Math 1710. Tutorial 5. Solutions

Problem 1 Find the length of the curve in (a)-(c).

(a) $y = \frac{3}{2}\left(\sqrt[3]{x} - \frac{1}{5}\sqrt[3]{x^5}\right), \quad 1 \le x \le 8.$

Solution

$$L = \int_1^8 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{3}{2}\cdot\frac{1}{3}x^{-\frac{2}{3}} - \frac{3}{2}\cdot\frac{1}{5}\cdot\frac{5}{3}\cdot x^{\frac{2}{3}} = \frac{1}{2}\left(x^{-\frac{2}{3}} - x^{\frac{2}{3}}\right)$$

$$L = \int_1^8 \sqrt{1+\left[\frac{1}{2}\left(x^{-\frac{2}{3}} - x^{\frac{2}{3}}\right)^2\right]}\,dx = \int_1^8 \sqrt{1+\frac{1}{4}x^{-\frac{4}{3}}+\frac{1}{4}x^{\frac{4}{3}} - \frac{1}{2}}\,dx$$

$$= \int_1^8 \sqrt{\frac{1}{4}x^{-\frac{4}{3}}+\frac{1}{4}x^{\frac{4}{3}}+\frac{1}{2}}\,dx = \int_1^8 \sqrt{\left[\frac{1}{2}\left(x^{-\frac{2}{3}}+x^{\frac{2}{3}}\right)\right]^2}\,dx = \frac{1}{2}\int_1^8 \left(x^{-\frac{2}{3}}+x^{\frac{2}{3}}\right)dx$$

$$= \frac{1}{2}\left[3x^{\frac{1}{3}} + \frac{3}{5}x^{\frac{5}{3}}\right]\Bigg|_1^8 = \frac{1}{2}\left[3\sqrt[3]{8} + \frac{3}{5}\left(\sqrt[3]{8}\right)^5 - 3 - \frac{3}{5}\right]$$

$$= \frac{1}{2}\left[6 + \frac{96}{5} - 3 - \frac{3}{5}\right] = \frac{1}{2}\cdot\left[3 + \frac{93}{5}\right] = \boxed{\frac{54}{5}}$$

(b) $y = \frac{1}{2}\left(\ln(\sin x) + \ln(\cos x)\right), \quad \frac{\pi}{4} \le x \le \frac{\pi}{3}.$

Solution

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}\right) = \frac{1}{2}\left(\cot x - \tan x\right)$$

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1+\left[\frac{1}{2}(\cot x - \tan x)^2\right]}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1+\frac{1}{4}\cot^2 x + \frac{1}{4}\tan^2 x - \frac{1}{2}}\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\left[\frac{1}{2}(\cot x + \tan x)\right]^2}\,dx = \frac{1}{2}\int_{\frac{\pi}{4}}^{\pi/3} \left(\cot x + \tan x\right)dx =$$

$$= (\text{one can use substitution here}) = \frac{1}{2}\left[\ln(\sin x) - \ln(\cos x)\right]\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\ln(\tan x)\Bigg|_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \frac{1}{2}\ln\left(\tan\frac{\pi}{3}\right) - \frac{1}{2}\ln\left(\tan\frac{\pi}{4}\right) = \frac{\ln\sqrt{3}}{2} = \boxed{\frac{\ln 3}{4}}$$

(c) $3x = 2(y^2+1)^{3/2}$, from $\left(\frac{4\sqrt{2}}{3}, 1\right)$ to $(18, -2\sqrt{2})$.

Solution

$$x = \frac{2}{3}(y^2+1)^{3/2} \Rightarrow L = \int_{-2\sqrt{2}}^{1} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

$$\frac{dx}{dy} = \frac{2}{3}\cdot\frac{3}{2}(y^2+1)^{\frac{1}{2}}\cdot 2y = 2y\sqrt{y^2+1}$$

$$L = \int_{-2\sqrt{2}}^{1} \sqrt{1+\left(2y\sqrt{y^2+1}\right)^2}\, dy = \int_{-2\sqrt{2}}^{1} \sqrt{4y^4+4y^2+1}\, dy$$

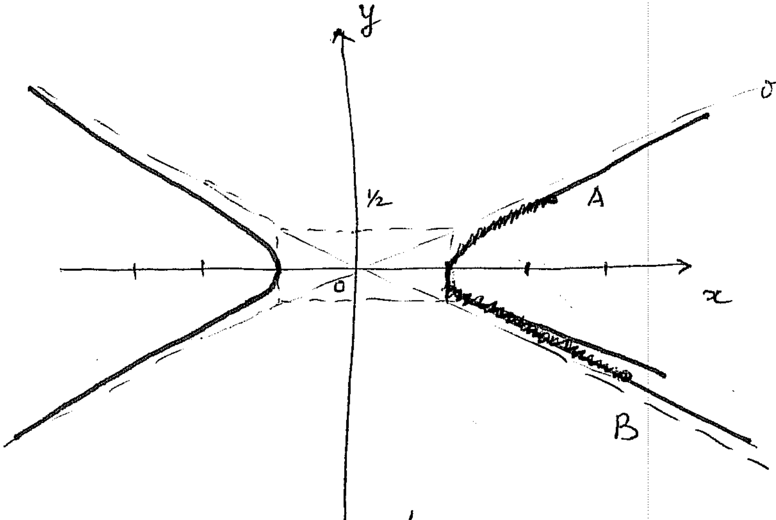
$$= \int_{-2\sqrt{2}}^{1} \sqrt{(2y^2+1)^2}\, dy = \int_{-2\sqrt{2}}^{1} (2y^2+1)\, dy = \left[\frac{2}{3}y^3+y\right]\Big|_{-2\sqrt{2}}^{1}$$

$$= \left[\frac{2}{3}+1\right] - \left[\frac{2}{3}(-2\sqrt{2})^3 - 2\sqrt{2}\right] = \frac{5}{3} + \frac{32\sqrt{2}}{3} + 2\sqrt{2} = \boxed{\frac{5}{3} + \frac{38\sqrt{2}}{3}}$$

(d) Set up a definite integral for the length of the curve $x^2 - 4y^2 = 1$ from $(\sqrt{5}, 1)$ to $(3, -\sqrt{2})$.

Solution

$x^2 - 4y^2 = 1 \iff \frac{x^2}{1^2} - \frac{y^2}{\left(\frac{1}{2}\right)^2} = 1$ – hyperbola. $A(\sqrt{5}, 1)$, $B(3, -\sqrt{2})$



The path from A to B is a graph of a function in the form $x = f(y)$:

$x^2 - 4y^2 = 1 \Rightarrow x^2 = 1 + 4y^2$

$\Rightarrow x = \sqrt{1+4y^2}$ ($x > 0$ along the path AB)

Hence,

$$L = \int_{-\sqrt{2}}^{1} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy = \boxed{\int_{-\sqrt{2}}^{1} \sqrt{1+\left(\frac{8y}{2\sqrt{1+4y^2}}\right)^2}\, dy}$$

$$\left(= \int_{-\sqrt{2}}^{1} \sqrt{1+\frac{16y^2}{1+4y^2}}\, dy \right)$$

(e) Find the length of the curve $y = \sqrt{2x - x^2} - 1$, $\frac{1}{2} \le x \le 1$.

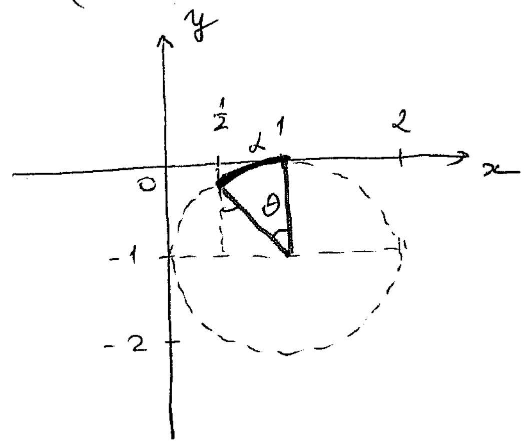
Solution

$y = \sqrt{2x - x^2} - 1$, $\frac{1}{2} \le x \le 1$ — arc of a circle:

$y + 1 = \sqrt{2x - x^2}$

$x^2 - 2x + (y+1)^2 = 0$

$(x-1)^2 + (y+1)^2 = 1$ - circle centered at $(1,-1)$ with radius 1.



The arc α corresponds to the angle $\theta = \frac{\pi}{6}$ (indeed, $\sin\theta = \frac{1}{2}$)

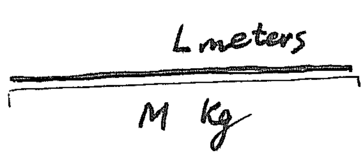
$(x-1)^2 + (y+1)^2 = 1$ is a circle of radius 1, and therefore, the length of the arc α is equal to the angle $\theta \Rightarrow$

$\Rightarrow \boxed{L = \frac{\pi}{6}.}$

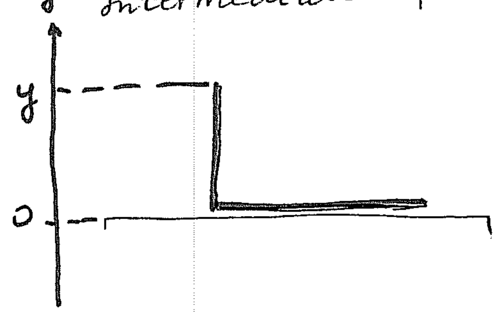
Remark It is also possible to set up an integral and evaluate it.

Problem 2 (sketch of the sol-n)

Initial position:



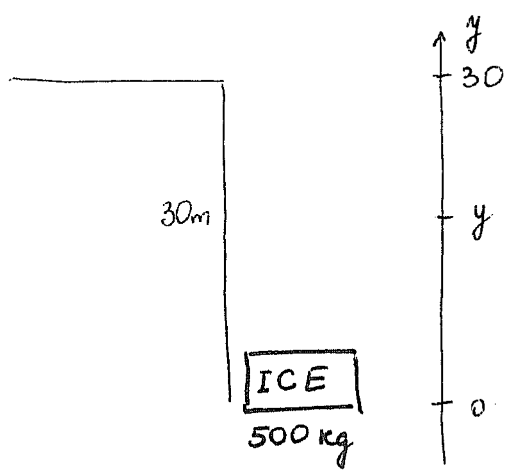
Intermediate pos-n



Work = {work to pull the chain all the way up} + {work to pull from $y = L$ to $y = L+5$}

$$= \int_0^L Mg\frac{y}{L}\,dy + Mg \cdot 5 = \frac{Mg}{L} \cdot \frac{L^2}{2} + 5Mg = \boxed{Mg\left(\frac{L}{2} + 5\right)}$$

Problem 3 (sketch)



At intermediate ~~time~~ position $0 \le y \le 30$, the mass of ~~the~~ the block is equal to

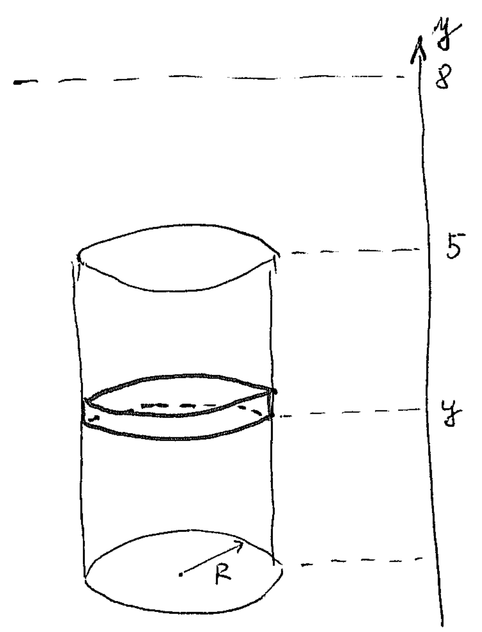
$$\left(500 - 12 \cdot \frac{y}{30}\right) \text{kg}$$

Thus, $$W = \int_0^{30} \left(500 - \frac{2}{5} y\right) g \, dy$$

$$= g\left(500y - \frac{y^2}{5}\right)\Big|_0^{30} = g(15000 - 180) = \boxed{14820g \ [J]}$$

* You can leave g in your final answer or plug $g = 10 \ m/s^2$.

Problem 4 (sketch)



$R = 6$ m

mass of a "small" slice:

$$m = \rho \cdot V = \rho \cdot \pi R^2 \cdot dy \quad , \quad F = mg$$

$$\Rightarrow W = \int_0^5 \rho \cdot \pi R^2 g (8 - y) \, dy$$

$$= \pi \rho \cdot R^2 g \int_0^5 (8 - y) \, dy$$

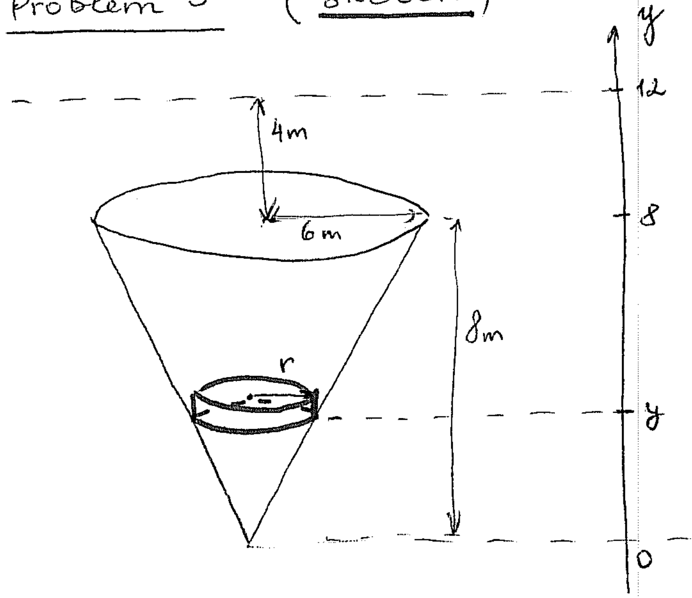
$$= 36 \pi \rho g \left[8y - \frac{y^2}{2} \right]\Big|_0^5$$

$$= 36 \pi \rho g \cdot \left[40 - \frac{25}{2} \right] = 18 \pi \rho g \cdot 55$$

$\rho = 1000 \ kg/m^3$

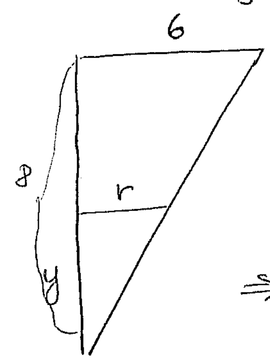
$$= 990000 \pi g \ [J]$$

Problem 5 (sketch)



Mass of a "small" slice (cylindrical):

$m = \rho \cdot V = \rho \pi r^2 dy$



Similar triangles:

$\frac{r}{6} = \frac{y}{8}$

$\Rightarrow r = \frac{6}{8} y = \frac{3}{4} y$

$\Rightarrow m = \rho \pi \left(\frac{3}{4} y\right)^2 dy, \quad F = mg$

$$W = \int_0^8 \rho \pi \left(\frac{3}{4} y\right)^2 g \, (12 - y) \, dy =$$

$$= \rho \pi g \cdot \frac{3}{4} \int_0^8 \left(12y^2 - y^3\right) dy = \frac{3}{4} \pi \rho g \left[4y^3 - \frac{y^4}{4} \right] \Big|_0^8$$

$$= \frac{3}{4} \pi \rho g \left[2048 - 1024 \right] = \frac{3}{4} \pi \cdot 817 \cdot g \cdot 1024 = \boxed{627456 \pi g \, [J]}$$