THE UNIVERSITY OF MANITOBA

DATE: June 22, 2013 **FINAL EXAMINATION**

DEPARTMENT & COURSE NO: MATH2132 **TIME**: 3 hours

EXAMINATION: Engineering Mathematical Analysis 2 **EXAMINER**: D. Trim

11 **1.** Find the interval of convergence for the power series

$$\sum_{n=3}^{\infty} \frac{(1+1/n)^3}{2^{n+2}}(x-2)^{2n}.$$

First we set $y=(x-2)^2$, in which case the series becomes

$$\sum_{n=3}^{\infty}\frac{(1+1/n)^3}{2^{n+2}}y^n$$

$$R_y = \lim_{n\to\infty}\left|\frac{\frac{(1+1/n)^3}{2^{n+2}}}{\frac{(1+1/(n+1))^3}{2^{n+3}}}\right| = 2.$$

Thus, $R_x=\sqrt{2}$. The open interval of convergence is

$|x-2|<\sqrt{2} \Rightarrow -\sqrt{2}<x-2<\sqrt{2} \Rightarrow 2-\sqrt{2}<x<2+\sqrt{2}.$

At the end points $x=2\pm\sqrt{2}$, the series becomes

$$\sum_{n=3}^{\infty} \frac{(1+1/n)^3}{2^{n+2}}(2\pm\sqrt{2}-2)^{2n} = \sum_{n=3}^{\infty}\frac{(1+1/n)^3}{2^{n+2}}(2^n) = \frac{1}{4}\sum_{n=3}^{\infty}\left(1+\frac{1}{n}\right)^3$$

Since $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^3=1\neq 0$, the series diverges (by the n^{th}-term test).

The interval of convergence is therefore $2-\sqrt{2}<x<2+\sqrt{2}$.

10 **2.** Find the Taylor series about $x=4$ for the function

$$f(x)=\frac{x}{x+3}.$$

Use a method that guarantees that the series converges to $f(x)$. Express your answer in sigma notation, simplified as much as possible. Determine the interval of convergence for the series.

$\boxed{x-4}$

$$\frac{x}{x+3} = 1-\frac{3}{x+3} = 1-\frac{3}{7+(x-4)} = 1-\frac{3/7}{1+\left(\frac{x-4}{7}\right)}$$

$$=1-\frac{3}{7}\sum_{n=0}^{\infty}\left[-\left(\frac{x-4}{7}\right)\right]^n = 1-\frac{3}{7}\sum_{n=0}^{\infty}\frac{(-1)^n}{7^n}(x-4)^n$$

$$=1-\frac{3}{7}+\sum_{n=0}^{\infty}\frac{3(-1)^{n+1}}{7^{n+1}}(x-4)^n$$

$$=\frac{4}{7}+\sum_{n=0}^{\infty}\frac{3(-1)^{n+1}}{7^{n+1}}(x-4)^n$$

The interval of convergence is

$$\left|-\left(\frac{x-4}{7}\right)\right|<1$$

$$|x-4|<7$$

$$-7<x-4<7$$

$$-3<x<11$$

15 **3.** Find a general solution for the differential equation

$$2y''' + 3y'' - 5y' - 3y = x + 3e^{2x}.$$

The auxiliary equation is

$$2m^3 + 3m^2 - 5m - 3 = 0 \quad 2$$

$$(2m+1)(m^2 + m - 3) = 0$$

$$m = -\tfrac{1}{2}, \; m = \frac{-1 \pm \sqrt{1+12}}{2} = \frac{-1 \pm \sqrt{13}}{2} \quad 3$$

$$y_h(x) = C_1 e^{-x/2} + C_2 e^{(-1+\sqrt{13})x/2} + C_3 e^{-(1+\sqrt{13})x/2}. \quad 2$$

A particular solution is $y_p(x) = Ax + B + Ce^{2x}$. Substituting into the differential equation, 1

$$2(8Ce^{2x}) + 3(4Ce^{2x}) - 5(A + 2Ce^{2x}) - 3(Ax + B + Ce^{2x}) = x + 3e^{2x}.$$

When we equate coefficients:

$$e^{2x} : 16C + 12C - 10C - 3C = 3 \Rightarrow C = \tfrac{1}{5} \quad 1$$

$$x : -3A = 1 \Rightarrow A = -\tfrac{1}{3} \quad 1$$

$$1 : -5A - 3B = 0 \Rightarrow B = \tfrac{5}{9} \quad 1$$

Thus, $y_p(x) = -\frac{x}{3} + \frac{5}{9} + \frac{1}{5}e^{2x}$

$$y(x) = C_1 e^{-x/2} + C_2 e^{(-1+\sqrt{13})x/2} + C_3 e^{-(1+\sqrt{13})x/2} - \frac{x}{3} + \frac{5}{9} + \frac{1}{5}e^{2x}$$

6 **4.** You are given that the roots of the auxiliary equation associated with the linear, differential equation

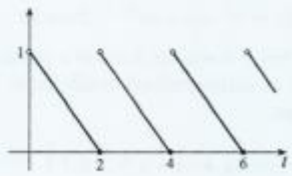
$$\phi(D)y = x^2 + 4 + xe^{4x} - 2\cos 3x$$

are $m = 0,\ 2 \pm 3i,\ 2 \pm 3i,\ \pm\sqrt{3},\ 4,\ 4$. Write down the form of a particular solution of the differential equation as predicted by the method of undetermined coefficients. Do **NOT** find the coefficients, just the form of the particular solution.

$$y_h(x) = C_1 + e^{2x}\left[(C_2 + C_3x)\cos 3x + (C_4 + C_5x)\sin 3x\right] + C_6 e^{\sqrt{3}x} + C_7 e^{-\sqrt{3}x} + (C_8 + C_9x)e^{4x} \quad 2$$

$$y_p(x) = Ax^3 + Bx^2 + Cx + Dx^3e^{4x} + Ex^2e^{4x} + F\cos 3x + G\sin 3x$$

8 **5.** Find the Laplace transform for the function shown below. You need **NOT** simplify your answer, but you must use a method that does not involve integration by parts.



$f(t) = -\frac{1}{2}(t-2), \quad 0 < t \le 2.$

$$F(s) = \frac{1}{1-e^{-2s}} \mathcal{L}\left\{ -\frac{1}{2}(t-2)[h(t) - h(t-2)] \right\} \sim 4$$

$$= \frac{-1}{2(1-e^{-2s})} \mathcal{L}\left\{ t-2-(t-2)h(t-2) \right\} \sim 1$$

$$= \frac{-1}{2(1-e^{-2s})} \left[\frac{1}{s^2} - \frac{2}{s} - e^{-2s}\mathcal{L}\{t+2-2\} \right] \sim 2$$

$$= \frac{-1}{2(1-e^{-2s})} \left[\frac{1}{s^2} - \frac{2}{s} - \frac{e^{-2s}}{s^2} \right] \sim 1$$

5 **6.** Find the Laplace transform of the function $f(t) = e^{2t} \sin t\, h(t - 2\pi)$.

$$F(s) = e^{-2\pi s} \mathcal{L}\left\{ e^{2(t+2\pi)} \sin(t+2\pi) \right\} - 2$$

$$= e^{-2\pi s} e^{4\pi} \mathcal{L}\left\{ e^{2t} \sin t \right\} - 1$$

$$= e^{2\pi(2-s)} \left[\frac{\cancel{s}\,1}{(s-2)^2+1} \right] - 2$$

8 **7.** Find inverse Laplace transforms for the functions:

(a) $F(s) = \frac{e^{-3s}}{s^3 - 3s^2 + 3s - 1}$ (b) $F(s) = \frac{s}{s^2 + 2s + 4}$.

(a) Since $\mathcal{L}^{-1}\left\{ \frac{1}{s^3-3s^2+3s-1} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^3} \right\} = e^t \mathcal{L}^{-1}\left\{ \frac{1}{s^3} \right\} = \frac{t^2}{2} e^t$, -1

$$\mathcal{L}^{-1}\left\{ \frac{e^{-3s}}{s^3-3s^2+3s-1} \right\} = \frac{(t-3)^2}{2} e^{t-3} h(t-3) - 1$$

(b) $$\mathcal{L}^{-1}\left\{ \frac{s}{s^2+2s+4} \right\} = \mathcal{L}^{-1}\left\{ \frac{(s+1)-1}{(s+1)^2+3} \right\} - 2$$

$$= e^{-t} \mathcal{L}^{-1}\left\{ \frac{s-1}{s^2+3} \right\} - 1$$

$$= e^{-t}\left[\cos\sqrt{3}t - \frac{1}{\sqrt{3}} \sin\sqrt{3}t \right] - 1$$

15 **8.** Solve the initial value problem

$$y'' + 4y = 3\delta(t-2) + h(t-1), \qquad y(0) = 1, \quad y'(0) = 0.$$

If we take Laplace transforms,

$$s^2Y - s + 4Y = 3e^{-2s} + \frac{e^{-s}}{s}$$ — 5

$$\therefore Y(s) = \frac{3e^{-2s}}{s^2+4} + \frac{e^{-s}}{s(s^2+4)} + \frac{s}{s^2+4}$$ — 2

$$= \frac{3e^{-2s}}{s^2+4} + \left(\frac{1/4}{s} - \frac{s/4}{s^2+4}\right)e^{-s} + \frac{s}{s^2+4}$$ — ~~4~~ 3

$$\therefore y(t) = \frac{3}{2}\sin 2(t-2)h(t-2) + \frac{1}{4}h(t-1) - \frac{1}{\cancel{8}4}\cancel{\sin}\cos 2(t-1)h(t-1) + \cos 2t$$ — ~~4~~ 5

11 **9.** A tank contains 10 kilograms of sugar disolved in 500 litres of water. Solution with 3 kilograms of sugar per 1000 litres of water is added to the tank at the rate of 100 millilitres per second. Well-stirred mixture is removed from the tank at 200 millilitres per second. Set up, but DO NOT SOLVE, an initial-value problem for the number of grams of sugar in the tank as a function of time. For how long is your model valid?

If $A(t)$ represents the number of grams of sugar in the tank, then

$$\frac{dA}{dt} = (\text{rate sugar enters}) - (\text{rate sugar leaves})$$ — 1

2 — 2

$$= \frac{3000(100)}{10^6} - \frac{200A}{500{,}000 - 100t}$$

$$= \frac{3}{10} - \frac{2A}{500{,}000 - 100t}$$ — 1

The initial condition is $A(0) = 10{,}000$.

The model is valid for $500{,}000 - 100t > 0 \Rightarrow t < 5000$ s. — 2

11 **10.** A mass of 100 grams is suspended from a spring with constant 600 newtons per metre. At time $t = 0$, it is 10 cm above its equilibrium position and is given velocity 2 metres per second downward. Find the amplitude and the period of the resulting motion of the mass.

$$\frac{1}{10}\frac{d^2x}{dt^2} + 600x = 0, \quad x(0) = \frac{1}{10}, \quad x'(0) = -2$$ — 2 — 1 — 1

The auxiliary equation is $m^2 + 6000 = 0 \Rightarrow m = \pm 20\sqrt{15}\,i$.

$$x(t) = C_1\cos 20\sqrt{15}t + C_2\sin 20\sqrt{15}t.$$ — 1

$$\frac{1}{10} = C_1, \quad -2 = 20\sqrt{15}\,C_2$$ — 2

$$x(t) = \frac{1}{10}\cos 20\sqrt{15}t - \frac{1}{10\sqrt{15}}\sin 20\sqrt{15}t \text{ m.}$$

The amplitude is $\sqrt{\frac{1}{100} + \frac{1}{1500}} = \frac{2}{5\sqrt{15}}$ m. — ~~2~~ 1

The period is $\frac{2\pi}{20\sqrt{15}} = \frac{\pi}{10\sqrt{15}}$ s. — 1

The following table of Laplace transforms may be used without proof.

$f(t)$		$F(s) = \mathcal{L}\{f(t)\}$
$t^n \quad (n = 0, 1, 2, \ldots)$	$\leftrightarrow$	$\dfrac{n!}{s^{n+1}}$
e^{at}	$\leftrightarrow$	$\dfrac{1}{s-a}$
$\sin at$	$\leftrightarrow$	$\dfrac{a}{s^2+a^2}$
$\cos at$	$\leftrightarrow$	$\dfrac{s}{s^2+a^2}$
$h(t-a)$	$\leftrightarrow$	$\dfrac{e^{-as}}{s}$
$\delta(t-a)$	$\leftrightarrow$	e^{-as}
$e^{at}f(t)$	$\leftrightarrow$	$F(s-a)$
$f(t)h(t-a)$	$\rightarrow$	$e^{-as}\mathcal{L}\{f(t+a)\}$
$f(t-a)h(t-a)$	$\leftarrow$	$e^{-as}F(s)$
$p-$periodic $f(t)$	$\rightarrow$	$\dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$
$\displaystyle\int_0^t f(u)g(t-u)\,du$	$\leftarrow$	$F(s)G(s)$
$f'(t)$	$\rightarrow$	$sF(s) - f(0)$
$f''(t)$	$\rightarrow$	$s^2F(s) - sf(0) - f'(0)$