MATH 1210 Tutorial # 5

Oct. 13 - 19, 2011

1. Given the vectors $\vec{u} = a\hat{i} - 2\hat{j} + \hat{k}$ and $\vec{v} = a\hat{i} + a\hat{j} - 3\hat{k}$. Determine for which values of a these two vectors are
   (a) parallel.
   (b) perpendicular.

2. Consider the vectors $\vec{u} = \hat{i} + \hat{j}$ and $\vec{v} = \hat{j} + \hat{k}$. Find
   (a) $\vec{u} \cdot \vec{v}$.
   (b) $\vec{u} \times \vec{v}$.
   (c) The angle between $\vec{u}$ and $\vec{v}$.

3. Show that the triangle in $\mathbb{E}^3$ with vertices $P_1(2,2,3), P_2(1,4,4), P_3(5,4,2)$ is a right triangle and find the length of its hypotenuse.

4. Show that the triangle with vertices $(1,1,0), (0,1,1), (1,0,1)$ is equilateral and find the coordinates of its center.
Hint: The distance of the center C of an equilateral triangle from a vertex V of the triangle equals $\frac{2}{3}|VM|$, where M is the midpoint of the side opposite the vertex V.

5. Show that
$$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$
for any 3 vectors $\vec{u}, \vec{v}, \vec{w} \in \mathbb{E}^3$.
Hint: First verify this for the special cases $\vec{u} = \hat{i}, \vec{u} = \hat{j}$, and $\vec{u} = \hat{k}$. Then use the representation $\vec{u} = u_1\hat{i} + u_2\hat{j} + u_3\hat{k}$.

1. (a) For $\vec{u}$ and $\vec{v}$ to be parallel, we must have that $\vec{u} = \lambda \vec{v}$ for some real number $\lambda \neq 0$. This leads to the equations $a = \lambda a$, $-2 = \lambda a$, $1 = \lambda(-3) = -3\lambda$. It follows from the first equation that $a = 0$, and hence that $-2 = 0$ by the second equation. Since this is impossible, $\vec{u}$ and $\vec{v}$ are never parallel.

(b) For $\vec{u} \perp \vec{v}$ we must have tha $\vec{u} \cdot \vec{v} = 0$, i.e. $(a)(a) + (-2)(a) + (1)(-3) = a^2 - 2a - 3 = 0$, from which it follows that $a = -1$ or 3.

2. (a) $\vec{u} \cdot \vec{v} = (\hat{\imath} + \hat{\jmath}) \cdot (\hat{\jmath} + \hat{k}) = (1)(0) + (1)(1) + (0)(1) = 1$

(b) $\vec{u} \times \vec{v} = (\hat{\imath} + \hat{\jmath}) \times (\hat{\jmath} + \hat{k}) = (\hat{\imath} \times \hat{\jmath}) + (\hat{\jmath} \times \hat{\jmath}) + (\hat{\imath} \times \hat{k}) + (\hat{\jmath} \times \hat{k})$
$= \hat{k} + \vec{0} + \vec{j} + \hat{\imath} = \hat{\imath} + \hat{\jmath} + \hat{k}$.

(c) $\cos(\measuredangle(\vec{u}, \vec{v})) = \vec{u} \cdot \vec{v} / (\|\vec{u}\| \|\vec{v}\|) = 1/(\sqrt{2} \cdot \sqrt{2}) = 1/2$, so $\theta = \measuredangle(\vec{u}, \vec{v}) = \pi/3$ or $\theta = 5\pi/3$. Which one? We also have that

$$\sin\theta = \|\vec{u} \times \vec{v}\| / (\|\vec{u}\| \|\vec{v}\|) = \sqrt{3}/2.$$

Since $\sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$, it follows that $\theta = \frac{\pi}{3}$.

3. Since $\overrightarrow{P_1P_2} \cdot \overrightarrow{P_1P_3} = (-1, 2, 1) \cdot (3, 2, -1) = (-1)(3) + (2)(2) + (1)(-1) = 0$, the vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ are perpendicular, so the triangle $\triangle P_1P_2P_3$ is a right triangle with the right angle at P_1 and its hypotenuse the line segment

P_2P_3, which has length

$$\|P_2P_3\| = \|4\hat{\imath} + 0\hat{\jmath} + (-2)\vec{k}\| = \sqrt{4^2+0^2+(-2)^2} = 2\sqrt{5}.$$

4. Setting $P_1 = (1,1,0)$, $P_2 = (0,1,1)$, $P_3 = (1,0,1)$ we get

$$\|\overrightarrow{P_1P_2}\| = \|(-1)\hat{\imath} + 0\hat{\jmath} + 1\cdot\vec{k}\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$
$$\|\overrightarrow{P_1P_3}\| = \|0\cdot\hat{\imath} + (-1)\hat{\jmath} + 1\vec{k}\| = \sqrt{(-1)^2+1^2} = \sqrt{2}$$
$$\|\overrightarrow{P_2P_3}\| = \|1\hat{\imath} + (-1)\hat{\jmath} + 0\vec{k}\| = \sqrt{1^2+(-1)^2} = \sqrt{2},$$

which shows that $\triangle P_1P_2P_3$ is equilateral.
Let M denote the midpoint of the line segment $\overline{P_2P_3}$. Then

$$\overrightarrow{OM} = \overrightarrow{OP_2} + \tfrac{1}{2}\overrightarrow{P_2P_3}$$
$$= [0,1,1] + \tfrac{1}{2}[1,-1,0] = [\tfrac{1}{2},\tfrac{1}{2},1]$$

so $M = (1/2, 1/2, 1)$. If C denotes the center of the triangle, then

$$\overrightarrow{OC} = \overrightarrow{OP_1} + \tfrac{2}{3}\overrightarrow{P_1M} = [1,1,0] + \tfrac{2}{3}[-\tfrac{1}{2},-\tfrac{1}{2},1]$$
$$= [\tfrac{2}{3},\tfrac{2}{3},\tfrac{2}{3}]$$

Therefore $C = (2/3, 2/3, 2/3)$.

5. Let $\vec{u} = \hat{\imath}$, $\vec{v} = v_1\hat{\imath} + v_2\hat{\jmath} + v_3\vec{k}$, $\vec{w} = w_1\hat{\imath} + w_2\hat{\jmath} + w_3\vec{k}$.
Then

$$\vec{u}\times(\vec{v}\times\vec{w}) = \hat{\imath}\times\left((v_2w_3 - v_3w_2)\hat{\imath} + (v_3w_1 - v_1w_3)\hat{\jmath} + (v_1w_2 - v_2w_1)\hat{k}\right)$$
$$= (v_2w_3 - v_3w_2)(\hat{\imath}\times\hat{\imath}) + (v_3w_1 - v_1w_3)(\hat{\imath}\times\hat{\jmath})$$
$$+ (v_1w_2 - v_2w_1)(\hat{\imath}\times\vec{k})$$

$$= (v_3 v_1 - v_1 w_3)\hat{k} + (v_2 w_1 - v_1 w_2)\hat{j},$$

where we have used $\hat{\imath} \times \hat{\imath} = \vec{0}$, $\hat{\imath} \times \hat{k} = -\hat{j}$.

On the other hand,

$$(\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w} = (\hat{\imath}\cdot\vec{w})\vec{v} - (\hat{\imath}\cdot v)\vec{w}$$

$$= w_1(v_1\hat{\imath} + v_2\hat{j} + v_3\hat{k}) - v_1(w_1\hat{\imath} + w_2\hat{j} + w_3\hat{k})$$

$$= (w_1 v_1 - v_1 w_1)\hat{\imath} + (w_1 v_2 - v_1 w_2)\hat{j} + (w_1 v_3 - v_1 w_3)\hat{k}$$

$$= (w_1 v_2 - v_1 w_2)\hat{j} + (w_1 v_3 - v_1 w_3)\hat{k}$$

This proves that the formula is true for $\vec{u} = \hat{\imath}$. A similar calculation shows that it also holds for the special cases $\vec{u} = \hat{j}$ and $\vec{u} = \hat{k}$. Finally, setting $\vec{u} = u_1\hat{\imath} + u_2\hat{j} + u_3\hat{k}$, we get

$$\vec{u}\times(\vec{v}\times\vec{w}) = (u_1\hat{\imath} + u_2\hat{j} + u_3\hat{k})\times(\vec{v}\times\vec{w})$$

$$= u_1(\hat{\imath}\times(\vec{v}\times\vec{w})) + u_2(\hat{j}\times(\vec{v}\times\vec{w})) + u_3(\hat{k}\times(\vec{v}\times\vec{w}))$$

$$= u_1[(\hat{\imath}\cdot\vec{w})\vec{v} - (\hat{\imath}\cdot\vec{v})\vec{w}]$$

$$+ u_2[(\hat{j}\cdot\vec{w})\vec{v} - (\hat{j}\cdot\vec{v})\vec{w}]$$

$$+ u_3[(\hat{k}\cdot\vec{w})\vec{v} - (\hat{k}\cdot\vec{v})\vec{w}]$$

$$= ((u_1\hat{\imath} + u_2\hat{j} + u_3\hat{k})\cdot\vec{w})\vec{v} - ((u_1\hat{\imath} + u_2\hat{j} + u_3\hat{k})\cdot\vec{v})\vec{w}$$

$$= (\vec{u}\cdot\vec{w})\vec{v} - (\vec{u}\cdot\vec{v})\vec{w}$$