MATH 1210 Assignment 4

March 21, 2014

Due: March 28, 2014, in class

Question 1. Consider the following system of linear equations:

$$\begin{cases} w - x + 2y - 3z = 0 \\ 3w - 3x + 8y - 5z = 0 \\ 2w - 2x + 5y - 4z = 0 \\ 3w - 3x + 7y - 7z = 0 \end{cases}$$

(a) Find the reduced row-echelon form of the augmented matrix of this system.

Solution: (For a homogeneous system, we don't need to write down a column of constants in the augmented matrix since it can never change from 0.)

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 3 & -3 & 8 & -5 \\ 2 & -2 & 5 & -4 \\ 3 & -3 & 7 & -7 \end{pmatrix} \quad \begin{array}{l} \\ R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad R_2 \to \tfrac{1}{2}R_2$$

$$\begin{pmatrix} 1 & -1 & 2 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array}$$

$$\begin{pmatrix} 1 & -1 & 0 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(b) Find all the basic solutions of this system.

Solution: The variables x and z become parameters, and we write the other variables in terms of them:

$$\begin{cases} w = s + 7t \\ x = s \\ y = -2t \\ z = t \end{cases}$$

We could, if we wanted, write this in vector form, but this is not required for the solution of this problem:

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 7 \\ 0 \\ -2 \\ 1 \end{pmatrix}$$

The basic solutions are the columns of coefficients of the parameters: $\langle 1, 1, 0, 0 \rangle$ and $\langle 7, 0, -2, 1 \rangle$, which we see written out explicitly in the vector form of the solution.

(c) Use part (b) to find a solution in which $w = -4$ and $y = 4$.

Solution: If $y = 4$ then since $y = -2t$, $t = -2$. Then $-4 = w = s + 7t = s - 14$, so $s = 10$.

Then the required solution is $\langle -4, 10, 4, -2 \rangle$.

Question 2. Suppose that $AX = B$ is the matrix representation of a system of linear equations.

(a) Suppose that Y is a solution of this system, and that Z_1 and Z_2 are solutions of the associated homogeneous system. Show that $Y - 17Z_1 + 16Z_2$ is also a solution of $AX = B$.

Solution:

$$\begin{aligned} A(Y - 17Z_1 + 16Z_2) &= AY - 17AZ_1 + 16AZ_2 \\ &= B - 17\mathbf{0} + 16\mathbf{0} \\ &= B \end{aligned}$$

(b) Suppose that Y_1 and Y_2 are solutions of $AX = B$. Show that $2Y_1 - Y_2$ is also a solution of $AX = B$.

Solution:

$$\begin{aligned} A(2Y_1 - Y_2) &= 2AY_1 - AY_2 \\ &= 2B - B \\ &= B \end{aligned}$$

Question 3. Let $A = \begin{pmatrix} 1 & 2 & c \\ 3 & 4c & 12 \\ c & -1 & 2 \end{pmatrix}$ and let X be the column vector of variables x, y, and z. For which values of c, if any, does the system $AX = \mathbf{0}$ have non-trivial solutions?

Solution: [The solution which we hoped to see] The system has non-trivial solutions if $|A| = 0$.

So we calculate $|A|$ by any method, for instance

$$\begin{vmatrix} 1 & 2 & c \\ 3 & 4c & 12 \\ c & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & c \\ 0 & 4c-6 & 12-3c \\ 0 & -1-2c & 2-c^2 \end{vmatrix} = \begin{vmatrix} 4c-6 & 12-3c \\ -1-2c & 2-c^2 \end{vmatrix} =$$

$$(4c-6)(2-c^2) - (12-3c)(-1-2c) = -4c^3 + 29c = -c(4c^2-29)\,,$$

so $|A| = 0$ when $c = 0$ or $c = \pm\frac{\sqrt{29}}{2}$.

Solution: [A more difficult method] Use row reduction of the augmented matrix for $AX = \mathbf{0}$ to determine the values of c for which the row-echelon form has a row of zeroes.

Question 4. Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & -5 & 3 \\ -3 & 2 & 0 \\ 1 & -3 & 2 \end{pmatrix}$$

Evaluate the determinant of A in two ways:

(a) by expansion along column 3;

Solution:

$$|\,A\,| = 3\begin{vmatrix} -3 & 2 \\ 1 & -3 \end{vmatrix} + 2\begin{vmatrix} 2 & -5 \\ -3 & 2 \end{vmatrix} = 3(9-2) + 2(4-15) = -1$$

(b) by row reduction.

Comment: There is no hard-and-fast rule that says that you have to do row reductions in a particular order when solving determinants, so there are quite a few correct pathways to the answer. The goal is to get a triangular (upper or lower) matrix, since the determinant of a triangular matrix is just the product of the diagonal entries. We give only one version here. Our strategy is to avoid fractions as much as possible, so we start with a row interchange.

Solution:

$$|\,A\,| = \begin{vmatrix} 2 & -5 & 3 \\ -3 & 2 & 0 \\ 1 & -3 & 2 \end{vmatrix} \quad (R_1 \leftrightarrow R_3) \quad = -\begin{vmatrix} 1 & -3 & 2 \\ -3 & 2 & 0 \\ 2 & -5 & 3 \end{vmatrix} \quad \begin{matrix} \\ (R_2 \to R_2 + 3R_1) \\ (R_3 \to R_2 - 2R_1) \end{matrix}$$

$$= -\begin{vmatrix} 1 & -3 & 2 \\ 0 & -7 & 6 \\ 0 & 1 & -1 \end{vmatrix} \quad (R_2 \leftrightarrow R_3) \; = \begin{vmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & -7 & 6 \end{vmatrix} \quad R_3 \to R_3 + 7R_2$$

$$= \begin{vmatrix} 1 & -3 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{vmatrix} = (1)(1)(-1) = -1$$

Note that we introduce a minus sign twice, once at each row interchange. Of course they cancel each other out.

Question 5. For $n \geq 1$, let H_n be the $n \times n$ matrix with (i,j)-entry $\dfrac{1}{i+j-1}$.

Showing all your work carefully, evaluate $|H_3|$ by any method.

Solution: Evaluation by row-reduction is no worse than any other method.

$$|H_3| = \begin{vmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{vmatrix} \quad \begin{matrix} (R_2 \to R_2 - \frac{1}{2}R_1) \\ (R_3 \to R_3 - \frac{1}{3}R_1) \end{matrix} = \begin{vmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 4/45 \end{vmatrix} \quad (R_3 \to R_3 - R_2)$$

$$= \begin{vmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 0 & 1/180 \end{vmatrix} = (1)\left(\frac{1}{12}\right)\left(\frac{1}{180}\right) = \frac{1}{2160}$$

Question 6. Suppose that A, B, and C are 5×5 matrices with $|A| = -2$ and $|B| = 3$.

Comments for the solutions The basic facts being tested here are (for M, M' $n \times n$ matrices):

$$|cM| = c^n |M|, \qquad |M^T| = |M|, \qquad |MM'| = |M||M'|$$

(a) Find $\left|\frac{1}{6}B^3A^5\right|$.

Solution: $\left|\frac{1}{6}B^3A^5\right| = \left(\frac{1}{6}\right)^5 |B|^3 |A^5| = \frac{1}{2^5 3^5} 3^3(-2)^5 = -\frac{1}{9}$.

(b) If $\left|A^3C^2B^T\right| = -96$, find $|C|$.

Solution: $-96 = \left|A^3C^2B^T\right| = |A|^3 |C|^2 \left|B^T\right| = (-8)|C|^2 3$, so $|C|^2 = \frac{-96}{-24} = 4$ and therefore $|C| = \pm 2$.

Question 7. Use Cramer's rule to find the value of z such that

$$\begin{cases} 2x + 3y & = & 3 \\ x - 2y - 5z & = & -2 \\ 4x - y + z & = & 1 \end{cases}$$

Solution: The determinant of the matrix of coefficients is

$$\begin{vmatrix} 2 & 3 & 0 \\ 1 & -2 & -5 \\ 4 & -1 & 1 \end{vmatrix} = 2\begin{vmatrix} -2 & -5 \\ -1 & 1 \end{vmatrix} - 3\begin{vmatrix} 1 & -5 \\ 4 & 1 \end{vmatrix} = 2(-2-5) - 3(1+20) = -77$$

The variable z corresponds to column 3 of this matrix, so we replace column 3 by the column of constants and calculate the determinant:

$$\begin{vmatrix} 2 & 3 & 3 \\ 1 & -2 & -2 \\ 4 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 3 & 0 \\ 1 & -2 & 0 \\ 4 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ 1 & -2 \end{vmatrix} = 2(-4-3) = -14$$

Therefore $z = \dfrac{-14}{-77} = \dfrac{2}{11}$.

Remark: You may use any method of evaluating the determinants that you wish. For the first one, we did a straightforward expansion by cofactors along row 1. For the second, we took advantage of the obvious similarity between column 2 and column 3 and used an elementary column operation. Then we expanded by cofactors along column 3. But there are many other equally direct ways of computing these two determinants.

Question 8.

(a) Determine whether the following set of vectors is linearly dependent or linearly independent, by the method of determinants.

$$\{ \langle 2, 3, -1 \rangle , \langle -1, 2, -10 \rangle , \langle 3, -1, 9 \rangle \}$$

Solution: We write the vectors as columns of a matrix, and then compute the determinant.

Again, we did not specify a method that you had to use, so you can compute the determinant by any method with which you are comfortable. We chose to first simplify column 1 by elementary row operations, then expand by cofactors along column 1.

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ -1 & -10 & 9 \end{vmatrix} = \begin{vmatrix} 0 & -21 & 24 \\ 0 & -28 & 26 \\ -1 & -10 & 9 \end{vmatrix} = (-1) \begin{vmatrix} -21 & 24 \\ -28 & 26 \end{vmatrix} = (-1)((-21)26-24(-21)) \neq 0,$$

which is enough: the vectors are linearly independent because the determinant is not 0.

(b) For what values (if any) of a are the following vectors linearly independent?

$$\{ \langle 1, 2, 2 \rangle , \langle 1, a, 2 \rangle , \langle 1, a, 1 \rangle \}$$

Solution: We use the same method as in part(a); this one is so simple we just expand along row 1.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & a & a \\ 2 & 2 & 1 \end{vmatrix} = 1 \begin{vmatrix} a & a \\ 2 & 1 \end{vmatrix} - 1 \begin{vmatrix} 2 & a \\ 2 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & a \\ 2 & 2 \end{vmatrix} = (a-2a)-(2-2a)+(4-2a) = -a+2$$

The vectors are linearly independent if the determinant is not 0, that is, if $a \neq 2$.

(c) Show that the following three vectors are linearly dependent, and write one as a linear combination of the other two.

$$\{ \mathbf{v}_1 = \langle 2, -1, 3, 4 \rangle , \mathbf{v}_2 = \langle -3, 2, -5, -7 \rangle , \mathbf{v}_3 = \langle 1, 1, 0, -1 \rangle \}$$

Solution: We can't solve this by determinants! Instead, we solve a system of linear equations to find the coefficients of a suitable linear combination. We try to find scalars c_1, c_2, and c_3, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$. Again, we put the vectors as the columns of a matrix, then row-reduce. We have left out the column of constants (all zero).

$$\begin{pmatrix} 2 & -3 & 1 \\ -1 & 2 & 1 \\ 3 & -5 & 0 \\ 4 & -7 & -1 \end{pmatrix} \quad R_1 \leftrightarrow R_2 \quad \begin{pmatrix} -1 & 2 & 1 \\ 2 & -3 & 1 \\ 3 & -5 & 0 \\ 4 & -7 & -1 \end{pmatrix} \quad \begin{array}{l} \\ R_2 \to R_2 + 2R_1 \\ R_3 \to R_3 + 3R_1 \\ R_4 \to R_4 + 4R_1 \end{array}$$

$$\begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{pmatrix} \quad \begin{array}{l} R_1 \to R_1 + 2R_2 \\ \\ R_3 \to R_3 - R_2 \\ R_4 \to R_4 - R_2 \end{array} \quad \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

So there is one parameter. To get a non-trivial solution, we set it to something different from 0, say 1, and we get $c_1 = -5, \quad c_2 = -3, \quad c_3 = 1$, and so

$$\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2 \,,$$

that is,

$$\langle\, 1, 1, 0, -1 \,\rangle = 5 \,\langle\, 2, -1, 3, 4 \,\rangle + 3 \,\langle\, -3, 2, -5, -7 \,\rangle$$

This is the "natural" solution, taking the vectors into the matrix in the same order as they were presented in the question, and taking advantage of a choice of parameter to make it easy to express the third vector as a combination of the first two, but there are many other correct paths to a solution, and of course, two other correct solutions:

$$\mathbf{v}_1 = -\frac{3}{5}\mathbf{v}_2 + \frac{1}{5}\mathbf{v}_3$$

$$\mathbf{v}_3 = -\frac{5}{3}\mathbf{v}_1 + \frac{1}{3}\mathbf{v}_2$$

Question 9.

(a) Show that if $\mathbf{u}_1$ and $\mathbf{u_2}$ are a pair of linearly independent vectors, then $\mathbf{v}_1 = \mathbf{u}_1 + \mathbf{u}_2$ and $\mathbf{v}_2 = \mathbf{u}_1 - \mathbf{u}_2$ are also linearly independent.

Solution: Suppose that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}$. To show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, it is enough to show that c_1 and c_2 are both 0. But all we have to do is rewrite this equation in terms of $\mathbf{u}_1$ and $\mathbf{u_2}$:

$$c_1(\mathbf{u}_1 + \mathbf{u}_2) + c_2(\mathbf{u}_1 - \mathbf{u}_2) = \mathbf{0}\,,$$

that is, (regrouping)

$$(c_1 + c_2)\mathbf{u}_1 + (c_1 - c_2)\mathbf{u}_2 = 0\,.$$

Since $\mathbf{u}_1$ and $\mathbf{u_2}$ are linearly independent, it follows that $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. But then it follows immediately from the second equation that $c_1 = c_2$, and so from the first equation $c_1 = c_2 = 0$. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

(b) Write each of $\mathbf{u}_1$ and $\mathbf{u}_2$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

Solution: We are asked to find constants d_1 and d_2 so that $\mathbf{u}_1 = d_1\mathbf{v}_1 + d_2\mathbf{v}_2$, (and a similar problem for $\mathbf{u}_2$). That is,

$$\mathbf{u}_1 = d_1(\mathbf{u}_1 + \mathbf{u}_2) + d_2(\mathbf{u}_1 - \mathbf{u}_2)\,.$$

Regrouping, we get

$$(d_1 + d_2 - 1)\mathbf{u}_1 + (d_1 - d_2)\mathbf{u}_2 = \mathbf{0}\,,$$

and since $\mathbf{u}_1$ and $\mathbf{u_2}$ are linearly independent, it follows that $d_1 + d_2 - 1 = 0$ and $d_1 - d_2 = 0$. From the second equation, $d_1 = d_2$, and so from the first, $d_1 = d_2 = 1/2$. Hence $\mathbf{u}_1 = \frac{1}{2}\mathbf{v}_1 + \frac{1}{2}\mathbf{v}_2$.

Working through the same calculations for $\mathbf{u}_2$ instead of $\mathbf{u}_1$, we get $\mathbf{u}_2 = \frac{1}{2}\mathbf{v}_1 - \frac{1}{2}\mathbf{v}_2$.