Summer 2012 Exam

1. Find the distance between the lines

$$x = t, y = 3t + 1, z = 1 - 2t \qquad x = 2u + 1, y = 1 + u, z = 4 - 2u.$$

2. Let $f(x, y, z) = -\cos(\pi yz) + y\ln(x^2 + z^2)$, and the curve C be the intersection of

$$y^2 + z^2 = 4 \qquad \text{and} \qquad y = -x.$$

   (a) Find a parametrization of C such that z is increasing when x is positive.
   (b) Find the tangent line to the curve C at the point $(\sqrt{2}, -\sqrt{2}, \sqrt{2})$.
   (c) SET UP BUT DO NOT EVALUATE a definite integral to determine the length of the part of the curve C from $(\sqrt{2}, -\sqrt{2}, \sqrt{2})$ to $(0, 0, 2)$.
   (d) Find the rate of change of f with respect to length, along the curve C directed so that x increases at the point $(\sqrt{2}, -\sqrt{2}, \sqrt{2})$.

3. (a) Find a chain rule for $\left.\frac{\partial u}{\partial r}\right)_t$ if $u = f(x, y, s), x = g(t), y = h(s), s = k(r, t)$.
   (b) Use your chain rule in part (a) to find $\left.\frac{\partial u}{\partial r}\right)_t$ if

$$u = \sqrt{x^2 + ys}, x = \frac{t}{e^{t^2}}, y = \tan(s^2 + 1), s = \ln(rt).$$

4. For the function $f(x, y) = xy(5 - y - x^2)$.
   (a) Find the critical points of f.
   (b) Classify the critical points $(1, 2)$ and $(0, 0)$.
   (c) Find the absolute maximum and minimum of f on the region in the first quadrant bounded by $x = 0, y = 0$ and $y = 5 - x^2$.

5. Evaluate the double iterated integral

$$\int_0^1 \int_y^1 6y^2 \cos(x^2)dxdy$$

6. Let R be the region outside the circle $x^2 + y^2 = 1$ and inside the circle $x^2 + y^2 = 2x$. SET UP BUT DO NOT INTEGRATE double iterated integrals in polar coordinates to determine the volume of R rotated about the line $x = -1$.

7. Find the surface area of the paraboloid $z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 1$.

8. For the region bounded by the curves

$$z = (x - 1)^2 + 4(y - 1)^2 - 5 \qquad \text{and } 2x + 8y + z = 4$$

   with density equal to the distance to the origin:
   (a) SET UP BUT DO NOT EVALUATE triple iterated integrals in cartesian coordinates to determine $\overline{x}$, the x-coordinate of the center of mass.
   (b) SET UP BUT DO NOT EVALUATE triple iterated integrals in cartesian coordinates to determine the moment of inertia about the x-axis.

9. SET UP BUT DO NOT EVALUATE a triple iterated integral to determine the volume of the solid bounded by $z = 1$ and $z = \sqrt{4 - x^2 - y^2}$ using:
   (a) cylindrical coordinates.
   (b) spherical coordinates.

10. Use spherical coordinates to evaluate the integral

$$\iiint_V \sqrt{x^2 + y^2}\, dV$$

   where V is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

Answers

1. $\dfrac{19}{\sqrt{45}}$

2. (a) (One possible answer is) $x = -2\sin t, \quad y = 2\sin t, \quad z = 2\cos t$

   (b) $\langle -\sqrt{2}s + \sqrt{2}, \sqrt{2}s - \sqrt{2}, \sqrt{2}s + \sqrt{2}\rangle$

   (c) $\displaystyle\int_{-\pi/4}^{0} \sqrt{8\cos^2 t + 4\sin^2 t}\, dt$

   (d) $-\dfrac{\ln 4}{\sqrt{3}}$

3. (a) $\left.\dfrac{\partial u}{\partial r}\right)_t = \left.\dfrac{\partial u}{\partial y}\right)_{x,s} \left.\dfrac{dy}{ds}\dfrac{\partial s}{\partial r}\right)_t + \left.\dfrac{\partial u}{\partial s}\right)_{x,y} \left.\dfrac{\partial s}{\partial r}\right)_t$

   (b) $\left.\dfrac{\partial u}{\partial r}\right)_t = \left(\dfrac{s}{2\sqrt{x^2+ys}}\right)\left(2s\sec^2(s^2+1)\right)\left(\dfrac{1}{r}\right) + \left(\dfrac{y}{2\sqrt{x^2+ys}}\right)\left(\dfrac{1}{r}\right)$

4. (a) $(0,0), (0,5), (\pm\sqrt{5}, 0), (\pm 1, 2)$

   (b) $(0,0)$ is a saddle point and $(1,2)$ is a relative maximum.

   (c) The maximum is 4 at $(1,2)$ and the minimum is 0 anywhere along the boundry.

5. $\sin 1 + \cos 1 - 1$.

6. $\displaystyle\int_{-\pi/3}^{\pi/3}\int_{1}^{2\cos\theta} 2\pi(r\cos\theta + 1) r\, dr\, d\theta$

7. $\dfrac{\pi}{6}(5^{3/2} - 1)$

8. (a) $M = \displaystyle\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{(x-1)^2+4(y-1)^2-5}^{4-2x-8y} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

   $M_{yz} = \displaystyle\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{(x-1)^2+4(y-1)^2-5}^{4-2x-8y} x\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

   $\overline{x} = \dfrac{M_{yz}}{M}$

   (b) $I_x = \displaystyle\int_{-1}^{1}\int_{-\sqrt{4-4y^2}}^{\sqrt{4-4y^2}}\int_{(x-1)^2+4(y-1)^2-5}^{4-2x-8y} (y^2+z^2)\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

9. SET UP BUT DO NOT EVALUATE a triple iterated integral to determine the volume of the solid bounded by $z = 1$ and $z = \sqrt{4 - x^2 - y^2}$ using:

   (a) $\displaystyle\int_{0}^{2\pi}\int_{0}^{\sqrt{3}}\int_{1}^{\sqrt{4-r^2}} r\, dz\, dr\, d\theta$

   (b) $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/3}\int_{\sec\phi}^{2} \mathbb{R}^2 \sin\phi\, d\mathbb{R}\, d\phi\, d\theta$

10. $\dfrac{\pi^2}{16} - \dfrac{\pi}{8}$