MATH 1210 Fall 2013 Assignment 1 Solutions

1. *Show the following are true by induction:*

   (a) $3^3 + 3^5 + \cdots + 3^{2n-1} = \frac{1}{8}\big(3^{2n+1} - 27\big)$ *for all* $n \geq 2$.

   Let $P(n)$ be the statement $3^3 + 3^5 + \cdots + 3^{2n-1} = \frac{1}{8}\big(3^{2n+1} - 27\big)$.
   For $n = 2$,

$$LHS = 3^3 = 27, \qquad RHS = \frac{1}{8}\big(3^5 - 27\big) = \frac{216}{8} = 27.$$

   Thus $P(2)$ is true.
   Suppose $P(k)$ is true for some integer $k \geq 2$. That is $3^3 + 3^5 + \cdots + 3^{2k-1} = \frac{1}{8}\big(3^{2k+1} - 27\big)$. We must show $P(k+1)$ is true. Thus we are attempting to show $3^3 + 3^5 + \cdots + 3^{2(k+1)-1} = \frac{1}{8}\big(3^{2(k+1)+1} - 27\big)$

$$\begin{aligned}
LHS &= 3^3 + 3^5 + \cdots + 3^{2(k+1)-1} \\
&= 3^3 + 3^5 + \cdots + 3^{2k+1} \\
&= 3^3 + 3^5 + \cdots + 3^{2k-1} + 3^{2k+1} \\
&= \frac{1}{8}\big(3^{2k+1} - 27\big) + 3^{2k+1} \\
&= \frac{1}{8}\big(3^{2k+1} - 27 + 8 \cdot 3^{2k+1}\big) \\
&= \frac{1}{8}\big(9(3^{2k+1}) - 27\big) \\
&= \frac{1}{8}\big(3^{2k+3} - 27\big) \\
&= \frac{1}{8}\big(3^{2(k+1)+1} - 27\big) \\
&= RHS
\end{aligned}$$

   Thus $P(k+1)$ is true.
   Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 2$.

   (b) 6 *divides* $n(n^2 + 5)$ *for all* $n \geq 1$.

   Let $P(n)$ be the statement 6 divides $n(n^2 + 5)$
   For $n = 1$,

$$n(n^2 + 5) = 1(1^2 + 5) = 6$$

   which is divisible by 6.

Thus $P(1)$ is true.

Suppose $P(k)$ is true for some integer $k \geq 1$. That is 6 divides $k^3 + 5k$ or equivalently $k^3 + 5k = 6l$ for some integer l. We must show $P(k+1)$ is true. Thus we are attempting to show $(k+1)^3 + 5(k+1) = 6L$ for some integer L.

$$\begin{aligned}(k+1)^3 + 5(k+1) &= k^3 + 3k^2 + 3k + 1 + 5k + 5 \\ &= 6l + 3k^2 + 3k + 6 \\ &= 3(2l + 2 + k^2 + k)\end{aligned}$$

Now 2 divides $2l$ and 2, $k^2 + k$ is always even, hence 2 divides $2l + 2 + k^2 + k$. Putting this together shows 6 divides $(k+1)^3 + 5(k+1)$. Thus $P(k+1)$ is true.

Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

(c) *For* $n \geq 1$,

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

Let $P(n)$ be the statement

$$\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^2} \leq 2 - \frac{1}{n}.$$

For $n = 1$,

$$LHS = \frac{1}{1^2} = 1, \qquad RHS = 2 - \frac{1}{1} = 1.$$

Hence the $LHS \leq RHS$ and so $P(1)$ is true.

Suppose $P(k)$ is true for some integer $k \geq 1$. That is $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$.

We must show $P(k+1)$ is true. Thus we are attempting to show $\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$.

$$
\begin{aligned}
LHS &= \frac{1}{1^2} + \frac{1}{2^2} + \dots \frac{1}{(k+1)^2} \\
&= \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \\
&\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&= 2 - \frac{k^2 + 2k + 1 - k}{k(k+1)^2} \\
&= 2 - \frac{k^2 + k + 1}{k(k+1)^2} \\
&\leq 2 - \frac{k^2 + k}{k(k+1)^2} \\
&= 2 - \frac{k+1}{(k+1)^2} \\
&= 2 - \frac{1}{k+1} \\
&= RHS
\end{aligned}
$$

Thus $P(k+1)$ is true.

Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

2. *Show for all $n \geq 1$ that*

$$\sum_{i=n}^{3n} i^2 = \frac{26n^3 + 15n^2 + n}{3}:$$

(a) *by using summation formulas* $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ *and/or* $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6.}$

$$
\begin{aligned}
S &= \sum_{i=n}^{3n} i^2 \\
&= \sum_{i=1}^{3n} i^2 - \sum_{i=1}^{n-1} i^2 \\
&= \frac{3n(3n+1)(2(3n)+1)}{6} - \frac{(n-1)((n-1)+1)(2(n-1)+1)}{6} \\
&= \frac{3n(3n+1)(6n+1)}{6} - \frac{(n-1)(n)(2n-1)}{6} \\
&= \frac{54n^3+27n^2+3n}{6} - \frac{2n^3-3n^2+n}{6} \\
&= \frac{52n^3+30n^2+2n}{6} \\
&= \frac{26n^3+15n^2+n}{3}
\end{aligned}
$$

(b) *by induction.*
Let $P(n)$ be the statement $n^2+(n+1)^2+\cdots+(3n)^2 = \frac{26n^3+15n^2+n}{3}$.
For $n=1$,

$$
LHS = 1^2+2^2+3^2 = 14, \qquad RHS = \frac{26(1)^3+15(1)^2+1}{3} = \frac{42}{3} = 14.
$$

Thus $P(1)$ is true.
Suppose $P(k)$ is true for some integer $k \geq 2$. That is $k^2+(k+1)^2+\cdots+(3k)^2 = \dfrac{26k^3+15k^2+k}{3}$. We must show $P(k+1)$ is true. Thus we are attempting to show $(k+1)^2+(k+2)^2+\ldots(3(k+1))^2 = \dfrac{26(k+1)^3+15(k+1)^2+(k+1)}{3}$.

$$
\begin{aligned}
LHS &= (k+1)^2+(k+2)^2+\cdots+(3(k+1))^2 \\
&= (k+1)^2+(k+2)^2+\cdots+(3k+3)^2 \\
&= k^2+(k+1)^2+(k+2)^2+\cdots+(3k+1)^2+(3k+2)^2+(3k+3)^2-k^2 \\
&= \frac{26k^3+15k^2+k}{3} + (9k^2+6k+1)+(9k^2+12k+4)+(9k^2+18k+9)-k^2 \\
&= \frac{26k^3+15k^2+k}{3} + 26k^2+36k+14 \\
&= \frac{26k^3+93k^2+109k+42}{3}
\end{aligned}
$$

$$
\begin{aligned}
RHS &= \frac{26(k+1)^3 + 15(k+1)^2 + (k+1)}{3} \\
&= \frac{26(k^3 + 3k^2 + 3k + 1) + 15(k^2 + 2k + 1) + (k+1)}{3} \\
&= \frac{26k^3 + 78k^2 + 78k + 26 + 15k^2 + 30k + 15 + k + 1}{3} \\
&= \frac{26k^3 + 93k^2 + 109k + 42}{3} \\
&= LHS
\end{aligned}
$$

Thus $P(k+1)$ is true.

Therefore by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

3. *Possibly using the summation formulas in question 2a, find the value of the following summations.*

   (a) $\sum_{i=10}^{40} (2i - 19)^2$

   Using a substitution $j = i - 9 \Rightarrow i = j + 9$ we get

$$
\begin{aligned}
S &= \sum_{i=10}^{40} (2i - 19)^2 \\
&= \sum_{j=1}^{31} (2(j+9) - 19)^2 \\
&= \sum_{j=1}^{31} (2j - 1)^2 \\
&= \sum_{j=1}^{31} (4j^2 - 4j + 1) \\
&= 4\sum_{j=1}^{31} j^2 - 4\sum_{j=1}^{31} j + \sum_{j=1}^{31} 1 \\
&= 4\left(\frac{31 \cdot 32 \cdot 63}{6}\right) - 4\left(\frac{31 \cdot 32}{2}\right) + 31 \\
&= 4 \cdot 10416 - 4 \cdot 496 + 31 \\
&= 39711
\end{aligned}
$$

(b) $\sum\limits_{j=-20}^{59}\big((j+21)^2-4(j+21)\big)$ Using a substitution $k=j+21 \Rightarrow j=k-21$ we get

$$\begin{aligned}
S &= \sum_{j=-20}^{59}\big((j+21)^2-4(j+21)\big)\\
&= \sum_{j=1}^{80}\big(k^2-4k\big)\\
&= \sum_{j=1}^{80}k^2-4\sum_{j=1}^{80}k\\
&= \left(\frac{80\cdot 81\cdot 161}{6}\right)-4\left(\frac{80\cdot 81}{2}\right)\\
&= 173880-4\cdot 3240\\
&= 160920
\end{aligned}$$

4. *Simplify* $\dfrac{(2i-5)(3-2i)}{(3+i)^2}$ *in Cartesian form.*

$$\begin{aligned}
\frac{(2i-5)(3-2i)}{(3+i)^2} &= \frac{(2i-5)(3-2i)}{(3+i)(3+i)}\\
&= \frac{6i-15+10i-4i^2}{9+6i+i^2}\\
&= \frac{6i-15+10i+4}{9+6i-1}\\
&= \frac{-11+16i}{8+6i}\\
&= \frac{(-11+16i)(8-6i)}{(8+6i)(8-6i)}\\
&= \frac{-88+66i+128i-96i^2}{64-36i^2}\\
&= \frac{-88+66i+128i+96}{64+36}\\
&= \frac{8+194i}{100}\\
&= \frac{2}{25}+\frac{97}{50}i
\end{aligned}$$

5. *Find all* 5^{th} *roots of* $4-4i$. *Leave your answers in Polar form.*

If we convert $4-4i$ to exponential form

$$|4-4i| = \sqrt{4^2+(-4)^2} = \sqrt{32}$$
$$\tan\theta = \frac{y}{x} = -1 \Rightarrow \theta = -\frac{\pi}{4}.$$

Therefore
$$4-4i = \sqrt{32}e^{-i(\pi/4+2k\pi)}$$
for any integer k.

Therefore the fifth roots are
$$(4-4i)^{1/5} = \left(\sqrt{32}e^{-i(\pi/4+2k\pi)}\right)^{1/5} = \sqrt{2}e^{-i(\pi/20+2k\pi/5)}$$

For $k=0$ we get
$$\sqrt{2}e^{-i(\pi/20)} = \sqrt{2}\big(\cos(-\pi/20) + i\sin(-\pi/20)\big).$$

For $k=1$ we get
$$\sqrt{2}e^{i(3\pi/20)} = \sqrt{2}\big(\cos(3\pi/20) + i\sin(3\pi/20)\big).$$

For $k=2$ we get
$$\sqrt{2}e^{i(11\pi/20)} = \sqrt{2}\big(\cos(11\pi/20) + i\sin(11\pi/20)\big).$$

For $k=3$ we get
$$\sqrt{2}e^{i(19\pi/20)} = \sqrt{2}\big(\cos(19\pi/20) + i\sin(19\pi/20)\big).$$

For $k=4$ we get
$$\sqrt{2}e^{i(27\pi/20)} = \sqrt{2}\big(\cos(27\pi/20) + i\sin(27\pi/20)\big).$$

6. *Find z^{20} if $z = \sqrt{3} - i$. Leave your answers in Cartesian form.*

If we convert $\sqrt{3} - i$ to exponential form
$$\left|\sqrt{3} - i\right| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2$$
$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6}.$$

Therefore
$$z = \sqrt{3} - i = 2e^{-i\pi/6}.$$

Hence

$$z^{20} = \left(2e^{-i\pi/6}\right)^{20} = 2^{20}e^{-20i\pi/6} = 2^{20}e^{-10i\pi/3}.$$

Converting to Cartesian form

$$\begin{aligned}
z^{20} &= 2^{20}\big(\cos(-10\pi/3) + i\,\sin(-10\pi/3)\big)\\
&= 2^{20}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\,i\right)\\
&= -2^{19} + 2^{19}\sqrt{3}\,i.
\end{aligned}$$

7. *Solve the following equations. Leave your answers in Cartesian form.*

(a)

$$\begin{aligned}
(\overline{3+2i})z &= i^6(1+2i)(3-4i)\\
\Rightarrow (3-2i)z &= (-1)(1+2i)(3-4i)\\
\Rightarrow (3-2i)z &= (-1-2i)(3-4i)\\
\Rightarrow (3-2i)z &= -3-6i+4i+8i^2\\
\Rightarrow (3-2i)z &= -3-6i+4i-8\\
\Rightarrow (3-2i)z &= -11-2i\\
\Rightarrow z &= \frac{-11-2i}{3-2i}\\
\Rightarrow z &= \frac{(-11-2i)(3+2i)}{(3-2i)(3+2i)}\\
\Rightarrow z &= \frac{-33-6i-22i-4i^2}{9-4i^2}\\
\Rightarrow z &= \frac{-33-6i-22i+4}{9+4}\\
\Rightarrow z &= \frac{-29-28i}{13}\\
\Rightarrow z &= -\frac{29}{13} - \frac{28}{13}i
\end{aligned}$$

(b) $z^4 - 2z^2 + 4 = 0$

Let $u = z^2$. Therefore $u^2 - 2u + 4 = 0$. Hence

$$u = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(4)}}{2} = \frac{2 \pm \sqrt{-12}}{2} = 1 \pm \sqrt{3}\,i.$$

Next we solve $z^2 = 1 \pm \sqrt{3}\,i$.

Converting to exponential leads to

$$\left|1 \pm \sqrt{3}\, i\right| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

$$\tan\theta = \frac{y}{x} = \pm\sqrt{3} \Rightarrow \pm\frac{\pi}{3}$$

Hence

$$z = \left(2e^{i(\pm\frac{\pi}{3}+2k\pi)}\right)^{1/2} = \sqrt{2}e^{i(\pm\frac{\pi}{6}+k\pi)}.$$

Hence for $k = 0$.

$$z = \sqrt{2}e^{i\frac{\pm\pi}{6}} = \sqrt{2}\big(\cos(\pm\pi/6) + i\sin(\pm\pi/6)\big) = \sqrt{2}\left(\frac{\sqrt{3}}{2} \pm i\frac{1}{2}\right) = \frac{\sqrt{3}}{\sqrt{2}} \pm \frac{1}{\sqrt{2}}\, i$$

Hence for $k = 1$.

$$z = \sqrt{2}e^{i\frac{\pm 7\pi}{6}} = \sqrt{2}\big(\cos(\pm 7\pi/6) + i\sin(\pm 7\pi/6)\big) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} \mp i\frac{1}{2}\right) = -\frac{\sqrt{3}}{\sqrt{2}} \mp \frac{1}{\sqrt{2}}\, i$$

8. *Show that*

$$1 + e^{2\pi i/5} + e^{4\pi i/5} + e^{6\pi i/5} + e^{8\pi i/5} = 0.$$

(Hint: If $z = e^{2\pi i/5}$, what is $z^5 - 1$?)

Following the hint, If $z = e^{2\pi i/5}$, then

$$z^5 - 1 = \left(e^{2\pi i/5}\right)^5 - 1 = e^{2\pi i} - 1 = 1 - 1 = 0$$

Factoring $z^5 - 1 = 0$ leads to

$$(z-1)(z^4 + z^3 + z^2 + z + 1) = 0$$

Since $z \neq 1$ we know that

$$0 = 1 + z + z^2 + z^3 + z^4 = 1 + e^{2\pi i/5} + e^{4\pi i/5} + e^{6\pi i/5} + e^{8\pi i/5}.$$