MATH 2132 Problem Workshop 7

1. Solve the following differential equations/inital value problems/boundary value problems, using Laplace Transforms.

   (a) $y'' + 3y' + 7y = 3\sin 2t, \quad y(0) = 1,\ y'(0) = 2.$

Solution:
Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 7Y(s) = \frac{6}{s^2+4}$$

Plugging in the initial values yields and simplifying yields

$$(s^2+3s+7)Y(s) - s - 2 - 3 = \frac{6}{s^2+4} \Rightarrow (s^2+3s+7)Y(s) = s + 5 + \frac{6}{s^2+4}$$

Therefore

$$Y(s) = \frac{s+5}{s^2+3s+7} + \frac{6}{(s^2+4)(s^2+3s+7)}$$

The first term leads to

$$\frac{(s+3/2)+7/2}{(s+3/2)^2+19/4}$$

which has inverse Laplace transforms of $e^{-3t/2}\cos\frac{\sqrt{19}}{2}t + \frac{7}{\sqrt{19}}e^{-3t/2}\sin\frac{\sqrt{19}}{2}t$ respectively. As for the third term, we require some partial fractions.

$$\begin{aligned}\frac{6}{(s^2+4)(s^2+3s+7)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+3s+7}\\ &= \frac{(As+B)(s^2+3s+7)+(Cs+D)(s^2+4)}{(s^2+4)(s^2+3s+7)}\end{aligned}$$

Therefore

$$6 = s^3(A+C) + s^2(3A+B+D) + s(3B+7A+4C) + (7B+4D)$$

Therefore $A + C = 0 \Rightarrow C = -A$. $0 = 3B + 7A + 4C = 3B + 3A \Rightarrow B = -A$. $0 = 3A + B + D = 2A + D \Rightarrow D = -2A$.

Hence $6 = -7A - 8A \Rightarrow A = -\frac{2}{5} \Rightarrow B = \frac{2}{5}$, $C = \frac{2}{5}$, $D = \frac{4}{5}$.
Hence the inverse Laplace transform is

$$\begin{aligned}
&\frac{1}{5}\mathcal{L}^{-1}\left(\frac{-2s+2}{s^2+4} + \frac{2s+4}{(s+3/2)^2+19/4}\right)\\
&= \frac{1}{5}\mathcal{L}^{-1}\left(\frac{-2s}{s^2+4} + \frac{2}{s^2+4} + \frac{2(s+3/2)}{(s+3/2)^2+19/4} + \frac{1}{(s+3/2)^2+19/4}\right)\\
&= \frac{1}{5}\left(-2\cos 2t + \sin 2t + e^{3/2t}\left(2\cos\frac{\sqrt{19}}{2}t + \frac{2}{\sqrt{19}}\sin\frac{\sqrt{19}}{2}t\right)\right)
\end{aligned}$$

Thus the solution is

$$y(t) = -\frac{2}{5}\cos 2t + \frac{1}{5}\sin 2t + e^{3/2t}\left(\frac{7}{5}\cos\frac{\sqrt{19}}{2}t + \frac{37}{5\sqrt{19}}\sin\frac{\sqrt{19}}{2}t\right)$$

(b) $y'' + 4y' + 4y = te^{-2t}, \quad y(0) = -1,\ y'(0) = 0.$

Solution:
Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{1}{(s+2)^2}$$

Plugging in the initial values yields and simplifying yields

$$(s^2+4s+4)Y(s) + s + 4 = \frac{1}{(s+2)^2} \Rightarrow (s+2)^2)Y(s) = -s - 4 + \frac{1}{(s+2)^2}$$

Therefore

$$Y(s) = \frac{-s-4}{(s+2)^2} + \frac{1}{(s+2)^4}$$

which simplifies to

$$\frac{-(s+2)-2}{(s+2)^2} + \frac{1}{(s+2)^4} = -\frac{1}{s+2} - \frac{2}{(s+2)^2} + \frac{1}{(s+2)^4}$$

which has inverse Laplace transforms

$$y(t) = -e^{-2t} - 2te^{-2t} + \frac{1}{6}t^3e^{-2t}.$$

(c) $y'' + 20y = t^2 - e^{-t}, \quad y(0) = 0,\, y'(0) = 2.$

Solution:
Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 20Y(s) = \frac{2}{s^3} - \frac{1}{s+1}$$

Plugging in the initial values yields and simplifying yields

$$(s^2+20)Y(s) - 2 = \frac{1}{s+1} \Rightarrow (s^2+3s+7)Y(s) = 2 + \frac{2}{s^3} - \frac{1}{s+1}$$

Therefore

$$Y(s) = \frac{2}{s^2+20} + \frac{2}{s^3(s^2+20)} - \frac{1}{(s+1)(s^2+20)}$$

The first term has inverse Laplace transform of $\frac{2}{\sqrt{20}} \sin \sqrt{20}t$.
As for the second term, we require some partial fractions.

$$\begin{aligned}\frac{2}{s^3(s^2+20)} &= \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{Ds+E}{s^2+20} \\ &= \frac{(As^2+Bs+C)(s^2+20) + (Ds+E)s^3}{s^3(s^2+20)}\end{aligned}$$

Therefore

$$2 = s^4(A+D) + s^3(B+E) + s^2(20A+C) + s(20B) + (20C)$$

Therefore $B = 0 \Rightarrow E = 0$. $20C = 2 \Rightarrow C = \frac{1}{10} \Rightarrow A = -\frac{1}{200} \Rightarrow D = \frac{1}{200}$.
Hence the inverse Laplace transform is

$$\begin{aligned}&\mathcal{L}^{-1}\left(-\frac{1/200}{s} + \frac{1}{10}\frac{1}{s^3} + \frac{1}{200}\frac{s}{s^2+20}\right) \\ &= -\frac{1}{200} + \frac{1}{20}t^2 + \frac{1}{200}\cos\sqrt{20}t.\end{aligned}$$

As for the third term, we also require some partial fractions.

$$\frac{1}{(s+1)(s^2+20)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+20}$$
$$= \frac{A(s^2+20)+(Bs+C)(s+1)}{(s+1)(s^2+20)}$$

Therefore

$$1 = s^2(A+B) + s(B+C) + (20A+C)$$

Therefore $A+B=0 \Rightarrow B=-A$. $0 = B+C \Rightarrow C = -B = A$. Hence $1 = 20A + A \Rightarrow A = \frac{1}{21} \Rightarrow B = -\frac{1}{21}$, $C = \frac{1}{21}$.
Hence the inverse Laplace transform is

$$\frac{1}{21}\mathcal{L}^{-1}\left(\frac{1}{s+1} + \frac{-s+1}{s^2+20}\right)$$
$$= \frac{1}{21}\left(e^{-t} - \cos\sqrt{20}t + \frac{1}{\sqrt{20}}\sin\sqrt{20}t\right)$$

Thus the solution is

$$y(t) = \frac{2}{\sqrt{20}}\sin\sqrt{20}t - \frac{1}{200} + \frac{1}{20}t^2 + \frac{1}{200}\cos\sqrt{20}t - \frac{1}{21}\left(e^{-t} - \cos\sqrt{20}t + \frac{1}{\sqrt{20}}\sin\sqrt{20}t\right).$$

(d) $y'' - 2y' + y = \cos 2t, \quad y(0) = 1,\ y(\pi) = 1.$

Solution:
Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = \frac{s}{s^2+4}$$

However, we do not know $y'(0)$. Therefore, let $a = y'(0)$.
Plugging in the initial values yields and simplifying yields

$$(s^2 - 2s + 1)Y(s) - s - a + 2 = \frac{s}{s^2+4} \Rightarrow (s-1)^2Y(s) = s + a - 2 + \frac{s}{s^2+4}$$

Therefore

$$Y(s) = \frac{s-1}{(s-1)^2} + \frac{a-1}{(s-1)^2} + \frac{s}{(s-1)^2(s^2+4)}.$$

The first two terms have inverse Laplace transform of $e^t + (a-1)te^t$.
As for the last term, we require some partial fractions.

$$\begin{aligned}
\frac{s}{(s-1)^2(s^2+4)} &= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{Cs+D}{s^2+4} \\
&= \frac{(A(s-1)(s^2+4) + B(s^2+4) + (Cs+D)(s-1)^2}{(s-1)^2(s^2+4)} \\
&= \frac{A(s^3-s^2+4s-4) + B(s^2+4) + C(s^3-2s^2+s) + D(s^2-2s+1)}{(s-1)^2(s^2+4)}
\end{aligned}$$

Therefore

$$s = s^3(A+C) + s^2(-A+B-2C+D) + s(4A+C-2D) + (-4A+4B+D)$$

Therefore

$$\begin{aligned}
0 &= A + C \\
0 &= -A + B - 2C + D \\
1 &= 4A + C - 2D \\
0 &= -4A + 4B + D
\end{aligned}$$

Therefore $C = -A \Rightarrow A + B + D = 0 \Rightarrow D = -A - B$. Thus $-5A + 3B = 0 \Rightarrow B = \frac{5}{3}A \Rightarrow D = -\frac{8}{3}A$.
Hence

$$1 = 4A - A + \frac{16}{3}A \Rightarrow A = \frac{3}{25} \Rightarrow B = \frac{1}{5},\ C = -\frac{3}{25},\ D = \frac{8}{25}.$$

Hence the inverse Laplace transform is

$$\begin{aligned}
&\frac{1}{25}\mathcal{L}^{-1}\left(\frac{3}{s-1} + \frac{5}{s-1} + \frac{-3s-8}{s^2+4}\right) \\
&= \frac{1}{25}\left(3e^t + 5te^t - 3\cos 2t + 4\sin 2t\right)
\end{aligned}$$

Thus the solution is

$$y(t) = \frac{28}{25}e^t + \left(a - \frac{4}{5}\right)te^t - \frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t$$

However, we still need to solve for a. Using $y(\pi) = 1$ we have
$1 = \frac{28}{25}e^{\pi} + \left(a - \frac{4}{5}\right)\pi e^{\pi} - \frac{3}{25}\cos 2\pi - \frac{4}{25}\sin 2\pi$
Hence $\left(a - \frac{4}{5}\right) = \frac{28}{25\pi e^{\pi}} - \frac{28}{25\pi}$. Therefore

$$y(t) = \frac{28}{25}e^t + \left(\frac{28}{25\pi e^{\pi}} - \frac{28}{25\pi}\right)te^t - \frac{3}{25}\cos 2t - \frac{4}{25}\sin 2t$$

(e) $y'' + 2y' + 7y = e^t \sin t$.

Solution:
We are finding the general solution. First finding the homogeneous: The auxiliary equation is

$$m^2 + 2m + 7 = 0 \Rightarrow m = -1 \pm \sqrt{6}\,i.$$

Thus the homogenous solution is

$$y_h = c_1 e^{-t}\cos\sqrt{6}t + c_2 e^{-t}\sin\sqrt{6}t$$

For a particular solution, we can choose the initial conditions, so we use
$y'' + 2y' + 7y = e^t \sin t;$, $y(0) = 0$, $y'(0) = 0$.
Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 7Y(s) = \frac{1}{(s-1)^2 + 1}$$

Plugging in the initial values and simplifying yields

$$(s^2 + 2s + 7)Y(s) = \frac{1}{(s-1)^2 + 1} \Rightarrow Y(s) = \frac{1}{(s^2 + 2s + 7)(s^2 - 2s + 2)}.$$

We require some partial fractions.

$$\begin{aligned}\frac{1}{(s^2+2s+7)(s^2-2s+2)} &= \frac{As+B}{s^2+2s+7} + \frac{Cs+D}{s^2-2s+2} \\ &= \frac{(As+B)(s^2-2s+2)+(Cs+D)(s^2+2s+7)}{(s^2+2s+7)(s^2-2s+2)}\end{aligned}$$

Therefore

$$\begin{aligned}1 &= A(s^3-2s^2+2s)+B(s^2-2s+2)+C(s^3+2s^2+7s)+D(s^2+2s+7) \\ &= s^3(A+C)+s^2 -2A+B+2C+D+s(2A-2B+7C+2D)+(2B+7D)\end{aligned}$$

Therefore

$$\begin{aligned}0 &= A+C \\ 0 &= -2A+B+2C+D \\ 0 &= 2A-2B+7C+2D \\ 1 &= 2B+7D\end{aligned}$$

which has solutions

$$A = \frac{4}{97} B = \frac{3}{97},\ C = -\frac{4}{97},\ D = \frac{13}{97}.$$

Hence the inverse Laplace transform is

$$\begin{aligned}&\frac{1}{97}\mathcal{L}^{-1}\left(\frac{4s+3}{s^2+2s+7} + \frac{-4s+13}{s^2-2s+2}\right) \\ &\frac{1}{97}\mathcal{L}^{-1}\left(\frac{4(s+1)-1}{(s+1)^2+6} + \frac{-4(s-1)+9}{(s-1)^2+1}\right) \\ &= \frac{1}{97}\left(e^{-t}\left(4\cos\sqrt{6}t - \frac{1}{\sqrt{6}}\sin\sqrt{6}t\right) + e^t\left(-4\cos t + 9\sin t\right)\right)\end{aligned}$$

Thus the solution is

$$\begin{aligned}y(t) &= c_1 e^{-t}\cos\sqrt{6}t + c_2 e^{-t}\sin\sqrt{6}t \\ &\quad + \frac{1}{97}\left(e^{-t}\left(4\cos\sqrt{6}t - \frac{1}{\sqrt{6}}\sin\sqrt{6}t\right) + e^t\left(-4\cos t + 9\sin t\right)\right) \\ &= C_1 e^{-t}\cos\sqrt{6}t + C_2 e^{-t}\sin\sqrt{6}t + e^t(-4\cos t + 9\sin t)\end{aligned}$$

2. A 200 gram mass is at rest on the end of a spring with constant 10 Newtons per metre. At $t = 0$, a force $f(t) = 4\sin 10t$ begins to act on the mass. Find the displacement of the mass as a function of time.

Solution:

The differential equation is

$$0.2y'' + 10y = 4\sin 10t \Rightarrow y'' + 50y = 20\sin 10t.$$

Since the spring is at rest and given no displacement, $y(0) = y'(0) = 0$.

Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 50Y(s) = \frac{20}{s^2+100}$$

Plugging in the initial values and simplifying yields

$$(s^2+50)Y(s) = \frac{20}{s^2+100} \Rightarrow Y(s) = \frac{20}{(s^2+100)(s^2+50)}.$$

We require some partial fractions.

$$\begin{aligned}\frac{20}{(s^2+100)(s^2+50)} &= \frac{As+B}{s^2+100} + \frac{Cs+D}{s^2+50} \\ &= \frac{(As+B)(s^2+50)+(Cs+D)(s^2+100)}{(s^2+100)(s^2+50)}\end{aligned}$$

Therefore

$$\begin{aligned}20 &= A(s^3+50s) + B(s^2+50) + C(s^3+100s) + D(s^2+50) \\ &= s^3(A+C) + s^2B + D + s(50A+100C) + (50B+100D)\end{aligned}$$

Therefore

$$\begin{aligned}0 &= A + C \\ 0 &= B + D \\ 0 &= 50A + 100C \\ 20 &= 50B + 100D\end{aligned}$$

which has solutions

$$A = C = 0B = -\frac{2}{5},\ D = \frac{2}{5}.$$

Hence the inverse Laplace transform is

$$\begin{aligned}&\frac{2}{5}\mathcal{L}^{-1}\left(\frac{-1}{s^2+100} + \frac{1}{s^2+50}\right)\\ &= \frac{2}{5}\left(-\frac{1}{10}\sin 10t + \frac{1}{\sqrt{50}}\sin\sqrt{50}t\right)\text{m.}\end{aligned}$$

3. A 100 gram mass is suspended from a spring with constant 25 Newtons per metre. At $t = 0$, the mass is lifted 10 centimetres above its rest position and given velocity 2 metres per second downward. During its subsequent motion, damping equal to the velocity also acts on the mass. In addition after 4 seconds, a constant force of 50 Newtons acts vertically upwards on the mass. Find the displacement of the mass as a function of time.

Solution:

The differential equation is

$$0.1y'' + y' + 25y = 50h(t-4) \Rightarrow y'' + 10y' + 250y = 500h(t-4).$$

$y(0) = 0.1$, $y'(0) = -2$.

Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 10(sY(s) - y(0)) + 250Y(s) = \frac{500e^{-4s}}{s}$$

Plugging in the initial values and simplifying yields

$$(s^2+10s+250)Y(s) - 0.1s + 2 - 1 = \frac{500e^{-4s}}{s} \Rightarrow Y(s) = \frac{0.1(s+5) - 1.5}{(s+5)^2 + 225} + \frac{500e^{-4s}}{s(s^2+10s+250)}.$$

The first two terms yields

$e^{-5t}(0.1\cos 15t - 0.1\sin 15t)$

For the last term require some partial fractions to find $f(t)$ which is the Laplace transform of $F(s) = \dfrac{500}{s(s^2+10s+250)}$

$$\begin{aligned}\frac{500}{s(s^2+10s+250)} &= \frac{A}{s} + \frac{Bs+C}{s^2+10s+250} \\ &= \frac{As^2+10As+250A+Bs^2+Cs}{s(s^2+10s+250)}\end{aligned}$$

Therefore

$$500 = s^2(A+B) + s(C+10A) + (250A)$$

Hence $A = 2 \Rightarrow B = -2$, $C = -20$.

Hence

$$\begin{aligned}f(t) &= \mathcal{L}^{-1}\left(\frac{2}{s} - \frac{2s+20}{s^2+10s+250}\right) \\ &= \mathcal{L}^{-1}\left(\frac{2}{s} - \frac{2(s+5)+10}{(s+5)^2+225}\right) \\ &= 2 - e^{-5t}\left(2\cos 15t + \frac{2}{3}\sin 15t\right)\end{aligned}$$

Hence the displacement is

$$e^{-5t}(0.1\cos 15t - 0.1\sin 15t) + h(t-4)\left(2 - e^{-5(t-4)}\left(2\cos 15(t-4) + \frac{2}{3}\sin 15(t-4)\right)\right)$$

4. Do the previous example except change the force to an instantaneous impulse force.

Solution:

The differential equation is

$$0.1y'' + y' + 25y = 50\delta(t-4) \Rightarrow y'' + 10y' + 250y = 500h(t-4).$$

$y(0) = 0.1$, $y'(0) = -2$.

Taking the Laplace Transform yields

$$s^2Y(s) - sy(0) - y'(0) + 10(sY(s) - y(0)) + 250Y(s) = 500e^{-4s}$$

Plugging in the initial values and simplifying yields

$$(s^2+10s+250)Y(s)-0.1s+2-1 = \frac{500e^{-4s}}{s} \Rightarrow Y(s) = \frac{0.1(s+5) - 1.5}{(s+5)^2 + 225} + \frac{500e^{-4s}}{s^2 + 10s + 250)}.$$

The first two terms yields

$e^{-5t}(0.1\cos 15t - 0.1\sin 15t)$

For the last term require some partial fractions to find $f(t)$ which is the Laplace transform of

$$F(s) = \frac{500}{(s+5)^2 + 225)}$$

Therefore $f(t) = \dfrac{500}{15}e^{-5t}\sin 15t = \dfrac{100}{3}e^{-5t}\sin 15t$

Hence the displacement is

$$e^{-5t}(0.1\cos 15t - 0.1\sin 15t) + h(t-4)\left(\frac{100}{3}e^{-5(t-4)}\sin 15(t-4)\right))$$