MAT2130: Engineering Mathematical Analysis 1

Midterm 1 Solutions

1. Find the equation of the plane that contains the lines

$$\ell: \quad (1,2,4) + t\left(-\frac{1}{2}, 0, 1\right), \quad t \in \mathbb{R}$$

and

$$m: \quad (1,3,6) + t(1,0,-2), \quad t \in \mathbb{R}.$$

The line ℓ is parallel to the vector $\mathbf{u} = \left(-\frac{1}{2}, 0, 1\right)$, and the line m is parallel to the vector $\mathbf{v} = (1,0,-2)$. Since $\mathbf{v} = -2\mathbf{u}$, we see that ℓ and m are parallel.

One vector parallel to the plane containing ℓ and m is $\mathbf{v} = (1,0,-2)$. We need to find a second vector parallel to this plane. Note that a point on line ℓ is $(1,2,4)$, and a point on line m is $(1,3,6)$. Therefore the vector

$$\mathbf{w} = (1,3,6) - (1,2,4) = (0,1,2)$$

is also parallel to the plane containing these lines.

A normal vector for the plane is then

$$(1,0,-2) \times (0,1,2) = (2,-2,1).$$

Using this normal vector, and the point $(1,2,4)$ on the plane, we find that the equation of the plane is

$$2(x-1) - 2(y-2) + (z-4) = 0,$$

which reduces to

$$2x - 2y + z - 2 = 0.$$

Alternative: Using the normal vector, we find that the equation of the plane takes the form

$$2x - 2y + z + D = 0$$

for some $D \in \mathbb{R}$. To find D, we substitute the point $(1,3,6)$:

$$2 - 6 + 6 + D = 0,$$

which implies that $D = -2$. Therefore the equation of the plane is

$$2x - 2y + z - 2 = 0.$$

2. Find the distance between the point $(-1,0,3)$ and the line

$$x = 1, \quad y - 2 = \frac{1-z}{5}.$$

First, we convert the line into parametric form by setting $y - 2 = \frac{1-z}{5} = t$. The result is

$$x = 1, \quad y = 2 + t, \quad z = 1 - 5t, \quad t \in \mathbb{R}.$$

A point on this line is $Q = (1,2,1)$, and a vector parallel to the line is $\mathbf{v} = (0,1,-5)$. Since $|\mathbf{v}| = \sqrt{1^2 + (-5)^2} = \sqrt{26}$, the unit vector corresponding to $\mathbf{v}$ is $\widehat{\mathbf{v}} = \left(0, \frac{1}{\sqrt{26}}, -\frac{5}{\sqrt{26}}\right)$.

Let $P = (-1,0,3)$. Then

$$\mathbf{PQ} = (1,2,1) - (-1,0,3) = (2,2,-2)$$

is a vector from the point to the line. Using the procedure discussed in class, the distance from the point to the line is

$$\begin{aligned}
|\mathbf{PQ} \times \widehat{\mathbf{v}}| &= \left|(2,2,-2) \times \left(0, \frac{1}{\sqrt{26}}, -\frac{5}{\sqrt{26}}\right)\right| \\
&= \frac{2}{\sqrt{26}} \left|(1,1,-1) \times (0,1,-5)\right| \\
&= \frac{2}{\sqrt{26}} \left|(-4,5,1)\right| \\
&= \frac{2\sqrt{42}}{\sqrt{26}} = 2\sqrt{\frac{21}{13}}.
\end{aligned}$$

Alternative: A vector in the direction of the line is $\mathbf{v} = (0,1,-5)$, and a vector from the point to the line is $\mathbf{PQ} = (2,2,-2)$. A vector perpendicular to the plane containing the point and the line is

$$\mathbf{PQ} \times \mathbf{v} = (2,2,-2) \times (0,1,-5) = 2(-4,5,1).$$

Then a vector perpendicular to the line, lying in the plane defined by the point and the line, is

$$\mathbf{w} = (\mathbf{PQ} \times \mathbf{v}) \times \mathbf{v} = 2(-4,5,1) \times (0,1,-5) = 2(-26,-20,-4) = -4(13,10,2).$$

The corresponding unit vector is $\widehat{\mathbf{w}} = \frac{1}{\sqrt{273}}(13,10,2)$. Finally, the distance from the point to the line is

$$\begin{aligned}
|\mathbf{PQ} \cdot \widehat{\mathbf{w}}| &= \frac{2}{\sqrt{273}} \left|(1,1,-1) \cdot (13,10,2)\right| \\
&= \frac{42}{\sqrt{273}} = 2\sqrt{\frac{21}{13}}.
\end{aligned}$$

3. On a large, clearly labeled diagram, sketch the surface $3(z-1)^2 - 4(y+2)^2 = 12$ in 3D space. Label any important points and cross sections.

Observe that the variable x does not appear in the given equation. We therefore sketch the curve in the yz-plane, then obtain the surface by translating the curve in the $\pm x$-directions.

In the yz-plane, the equation can be written as

$$\frac{(z-1)^2}{4} - \frac{(y+2)^2}{3} = 1,$$

which describes a hyperbola, center $(y=-2, z=1)$. Since the quadratic term in z has the positive sign, the hyperbola opens in the $\pm z$ directions.

If we set $y=-2$, then the solutions for z are 3 and -1. The points $(y=-2, z=3)$ and $(y=-2, z=-1)$ are on the curve, and represent the local max and min of the two branches of the hyperbola.

Other possible pieces of information: if we set $z=0$, then the equation reduces to

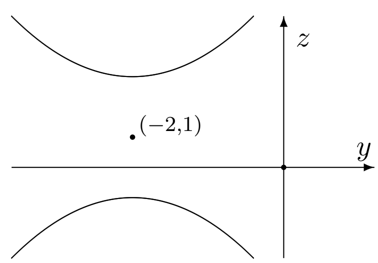
$$\frac{1}{4} - \frac{(y+2)^2}{3} = 1,$$

which has no solutions. This hyperbola does not intersect the y-axis. If we set $y=0$, then the equation reduces to

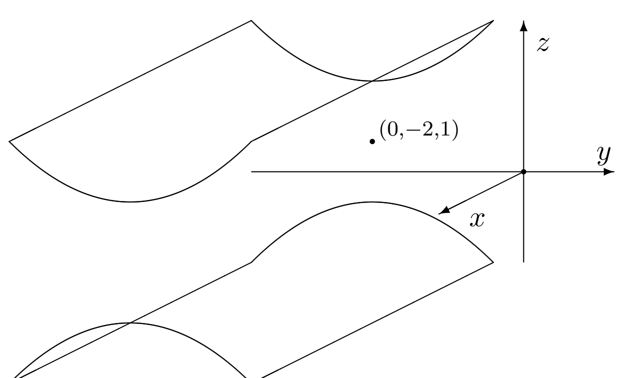
$$\frac{(z-1)^2}{4} - \frac{4}{3} = 1,$$

whose solutions are $z = 1 \pm \sqrt{\frac{28}{3}}$. The z-intercepts for the upper and lower branches of the hyperbola are $z = 1 + \sqrt{\frac{28}{3}}$ and $z = 1 - \sqrt{\frac{28}{3}}$, respectively.

Sketch in yz-plane



Rough sketch 3D space



4. Let $\mathcal{C}$ be the curve in 3D space formed by the intersection of the surfaces

$$y = (x-1)^2 + z^2$$

and

$$x - 2z = 1.$$

Find a parametric representation of $\mathcal{C}$ such that y is increasing when z is negative.

The equation $y = (x-1)^2 + z^2$ can only be satisfied when $y \geq 0$. There are no constraints on x or z.

Let $z = t$, $t \in \mathbb{R}$. Then

$$x = 1 + 2z = 1 + 2t,$$

and

$$y = (x-1)^2 + z^2 = 5t^2.$$

Thus a parametrization of the curve is

$$x = 1 + 2t, \quad y = 5t^2, \quad z = t, \quad t \in \mathbb{R}.$$

We check if the constraint is satisfied. The derivative of y is

$$y' = 10t.$$

Therefore y is increasing when t is positive, and decreasing where t is negative. Since $z = t$, we see that y is decreasing where z is negative. We have not met the constraint, so we must reverse direction.

Let $t' = -t$. With this substitution, the parametrization becomes

$$x = 1 - 2t', \quad y = 5{t'}^2, \quad z = -t', \quad t' \in \mathbb{R}.$$

This is a parametrization of the curve with the desired properties.

Alternative 1: If we let $x = t$, $t \in \mathbb{R}$, we obtain a similar parametrization that must also be reversed in order to satisfy the constraint. The final answer is

$$x = -t, \quad y = \frac{5}{4}(t+1)^2, \quad z = -\frac{t+1}{2}, \quad t \in \mathbb{R}.$$

Alternative 2: Let $y = t$, $t \geq 0$. Then $(x-1)^2 + z^2 = t$. If we substitute $x = 1 + 2z$ in this equation, we find that $5z^2 = t$, which has two solutions: $z = \sqrt{\frac{t}{5}}$ and $z = -\sqrt{\frac{t}{5}}$. These expressions give us two corresponding solutions for x, namely $x = 1 + 2\sqrt{\frac{t}{5}}$ and $x = 1 - 2\sqrt{\frac{t}{5}}$, respectively. Therefore the curve can be parametrized in two pieces:

$$x = 1 + 2\sqrt{\frac{t}{5}}, \quad y = t, \quad z = \sqrt{\frac{t}{5}}, \quad t \geq 0$$

and

$$x = 1 - 2\sqrt{\frac{t}{5}}, \quad y = t, \quad z = -\sqrt{\frac{t}{5}}, \quad t \geq 0.$$

With this parametrization, y is always increasing. In particular, it is increasing on the second piece, where z is negative.

5. Let $\mathcal{D}$ be the curve in 3D space with vector equation

$$\mathbf{r}(t) = t^3\widehat{\mathbf{i}} + \left(\frac{1}{\pi}\sin\pi t - t\cos\pi t\right)\widehat{\mathbf{j}} + \left(t\sin\pi t + \frac{1}{\pi}\cos\pi t\right)\widehat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

(a) Verify that the point $\left(1, 1, -\frac{1}{\pi}\right)$ is on $\mathcal{D}$. Find a unit vector $\widehat{\mathbf{T}}$ tangent to $\mathcal{D}$ at $\left(1, 1, -\frac{1}{\pi}\right)$, pointing in the direction of increasing t.

We need to find a value of t such that $\mathbf{r}(t) = \left(1, 1, -\frac{1}{\pi}\right)$. Looking at the x-coordinate, the only possibility is $t = 1$. We check by substitution:

$$\mathbf{r}(1) = \left(1, \frac{1}{\pi}\sin\pi - \cos\pi, \sin\pi + \frac{1}{\pi}\cos\pi\right) = \left(1, 1, -\frac{1}{\pi}\right),$$

as needed.

To find a tangent vector to the curve, we take the derivative of $\mathbf{r}(t) = (x(t), y(t), z(t))$. The results for each component are

$$\begin{aligned} x'(t) &= 3t^2, \\ y'(t) &= \frac{1}{\pi}\pi\cos\pi t - \cos\pi t + t\pi\sin\pi t \\ &= \pi t\sin\pi t, \\ z'(t) &= \sin\pi t + t\pi\cos\pi t - \frac{1}{\pi}\pi\sin\pi t \\ &= \pi t\cos\pi t. \end{aligned}$$

That is,

$$\mathbf{r}'(t) = \left(3t^2, \pi t\sin\pi t, \pi t\cos\pi t\right).$$

A tangent vector to the curve at the point corresponding to $t = 1$ is then

$$\mathbf{T} = \mathbf{r}'(1) = (3, 0, -\pi).$$

This vector points in the direction of increasing t. The corresponding unit vector is

$$\widehat{\mathbf{T}} = \left(\frac{3}{\sqrt{9+\pi^2}}, 0, -\frac{\pi}{\sqrt{9+\pi^2}}\right).$$

(b) Find the length of $\mathcal{D}$ over the interval $1 < t < 2$.

The length of the curve over the given interval is

$$L = \int_1^2 |\mathbf{r}'(t)|\,dt.$$

Recall that

$$\mathbf{r}'(t) = \left(3t^2, \pi t \sin \pi t, \pi t \cos \pi t\right).$$

Then

$$\left|\mathbf{r}'(t)\right|^2 = 9t^4 + (\pi t)^2 \sin^2 \pi t + (\pi t)^2 \cos^2 \pi t = 9t^4 + \pi^2 t^2,$$

which implies that

$$\left|\mathbf{r}'(t)\right| = \sqrt{9t^4 + \pi^2 t^2} = t\sqrt{9t^2 + \pi^2}.$$

We must evaluate the integral

$$L = \int_1^2 t\sqrt{9t^2 + \pi^2}\,dt.$$

Let $u = 9t^2 + \pi^2$. Then $du = 18t\,dt$, which implies that $t\,dt = \frac{1}{18}du$. With this change of variables, the integral becomes

$$\begin{aligned}
L &= \frac{1}{18}\int_{t=1}^{2} \sqrt{u}\,du \\
&= \frac{1}{18}\left[\frac{2}{3}u^{3/2}\right]_{t=1}^{2} \\
&= \frac{1}{27}\left[\left(9t^2 + \pi^2\right)^{3/2}\right]_1^2 \\
&= \frac{1}{27}\left[\left(36 + \pi^2\right)^{3/2} - \left(9 + \pi^2\right)^{3/2}\right].
\end{aligned}$$