UNIVERSITY OF MANITOBA

DATE: October 15, 2015 | TERM TEST 1



EXAMINATION: Engineering Mathematical Analysis 2 | TIME: 60 minutes

COURSE: MATH 2132 | EXAMINER: M. Virgilio

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1. Determine whether the sequence converges or diverges, and if it converges, find the limit.

[6] (a) $\left\{\left(\frac{n-4}{n+2}\right)^n\right\}$

$$\lim_{n\to\infty}\left(\frac{n-4}{n+2}\right)^n = \lim_{x\to\infty}\left(\frac{x-4}{x+2}\right)^x = L$$

$$\ln L = \ln \lim_{x\to\infty}\left(\frac{x-4}{x+2}\right)^x = \lim_{x\to\infty} \ln\left(\frac{x-4}{x+2}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x-4}{x+2}\right)$$

$$\left(\frac{0}{0}\text{ form}\right) = \lim_{x\to\infty}\frac{\ln\left(\frac{x-4}{x+2}\right)}{1/x} \overset{H}{=} \lim_{x\to\infty}\frac{\frac{x+2}{x-4}\left(\frac{6}{(x+2)^2}\right)}{-1/x^2}$$

$$= \lim_{x\to\infty}\frac{\frac{6}{(x-4)(x+2)}}{-1/x^2} = \lim_{x\to\infty}\frac{-6x^2}{(x-4)(x+2)} = \lim_{x\to\infty}\frac{-6x^2}{x^2-2x-8}$$

$\ln L = -6$. Therefore $\boxed{L = e^{-6}}$

[5] (b) $\{\sqrt{n^2+3n-2}-\sqrt{n^2-2n-1}\}$.

$$\sqrt{n^2+3n-2}-\sqrt{n^2-2n-1}$$

$$= \frac{\left(\sqrt{n^2+3n-2}-\sqrt{n^2-2n-1}\right)\left(\sqrt{n^2+3n-2}+\sqrt{n^2-2n-1}\right)}{\left(\sqrt{n^2+3n-2}+\sqrt{n^2-2n-1}\right)}$$

$$= \frac{n^2+3n-2-(n^2-2n-1)}{\sqrt{n^2+3n-2}+\sqrt{n^2-2n-1}} = \frac{5n-1}{\sqrt{n^2}\left(\sqrt{1+\frac{3}{n}-\frac{2}{n^2}}+\sqrt{1-\frac{2}{n}-\frac{1}{n^2}}\right)}$$

$$= \frac{5-\frac{1}{n}}{\sqrt{1+\frac{3}{n}-\frac{2}{n^2}}+\sqrt{1-\frac{2}{n}-\frac{1}{n^2}}} \xrightarrow[n\to\infty]{} \boxed{\frac{5}{2}}.$$

[5] 2. Express 5.7463463463... as a series, and find the rational number it represents.

$$5.7\overline{463} = 5 + 0.7 + 0.0463 + 0.0000463 + 0.0000000463 + \cdots$$

$$= 5 + \frac{7}{10} + \frac{463}{10^4} + \frac{463}{10^7} + \frac{463}{10^{10}} + \cdots$$

$$= 5 + \frac{7}{10} + \frac{\frac{463}{10{,}000}}{1 - \frac{1}{1000}} = 5 + \frac{7}{10} + \frac{463}{9990}$$

[4] 3. Find the value of "$m$" for which the sum of the series

$$\frac{2m}{\sqrt{7}}x^3 - \frac{6m}{7}x^5 + \frac{18m}{7\sqrt{7}}x^7 - \frac{54m}{49}x^9 + \cdots$$

is $\frac{8x^3}{\sqrt{7}+3x^2}$.

$$\frac{2m}{\sqrt{7}}x^3 - \frac{6m}{7}x^5 + \frac{18m}{7\sqrt{7}}x^7 - \frac{54m}{49}x^9 + \cdots$$

$$= \frac{2m}{\sqrt{7}}x^3 + \left(\frac{2m}{\sqrt{7}}x^3\right)\left(-\frac{3}{\sqrt{7}}x^2\right) + \left(\frac{2m}{\sqrt{7}}x^3\right)\left(-\frac{3}{\sqrt{7}}x^2\right)^2 + \cdots$$

This is a geometric series of the form $a + ar + ar^2 + \cdots$ with sum $\frac{a}{1-r}$, $|r| < 1$.

Here, $a = \frac{2m}{\sqrt{7}}x^3$ and $r = -\frac{3}{\sqrt{7}}x^2$, then the sum is

$$\frac{\frac{2m}{\sqrt{7}}x^3}{1 + \frac{3}{\sqrt{7}}x^2} = \frac{2mx^3}{\sqrt{7}+3x^2} = \frac{8x^3}{\sqrt{7}+3x^2}.$$ Therefore $\boxed{m=4}$.

4. Let $f(x) = \cos 6x$ for $-\infty < x < \infty$.

[6] (a) Find the first 3 terms of the Taylor series of $f(x)$ about $x = \pi$.

$f(x) = \cos 6x$ $\quad f(\pi) = 1$

$f'(x) = -6\sin 6x$ $\quad f'(\pi) = 0$

$f''(x) = -6^2\cos 6x$ $\quad f''(\pi) = -6^2$

$f^{(3)}(x) = 6^3 \sin 6x$ $\quad f^{(3)}(\pi) = 0$

$f^{(4)}(x) = 6^4\cos 6x$ $\quad f^{(4)}(\pi) = 6^4$

$$f(x) = 1 - \frac{6^2}{2!}(x-\pi)^2 + \frac{6^4}{4!}(x-\pi)^4 + \cdots$$

$$f^{(k)}(x) = \begin{cases} \pm 6^k \sin 6x, & k \text{ is odd} \\ \pm 6^k \cos 6x, & k \text{ is even} \end{cases}$$

[4] (b) Find the nth-remainder $R_n(x)$ with $c = \pi$.

$$R_n(x) = \frac{f^{(n+1)}(z_n)}{(n+1)!}(x-\pi)^{n+1} = \begin{cases} \dfrac{\pm 6^{n+1}\sin 6z_n}{(n+1)!}(x-\pi)^{n+1}, & (n+1) \text{ odd} \\ \dfrac{\pm 6^{n+1}\cos 6z_n}{(n+1)!}(x-\pi)^{n+1}, & (n+1) \text{ even.} \end{cases}$$

[8] (c) Show that $\lim_{n\to\infty} R_n(x) = 0$ for all $x$.

Since $f^{(n+1)}(z_n) = \begin{cases} \pm 6^{n+1}\sin 6z_n, & (n+1) \text{ odd} \\ \pm 6^{n+1}\cos 6z_n, & (n+1) \text{ even} \end{cases}$, $|\sin 6z_n| \le 1$

and $|\cos 6z_n| \le 1$ for all $z_n$, then $|f^{(n+1)}(z_n)| \le 6^{n+1}$ for all $z_n$. Using the formula for $R_n(x)$, we obtain

$$|R_n(x)| \le 6^{n+1}\frac{|x-\pi|^{n+1}}{(n+1)!} = \frac{(6|x-\pi|)^{n+1}}{(n+1)!}$$

Since $\dfrac{(6|x-\pi|)^{n+1}}{(n+1)!} \to 0$ as $n\to\infty$, then $\lim_{n\to\infty} R_n(x) = 0$.

[12] 5. Find the open interval of convergence for the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-1)^5 3^{2n}}{n^2+2n}(2x+1)^n.$$

let $X = 2x+1$, then

$$a_n = \frac{(-1)^{n+1}(n-1)^5 3^{2n}}{n^2+2n}, \quad a_{n+1} = \frac{(-1)^{n+2} n^5 3^{2n+2}}{(n+1)^2+2(n+1)}$$

$$a_{n+1} = \frac{(-1)^{n+2} n^5 3^{2n+2}}{n^2+4n+3}$$

$$R_X = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n-1)^5 3^{2n}}{n^2+2n}\cdot\frac{n^2+4n+3}{(-1)^{n+2} n^5 3^{2n+2}}\right|$$

$$= \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^5 \frac{3^{2n}}{3^{2n}3^2}\,\frac{n^2+4n+3}{n^2+2n}$$

$$= \lim_{n\to\infty} \frac{1}{9}\left(1-\frac{1}{n}\right)^5 \frac{n^2+4n+3}{n^2+2n}$$

$$= \lim_{n\to\infty} \frac{1}{9}\left(1-\frac{1}{n}\right)^5 \lim_{n\to\infty}\frac{n^2+4n+3}{n^2+2n}$$

∘) $\displaystyle\lim_{n\to\infty}\frac{1}{9}\left(1-\frac{1}{n}\right)^5 = \frac{1}{9}$

∘) $\displaystyle\lim_{n\to\infty}\frac{n^2+4n+3}{n^2+2n} = \lim_{n\to\infty}\frac{1+\frac{4}{n}+\frac{3}{n^2}}{1+\frac{2}{n}} = 1$

So $R_X = \frac{1}{9}$ and $-\frac{1}{9} < 2x+1 < \frac{1}{9}$ or $-\frac{10}{18} < x < -\frac{8}{18}$.