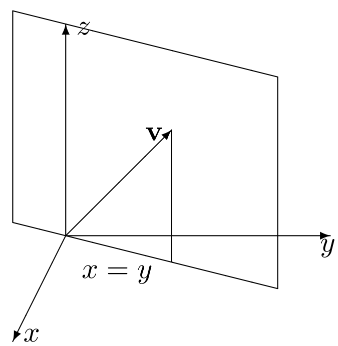
MATH 2130 – Tutorial Problem Solutions, Thu Jan 11

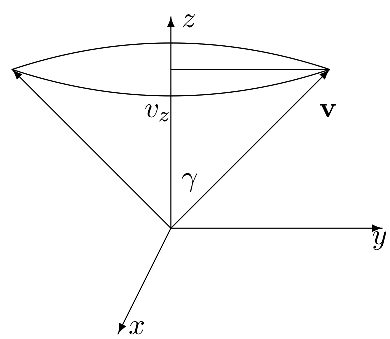
Cartesian components of a vector

Example. Let γ be an angle, $0 < \gamma < \pi$, and let $r > 0$ be a constant. Find the Cartesian components of all vectors that have length r, that make angle γ with the positive z-axis, and that have equal x- and y-components.

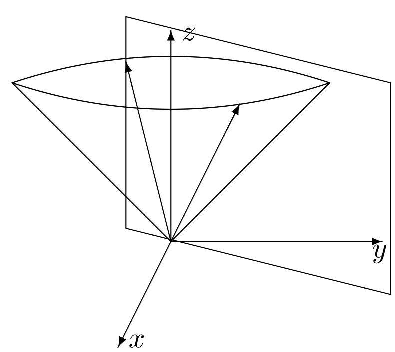
Solution. The vectors with equal x- and y-components lie in the plane $x = y$:



The vectors that make angle γ with the positive z-axis form a cone.



The intersection of the cone and the plane forms two rays. The desired vectors must lie along those rays, and have length r. There appear to be two solutions.



Let the desired vector be $\mathbf{v} = (v_x, v_x, v_z)$, where we note that the x- and y-components are equal. Since $\mathbf{v}$ has length r, we see that

$$2v_x^2 + v_z^2 = r^2.$$

Recall that $\widehat{\mathbf{k}} = (0,0,1)$ is a unit vector in the direction of the positive z-axis. The angle γ satisfies $\mathbf{v} \cdot \widehat{\mathbf{k}} = |\mathbf{v}||\widehat{\mathbf{k}}| \cos\gamma = r\cos\gamma$, where $\mathbf{v} \cdot \widehat{\mathbf{k}} = v_z$. Therefore

$$v_z = r\cos\gamma.$$

Substituting this into the equation obtained from the length of $\mathbf{v}$ yields

$$2v_x^2 + r^2\cos^2\gamma = r^2,$$

which implies that

$$v_x^2 = \frac{r^2}{2}\left(1 - \cos^2\gamma\right) = \frac{r^2}{2}\sin^2\gamma.$$

There are two solutions:

$$\mathbf{v} = \left(\frac{r}{\sqrt{2}}\sin\gamma, \frac{r}{\sqrt{2}}\sin\gamma, r\cos\gamma\right)$$

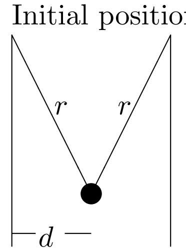
or

$$\mathbf{v} = \left(-\frac{r}{\sqrt{2}}\sin\gamma, -\frac{r}{\sqrt{2}}\sin\gamma, r\cos\gamma\right).$$

Sum of forces

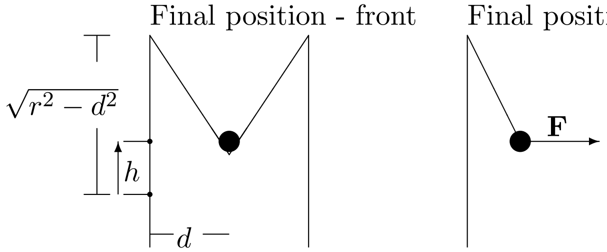
Example. A mass M is suspended from two ropes, each of length r. The ropes are attached to identical vertical poles that are a distance $2d$ apart.

Initial position

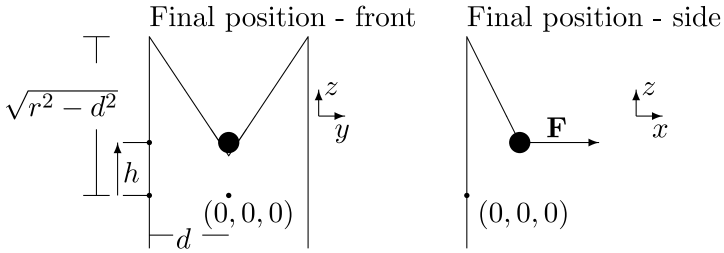


The mass is pulled outward by a horizontal force $\mathbf{F}$ until it has been lifted vertically by h. Find the horizontal force $\mathbf{F}$ required to hold this new position.

Final position - front   Final position - side



Solution. Choose coordinates so that the original position of the mass M is the origin, the force $\mathbf{F}$ is in the positive x-direction, and the tops of the poles are in the positive z-direction.



Before the force was applied, the mass hung a vertical distance $\sqrt{r^2-d^2}$ below the tops of the poles. With respect to our coordinate system, the tops of the poles have z-coordinate $\sqrt{r^2-d^2}$, and they have y-coordinates $\pm d$ and x-coordinate 0. That is, these points are

$$\left(0, \pm d, \sqrt{r^2-d^2}\right).$$

After the force is applied, the mass has z-coordinate h and y-coordinate 0. Let x be its x-coordinate, where $x > 0$. Then this point is

$$(x, 0, h).$$

Let $\mathbf{R}_1, \mathbf{R}_2$ be vectors from the mass in its new position to the tops of the two poles. Then $|\mathbf{R}_1| = |\mathbf{R}_2| = r$, and

$$\mathbf{R}_1 = (-x, -d, \sqrt{r^2-d^2} - h), \quad \mathbf{R}_2 = (-x, d, \sqrt{r^2-d^2} - h).$$

We solve for x:

$$x^2 + d^2 + \left(\sqrt{r^2-d^2} - h\right)^2 = r^2,$$

which implies that

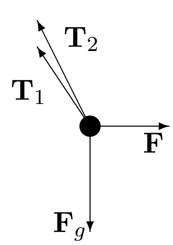
$$x^2 + h^2 - 2h\sqrt{r^2-d^2} = 0,$$

and so

$$x = \sqrt{2h\sqrt{r^2-d^2} - h^2}.$$

Forces on the mass:

- $\mathbf{F} = (F, 0, 0)$ for some $F > 0$,
- gravity, $\mathbf{F}_G = (0, 0, -Mg)$, where g is the acceleration due to gravity,
- the tensions in the ropes, $\mathbf{T}_1 = T_1\widehat{\mathbf{R}}_1$ and $\mathbf{T}_2 = T_2\widehat{\mathbf{R}}_2$.



To hold this configuration steady, we must have

$$\mathbf{F}_g + \mathbf{F} + \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}.$$

Note that

$$\begin{aligned}
\mathbf{T}_1 &= T_1 \widehat{\mathbf{R}}_1 \\
&= \frac{T_1}{r} \mathbf{R}_1 \\
&= \left(-\frac{T_1 x}{r}, -\frac{T_1 d}{r}, \frac{T_1}{r} \left(\sqrt{r^2 - d^2} - h \right) \right), \\
\mathbf{T_2} &= \left(-\frac{T_2 x}{r}, \frac{T_2 d}{r}, \frac{T_2}{r} \left(\sqrt{r^2 - d^2} - h \right) \right).
\end{aligned}$$

Thus

$$\begin{aligned}
\mathbf{0} &= \mathbf{F}_G + \mathbf{F} + \mathbf{T}_1 + \mathbf{T}_2 \\
&= \left(F - \frac{T_1 x}{r} - \frac{T_2 x}{r}, -\frac{T_1 d}{r} + \frac{T_2 d}{r}, \frac{T_1}{r} \left(\sqrt{r^2 - d^2} - h \right) + \frac{T_2}{r} \left(\sqrt{r^2 - d^2} - h \right) - Mg \right).
\end{aligned}$$

Each component must be zero. From the y-component, we find that $T_1 = T_2$. With this substitution in the z-component, we get

$$\frac{2T_1}{r} \left(\sqrt{r^2 - d^2} - h \right) - Mg = 0,$$

which implies that

$$T_1 = \frac{Mgr}{2 \left(\sqrt{r^2 - d^2} - h \right)}.$$

Finally, from the x-component, we get

$$\begin{aligned}
F =& \frac{2T_1 x}{r} \\
=& \frac{Mg}{\left(\sqrt{r^2 - d^2} - h \right)} \sqrt{2h\sqrt{r^2 - d^2} - h^2}.
\end{aligned}$$