf{k}}$ at the point $(2, -16, 6)$, pointing in the direction of increasing t.

16. Find the length of each curve over the specified interval.

   (a) $x = -3t$, $y = \cos t$, $z = \sin t$, $0 \leq t \leq 4\pi$.

   (b) $x = t^2$, $y = \frac{1}{3}t^3$, $z = 4$, $1 \leq t \leq 2$.

   (c) $x = t\cos t$, $y = t\sin t$, $z = \frac{2\sqrt{2}}{3}t^{3/2}$, $0 \leq t \leq 2$.

17. Find a parametric representation for the line through the point $P = (P_x, P_y, P_z)$, parallel to the vector $\mathbf{v} = (v_x, v_y, v_z)$. Let $T \in \mathbb{R}$ be arbitrary. Find the length of the segment of this line between $t = 0$ and $t = T$:

   (a) by finding its endpoints and using the formula for the length of a line segment;

   (b) using the formula for the length of a curve.

18. Consider the curve $\mathbf{r}(t) = t\widehat{\mathbf{i}} + 2t^{3/2}\widehat{\mathbf{j}} + (2\sqrt{2}t + 1)\widehat{\mathbf{k}}$, where $t \geq 0$.

   (a) Find the length of the curve from $t = 0$ to an arbitrary $t > 0$. Let this length be $s(t)$.

   (b) Solve for t in terms of s, and reparametrize the curve in terms of s.

   (c) Verify that the new parametrization $\mathbf{r}(s)$ satisfies $|\mathbf{r}'(s)| = 1$ everywhere.