UNIVERSITY OF MANITOBA

DATE: December 10, 2010
DEPARTMENT & COURSE NO: MATH 1510
EXAMINATION: Applied Calculus I

FINAL EXAMINATION

TIME: 2 hours
EXAMINER: various

---

[12] **1.** Find $f'(x)$. DO NOT SIMPLIFY YOUR ANSWER.

(a) $f(x) = (\cos(\tan 3x) + \ln x)^5$

$$f'(x) = 5(\cos(\tan 3x) + \ln x)^4 \left(-\sin(\tan 3x)\sec^2(3x)3 + \frac{1}{x}\right)$$

(b) $f(x) = \sqrt{x}^{\sqrt{x}} e^x$

$$\ln f(x) = \sqrt{x}\ln\sqrt{x} + \ln e^x$$

$$\ln f(x) = \frac{1}{2}\sqrt{x}\ln x + x$$

$$\frac{f'(x)}{f(x)} = \frac{1}{2}\cdot\frac{1}{2\sqrt{x}}\ln x + \frac{1}{2}\sqrt{x}\cdot\frac{1}{x} + 1$$

$$f'(x) = \sqrt{x}^{\sqrt{x}} e^x \left(\frac{\ln x}{4\sqrt{x}} + \frac{1}{2\sqrt{x}} + 1\right)$$

[8] **2.** Find each limit, if it exists. If the limit does not exist, indicate whether it tends to ∞ or $-\infty$, or neither.

(a) $\displaystyle\lim_{x\to 2^-} \frac{-x^2+3x-2}{x^2-4x+4}$

$$= \lim_{x\to 2^-} \frac{-(x-2)(x-1)}{(x-2)^2}$$

$$= \lim_{x\to 2^-} \frac{-(x-1)}{x-2} = \infty \qquad \left\{ \frac{-1}{0^-} \right.$$

(b) $\displaystyle\lim_{x\to 2} \frac{2-x}{\sqrt{4-x}-\sqrt{x}}$ $\cdot \dfrac{\sqrt{4-x}+\sqrt{x}}{\sqrt{4-x}+\sqrt{x}}$

$$= \lim_{x\to 2} \frac{(2-x)(\sqrt{4-x}+\sqrt{x})}{4-x-x}$$

$$= \lim_{x\to 2} \frac{(2-x)(\sqrt{4-x}+\sqrt{x})}{2(2-x)}$$

$$= \lim_{x\to 2} \frac{\sqrt{4-x}+\sqrt{x}}{2} = \frac{\sqrt{2}+\sqrt{2}}{2} = \sqrt{2}$$

[30] **3.** Evaluate each integral.

(a) $\displaystyle\int \left(\frac{3}{\sqrt[3]{x^2}} + e^{2x} + e^{\pi} + x^{\pi} + \sin(3x) \right) dx$

$= 3 \cdot 3 \cdot x^{\frac{1}{3}} + \frac{1}{2} e^{2x} + e^{\pi} x + \frac{1}{\pi+1} x^{\pi+1} - \frac{1}{3}\cos(3x) + C, \ C \in \mathbb{R}$

(b) $\displaystyle\int \frac{1}{x(\ln x)^2}\, dx = \left\{ \begin{matrix} \ln x = t \\ \frac{dt}{dx} = \frac{1}{x} \end{matrix} \right\} = \int \frac{1}{t^2}\, dt = -\frac{1}{t} + c$

$= -\frac{1}{\ln x} + c, \quad c \in \mathbb{R}$

(c) $\displaystyle\int \cos^6 x \cdot \sin x \, dx = \left\{ \begin{matrix} \cos x = t \\ -\sin x = \frac{dt}{dx} \end{matrix} \right\} = -\int t^6 \, dt$

$= -\frac{t^7}{7} + C = -\frac{\cos^7 x}{7} + c, \quad c \in \mathbb{R}$

(d) $\int_{-4}^{-2} x(x+3)^{20}\,dx$ $= \left\{ \begin{array}{ll} t = x+3 & x=-4:\ t=-1 \\ dt = dx & x=-2:\ t=1 \\ x = t-3 & \end{array} \right\}$

$$= \int_{-1}^{1} (t-3)t^{20}\,dt = \int_{-1}^{1} (t^{21} - 3t^{20})\,dt$$

$$= \left(\frac{1}{22} t^{22} - \frac{1}{7} t^{21} \right) \Big|_{-1}^{1}$$

$$= \frac{1}{22} - \frac{1}{7} - \frac{1}{22} + \frac{1}{7} = \frac{2}{7}$$

(e) $\int_1^e \frac{3x^2+4x+2}{x}\,dx$ [Simplify your answer as much as possible.]

$$= \int_1^e (3x + 4 + 2x^{-1})\,dx = \left(\frac{3x^2}{2} + 4x + 2\ln|x| \right) \Big|_1^e$$

$$= \frac{3e^2}{2} + 4e + 2\ln e - \frac{3}{2} - 4 - 0 = \frac{3e^2}{2} + 4e - \frac{7}{2}$$

[10] **4.** (a) Find all intervals on which $f(x) = \dfrac{1}{x^2+3}$ is increasing.

$$f'(x) = -\frac{1}{(x^2+3)^2} \cdot 2x$$

	$x<0$	$x>0$
f':	$+$	$-$
f:	$\nearrow$	$\searrow$

Ans: $(-\infty, 0)$

(b) Find all intervals on which $f(x) = \dfrac{1}{x^2+3}$ is concave down.

$$f''(x) = -2\,\frac{(x^2+3)^2 - x\cdot 2(x^2+3)2x}{(x^2+3)^4}$$

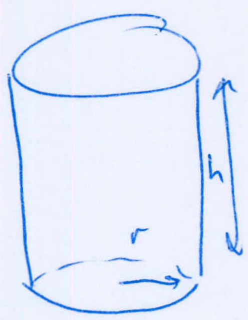
$$= -2\,\frac{x^2+3-4x^2}{(x^2+3)^3} = 6\,\frac{x^2-1}{(x^2+3)^3}$$

	$x<-1$	$-1<x<1$	$x>1$
f''	$+$	$-$	$+$
f	$\cup$	$\cap$	$\cup$

Ans: $(-1, 1)$

[10] **5.** For the following question it is sufficient to write down a **function** and indicate an **interval** on which it is defined. The function and interval should be chosen so that in order to solve the problem you would find the maximum or minimum value of the function on that interval. IT IS NOT NECESSARY TO ACTUALLY FIND THE MAXIMUM OR MINIMUM VALUE.

The material for the top and bottom of a cylindrical can costs \$8 per square meter and material for the side costs \$5 per square meter. What is the largest volume the can may contain if the total price of the material is \$100?



h - height
r - radius

cost: $C(h,r) = \overbrace{8\cdot 2\pi r^2}^{\text{top \& bottom}} + 5\cdot\underbrace{2\pi r h}_{\text{side}}$

$= 16\pi r^2 + 10\pi r h$

$V(h,r) = \pi r^2 h$

$100 = 16\pi r^2 + 10\pi r h$

$h = \dfrac{100 - 16\pi r^2}{10\pi r}$ $\qquad r, h > 0$

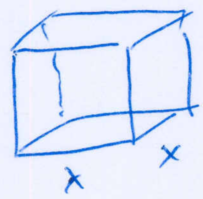
$$V(r) = \pi r^2 \frac{100 - 16\pi r^2}{10\pi r} = \frac{1}{5} r\left(50 - 8\pi r^2\right)$$

$h > 0$: $\quad 100 - 16\pi r^2 > 0$

$r < \dfrac{5}{2\sqrt{\pi}}$

interval: $\quad 0 < r < \dfrac{5}{2\sqrt{\pi}}$

[11] **6.** A rectangular solid has a square base. The height is 2 cm longer than the width (or length) of the base. The volume of the solid is increasing at a rate 10 cm^3/s when the height is 3 cm. What is the rate of change of its surface area at this moment?



$h = x + 2$

$\left.\frac{dV}{dt}\right|_{h=3} = 10 \quad cm^3/s$

$V = x^2(x+2)$

$V = x^3 + 2x^2$

$\frac{dV}{dt} = (3x^2 + 4x)\frac{dx}{dt}$

when $h = 3$ cm $= x+2$

$\underline{x = 1}$ cm

$10 = (3+4)\frac{dx}{dt}$

$\frac{dx}{dt} = \frac{10}{7}$

$A = 2x^2 + 4x(x+2)$

$A = 6x^2 + 8x$

$\frac{dA}{dt} = (12x + 8)\frac{dx}{dt}$

when $x = 1$:

$\frac{dA}{dt} = (12 + 8)\cdot\frac{10}{7} = \frac{200}{7}$

Ans: the surface area is increasing at $\frac{200}{7}$ cm^2/s

[11] **7.** The velocity of a particle moving along the x-axis is given in meters per second by $v(t) = t - t^3$ where $t \geq 0$ is time in seconds. If the particles starts from position $x = 4m$ at $t = 0s$ determine:

(a) the position of the particle at $t = 2s$,

$$x(t) = \int v(t)\,dt = \int (t - t^3)\,dt = \frac{t^2}{2} - \frac{t^4}{4} + C$$

$x(0) = 4$

$4 = 0 - 0 + C$

$C = 4$

$$x(t) = \frac{t^2}{2} - \frac{t^4}{4} + 4$$

$$x(2) = \frac{4}{2} - \frac{16}{4} + 4 = 2 - 4 + 4 = \underline{2\ m}$$

(b) the acceleration of the particle at $t = 3s$,

$a(t) = v'(t) = 1 - 3t^2$

$a(3) = 1 - 3(3)^2 = 1 - 27 = \underline{-26 \quad m/s^2}$

(c) if the particle is speeding up or slowing down at $t = 2s$. Justify your answer.

$v(2) = 2 - 2^3 = -6 \quad m/s$

$a(2) = 1 - 3 \cdot 2^2 = -11 \quad m/s^2$

$a(2) \cdot v(2) > 0$

<u>Ans</u>: <u>Speeding up</u>.

[8] **8.** Find $f(1)$ if $f''(x) = 12x^2 - 2$, $f(0) = 2$ and $f'(0) = 3$.

$$f'(x) = \int (12x^2 - 2)\,dx = 4x^3 - 2x + c$$

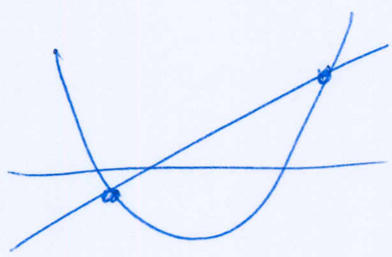
$$3 = f'(0) = c$$

$$f(x) = \int (4x^3 - 2x + 3)\,dx = x^4 - x^2 + 3x + k$$

$$2 = f(0) = k$$

$$f(1) = 1^4 - 1^2 + 3 + 2 = \underline{5}$$

[5 (bonus)] **9.** Find the area of the region bounded by the curves $y = x^2 - 2x - 9$ and $y = x - 5$.



$$x^2 - 2x - 9 = x - 5$$

$$x^2 - 3x - 4 = 0$$

$x = -1 \quad x = 4$ – intersection points

line is above

$$\int_{-1}^{4} \left((x-5) - (x^2 - 2x - 9)\right)dx = \int_{-1}^{4} (4 + 3x - x^2)\,dx$$

$$= \left(4x + \frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg|_{-1}^{4} = 16 + 24 - \frac{64}{3} + 4 - \frac{3}{2} - \frac{1}{3}$$

$$= \underline{\frac{125}{6}}.$$