Question 1. *Prove using mathematical induction that for all* $n \geq 1$,

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Solution.

For any integer $n \geq 1$, let P_n be the statement that

$$1 + 4 + 7 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}.$$

Base Case. The statement P_1 says that

$$1 = \frac{1(3-1)}{2},$$

which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1 + 4 + 7 + \cdots + (3k - 2) = \frac{k(3k-1)}{2}.$$

It remains to show that P_{k+1} holds, that is,

$$1 + 4 + 7 + \cdots + (3(k+1) - 2) = \frac{(k+1)(3(k+1)-1)}{2}.$$

$$\begin{aligned}
1 + 4 + 7 + \cdots + (3(k+1) - 2) &= 1 + 4 + 7 + \cdots + (3(k+1) - 2) \\
&= 1 + 4 + 7 + \cdots + (3k + 1) \\
&= 1 + 4 + 7 + \cdots + (3k - 2) + (3k + 1) \\
&= \frac{k(3k-1)}{2} + (3k+1) \\
&= \frac{k(3k-1) + 2(3k+1)}{2} \\
&= \frac{3k^2 - k + 6k + 2)}{2} \\
&= \frac{3k^2 + 5k + 2)}{2} \\
&= \frac{(k+1)(3k+2)}{2} \\
&= \frac{(k+1)(3(k+1)-1)}{2}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

Question 2. *Use the Principle of Mathematical Induction to verify that, for n any positive integer, $6^n - 1$ is divisible by 5.*

Solution.

For any $n \geq 1$, let P_n be the statement that $6^n - 1$ is divisible by 5.

Base Case. The statement P_1 says that

$$6^1 - 1 = 6 - 1 = 5$$

is divisible by 5, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is, $6^k - 1$ is divisible by 5.

It remains to show that P_{k+1} holds, that is, that $6^{k+1} - 1$ is divisible by 5.

$$\begin{aligned} 6^{k+1} - 1 &= 6(6^k) - 1 \\ &= 6(6^k - 1) - 1 + 6 \\ &= 6(6^k - 1) + 5. \end{aligned}$$

By P_k, the first term $6(6^k - 1)$ is divisible by 5, the second term is clearly divisible by 5. Therefore the left hand side is also divisible by 5. Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Question 3. *Verify that for all $n \geq 1$, the sum of the squares of the first $2n$ positive integers is given by the formula*

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Solution.

For any integer $n \geq 1$, let P_n be the statement that

$$1^2 + 2^2 + 3^2 + \cdots + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}.$$

<u>Base Case.</u> The statement P_1 says that

$$1^2 + 2^2 = \frac{(1)(2(1)+1)(4(1)+1)}{3} = \frac{3(5)}{3} = 5,$$

which is true.

<u>Inductive Step.</u> Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}.$$

It remains to show that P_{k+1} holds, that is,

$$1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 = \frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3}.$$

$$\begin{aligned}
1^2 + 2^2 + 3^2 + \cdots + (2(k+1))^2 &= 1^2 + 2^2 + 3^2 + \cdots + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + \cdots + (2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 && \text{(by } P_k\text{)} \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{k(8k^2+6k+1) + 3(4k^2+4k+1) + 3(4k^2+8k+4)}{3} \\
&= \frac{(8k^3+6k^2+k) + (12k^2+12k+3) + (12k^2+24k+12)}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3}
\end{aligned}$$

On the other side of P_{k+1},

$$\frac{(k+1)(2(k+1)+1)(4(k+1)+1)}{3} = \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$$

$$
\begin{aligned}
&= \frac{(k+1)(2k+3)(4k+5)}{3} \\
&= \frac{(2k^2+5k+3)(4k+5)}{3} \\
&= \frac{8k^3+30k^2+37k+15}{3}.
\end{aligned}
$$

Therefore P_{k+1} holds.

Thus, by the principle of mathematical induction, for all $n \geq 1$, P_n holds. $\square$

Question 4. *Consider the sequence of real numbers defined by the relations*

$$x_1 = 1 \text{ and } x_{n+1} = \sqrt{1 + 2x_n} \text{ for } n \geq 1.$$

Use the Principle of Mathematical Induction to show that $x_n < 4$ *for all* $n \geq 1$*.*

Solution.

For any $n \geq 1$, let P_n be the statement that $x_n < 4$.

Base Case. The statement P_1 says that $x_1 = 1 < 4$, which is true.

Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is, $x_k < 4$.

It remains to show that P_{k+1} holds, that is, that $x_{k+1} < 4$.

$$\begin{aligned} x_{k+1} &= \sqrt{1 + 2x_k} \\ &< \sqrt{1 + 2(4)} \\ &= \sqrt{9} \\ &= 3 \\ &< 4. \end{aligned}$$

Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Question 5. *Show that* $n! > 3^n$ *for* $n \geq 7$.

Solution.

For any $n \geq 7$, let P_n be the statement that $n! > 3^n$.

Base Case. The statement P_7 says that $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040 > 3^7 = 2187$, which is true.

Inductive Step. Fix $k \geq 7$, and suppose that P_k holds, that is, $k! > 3^k$.

It remains to show that P_{k+1} holds, that is, that $(k+1)! > 3^{k+1}$.

$$\begin{aligned}
(k+1)! &= (k+1)k! \\
&> (k+1)3^k \\
&\geq (7+1)3^k \\
&= 8 \times 3^k \\
&> 3 \times 3^k \\
&= 3^{k+1}.
\end{aligned}$$

Therefore P_{k+1} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Question 6. *Let $p_0 = 1$, $p_1 = \cos\theta$ (for θ some fixed constant) and $p_{n+1} = 2p_1p_n - p_{n-1}$ for $n \geq 1$. Use an extended Principle of Mathematical Induction to prove that $p_n = \cos(n\theta)$ for $n \geq 0$.*

Solution.

For any $n \geq 0$, let P_n be the statement that $p_n = \cos(n\theta)$.

Base Cases. The statement P_0 says that $p_0 = 1 = \cos(0\theta) = 1$, which is true. The statement P_1 says that $p_1 = \cos\theta = \cos(1\theta)$, which is true.

Inductive Step. Fix $k \geq 0$, and suppose that both P_k and P_{k+1} hold, that is, $p_k = \cos(k\theta)$, and $p_{k+1} = \cos((k+1)\theta)$.

It remains to show that P_{k+2} holds, that is, that $p_{k+2} = \cos((k+2)\theta)$.

We have the following identities:

$$\cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$\cos(a-b) = \cos a \cos b + \sin a \sin b$$

Therefore, using the first identity when $a = \theta$ and $b = (k+1)\theta$, we have

$$\cos(\theta + (k+1)\theta) = \cos\theta\cos(k+1)\theta - \sin\theta\sin(k+1)\theta,$$

and using the second identity when $a = (k+1)\theta$ and $b = \theta$, we have

$$\cos((k+1)\theta - \theta) = \cos(k+1)\theta\cos\theta + \sin(k+1)\theta\sin\theta.$$

Therefore,

$$\begin{aligned}
p_{k+2} &= 2p_1p_{k+1} - p_k \\
&= 2(\cos\theta)(\cos((k+1)\theta)) - \cos(k\theta) \\
&= (\cos\theta)(\cos((k+1)\theta)) + (\cos\theta)(\cos((k+1)\theta)) - \cos(k\theta) \\
&= \cos(\theta + (k+1)\theta) + \sin\theta\sin(k+1)\theta + (\cos\theta)(\cos((k+1)\theta)) - \cos(k\theta) \\
&= \cos((k+2)\theta) + \sin\theta\sin(k+1)\theta + (\cos\theta)(\cos((k+1)\theta)) - \cos(k\theta) \\
&= \cos((k+2)\theta) + \sin\theta\sin(k+1)\theta + \cos((k+1)\theta - \theta) - \sin(k+1)\theta\sin\theta - \cos(k\theta) \\
&= \cos((k+2)\theta) + \cos(k\theta) - \cos(k\theta) \\
&= \cos((k+2)\theta).
\end{aligned}$$

Therefore P_{k+2} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

Question 7. *Consider the famous Fibonacci sequence $\{x_n\}_{n=1}^{\infty}$, defined by the relations $x_1 = 1$, $x_2 = 1$, and $x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$.*

(a) Compute x_{20}.

(b) Use an extended Principle of Mathematical Induction in order to show that for $n \geq 1$,

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

(c) Use the result of part (b) to compute x_{20}.

Solution.

(a)

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765$$

(b) For any $n \geq 1$, let P_n be the statement that

$$x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right].$$

Base Case. The statement P_1 says that

$$\begin{aligned} x_1 &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] \\ &= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right] \\ &= \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}-1+\sqrt{5}}{2}\right] \\ &= \frac{1}{\sqrt{5}}\left[\frac{2\sqrt{5}}{2}\right] \\ &= 1, \end{aligned}$$

which is true. The statement P_2 says that

$$x_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+2\sqrt{5}+5}{4}\right)-\left(\frac{1-2\sqrt{5}+5}{4}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+2\sqrt{5}+5-1+2\sqrt{5}-5}{4}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] \\
&= 1,
\end{aligned}
$$

which is again true.

<u>Inductive Step.</u> Fix $k \geq 1$, and suppose that P_k and P_{k+1} both hold, that is,

$$x_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right],$$

and

$$x_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$$

It remains to show that P_{k+2} holds, that is, that

$$x_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right].$$

$$
\begin{aligned}
x_{k+2} &= x_k + x_{k+1} \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^k\right]+\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k+\left(\frac{1+\sqrt{5}}{2}\right)^{k+1}-\left(\frac{1-\sqrt{5}}{2}\right)^k-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(1+\frac{1+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(1+\frac{1-\sqrt{5}}{2}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{3+\sqrt{5}}{2}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{3-\sqrt{5}}{2}\right)\right] \\
&= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{6+2\sqrt{5}}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{6-2\sqrt{5}}{4}\right)\right]
\end{aligned}
$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+2\sqrt{5}+5}{4}\right)-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1-2\sqrt{5}+5}{4}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k\left(\frac{1+\sqrt{5}}{2}\right)^2-\left(\frac{1-\sqrt{5}}{2}\right)^k\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right].$$

Therefore P_{k+2} holds.

Thus by the principle of mathematical induction, for all $n \geq 1$, P_n holds.

(c) Plugging $n = 20$ in a calculator yields the answer quickly.