Math 1710. Homework Problems IV (January 11, 2012)

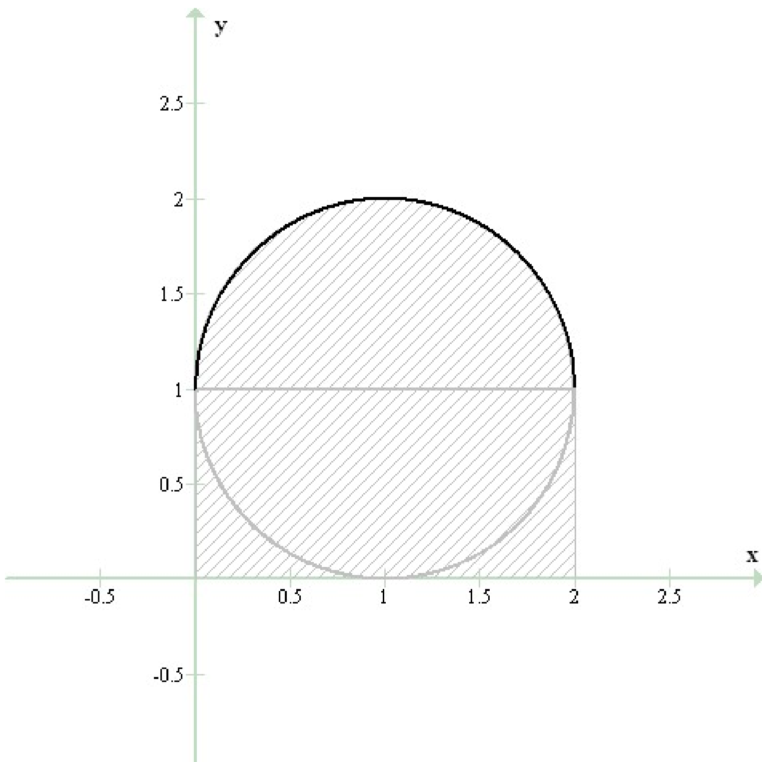
1. Evaluate $\int_0^2 \left(1+\sqrt{2x-x^2}\right)\,dx$.

Solution: The graph of the function $y = 1+\sqrt{2x-x^2}$ is a semicircle. Indeed,

$$y = 1+\sqrt{2x-x^2} \Leftrightarrow y-1 = \sqrt{2x-x^2} \Leftrightarrow (y-1)^2 = 2x-x^2 \Leftrightarrow x^2-2x+(y-1)^2 = 0$$

$$\Leftrightarrow x^2-2x+1+(y-1)^2 = 1 \Leftrightarrow (x-1)^2+(y-1)^2 = 1,$$

which is equation of a circle with center at $(1,1)$ and radius $r = 1$. However, since $y = 1+\sqrt{2x-x^2} \geq 1$, for any value of x, this equation only represents the top semicircle (see the picture below):



Therefore,

$$\int_0^2 \left(1+\sqrt{2x-x^2}\right)\,dx = \text{Area under the curve} \quad y = 1+\sqrt{2x-x^2}$$

$$= \text{Area of the semicircle} + \text{Area of the rectangle} = \frac{\pi\cdot 1^2}{2}+2\times 1 = 2+\frac{\pi}{2}$$

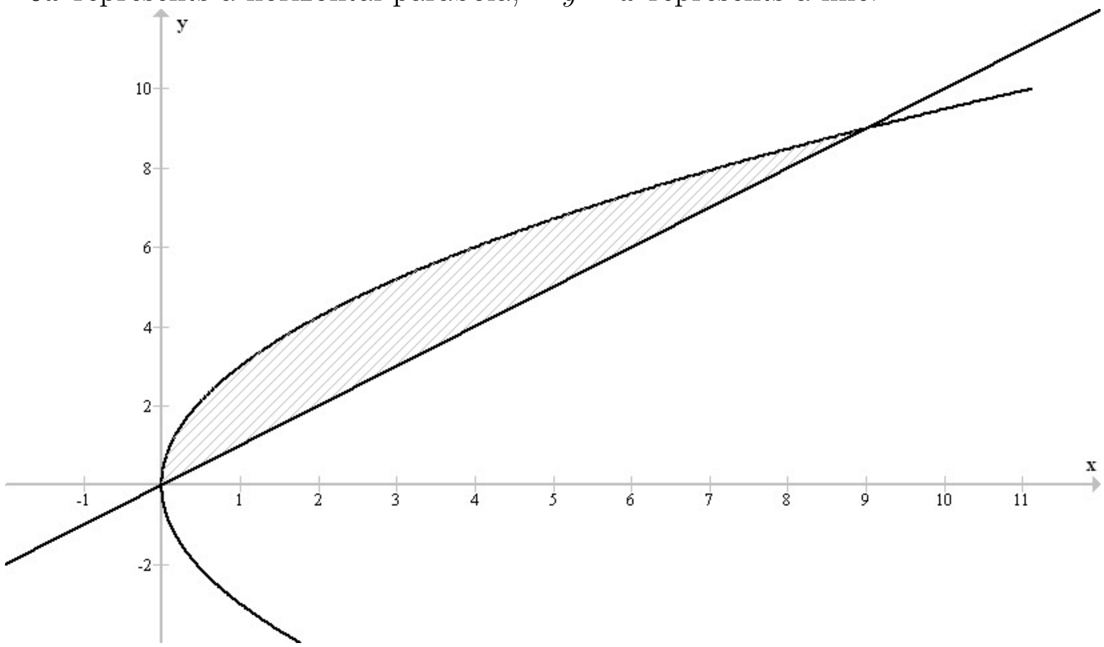
2. Find the area of the region bounded by the curves $y^2 = 9x$ and $y = x$.

Solution: First, we find the intersection points and sketch the graphs:

$$\begin{cases} y^2 = 9x, \\ y = x \end{cases} \Leftrightarrow x^2 = 9x \Leftrightarrow x(x-9) = 0 \Rightarrow x = 0, 9$$

So, the points of intersection are $x = y = 0$ and $x = y = 9$.

$y^2 = 9x$ represents a horizontal parabola; $y = x$ represents a line.



Hence, we write both functions in the form $y = f(x)$ and $y = g(x)$:

- the top branch of parabola $y^2 = 9x$ has equation $y = 3\sqrt{x}$ – top function;
- $y = x$ – bottom function

$$Area = \int_0^9 \left(3\sqrt{x} - x\right)\, dx = \left(\frac{3x^{3/2}}{3/2} - \frac{x^2}{2}\right)\Bigg|_0^9 = \left(2x^{3/2} - \frac{x^2}{2}\right)\Bigg|_0^9 = 2 \cdot 9^{3/2} - \frac{9^2}{2} - 0 = 54 - \frac{81}{2} = \boxed{\frac{27}{2}}$$