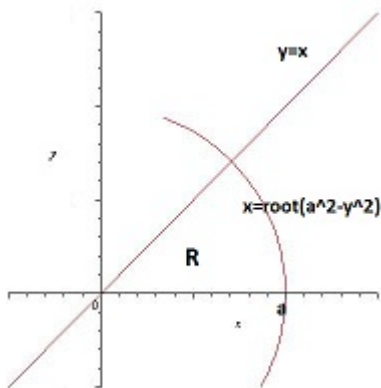


# MATH 2130 Summer Evening 2013 Problem Workshop 6 Solutions

1. The region in question is given below



(a) The formula for mass is

$$\iint_R \rho dA.$$

Since density is constant, this is the same as

$$\rho \iint_R dA = \rho(\text{area of } R).$$

To find the area of  $R$ , we can note that it is just one-eighth of the area of the circle of radius  $a$ . Hence the mass is

$$\rho \frac{\pi a^2}{8}.$$

Failing that we can use polar coordinates to re-write the region as

$$0 \leq r \leq a, 0 \leq \theta \leq \frac{\pi}{4}.$$

(Note that we can get  $\pi/4$  since when  $y = x$ ,  $r \cos \theta = r \sin \theta \Rightarrow \tan \theta = 1 \Rightarrow \theta = \pi/4$ )

Therefore

$$M = \rho \iint_R dA = \rho \int_0^{\pi/4} \int_0^a r dr d\theta = \rho \int_0^{\pi/4} \left[ \frac{r^2}{2} \right]_0^a d\theta = \rho \int_0^{\pi/4} \frac{a^2}{2} d\theta = \rho \frac{\pi}{4} \frac{a^2}{2} = \rho \frac{\pi a^2}{8}.$$

(b) To find the moment about the  $x$ -axis, we use the formula for first moment

$$\iint_R \rho y dA$$

where  $d$  is the distance to whatever line we are find the first moment about. For the  $x$ -axis,  $d = y$ , and so we get

$$M_x = \iint_R y dA.$$

From here we have a couple different options. The region is part of a circle, so polar coordinates might be useful. I'll do it both ways Solution 1: Cartesian

The region is awful if  $y$  is in terms of  $x$ , so we'll do it the other way. Setting the curves equal to each other to find bounds on  $y$  gives

$$y = \sqrt{a^2 - y^2} \Rightarrow y^2 = a^2 - y^2 \Rightarrow y^2 = \frac{a^2}{2} \Rightarrow y = \frac{a}{\sqrt{2}}$$

using  $a, y \geq 0$ . Hence

$$y \leq x \leq \sqrt{a^2 - y^2}, \quad 0 \leq y \leq \frac{a}{\sqrt{2}}.$$

Therefore

$$\begin{aligned} M_x &= \rho \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2 - y^2}} y dx dy \\ &= \rho \int_0^{a/\sqrt{2}} xy \Big|_y^{\sqrt{a^2 - y^2}} dy \\ &= \rho \int_0^{a/\sqrt{2}} (y\sqrt{a^2 - y^2} - y^2) dy \\ &= \rho \left( -\frac{1}{3}(a^2 - y^2)^{3/2} - \frac{y^3}{3} \right) \Big|_0^{a/\sqrt{2}} \\ &= \rho \left[ \left( -\frac{1}{3}(a^2 - (a/\sqrt{2})^2)^{3/2} - \frac{(a/\sqrt{2})^3}{3} \right) - \left( -\frac{1}{3}(a^2 - 0^2)^{3/2} - \frac{0^3}{3} \right) \right] \\ &= \rho \left[ \left( -\frac{1}{3} \left( \frac{a^2}{2} \right)^{3/2} - \frac{a^3}{6\sqrt{2}} \right) - \left( -\frac{a^3}{3} \right) \right] \\ &= \rho \left( -\frac{a^3}{6\sqrt{2}} - \frac{a^3}{6\sqrt{2}} + \frac{a^3}{3} \right) \\ &= \rho \frac{(\sqrt{2} - 1)a^3}{3\sqrt{2}} \end{aligned}$$

Solution 2: Polar

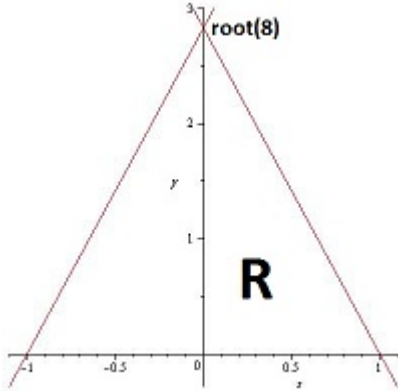
The region is as in part (a). The only change is we have to replace  $y = r \sin \theta$  in the integrand. Hence we get

$$\begin{aligned}
M_x &= \rho \iint_R y dA \\
&= \rho \int_0^{\pi/4} \int_0^a r^2 \sin \theta dr d\theta \\
&= \rho \int_0^{\pi/4} \left[ \frac{r^3}{3} \sin \theta \right]_0^a d\theta \\
&= \rho \int_0^{\pi/4} \frac{a^3 \sin \theta}{3} d\theta \\
&= -\left[ \frac{\rho a^3 \cos \theta}{3} \right]_0^{\pi/4} \\
&= -\frac{\rho a^3 \cos(\pi/4)}{3} + \frac{\rho a^3 \cos 0}{3} \\
&= \frac{\rho a^3 (-1/\sqrt{2} + 1)}{3} \\
&= \frac{\rho a^3 (\sqrt{2} - 1)}{3\sqrt{2}}
\end{aligned}$$

(c)

$$M\bar{y} = M_x \Rightarrow \bar{y} = \frac{M_x}{M} = \frac{\rho \frac{(\sqrt{2}-1)a^3}{3\sqrt{2}}}{\rho \frac{\pi a^2}{8}} = \frac{8(\sqrt{2}-1)a}{3\pi\sqrt{2}}$$

2. We take the isosceles triangle and put the base on the  $x$ -axis (symmetrically). Using the pythagorean theorem, we get that the triangle has height  $\sqrt{8}$ . Hence we get the region



from which we'll find  $I_x$  the moment of inertia about the  $x$ -axis. Since the distance to the  $x$ -axis is  $d = y$ , we get the moment of inertia is

$$\iint_R \rho d^2 dA = \rho \iint_R y^2 dA.$$

Using some symmetry, since the region is symmetric and  $y^2$  is even, we get

$$\begin{aligned}
I &= \rho \iint_R y^2 dA \\
&= 2\rho \int_0^1 \int_0^{\sqrt{8}-\sqrt{8}x} y^2 dy dx \\
&= 2\rho \int_0^1 \left[ \frac{y^3}{3} \right]_0^{\sqrt{8}-\sqrt{8}x} dx \\
&= 2\rho \int_0^1 \left( \frac{(\sqrt{8}-\sqrt{8}x)^3}{3} - 0 \right) dx \\
&= 2(\sqrt{8})^3 \rho \int_0^1 \left( \frac{(1-x)^3}{3} \right) dx \\
&= 32\sqrt{2}\rho \left( -\frac{(1-x)^4}{12} \right) \Big|_0^1 \\
&= 32\sqrt{2}\rho \left[ 0 - \left( -\frac{1}{12} \right) \right] \\
&= \frac{8\sqrt{2}\rho}{3}
\end{aligned}$$

3. We are finding surface area which has the formula

$$S = \iint_R \sqrt{1 + z_x^2 + z_y^2} dA.$$

The region  $R$ , is the circle  $x^2 + y^2 = a^2$  in the  $xy$ -plane. The integrand is

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + y^2 + x^2}.$$

Therefore we get

$$S = \iint_R \sqrt{1 + y^2 + x^2} dA.$$

Since we have a circle and  $x^2 + y^2$  in the integrand, this is a clear case for polar coordinates. The region has  $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$ . Hence the surface area is

$$\begin{aligned}
S &= \int_0^{2\pi} \int_0^a \sqrt{1+r^2} r dr d\theta \\
&= \frac{1}{3} \int_0^{2\pi} (1+r^2)^{3/2} \Big|_0^a d\theta \\
&= \frac{1}{3} \int_0^{2\pi} ((1+a^2)^{3/2} - 1) d\theta \\
&= \frac{2\pi}{3} ((1+a^2)^{3/2} - 1)
\end{aligned}$$

4. The integrand is

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+(4x)^2+(2y)^2} = \sqrt{1+16x^2+4y^2}.$$

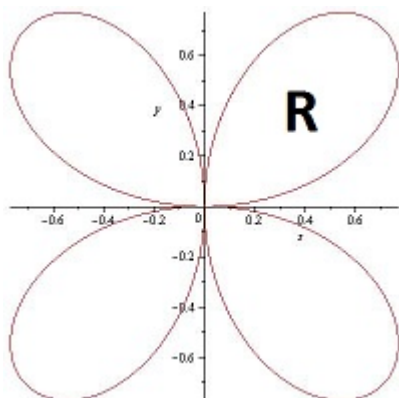
The region  $R$  is the triangle bounded by the lines  $y = 0, x = 0, y = 1 - x$ . Hence we get

$$S = \iint_R \sqrt{1+z_x^2+z_y^2} dA = \int_0^1 \int_0^{1-x} \sqrt{1+16x^2+4y^2} dy dx.$$

5. Determining what this region looks like, we note that polar is helpful since we see  $x^2 + y^2$ . Hence we get

$$(r^2)^3 = 4a^2(r^4 \cos^2 \theta \sin^2 \theta) \Rightarrow r^2 = 4a^2 \cos^2 \theta \sin^2 \theta \Rightarrow r = \pm 2a \sin \theta \cos \theta.$$

The picture of which (with  $a = 1$ ) is



Symmetry allows us to take 4 times the leaf in the first quadrant, hence using  $R$  as the region in the first quadrant, we get the area is

$$\begin{aligned}
A &= 4 \iint_R dA \\
&= 4 \int_0^{\pi/2} \int_0^{2a \sin \theta \cos \theta} r dr d\theta \\
&= 4 \int_0^{\pi/2} \left[ \frac{r^2}{2} \right]_0^{2a \sin \theta \cos \theta} d\theta \\
&= 4 \int_0^{\pi/2} \left( \frac{(2a \sin \theta \cos \theta)^2}{2} - 0 \right) d\theta \\
&= 8a^2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta
\end{aligned}$$

Using  $\sin^2 \theta = \frac{1 - \cos 2\theta}{2}$  and  $\cos^2 \theta = \frac{1 + \cos 2\theta}{2}$ , we get the area is

$$\begin{aligned}
A &= 8a^2 \int_0^{\pi/2} \sin^2 \theta \cos^2 \theta d\theta \\
&= 8a^2 \int_0^{\pi/2} \left( \frac{1 - \cos 2\theta}{2} \right) \left( \frac{1 + \cos 2\theta}{2} \right) d\theta \\
&= 2a^2 \int_0^{\pi/2} (1 - \cos^2 2\theta) d\theta \\
&= 2a^2 \int_0^{\pi/2} \left( 1 - \frac{1 + \cos 4\theta}{2} \right) d\theta \\
&= a^2 \int_0^{\pi/2} (1 - \cos 4\theta) d\theta \\
&= a^2 \left( \theta - \frac{\sin 4\theta}{4} \right) \Big|_0^{\pi/2} \\
&= a^2 \left( \frac{\pi}{2} - \frac{\sin 2\pi}{4} - (0 - 0) \right) \\
&= \frac{\pi a^2}{2}.
\end{aligned}$$

6. The region strongly suggests that we use polar coordinates, even though the function itself doesn't. We are basically forced into polar by the region

$$R = \{(r, \theta) | 1 \leq r \leq 2, 0 \leq \theta \leq \pi/2\}.$$

Hence we get

$$\begin{aligned}
I &= \iint_R xy(x+y)dA \\
&= \int_0^{\pi/2} \int_1^2 (r \cos \theta)(r \sin \theta)(r \cos \theta + r \sin \theta) r dr d\theta \\
&= \int_0^{\pi/2} \int_1^2 r^4 (\cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta) dr d\theta \\
&= \int_0^{\pi/2} \frac{r^5}{5} (\cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta) \Big|_1^2 d\theta \\
&= \int_0^{\pi/2} \left( \frac{32}{5} - \frac{1}{5} \right) (\cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta) d\theta \\
&= \frac{31}{5} \int_0^{\pi/2} (\cos^2 \theta \sin \theta + \cos \theta \sin^2 \theta) d\theta \\
&= \frac{31}{5} \left( -\frac{\cos^3 \theta}{3} + \frac{\sin^3 \theta}{3} \right) \Big|_0^{\pi/2} \\
&= \frac{31}{15} [(-\cos^3(\pi/2) + \sin^3(\pi/2)) - (-\cos^3(0) + \sin^3(0))] \\
&= \frac{31}{15} [(0 + 1) - (-1 + 0)] \\
&= \frac{62}{15}.
\end{aligned}$$

7. The region is above the  $xy$ -plane and below the plane  $z = 6 - 2x - 3y$ . Hence we get the bounds on  $z$  to be

$$0 \leq z \leq 6 - 2x - 3y.$$

Projecting the region into the  $xy$ -plane makes the triangle bounded by  $x = 0, y = 0$  and  $2x + 3y = 6$ . Hence we get bounds on  $y, x$  to be

$$0 \leq y \leq 2 - \frac{2}{3}x, \quad 0 \leq x \leq 3.$$

Putting this together leads to the integral

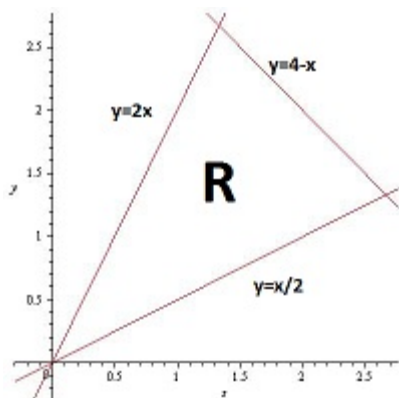
$$\begin{aligned}
I &= \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} x dz dy dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} x z \Big|_0^{6-2x-3y} dy dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} x(6-2x-3y-0) dy dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} (6x-2x^2-3xy) dy dx \\
&= \int_0^3 \left( 6xy - 2x^2y - \frac{3xy^2}{2} \right) \Big|_0^{2-\frac{2}{3}x} dx \\
&= \int_0^3 \left( 6x\left(2-\frac{2}{3}x\right) - 2x^2\left(2-\frac{2}{3}x\right) - \frac{3x\left(2-\frac{2}{3}x\right)^2}{2} - 0 \right) dx \\
&= \int_0^3 \left( 12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 - \frac{3x\left(4-\frac{8}{3}x+\frac{4}{9}x^2\right)}{2} \right) dx \\
&= \int_0^3 \left( 12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 - \left(6x - 4x^2 + \frac{2}{3}x^3\right) \right) dx \\
&= \int_0^3 \left( 6x - 4x^2 + \frac{2}{3}x^3 \right) dx \\
&= \left( 3x^2 - \frac{4x^3}{3} + \frac{1}{6}x^4 \right) \Big|_0^3 \\
&= \left( 3(3)^2 - \frac{4(3)^3}{3} + \frac{1}{6}(3)^4 \right) - 0 \\
&= 27 - 36 + \frac{27}{2} \\
&= \frac{9}{2}.
\end{aligned}$$

8. The volume is

$$\iiint_V dV.$$

We need to figure out bounds for our region. Clearly  $z$  is bounded below by  $z = 0$  and is bounded above by  $z = 16 - 4x - 4y$ . This leaves us with the region in the  $xy$ -plane bounded by  $4x + 4y = 16$ ,  $y = x/2$ ,  $y = 2x$  graphed below.





The intersection of  $y = x/2$  and  $y = 2x$  is clearly  $(0, 0)$ . The intersection of  $y = 2x$  and  $4x + 4y = 16$  is

$$4x + 8x = 16 \Rightarrow x = \frac{16}{12} = \frac{4}{3} \Rightarrow \left(\frac{4}{3}, \frac{8}{3}\right).$$

The intersection of  $y = x/2$  and  $4x + 4y = 16$  is

$$4x + 2x = 16 \Rightarrow x = \frac{16}{6} = \frac{8}{3} \Rightarrow \left(\frac{8}{3}, \frac{4}{3}\right).$$

Since the upper function changes from  $0 \leq x \leq 4/3$  to  $4/3 \leq x \leq 8/3$  we will need to split the function up into two integrals.

$$I_1 = \int_0^{4/3} \int_{x/2}^{2x} \int_0^{16-4x-4y} dz dy dx, \quad I_2 = \int_{4/3}^{8/3} \int_{x/2}^{4-x} \int_0^{16-4x-4y} dz dy dx$$

$$\begin{aligned}
I_1 &= \int_0^{4/3} \int_{x/2}^{2x} \int_0^{16-4x-4y} dz dy dx \\
&= \int_0^{4/3} \int_{x/2}^{2x} (16 - 4x - 4y) dy dx \\
&= \int_0^{4/3} [16y - 4xy - 2y^2]_{x/2}^{2x} dx \\
&= \int_0^{4/3} \left[ (32x - 8x^2 - 8x^2) - \left( 8x - 2x^2 - \frac{1}{2}x^2 \right) \right] dx \\
&= \int_0^{4/3} \left( 24x - \frac{27}{2}x^2 \right) dx \\
&= \left( 12x^2 - \frac{9}{2}x^3 \right) \Big|_0^{4/3} \\
&= \left( 12(4/3)^2 - \frac{9}{2}(4/3)^3 \right) - \left( 12(0)^2 - \frac{9}{2}(0)^3 \right) \\
&= \left( \frac{64}{3} - \frac{32}{3} \right) - 0 \\
&= \frac{32}{3}.
\end{aligned}$$

$$\begin{aligned}
I_2 &= \int_{4/3}^{8/3} \int_{x/2}^{4-x} \int_0^{16-4x-4y} dz dy dx \\
&= \int_{4/3}^{8/3} \int_{x/2}^{4-x} (16 - 4x - 4y) dy dx \\
&= \int_{4/3}^{8/3} 16y - 4xy - 2y^2 \Big|_{x/2}^{4-x} dx \\
&= \int_{4/3}^{8/3} \left[ (64 - 16x - 4x(4-x) - 2(16 - 8x + x^2)) - \left( 8x - 2x^2 - \frac{1}{2}x^2 \right) \right] dx \\
&= \int_{4/3}^{8/3} \left[ (64 - 16x - (16x - 4x^2) - (32 - 16x + 2x^2) - \left( 8x - 2x^2 - \frac{1}{2}x^2 \right)) \right] dx \\
&= \int_{4/3}^{8/3} \left( 32 - 24x + \frac{9}{2}x^2 \right) dx \\
&= \left( 32x - 12x^2 - \frac{3}{2}x^3 \right) \Big|_{4/3}^{8/3} \\
&= \left( 32(8/3) - 12(8/3)^2 + \frac{3}{2}(8/3)^3 \right) - \left( 32(4/3) - 12(4/3)^2 + \frac{3}{2}(4/3)^3 \right) \\
&= \left( \frac{256}{3} - \frac{256}{3} + \frac{256}{9} \right) - \left( \frac{128}{3} - \frac{64}{3} + \frac{32}{9} \right) \\
&= \left( \frac{256}{9} \right) - \left( \frac{224}{9} \right) \\
&= \frac{32}{9}
\end{aligned}$$

Hence the total volume is

$$\frac{32}{3} + \frac{32}{9} = \frac{128}{9}$$