MATH 1210 Problem Workshop 11 Solutions

1. We have to see if there are any constants c_1, c_2, c_3 such that $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$. Has just the trivial solution $c_1 = c_2 = c_3 = 0$ or if there are more than just the trivial solution

   (a)

$$\begin{aligned}\mathbf{0} &= c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w}\\ &= c_1\langle 2, 1, -3, 0\rangle + c_2\langle 5, -1, 2, 3\rangle + c_3\langle 0, 3, 2, -4\rangle\\ &= \langle 2c_1 + 5c_2, c_1 - c_2 + 3c_3, -3c_1 + 2c_2 + 2c_3, 3c_2 - 4c_3\rangle\end{aligned}$$

   Hence

$$\begin{aligned}2c_1 + 5c_2 &= 0\\ c_1 - c_2 + 3c_3 &= 0\\ -3c_1 + 2c_2 + 2c_3 &= 0\\ 3c_2 - 4c_3 &= 0\end{aligned}$$

   Putting this into an augmented matrix to solve yields

$$\left[\begin{array}{ccc|c} 2 & 5 & 0 & 0\\ 1 & -1 & 3 & 0\\ -3 & 2 & 2 & 0\\ 0 & 3 & -4 & 0\end{array}\right] \text{Using } R_1 \leftrightarrow R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0\\ 2 & 5 & 0 & 0\\ -3 & 2 & 2 & 0\\ 0 & 3 & -4 & 0\end{array}\right] \text{Using } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_2 + 3R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0\\ 0 & 7 & -6 & 0\\ 0 & -1 & 11 & 0\\ 0 & 3 & -4 & 0\end{array}\right] \text{Using } R_3 \to -R_3 \text{ and } R_2 \leftrightarrow R_3 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & 0\\ 0 & 1 & -11 & 0\\ 0 & 7 & -6 & 0\\ 0 & 3 & -4 & 0\end{array}\right] \text{Using } R_1 \to R_1 + R_2,\ R_3 \to R_3 - 7R_2 \text{ and } R_4 \to R_4 - 3R_2$$

   yields

$$\left[\begin{array}{ccc|c} 1 & 0 & -8 & 0\\ 0 & 1 & -11 & 0\\ 0 & 0 & 71 & 0\\ 0 & 0 & 29 & 0\end{array}\right] \text{Using } R_3 \to \frac{1}{71}R_3 \text{ yields}$$

$\left[\begin{array}{ccc|c} 1 & 0 & -8 & 0 \\ 0 & 1 & -11 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 29 & 0 \end{array}\right]$ Using $R_1 \to R_1 + 8R_3$, $R_1 \to R_2 + 11R_3$ and $R_4 \to R_4 - 29R_3$ yields

$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Hence $c_1 = c_2 = c_3 = 0$ meaning the only solution is the trivial solution. Therefore the vectors are linearly independent.

(b)

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} \\ &= c_1\langle 3, 2, -1\rangle + c_2\langle -4, 2, 6\rangle + c_3\langle 5, -1, 2\rangle \\ &= \langle 3c_1 - 4c_2 + 5c_3, 2c_1 + 2c_2 - c_3, -c_1 + 6c_2 + 2c_3\rangle \end{aligned}$$

Hence

$$\begin{aligned} 3c_1 - 4c_2 + 5c_3 &= 0 \\ 2c_1 + 2c_2 - c_3 &= 0 \\ -c_1 + 6c_2 + 2c_3 &= 0 \end{aligned}$$

Since the coefficient matrix is square, Cramer's rule applies and there is only the trivial solution if and only if the determinant the coefficient matrix is non-zero.

$$\begin{aligned} |A| &= \begin{vmatrix} 3 & -4 & 5 \\ 2 & 2 & -1 \\ -1 & 6 & 2 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -1 \\ 6 & 2 \end{vmatrix} - (-4)\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & 2 \\ -1 & 6 \end{vmatrix} \\ &= 3(4 - (-6)) + 4(4 - 1) + 5(12 - (-2)) \\ &= 3(10) + 4(3) + 5(14) \\ &= 112. \end{aligned}$$

Hence the trivial solution is the only solutions and therefore the vectors are linearly independent.

(c)

$$\begin{aligned} \mathbf{0} &= c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} \\ &= c_1\langle 3, 2\rangle + c_2\langle 5, -1\rangle + c_3\langle 6, -23\rangle \\ &= \langle 3c_1 + 5c_2 + 6c_3, 2c_1 - 2c_2 - 23c_3\rangle \end{aligned}$$

Hence

$$\begin{aligned} 3c_1 + 5c_2 + 6c_3 &= 0 \\ 2c_1 - 2c_2 - 23c_3 &= 0 \end{aligned}$$

Since there are more variables then equations, the *homogeneous* system has infinitely many solutions. Therefore the vectors are linearly dependent.

2. We need to solve the equation.

$$c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$$

which leads to the system

$$\begin{aligned} 2c_1 + 5c_2 &= 0 \\ c_1 - c_2 - 7c_3 &= 0 \\ -3c_1 + 2c_2 + 19c_3 &= 0 \\ 3c_2 + 6c_3 &= 0 \end{aligned}$$

$$\mathbf{u} = \langle 2, 1, -3, 0\rangle, \quad \mathbf{v} = \langle 5, -1, 2, 3\rangle, \quad \mathbf{w} = \langle 0, -7, 19, 6\rangle$$

Putting this into an augmented matrix to solve yields

$$\left[\begin{array}{ccc|c} 2 & 5 & 0 & 0 \\ 1 & -1 & -7 & 0 \\ -3 & 2 & 19 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \text{Using } R_1 \leftrightarrow R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -7 & 0 \\ 2 & 5 & 0 & 0 \\ -3 & 2 & 19 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \text{Using } R_2 \to R_2 - 2R_1 \text{ and } R_3 \to R_2 + 3R_1 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -7 & 0 \\ 0 & 7 & 14 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \text{Using } R_2 \to \frac{1}{7}R_2 \text{ yields}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & -7 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & 3 & 6 & 0 \end{array}\right] \text{Using } R_1 \to R_1 + R_2,\ R_3 \to R_3 + R_2 \text{ and } R_4 \to R_4 - 3R_2 \text{ yields}$$

$\left[\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ Hence we get infinitely many solutions, implying that the vectors are linearly dependent. As for a way to write one as a linear combination of the others. Note the general solutions is

$$c_2 = -2c_3 \text{ and } c_1 = 5c_3$$

where c_3 is arbitrary. Letting $c_3 = 1$ we can get $c_2 = -2$ and $c_1 = 5$ leading to

$$5\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0} \Rightarrow \mathbf{w} = -5\mathbf{u} + 2\mathbf{v}.$$

3. For B^{-1}

$$\left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right] \Rightarrow^{R_2-2R_1} \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & -8 & -2 & 1 \end{array}\right] \Rightarrow^{-R_2/8}$$
$$\left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/4 & -1/8 \end{array}\right] \Rightarrow^{R_1-3R_2} \left[\begin{array}{cccc} 1 & 0 & 1/4 & 3/8 \\ 0 & 1 & 1/4 & -1/8 \end{array}\right]$$

Therefore

$$B^{-1} = \left[\begin{array}{cc} 1/4 & 3/8 \\ 1/4 & -1/8 \end{array}\right]$$

C^{-1} does not exist since C is not square.

For D^{-1} we augment with a 3×3 identity matrix

$$\left[\begin{array}{cccccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow^{R_2-R_1} \left[\begin{array}{cccccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow^{R_2 \leftrightarrow R_3}$$
$$\left[\begin{array}{cccccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -2 & -1 & 1 & 0 \end{array}\right] \Rightarrow^{-1/2R_3} \left[\begin{array}{cccccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right] \Rightarrow^{\substack{R_2-2R_3 \\ R_1-2R_3}}$$
$$\left[\begin{array}{cccccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right] \Rightarrow^{R_1-R-2} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 & -1/2 & 0 \end{array}\right]$$

Therefore

$$D^{-1} = \left[\begin{array}{ccc} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1/2 & -1/2 & 0 \end{array}\right]$$

For E^{-1}

$$\left[\begin{array}{cccccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow^{R_2 \leftrightarrow R_1} \left[\begin{array}{cccccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow^{R_3 - 2R_1}$$

$$\left[\begin{array}{cccccc} 1 & 1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -4 & 0 & -2 & 1 \end{array}\right] \Rightarrow^{R_1 - R_2}_{R_3 + R_2} \left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right] \Rightarrow^{R_2 + R_3}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & -2 & 1 & -2 & 1 \end{array}\right] \Rightarrow^{-1/2 R_3} \left[\begin{array}{cccccc} 1 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right] \Rightarrow^{R_1 - R_3}$$

$$\left[\begin{array}{cccccc} 1 & 0 & 0 & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 2 & -2 & 1 \\ 0 & 0 & 1 & -1/2 & 1 & -1/2 \end{array}\right]$$

Therefore

$$E^{-1} = \left[\begin{array}{ccc} -1/2 & 0 & 1/2 \\ 2 & -2 & 1 \\ -1/2 & 1 & -1/2 \end{array}\right]$$