UNIVERSITY OF MANITOBA
DEPARTMENT OF MATHEMATICS
MATH 1710 Applied Calculus II
SECOND MIDTERM EXAMINATION
March 13, 2012 5:30 pm

LAST NAME: (Print in ink) ____________________

FIRST NAME: (Print in ink) Solutions

STUDENT NUMBER: (in ink) ____________________

SIGNATURE: (in ink) ____________________
(I understand that cheating is a serious offense)

Please indicate your instructor and section by checking the appropriate box below:

DO NOT WRITE IN THIS TABLE

Total	**/50**
1	/7
2	/10
3	/6
4	/10
5	/10
6	/7
Bonus	/3

☐ A01 MWF (8:30am – 9:20am) O. Maizlish

☐ A02 TR (1pm – 2:15pm) A. Gumel

INSTRUCTIONS TO STUDENTS:

Fill in all the information above. This is a 1 hour exam.

You are permitted to bring one information page (21.6cm. by 28.0 cm. or 8.5 in. by 11 in.) which may contain information on one side only, must be hand-written (not mechanically reproduced) and must bear your name and the student identification number. Information pages not meeting these criteria will be confiscated. ***No*** *other aids, calculators, texts, notes, cellphones, pagers or translators are permitted.*

This exam has a title page, 6 pages of questions and 1 blank page at the end for rough work. Please check that you have all pages.

The value of each question is indicated in the left-hand margin beside the statement of the question. The total value of all questions is 50.

Answer all questions on the exam paper in the space provided. If you need more room, you may continue your work on the ***reverse*** *side of the page, but* ***clearly indicate*** *that your work is continued there.*

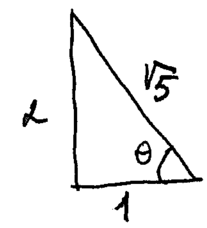
Show all your work clearly and justify your answers (unless it is explicitly stated that you do not have to do that). **Unjustified answers will receive LITTLE or NO CREDIT.**

[7] **1.** Simplify the following expressions

(a) $\tan\left(\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)\right)$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right) \Rightarrow \cos\theta = \frac{1}{\sqrt{5}}$

θ is in the first quadrant

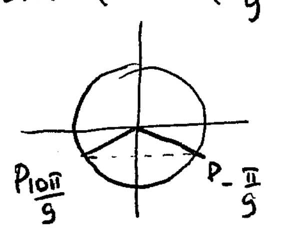


Pythagorean theorem $\Rightarrow$ opposite side $= 2$

$\tan\theta = \frac{2}{1} = 2$ ($\tan\theta$ is positive when θ is in the 1st quadrant)

(b) $\sin^{-1}\left(\sin\frac{10\pi}{9}\right)$

$$\sin^{-1}\left(\sin\left(\frac{10\pi}{9}\right)\right) = \sin^{-1}\left(\sin\left(-\frac{\pi}{9}\right)\right) = -\frac{\pi}{9}$$



since $\sin^{-1}(\sin x) = x$, for $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

[10] **2.** The curve C is given by the equation

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) + y^3 = 1 + \frac{\pi}{3}.$$

(a) Find the point on this curve corresponding to $x = \sqrt{3}$.

Plug $x = \sqrt{3} \Rightarrow$

$$\sin^{-1}\left(\frac{2\sqrt{3}}{1+3}\right) + y^3 = 1 + \frac{\pi}{3}$$

$$\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + y^3 = 1 + \frac{\pi}{3}$$

$$\frac{\pi}{3} + y^3 = 1 + \frac{\pi}{3} \Rightarrow y^3 = 1 \Rightarrow y = 1$$

(b) Find the equation of the tangent line to the curve C at the point corresponding to $x = \sqrt{3}$.

1) Slope : use implicit differentiation to find $\left.\frac{dy}{dx}\right|_{x=\sqrt{3}}$:

$$\frac{d}{dx}\left(\sin^{-1}\left(\frac{2x}{1+x^2}\right) + y^3\right) = 0$$

$$\frac{1}{\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}} \cdot \frac{2(1+x^2) - 2x\cdot(2x)}{(1+x^2)^2} + 3y^2 \cdot \frac{dy}{dx} = 0$$

Plug $x = \sqrt{3}$, $y = 1$: $\frac{1}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2}} \cdot \frac{2\cdot 4 - 2\cdot\sqrt{3}\cdot 2\cdot\sqrt{3}}{4^2} + 3\frac{dy}{dx} = 0$

$$2 \cdot \frac{8-12}{4^2} + 3\frac{dy}{dx} = 0 \Rightarrow -\frac{1}{2} + 3\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{6}$$

2) Equation of the tangent line :

$(y-1) = \frac{1}{6}(x - \sqrt{3})$, or $y = \frac{1}{6}x - \frac{\sqrt{3}}{6} + 1$.

[6] **3.** Find the second moment of the area of the "submarine" cross-section (see Figure 1) about the x-axis. Do **NOT** simplify your answer.

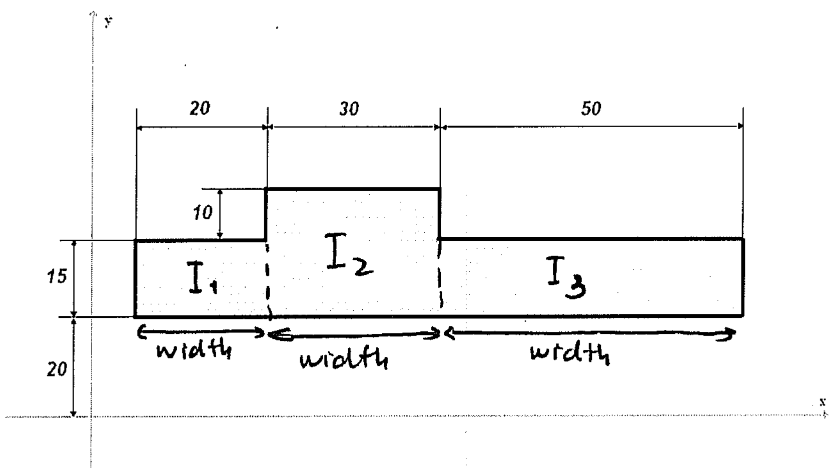


Figure 1

Moment of inertia = moment of inertia of I_1
+ moment of inertia of I_2
+ moment of inertia of I_3

$= \{$ use formula $\frac{1}{3}$ width $\times \left[d_2^3 - d_1^3\right] \}$

$= \frac{1}{3} \cdot 20 \left[35^3 - 20^3\right] + \frac{1}{3} \cdot 30 \cdot \left[45^3 - 20^3\right] + \frac{1}{3} \cdot 50 \left[35^3 - 20^3\right]$

[10] **4.** A dam has the shape of an isosceles trapezoid with height $20m$, width $50m$ at the top and $30m$ at the bottom (see Figure 2 below). **Set up** (but do **NOT** evaluate) a definite integral to find the force on the dam due to water pressure if the water level is $4m$ from the top of the dam.

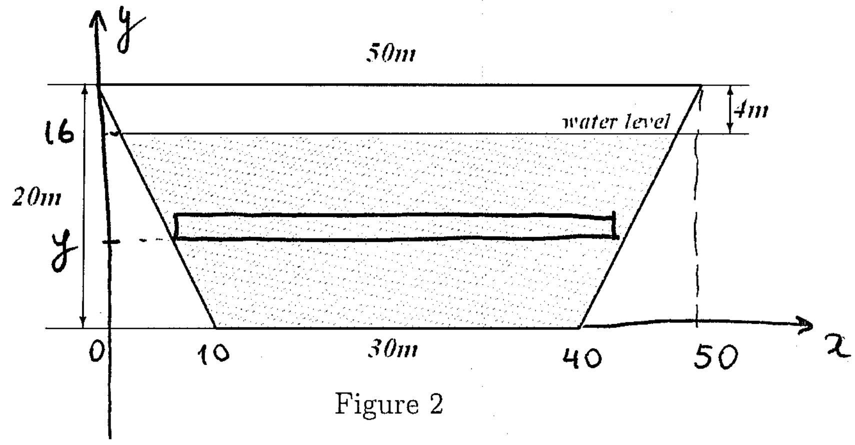


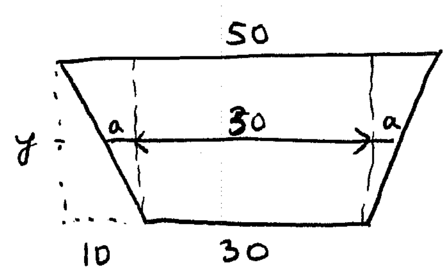
Figure 2

Use horizontal rectangles:

Force acting on a small rectangle:

$\rho \cdot g \cdot h \times \text{Area} = \rho \cdot g \cdot (16 - y) \times \text{length} \times dy$

Find length:



length $= 30 + 2a$

Similar triangles $\Rightarrow$

$\frac{a}{10} = \frac{y}{20} \Rightarrow a = y/2$

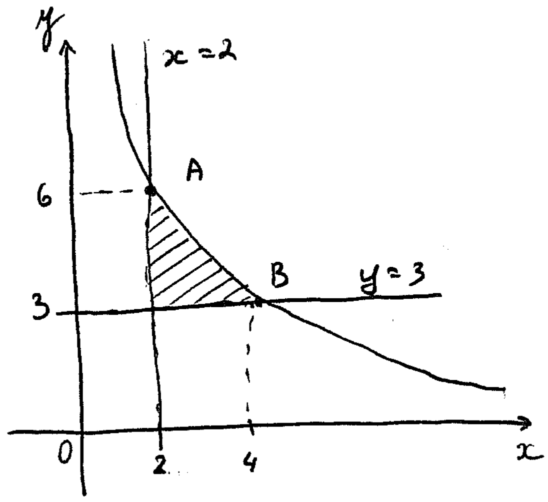
$\Rightarrow \text{length} = 30 + 2 \cdot \frac{y}{2} = 30 + y$

Hence, the total force $=$

$$\int_0^{16} \rho g (16-y)(y+30)\, dy \qquad (N)$$

[10] **5.** Find the centroid of the region in the first quadrant bounded by the curves

$$y = \frac{12}{x},\ x = 2,\ y = 3.$$



Pts of intersection:

A: $\begin{cases} y = \frac{12}{x} \\ x = 2 \end{cases} \Rightarrow \begin{matrix} x_A = 2 \\ y_A = 6 \end{matrix}$

B: $\begin{cases} y = \frac{12}{x} \\ y = 3 \end{cases} \Rightarrow \begin{matrix} x_B = 4 \\ y_B = 3 \end{matrix}$

First moment about the y-axis: (use vertical rectangles)

$$\int_2^4 x \cdot \left(\frac{12}{x} - 3\right) dx = \int_2^4 (12 - 3x)\, dx = \left(12x - \frac{3x^2}{2}\right)\Big|_2^4 =$$

$$= 48 - 24 - 24 + 6 = 6$$

First moment about the x-axis: (use horizontal rectangles)

$x = \frac{12}{y}$,

$$\int_3^6 y\left(\frac{12}{y} - 2\right) dy = \int_3^6 (12 - 2y)\, dy = \left(12y - y^2\right)\Big|_3^6 =$$

$$= 72 - 36 - 36 + 9 = 9$$

Mass (Area) of the region: (use vertical rectangles)

$$\int_2^4 \left(\frac{12}{x} - 3\right) dx = \left[12\ln|x| - 3x\right]\Big|_2^4 = 12\ln 4 - 12 - 12\ln 2 + 6$$

$$= 12\ln 2 - 6$$

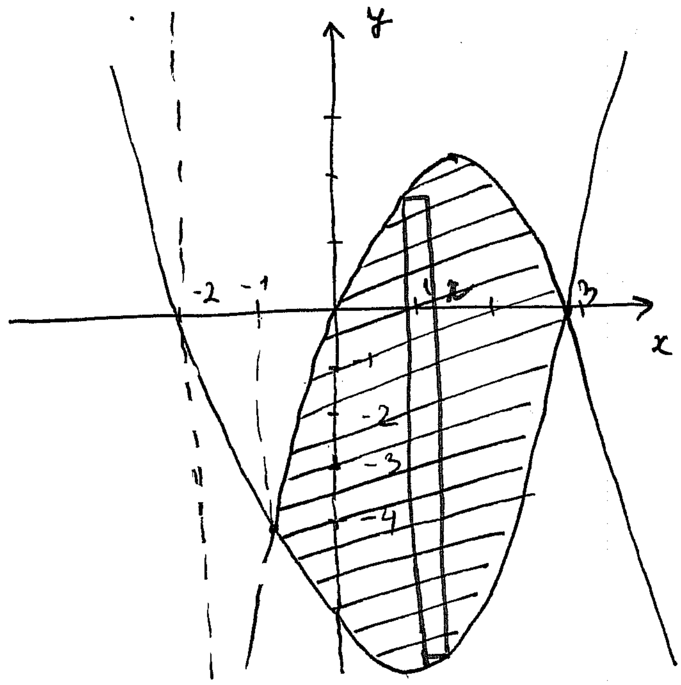
Coordinates of the centroid:

$$\bar{x} = \frac{6}{12\ln 2 - 6} = \frac{1}{2\ln 2 - 1}, \qquad \bar{y} = \frac{9}{12\ln 2 - 6} = \frac{3}{4\ln 2 - 2}$$

[7] **6.** The curves

$$y = x^2 - x - 6, \; y = 3x - x^2$$

define a thin plate with constant mass per area ρ. **Set up** (but do **NOT** evaluate) a definite integral to find its moment of inertia about the line $x = -2$.



Pts of intersection:

$$\begin{cases} y = x^2 - x - 6 \\ y = 3x - x^2 \end{cases}$$

$x^2 - x - 6 = 3x - x^2$

$2x^2 - 4x - 6 = 0$

$x^2 - 2x - 3 = 0$

$x = -1, 3$

Use vertical rectangles.

Moment of inertia $= \displaystyle\int_{-1}^{3} \underbrace{\rho\left[(3x - x^2) - (x^2 - x - 6)\right]}_{\text{mass}} \underbrace{(x+2)}_{\text{distance}}{}^2 \, dx$

mass $= \rho\left[(3x - x^2) - (x^2 - x - 6)\right]$

distance $= (x + 2)$

$= \displaystyle\int_{-1}^{3} \rho\left[-2x^2 + 4x + 6\right](x + 2)^2 \, dx$

[3] **7. Bonus:** Evaluate $\cos^{-1}\left(\sin\left(\frac{\pi}{5}\right)\right)$.

$$\cos^{-1}x + \sin^{-1}x = \frac{\pi}{2} \Rightarrow \cos^{-1}\left(\sin\frac{\pi}{5}\right) = \frac{\pi}{2} - \sin^{-1}\left(\sin\frac{\pi}{5}\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{5} = \frac{3\pi}{10}$$

Blank Page