MAT2130: Engineering Mathematical Analysis 1

Final Exam Practice Problems – Selected Solutions

1. (a) The integral is over a semicircle of radius 1, lying in the half-plane where $y \geq 0$. We convert to polar coordinates.

$$\begin{aligned}
\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{4-x^2-y^2}\,dx\,dy = &\int_0^1 \int_0^\pi \left(\sqrt{4-r^2}\right) r\,d\theta\,dr \\
=&\pi \int_0^1 r\sqrt{4-r^2}\,dr \\
=&\pi \left[-\frac{1}{3}(4-r^2)^{3/2}\right]_{r=0}^1 \quad d\theta \\
=&\frac{\pi}{3}\left(8-3^{3/2}\right).
\end{aligned}$$

(b) The integral can be evaluated as written.

$$\begin{aligned}
\int_0^1 \int_{x^2}^{2-x^2} x\sqrt{y}\,dy\,dx = &\int_0^1 x\left[\frac{2}{3}y^{3/2}\right]_{y=x^2}^{2-x^2} \quad dx \\
=&\frac{2}{3}\int_0^1 \left[x(2-x^2)^{3/2}-x^4\right]\,dx \\
=&\frac{2}{3}\left[-\frac{1}{5}(2-x^2)^{5/2}-\frac{1}{5}x^5\right]_{x=0}^1 \\
=&\frac{2}{15}\left(2^{5/2}-2\right).
\end{aligned}$$

(c) We need to reverse the order of integration.

$$\begin{aligned}
\int_0^1 \int_{2x}^2 e^{y^2}\,dy\,dx = &\int_0^2 \int_0^{y/2} e^{y^2}\,dx\,dy \\
=&\int_0^2 \frac{y}{2}e^{y^2}\,dy \\
=&\left[\frac{1}{4}e^{y^2}\right]_{y=0}^2 \\
=&\frac{1}{4}\left(e^4-1\right).
\end{aligned}$$

(d) The integrals in x and y have to be evaluated using trig functions. First, we can separate x and y:

$$\int_0^1 \int_0^3 \frac{\sqrt{9-x^2}}{y^2+1}\,dx\,dy = \int_0^1 \frac{1}{y^2+1}\int_0^3 \sqrt{9-x^2}\,dx\,dy.$$

The innermost integral is

$$
\begin{aligned}
\int_0^3 \sqrt{9-x^2}\,dx =& \int_0^{\pi/2} 9\cos^2\theta\,d\theta \quad (\text{using } x = 3\sin\theta) \\
=& 9\int_0^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta \\
=& \frac{9\pi}{4}.
\end{aligned}
$$

(Alternatively, we can recognize this integral as the area of one quarter of a circle with radius 3.)

With this result, the remaining integral becomes

$$
\begin{aligned}
\frac{9\pi}{4}\int_0^1 \frac{1}{y^2+1}\,dy =& \frac{9\pi}{4}\left[\tan^{-1} y\right]_{y=0}^1 \\
=& \frac{9\pi}{4}\left(\frac{\pi}{4}-0\right) \\
=& \frac{9\pi^2}{16}.
\end{aligned}
$$

Alternatively, if we don't recognize the integrand $\frac{1}{y^2+1}$ as the derivative of $\tan^{-1} y$, we can use another trig substitution:

$$
\begin{aligned}
\int_0^1 \frac{1}{y^2+1}\,dy =& \int_0^{\pi/4} \frac{1}{1+\tan^2\theta}\sec^2\theta\,d\theta \quad (\text{using } y = \tan\theta) \\
=& \int_0^{\pi/4} d\theta \\
=& \frac{\pi}{4}.
\end{aligned}
$$

(e) We need to reverse the order of integration.

$$
\begin{aligned}
\int_2^9 \int_{\sqrt{y}}^3 \cos(x^3-6x)\,dx\,dy =& \int_{\sqrt{2}}^3 \int_2^{x^2} \cos(x^3-6x)\,dy\,dx \\
=& \int_{\sqrt{2}}^3 \left(x^2-2\right)\cos(x^3-6x)\,dx \\
=& \left[\frac{1}{3}\sin(x^3-6x)\right]_{x=\sqrt{2}}^3 \\
=& \frac{1}{3}\left(\sin(9)-\sin(2^{3/2}-6\sqrt{2})\right).
\end{aligned}
$$

(f) The integral is over the region in the first quadrant outside the circle $x^2+y^2=1$ and inside the circle $x^2+y^2=4$. We convert to polar coordinates.

$$\begin{aligned}\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx+\int_1^2\int_0^{\sqrt{4-x^2}}\frac{x}{\sqrt{x^2+y^2}}\,dy\,dx&=\int_1^2\int_0^{\pi/2}\frac{r\cos\theta}{r}r\,d\theta\,dr\\&=\int_1^2 r\int_0^{\pi/2}\cos\theta\,d\theta\,dr\\&=\int_1^2 r\Big[\sin\theta\Big]_{\theta=0}^{\pi/2}dr\\&=\int_1^2 r\,dr\\&=\left[\frac{1}{2}r^2\right]_{r=1}^2\\&=\frac{3}{2}.\end{aligned}$$

2. (a) Let (x,y) be an arbitrary point within the region R. A point on the line $x+2y=2$ is $(2,0)$, so a vector from the line to (x,y) is $(x-2,y)$. A vector perpendicular to the line is $(1,2)$, and the corresponding unit vector is $\frac{1}{\sqrt{5}}(1,2)$. Therefore the directed distance from (x,y) to the line is

$$d(x,y)=\frac{1}{\sqrt{5}}(1,2)\cdot(x-2,y)=\frac{x+2y-2}{\sqrt{5}}.$$

Now we need to set up a double iterated integral for the integral over R. The disk $x^2+y^2\le 1$ lies within $-1\le x\le 1$. At each value of x, we have $-\sqrt{1-x^2}\le y\le\sqrt{1-x^2}$. Thus an integral for the first moment of area of the region R is

$$\iint_R d(x,y)\,dA=\int_{-1}^1\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{x+2y-2}{\sqrt{5}}\,dy\,dx.$$

Alternatively, we could have put dy on the outside. Notice, however, that we cannot use symmetry arguments to reduce the integral to $2\int_0^1 2\int_0^{\sqrt{1-x^2}} d(x,y)\,dy\,dx$, because the integrand is not symmetric under reflection about either axis.

(b) The region R is a triangle with vertices $(3,0)$, $(0,-3)$ and $(-1,2)$. This triangle lies within $-1\le x\le 3$. When $-1\le x\le 0$, we have $-5x-3\le y\le -\frac{x}{2}+\frac{3}{2}$. When $0\le x\le 3$, we have $x-3\le y\le -\frac{x}{2}+\frac{3}{2}$.

To calculate the center of mass for R, we need the total mass contained in R, which is

$$M=\iint_R\rho(x,y)\,dA=\int_{-1}^0\int_{-5x-3}^{-x/2+3/2}(x^2+y^2)\,dy\,dx+\int_0^3\int_{x-3}^{-x/2+3/2}(x^2+y^2)\,dy\,dx.$$

Then the center of mass is the point $(\overline{x},\overline{y})$, where

$$\begin{aligned}\overline{x}&=\frac{1}{M}\iint_R x\rho(x,y)\,dA=\frac{1}{M}\int_{-1}^0\int_{-5x-3}^{-x/2+3/2}x(x^2+y^2)\,dy\,dx+\frac{1}{M}\int_0^3\int_{x-3}^{-x/2+3/2}x(x^2+y^2)\,dy\,dx,\\ \overline{y}&=\frac{1}{M}\iint_R y\rho(x,y)\,dA=\frac{1}{M}\int_{-1}^0\int_{-5x-3}^{-x/2+3/2}y(x^2+y^2)\,dy\,dx+\frac{1}{M}\int_0^3\int_{x-3}^{-x/2+3/2}y(x^2+y^2)\,dy\,dx.\end{aligned}$$

(c) Let (x, y) be an arbitrary point within R. Then the distance between (x, y) and the line $y = -1$ is $d(x, y) = y + 1$.

The region R lies within $-1 \le y \le 0$. At each value of y, $-\sqrt{-y/2} \le x \le \sqrt{-y/2}$. A double iterated integral for the moment of inertia of R about the line $y = -1$ is

$$I = \iint_R d(x,y)^2 \rho(x,y)\, dA = \int_{-1}^{0} \int_{-\sqrt{-y/2}}^{\sqrt{-y/2}} \rho (y+1)^2 \, dx \, dy.$$

(d) Let (x, y) be a point within R. The distance from (x, y) to the line $x = 0$ is x. The contribution to the volume of revolution due to an element of area $dx\, dy$ located at the point (x, y) is $2\pi x \, dx \, dy$.

The curves $y = 6 - \frac{1}{4}x^4$ and $y = \frac{1}{2}x^2$ intersect at the point $(2, 2)$. The line $x = 3$ intersects $y = 6 - \frac{1}{4}x^4$ at $(3, -\frac{57}{4})$, and it intersects $y = \frac{1}{2}x^2$ at $(3, \frac{9}{2})$.

The region R lies within $2 \le x \le 3$. At each value of x, $6 - \frac{1}{4}x^4 \le y \le \frac{1}{2}x^2$. An integral for the volume of revolution is then

$$V = \iint_R 2\pi x \, dA = \int_2^3 \int_{6 - x^4/4}^{x^2/2} 2\pi x \, dy \, dx.$$

(e) Set up xy-axes so that the center of the ellipse is at $(0, 0)$, and so that the surface of the fluid lies at $y = 2a$. Then a point (x, y) within the ellipse is at the depth $2a - y$, so the pressure at this point is $\rho g(2a - y)$.

The equation of the boundary of the ellipse is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$. The region contained within this ellipse lies within $-b \le x \le b$. At each value of b, $-\sqrt{1 - (x/b)^2} \le y \le \sqrt{1 - (x/b)^2}$. Therefore the total force due to fluid pressure is

$$F = \int_{-b}^{b} \int_{-\sqrt{1-(x/b)^2}}^{\sqrt{1-(b/x)^2}} \rho g(2a - y) \, dy \, dx.$$

(f) The curve $r = \sin\theta - \cos\theta$ lies within the region where $\sin\theta \ge \cos\theta$ (in other words, where $y \ge x$). In polar coordinates, this corresponds to $\frac{\pi}{4} \le \theta \le \frac{5\pi}{4}$. For every such θ, the region R enclosed by the curve satisfies $0 \le r \le \sin\theta - \cos\theta$.

To find the centroid, we first need the total area of the region, which is

$$A = \iint_R dA = \int_{\pi/4}^{5\pi/4} \int_0^{\sin\theta - \cos\theta} r \, dr \, d\theta.$$

The centroid is the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{A} \iint_R x \, dA = \frac{1}{A} \int_{\pi/4}^{5\pi/4} \int_0^{\sin\theta - \cos\theta} (r\cos\theta) r \, dr \, d\theta,$$

$$\overline{y} = \frac{1}{A} \iint_R y \, dA = \frac{1}{A} \int_{\pi/4}^{5\pi/4} \int_0^{\sin\theta - \cos\theta} (r\sin\theta) r \, dr \, d\theta.$$

3. (a) We are given z as a function of x and y, so we will set up an integral over a region in the xy-plane.

A point on the surface $z = 3x^2 + y^2$ lies below the plane $3x + y + z = 1$ provided $z \le 1 - 3x - y$. This occurs when

$$3x^2 + y^2 \le 1 - 3x - y,$$

which can be rearranged to

$$3\left(x + \frac{1}{2}\right)^2 + \left(y + \frac{1}{2}\right)^2 \le 2.$$

Thus the desired portion of the surface lies over the region in the xy-plane described by this inequality, which describes an elliptical disk in the xy-plane with center $(-\frac{1}{2}, -\frac{1}{2})$ and semi-axes of lengths $\sqrt{\frac{2}{3}}$ and $\sqrt{2}$ in the x- and y-directions, respectively.

The disk lies within

$$-\frac{1}{2} - \sqrt{\frac{2}{3}} \le x \le -\frac{1}{2} + \sqrt{\frac{2}{3}}.$$

At each value of x, we have

$$-\frac{1}{2} - \sqrt{2 - 3\left(x + \frac{1}{2}\right)^2} \le y \le -\frac{1}{2} + \sqrt{2 - 3\left(x + \frac{1}{2}\right)^2}.$$

Lastly, the integrand for the surface area calculation is

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + 36x^2 + 4y^2}.$$

Therefore the surface area is

$$S = \int_{-\frac{1}{2}-\sqrt{\frac{2}{3}}}^{-\frac{1}{2}+\sqrt{\frac{2}{3}}} \int_{-\frac{1}{2}-\sqrt{2-3\left(x+\frac{1}{2}\right)^2}}^{-\frac{1}{2}+\sqrt{2-3\left(x+\frac{1}{2}\right)^2}} \sqrt{1 + 36x^2 + 4y^2}\, dy\, dx.$$

(b) In the xy-plane, the constraints $x = 0$, $y = 0$ and $x + y = 1$ describe a right triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Let this triangle be R'. Since none of the constraints depend on z, the volume in 3D space bounded by these planes is obtained by translating the triangle in the $\pm z$ directions. We will set up the surface area calculation as an integral over a region in the xy-plane.

Notice that the surface $z^2 = x - y^2$ only exists in the region where $x \ge y^2$. Not every point inside the triangle R' satisfies this condition, so our surface area integral is not over all of R', but over the portion of R' lying below the curve $y = x^2$.

Let the region inside R', lying below $y = x^2$, be R. Then R has vertices at $(0, 0)$, $(1, 0)$ and $\left(\frac{\sqrt{5}-1}{2}, \frac{3-\sqrt{5}}{2}\right)$. We see that R lies within $0 \le y \le \frac{3-\sqrt{5}}{2}$, and at each value of y, $\sqrt{y} \le x \le 1 - y$.

Notice that the surface $z^2 = x - y^2$ is symmetric under the reflection $z \leftrightarrow -z$. That is, over R, there are two branches of the surface, one with $z \ge 0$ and the other with $z \le 0$. We restrict our attention to the branch where $z \ge 0$, and calculate the total surface area by doubling the result.

If we assume that $z \ge 0$, then $z = \sqrt{x - y^2}$. The two partial derivatives of z are

$$\frac{\partial z}{\partial x} = \frac{1}{2\sqrt{x - y^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{x - y^2}},$$

so the integrand for the surface area calculation is

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{1 + 4y^2}{4(x - y^2)}}.$$

Finally, the surface area is

$$S = 2\int_0^{(3-\sqrt{5})/2}\int_{\sqrt{y}}^{1-y}\sqrt{1+\frac{1+4y^2}{x-y^2}}\,dx\,dy.$$

(c) The surface is a sphere with radius $\sqrt{2}$ and center $(1,2,0)$. We want to find the portion of it lying inside the cylinder $x^2+y^2=1$. We will set up an integral over a region in xy-plane.

Since $z^2 \geq 0$, the projection of the sphere to the xy-plane is the disk $(x-1)^2+(y-2)^2 \leq 2$, which can be rewritten as

$$x^2+y^2 \leq 2x+4y-3.$$

The boundary of the disk is the circle $x^2+y^2=2x+4y-3$. It intersects the circle $x^2+y^2=1$ at two points on the line $x+2y=2$. If we substitute $x=2-2y$ into $x^2+y^2=1$, we obtain a quadratic equation that can be factored. The solutions are $y=\frac{3}{5}$ and $y=1$, which yield intersection points $\left(\frac{4}{5},\frac{3}{5}\right)$ and $(0,1)$, respectively.

The surface area integral takes place over the region R contained within the circles $x^2+y^2=1$ and $(x-1)^2+(y-2)^2=2$. Notice that the segment of the circle $x^2+y^2=1$ that bounds R satisfies $x \geq 0$, which means that $x=\sqrt{1-y^2}$. The segment of the circle $(x-1)^2+(y-2)^2=2$ that bounds R satisfies $x-1 \leq 0$, which means that $x=1-\sqrt{2-(y-2)^2}$.

From these observations and the vertices found above, we see that R lies within $\frac{3}{5} \leq y \leq 1$. At each value of y, $1-\sqrt{2-(y-2)^2} \leq x\sqrt{1-y^2}$.

Lastly, we need to find the integrand. The sphere is symmetric under $z \leftrightarrow -z$, so we can restrict our attention to the region where $z \geq 0$, and double the result to get the total surface area. If $z \geq 0$, then

$$z=\sqrt{2-(x-1)^2-(y-2)^2},$$

which has partial derivatives

$$\frac{\partial z}{\partial x} = -\frac{x-1}{\sqrt{2-(x-1)^2-(y-2)^2}}, \quad \frac{\partial z}{\partial y} = -\frac{y-2}{\sqrt{2-(x-1)^2+(y-2)^2}}.$$

Thus the integrand for the surface area integral is

$$\sqrt{1+\frac{(x-1)^2+(y-2)^2}{2-(x-1)^2-(y-2)^2}}.$$

The desired surface area is

$$S = 2\int_{3/5}^{1}\int_{1-\sqrt{2-(y-2)^2}}^{\sqrt{1-y^2}}\sqrt{1+\frac{(x-1)^2+(y-2)^2}{2-(x-1)^2-(y-2)^2}}\,dx\,dy.$$

(d) The surface $x^2=y^3-z^3$ only exists where $y \geq z$. We will set up an integral over a region in the yz-plane.

The projection of the sphere $x^2+y^2+z^2=1$ to the yz-plane is the disk $y^2+z^2 \leq 1$. Our integral for the surface area takes place over the portion of this disk where $y \geq z$. The line $y=z$ and the circle $y^2+z^2=1$ intersect at the point $\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$. Our region of integration lies within $-1 \leq z \leq \frac{1}{\sqrt{2}}$. If $-1 \leq z \leq -\frac{1}{\sqrt{2}}$, then $-\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}$. If $-\frac{1}{\sqrt{2}} \leq z \leq \frac{1}{\sqrt{2}}$, then $z \leq y\sqrt{1-z^2}$.

The surface $x^2 = y^3 - z^3$ is symmetric under $x \leftrightarrow -x$. We consider the portion of the surface where $x \geq 0$, and double the result to get the total surface area. If $x \geq 0$, then $x = \sqrt{y^3 - z^3}$, which has partial derivatives

$$\frac{\partial x}{\partial y} = \frac{3y^2}{2\sqrt{y^3 - z^3}}, \qquad \frac{\partial x}{\partial z} = -\frac{3z^2}{2\sqrt{y^3 - z^3}}.$$

Thus the integrand for the surface area calculation is

$$\sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2} = \sqrt{1 + \frac{9y^4 + 9z^4}{4(y^3 - z^3)}}.$$

Hence the surface area is

$$S = 2\int_{-1}^{-1/\sqrt{2}} \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \sqrt{1 + \frac{9y^4 + 9z^4}{4(y^3 - z^3)}}\, dy\, dz + \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{z}^{\sqrt{1-z^2}} \sqrt{1 + \frac{9y^4 + 9z^4}{4(y^3 - z^3)}}\, dy\, dz.$$

Alternative: the region of integration is simpler in polar coordinates, but the integrand is more complicated.

4. (a) If $z \geq h$, then points on the sphere satisfy $x^2 + y^2 \leq R^2 - h^2$. The projection of this portion of the sphere to the xy-plane is the disk $x^2 + y^2 \leq R^2 - h^2$, which becomes $r \leq \sqrt{R^2 - h^2}$ in polar coordinates. There is no constraint on θ, so $0 \leq \theta \leq 2\pi$.

If $z \geq h$, then $z \geq 0$, so the desired portion of the sphere satisfies $z = \sqrt{R^2 - x^2 - y^2}$. The partial derivatives are

$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}, \qquad \frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}},$$

so the integrand for the surface area calculation is

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1 + \frac{x^2 + y^2}{R^2 - x^2 - y^2}} = \frac{R}{\sqrt{R^2 - x^2 - y^2}}.$$

In polar coordinates, this becomes

$$\frac{R}{\sqrt{R^2 - r^2}}.$$

The surface area is given by

$$\begin{aligned} S &= \int_0^{\sqrt{R^2-h^2}} \int_0^{2\pi} \frac{R}{\sqrt{R^2 - r^2}} r\, d\theta\, dr \\ &= 2\pi R \int_0^{\sqrt{R^2-h^2}} \frac{r}{\sqrt{R^2 - r^2}}\, dr \\ &= 2\pi R \left[-\sqrt{R^2 - r^2}\right]_{r=0}^{\sqrt{R^2-h^2}} \\ &= 2\pi R\,(R - h). \end{aligned}$$

(b) If $z \leq h$, then a point on the cone satisfies $a\sqrt{x^2 + y^2} \leq h$, which implies that $x^2 + y^2 \leq \left(\frac{h}{a}\right)^2$, which is a solid disk of radius $\frac{h}{a}$ in the xy-plane. In polar coordinates, this region is described by $0 \le \theta \le 2\pi$ and $0 \le r \le \frac{h}{a}$.

We are given z as a function of x and y. The partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{ax}{\sqrt{x^2+y^2}}, \quad \frac{\partial z}{\partial y} = \frac{ay}{\sqrt{x^2+y^2}},$$

so the integrand for the surface area calculation is

$$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{1+\frac{a^2(x^2+y^2)}{x^2+y^2}} = \sqrt{1+a^2}.$$

Thus the surface area is

$$\begin{aligned} S = & \int_0^{h/a}\int_0^{2\pi} \sqrt{1+a^2}r\,d\theta\,dr \\ =& 2\pi\sqrt{1+a^2}\int_0^{h/a} r\,dr \\ =& 2\pi\sqrt{1+a^2}\frac{1}{2}\left(\frac{h}{a}\right)^2 = \frac{\pi h^2}{a^2}\sqrt{1+a^2}. \end{aligned}$$

5. (a) The plane $2x+y+3z=6$ has intercepts $(3,0,0)$, $(0,6,0)$ and $(0,0,2)$. The volume in the first octant bounded by the plane lies within $0 \le z \le 2$.

Fix a value of z, $0 < z < 2$. Then the corresponding 2D region is in the first quadrant, bounded by the line $2x+y=6-3z$. The intercepts of the line are $(x=0, y=6-3z)$ and $\left(x=\frac{6-3z}{2}, y=0\right)$. The region lies within $0 \le x \le \frac{6-3z}{2}$. At each value of x, we have $0 \le y \le 6-3z-2x$. Thus the volume is

$$\begin{aligned} V = \int_0^2\int_0^{(6-3z)/2}\int_0^{6-3z-2x} dy\,dx\,dz &= \int_0^2\int_0^{(6-3z)/2} (6-3z-2x)\,dx\,dz \\ &= \int_0^2 \left[6x-3xz-x^2\right]_{x=0}^{(6-3z)/2}\,dz \\ &= \int_0^2 \frac{(6-3z)^2}{4}\,dz \\ &= \left[-\frac{(6-3z)^3}{36}\right]_{z=0}^2 \\ &= 6. \end{aligned}$$

We can verify this answer using the formula for the volume of a triangle-based pyramid.

(b) In cylindrical coordinates, the boundaries are $z=1-r^2$ and $z=0$. The region lies within

$0 \leq \theta \leq 2\pi$, and within $0 \leq r \leq 1$. At any value of r, $0 \leq z \leq 1 - r^2$. Thus the volume is

$$
\begin{aligned}
V = \int_0^1 \int_0^{1-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr =& 2\pi \int_0^1 \int_0^{1-r^2} r \, dz \, dr \\
=& 2\pi \int_0^1 \left(r - r^3\right) \, dr \\
=& 2\pi \left[\frac{1}{2} r^2 - \frac{1}{4} r^4\right]_{r=0}^1 \\
=& \frac{\pi}{2}.
\end{aligned}
$$

(c) The volume lies within $0 \leq y \leq 4$, independent of the values of x and z. In the xz-plane, the curves $x = z^2$ and $x = z^3$ intersect at the points $(0,0)$ and $(1,1)$. The 2D region lies within $0 \leq z \leq 1$. At each value of z, $z^3 \leq x \leq z^2$. Therefore the volume is

$$
\begin{aligned}
V = \int_0^1 \int_{z^3}^{z^2} \int_0^4 dy \, dx \, dz =& 4 \int_0^1 \int_{z^3}^{z^2} dx \, dz \\
=& 4 \int_0^1 \left(z^2 - z^3\right) \, dz \\
=& 4 \left[\frac{1}{3} z^3 - \frac{1}{4} z^4\right]_{z=0}^1 \\
=& \frac{1}{3}.
\end{aligned}
$$

(d) In cylindrical coordinates, the boundaries are $z = 2 - r^2$ and $z = r$. Since there is no θ dependence, we have $0 \leq \theta \leq 2\pi$, and we can consider a plot in the rz-plane. The two curves in the rz-plane intersect at $(r = 1, z = 1)$. The region lies within $0 \leq r \leq 1$. At each value of r, $r \leq z \leq 2 - r^2$. Thus the volume is

$$
\begin{aligned}
V = \int_0^1 \int_r^{2-r^2} \int_0^{2\pi} r \, d\theta \, dz \, dr =& 2\pi \int_0^1 \int_r^{2-r^2} r \, dz \, dr \\
=& 2\pi \int_0^1 \left(2r - r^3 - r^2\right) \, dr \\
=& 2\pi \left[r^2 - \frac{1}{4} r^4 - \frac{1}{3} r^3\right]_{r=0}^1 \\
=& \frac{5\pi}{6}.
\end{aligned}
$$

(e) We put the integral over y on the outside, and consider cross sections in the xz-plane. Let $y \geq 0$ be fixed. Then the 2D region is in the first quadrant, bounded by the horizontal line $z = 2 - y$ and by the diagonal line $z = x - y$. As y increases, the upper boundary line $z = 2 - y$ moves down, and the diagonal line $z = x - y$ also moves down. The region vanishes when $y = 2$, because the upper boundary line has descended all the way to the x-axis. Thus $0 \leq y \leq 2$.

The region lies within $0 \le z \le 2-y$. At each value of z, $0 \le x \le y+z$. Thus the volume is

$$\begin{aligned}
V = \int_0^2 \int_0^{2-y} \int_0^{y+z} dx\,dz\,dy &= \int_0^2 \int_0^{2-y} (y+z)\,dz\,dy \\
&= \int_0^2 \left[yz + \frac{1}{2}z^2 \right]_{z=0}^{2-y} dy \\
&= \frac{1}{2} \int_0^2 \left(4 - y^2\right)\, dy \\
&= \frac{1}{2} \left[4y - \frac{1}{3}y^3 \right]_{y=0}^{2} \\
&= \frac{8}{3}.
\end{aligned}$$

(f) In spherical coordinates, the surface is $\frac{1}{2}r^5 = (r\cos\theta\sin\phi)(r\sin\theta\sin\phi)$, which simplifies to

$$r^3 = 2\sin\theta\cos\theta\sin^2\phi.$$

We are restricted to the first octant, so $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$. At every such (θ, ϕ), the expression $2\sin\theta\cos\theta\sin^2\phi$ is nonnegative, so a point on the surface exists at radius $r = \left(2\sin\theta\cos\theta\sin^2\phi\right)^{1/3}$. The solid enclosed by the surface consists of all points with $0 \le r \le \left(2\sin\theta\cos\theta\sin^2\phi\right)^{1/3}$. Therefore the volume is

$$\begin{aligned}
V = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{\left(2\sin\theta\cos\theta\sin^2\phi\right)^{1/3}} r^2 \sin\phi\,dr\,d\phi\,d\theta &= \int_0^{\pi/2} \int_0^{\pi/2} \frac{1}{3} \left(2\sin\theta\cos\theta\sin^2\phi\right) \sin\phi\,d\phi\,d\theta \\
&= \frac{1}{3} \int_0^{\pi/2} \sin(2\theta) \int_0^{\pi/2} \sin^3\phi\,d\phi\,d\theta.
\end{aligned}$$

The integral over ϕ can be evaluated by writing

$$\sin^3\phi = (1 - \cos^2\phi)\sin\phi.$$

Then

$$\begin{aligned}
\int_0^{\pi/2} \sin^3\phi\,d\phi &= \int_0^{\pi/2} \sin\phi\,d\phi - \int_0^{\pi/2} \cos^2\phi\sin\phi, d\phi \\
&= \Big[-\cos\phi \Big]_{\phi=0}^{\pi/2} - \left[-\frac{1}{3}\cos^3\phi \right]_{\phi=0}^{\pi/2} \\
&= \frac{2}{3}.
\end{aligned}$$

Thus the volume is

$$\begin{aligned}
V &= \frac{2}{9} \int_0^{\pi/2} \sin(2\theta)\,d\theta \\
&= \frac{2}{9} \left[-\frac{1}{2}\cos(2\theta) \right]_{\theta=0}^{\pi/2} \\
&= \frac{2}{9}.
\end{aligned}$$

6. Let the ellipse be centered at the origin in the xy-plane, with axes along the x- and y-axes. The equation of the ellipse is

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1.$$

By symmetry, we can restrict our attention to $x \geq 0$ and $y \geq 0$, and multiply the resulting area by 4. The portion of the ellipse in the first quadrant lies within $0 \leq x \leq a$. At each value of x, $0 \leq y \leq b\sqrt{1-(x/a)^2}$. Therefore the area is

$$\begin{aligned} A = 4 \int_0^a \int_0^{b\sqrt{1-(x/a)^2}} dy\, dx =& 4 \int_0^a b\sqrt{1 - \left(\frac{x}{a}\right)^2}\, dx \\ =& 4b \int_0^{\pi/2} a\cos^2\theta\, d\theta \quad (\text{using } x = a\sin\theta) \\ =& 4ab \int_0^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right) d\theta \\ =& 4ab\left(\frac{\pi}{4}\right) = \pi ab. \end{aligned}$$

In the case where $a = b$, we have a circle of radius a. The formula gives an area of πa^2, which is correct.

7. In polar coordinates, the circle $x^2 + y^2 = 1$ becomes $r = 1$. The circle $(x-a)^2 + y^2 = 1$ rearranges to $x^2 + y^2 = 1 - a^2 + 2ax$. In polar coordinates, this equation is $r^2 = 1 - a^2 + 2ar\cos\theta$.

To find the intersection points of the two circles, we set $r = 1$ in the second equation. It becomes $2a\cos\theta = a^2$, which implies that $\cos\theta = \frac{a}{2}$. If $a = 2$, then the only solution is $\theta = 0$, and the two circles are tangent. If $1 \leq a < 2$, then there are two solutions, $\theta = \pm\cos^{-1}\left(\frac{a}{2}\right)$. Let the positive solution be θ_0. By symmetry, the total area of intersection is double the area lying within $0 \leq \theta \leq \theta_0$.

At any value of θ within this range, the radius is bounded above by $r = 1$, and it is bounded below by the smaller of the two solutions to

$$r^2 - 2ar\cos\theta + a^2 - 1 = 0.$$

Using the quadratic formula, we get

$$r = \frac{2a\cos\theta - \sqrt{4a^2\cos^2\theta - 4(a^2-1)}}{2} = a\cos\theta - \sqrt{1 - a^2\sin^2\theta}.$$

Thus the region of intersection lies within $a\cos\theta - \sqrt{1-a^2\sin^2\theta} \leq r \leq 1$. The area is

$$A = 2\int_0^{\theta_0} \int_{a\cos\theta - \sqrt{1-a^2\sin^2\theta}}^1 r\, dr\, d\theta.$$

8. (a) This is an integral over the volume contained beneath the surface $z = 2 - \sqrt{x^2+y^2}$ and above the xy-plane. In cylindrical coordinates, the equation of the surface is $z = 2 - r$. There is no constraint on θ. The volume of integration lies within $0 \leq r \leq 2$, and at each value of r, $0 \leq z \leq 2 - r$. Thus the integral is

$$\int_0^{2\pi} \int_0^2 \int_0^{2-r} r\, dz\, dr\, d\theta.$$

(b) This is an integral over the volume in the first octant contained within $z = \sqrt{1 - x^2 - y^2}$; that is, within the sphere $x^2 + y^2 + z^2 = 1$. Since we are in the first octant, we have $0 \le \theta \le \frac{\pi}{2}$. The region lies within $0 \le r \le 1$, and at each value of r, $0 \le z \le \sqrt{1 - r^2}$. The integral is

$$\int_0^{\pi/2} \int_0^1 \int_0^{\sqrt{1-r^2}} r \, dz \, dr \, d\theta.$$

(c) This is an integral over a cube, which is quite annoying in cylindrical coordinates. The volume lies within $-1 \le z \le 1$. At each value of z, we have to integrate over the square with $-1 \le x \le 1$ and $-1 \le y \le 1$.

In polar coordinates, the four lines $x = 1$, $y = 1$, $x = -1$ and $y = -1$ that bound the square become $r = \sec\theta$, $r = \csc\theta$, $r = -\sec\theta$ and $r = -\csc\theta$, respectively. We break up the square along its diagonals:

$$\begin{aligned} \text{when } -\frac{\pi}{4} \le \theta \le \frac{\pi}{4}, &\quad r \le \sec\theta, \\ \text{when } \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}, &\quad r \le \csc\theta, \\ \text{when } \frac{3\pi}{4} \le \theta \le \frac{5\pi}{4}, &\quad r \le -\sec\theta, \\ \text{when } \frac{5\pi}{4} \le \theta \le \frac{7\pi}{4}, &\quad r \le -\csc\theta. \end{aligned}$$

Therefore the triple integral becomes

$$\begin{aligned} &\int_{-1}^1 \int_{-\pi/4}^{\pi/4} \int_0^{\sec\theta} r \, dr \, d\theta \, dz + \int_{-1}^1 \int_{\pi/4}^{3\pi/4} \int_0^{\csc\theta} r \, dr \, d\theta \, dz \\ &+ \int_{-1}^1 \int_{3\pi/4}^{5\pi/4} \int_0^{-\sec\theta} r \, dr \, d\theta \, dz + \int_{-1}^1 \int_{5\pi/4}^{7\pi/4} \int_0^{-\csc\theta} r \, dr \, d\theta \, dz. \end{aligned}$$

Note: since we were only given an integrand of 1, we could have said that the total integral is four times the integral over one of these regions. This would not have worked if we were integrating an arbitrary function f over the region.

(d) This is an integral over the volume contained within $-1 \le z \le 1$, where at every value of z, the 2D region in the xy-plane is bounded by $x = |y|$ and $x^2 + y^2 = 1$.

In cylindrical coordinates, we still have $-1 \le z \le 1$. At any value of z, the constraint $x = |y|$ becomes $\theta = \frac{\pi}{4}$ or $\theta = -\frac{\pi}{4}$. The region of integration lies within $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. At each θ, $0 \le r \le 1$. Therefore the integral is

$$\int_{-1}^1 \int_{-\pi/4}^{\pi/4} \int_0^1 r \, dr \, d\theta \, dz.$$

9. (a) This is an integral over the volume contained within the hemisphere of radius 1, centered at the origin, that is bounded by the yz-plane and lies within the region $x \le 0$. In spherical coordinates, this hemisphere is given by $r = 1$, $0 \le \phi \le \pi$, and $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$. Therefore the integral is

$$\int_0^1 \int_0^{\pi} \int_{\pi/2}^{3\pi/2} r^2 \sin\phi \, d\theta \, d\phi \, dr.$$

(b) This is an integral over the cylinder $x^2 + y^2 = 1$, lying between $z = 0$ and $z = 1$. In spherical coordinates, the cylinder becomes $r \sin\phi = 1$, and the plane $z = 1$ becomes $r \cos\phi = 1$.

The volume lies within $0 \le \theta \le 2\pi$, and within $0 \le \phi \le \frac{\pi}{2}$. If $0 \le \phi \le \frac{\pi}{4}$, then the radial direction is bounded above by the plane $r \cos\phi = 1$, so $0 \le r \le \sec\phi$. If $\frac{\pi}{4} \le \phi \le \frac{\pi}{2}$, then the radial direction is bounded above by the cylinder $r \sin\phi = 1$, so $0 \le r \le \csc\phi$. Thus the integral becomes

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sec\phi} r^2 \sin\phi \, dr \, d\phi \, d\theta + \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{\csc\phi} r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

(c) This is an integral over the sphere $x^2 + y^2 + (z-1)^2 = 1$. We rewrite this equation as $x^2 + y^2 + z^2 = 2z$, which in spherical coordinates becomes $r^2 = 2r\cos\phi$, or $r = 2\cos\phi$.

The sphere lies within $0 \le \theta \le 2\pi$. It is contained within $z \ge 0$, so $0 \le \phi \le \frac{\pi}{2}$. At any value of ϕ, $0 \le r \le 2\cos\phi$. Therefore the integral is

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^{2\cos\phi} r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

(d) This is an integral over the volume contained within the sphere $x^2 + y^2 + z^2 = 9$ and the cone $z = \sqrt{2(x^2 + y^2)}$. In spherical coordinates, the sphere is $r = 3$ and the cone is $\cos\phi = \sqrt{2}\sin\phi$, or $\tan\phi = \frac{1}{\sqrt{2}}$. Let $\phi_0 = \tan^{-1}\frac{1}{\sqrt{2}}$.

The volume lies within $0 \le \theta \le 2\pi$, and within $0 \le \phi \le \phi_0$. At any ϕ and θ, $0 \le r \le 3$. Therefore the integral is

$$\int_0^{2\pi} \int_0^{\phi_0} \int_0^{3} r^2 \sin\phi \, dr \, d\phi \, d\theta.$$

10. (a) The integral can be performed as written.

$$\begin{aligned}
\int_0^{\pi/2} \int_0^{\cos\theta} \int_0^{\sin\theta} rz \, dr \, dz \, d\theta &= \int_0^{\pi/2} \int_0^{\cos\theta} z \left[\frac{1}{2} r^2\right]_{r=0}^{\sin\theta} dz \, d\theta \\
&= \frac{1}{2} \int_0^{\pi/2} \int_0^{\cos\theta} z \sin^2\theta \, dz \, d\theta \\
&= \frac{1}{2} \int_0^{\pi/2} \sin^2\theta \left[\frac{1}{2} z^2\right]_{z=0}^{\cos\theta} d\theta \\
&= \frac{1}{4} \int_0^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta \\
&= \frac{1}{16} \int_0^{\pi/2} \sin^2(2\theta) \, d\theta \\
&= \frac{1}{16} \int_0^{\pi/2} \left(\frac{1 - \cos(4\theta)}{2}\right) d\theta \\
&= \frac{1}{16} \cdot \frac{\pi}{4} = \frac{\pi}{64}.
\end{aligned}$$

(b) Notice that the condition $r \le 2\sec\phi$ rearranges to $r\cos\phi \le 2$, or $z \le 2$. The other two limits of integration, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \frac{\pi}{6}$, describe a cone. Therefore this integral represents the volume contained within the cone $\phi = \frac{\pi}{6}$, below $z = 2$.

One option: the height of the cone is 2, and its radius R satisfies $\frac{R}{2} = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$, which implies that $R = \frac{2}{\sqrt{3}}$. The integral is the volume of the cone, which is

$$\frac{1}{3}\pi\left(\frac{2}{\sqrt{3}}\right)^2 2 = \frac{8\pi}{9}.$$

Another option: we can rewrite the cone in cylindrical coordinates. In general, the equation of a cone is $z = ar$ for some $a \geq 0$, where the angle ϕ between the z-axis and the slanted surface of the cone satisfies $\tan\phi = \frac{r}{z} = \frac{1}{a}$. In particular, we have $\phi = \frac{\pi}{6}$, so $a = \frac{1}{\tan\phi} = \sqrt{3}$. Therefore the equation of this cone, in cylindrical coordinates, is $z = \sqrt{3}r$. The volume lies within $0 \leq \theta \leq 2\pi$ and $0 \leq z \leq 2$. At each value of z, $0 \leq r \leq \frac{1}{\sqrt{3}}z$. Therefore the integral is

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_0^{z/\sqrt{3}} r\,dr\,dz\,d\theta &= \int_0^{2\pi}\int_0^2 \frac{1}{6}z^2\,dz\,d\theta \\ &= \frac{1}{6}\int_0^{2\pi}\frac{8}{3}\,d\theta \\ &= \frac{8\pi}{9}.\end{aligned}$$

Another option: we can perform the integral as written.

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi/6}\int_0^{2\sec\phi} r^2\sin\phi\,dr\,d\phi\,d\theta &= \int_0^{2\pi}\int_0^{\pi/6}\frac{8}{3}(\cos\phi)^{-3}\sin\phi\,d\phi\,d\theta \\ &= \frac{8}{3}\int_0^{2\pi}\left[\frac{1}{2}(\cos\phi)^{-2}\right]_{\phi=0}^{\pi/6} d\theta \\ &= \frac{4}{3}\int_0^{2\pi}\left[\left(\frac{2}{\sqrt{3}}\right)^2 - 1\right] d\theta \\ &= \frac{4}{9}(2\pi) = \frac{8\pi}{9}.\end{aligned}$$

(c) The integral looks like it would be easier if we perform the integration over y first. At each fixed value of x, we are integrating over the region in the yz-plane bounded by the lines $z = y$, $z = x$ and $y = 0$. If we reverse the order of integration in z and y, we get

$$\int_0^x\int_y^x dz\,dy = \int_0^x\int_0^z dy\,dz.$$

Therefore the original integral is

$$\begin{aligned}\int_0^1\int_0^x\int_0^z \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dz\,dx &= \int_0^1\int_0^x x\left[\sqrt{x^2+y^2}\right]_{y=0}^z dz\,dx \\ &= \int_0^1\int_0^x x\left(\sqrt{x^2+z^2} - x\right) dz\,dx \\ &= \int_0^1\int_0^x x\sqrt{x^2+z^2}\,dz\,dx - \int_0^1\int_0^x x^2\,dz\,dx.\end{aligned}$$

The second term is

$$\int_0^1 \int_0^x x^2\, dz\, dx = \int_0^1 x^3\, dx = \frac{1}{3}.$$

The first term looks easier if we reverse the order again:

$$\begin{aligned}
\int_0^1 \int_0^x x\sqrt{x^2+z^2}\, dz\, dx &= \int_0^1 \int_z^1 x\sqrt{x^2+z^2}\, dx\, dz \\
&= \int_0^1 \left[\frac{1}{3}(x^2+z^2)^{3/2}\right]_{x=z}^1 dz \\
&= \frac{1}{3}\int_0^1 \left[(1+z^2)^{3/2} - 2\sqrt{2}z^3\right] dz \\
&= \frac{1}{3}\int_0^1 (1+z^2)^{3/2}\, dz - \frac{\sqrt{2}}{6}.
\end{aligned}$$

For the remaining integral, we have no choice but to perform a trig substitution:

$$\begin{aligned}
\int_0^1 (1+z^2)^{3/2}\, dz &= \int_0^{\pi/4} \sec^5\theta\, d\theta \quad (\text{using } z=\tan\theta) \\
&= \int_0^{\pi/4} \sec^3\theta \sec^2\theta\, d\theta \\
&= \left[\sec^3\theta \tan\theta\right]_{\theta=0}^{\pi/4} - \int_0^{\pi/4} 3\sec^3\theta\tan^2\theta\, d\theta \\
&\quad (\text{using int. by parts, } u=\sec^3\theta,\ dv=\sec^2\theta\, d\theta) \\
&= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^3\theta\left(\sec^2\theta - 1\right) d\theta \\
&= 2\sqrt{2} - 3\int_0^{\pi/4} \sec^5\theta\, d\theta + 3\int_0^{\pi/4}\sec^3\theta\, d\theta.
\end{aligned}$$

That is,

$$\int_0^{\pi/4} \sec^5\theta\, d\theta = 2\sqrt{2} - 3\int_0^{\pi/4}\sec^5\theta\, d\theta + 3\int_0^{\pi/4}\sec^3\theta\, d\theta,$$

which implies that

$$\int_0^{\pi/4} \sec^5\theta\, d\theta = \frac{\sqrt{2}}{2} + \frac{3}{4}\int_0^{\pi/4}\sec^3\theta\, d\theta.$$

We have to apply the same trick again to evaluate the integral of $\sec^3\theta$. The result is

$$\begin{aligned}
\int_0^{\pi/4} \sec^3\theta\, d\theta &= \frac{\sqrt{2}}{2} + \frac{1}{2}\int_0^{\pi/4}\sec\theta\, d\theta \\
&= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln|\sec\theta + \tan\theta|\Big|_{\theta=0}^{\pi/4} \\
&= \frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right).
\end{aligned}$$

Therefore

$$\int_0^1 (1+z^2)^{3/2}\, dz = \int_0^{\pi/4}\sec^5\theta\, d\theta = \frac{\sqrt{2}}{2} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)\right],$$

and finally,

$$\int_0^1 \int_0^x \int_0^z \frac{xy}{\sqrt{x^2+y^2}}\,dy\,dz\,dx = \frac{1}{3}\left\{\frac{\sqrt{2}}{2} + \frac{3}{4}\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln\left(\sqrt{2}+1\right)\right]\right\} - \frac{\sqrt{2}}{6} - \frac{1}{3}.$$

(d) We can evaluate the integral as written.

$$\begin{aligned}
\int_0^{\pi/2}\int_0^{\phi}\int_0^{\cos\theta} r^2\sin\phi\,dr\,d\theta\,d\phi &= \int_0^{\pi/2}\int_0^{\phi}\frac{1}{3}\cos^3\theta\sin\phi\,d\theta\,d\phi \\
&=\frac{1}{3}\int_0^{\pi/2}\sin\phi\int_0^{\phi}\left(1-\sin^2\theta\right)\cos\theta\,d\theta\,d\phi \\
&=\frac{1}{3}\int_0^{\pi/2}\sin\phi\left[\sin\theta-\frac{1}{3}\sin^3\theta\right]_{\theta=0}^{\phi}d\phi \\
&=\frac{1}{3}\int_0^{\pi/2}\left(\sin^2\phi-\frac{1}{3}\sin^4\phi\right)d\phi \\
&=\frac{1}{3}\int_0^{\pi/2}\left(\sin^2\phi-\frac{1}{3}(1-\cos^2\phi)\sin^2\phi\right)d\phi \\
&=\frac{2}{9}\int_0^{\pi/2}\sin^2\phi\,d\phi+\frac{1}{9}\int_0^{\pi/2}\cos^2\phi\sin^2\phi\,d\phi \\
&=\frac{2}{9}\int_0^{\pi/2}\left(\frac{1-\cos(2\phi)}{2}\right)d\phi+\frac{1}{36}\int_0^{\pi/2}\sin(2\phi)\,d\phi \\
&=\frac{\pi}{18}+\frac{1}{36}.
\end{aligned}$$

(e) The integral looks easier if we integrate over r first, then z. In the rz-plane, the double integral is over the region bounded by $z = r^2+1$, $r = 1$ and $z = 2$. If we reverse the order of integration, we get

$$\int_0^1\int_{r^2+1}^2 dz\,dr = \int_1^2\int_0^{\sqrt{z-1}} dr\,dz.$$

Therefore the original integral is

$$\begin{aligned}
\int_0^{\pi}\int_0^1\int_{r^2+1}^2 \frac{r(2z+1)}{(z^2+r^2)}\,dz\,dr\,d\theta &= \int_0^{\pi}\int_1^2\int_0^{\sqrt{z-1}}\frac{r(2z+1)}{(z^2+r^2)^2}\,dr\,dz\,d\theta \\
&=\int_0^{\pi}\int_1^2(2z+1)\left[-\frac{1}{2(z^2+r^2)}\right]_{r=0}^{\sqrt{z-1}}dz\,d\theta \\
&=\frac{1}{2}\int_0^{\pi}\int_1^2\left(\frac{2z+1}{z^2}-\frac{2z+1}{z^2+z-1}\right)dz\,d\theta \\
&=\frac{1}{2}\int_0^{\pi}\left[2\ln z-\frac{1}{z}-\ln\left(z^2+z-1\right)\right]_{z=1}^2 d\theta \\
&=\frac{1}{2}\int_0^{\pi}\left[2\ln 2-\frac{1}{2}-\ln(4)+1\right]d\theta \\
&=\frac{\pi}{4}.
\end{aligned}$$