PARTIAL FRACTIONS

Given a rational function, i.e., one of the form: $\dfrac{N(x)}{D(x)}$ where $N(x)$ and $D(x)$ are polynomials, first perform a long division (if necessary), so that we may assume that the degree of $N(x)$ is *less than* the degree of $D(x)$. Then write $\dfrac{N(x)}{D(x)}$ as a sum of terms as follows:

1. Factor $D(x)$ into real linear factors [of the form $(ax+b)$] and irreducible real quadratic factors [of the form (ax^2+bx+c) where $b^2-4ac<0$].

   Note: In the terms which are developed below, $A, B, C,$ and the $A_j, B_j,$ and C_j are constants which are to be determined.

2. For a factor $(ax+b)$ in $D(x)$, include a term: $\dfrac{A}{ax+b}$.

   For a repeated factor $(ax+b)^k$ in $D(x)$, include a sum of terms:

$$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}.$$

3. For an irreducible real quadratic factor (ax^2+bx+c) in $D(x)$, include a term: $\dfrac{Bx+C}{ax^2+bx+c}$.

   For a repeated irreducible real quadratic factor $(ax^2+bx+c)^k$ in $D(x)$, include a sum of terms:

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_kx+C_k}{(ax^2+bx+c)^k}.$$

4. Add all the terms obtained above and solve for all the undetermined constants.

5. Method (1): If $D(x)$ is a product of *distinct* linear factors, then use the cover-up rule (see page 3).

   Method (2): Multiply both sides by $D(x)$, and then substitute various values for x (for example, to make the linear factors zero).

   Method (3): Multiply both sides by $D(x)$, expand, and equate the coefficients on the two sides for each of the powers of x and solve for the constants.

PARTIAL FRACTIONS – Examples:

1. $$\frac{5x^2-4x+3}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Multiply by $(x-1)(x^2+1)$: $5x^2-4x+3 = A(x^2+1)+(Bx+C)(x-1)$.

Let $x = 1$ (to make $x - 1 = 0$): $4 = 2A$. Thus, $A = 2$.

$(Bx+C)(x-1) = 5x^2-4x+3-2(x^2+1) = 3x^2-4x+1 = (x-1)(3x-1)$.

$Bx+C = 3x-1$. Thus, $B = 3$, $C = -1$.

Therefore, $\dfrac{5x^2-4x+3}{(x-1)(x^2+1)} = \dfrac{2}{x-1} + \dfrac{3x-1}{x^2+1}$.

2. $$\frac{5x^2-x-2}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}.$$

Multiply by $x^2(x+1)$: $5x^2-x-2 = Ax(x+1)+B(x+1)+Cx^2$.

Let $x = 0$: $-2 = B$; let $x = -1$: $4 = C$;
let $x = 1$ (this is arbitrary - any other choice would be just as good):
$2 = 2A + 2B + C$. Thus, $A = 1 - B - C/2 = 1$.

Therefore, $\dfrac{5x^2-x-2}{x^2(x+1)} = \dfrac{1}{x} - \dfrac{2}{x^2} + \dfrac{4}{x+1}$.

3. $$\frac{2x^3+3x^2-3x+8}{(x-1)^3(x^2+4)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{Dx+E}{x^2+4}.$$

(After much algebra: $A = 1$, $B = 1$, $C = 2$, $D = -1$, $E = 0$).

4. $$\frac{1}{x(x^2+3)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}.$$

(After more algebra: $A = \dfrac{1}{9}$, $B = -\dfrac{1}{9}$, $C = 0$, $D = -\dfrac{1}{3}$, $E = 0$).

COVER-UP RULE

This applies to a ratio of polynomials, $\dfrac{N(x)}{D(x)}$, where $D(x)$ is a product of *distinct* linear factors, and the degree of $N(x)$ is less than that of $D(x)$:

$$\frac{N(x)}{D(x)} = \frac{N(x)}{(a_1 x + b_1)(a_2 x + b_2)\cdots(a_n x + b_n)} = \frac{A_1}{a_1 x + b_1} + \frac{A_2}{a_2 x + b_2} + \cdots + \frac{A_n}{a_n x + b_n}.$$

Multiply by $(a_1 x + b_1)$: $\dfrac{N(x)}{(a_2 x + b_2)\cdots(a_n x + b_n)} = A_1 + \dfrac{A_2(a_1 x + b_1)}{a_2 x + b_2} + \cdots + \dfrac{A_n(a_1 x + b_1)}{a_n x + b_n}$.

Now, let $x = -\dfrac{b_1}{a_1}$, in order to make $(a_1 x + b_1) = 0$.

Then, $A_1 = \dfrac{N(x)}{(a_2 x + b_2)\cdots(a_n x + b_n)}\Bigg|_{\left(x=-\frac{b_1}{a_1}\right)}$.

We see that $A_1 = \dfrac{N(x)}{D(x)}$ evaluated at $x = -\dfrac{b_1}{a_1}$, with the factor $(a_1 x + b_1)$ "covered up". Similarly for $A_2, \ldots, A_n$.

Examples: (In # 1 - 3, the calculation is shown only for A_1; the others are done in a similar manner).

1. $\dfrac{4x-5}{(x-1)(x-2)} = \dfrac{1}{x-1} + \dfrac{3}{x-2}$.

   $A_1 = \dfrac{4x-5}{(x-2)}\Big|_{(x=1)} = 1$, i.e., the factor $(x-1)$ has been "covered up".

2. $\dfrac{x+4}{x^3 - x} = \dfrac{x+4}{x(x-1)(x+1)} = \dfrac{-4}{x} + \dfrac{\frac{5}{2}}{x-1} + \dfrac{\frac{3}{2}}{x+1}$. $\quad A_1 = \dfrac{x+4}{(x-1)(x+1)}\Big|_{(x=0)} = -4$.

3. $\dfrac{x+1}{6x^2 - 5x + 1} = \dfrac{x+1}{(2x-1)(3x-1)} = \dfrac{3}{2x-1} + \dfrac{-4}{3x-1}$. $\quad A_1 = \dfrac{x+1}{3x-1}\Big|_{(x=\frac{1}{2})} = 3$.

4. $\dfrac{1}{x(px+q)} = \dfrac{\left(\frac{1}{q}\right)}{x} - \dfrac{\left(\frac{p}{q}\right)}{px+q}$, provided p and q are constants and $p, q \neq 0$.