MATH 2132 Problem Workshop 4

1. Find a general solution to the following differential equations

   (a) $(y-1)\dfrac{dy}{dx} = yx^2$

   Solution:
   This is a separable equation and can be separated to be

   $$\frac{y-1}{y}\frac{dy}{dx} = x^2$$

   Note that we have divided by y which may eliminate $y = 0$ from being a solution. Hence we should check that separately to see that $y = 0$ is a solution.
   Integrating both sides leads to

   $$\int \left(1 - \frac{1}{y}\right) dy = \int x^2\, dx$$
   $$y - \ln|y| = \frac{x^3}{3} + C$$

   which can't easily be solved for y.
   Hence the solutions are

   $$y = 0,\ y - \ln|y| = \frac{x^3}{3} + C$$

   for any constant C.

   (b) $\dfrac{y-1}{y}\dfrac{dy}{dx} = x^2$

   Solution:
   The only difference between this question and the previous is that while $y = 0$ was a solution of the previous question, it is not a solution of this. Hence the solution is

   $$y - \ln|y| = \frac{x^3}{3} + C$$

   for any constant C.

   (c) $x^2\dfrac{dy}{dx} = y^2 - 1$

Solution:

Separating again leads to

$$\frac{1}{y^2-1}\frac{dy}{dx} = \frac{1}{x^2}.$$

Similar to the first question we divided leaving $y = 1$ and $y = -1$ in the denominator. Looking at the original question, these can be verified as solutions.

Integrating the left side requires $\displaystyle\int \frac{1}{(y-1)(y+1))}\, dy$ which can be done by partial fractions.

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y+1} + \frac{B}{y-1} = \frac{A(y-1)+B(y+1)}{(y-1)(y+1)}$$

Hence $1 = A(y-1) + B(y+1)$.

Using $y = 1$ we can get $2B = 1 \Rightarrow B = \dfrac{1}{2}$.

Using $y = -1$ we can get $-2A = 1 \Rightarrow A = -\dfrac{1}{2}$.

Thus the integral is

$$\begin{aligned}\int \frac{1}{(y-1)(y+1)}\, dy &= \int \frac{-1/2}{y+1} + \frac{1/2}{y-1},\, dy \\ &= -\frac{1}{2}\ln|y+1| + \frac{1}{2}|y-1| \\ &= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|\end{aligned}$$

Note that we'll leave the $+C$ for the other side.

Hence

$$\frac{1}{2}\ln\left|\frac{y+1}{y-1}\right| = -\frac{1}{x} + C$$

Solving for x leads to

$$
\begin{aligned}
\ln\left|\frac{y-1}{y+1}\right| &= -\frac{2}{x} + C_1 \quad C_1 = 2C \\
\Rightarrow \left|\frac{y-1}{y+1}\right| &= e^{C_1}e^{-2/x} \\
\Rightarrow \frac{y-1}{y+1} &= Ke^{-2/x} \qquad K = \pm e^{C_1},\ \text{Thus } K \text{ is any constant but } 0. \\
\Rightarrow y-1 &= yKe^{-2x} + Ke^{-2/x} \\
\Rightarrow -1 - Ke^{-2/x} &= yKe^{-2/x} - y \\
\Rightarrow y &= \frac{1+Ke^{-2/x}}{1-Ke^{-2/x}}
\end{aligned}
$$

Note that when $K = 0$ we get $y = 1$ which was a solution. Hence our solutions are

Thus the solutions are

$$
y = -1,\ y = \frac{1+Ke^{-2/x}}{1-Ke^{-2/x}}
$$

for any constant K.

(d) $x\frac{dy}{dx} = 3y + x^5\sqrt{1+x^2}$, x¿0

Solution:

This is not separable, but it is linear. Putting it into the proper form.

$$
\frac{dy}{dx} - \frac{3}{x}y = x^4\sqrt{1+x^2}
$$

An integrating factor is

$$
\mu(x) = e^{-\int 3/x\,dx} = e^{-3\ln x} = x^{-3}
$$

Thus we get $(x^{-3}y)' = x\sqrt{1+x^2}$.

Integrating yields

$$
\begin{aligned}
x^{-3}y &= \frac{1}{3}(1+x^2)^{3/2} + C \\
\Rightarrow y &= \frac{1}{3}x^3(1+x^2)^{3/2} + Cx^3 \\
\Rightarrow y &= \frac{1}{3}x^3(1+x^2)^{3/2} + Cx^3
\end{aligned}
$$

2. Find an explicit solution of the initial-value problem and where is the solution valid?

$$\frac{dy}{dx} = \frac{x^4}{y+1}, \qquad y(1) = 2.$$

Solution:

This is separable, hence we get

$$(y+1)\frac{dy}{dx} = x^4.$$

Integrating leads to

$$\int (y+1)\,dy = x^4\,dx\frac{(y+1)^2}{2} = \frac{x^5}{5} + C$$

We can plug in the initial value at this point or later. Doing it now would lead to

$$\frac{9}{2} = \frac{1}{5} + C \Rightarrow C = \frac{43}{10}$$

Hence

$$(y+1)^2 = \frac{2x^5+43}{5} \Rightarrow y = -1 \pm \sqrt{\frac{2x^5+43}{5}}.$$

Since $y(1) = 2 > -1$ we must use the positive square root and get

$$y = -1 + \sqrt{\frac{2x^5+43}{5}}.$$

The solution is valid when $2x^5 \geq -43 \Rightarrow x \geq \sqrt[5]{-43/2}$.

3. We will be solve the following word problem:

   A tank originally contains 1000 litres of water in which 10 kilograms of sugar has been dissolved (uniformly). A mixture containing 2 kilograms of sugar per 100 litres of water is added to the tank at 15 millilitres per minute. At the same time, 20 millitires of well-stirred mistures is removed from the tank each minute. Find the amount of sugar in the tank as a function of time t. For how long is the solution valid?

   (a) Let $Q(t)$ be the quantity of sugar in the tank. What is $Q(0)$ in kilograms.

Solution:
$Q(0) = 10$ since the tank originally contained 10 kilograms of sugar

(b) What is the rate which the sugar is entering the tank? Include units.

Solution:
The mixture coming in contains 2/100 kg/L of sugar. Since the mixture is coming in at 0.015 L/min, the sugar enters the tank at

$$0.0003\text{kg/min.}$$

(c) What is the rate which the sugar is leaving the tank? Include units.

Solution:
First we need to know how much is in the tank at any time. Since 0.015 L comes in per minute and 0.002 comes out each minute, we lose 0.005 L per minute. Hence the amount of liquid in the tank is $1000 - 0.005t$

There is $Q(t)$ kg of sugar at any time and there is $1000 - 0.005t$ L in the tank. Hence the concentration of sugar is $Q(t)/(1000 - 0.005t)$kg/L. The liquid leaves at 0.02 L/min, hence the sugar enters the tank at

$$\frac{0.02Q}{1000 - 0.005t}\text{kg/min.}$$

Note that this will only work until there is nothing left in the tank which is when $1000 - 0.005t = 0 \Rightarrow t = 200000$ minutes.

(d) Set up an initial value problem which solves the question.

Solution:
$\frac{dQ}{dt}$ is the rate of change which is the rate in minus the rate out. Thus

$$\frac{dQ}{dt} = 0.0003 - \frac{0.02Q}{1000 - 0.005t}, \qquad Q(0) = 10. \qquad t < 200000.$$

(e) Solve the differential equation and thus the word problem.

Solution: The equation is linear, and thus
$\dfrac{dQ}{dt} + \dfrac{4}{200000-t}Q = \frac{3}{10000}$
The integrating factor is

$$\mu(t) = e^{\int} \frac{4}{200000-t} = e^{-4\ln(200000-t)} = (200000-t)^{-4}.$$

Hence

$$\begin{aligned} \left((200000-t)^{-4}Q\right)' &= \frac{3}{10000}(200000-t)^{-4} \\ \Rightarrow (200000-t)^{-4}Q &= \frac{1}{10000}(200000-t)^{-3} + C \\ \Rightarrow Q &= \frac{1}{10000}(200000-t) + C(200000-t)^4 \end{aligned}$$

Since $Q(0) = 10$

$$10 = \frac{1}{10000}(200000) + C(200000)^4 \Rightarrow -10 = 16 \cdot 10^{-20}C \Rightarrow C = -\frac{10}{16} \cdot 10^{-20}.$$

Thus
$$Q(t) = \frac{200000-t}{10000} - \frac{10}{16} \cdot 10^{-20}(200000-t)^4$$

4. Find a general solution of the differential equation $xy'' = x^3 - y'$, $x < 0$

Solution:

There is no y term, so we can reduce the order. Let $v = \dfrac{dy}{dx}$, then $\dfrac{dv}{dx} = y''$. Thus

$$xv = x^3 - v \Rightarrow v + \frac{1}{x} = x^2$$

The integrating factor is

$$\mu(x) = e^{\int 1/x\,dx} = e^{\ln|x|} = -x.$$

Hence

$$(-xv)' = \pm - x^3 \Rightarrow -xv = \pm - \frac{x^4}{4} + C \Rightarrow v = \frac{x^3}{4} - \frac{C}{x}.$$

Thus

$$y = \int v\,dx = \frac{x^4}{16} - C\ln|x| + D$$

where C, D are constants.

5. Solve the initial value problem

$$y'' = 4yy', \qquad y(0) = 1,\ y'(0) = 0$$

Solution:

There is no x term, so we change the differential equation to an equation for y

Let $v = \frac{dy}{dx}$. thus $\frac{dy}{dx} = v$, $\frac{d^y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v$.

Hence the equation becomes

$$\frac{dv}{dy}v = 4yv \Rightarrow \frac{dv}{dy} = 4y$$

Integrating yields $v = 2y^2 + C$ and thus $\frac{dy}{dx} = 2y^2 + C$.

Solving for C yields, $0 = 2(1)^2 + C \Rightarrow C = -2$

Thus we get $\frac{dy}{dx} = 2y^2 - 2$.

Separating we get

$$\frac{1}{2}\int \frac{1}{y^2-1}\,dy = \int 1\,dx$$

Note that if $y = \pm 1$ we have divided by 0. Testing those solutions separately, we get that they are both solutions.

Integrating the left side requires some partial fractions.

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y+1} + \frac{B}{y-1} = \frac{A(y-1)+B(y+1)}{(y-1)(y+1)}$$

Hence $1 = A(y-1) + B(y+1)$.

Using $y = 1$ we can get $2B = 1 \Rightarrow B = \dfrac{1}{2}$.

Using $y = -1$ we can get $-2A = 1 \Rightarrow A = -\dfrac{1}{2}$.

Thus the integral is

$$\begin{aligned}
\frac{1}{2}\int \frac{1}{(y-1)(y+1)}\,dy &= \frac{1}{2}\int \frac{-1/2}{y+1} + \frac{1/2}{y-1},\,dy \\
&= -\frac{1}{4}\ln|y+1| + \frac{1}{4}|y-1| \\
&= \frac{1}{4}\ln\left|\frac{y-1}{y+1}\right|
\end{aligned}$$

Hence

$$\frac{1}{4}\ln\left|\frac{y+1}{y-1}\right| = x + D$$

Solving for x leads to

$$\begin{aligned}
\ln\left|\frac{y-1}{y+1}\right| &= 4x + D_1 \quad D_1 = 4D \\
\Rightarrow \left|\frac{y-1}{y+1}\right| &= e^{D_1}e^{4x} \\
\Rightarrow \frac{y-1}{y+1} &= Ke^{4x} \qquad K = \pm e^{D_1},\ \text{Thus } K \text{ is any constant but } 0. \\
\Rightarrow y - 1 &= yKe^{4x} + Ke^{4x} \\
\Rightarrow -1 - Ke^{4x} &= yKe^{4x} - y \\
\Rightarrow y &= \frac{1 + Ke^{4x}}{1 - Ke^{4x}}
\end{aligned}$$

Solving the initial value problem we can either note that $y(0) = 1$ and $y'(0) = 0$ is satisfied by the solution $y = 1$ that we verified above. However we can also get it from

$$1 = \frac{1 + Ke^0}{1 - Ke^0} \Rightarrow 1 = \frac{1+K}{1-K} \Rightarrow 1 - K = 1 + K \Rightarrow K = 0$$

Hence $y = 1$.

6. Find general solutions for the following differential equations

(a) $y''' + 8y'' + 19y' + 10y = 0$

Solution:
The auxiliary equation of this homogeneous D.E. with constant coefficients is

$$m^3 + 8m^2 + 19m + 10 = 0$$

Solving for m can use techniques from classical algebra (Rational root theorem etc.) We can test and see that $m = -1$ is a solution. Factoring yields

$$(m+1)(m^2 + 7m + 10) = (m+1)(m+2)(m+5) = 0$$

which has solutions $m = -1,\ -2,\ -5$.
Thus the general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-5x}.$$

(b) $6y''' + y'' - y' = 0$

Solution:
The auxiliary equation of this homogeneous D.E. with constant coefficients is

$$6m^3 + m^2 - m = 0$$

Factoring yields

$$m(6m^2 + m - 1) = m(3m-1)(2m+1) = 0$$

which has solutions $m = 0,\ -\frac{1}{2},\ \frac{1}{3}$.
Thus the general solution is

$$y = c_1 + c_2 e^{-x/2} + c_3 e^{x/3}.$$