MATH2132 Test 3 - Collection

Student Name -

Student Number -

Values

14 **1.** (a) Find a 2-parameter family of solutions of the differential equation

$$xy'' - 3y' = x^5.$$

(b) Can there be any singular solutions to your family of solutions in part (a)? Explain.

ans)

a) $y = \frac{x^6}{12} + \frac{C}{4}x^4 + D$

b) NO, DE is linear

7 **4.** (a) You are given that a general solution of the homogeneous equation associated with the linear, constant coefficient differential equation

$$\phi(D)y = x^2 + e^x \cos 2x$$

is

$$y_h(x) = C_1 + C_2 x + C_3 \cos 2x + C_4 \sin 2x.$$

What is the differential equation?

(b) What is the form for a particular solution of the differential equation as predicted by the method of undetermined coefficients? Do **NOT** evaluate the coefficients.

ans)

a) $y^{(IV)} + 4y'' = x^2 + e^x \cos 2x$

b) $y_p(x) = Ax^4 + Bx^3 + Cx^2 + e^x(D\cos 2x + E\sin 2x)$

Values

10 **1.** Find a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 5y = 2x - e^{-x}.$$

Ans) $y(x) = e^{-x}(C_1 \cos 4x + C_2 \sin 4x) + \frac{2}{5}x - \frac{4}{25} - \frac{1}{4}e^{-x}$

8 **2.** The roots of the auxiliary equation associated with the differential equation

$$\phi(D)y = x^2 e^{3x} + \cos 4x + x^2$$

are

$$m = 2 \pm \sqrt{7}, 3, 3, 3, 4 \pm \sqrt{6}i, 0.$$

What is the form of a particular solution of the differential equation? Eliminate all unnecesary terms. Do **NOT** attempt to determine the unknown coefficients.

ans) $y_p(x) = (Ax^5 + Bx^4 + Cx^3)e^{3x} + D\cos 4x + E\sin 4x + Fx^3 + Gx^2 + Hx$

14 **2.** Two substances A and B react to form a third substance C in such a way that 1 gram of A reacts with 1 gram of B to produce 2 grams of C. The rate at which C is formed is proportional to the product of the amounts of A and B present in the mixture. If 10 grams of A and 10 grams of B are originally brought together at time $t = 0$, find the amount of C present in the mixture as a function of time.

ans) $C(t) = \frac{100kt}{1 + 5kt}$

7 **3.** Let $\phi(m) = 0$ be the auxiliary equation associated with the differential equation $\phi(D)y = 0$. It is known that

$$\phi(m) = (m+1)(m-7)^3(m^2 - 4m + 13)^2.$$

What is a general solution of the differential equation?

ans) $y(x) = C_1 e^{-x} + (C_2 + C_3 x + C_4 x^2)e^{7x} + [(C_5 + C_6 x)\cos 3x + (C_7 + C_8 x)\sin 3x]$

14 **3.** Find a general solution of the differential equation

$$3y''' + 5y'' + 4y' - 2y = 2e^x + x.$$

ans) $y(x) = C_1 e^{x/3} + e^{-x}(C_2 \cos x + C_3 \sin x) + \frac{1}{5}e^x - \frac{x}{2} - 1$

15 **4.** Find a general solution of the differential equation

$$y'' - 4y' - 5y = 8xe^x.$$

ans) $y(x) = C_1 e^{-x} + C_2 e^{5x} - xe^x + \frac{1}{4}e^x$

10 **4.** (a) A 500 gram mass is suspended from a spring with constant 50 newtons per metre. It is set into motion by releasing it from a position 10 centimetres above its equilibrium position. If a damping force proportional to velocity with coefficient $\beta = 2$ acts on the mass, find its position as a function of time.

(b) If your solution is expressed in the form $Ae^{-\beta t}\sin(\omega t + \phi)$, where A and ω are constants, what is A?

ans) a) $x(t) = e^{-2t}\left(\frac{1}{10}\cos 4\sqrt{6}t + \frac{1}{20\sqrt{6}}\sin 4\sqrt{6}t\right)$ b) $\omega = 4\sqrt{6}$, $A = \left[\left(\frac{1}{10}\right)^2 + \left(\frac{1}{20\sqrt{6}}\right)^2\right]^{1/2}$

G1) Given that the roots of the auxiliary equation $\varphi(m)=0$ associated with the linear differential equation

$$\varphi(D)y = 2e^{3x} + e^{4x}\cos x + 7x^2$$

are $m = 1,\ \pm 3,\ \pm 3,\ 4 \pm 2i$, what is the form of a particular solution $y_p(x)$?
DO NOT EVALUATE THE COEFFICIENTS IN $y_p(x)$.

G2) Find all solutions of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 3\sin x - x^2.$$

G3) The roots of the auxiliary equation $\phi(m) = 0$ associated with the linear differential equation

$$\phi(D)y = x^2e^x + \sin x$$

are $m = 1, 1, -1 \pm i, -3$. What form would you assume for a particular solution of the differential equation. Do NOT attempt to evaluate the coefficients in your particular solution.

G4) Given that $y(x) = C_1e^x + C_2\cos 3x + C_3\sin 3x$ is the general solution of the homogeneous differential equation associated with

$$y''' - y'' + 9y' - 9y = 2e^{-x} + 4x,$$

find a particular solution, and hence a general solution of this differential equation.

G5) Find a general solution for the differential equation

$$y'' + 2y' + 3y = xe^{-x} + 2x.$$

Answers.

G1) $y_p(x) = Ax^2e^{3x} + B + Cx + Dx^2 + e^{4x}(G\cos x + F\sin x)$

G2) $y(x) = e^{-x}(c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x) + \frac{9}{13}\sin x - \frac{6}{13}\cos x - \frac{x^2}{4} + \frac{x}{4}$

G3) $y_p(x) = Ax^4e^x + Bx^3e^x + Cx^2e^x + D\sin x + E\cos x$

G4) $y(x) = c_1e^x + c_2\cos 3x + c_3\sin 3x - \frac{1}{10}e^{-x} - \frac{4}{9}(x+1)$

G5) $y(x) = e^{-x}(c_1\cos\sqrt{2}x + c_2\sin\sqrt{2}x) + \frac{x}{2}e^{-x} + \frac{2}{3}x - \frac{4}{9}$

H1) Set up (BUT *DO NOT SOLVE*) an initial-value problem to determine the amount of salt $x(t)$ at time t in the tank described below.

The tank initially contains 1000 litres of water into which 3 kg. of salt have been dissolved. Brine, containing 4 kg. of salt per litre, is pumped into the tank at a constant rate of 3 litres per minute. At the same time, the well-mixed brine in the tank is removed at a constant rate of $\frac{5}{2}$ litres per minute. You may assume that the tank is sufficiently large that we do not have to concern ourselves with the possibility of overflow.

REMEMBER: IT IS NOT NECESSARY TO SOLVE THIS INITIAL-VALUE PROBLEM.

H2) A tank originally contains 1000 L of water in which 5 kg of salt have been dissolved. A mixture containing 2 kg of salt for each 100 L of solution is added to the tank at 10 mL/s. At the same time, the well-stirred mixture in the tank is removed at the rate of 5 mL/s.

(a) Show that the initial-value problem for the number of grams $S(t)$ of salt in the tank at any given time is

$$\frac{dS}{dt} = \frac{1}{5} - \frac{5S}{10^6 + 5t}, \qquad S(0) = 5000.$$

(b) Solve the problem in (a) for $S(t)$.

H3)

At time $t = 0$, a cup of coffee at temperature 95°C is placed in a room whose temperature as a function of time is $f(t)$. Newton's law of cooling states that the time rate of change of the temperature of the coffee is proportional to the difference between the temperature of the coffee and room temperature.

(a) Show that temperature $T(t)$ of the coffee must satisfy an initial value problem of the form

$$\frac{dT}{dt} - kT = -kf(t), \quad T(0) = 95,$$

where k is a constant. Is k positive or negative? Explain.

(b) Show that the general solution of the differential equation (but not the initial condition) can be expressed in the form

$$T(t) = -ke^{kt}\int e^{-kt}f(t)\,dt + Ce^{kt}.$$

(c) Find $T(t)$ when $f(t)$ is a constant 15°C.

Answers

H1) $\frac{dx}{dt} = 12 - \frac{x}{1000 + t/2}$, $x(0) = 3$

H2) a) proof → Hint: use the fact → $\begin{pmatrix}\text{rate of change}\\ \text{of Salt}\end{pmatrix} = \begin{pmatrix}\text{rate of Sail}\\ \text{Coming in}\end{pmatrix} - \begin{pmatrix}\text{rate of Salt}\\ \text{Coming out}\end{pmatrix}$

b) $S(t) = \frac{1}{50}(10^6 + 5t) - \frac{15 \times 10^9}{10^6 + 5t}$ (grams)

H3) a) proof
b) proof

c) $T(t) = 15 + 80e^{kt}$