MATH 1210 Assignment 2

Due: 1:30 pm Friday 6 February 2009 (at your instructor's office)

NOTES:

1. **Late assignments will NOT be accepted**.
2. **If your assignment is not accompanied by a signed Faculty of Science "Honesty Declaration", it will NOT be graded**.

Provide a complete solution to each of the following problems:

1. In each of the following cases, evaluate the given complex number, writing your answer in (simplified) Cartesian form:
   (a) $(1+i)-(7+2i)$
   (b) i^{57}
   (c) $i^2(-1+3i)$
   (d) $\dfrac{2-3i}{4+2i}$
   (e) $\overline{\left(7-2\bar{i}\right)^2}$
   (f) $\left(1-\sqrt{3}i\right)^4$
   (g) $(4-2i)^6$
   (h) $\left(2+\overline{(-1+i)}\right)^2$
   (i) $\dfrac{(2+i)(4-6i)}{(3-i)i^3}$
   (j) $\dfrac{2+i}{\left(1+\dfrac{1}{1-i}\right)}$

2. (a) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be any two given complex numbers, written in polar form. Show **by direct calculation** that
$$|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|} \text{ and } arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$$
where the latter equation must be interpreted in the manner described in the paragraph following equation (2.14b) on page 19 of the course notes.

   (b) Let $z_1 = -\dfrac{1}{\sqrt{2}}(1+i)$ and $z_2 = \dfrac{-1+\sqrt{3}i}{2}$.
Evaluate $\dfrac{z_1}{z_2}$ directly (i.e., in Cartesian form) and determine its argument,

and confirm that $arg\left(\frac{z_1}{z_2}\right) = arg(z_1) - arg(z_2)$ (subject to the interpretation of part(a)).

HINT: You will need to use a calculator to compute $arg\left(\frac{z_1}{z_2}\right)$, and will probably need to include a "correction " involving some multiple of 2π in order to confirm this relation.

In addition, find the principal value of $arg\left(\frac{z_1}{z_2}\right)$, $arg(z_1)$ and $arg(z_2)$ and show that

$$p.v.\left(arg\left(\frac{z_1}{z_2}\right)\right) \neq p.v.(arg(z_1)) - p.v.(arg(z_2)).$$

3. Let a and b be any real numbers and consider the complex number $a + ib$.

   (a) Use the Binomial Theorem to evaluate $(a + ib)^n$ for $n = 2, 3$ and 4, **in Cartesian form**, simplifying your answers as far as possible.

   (b) Let $a = \cos\theta$ and $b = \sin\theta$, so that $a + ib = e^{i\theta}$.
   Use the Theorem of de Moivre, written in the form

   $$\left(e^{i\theta}\right)^n = e^{i(n\theta)},$$

   in order to derive the following **multiple-angle formulae** for the sine and cosine functions:

   $$\begin{aligned}
   \sin(2\theta) &= 2\sin\theta\cos\theta, \\
   \cos(2\theta) &= \cos^2\theta - \sin^2\theta, \\
   \sin(3\theta) &= 3\cos^2\theta\sin\theta - \sin^3\theta, \\
   \cos(3\theta) &= \cos^3\theta - 3\cos\theta\sin^2\theta, \\
   \sin(4\theta) &= 4\cos^3\theta\sin\theta - 4cos\theta\sin^3\theta, \\
   \cos(4\theta) &= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.
   \end{aligned}$$

4. In each of the following cases, find **all solutions** of the given equation, writing your answers in both Cartesian and polar (or exponential) form, and graphically displaying these solutions in the complex plane:

   (a) $x^6 + 1 = 0$,

   (b) $x^2 + i = 0$,

   (c) $x^3 + 4\sqrt{2}(1 + i) = 0$.

   Comment: The above problems could be rephrased as

   (a) Find all sixth roots of -1,

(b) Find all square roots of $-i$,

(c) Find all cube roots of $-4\sqrt{2}(1+i)$.

5. Consider the function $w = iz$, in which both z and w are complex numbers and $i = \sqrt{-1}$.

   (a) Suppose that we write z and w in Cartesian form as $z = x + iy$ and $w = u + iv$, in which x, y, u and v are all **real**.
   Show that the given complex function is equivalent to the two (real) equations

   $$u = -y \text{ and } v = x.$$

   (b) Regard the latter two equations as defining a transformation of variables between the xy-plane and the uv-plane. Illustrate graphically the effect of this transformation on an arbitrary point (x, y) in the xy-plane.

   (c) Use the above results to geometrically describe (in words) the effect of the function $w = iz$, if it is regarded as a transformation between two copies of the complex plane (with z being any point in the first plane and w being the corresponding point (as determined by the function $w = iz$) on the second plane).

   (d) Use the exponential representation for complex numbers in order to confirm the result of part (c).
   HINT: Let $z = re^{i\theta}$ and $w = Re^{i\phi}$ and determine the relationships between R and r, and between ϕ and θ.