[6] 1. Find two non-zero perpendicular vectors which are parallel to the plane $x+2y+3z+4=0$. Explicitly show that your vectors are parallel to the plane. Hint for the first part: Find one non-zero vector **u** parallel to the plane; for the second vector, take $\mathbf{u}\times\mathbf{n}$, where **n** is a normal to the plane.

Ans. $\vec{u}_1 = (-2,1,0)$
$\vec{u}_2 = (3,6,-5)$
(Many answers are possible)

[6] 2. Find all values of the constant α so that the plane $\alpha x+y+z=0$ is at a distance 1 from the point $(2,0,0)$.

Ans.
$\alpha = \pm\sqrt{2/3} = \pm\frac{\sqrt{6}}{3}$

[6] 3. Let $f(x,y)=e^{2x}\cos\alpha y$, where α is a constant. Find all values of α so that f is harmonic (i.e., $f_{xx}+f_{yy}=0$).

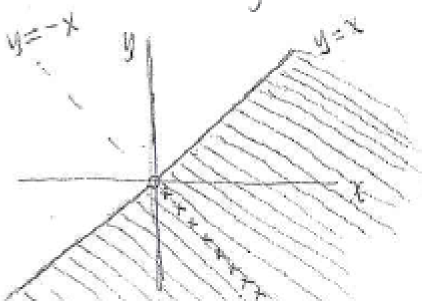
Ans.
$\alpha = \pm 2$

[7] 4. Find a point $P(x_0,y_0,z_0)$ so that a normal to the surface $f(x,y,z)=x^2+2yz-52=0$ at P is in the direction $(1,2,3)$. Find the equation of the tangent plane to the surface at P.

Ans.
$P(2,6,4)$
$x+2y+3z-26=0$

[5] 5. Let $f(x,y)=\dfrac{\sqrt{x-y}}{x+y}$. Find the domain of f and sketch it.

Ans.
$x \geq y$
$x \neq -y$



[6] 6. Let $f(x,y,z) = x^2y + 2e^{2z}$. Find the unit vector v so that the directional derivative of f in direction v at the point $(1,0,0)$ is minimum. What is the value of this minimum directional derivative?

Ans.

$$\hat{v} = \left(0, \frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}}\right) = \frac{(0,-1,-4)}{\sqrt{17}}$$

$$\left(D_{\hat{v}} f\right)_{min} = -\sqrt{17}$$

[6] 7. Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x^2y^2 + 3x^2 + 3y^2}{x^2+y^2}$.

Answer
3

[5] 8. Suppose $z = f(x,y,t,u,v)$, $u = g(y,t)$ and $v = h(x,t)$. Express $\left.\dfrac{\partial z}{\partial y}\right)_{x,t}$ only in terms of f, g and h using the chain rule.

Ans.

$$\left.\frac{\partial f}{\partial y}\right)_{x,t,u,v} + \left.\frac{\partial f}{\partial u}\right)_{x,y,t,v} \left.\frac{\partial g}{\partial y}\right)_{t}$$

[10] 9. Define $f(x,y) = x^2 + 4xy + 5y^2 + 2y$. Determine all critical point(s) and classify them as local min, local max or saddle point.

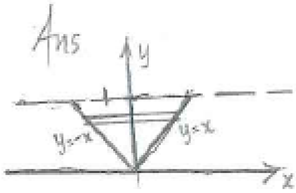
Ans.
CP : $(2,-1)$
local min

[10] 10. Define $f(x,y) = 4(x^2+y^2)^2 + x^2$. Find the maximum and minimum values of f in the annulus $1 \le x^2+y^2 \le 4$.

Ans.
Min = 4
Max = 80

[7] 11. Evaluate $\displaystyle\int_0^1 \int_{-y}^{y} \left(\frac{x^3 y e^{x^2}}{1+x^2+\sin(x^2)+e^y} + y\right) dx\, dy$. Be sure to justify all steps. Sketch the region of integration in the xy-plane.

Ans



Ans
2/3

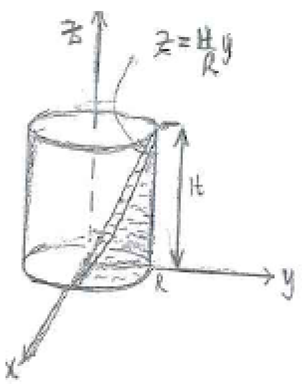
Hint: Consider Symmetry

[10] 12. Set up, but do NOT evaluate, a triple iterated integral for the volume of the solid in the first octant bounded by the surface $x = 4 - y^2 - z^2$ and lying below the plane $z = 1$. Sketch the solid.

Ans.)
$$\int_0^1 \int_0^{\sqrt{4-z^2}} \int_0^{4-(y^2+z^2)} dx\,dy\,dz$$

[12] 13. Find the surface area of $z = 12 - x^2 - y^2$ that lies above (inside) $z = \sqrt{x^2+y^2}$.

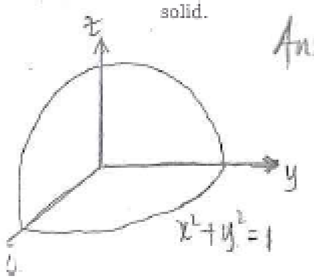
Ans.
$$\frac{\pi}{6}\left(37\sqrt{37}-1\right)$$



[12] 14. Consider a glass which is a circular cylinder of radius R and height H. It is filled with water and is tipped until the water surface, which is a plane, runs through the centre of the base of the glass. Find the volume of water at this instance. Hint: Set up the problem so that the axis of the cylinder is along the z-axis.

Ans.
$$V = \int_0^{\pi}\int_0^{R}\int_0^{\frac{H}{R}r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{R} \frac{H}{R} r^2 \sin\theta\,dr\,d\theta$$
$$= \frac{H}{R}\int_0^{\pi} \frac{R^3}{3}\sin\theta\,d\theta = \frac{HR^2}{3}\int_0^{\pi}\sin\theta\,d\theta = \frac{2}{3}HR^2$$

[12] 15. Set up, but do NOT evaluate, triple iterated integrals using spherical coordinates for the x coordinate of the centre of mass of the solid which is the first octant part of the solid bounded by the sphere of radius one with centre at the origin. Assume that the density of the solid is a known positive function $\delta = \delta(x, y, z)$. Also, set up, but do NOT evaluate, a triple iterated integral using Cartesian coordinates for the mass of the above solid.



Ans.)
$$M = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \delta(R,\phi,\theta)\,R^2\sin\phi\,dR\,d\phi\,d\theta$$
$$\bar{x} = \frac{1}{M}\int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 \delta(R,\phi,\theta)\,R^3\sin^2\phi\cos\theta\,dR\,d\phi\,d\theta$$

$$M = \int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-(x^2+y^2)}} \delta(x,y,z)\,dz\,dy\,dx$$