MATH 1210 Problem Workshop 4 Solutions

Solutions to 4–7 also include Descartes' Rule of Signs and the Bounds Theorem.

1. (a) Let $P(x)$ be the polynomial, $x^4 + (3+i)x^3 - 2ix + 5$, then the remainder when $P(x)$ is divided by $x - 4i$ is

$$\begin{aligned} P(4i) &= (4i)^4 + (3+i)(4i)^3 - 2i(4i) + 5 \\ &= 256i^4 + (3+i)(64i^3) - 8i^2 + 5 \\ &= 256i^4 + 192i^3 + 64i^4 - 8i^2 + 5 \\ &= 256 - 192i + 64 + 8 + 5 \\ &= 333 - 192i \end{aligned}$$

(b) Let $P(x)$ be the polynomial, $x^3 - 2x^2 + 3x + 6$, then the remainder when $P(x)$ is divided by $3x + 2$ is

$$\begin{aligned} P\left(-\frac{2}{3}\right) &= \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 + 3\left(-\frac{2}{3}\right) + 6 \\ &= -\frac{8}{27} - \frac{8}{9} - 2 + 6 \\ &= \frac{76}{27} \end{aligned}$$

2. If $3 - 2i$ is a zero of a polynomial with real coefficients, then so must its conjugate. Hence $3 + 2i$ is also a solution.

Therefore by the factor theorem, $(x - (3-2i))(x - (3+2i)) = (x^2 - 6x + 13)$ is a factor of $3x^3 - 17x^2 + 33x + 13$. Long division tells us that

$$P(x) = 3x^3 - 17x^2 + 33x + 13 = (x^2 - 6x + 13)(3x + 1)$$

and hence the zeros are

$$3 + 2i, 3 - 2i \text{ and } -\frac{1}{3}.$$

3. From the remainder theorem we know that if $P(x) = 4x^4 + hx^3 - 3x^2 + kx + 5$,

$$P(2) = 141 \text{ and } P\left(-\frac{1}{3}\right) = \frac{298}{81}.$$

From the first equation we have that

$$141 = 4(2)^4 + h(2)^3 - 3(2)^2 + k(2) + 5 = 8h + 2k + 57$$

This reduces to

$$8h + 2k = 84 \Rightarrow 4h + k = 42 \Rightarrow k = 42 - 4h$$

From the second we get that

$$\begin{aligned}\frac{298}{81} &= 4\left(-\frac{1}{3}\right)^4 + h\left(-\frac{1}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^2 + k\left(-\frac{1}{3}\right) + 5 \\ &= \frac{4}{81} - \frac{1}{27}h - \frac{1}{3} - \frac{1}{3}k + 5.\end{aligned}$$

By multiplying through by 81 this implies

$$\begin{aligned}298 &= 4 - 3h - 27 - 27k + 405 \\ \Rightarrow -84 &= -3h - 27k \\ \Rightarrow h + 9k &= 28 \\ \Rightarrow h + 9(42 - 4h) &= 28 \\ \Rightarrow h + 378 - 36h &= 28 \\ \Rightarrow 35h &= 350 \\ \Rightarrow h &= 10 \\ \Rightarrow k &= 2.\end{aligned}$$

4. $P(x) = 2x^4 - 13x^3 + 24x^2 - 9x$ has 3 sign changes and $P(-x) = 2x^4 + 13x^3 + 24x^2 + 9x$ has no sign changes, so there are 3 or 1 positive real zeros and 0 negative real zeros.

   The bounds theorem says

   $$|x| < \frac{M}{|a_n|} + 1 = \frac{24}{2} + 1 = 13.$$

   The rational root thereom says any rational root p/q has p dividing the constant term which is zero. Since every integer satisfies this, it isn't helpful. However we can factor out the x to get $P(x) = x(2x^3 - 13x^2 + 24x - 9)$. Let $Q(x) = 2x^3 - 13x^2 + 24x - 9$. Hence p divides -9 and q divides $a_n = 2$. Hence $p = \pm 1, \pm 3$ or ± 9 and $q = \pm 1, \pm 2$.

   Hence the possible rational roots are

   $$\pm 1, \pm 3, \pm 9, \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}.$$

   However since we know there are no negative roots from (a) we get the possible roots are

   $$1, 3, 9, \frac{1}{2}, \frac{3}{2}, \frac{9}{2}.$$

Checking values eventually gets

$$\begin{aligned}
Q\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^3 - 13\left(\frac{1}{2}\right)^2 + 24\left(\frac{1}{2}\right) - 9 \\
&= \frac{2}{8} - \frac{13}{4} + 12 - 9 \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

(Note: you could also plug 3 into Q and get it to work)

Using division we can get that

$$P(x) = x(2x-1)(x^2-6x+9) = x(2x-1)(x-3)^2.$$

Therefore the zeros are $0, 1/2, 3$ (with multiplicity 2)

5. $P(x) = 3x^4 - 10x^3 - 20x^2 - 23x - 10$ has 1 sign changes and $P(-x) = 3x^4 + 10x^3 - 20x^2 + 23x - 10$ has three sign changes, so there is 1 positive real zeros and 3 or 1 negative real zeros.

   The bounds theorem says

   $$|x| < \frac{M}{|a_n|} + 1 = \frac{23}{3} + 1 = \frac{26}{3}.$$

   The rational root thereom says any rational root p/q has p dividing the constant term. Hence p divides -10 and q divides $a_n = 3$. Hence $p = \pm 1, \pm 2 \pm 5$ or ± 10 and $q = \pm 1, \pm 3$.

   Hence the possible rational roots are

   $$\pm 1, \pm 2, \pm 5, \pm 10, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}.$$

   However since $|x| < \dfrac{26}{3}$ from (b) we get the possible roots are

   $$\pm 1, \pm 2, \pm 5, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, , \pm\frac{10}{3}.$$

   Checking values eventually gets

   $$\begin{aligned}
   P(5) &= 3(5)^4 - 10(5)^3 - 20(5)^2 - 23(5) - 10 \\
   &= 1875 - 1250 - 250 - 115 - 10 \\
   &= 0.
   \end{aligned}$$

Using division we can get that

$$P(x) = (x-5)(3x^3 + 5x^2 + 5x + 2).$$

Let $Q(x) = 3x^3 + 5x^2 + 5x + 2$. Plugging in values again (although only negative) lead to

$$\begin{aligned}
Q\Big(-\frac{2}{3}\Big) &= 3\Big(-\frac{2}{3}\Big)^3 + 5\Big(-\frac{2}{3}\Big)^2 + 5\Big(-\frac{2}{3}\Big) + 2 \\
&= -\frac{8}{9} + \frac{20}{9} - \frac{10}{3} + 2 \\
&= \frac{4}{3} - \frac{10}{3} + 2 \\
&= -2 + 2 \\
&= 0
\end{aligned}$$

Using division we can get that

$$P(x) = (x-5)(3x+2)(x^2 + x + 1).$$

Since the solutions to $x^2 + x + 1 = 0$ are

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}i}{2}.$$

Hence all the zeros of $P(x)$ are

$$5, -\frac{2}{3}, \frac{-1 \pm \sqrt{3}i}{2}.$$

6. $P(x) = 12x^4 - 11x^3 + 50x^2 - 44x + 8$ has 4 sign changes and $P(-x) = 12x^4 + 11x^3 + 50x^2 + 44x + 8$ has no sign changes, so there is 4,2 or 0 positive real zeros and no negative real zeros.

   The bounds theorem says

   $$|x| < \frac{M}{|a_n|} + 1 = \frac{50}{12} + 1 = \frac{31}{6}.$$

   The rational root thereom says any rational root p/q has p dividing the constant term. Hence p divides 8 and q divides $a_n = 12$. Hence $p = \pm 1, \pm 2 \pm 4$ or ± 8 and $q = \pm 1, \pm 2, \pm 3, \pm 4 \pm 6, \pm 12$.

   Hence the possible rational roots are

   $$\pm 1, \pm 2 \pm 4, \pm 8, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}, \pm\frac{8}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{12}.$$

However since $|x| < \dfrac{31}{6}$ from (b) and there are no negative roots from (a), we get the possible roots are

$$1, 2, 4, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}.$$

Checking values eventually gets

$$\begin{aligned}
P\left(\frac{2}{3}\right) &= 12\left(\frac{2}{3}\right)^4 - 11\left(\frac{2}{3}\right)^3 + 50\left(\frac{2}{3}\right)^2 - 44\left(\frac{2}{3}\right) + 8 \\
&= \frac{64}{27} - \frac{88}{27} + \frac{200}{9} - \frac{88}{3} + 8 \\
&= -\frac{8}{9} + \frac{200}{9} - \frac{88}{3} + 8 \\
&= \frac{64}{3} - \frac{88}{3} + 8 \\
&= -8 + 8 \\
&= 0.
\end{aligned}$$

Using division we can get that

$$P(x) = (3x-2)(4x^3 - x^2 + 16x - 4).$$

Let $Q(x) = 4x^3 - x^2 + 16x - 4$. Plugging in values again leads to

$$\begin{aligned}
Q\left(\frac{1}{4}\right) &= 4\left(\frac{1}{4}\right)^3 - \left(\frac{1}{4}\right)^2 + 16\left(\frac{1}{4}\right) - 4 \\
&= \frac{1}{16} - \frac{1}{16} + 4 - 4 \\
&= 0.
\end{aligned}$$

Using division we can get that

$$P(x) = (3x-2)(4x-1)(x^2+4).$$

Since the solutions to $x^2 + 4 = 0$ are $\pm 2i$, the zeros of $P(x)$ are

$$\frac{2}{3}, \frac{1}{4}, \pm 2i.$$

7. $P(x) = 2x^5 - x^4 + 2x - 1$ has 3 sign changes and $P(-x) = -2x^5 - x^4 - 2x - 1$ has no sign changes, so there is 3 or 1 positive real zeros and no negative real zeros.

The bounds theorem says

$$|x| < \frac{M}{|a_n|} + 1 = \frac{2}{2} + 1 = 2.$$

The rational root thereom says any rational root p/q has p dividing the constant term. Hence p divides -1 and q divides $a_n = 2$. Hence $p = \pm 1$ and $q = \pm 1, \pm 2$.

Hence the possible rational roots are

$$\pm 1, \pm\frac{1}{2}$$

However since there are no negative roots from (a), we get the possible roots are

$$1, \frac{1}{2}.$$

Checking values eventually gets

$$\begin{aligned} P\left(\frac{1}{2}\right) &= 2\left(\frac{1}{2}\right)^5 - \left(\frac{1}{2}\right)^4 + 2\left(\frac{1}{2}\right) - 1 \\ &= \frac{1}{16} - \frac{1}{16} + 1 - 1 \\ &= 0. \end{aligned}$$

Using division we can get that

$$P(x) = (2x - 1)(x^4 + 1).$$

Now we could keep tring to work with the $x^4 + 1$, but we'll notice that is has no real solutions and therefore looking at the rational possibilities is pointless. We can solve this a few ways. We can either find the fourth roots of -1 (done similar to last week's tutorial), or we can find that $x^2 = \pm i$ and hence find the square roots of both i and $-i$.

The latter can also be done a couple of ways. We'll do them each a different way

For $x^2 = i$ we can change it to exponential form. (This is also how we could solve $x^4 = -1$.) So

$$r^2 e^{i(2\theta)} = e^{i(\pi/2 + 2k\pi)}$$

using that $|i| = 1$ and $arg(i) = \pi/2$.

Hence $r = 1$ and $\theta = \pi/4$ for $k = 0$ or $5\pi/4$ for $k = 1$. Hence the solutions in exponential form are

$$x = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$$

or

$$x = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

For $x^2 = -i$ we could do the same thing ($arg(-i) = -\pi/2$) or we could do it the cartesian way.

Let $x = a + bi$. Then $x^2 = -i$ becomes $(a^2 - b^2) + (2ab)i = -i$. Hence

$$2ab = -1 \text{ and } a^2 - b^2 = 0.$$

From the second equation, we get that $a = b$ or $a = -b$ yielding

$$2b^2 = -1$$

which has no real solutions or

$$-2b^2 = -1 \Rightarrow b = \pm\frac{1}{\sqrt{2}}.$$

Using the two possible values of b yields the two solutions

$$-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \text{ or } \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i.$$

Hence the five solutions are

$$\frac{1}{2}, \frac{\pm 1 \pm i}{\sqrt{2}}$$