Extrema Problems

1. Divide 20 into two parts (not necessarily integers) such that the product of one part with the square of the other shall be a ~~minimum.~~ maximum

2. Find the number which most exceeds its square.

3. An open rectangular box is to be made from a piece of cardboard 8 in. wide and 15 in. long by cutting a square from each corner and bending up the sides. Find the dimensions of the box of largest volume.

4. A poster is to contain 50 in^2 of printed matter with margins of 4 in. each at top and bottom and 2 in. at each side. Find the overall dimensions if the total area of the poster is to be a minimum.

5. An oil can is to be made in the form of a right circular cylinder to contain 16π in^3. What dimensions of the can require the least amount of material?

6. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the curve $y = 12 - x^2$.

7. A box with square base and open top is to hold 32 in^3. Find the dimensions that require the least amount of material.

8. Find the height and radius of the right circular cylinder of maximum volume which can be inscribed in a sphere of radius $\sqrt{3}$ ft.

9. A right circular cone has altitude 12 ft and radius of base 6 ft. A cone is inscribed with its vertex at the center of the base of the given cone and its base parallel to the base of the given cone. Find the dimensions of the cone of maximum volume that can be so inscribed.

10. An aquarium is to be made 6 ft high and is to have a volume of 750 cu. ft. The base, ends, and back are to be made of slate, but the front is to be made of plate glass, which costs 3/2 times as much as slate per sq. ft. What dimensions should be chosen to make the cost of raw materials a minimum?

11. A playing field is to be built in the shape of a rectangle plus a semicircular area at each end. A 440 yd race track is to form the perimeter of the field. Find the dimensions of the field if the rectangular part is to have as large an area as possible.

12. The strength of a rectangular beam is proportional to the product of its width and the square of its depth. Find the dimensions of the strongest beam that can be cut from a cylindrical log of radius r.

13. A motorist is stranded in a desert 5 mi from a point A, which is the point on a long straight road nearest to him. He wishes to get to a point B on the road. If he can travel at 15 mi/hr on the desert and 39 mi/hr on the road, find the point at which he must meet the road to get to B in the shortest possible time if

    (a) B is 5 mi from A,
    (b) B is 10 mi from A,
    (c) B is 1 mi from A.

14. At midnight, ship B was 90 mi due south of ship A. Ship A sailed east at 15 mi/hr and ship B sailed north at 20 mi/hr. At what time were they closest together?

15. A farmer wishes to construct 9 pig pens by fencing a rectangular region and then subdividing the region by 8 fences parallel to one of the sides. If the farmer has 400 m of fencing, what dimensions of the region will give the largest total area?

16. A power company wishes to lay a cable from point A on one side of a river (100 m wide) to point B on the other side which is 500 m downstream from A. If underwater cable costs \$10 per meter and land cable costs \$5 per meter, how should the cable be laid in order to minimize the cost?

17. A real estate company owns 200 apartments which are fully occupied when the rent is \$300 per month. The company estimates that for each \$2 increase in rent, one less apartment will be rented. Each occupied apartment requires \$20 of maintenance per month. What rent should be charged to obtain the largest net income?

SOLUTIONS.

(1.) $x \mid 20-x$

$P = (20-x)x^2$

$P = 20x^2 - x^3$

$\frac{dP}{dx} = 40x - 3x^2$

$0 = 40x - 3x^2 = x(40-3x)$

C.P.'s: $x=0$ $x = 40/3$

Domain: $[0, 20]$

x	0	20	$40/3$
P	0	0	$32000/27$

$\therefore$ Max product if the squared part is $40/3$ and the other is $20/3$.

(2.) $E = x - x^2$ $\frac{dE}{dx} = 1 - 2x$

$0 = 1 - 2x$

C.P: $x = 1/2$

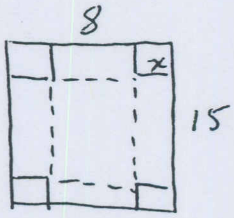
Domain: $\mathbb{R}$.

Since only one C.P.

x		$1/2$	
E'	+		−
		MAX.	

When $x = 1/2$, it exceeds its square by the mo

(3.)



$V = lwh$

$V = (15-2x)(8-2x)x$

$V = (15-2x)(8x-2x^2)$

$\frac{dV}{dx} = -2(8x-2x^2) + (15-2x)(8-4x)$

$0 = 12x^2 - 92x + 120$

$0 = 4(3x-5)(x-6)$

$x = 5/3$ $\cancel{x=6}$

Domain: $[0, 4]$

x	0	4	$5/3$
V	0	0	+ve

$\therefore$ Max. volume when $x = 5/3$

dimensions are $\frac{5}{3}$ in $\times \frac{14}{3}$ in $\times \frac{5}{3}$

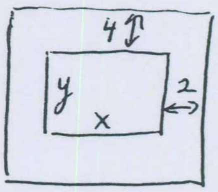
$\boxed{\text{or}}$

in $[0, 4]$ only one C.P. $x = 5/3$

x		$5/3$	
$\frac{dV}{dx}$	+		−

$\therefore$ MAX (state dimensions as above.)

(4.)



Area $= lw$

$= (x+4)(y+8)$

But $xy = 50$

$\therefore y = \frac{50}{x}$

Domain: $(0, \infty)$

$A = (x+4)\left(\frac{50}{x} + 8\right)$

$\frac{dA}{dx} = 1\left(\frac{50}{x} + 8\right) + (x+4)\left(\frac{-50}{x^2}\right)$

$= \frac{50}{x} + 8 - \frac{50}{x} - \frac{200}{x^2}$

$0 = 8 - \frac{200}{x^2}$

$0 = \frac{8x^2 - 200}{x^2} = \frac{8(x-5)(x+5)}{x^2}$

$x = 5$ $\cancel{x = -5}$

only one C.P. in $(0, \infty)$

x		5	
$\frac{dA}{dx}$	−		+
		MIN.	

$\therefore$ Minimum area occurs when dimensions are:

$x = 5$ $y = \frac{50}{5} = 10$

dimensions are: 9 in $\times$ 18 in.

(5.) 

surface area $= 2\pi r^2 + (2\pi r)h$

But $V = \pi r^2 h$ $\quad \therefore\ A = 2\pi r^2 + 2\pi r\left(\frac{16}{r^2}\right) = 2\pi r^2 + \frac{32\pi}{r}$

$16\pi = \pi r^2 h$

$\frac{16}{r^2} = h$

Domain: $(0, \infty)$

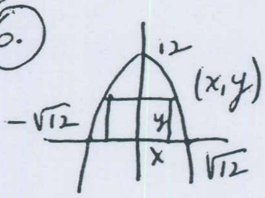
$\frac{dA}{dr} = 4\pi r - \frac{32\pi}{r^2}$

$0 = \frac{4\pi r^3 - 32\pi}{r^2} = \frac{4\pi(r^3 - 8)}{r^2} = \frac{4\pi(r-2)(r^2+2r+}{r^2}$

only c.P. is $r = 2$

r		2	
$\frac{dA}{dr}$	$-$		$+$
		MIN.	

$\therefore$ Least material is used if $r = 2$ in. and $h = 4$ in.

(6.) 

Domain: $[0, 2\sqrt{3}]$

Area $= lw$

$= 2xy$

$= 2x(12 - x^2)$

$= 24x - 2x^3$

$\frac{dA}{dx} = 24 - 6x^2$

$0 = 6(4 - x^2) = 6(2-x)(2+x)$

$x = 2 \quad \cancel{x = -2}$

x	0	$2\sqrt{3}$	2
A	0	0	32

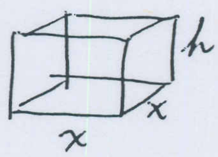
Max. area $= 32$.

$\boxed{\text{or}}$

only one c.P. in $[0, 2\sqrt{3}]$

x		2	
$\frac{dA}{dx}$	$+$	MAX	$-$

(as above.)

(7.) 

Surface area $= x^2 + 4xh$

But $V = 32\ in^3$

$\therefore\ 32 = x^2 h$

$\frac{32}{x^2} = h$

Domain: $(0, \infty)$

$\therefore\ A = x^2 + 4x\left(\frac{32}{x^2}\right) = x^2 + \frac{128}{x}$

$\frac{dA}{dx} = 2x - \frac{128}{x^2}$

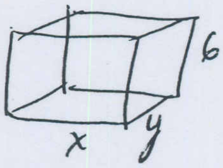
$= \frac{2x^3 - 128}{x^2} = \frac{2(x^3 - 64)}{x^2}$

$0 = \frac{2(x-4)(x^2+4x+16)}{x^2}$

only c.P $\cancel{is}$ is: $x = 4$

x		4	
$\frac{dA}{dx}$	$-$		$+$
		MIN.	

$\therefore$ Minimum surface area occurs when base is 4in X 4in and height $= \frac{32}{4^2} = \frac{32}{16} = 2$ in.

(10.) 

Cost $=$ area x price

$= 2(6y) + 6x + \frac{3}{2}(6x) + xy$ (front)

But $V = 750$

$750 = xy6$

$\frac{125}{x} = y$

Domain: $(0, \infty)$

$\therefore\ C = 12\left(\frac{125}{x}\right) + 6x + 9x + x\left(\frac{125}{x}\right)$

$C = \frac{1500}{x} + 6x + 9x + 125$

$\frac{dC}{dx} = -\frac{1500}{x^2} + 6 + 9$

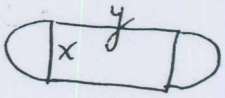
$0 = \frac{-1500 + 15x^2}{x^2} = \frac{15(x^2 - 100)}{x^2} = \frac{15(x-10)(x+10)}{x^2}$

only c.P. in domain is: $x = 10 \quad \cancel{x = -10}$

x		10	
$\frac{dC}{dx}$	$-$		$+$
		MIN.	

$\therefore x = 10 \quad y = \frac{125}{10} = 12.5$ Min. material used if base is 10ft X 12.5ft and height $= 6$ ft.

(11.)



Area = lw
$A = xy$

But Perimeter = 440 yd.

Domain $\left[0, \frac{440}{\pi}\right]$

$2y + 2\pi\left(\frac{x}{2}\right) = 440$

$2y + x\pi = 440$

$\therefore y = 220 - \frac{\pi x}{2}$

$\therefore A = x\left(220 - \frac{\pi x}{2}\right)$

$A = 220x - \frac{\pi x^2}{2}$

$\frac{dA}{dx} = 220 - \pi x$

$0 = 220 - \pi x$

$x = \frac{220}{\pi}$

Since only one C.P.

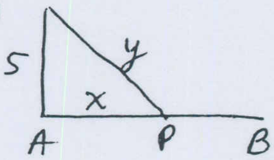
$$\begin{array}{c|c|c} x & \frac{220}{\pi} & \\ \hline \frac{dA}{dx} & + & - \end{array}$$

$\therefore x = \frac{220}{\pi}$ $\quad y = 220 - \frac{\pi\left(\frac{220}{\pi}\right)}{2}$

$y = 220 - 110 = 110$

$\therefore$ Dimension of Max. rectangle area are $\frac{220}{\pi}$ yd $\times$ 110 yd.

(13.)



(a) If AB = 5 miles.

$T = \frac{y}{15} + \frac{5-x}{39}$ But $y^2 = x^2 + 5^2$

$\therefore y = \sqrt{x^2+25}$ (+ve since y is a distance.)

$\therefore T = \frac{\sqrt{x^2+25}}{15} + \frac{5}{39} - \frac{x}{39}$

Domain: $[0,5]$

$\frac{dT}{dx} = \frac{\frac{1}{2}(x^2+25)^{-1/2}(2x)}{15} - \frac{1}{39} = \frac{x}{15\sqrt{x^2+25}} - \frac{1}{39} = \frac{39x - 15\sqrt{x^2+25}}{15(39)\sqrt{x^2+25}}$

$0 = \frac{39x - 15\sqrt{x^2+25}}{15(39)\sqrt{x^2+25}}$ $\quad \therefore 0 = 39x - 15\sqrt{x^2+25}$

$15\sqrt{x^2+25} = 39x$

$225(x^2+25) = 1521x^2$

$225(25) = 1296x^2$

$x^2 = \frac{225(25)}{1296}$

$x = \pm\frac{15(5)}{36} = \pm\frac{25}{12}$

But x is a length so $x = \frac{25}{12}$

only one C.P. in $[0,5]$

$$\begin{array}{c|c|c} x & \frac{25}{12} & \\ \hline \frac{dT}{dx} & - & + \\ & \text{MIN.} & \end{array}$$

Shortest time if P is $\frac{25}{12}$ miles from A towards

(b) Same as above except.

Domain: $[0,10]$

and $T = \frac{y}{15} + \frac{10-x}{39} = \frac{\sqrt{x^2+25}}{15} + \frac{10}{39} - \frac{x}{39}$

$\therefore \frac{dT}{dx} = \frac{x}{15\sqrt{x^2+25}} - \frac{1}{39}$ $\quad \therefore$ solution is again $\frac{25}{12}$ miles from A

(c) Same as above except

Domain: $[0,1]$

and $T = \frac{y}{15} + \frac{1-x}{39} = \frac{\sqrt{x^2+25}}{15} + \frac{1}{39} - \frac{x}{39}$

$\therefore \frac{dT}{dx} = \frac{x}{15\sqrt{x^2+25}} - \frac{1}{39}$ and $x = 5$ is not in domain.

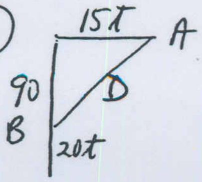
$\therefore$ check the endpoints

$x = 0 \quad T = \frac{\sqrt{25}}{15} + \frac{1}{39} = \frac{5}{15} + \frac{1}{39} = \frac{14}{39}$

$x = 1 \quad T = \frac{\sqrt{26}}{15} + \frac{1}{39} - \frac{1}{39} = \frac{\sqrt{26}}{15}$

But $\frac{\sqrt{26}}{15} < \frac{14}{39}$ $\therefore$ Min occurs when P is 1 mile from A towards B.

(14.)



$D^2 = (15t)^2 + (90-20t)^2 = 625t^2 - 3600t + 8100$

$2D\frac{dD}{dt} = 1350t - 3600$

$0 = 450(3t - 8)$

$t = 8/3$

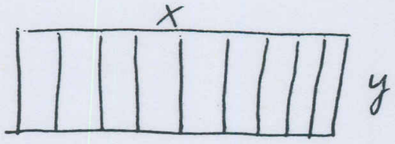
Domain: $[0, \infty)$

Since only on C.P.

$$\begin{array}{c|c|c} t & 8/3 & \\ \hline \frac{dD}{dt} & - & + \\ & \text{min.} & \end{array}$$

The ships are closest at 8/3 hrs after midnight or 2:40 A.M.

(15.)



Area = lw

$A = xy$

$A = (200-5y)y$

$= 200y - 5y^2$

$\frac{dA}{dy} = 200 - 10y$

$0 = 10(20-y)$

$y = 20$

But $400 = 2x + 10y$

$200 = x + 5y.$

$x = 200 - 5y.$

Domain:

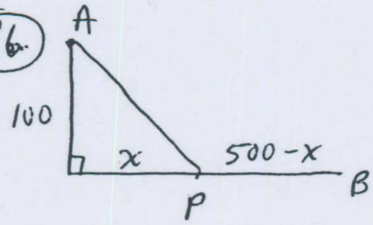
$(0, 40)$

Since only one C.P. $y = 20$

y		20	
$\frac{dA}{dy}$	+		−
		MAX	

Dimensions of Max area are: 20m × 100m

(16.)



Domain: $[0, 500]$

Cost = length × price

$C = 10(AP) + 5(PB)$

$(AP)^2 = 100^2 + x^2$

$AP = \sqrt{10000 + x^2}$

$C = 10\sqrt{10000+x^2} + 5(500-x)$

$\frac{dC}{dx} = 10\left[\frac{1}{2}(10000+x^2)^{-1/2}(2x)\right] - 5$

$= \frac{10x}{\sqrt{10000+x^2}} - 5$

$0 = \frac{10x - 5\sqrt{10000+x^2}}{\sqrt{10000+x^2}}$

$\therefore 0 = 10x - 5\sqrt{10000+x^2}$

$5\sqrt{10000+x^2} = 10x$

$25(10000+x^2) = 100x^2$

$25(10000) = 75x^2$

$\frac{10000}{3} = x^2$

$x = \frac{100}{\sqrt{3}}$ (+ve since x is a distance)

Since only one C.P. in $[0, 500]$

x		$100/\sqrt{3}$	
$\frac{dC}{dx}$	−		+
		Min	

Min. cost occurs when the cable hits the shore $\left(500 - \frac{100}{\sqrt{3}}\right)$ m before B.

(17.) Income = (number of apartments rented) × rent/apt − maintenance cost

Let x = number of \$2 increases in rent then.

Domain:

$(0, 200)$

$I = (200-x)(300+2x) - 20(200-x) = -2x^2 + 120x + 56000$

$\frac{dI}{dx} = -4x + 120$

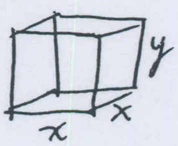
$0 = -4(x-30)$

Since only once C.P.

x		30	
$\frac{dI}{dx}$	+		−

Max. Income occurs when the rent is $300 + 2(30) =$ \$360.

(18.)



Volume $= x^2 y$. But max. "dimensions" are: $96 = 4x + y$

$\therefore y = 96 - 4x$

$V = x^2(96-4x)$

$V = 96x^2 - 4x^3$

$\frac{dV}{dx} = 192x - 12x^2$

$0 = 12x(16-x)$

$x = 0 \quad x = 16$

Domain:

$[0, \infty)$

x	0	$x \to \infty$	16
V	0	$-\infty$	$96(16)^2 - 4(16)^3 = 16^2(96-64) > 0$

$\therefore$ Dimensions of Max. volume are:

16 in × 16 in × $[96 - 4(16)]$ in

= 16 in × 16 in × 32 in.