MATH 1210 Tutorial # 2 Solutions for TAs

1. Prove that

$$\sum_{k=1}^{n}((2k-1)\sqrt{3})^2 = n(4n^2-1)$$

for every positive integer n.

Solution: For $n=1$ we get

$LHS = ((2(1)-1)\sqrt{3})^2 = 3 = 1(4(1^2)-1) = RHS$.

Suppose that the statement holds for some $m \geq 1$, i.e.,

$$\sum_{k=1}^{m}((2k-1)\sqrt{3})^2 = m(4m^2-1).$$

Then
$$\sum_{k=1}^{m+1}((2k-1)\sqrt{3})^2 = \sum_{k=1}^{m}((2k-1)\sqrt{3})^2 + (2(m+1)-1)\sqrt{3})^2$$

$$= m(4m^2-1) + 3(2m+1)^2 = 4m^3+12m^2+11m+3.$$

On the other hand, we have that also

$$(m+1)(4(m+1)^2-1) = (m+1)(4m^2+8m+4-1)$$

$$= 4m^3+12m^2+11m+3$$

Therefore, the statement

$$\sum_{k=1}^{n}((2k-1)\sqrt{3})^2 = n(4n^2-1)$$

is valid for all integers ≥ 1

2. Decide whether or not the equalities

$$\text{(a)} \quad \sum_{k=1}^{n}(k+1)^3 = \Big(\sum_{k=1}^{n}(k+1)\Big)^2$$

and

$$\text{(b)} \quad \sum_{k=0}^{n}(k+1)^3 = \Big(\sum_{k=0}^{n}(k+1)\Big)^2$$

hold for all positive integers n.

(a) The equality does <u>not</u> hold for $n=1$:

$$\sum_{k=1}^{1}(k+1)^3 = (1+1)^3 = 2^3 = 8, \text{ wheras}$$

$$\left(\sum_{k=1}^{1}(k+1)\right)^2 = (1+1)^2 = 2^2 = 4 \neq 8$$

(b) The equality holds since

$$\left.\begin{array}{l} \displaystyle\sum_{k=0}^{n}(k+1)^3 = \sum_{k=1}^{n+1}k^3 = \frac{(n+1)^2(n+2)^2}{4} \\ \text{and} \\ \displaystyle\left(\sum_{k=0}^{n}(k+1)\right)^2 = \left(\sum_{k=1}^{n+1}k\right)^2 = \left[\frac{(n+1)(n+2)}{2}\right]^2 \end{array}\right] =$$

by using formulas proved in class

3. Rewrite the sum

$$\sum_{r=12}^{122}\frac{r-6}{r+9}$$

using an index whose initial and terminal values are 1 and 111, respectively (HINT: use a change of variables).

Introduce the new variable $s = r - 11$. Then the sum above becomes $\displaystyle\sum_{s=1}^{111}\frac{s+5}{s+20}$.

4. Simplify

$$\frac{169}{5+12i} + \left(\overline{(1-2i)^3+4}\right)^2$$

and express in Cartesian form.

$$= \frac{169\,(5-12i)}{(5+12i)(5-12i)} + \left(\overline{1-3\cdot 2i + 3\cdot 4i^2 - 8i^3 + 4}\right)^2$$

$$= \frac{169\,(5-12i)}{169} + \left(\overline{-7+2i}\right)^2 = (5-12i) + (-7-2i)^2$$

$$= (5-12i) + (45+28i) = 50+16i$$

5. Given that $i^2 = -1$, show that

$$\sum_{k=0}^{4n} i^k = 1$$

for all integers $n \geq 0$.

Note that the sum above is a finite geometric series with starting term $a=1$ and ratio $r=i$. According to Example 1.3 of the text, its value equals

$$a\,\frac{1-r^{4n+1}}{1-r} = 1\cdot\frac{1-i^{4n+1}}{1-i} = \frac{1-(i^4)^n\cdot i}{1-i}$$

$$= \frac{1-1^n\cdot i}{1-i} = 1$$