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[8] 1. Use the principle of mathematical induction on the positive integers $n \geq 1$ to prove that

$$1 + 3 + 5 + \cdots + (4n + 1) = (2n + 1)^2.$$

Solution: Let $P(n)$ be the statement $1 + 3 + 5 + \cdots + (4n + 1) = (2n + 1)^2$.

Show $P(1)$ is true.

$LHS = 1 + 3 + 5 = 9$ and $RHS = (2(1) + 1)^2 = 3^2 = 9$. Therefore the $LHS = RHS$ and so $P(1)$ is true.

Suppose $P(k)$ is true for an integer $n = k \geq 1$. That is

$$1 + 3 + 5 + \cdots + (4k + 1) = (2k + 1)^2.$$

We need to show $P(k + 1)$ is true. That is

$$1 + 3 + 5 + \cdots + (4(k + 1) + 1) = (2(k + 1) + 1)^2 = (2k + 3)^2.$$

$$\begin{aligned}
LHS &= 1 + 3 + 5 + \cdots + (4(k + 1) + 1) \\
&= 1 + 3 + 5 + \cdots + (4k + 1) + (4k + 3) + (4k + 5) \\
&= (2k + 1)^2 + (4k + 3) + (4k + 5) \\
&= 4k^2 + 4k + 1 + (4k + 3) + (4k + 5) \\
&= 4k^2 + 12k + 9 \\
&= (2k + 3)^2 \\
&= RHS.
\end{aligned}$$

Hence $P(k+1)$ is true. Therefore by the principle of mathematical induction, $P(n)$ is true for $n \geq 1$.

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[6] 2. Find all fourth roots of $-2-2\sqrt{3}\,i$. Leave your answer(s) in exponential form.

Solution:

We first put the complex number into exponential form

$$\left|-2-2\sqrt{3}\,i\right| = \sqrt{(-2)^2+(-2\sqrt{3})^2} = \sqrt{4+12} = \sqrt{16} = 4.$$

$$\tan\theta = \frac{y}{x} = \sqrt{3} \Rightarrow \theta = -\frac{2\pi}{3} + 2k\pi.$$

since $x, y < 0$

Hence $-2+2\sqrt{3}\,i = 4e^{\left(-2\pi/3+2k\pi\right)i}$. Therefore the fourth roots of $-2-2\sqrt{3}\,i$ are of the form

$$\left(4e^{\left(-2\pi/3+2k\pi\right)i}\right)^{1/4} = \sqrt{2}e^{\left(-\pi/6+k\pi/2\right)i}.$$

By the fundamental theroem of algebra, we know there are four fourth roots and therefore we let $k=0,1,2,3$. Hence the solutions are

$$\begin{aligned} k=0 &\Rightarrow \sqrt{2}e^{-\pi i/6} \\ k=1 &\Rightarrow \sqrt{2}e^{\pi i/3} \\ k=2 &\Rightarrow \sqrt{2}e^{5\pi i/6} \\ k=3 &\Rightarrow \sqrt{2}e^{4\pi i/3} \end{aligned}$$

[6] 3. Find, in Cartesian form $x + yi$, the complex number z that satisfies the equation

$$(1+i)^2 + \overline{(3+i)}\,z = 1+4i\,.$$

Solution:

$(1+i)^2 = (1+i)(1+i) = 1+2i+i^2 = 1+2i-1 = 2i$. and

$\overline{(3+i)} = 3-i$. Hence the equation becomes

$$\begin{aligned} 2i + (3-i)z &= 1+4i \\ (3-i)z &= 1+2i \\ z &= \frac{1+2i}{3-i}. \end{aligned}$$

Multiplying the top and the bottom of the fraction by the conjugate of the denominator leads to

$$\begin{aligned} z &= \frac{(1+2i)(3+i)}{(3-i)(3+i)} \\ &= \frac{(1+2i)(3+i)}{(3-i)(3+i)} \\ &= \frac{3+i+6i+2i^2}{9+3i-3i-i^2} \\ &= \frac{3+7i-2}{9+3i-3i+1} \\ &= \frac{1+7i}{10} \\ &= \frac{1}{10} + \frac{7}{10}i \end{aligned}$$

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4. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = 4x^4 - 9x^3 + 24x^2 - 9x + 10\,.$$

[3] (a) What are the possible rational zeros of $P(x)$?

> **Solution:**
> The rational solutions must be of the form $\frac{p}{q}$ where p divides $a_0 = 10$ and q divides $a_n = 4$. Hence p is one of ±1, ±2, ±5, ±10 and q is one of ±1, ±2, ±4. Therefore the possible rational solutions are
>
> $$\pm1,\ \pm2,\ \pm5,\ \pm10,\ \pm\frac{1}{2},\ \pm\frac{5}{2},\ \pm\frac{1}{4},\ \pm\frac{5}{4}.$$

[3] (b) Use Descartes' rule of signs to find the possible number of positive and negative roots of $P(x)$.

> **Solution:**
> $P(x)$ has 4 sign changes and therefore there are an even number less than or equal to 4 of positive solutions. Therefore there are $4, 2$ or 0 positive solutions.
> $P(-x) = 4(-x)^4 - 9(-x)^3 + 24(-x)^2 - 9(-x) + 10 = 4x^4 + 9x^3 + 24x^2 + 9x + 12$ has no sign changes and therefore there are no negative solutions.

[2] (c) Use the Bounds Theorem to find a bound on the roots of $P(x)$.

> **Solution:**
> The solutions must satisfy
>
> $$|z| < \frac{M}{|a_n|} + 1 = \frac{24}{4} + 1 = 7.$$

[1] (d) Update the list from part (a) using the information from parts (b) and (c).

> **Solution:**
>
> $$1,\ 2,\ 5,\ \frac{1}{2},\ \frac{5}{2},\ \frac{1}{4},\ \frac{5}{4}.$$

[6] (e) Given that $1 + 2\,i$ is a root of $P(x)$, find all the roots of $P(x)$.

> **Solution:**
> Since $P(x)$ has real coefficients, we know since $1 + 2i$ is a solution, so is the the conjugate $1 - 2i$. By the factor theorem, we know that both $(x - (1 + 2i))$ and $(x - (1 - 2i))$ are factors. Multiplying them together shows that
>
> $$(x - (1 + 2i))(x - (1 - 2i)) = x^2 - 2x + 5$$
>
> is a factor. Long division can show that
>
> $$P(x) = (x^2 - 2x + 5)(4x^2 - x + 2)$$

Solving for $4x^2 - x + 2 = 0$ leads to

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4(4)(2)}}{2(4)} = \frac{1 \pm \sqrt{-31}}{8}$$

Hence the roots of $P(x)$ are

$$1 + 2i,\ 1 - 2i,\ \frac{1 \pm \sqrt{31}\,i}{8},\ \frac{1 \pm \sqrt{31}\,i}{8}.$$

5. Let $A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 4 & 1 & -1 \end{pmatrix}$. Calculate the following if they are defined. If they are not defined, state why not. Be specific.

[2] (a) $2A - B$

Solution:

$$\begin{aligned} 2A - B &= 2\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 4 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 & -2 & 0 \\ 4 & 0 & 2 & -6 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 4 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & -3 & 0 \\ 5 & -4 & 1 & -5 \end{pmatrix} \end{aligned}$$

[2] (b) A^2

Solution:

$$A^2 = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix}.$$

Since the number of columns of the first matrix doesn't match the number of rows of the second matrix, the product is not defined.

[3] (c) $AB^T - I$

Solution:

$$\begin{aligned} AB^T - I &= \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 0 \\ -1 & 4 & 1 & -1 \end{pmatrix}^T - I \\ &= \begin{pmatrix} 1 & 1 & -1 & 0 \\ 2 & 0 & 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 4 \\ 1 & 1 \\ 0 & -1 \end{pmatrix} - I \\ &= \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \end{aligned}$$

[4] 6. Let $\mathbf{v} = \langle 1, 1, 2\rangle$ and $\mathbf{w} = \langle 2, -1, 1\rangle$. Calculate the smallest positive angle between $\mathbf{v}$ and $\mathbf{w}$.

Solution: Let θ be the angle between the two vectors. Then

$$\begin{aligned}\cos\theta &= \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}|\,|\mathbf{w}|}\\ &= \frac{\langle 1,1,2\rangle\cdot\langle 2,-1,1\rangle}{|\langle 1,1,2\rangle|\,|\langle 2,-1,1\rangle|}\\ &= \frac{1(2)+1(-1)+2(1)}{\sqrt{1^2+1^2+2^2}\sqrt{2^2+(-1)^2+1^2}}\\ &= \frac{3}{\sqrt{6}\sqrt{6}}\\ &= \frac{1}{2}.\end{aligned}$$

Hence $\theta = \dfrac{\pi}{3}$.

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[9] 7. Consider the point $P(2, 3, -2)$ and the two planes $\Pi_1 :\ x + 3y - z = 0$ and $\Pi_2 :\ -x + y + 2z = 0\,.$

(a) Find parametric equations of the line through the point P and parallel to the line of intersection of the planes Π_1 and Π_2.

Solution:

We need to find a vector parallel to the line and parallel to both planes. Hence the vector must be perpendicular to both normal vectors

$$\mathbf{n_1} = \langle 1, 3, -1\rangle \text{ and } \mathbf{n_2} = \langle -1, 1, 2\rangle.$$

Hence the vector $\mathbf{v}$ is

$$\begin{aligned}\mathbf{v} &= \langle 1, 3, -1\rangle \times \langle -1, 1, 2\rangle \\ &= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} \hat{\mathbf{i}} - \begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix} \hat{\mathbf{j}} + \begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} \hat{\mathbf{k}} \\ &= (6 - (-1))\hat{\mathbf{i}} - (2 - 1)\hat{\mathbf{j}} + (1 - (-3))\hat{\mathbf{k}} \\ &= 7\hat{\mathbf{i}} - 1\hat{\mathbf{j}} + 4\hat{\mathbf{k}}.\end{aligned}$$

Hence the parametric equations through the point $P(2, 3, -2)$ are

$$\begin{aligned} x &= 2 + 7t \\ y &= 3 - t \\ z &= -2 + 4t \end{aligned}$$

(b) Find an equation of the plane through P and perpendicular to both Π_1 and Π_2.

Solution:

We need to find the normal to the plane. Since the plane must be perpendicular to both Π_1 and Π_2, the normal vector must be perpendicular to

$$\mathbf{n_1} = \langle 1, 3, -1\rangle \text{ and } \mathbf{n_2} = \langle -1, 1, 2\rangle.$$

Hence

$$\mathbf{n} = \langle 1, 3, -1\rangle \times \langle -1, 1, 2\rangle = \langle 7, -1, 4\rangle.$$

Hence the equation of the plane through the point $P(2, 3, -2)$ is

$$7(x - 2) - 1(y - 3) + 4(z - (-2)) = 0$$

or

$$7x - y + 4z + 3 = 0$$

or

$$7x - y + 4z = -3.$$