Math 2130 - Engineering Mathematical Analysis 1

Tutorial Questions for §12.2 - 12.6.

12.2.1. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2+1}$.

12.2.2. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$.

12.2.3. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$.

Show that the limit exists along every straight line through the origin, and these give the same value. Then consider the limit along parabolas through the origin.

12.2.4. Let $f(x,y) = \begin{cases} \dfrac{e^{3x+2y} - 1 - (3x+2y)}{(3x+2y)^2}, & \text{if} \quad \mathrm{3x+2y} \neq 0; \\ \beta, & \text{if} \quad \mathrm{3x+2y} = 0. \end{cases}$

Note that the first line of the definition of f is undefined whenever $3x+2y=0$.

Can we find a value for β so that f is continuous on the whole xy plane?

12.3.1. If $f(x,y) = x^3 y \sin(y^2/x^2)$, find and simplify $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y}$.

12.4.1. (a) Given a surface with equation $z = f(x,y)$, the two vectors $\vec{u} = (1,0,f_x)$ and $\vec{v} = (0,1,f_y)$ are tangent to the surface. Note: f_x and f_y denote partial derivatives.

Define $G(x,y,z) = f(x,y) - z$. Then, the surface is given by $G = 0$.

Show that ∇G is normal to the surface by showing that it is perpendicular to each of $\vec{u}$ and $\vec{v}$.

(b) For the surface, $z = x^2 + y^2$, form $\vec{N} = \nabla G$, as above. Then, on the intersection of this surface with the plane $y = 0$, find a tangent vector, $\vec{T}$, to the resulting curve. Show that $\vec{N}$ and $\vec{T}$ are perpendicular and draw this curve and a few examples of the $\vec{N}$ and $\vec{T}$ vectors.

12.5.1. Let $f(x,y) = xy^2$. Calculate $\dfrac{\partial f}{\partial x}$ at the point $(0,2)$.

12.5.2. Let $f(x,y) = e^{x^2 y} + x^2 y^3$. Calculate all its first-order and second-order partial derivatives.

12.5.3. For what value, or values, of the constant b is the function $f(x,y) = e^{bx}\cos 5y$ harmonic in the entire xy-plane?

12.6.1. (a) You are told that $z = f(u,v,t)$, $u = g(x,y,t)$, $v = h(x,y,t)$, and $y = k(t)$. What is the chain rule for $\left.\dfrac{\partial z}{\partial t}\right)_x$?

(b) Verify your answer to (a) if $z = uvt, u = xyt, v = 2xyt$, and $y = t^2$.

Answers:

12.2.1: 0.

12.2.2: On $x = 0$ or $y = 0$, the limit is 1; on $y = x$, the limit is 2.

Alternatively, on $y = -x$, the limit is 0.

Thus, the limit DNE (does not exist).

12.2.3: On $x = 0$ or $y = mx$, the limit is 0; on $x = y^2$, the limit is $\frac{1}{2}$. Thus, the limit DNE.

12.2.4: Yes. Let $u = 3x + 2y$. Then, $\beta = \lim\limits_{u \to 0} \dfrac{e^u - 1 - u}{u^2}$.

Use L'Hospital's Rule twice to obtain: $\beta = \dfrac{1}{2}$.

12.3.1: $4x^3 y \sin(y^2/x^2)$.

12.4.1: (a) $\nabla G = (f_x, f_y, -1)$, so that $\nabla G \cdot \vec{u} = \nabla G \cdot \vec{v} = 0$.

(b) $\vec{N} = (2x, 2y, -1)$. The curve is: $x = t, y = 0, z = t^2$. Thus, $\vec{T} = (1, 0, 2t)$.

Thus, $\vec{N} = (2t, 0, -1)$ and $\vec{N} \cdot \vec{T} = 0$.

12.5.1: 4.

12.5.2: $f_x = 2xye^{x^2y} + 2xy^3$,

$f_y = x^2 e^{x^2y} + 3x^2y^2$,

$f_{xx} = e^{x^2y}[2y + 4x^2y^2] + 2y^3$,

$f_{xy} = f_{yx} = e^{x^2y}[2x + 2x^3y] + 6xy^2$,

$f_{yy} = x^4 e^{x^2y} + 6x^2y$.

12.5.3: $b = \pm 5$.

12.6.1: (a): $\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} + \dfrac{\partial z}{\partial t}$.

(b): Answer (a) yields: $2xvt^3 + xyvt + 4xut^3 + 2xyut + uv$

$= (4 + 2 + 4 + 2 + 2)x^2t^6 = 14x^2t^6$.

Direct substitution gives: $z = 2x^2t^7$. Thus, $\left.\dfrac{\partial z}{\partial t}\right)_x = 14x^2t^6$.