EXERCISES FOR CHAPTER 1: Sequences and Limits

1. (a) $\left(-\frac{1}{2}\right)^n$ (b) $\left(\frac{2}{3}\right)^n$ (c) $\frac{1+n}{1-2n}$

Solution

(a) $\lim\limits_{n\to\infty}\left(-\frac{1}{2}\right)^n = \lim\limits_{n\to\infty}\frac{(-1)^n}{2^n} = 0$

(b) $\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \lim\limits_{n\to\infty}\frac{2^n}{3^n} = 0$

(c) $\lim\limits_{n\to\infty}\frac{1+n}{1-2n} = \lim\limits_{n\to\infty}\frac{1}{-2} = -\frac{1}{2}$

2. (a) $\frac{1+(-1)^n}{2}$ (b) $\frac{1+n^2}{1+n}$ (c) $\frac{n+2}{n^2+3}$

Solution

(a) Limit does not exist.

(b) $\lim\limits_{n\to\infty}\frac{1+n^2}{1+n} = \lim\limits_{n\to\infty}\frac{2n}{1} = \infty$

(c) $\lim\limits_{n\to\infty}\frac{n+2}{n^2+3} = \lim\limits_{n\to\infty}\frac{1}{2n} = 0$

3. (a) $\frac{6n^5+n}{3n^5+1}$ (b) $\frac{\sin 2n}{\sqrt{n}}$ (c) $\frac{\ln 2n}{\ln n}$

Solution

(a) $\lim\limits_{n\to\infty}\frac{6n^5+n}{3n^5+1} = \lim\limits_{n\to\infty}\frac{6n^5(1+\frac{1}{6n^4})}{3n^5(1+\frac{1}{3n^5})} = 2$

(b)

$-\frac{1}{\sqrt{n}} \le \frac{\sin 2n}{\sqrt{n}} \le \frac{1}{\sqrt{n}}$ since $\lim\limits_{n\to\infty}\frac{1}{\sqrt{n}} = 0$

$\Rightarrow \lim\limits_{n\to\infty}\frac{\sin 2n}{\sqrt{n}} = 0$

(c) $\lim\limits_{n\to\infty}\frac{\ln 2n}{\ln n} = \lim\limits_{n\to\infty}\frac{\ln 2}{\ln n} + \lim\limits_{n\to\infty}\frac{\ln n}{\ln n} = 1$

4. (a) $7^{3/n}$ (b) $\sqrt[n]{5n^3}$ (c) $\frac{2^n}{n^2}$

Solution

(a) $y = \lim_{n\to\infty} 7^{3/n} \Rightarrow \ln y = \frac{3}{n}\lim_{n\to\infty} \ln 7 = 0 \Rightarrow y = 1$

(b) $\lim_{n\to\infty} \sqrt[n]{5n^3} = \lim_{n\to\infty} 5^{1/n} \times \lim_{n\to\infty} n^{3/n} = 1$

(c) $\lim_{n\to\infty} \frac{2^n}{n^2} = \lim_{n\to\infty} \frac{2^n \ln 2}{2n} = \lim_{n\to\infty} \frac{2^n (\ln 2)^2}{2} = \infty$

5. (a) $\frac{n^3}{3^n}$ (b) $\frac{n!+n^2}{2n!+n}$ (c) $\frac{n!+3^n}{1+n}$

Solution

(a) $\lim_{n\to\infty} \frac{n^3}{e^n} = \lim_{n\to\infty} \frac{3n^2}{3^n \ln 3} = \lim_{n\to\infty} \frac{6n}{3^n (\ln 3)^2} = \lim_{n\to\infty} \frac{6}{3^n (\ln 3)^3} = 0$

(b) $\lim_{n\to\infty} \frac{n!+n^2}{2n!+n} = \lim_{n\to\infty} \frac{n!(1+\frac{n^2}{n!})}{2n!(1+\frac{n}{2n!})} = \frac{1}{2}$

(c) $\lim_{n\to\infty} \frac{n!+3^n}{1+n} = \lim_{n\to\infty} \frac{n!(1+\frac{3^n}{n!})}{n(1+\frac{1}{n})} = \infty$

6. (a) $\frac{e^n}{1+2e^n}$ (b) $\frac{e^{2n}}{(1+2e^n)^2}$ (c) $n \sin\frac{\pi}{2n}$

Solution

(a) $\lim_{n\to\infty} \frac{e^n}{1+2e^n} = \lim_{n\to\infty} \frac{e^n}{2e^n} = \frac{1}{2}$

(b) $\lim_{n\to\infty} \frac{e^{2n}}{(1+2e^n)^2} = \lim_{n\to\infty} \frac{e^{2n}}{4e^{2n}(1+\frac{1}{2e^n})^2} = \frac{1}{4}$

(c) $\lim_{n\to\infty} n \sin\frac{\pi}{2n} = \lim_{n\to\infty} \frac{\sin\frac{\pi}{2n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{-\frac{\pi}{2n^2}\cos\frac{\pi}{2n}}{-\frac{1}{n^2}} = \frac{\pi}{2}\lim_{n\to\infty} \cos\frac{\pi}{2n} = \frac{\pi}{2}$

7. (a) $\ln\left(\frac{e^n+1}{e^n-1}\right)$ (b) $\frac{\sqrt{2n+1}}{\sqrt{n}}$ (c) $\left(\frac{3}{2}\right)^n$

Solution

(a) $\lim_{n\to\infty} \ln\left(\frac{e^n+1}{e^n-1}\right) = \ln \lim_{n\to\infty} \frac{e^n+1}{e^n-1} = \ln \lim_{n\to\infty} \frac{e^n}{e^n} = \ln 1 = 0$

(b) $\lim_{n\to\infty} \frac{\sqrt{2n+1}}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}(\sqrt{2+\frac{1}{n}})}{\sqrt{n}} = \sqrt{2}$

(c) $\lim_{n\to\infty} \left(\frac{3}{2}\right)^n = \infty$ since $\frac{3}{2} > 1$

8. Show that (a) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$, (b) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n+\alpha} = e^x$ and (c) $\lim_{n\to\infty}\left(1+\frac{x}{n+\alpha}\right)^n = e^x$ where α is a constant.

Solution

(a) Let $y=\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$. Call $\frac{x}{n}=\frac{1}{m}$ so that $m\to\infty$ as $n\to\infty$. Then $n=mx$ and so

$$\begin{aligned} y &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} \\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x \\ &= e^x \end{aligned}$$

(b)

$$\begin{aligned} \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n+\alpha} &= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\alpha} \times \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} \\ &= 1\times\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n} \\ &= 1\times e^x = e^x \end{aligned}$$

(c) Call $m=n+\alpha$. Then as $n\to\infty$, $m\to\infty$. So

$$\begin{aligned} \lim_{n\to\infty}\left(1+\frac{x}{n+\alpha}\right)^{n} &= \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m-\alpha} \\ &= \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{-\alpha} \times \lim_{m\to\infty}\left(1+\frac{x}{m}\right)^{m} \\ &= 1\times e^x = e^x \end{aligned}$$

9. (a) $\left(1-\frac{1}{n^2}\right)^n$ (b) $\left(1+\frac{\pi}{n}\right)^{\pi n}$ (c) $\left(\frac{n-1}{n}\right)^n$

Solution

(a)

$$\begin{aligned} \lim_{n\to\infty}\left(1-\frac{1}{n^2}\right)^n &= \lim_{n\to\infty}\left(\left(1-\frac{1}{n}\right)\left(1+\frac{1}{n}\right)\right)^n \\ &= \lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\ &= e^{-1}e^{1} \\ &= 1 \end{aligned}$$

(b)

$$\lim_{n\to\infty}\left(1+\frac{\pi}{n}\right)^{\pi n}=\left(\lim_{n\to\infty}\left(1+\frac{\pi}{n}\right)^{n}\right)^{\pi}$$
$$=\left(e^{\pi}\right)^{\pi}$$
$$=e^{\pi^2}$$

(c)

$$\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{n}=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{n}$$
$$=e^{-1}$$

10. (a) $\left(\dfrac{n+3}{n+2}\right)^{n}$ (b) $\left(\dfrac{n}{n+3}\right)^{n}$ (c) $\left(1-\dfrac{2}{n}\right)^{2n}$

Solution

(a)

$$\lim_{n\to\infty}\left(\frac{n+3}{n+2}\right)^{n}=\lim_{n\to\infty}\left(\frac{n+2+1}{n+2}\right)^{n}$$
$$=\lim_{n\to\infty}\left(1+\frac{1}{n+2}\right)^{n}$$
$$=e$$

(b)

$$\lim_{n\to\infty}\left(\frac{n}{n+3}\right)^{n}=\lim_{n\to\infty}\left(\frac{n+3-3}{n+3}\right)^{n}$$
$$=\lim_{n\to\infty}\left(1-\frac{3}{n+3}\right)^{n}$$
$$=e^{-3}$$

(c) $\lim_{n\to\infty}\left(1-\dfrac{2}{n}\right)^{2n}=\left(\lim_{n\to\infty}\left(1-\dfrac{2}{n}\right)^{n}\right)^{2}=(e^{-2})^2=e^{-4}$

11. (a) $\dfrac{(\ln n)^7}{n^2}$ (b) $\sqrt[n]{n3^n}$ (c) $n^{1+\frac{1}{n}}$

Solution

(a) $\lim_{n\to\infty}\dfrac{(\ln n)^7}{n^2}=\lim_{n\to\infty}\dfrac{7(\ln n)^6}{2n^2}=\lim_{n\to\infty}\dfrac{42(\ln n)^5}{4n^2}=\cdots=0$

(b) $\lim_{n\to\infty}\sqrt[n]{n3^n}=\lim_{n\to\infty}\sqrt[n]{n}\times\lim_{n\to\infty}\sqrt[n]{3^n}=1\times 3=3$

(c) $\lim_{n\to\infty}n^{1+\frac{1}{n}}=\lim_{n\to\infty}n\times\lim_{n\to\infty}n^{\frac{1}{n}}=\infty\times 1=\infty$

12. (a) $\sqrt[n]{n7^n}$ (b) $\left(\dfrac{1}{n}\right)^{1/\ln n}$ (c) $\arctan n$

Solution

(a) $\lim\limits_{n\to\infty}\sqrt[n]{n7^n} = \lim\limits_{n\to\infty}\sqrt[n]{n}\times\lim\limits_{n\to\infty}\sqrt[n]{7^n} = 1\times 7 = 7$.

(b) Let $y = \left(\dfrac{1}{n}\right)^{1/\ln n}$. Then $\lim\limits_{n\to\infty}\ln y = \lim\limits_{n\to\infty}\dfrac{\ln(\frac{1}{n})}{\ln n} = -\lim\limits_{n\to\infty}\dfrac{\ln n}{\ln n} = -1$.

Thus $\lim\limits_{n\to\infty}\left(\dfrac{1}{n}\right)^{1/\ln n} = e^{-1}$.

(c) $\lim\limits_{n\to\infty}\arctan n = \dfrac{\pi}{2}$.

13. Show that $\lim\limits_{n\to\infty}\left(1+x^n\right)^{1/n} = 1$ if $|x|<1$. Hence find $\lim\limits_{n\to\infty}\left(5^n+7^n\right)^{1/n}$.

Solution

$$y = \lim_{n\to\infty}\left(1+x^n\right)^{1/n}$$

$$\ln y = \lim_{n\to\infty}\frac{\ln(1+x^n)}{n} = 0$$

$$\Rightarrow y = 1$$

$$\lim_{n\to\infty}\left(5^n+7^n\right)^{1/n} = \lim_{n\to\infty}7\left(1+\left(\frac{5}{7}\right)^n\right)^{1/n}$$

$$= 7\times\lim_{n\to\infty}\left(1+\left(\frac{5}{7}\right)^n\right)^{1/n}$$

$$= 7$$

14. Find the limit (a) $\lim\limits_{n\to\infty}\left(2^n+4^n\right)^{-1/n}$.

Solution

$$\lim_{n\to\infty}\left(2^n+4^n\right)^{-1/n} = \lim_{n\to\infty}\frac{1}{\left(2^n+4^n\right)^{1/n}}$$

$$= \lim_{n\to\infty}\frac{1}{4\left(\left(\frac{2}{4}\right)^n+1\right)^{1/n}}$$

$$= \frac{1}{4}$$

15. (a) $\dfrac{2^n+4^n}{3^n+6^n}$ (b) $\left(\dfrac{2^n+4^n}{5^n+6^n}\right)^{1/n}$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{2^n+4^n}{3^n+6^n} = \lim\limits_{n\to\infty}\dfrac{4^n}{6^n}\dfrac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{1}{2}\right)^n+1} = \lim\limits_{n\to\infty}\dfrac{4^n}{6^n} = 0$

(b) $\lim\limits_{n\to\infty}\left(\dfrac{2^n+4^n}{5^n+6^n}\right)^{1/n} = \dfrac{4}{6}\lim\limits_{n\to\infty}\left(\dfrac{\left(\frac{2}{4}\right)^n+1}{\left(\frac{5}{6}\right)^n+1}\right)^{1/n} = \dfrac{2}{3}$

16. (a) $\dfrac{n^{137}}{137^n}$ (b) $\dfrac{n^{17}}{e^n}$ (c) $\dfrac{3^{2n+1}}{5n^2}$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{n^{137}}{137^n}=0$, standard result.

(b) $\lim\limits_{n\to\infty}\dfrac{n^{17}}{e^n}=0$, standard result.

(c) $\lim\limits_{n\to\infty}\dfrac{3^{2n+1}}{5n^2}=\lim\limits_{n\to\infty}\dfrac{3^{2n+1}2\ln 3}{10n}=\lim\limits_{n\to\infty}\dfrac{3^{2n+1}(2\ln 3)^2}{10}=\infty$

17. (a) $\dfrac{\pi}{\arctan n}$ (b) $\ln(n+1)-\ln n$ (c) $\ln(n^2+n)-\ln n^2$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{\pi}{\arctan n}=\dfrac{\pi}{\dfrac{\pi}{2}}=2$

(b) $\lim\limits_{n\to\infty}\big(\ln(n+1)-\ln n\big)=\lim\limits_{n\to\infty}\ln(\dfrac{n+1}{n})=\ln\lim\limits_{n\to\infty}(\dfrac{n+1}{n})=\ln 1=0$

(c) $\lim\limits_{n\to\infty}\big(\ln(n^2+n)-\ln n^2\big)=\lim\limits_{n\to\infty}\ln(\dfrac{n^2+n}{n^2})=\ln\lim\limits_{n\to\infty}(\dfrac{n^2+n}{n^2})=\ln 1=0$

18. (a) $\dfrac{8n!}{5n^n}$ (b) $\dfrac{n!}{10n!+(n-1)!}$ (c) $\dfrac{\ln 9n}{\sqrt[3]{n}}$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{8n!}{5n^n}=0$, standard result.

(b) $\lim\limits_{n\to\infty}\dfrac{n!}{10n!+(n-1)!}=\lim\limits_{n\to\infty}\dfrac{n!}{10n!(1+\dfrac{1}{10n})}=\dfrac{1}{10}$

(c) $\lim\limits_{n\to\infty}\dfrac{\ln 9n}{\sqrt[3]{n}}=\lim\limits_{n\to\infty}\dfrac{\ln 9}{\sqrt[3]{n}}+\lim\limits_{n\to\infty}\dfrac{\ln n}{\sqrt[3]{n}}=0+\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n}}{\dfrac{1}{3}n^{-2/3}}=3\lim\limits_{n\to\infty}\dfrac{1}{n^{1/3}}=0$

19. (a) $\dfrac{\sqrt{n^2+1}}{\sqrt{n}}$ (b) $\dfrac{\sin\sqrt{n}}{\sqrt{n}}$ (c) $\dfrac{n!}{5n^2}$

Solution

(a) $\lim\limits_{n\to\infty}\dfrac{\sqrt{n^2+1}}{\sqrt{n}}=\lim\limits_{n\to\infty}\dfrac{n\sqrt{1+\dfrac{1}{n}}}{\sqrt{n}}=\infty$

(b) $-\dfrac{1}{\sqrt{n}}\le\dfrac{\sin\sqrt{n}}{\sqrt{n}}\le\dfrac{1}{\sqrt{n}}\Rightarrow\lim\limits_{n\to\infty}\dfrac{\sin\sqrt{n}}{\sqrt{n}}=0$ since $\lim\limits_{n\to\infty}\dfrac{\pm 1}{\sqrt{n}}=0$

(c) $\lim\limits_{n\to\infty}\dfrac{n!}{5n^2}=\infty$, standard result.

20. (a) $\frac{n}{3^n}$ (b) n^3e^{-2n} (c) $\frac{n!+(n-1)!}{(n-1)!+(n-2)!}$

Solution

(a) $\lim_{n\to\infty}\frac{n}{3^n}=\lim_{n\to\infty}\frac{1}{3^n\ln 3}=0$

(b) $\lim_{n\to\infty}n^3e^{-2n}=\lim_{n\to\infty}\frac{n^3}{e^{2n}}=\lim_{n\to\infty}\frac{3n^2}{2e^{2n}}=\lim_{n\to\infty}\frac{6n}{4e^{2n}}=\lim_{n\to\infty}\frac{6}{8e^{2n}}=0$

(c) $\lim_{n\to\infty}\frac{n!+(n-1)!}{(n-1)!+(n-2)!}=\lim_{n\to\infty}\frac{n!(1+\frac{1}{n})}{(n-1)!(1+\frac{2}{n-1})}=\infty$

21. (a) $\frac{\ln n}{\sqrt{n}}$ (b) $n\ln(1+\frac{2}{n})$ (c) $\frac{3^{2n}}{7n}$

Solution

(a) $\lim_{n\to\infty}\frac{\ln n}{\sqrt{2n}}=\lim_{n\to\infty}\frac{\frac{1}{n}}{\sqrt{2}\frac{1}{2\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{2}}{\sqrt{n}}=0$

(b) $\lim_{n\to\infty}n\ln(1+\frac{2}{n})=\ln\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^n=\ln e^2=2$

(c) $\lim_{n\to\infty}\frac{3^{2n}}{7n}=\lim_{n\to\infty}\frac{9^n}{7n}=\lim_{n\to\infty}\frac{9^n\ln 9}{7}=\infty$

22. (a) $\frac{n!}{1+5n!}$ (b) $\frac{2n^n}{n^n+3n^{n-1}}$

Solution

(a) $\lim_{n\to\infty}\frac{n!}{1+5n!}=\lim_{n\to\infty}\frac{n!}{5n!(1+\frac{1}{5n!})}=\frac{1}{5}$

(b) $\lim_{n\to\infty}\frac{2n^n}{n^n+3n^{n-1}}=\lim_{n\to\infty}\frac{2n^n}{n^n(1+\frac{3}{n})}=2$

23. (a) $\frac{2n^n}{n^n+n!}$ (b) $\frac{n!}{e^{2n}+3n!}$ (c) $\frac{\ln 2n}{\sqrt{2n}}$

Solution

(a) $\lim_{n\to\infty}\frac{2n^n}{n^n+n!}=\lim_{n\to\infty}\frac{2n^n}{n^n(1+\frac{n!}{n^n})}=2$

(b) $\lim_{n\to\infty}\frac{n!}{e^{2n}+3n!}=\lim_{n\to\infty}\frac{n!}{3n!(1+\frac{e^{2n}}{3n!})}=\frac{1}{3}$

(c) $\lim_{n\to\infty}\frac{\ln 2n}{\sqrt{2n}}=\lim_{n\to\infty}\frac{\ln 2}{\sqrt{2n}}+\lim_{n\to\infty}\frac{\ln n}{\sqrt{2n}}=0+\lim_{n\to\infty}\frac{\frac{1}{n}}{\sqrt{2}\frac{1}{2\sqrt{n}}}=\lim_{n\to\infty}\frac{\sqrt{2}}{\sqrt{n}}=0$

24. James Stirling (1692-1770) showed that for large values of n, $n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Use this approximation for the factorial of n to show that $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

Solution

$\lim_{n\to\infty} \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n/e)^n \sqrt{2\pi n}}{n^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}}{e^n} = 0$.

(Notice that Stirling's result leads to the elegant limit $\lim_{n\to\infty} \frac{n! e^n}{n^n \sqrt{n}} = \sqrt{2\pi}$.)

25. (a) Show that $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. (b) Establish this limit by an $\varepsilon - N$ proof.

Solution

(a) $\lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{n^2}{n^2(1+\frac{1}{n^2})} = 1$

(b) $\left|\frac{n^2}{n^2+1} - 1\right| < \varepsilon \Rightarrow -\varepsilon < \frac{n^2}{n^2+1} - 1 < \varepsilon \Rightarrow -\varepsilon < \frac{1}{n^2+1} < \varepsilon$. Hence we have that $n^2 + 1 < \frac{1}{\varepsilon} \Rightarrow n > \sqrt{\frac{1}{\varepsilon} - 1}$. Thus $\left|\frac{n^2}{n^2+1} - 1\right| < \varepsilon$ for all $n > N$ where $N = \sqrt{\frac{1}{\varepsilon} - 1}$.

26. (a) Show that $\lim_{n\to\infty} \frac{1}{2^n} = 0$. (b) Establish this limit by an $\varepsilon - N$ proof.

Solution

(a) Let $y = \frac{1}{2^n}$. Then $\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \ln\left(\frac{1}{2^n}\right) = -\lim_{n\to\infty} n \ln 2 = -\infty$. Hence $y \to e^{-\infty} = 0$.

(b) $\left|\frac{1}{2^n}\right| < \varepsilon \Rightarrow -\varepsilon < \frac{1}{2^n} < \varepsilon$. Hence $\frac{1}{2^n} < \varepsilon \Rightarrow 2^n > \frac{1}{\varepsilon} \Rightarrow n > -\frac{\ln \varepsilon}{\ln 2}$. Thus $\left|\frac{1}{2^n}\right| < \varepsilon$ for all $n > N$ where $N = -\frac{\ln \varepsilon}{\ln 2}$.

27. (a) Show that $\lim_{n\to\infty} \frac{n!}{1+2n!} = \frac{1}{2}$. (b) Establish this limit by an $\varepsilon - N$ proof.

Solution

(a) $\lim_{n\to\infty} \frac{n!}{1+2n!} = \lim_{n\to\infty} \frac{n!}{2n!(1+\frac{1}{2n!})} = \frac{1}{2}$.

(b) $\left|\frac{n!}{1+2n!} - \frac{1}{2}\right| < \varepsilon \Rightarrow -\varepsilon < \frac{n!}{1+2n!} - \frac{1}{2} < \varepsilon$. I.e. $-\varepsilon < \frac{2n! - 1 - 2n!}{2(1+2n!)} < \varepsilon$ i.e. $-\varepsilon < \frac{-1}{2(1+2n!)} < \varepsilon$. We must then ensure that

$$\frac{-1}{2(1+2n!)} > -\varepsilon$$
$$\frac{1}{2(1+2n!)} < \varepsilon$$
$$2n! > \frac{2}{\varepsilon} - 1$$
$$n! > \frac{1}{\varepsilon} - \frac{1}{2}$$

Now $n! > n$ and so we if we demand $n > \frac{1}{\varepsilon} - \frac{1}{2}$ we automatically ensure $n! > \frac{1}{\varepsilon} - \frac{1}{2}$. Thus $\left|\frac{n!}{1+2n!} - \frac{1}{2}\right| < \varepsilon$ for all $n > N$ where $N > \frac{1}{\varepsilon} - \frac{1}{2}$. As a check take $\varepsilon = 10^{-1} \Rightarrow N > 10 - \frac{1}{2}$. With $N = 10$ we have that $\left|\frac{10!}{1+2\times 10!} - \frac{1}{2}\right| < 6.89\times 10^{-8} < 10^{-1}$. This shows that the choice of *N* is very loose. This is so because the inequality $n! > n$ is very easily satisfied so a much lower value of *N* than $N > \frac{1}{\varepsilon} - \frac{1}{2}$ could have done.

28. A sequence of positive numbers is defined recursively through $u_{n+1} = \sqrt{1-u_n}$, with $u_1 = \frac{1}{2}$. Given that the sequence is convergent, find the limit of the sequence, $\lim_{n\to\infty} u_n$.

Solution

Let the limit be *L*. Then $\lim_{n\to\infty} u_n = L$, $\lim_{n\to\infty} u_{n+1} = L$ and so $L = \sqrt{1-L} \Rightarrow L^2 + L - 1 = 0$. The positive root is $L = \frac{-1+\sqrt{5}}{2}$.

29. Find $\lim_{x\to 0} \frac{1-\cos x}{x^2}$.

Solution

Limit is of the 0/0 type. Hence $\lim_{x\to 0} \frac{1-\cos x}{x^2} = \lim_{x\to 0} \frac{\sin x}{2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2}$.

30. Find the limit $\lim_{x\to 0} \frac{\ln(1-2x)}{x}$.

Solution

Limit is of the 0/0 type. Hence $\lim_{x\to 0} \frac{\ln(1-2x)}{x} = \lim_{x\to 0} \frac{\frac{-2}{1-2x}}{1} = -2$.

31. Evaluate the limit $\lim_{x\to 0} \frac{\tan x}{x}$.

Solution

Limit is of the 0/0 type. Hence $\lim_{x\to 0} \frac{\tan x}{x} = \lim_{x\to 0} \frac{\sec^2 x}{1} = 1$.

32. Find $\lim_{x \to 0} \frac{\tan 3x}{x}$.

Solution

Limit is of the 0/0 type. Hence $\lim_{x \to 0} \frac{\tan 3x}{x} = \lim_{x \to 0} \frac{3\sec^2 3x}{1} = 3$.

33. Find $\lim_{x \to 0} \frac{\arcsin(3x) - 3x}{x^3}$.

Solution

Limit is of the 0/0 type. Hence

$$\lim_{x \to 0} \frac{\arcsin(3x) - 3x}{x^3} = \lim_{x \to 0} \frac{\frac{3}{\sqrt{1-9x^2}} - 3}{3x^2} = \lim_{x \to 0} \frac{3 \times (-18x)(-\frac{1}{2})(1-9x^2)^{-3/2}}{6x}$$

$$= \lim_{x \to 0} \frac{27(1-9x^2)^{-3/2} + 27x(-\frac{3}{2})(1-9x^2)^{-3/2}(-18x)}{6} = \frac{9}{2}$$

34. Evaluate $\lim_{x \to 0} (\frac{1}{x} - \frac{1}{\sin x})$.

Solution

$$\begin{aligned} \lim_{x \to 0} (\frac{1}{x} - \frac{1}{\sin x}) &= \lim_{x \to 0} (\frac{\sin x - x}{x \sin x}) \\ &= \lim_{x \to 0} (\frac{\cos x - 1}{\sin x + x\cos x}) \\ &= \lim_{x \to 0} (\frac{-\sin x}{\cos x + \cos x - x\sin x}) \\ &= 0 \end{aligned}$$

35. Find the limit $\lim_{x \to 0} \frac{\sin(x + 2\sin x)}{\sin x}$.

Solution

$$\begin{aligned} \lim_{x \to 0} \frac{\sin(x + 2\sin x)}{\sin x} &= \lim_{x \to 0} \frac{\cos(x + 2\sin x)(1 + 2\cos x)}{\cos x} \\ &= 3 \end{aligned}$$

36. Find $\lim_{x \to 0} (\frac{1}{x} - \frac{1}{\tan x})$.

Solution

We may write $\frac{1}{x} - \frac{1}{\tan x} = \frac{x - \tan x}{x \tan x}$ which is a 0/0 limit. Using L' Hôpital's rule we have

$$\begin{aligned} \lim_{x \to 0} (\frac{1}{x} - \frac{1}{\tan x}) &= \lim_{x \to 0} \frac{x - \tan x}{x \tan x} = \lim_{x \to 0} \frac{1 - \sec^2 x}{\tan x + x\sec^2 x} \\ &= \lim_{x \to 0} \frac{-2\sec x(\sec x \tan x)}{\sec^2 x + \sec^2 x + x 2\sec x(\sec x \tan x)} = 0 \end{aligned}$$

37. Find $\lim_{x\to 1}(\frac{1}{\ln x}-\frac{1}{x-1})$.

Solution

$\frac{1}{\ln x}-\frac{1}{x-1}=\frac{x-1-\ln x}{(x-1)\ln x}$ which results in a 0/0 limit. Using L' Hôpital's rule we have

$\lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\lim_{x\to 1}\frac{\frac{1}{x^2}}{\frac{1}{x}+\frac{1}{x^2}}=\frac{1}{2}$, using L' Hôpital's rule again.

38. Find $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}$.

Solution

The limit is of the 0/0 type. Using L' Hôpital's rule we have that

$$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x}=\lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}+\frac{1}{2\sqrt{1-x}}}{1}=1.$$

39. Consider the function $f(x)=1+e^x+e^{2x}+\cdots+e^{nx}$. (a) Show that $f(x)=\frac{e^{(n+1)x}-1}{e^x-1}$.

(b) Show that $\frac{df}{dx}=1e^x+2e^{2x}+\cdots+ne^{nx}$. (c) Hence show that $1+2+\cdots+n=\left.\frac{df}{dx}\right|_{x=0}$. (d) Hence evaluate $1+2+\cdots+n$. (e) Using this method find an expression for (but do not attempt to evaluate) $1^k+2^k+\cdots+n^k$ where k is a positive integer.

Solution

(a) $1+e^x+e^{2x}+\cdots+e^{nx}=\frac{e^{(n+1)x}-1}{e^x-1}$, by summing a geometric progression.

(b) $\frac{df}{dx}=1e^x+2e^{2x}+\cdots+ne^{nx}$

(c) Setting x=0 in the result above gives the answer.

(d)

$$\frac{df(x)}{dx}=\frac{(n+1)e^{(n+1)x}(e^x-1)-(e^{(n+1)x}-1)e^x}{(e^x-1)^2}$$

$$=\frac{e^x-e^{(n+1)x}-ne^{(n+1)x}+ne^{(n+2)x}}{(e^x-1)^2}$$

This gives a 0/0 limit if we put x=0. So by L' Hôpital's rule

$$\begin{aligned}
\frac{df(0)}{dx} &= \lim_{x\to 0}\frac{e^x-(n+1)e^{(n+1)x}-n(n+1)e^{(n+1)x}+n(n+2)e^{(n+2)x}}{2(e^x-1)e^x} \\
&= \lim_{x\to 0}\frac{e^x-(n+1)^2e^{(n+1)x}-n(n+1)^2e^{(n+1)x}+n(n+2)^2e^{(n+2)x}}{4e^{2x}-2e^x} \\
&= \frac{1-(n+1)^2-n(n+1)^2+n(n+2)^2}{2} \\
&= \frac{1-n^2-2n-1-n^3-2n^2-n+n^3+4n^2+4n}{2} \\
&= \frac{n^2+n}{2} \\
&= \frac{n(n+1)}{2}
\end{aligned}$$

(e) We saw that $\frac{df}{dx}=1e^x+2e^{2x}+\cdots+ne^{nx}$. Differentiating again gives

$$\frac{d^2f}{dx^2}=1^2e^x+2^2e^{2x}+\cdots+n^2e^{nx}$$

and so differentiating k times gives

$$\frac{d^kf}{dx^k}=1^ke^x+2^ke^{2x}+\cdots+n^ke^{nx}.$$

Thus at $x=0$ we have that

$$1^k+2^k+\cdots+n^k=\left.\frac{d^kf}{dx^k}\right|_{x=0}=\left.\frac{d^k}{dx^k}\left(\frac{e^{(n+1)x}-1}{e^x-1}\right)\right|_{x=0}.$$

Applying L' Hôpital's rule to this expression is hopeless. To make progress requires more advanced work.