# MATH 1210 Problem Workshop 7 Solutions

1. If $\mathbf{u} = \langle 3, -2, 4\rangle$ and $\mathbf{v} = \langle -3, 6, 2\rangle$, find:

   (a)

$$2\mathbf{u} - 4\mathbf{v} = 2\langle 3, -2, 4\rangle - 4\langle -3, 6, 2\rangle = \langle 6, -4, 8\rangle - \langle -12, 24, 8\rangle = \langle 18, -28, 0\rangle$$

   (b)

$$\begin{aligned} |\mathbf{u}| &= \sqrt{3^2 + (-2)^2 + 4^2} = \sqrt{9+4+16} = \sqrt{29}. \\ \hat{\mathbf{v}} &= \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\langle -3, 6, 2\rangle}{\sqrt{(-3)^2 + 6^2 + 2^2}} = \frac{1}{7}\langle -3, 6, 2\rangle \\ \mathbf{u}\cdot\mathbf{v} &= \langle 3, -2, 4\rangle \cdot \langle -3, 6, 2\rangle = 3(-3) + (-2)(6) + 4(2) = -13. \end{aligned}$$

   Therefore

$$\begin{aligned} |\mathbf{u}|\,\hat{\mathbf{v}} + 3(\mathbf{u}\cdot\mathbf{v})\mathbf{u} &= \sqrt{29}\left(\frac{1}{7}\langle -3, 6, 2\rangle\right) + 3(-13)\langle 3, -2, 4\rangle \\ &= \left\langle \frac{-3\sqrt{29}}{7} - 117, \frac{6\sqrt{29}}{7} + 78, \frac{2\sqrt{29}}{7} - 156 \right\rangle \end{aligned}$$

   (c)

$$\begin{aligned} 2\mathbf{u} \times (-3\mathbf{v}) &= \langle 6, -4, 8\rangle \times \langle 9, -18, -24\rangle \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 6 & -4 & 8 \\ 9 & -18 & -6 \end{vmatrix} \\ &= \begin{vmatrix} -4 & 8 \\ -18 & -6 \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} 6 & 8 \\ 9 & -6 \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} 6 & -4 \\ 9 & -18 \end{vmatrix}\hat{\mathbf{k}} \\ &= (24 - (-144))\hat{\mathbf{i}} - (-36 - 72)\hat{\mathbf{j}} + (-108 - (-36))\hat{\mathbf{k}} \\ &= 168\hat{\mathbf{i}} + 108\hat{\mathbf{j}} - 72\hat{\mathbf{k}} \end{aligned}$$

2. If $\mathbf{u} = \langle u_x, u_y, u_z\rangle, \mathbf{v} = \langle v_x, v_y, v_z\rangle$, then

$$\begin{aligned} \mathbf{v}\times\mathbf{u} &= \begin{vmatrix} v_y & v_z \\ u_y & u_z \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} v_x & v_z \\ u_x & u_z \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} v_x & v_y \\ u_x & u_y \end{vmatrix}\hat{\mathbf{k}} \\ &= (v_y u_z - v_z u_y)\hat{\mathbf{i}} - (v_x u_z - v_z u_x)\hat{\mathbf{j}} + (v_x u_y - v_y u_x)\hat{\mathbf{k}} \\ &= -\Big((v_z u_y - v_y u_z)\hat{\mathbf{i}} - (v_z u_x - v_x u_z)\hat{\mathbf{j}} + (v_y u_x - v_x u_y)\hat{\mathbf{k}}\Big) \\ &= -\left(\begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}\hat{\mathbf{i}} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}\hat{\mathbf{j}} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\hat{\mathbf{k}}\right) \\ &= -(\mathbf{u}\times\mathbf{v}) \end{aligned}$$

3. We need a point on the line and a vector parallel to the line. The point is the origin $(0,0,0)$ so we need to find the vector. Since the line is parallel to the line of intersection of the two planes, it must be perpendicular to the normal of both planes. Hence $\mathbf{v}$ is perpendicular to both $\langle 1,2,1\rangle$ and $\langle 1,-1,-3\rangle$. Hence

$$\begin{aligned}
\mathbf{v} &= \langle 1,2,1\rangle \times \langle 1,-1,-3\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 1 \\ 1 & -1 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 2 & 1 \\ -1 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 1 \\ 1 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} \hat{\mathbf{k}} \\
&= (-6-(-1))\hat{\mathbf{i}} - (-3-1)\hat{\mathbf{j}} + (-1-2)\hat{\mathbf{k}} \\
&= -5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}
\end{aligned}$$

Hence we have the line through $(0,0,0)$ and parallel to $\langle -5,4,-3\rangle$ yielding an answer of

$$x = -5t, \quad y = 4t, \quad z = -3t$$

4. We are finding the line through two points. The first point is given as $(-1,3,4)$. The second point isn't given however we need to find where the $z$-axis cuts the plane $x+2y-3z=6$. Since the $z$-axis has $x=y=0$, we get the second point has $-3z = 6 \Rightarrow z = -2$ Hence the second point is $(0,0,-2)$.

   Therefore we have two choice for the point and we can find the vector by taking the vector connecting the two points

$$\mathbf{v} = \langle -1-0, 3-0, 4-(-2)\rangle = \langle -1,3,6\rangle.$$

   We can use either point (the point $(0,0,-2)$ is nicer) to get parametric equations $x=-t, \quad y=3t, \quad z-(-2)=6t$. Solving for $t$ in each of these equations and setting them equal to each other yields the symmetric equations

$$-x = \frac{y}{3} = \frac{z+2}{6}.$$

   Note that if we used the point $(-1,3,4)$ instead we would get the equations

$$-(x+1) = \frac{y-3}{3} = \frac{z-4}{6}.$$

5. We need to find a point on both lines. This can be done in many different ways yielding many different points. One qay is to let $z=0$ and then solve for $x$ and $y$. Setting $z=0$ yields

$$x - 2y = 4 \Rightarrow x = 4+2y.$$

Inserting this into the second equation with $z = 0$ yields

$$2(4 + 2y) + y = -2 \Rightarrow 5y + 8 = -2 \Rightarrow y = -2 \Rightarrow x = 0.$$

Hence one point on both planes (and hence the line we are finding) is $(0, -2, 0)$.

Note that setting $y = 0$ would yield the point $\left(-\frac{2}{7}, 0, \frac{10}{7}\right)$ and setting $x = 0$ would yield $(0, -2, 0)$.

Now to find a vector parallel to the line. Since the line is on both planes, it must be perpendicular to both normals (similar to question 1). Hence the vector is

$$\begin{aligned}
\mathbf{v} &= \langle 1, -2, 3\rangle \times \langle 2, 1, -1\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 3 \\ 2 & 1 & -1 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 3 \\ 1 & -1 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & 3 \\ 2 & -1 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} \hat{\mathbf{k}} \\
&= (2 - 3)\hat{\mathbf{i}} - (-1 - 6)\hat{\mathbf{j}} + (1 - (-4))\hat{\mathbf{k}} \\
&= -\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}
\end{aligned}$$

Hence we have the line through $(0, -2, 0)$ and parallel to $\langle -1, 7, 5\rangle$ yielding an answer of

$$x = -t, \quad y = -2 + 7t, \quad z = 5t$$

6. Since we are given a line on the plane, we know the vector $\langle 1, 4, -2\rangle$ is a vector parallel to the plane. We need a second one. Two find a second vector, we can take any point on the given line (for example we could let $t = 0$ to get the point $(3, -2, 1)$) and find the vector to the given point $(1, 3, -2)$. The vector is therefore

$$\langle 1 - 3, 3 - (-2), -2 - 1\rangle = \langle -2, 5, -3\rangle.$$

Therefore the normal vector to the plane must be perpendicular to both $\langle 1, 4, -2\rangle$ and $\langle -2, 5, -3\rangle$ and is therefore

$$
\begin{aligned}
\mathbf{n} &= \langle 1,4,-2\rangle \times \langle -2,5,-3\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 4 & -2 \\ -2 & 5 & -3 \end{vmatrix} \\
&= \begin{vmatrix} 4 & -2 \\ 5 & -3 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & -2 \\ -2 & -3 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 4 \\ -2 & 5 \end{vmatrix} \hat{\mathbf{k}} \\
&= (-12-(-10))\hat{\mathbf{i}} - (-3-4)\hat{\mathbf{j}} + (5-(-8))\hat{\mathbf{k}} \\
&= -2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}}
\end{aligned}
$$

Hence we get the equation of the plane through the point $(1,3,-2)$ and normal to $\langle -2,7,13\rangle$ which is

$$-2(x-1) + 7(y-3) + 13(z-(-2)) = 0$$

or

$$-2x + 7y + 13z = -7.$$

7. We need to find a point on one of the lines and a vector perpendicular to the line. The point is easy, however note that we can't take the cross product of the vectors, since the vectors are parallel.

   The vector parallel to the first line is $\langle 6,-4,8\rangle$ and the vector parallel to the second line is $\langle 3,-2,4\rangle$ which are multiples of each other. (You may think that $v_z = -4$, but be careful. The numerator is $1-z$ not $z-1$)

   Okay, I lied. You can take the cross product, however the cross product will be $\langle 0,0,0\rangle$ and is therefore not useful.

   Hence while we can use one of these vectors, we'll need a different second vector which is not parallel. Let the first vector be $\langle 3,-2,4\rangle$. To find a second vector we can just take any point on one of the lines and connect it to a point on the other. For example on the first line we can get the point $(2,3,1)$ by letting $t=0$. For the second line we can let "$t=0$" to get

$$\frac{x-1}{3} = 0 \Rightarrow x = 1, \frac{y+5/2}{-2} = 0 \Rightarrow y = -\frac{5}{2}, \frac{1-z}{-4} = 0 \Rightarrow z = 1.$$

   Therefore the second vector is

$$\langle 2-1, 3-(-5/2), 1-1\rangle = \langle 1, 11/2, 0$$

   or equivalently

$$\langle 2,11,0\rangle.$$

(Note that we are only looking for parallel vectors, so we can multiply the final vector by a constant if we wish.

Hence the normal vector is

$$\begin{aligned}
\mathbf{n} &= \langle 3, -2, 4\rangle \times \langle 2, 11, 0\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & -2 & 4 \\ 2 & 11 & 0 \end{vmatrix} \\
&= \begin{vmatrix} -2 & 4 \\ 11 & 0 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 3 & 4 \\ 2 & 0 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 3 & -2 \\ 2 & 11 \end{vmatrix} \hat{\mathbf{k}} \\
&= (0-44)\hat{\mathbf{i}} - (0-8)\hat{\mathbf{j}} + (33-(-4))\hat{\mathbf{k}} \\
&= -44\hat{\mathbf{i}} + 8\hat{\mathbf{j}} + 37\hat{\mathbf{k}}
\end{aligned}$$

We can use any point on either line (for example $(2, 3, 1)$) Hence we get the equation of the plane through the point $(2, 3, 1)$ and normal to $\langle -44, 8, 37\rangle$ which is

$$-44(x-2) + 8(y-3) + 37(z-1) = 0$$

or

$$-44x + 8y + 37z = -27.$$

8. These lines have parallel vectors $\langle 2, -1, 3\rangle$ and $\langle 1, -2, 4\rangle$ which a clearly not parallel. However if the lines don't intersect, then there is no guarantee that there will be a plane which passes through both lines. Hence we first need to verify that the lines do in fact intersect. That would mean that there must be values of $s$ and $t$ for which the values of $x, y$ and $z$ on both lines match up.

$$1 + 2t = 1 + s, \quad 2 - t = 5 - 2s, \quad 3 + 3t = -2 + 4s.$$

From the first equation we can see that $s = 2t$. Inserting this into the second equation yields

$$2 - t = 5 - 2(2t) \Rightarrow 2 - t = 5 - 4t \Rightarrow 3t = 3 \Rightarrow t = 1 \Rightarrow s = 2.$$

These values of $s, t$ make $x$ and $y$ equal, however they also must make the $z$ values equal. Checking in the first equation and the second equation both yield $z = 6$. Hence the lines intersect at the point $(3, 1, 6)$.

To find the normal vector we can just find

$$
\begin{aligned}
\mathbf{n} &= \langle 2, -1, 3\rangle \times \langle 1, -2, 4\rangle \\
&= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 3 \\ 1 & -2 & 4 \end{vmatrix} \\
&= \begin{vmatrix} -1 & 3 \\ -2 & 4 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 2 & 3 \\ 1 & 4 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 2 & -1 \\ 1 & -2 \end{vmatrix} \hat{\mathbf{k}} \\
&= (-4 - (-6))\hat{\mathbf{i}} - (8 - 3)\hat{\mathbf{j}} + (-4 - (-1))\hat{\mathbf{k}} \\
&= 2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} - 3\hat{\mathbf{k}}
\end{aligned}
$$

We can use any point on either line (for example $(1, 2, 3)$ letting $t = 0$.) Hence we get the equation of the plane through the point $(1, 2, 3)$ and normal to $\langle 2, -5, -3\rangle$ which is

$$2(x - 1) - 5(y - 2) - 3(z - 3) = 0$$

or

$$2x - 5y - 3z = -17.$$