Math 1210 · Assignment 1 - Solutions Sept 24, 2008

1 a) Let $P_n : 5+11+17+\cdots+(6n-1) = n(3n+2)$

$(n=1)$ 5 $1(3(1)+2)=5$

$\therefore P_1$ is true.

Assume P_k is true; $5+11+17+\cdots+(6k-1) = k(3k+2)$

then $5+11+17+\cdots+(6k-1)+(6k+5)$

$= k(3k+2)+6k+5$

$= 3k^2+8k+5$

$= (k+1)(3k+5)$

$= (k+1)(3(k+1)+2)$ hence P_{k+1} is also true

Since P_1 is true and P_k implies P_{k+1} then by PMI, P_n is true for all $n \geq 1$

b) Let $P_n : 3^2+6^2+9^2+\cdots+(3n)^2 = \frac{3n(n+1)(2n+1)}{2}$

$(n=1)$ $3^2 = 9$ $\frac{3(1+1)(2+1)}{2} = 9$

So P_1 is true.

Assume P_k is true; $3^2+6^2+9^2+\cdots+(3k)^2 = \frac{3k(k+1)(2k+1)}{2}$

then $3^2+6^2+9^2+\cdots+(3k)^2+(3k+3)^2$

$= \frac{3k(k+1)(2k+1)}{2} + (3k+3)^2$

$= \frac{6k^3+9k^2+3k}{2} + \frac{18k^2+32k+18}{2}$

$= \frac{1}{2}(6k^3+27k^2+35k+18)$

$= \frac{1}{2}(3k+3)(k+2)(2k+3)$

$= \frac{(3(k+1))((k+1)+1)(2(k+1)+1)}{2}$ hence P_{k+1} is true

Since P_1 is true and P_k implies P_{k+1}, P_n is true by PMI for all $n \geq 1$

Q1

c) Let P_n : $5^{2n}-1$ is divisible by 8

$n=1$ $\quad 5^2-1 = 25-1 = 24 = 3\cdot 8$

So 24 is divisible by 8 and P_1 is true

Assume P_k is true ; $5^{2k}-1$ is divisible by 8

$$5^{2(k+1)}-1 = 5^{2k+2}-1 = 5^2\cdot 5^{2k} - 1$$
$$= 5^2\cdot 5^{2k} - 5^2 + 5^2 - 1$$
$$= 5^2(5^{2k}-1) + 24$$

Since $5^{2k}-1$ and 24 are divisible by 8, $5^{2(k+1)}-1$ is also divisible by 8, hence P_{k+1} is also true

Since P_1 is true and P_k implies P_{k+1}, by PMI P_n is true for all $n \geq 1$

d) Let P_n : $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \cdots + \frac{1}{3^{2n}} = \frac{1}{2}\left(1 - \frac{1}{9^n}\right)$

$(n=1)$ $\quad \frac{1}{3} + \frac{1}{3^2} = \frac{4}{9}$ $\qquad \frac{1}{2}(1 - \frac{1}{9}) = \frac{1}{2}(\frac{8}{9}) = \frac{4}{9}$

hence P_1 is true.

Assume P_k is true; $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k}} = \frac{1}{2}(1 - \frac{1}{9^k})$

then $\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \cdots + \frac{1}{3^{2k}} + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}}$

$$= \frac{1}{2}(1 - \frac{1}{9^k}) + \frac{1}{3^{2k+1}} + \frac{1}{3^{2k+2}}$$
$$= \frac{1}{2} - \frac{1}{2}\frac{1}{9^k} + \frac{1}{3}\frac{1}{9^k} + \frac{1}{9}\frac{1}{9^k}$$
$$= \frac{1}{2} + \left[-\frac{1}{2} + \frac{1}{3} + \frac{1}{9}\right]\frac{1}{9^k}$$
$$= \frac{1}{2} - \left[\frac{1}{18}\right]\frac{1}{9^k} = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{9}\cdot\frac{1}{9^k}$$
$$= \frac{1}{2}(1 - \frac{1}{9^{k+1}})$$

hence P_{k+1} is true.

Since P_1 is true and $P_k \Rightarrow P_{k+1}$, by PMI P_n is true for all $n \geq 1$

Q1

e) Let $P_n: n + (n+1) + (n+2) + \cdots + (3n) = 2n(2n+1)$

$(n=1)$ $\quad 1+2+3=6$ $\qquad\qquad 2(1)(2+1)=6$

Hence P_1 is true.

Assume P_k is true, $k+(k+1)+(k+2)+\cdots+3k = 2k(2k+1)$

$$\boxed{\text{want to show } (k+1)+(k+2)+(k+3)+\cdots+(3k+3) = (2k+2)(2k+3)}$$

then $(k+1)+(k+2)+(k+3)+\cdots+(3k+3)$

$= k+(k+1)+(k+2)+(k+3)+\cdots+(3k)+(3k+1)+(3k+2)+(3k+3)-k$

$= 2k(2k+1)+(3k+1)+(3k+2)+(3k+3)-k$

$= 4k^2+2k+3k+1+3k+2+3k+3-k$

$= 4k^2+10k+6$

$= (2k+2)(2k+3)$

$= 2(k+1)(2(k+1)+1)$, hence P_{k+1} is true

Since P_1 is true and $P_k \Rightarrow P_{k+1}$, by PMI P_n is true for all $n \geq 1$.

2 a) $\sum_{i=1}^{13}(3i)(2i-4) = \sum_{i=1}^{13}(6i^2-12i) = 6\sum_{i=1}^{13}i^2 - 12\sum_{i=1}^{13}i$

$= 6\left[\frac{13(14)(27)}{6}\right] - 12\left[\frac{13(14)}{2}\right] = 4914 - 1092 = 3822$

b) $\sum_{j=7}^{17}(7j^3-4j) = \sum_{j=1}^{17}(7j^3-4j) - \sum_{j=1}^{6}(7j^3-4j)$

$= 7\sum_{j=1}^{17}j^3 - 4\sum_{j=1}^{17}j - 7\sum_{j=1}^{6}j^3 + 4\sum_{j=1}^{6}j$

$= 7\left[\frac{(17^2)(18)^2}{4}\right] - 4\left[\frac{(17)(18)}{2}\right] - 7\left[\frac{6^2\,7^2}{4}\right] + 4\left[\frac{(6)(7)}{2}\right]$

$= 163863 - 612 - 3087 + 84$

$= 160248$

c) $\sum_{k=17}^{43} (k-16)^2 = \sum_{m=1}^{27} m^2$

$= \frac{(27)(28)(55)}{6} = 6930$

Q3 a) $-3+6-9+12-15+\cdots-51 = \sum_{j=1}^{17} (-1)^j 3j$

b) $\frac{1}{2} + \frac{3}{4} + \frac{5}{6} + \cdots + \frac{41}{42} = \sum_{j=1}^{21} \frac{2j-1}{2j}$

c) $\frac{\sqrt{3}}{2} - \frac{\sqrt{5}}{4} + \frac{\sqrt{7}}{6} - \frac{3}{8} + \cdots - \frac{5}{24} = \sum_{j=1}^{12} (-1)^{n+1} \frac{\sqrt{2j+1}}{2j}$

Q4 a) $2+4+6+\cdots+(2n) = \sum_{k=1}^{n} 2k$

$\sum_{k=1}^{n} 2k = 2\sum_{k=1}^{n} k = 2\left(\frac{n(n+1)}{2}\right) = n(n+1)$

b) $1^2+2^2+3^2+\cdots+(3n)^2 = \sum_{j=1}^{3n} n^2$

$\sum_{j=1}^{3n} n^2 = \frac{(3n)(3n+1)(6n+1)}{6} = \frac{n(3n+1)(6n+1)}{2}$

c) $1+3+5+7+\cdots+(4n-1) = \sum_{k=1}^{2n} (2k-1)$

$\sum_{k=1}^{2n} (2k-1) = 2\sum_{k=1}^{2n} k - \sum_{k=1}^{2n} 1 = 2\left(\frac{2n(2n+1)}{2}\right) - 2n$

$= 2n(2n+1) - 2n = 4n^2$

d) $n^2 + (n+1)^2 + (n+2)^2 + \cdots + (2n)^2 = \sum_{j=n}^{2n} j^2$

$\sum_{j=n}^{2n} j^2 = \sum_{j=1}^{2n} j^2 - \sum_{j=1}^{n-1} j^2 = \frac{(2n)(2n+1)(4n+1)}{6} - \frac{(n-1)(n)(2n-1)}{6} = \frac{n(14n+1)(n+1)}{6}$