SP

10 **1.** Find the interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{(-1)^n n}{4^{n+1}} (x-1)^{2n}.$$

If we set $y = (x-1)^2$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(-1)^n n}{4^{n+1}} y^n$$

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n n}{4^{n+1}}}{\frac{(-1)^{n+1}(n+1)}{4^{n+2}}} \right| = 4.$$

$R_x = 2.$

The open interval of convergence is

$|x-1| < 2 \Rightarrow -2 < x-1 < 2 \Rightarrow -1 < x < 3.$

At $x = 1$ and at $x = 3$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(-1)^n n}{4^{n+1}} (2)^{2n} = \frac{1}{4} \sum_{n=3}^{\infty} (-1)^n n$$

Since $\lim_{n\to\infty} (-1)^n n$ does not exist, this series diverges by the nth-term test. The interval of convergence is therefore $-1 < x < 3.$

14 **2.** Find the Maclaurin series for the function

$$f(x) = \frac{x}{x^2 - x - 2}.$$

Use a method that guarantees that the series converges to $f(x)$. Express your answer in sigma notation, simplified as much as possible. Determine the interval of convergence for the series.

$$f(x) = \frac{2/3}{x-2} + \frac{1/3}{x+1} \quad \checkmark 2$$

$$\frac{1/3}{1+x} = \frac{1}{3}\sum_{n=0}^{\infty}(-x)^n = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n x^n, \quad |x| < 1. \quad \checkmark 2 \quad \checkmark 1$$

$$\frac{2/3}{x-2} = -\frac{1}{3}\cdot\frac{1}{1-x/2} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n, \quad \left|\frac{x}{2}\right| < 1. \quad \checkmark 3 \quad \checkmark 1$$

$$f(x) = -\frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{2^n}x^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n x^n, \quad \checkmark 2$$

$$= \sum_{n=0}^{\infty}\frac{1}{3}\left[(-1)^n - \frac{1}{2^n}\right]x^n, \quad \checkmark 2$$

This is valid for $-1 < x < 1$. $\checkmark 1$

6 **3.** Find a maximum possible error when the function e^{-3x} is approximated by the first three terms in its Maclaurin series on the interval $0 \leq x \leq 0.2$.

$$e^{-3x} = 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \cdots$$

$$= 1 - 3x + \frac{9x^2}{2} - \frac{27x^3}{6} + \cdots \quad \sim 2$$

Since this series is alternating with absolute values of terms decreasing and approaching 0, when it is approximated by $1 - 3x + 9x^2/2$, the maximum error is

$$\left| -\frac{27x^3}{6} \right| \leq \frac{9}{2}(.02)^3 .$$

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THE UNIVERSITY OF MANITOBA

DATE: June 16, 2012 FINAL EXAMINATION

DEPARTMENT & COURSE NO: MATH2132 TIME: 3 hours

EXAMINATION: Engineering Mathematical Analysis 2 EXAMINER: D. Trim

15 **4.** Find a general solution for the differential equation

$$3y''' + 2y'' + 2y' - y = x - e^{-2x}.$$

The auxiliary equation is

$$0 = 3m^3 + 2m^2 + 2m - 1 = (3m-1)(m^2+m+1) \text{ with roots}$$

$$m = \frac{1}{3}, \quad m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$y_h(x) = C_1 e^{x/3} + e^{-x/2}\left[C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right]$$

When we substitute

$$y_p(x) = Ax + B + Ce^{-2x}$$

into the DE, we obtain

$$3[-8Ce^{-2x}] + 2[4Ce^{-2x}] + 2[A - 2Ce^{-2x}] - [Ax + B + Ce^{-2x}] = x - e^{-2x}$$

Equating coefficients gives:

$$e^{-2x}: \quad -24C + 8C - 4C - C = -1 \Rightarrow C = 1/21$$

$$x: \quad -A = 1 \Rightarrow A = -1$$

$$1: \quad 2A - B = 0 \Rightarrow B = -2$$

$$y_p(x) = -x - 2 + \frac{1}{21}e^{-2x}$$

$$y(x) = C_1 e^{x/3} + e^{-x/2}\left[C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right] - x - 2 + \frac{1}{21}e^{-2x}$$

6 **5.** You are given that the roots of the auxiliary equation associated with the linear, differential equation

$$\phi(D)y = 2xe^{4x} + x^3 - 2 + 3e^{2x}\cos 5x$$

are $m = 0,\ 2 \pm i,\ 2 \pm i,\ \pm 3,\ 4$. Write down the form of a particular solution of the differential equation as predicted by the method of undetermined coefficients. Do **NOT** find the coefficients, just the form of the particular solution.

$$y_h(x) = C_1 + e^{2x}\left[(C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x\right] + C_6 e^{3x} + C_7 e^{-3x} + C_8 e^{4x}$$ -1

-2 -2

$$y_p(x) = \underline{Ax^2 e^{4x} + Bx e^{4x}} + \underline{Cx^4 + Dx^3 + Ex^2 + Fx} + \underline{Ge^{2x}\cos 5x + He^{2x}\sin 5x}$$

-1

6 **6.** When a substance such as glucose is administered intravenously into the bloodstream, it is used up by the body at a rate proportional to the amount present at that time. If it is added at a variable rate $R(t)$, where t is time, and A_0 is the amount in the bloodstream when the intravenous feeding begins, set up, but **DO NOT SOLVE**, an initial value problem for the amount of glucose in the bloodstream at any time. Is the differential equation separable?

$$\frac{dA}{dt} = R(t) - kA, \quad A(0) = A_0$$

The DE is not separable.

7 7. Find an implicit definition for the solution of the initial value problem

$$y^2\frac{dy}{dx} = (x+1)(y^3+1), \qquad y(0) = 1.$$

$$\frac{y^2}{y^3+1}\,dy = (x+1)\,dx \sim 2$$

A 1-parameter family of solutions is defined implicitly by

$$\frac{1}{3}\ln|y^3+1| = \frac{x^2}{2}+x+C \sim 3$$

For $y(0)=1$,

$$\frac{1}{3}\ln 2 = C \sim 1$$

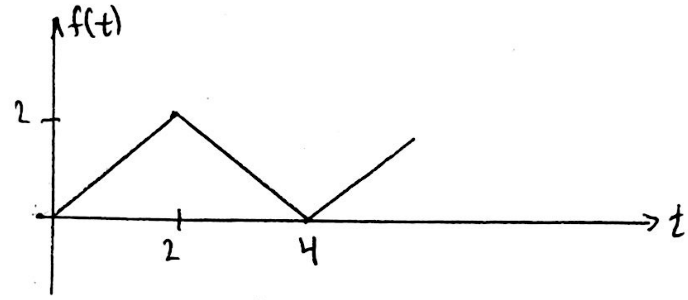
The solution is therefore defined implicitly by

$$\frac{1}{3}\ln|y^3+1| = \frac{x^2}{2}+x+\frac{1}{3}\ln 2 \sim 1$$

9 **8.** Find the Laplace transform for the function

$$f(t) = \begin{cases} t, & 0 \le t \le 2 \\ 4-t, & 2 < t \le 4 \end{cases} \qquad f(t+4) = f(t).$$

Simplify the transform as much as possible.



$$F(s) = \frac{1}{1-e^{-4s}} \int_0^4 f(t)\, e^{-st}\,dt \quad \sim 1$$

$$= \frac{1}{1-e^{-4s}} \mathcal{L}\{ t[h(t)-h(t-2)] + (4-t)[h(t-2)-h(t-4)] \}$$

$$= \frac{1}{1-e^{-4s}} \mathcal{L}\{ t + (4-2t)h(t-2) + (t-4)h(t-4) \} \quad \sim 1$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{1}{s^2} + e^{-2s}\mathcal{L}\{4-2(t+2)\} + e^{-4s}\mathcal{L}\{(t+4)-4\} \right] \quad \sim 2$$

$$= \frac{1}{1-e^{-4s}} \left[\frac{1}{s^2} + e^{-2s}\left(-\frac{2}{s^2}\right) + e^{-4s}\left(\frac{1}{s^2}\right) \right] \quad \sim 1$$

$$= \frac{1}{1-e^{-4s}} \left(\frac{1-2e^{-2s}+e^{-4s}}{s^2} \right)$$

$$= \frac{(1-e^{-2s})^2}{s^2(1+e^{-2s})(1-e^{-2s})} \quad \sim 1 \quad = \frac{1-e^{-2s}}{s^2(1+e^{-2s})} \quad \sim 1$$

9 **9.** Find the inverse Laplace transform for the function

$$F(s) = \frac{e^{-2s}(3s^2+2)}{s^3 - s^2 + 2}.$$

If we set

$$G(s) = \frac{3s^2+2}{s^3-s^2+2} = \frac{3s^2+2}{(s+1)(s^2-2s+2)}$$

$$= \frac{1}{s+1} + \frac{2s}{s^2-2s+2}, \quad -3$$

then

$$g(t) = e^{-t} \overset{-1}{+} 2\mathcal{L}^{-1}\left\{\frac{(s-1)+1}{(s-1)^2+1}\right\} -1$$

$$= e^{-t} + 2e^{t}\mathcal{L}^{-1}\left\{\frac{s+1}{s^2+1}\right\} -1$$

$$= e^{-t} + 2e^{t}(\cos t + \sin t) -1$$

$$\therefore f(t) = \left\{e^{-(t-2)} + 2e^{t-2}[\cos(t-2) + \sin(t-2)]\right\} h(t-2)$$

-2

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8 **10.** A mass of 1 kilogram is suspended from a spring with constant 400 newtons per metre. At time $t = 0$, it is at its equilibrium position and is given velocity 2 metres per second upward. During its subsequent motion, it is also subjected to a damping force that (in newtons) is equal to 40 times its velocity (in metres per second). Use Laplace transforms to find the position of the mass as a function of time.

$$\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 400x = 0 \sim 2, \quad x(0) = 0, \quad \underbrace{x'(0) = 2}_{1}.$$

When we take Laplace transforms,

$$[s^2X - 2] + 40[sX] + 400X = 0 \quad -\cancel{2}\,1$$

$$X(s) = \frac{2}{s^2+40s+400} = \frac{2}{(s+20)^2} \quad -\cancel{2}\,1$$

$$\therefore x(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{(s+20)^2}\right\} \sim 1$$

$$= 2e^{-20t}\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - 1$$

$$= 2te^{-20t} \text{ m} - 1$$

10 **11.** Solve the initial value problem

$$y'' - 3y' - 4y = 3\delta(t-2), \qquad y(0) = 0, \quad y'(0) = 1.$$

When we take Laplace transforms,

$$[s^2Y - 1] - 3[sY] - 4Y = 3e^{-2s} \quad \sim 2$$

$$Y(s) = \frac{1+3e^{-2s}}{s^2-3s-4} = \frac{1+3e^{-2s}}{(s-4)(s+1)} \quad \sim 1$$

$$= (1+3e^{-2s})\left[\frac{1/5}{s-4} + \frac{-1/5}{s+1}\right] \sim 2$$

$$y(t) = \frac{1}{5}e^{4t} - \frac{1}{5}e^{-t} + \frac{3}{5}\left[e^{4(t-2)} - e^{-(t-2)}\right]h(t-2).$$

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