UNIVERSITY OF MANITOBA

DATE: December 13, 2007, 9:00am — FINAL EXAMINATION

PAPER # 384 —

DEPARTMENT & COURSE NO: MATH 1510 — TIME: 2 hours

EXAMINATION: Applied Calculus I — EXAMINER: W. Korytowski, T. Kucera

[9] 1. Evaluate the following limits:

(a) $\lim_{x\to 2} \frac{x^2-4}{8-x^3}$ =

one way: $a^3-b^3=(a-b)(a^2+ab+b^2)$

$8-x^3=(2-x)(4+2x+x^2)$

or: $-x^3+0\cdot x^2+0\cdot x+8 \,|\, x-2$

$$\begin{array}{l} -x^3+0\cdot x^2+0\cdot x+8 \div (x-2) = -x^2-2x-4 \\ \underline{-x^3+2x^2} \\ -2x^2 \\ \underline{-2x^2+4x} \\ -4x+8 \\ \underline{-4x+8} \\ 0 \end{array}$$

$$= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(-x^2-2x-4)}$$

$$= \lim_{x\to 2} \frac{x+2}{-x^2-2x-4} = \frac{2+2}{-4-4-4} = -\frac{1}{3}$$

(b) $\lim_{x\to -\infty} \frac{\sqrt{3x^2+4}}{2x+5} = \lim_{x\to -\infty} \frac{\frac{1}{x}\sqrt{3x^2+4}}{2+\frac{5}{x}} =$

$\left\{ \begin{array}{l} \sqrt{x^2}=|x| \\ |x|=-x \text{ if } x\to-\infty \\ \sqrt{3x^2+4} \\ =-x\sqrt{3+\frac{4}{x^2}} \end{array} \right.$

$$= \lim_{x\to -\infty} \frac{-\sqrt{3+\frac{4}{x^2}}}{2+\frac{5}{x}} = -\frac{\sqrt{3}}{2}$$

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[15] 2. Find $\frac{dy}{dx}$ in each case (DO NOT SIMPLIFY YOUR ANSWERS):

(a) $y = \frac{\sec(x)}{x^4+10}$

$$\frac{dy}{dx} = \frac{\sec x \cdot \tan x \cdot (x^4+10) - 4x^3 \cdot \sec x}{(x^4+10)^2}$$

(b) $y = e^{-x}\cos\left(\frac{\pi}{4}x\right)$

$$\frac{dy}{dx} = \underbrace{-e^{-x}}_{\text{chain rule}} \cos\left(\frac{\pi}{4}x\right) + e^{-x} \cdot \left(-\sin\left(\frac{\pi}{4}x\right) \cdot \frac{\pi}{4}\right)$$

(c) $y = (x^3+3)^{10}$

$$\frac{dy}{dx} = 10\left(x^3+3\right)^9 \cdot 3x^2$$

(d) $y = \ln(3^x + x^2)$

$$\frac{dy}{dx} = \frac{1}{3^x + x^2} \cdot \left(3^x \cdot \ln 3 + 2x\right)$$

[19] 3. Evaluate the following indefinite and definite integrals:

(a) $\int (5x-14)^{10} dx$ $= \left\{ \begin{array}{l} 5x-14=t \\ 5=\frac{dt}{dx} \end{array} \right\} = \int t^{10} \cdot \frac{dt}{5}$

$= \frac{t^{11}}{55} + C = \frac{(5x-(14))^{11}}{55} + C, \quad C \in \mathbb{R}$

(b) $\int_0^1 \frac{x}{(x^2+4)^2} dx$ $= \left\{ \begin{array}{ll} x^2+4=t & x=1: t=5 \\ 2x=\frac{dt}{dx} & x=0: t=4 \\ x\,dx = \frac{dt}{2} & \end{array} \right\} = \int_4^5 \frac{dt}{2t^2}$

$= -\frac{1}{2t}\Big|_4^5 = -\frac{1}{2}\left(\frac{1}{5}-\frac{1}{4}\right) = \frac{1}{40}$

(c) $\int \frac{1}{x\ln(x)} dx$ $= \left\{ \begin{array}{l} \ln x = t \\ \frac{1}{x} = \frac{dt}{dx} \end{array} \right\} = \int \frac{dt}{t} = \ln|t| + c$

$= \ln|\ln x| + c, \quad c \in \mathbb{R}$

(d) $\int_0^5 x\sqrt{9-x}\, dx$ $= \left\{ \begin{array}{ll} 9-x=t & x=5: t=4 \\ -dx=dt & x=0: t=9 \\ x=9-t & \end{array} \right\} = \int_9^4 (9-t)\cdot\sqrt{t}\cdot(-dt)$

$= \int_9^4 (t^{3/2} - 9t^{1/2})\, dt = \left(\frac{2}{5}t^{5/2} - 9\cdot\frac{2}{3}t^{3/2}\right)\Big|_9^4$

$= \frac{64}{5} - 48 - \frac{2\cdot 243}{5} + 162 = \frac{-422}{5} + 114 = \frac{148}{5}$

$2\cdot 243 = 486 \qquad 486-64=422 \qquad 162-48=114 \qquad 5\cdot 114 = 570$

[14] 4. A particle moves on the x-axis with acceleration $a(t) = (4 - 6t)\text{m/s}^2$. At time $t = 0\text{s}$ the position is $x = 3\text{m}$ and the velocity is 4m/s.

(a) What is the velocity of the particle at $t = 1\text{s}$?

$v(t) = \int (4-6t)\,dt = 4t - 3t^2 + c$

$v(0) = 4$ $\quad c = 4$ $\quad v(t) = 4t - 3t^2 + 4$

$v(1) = 4 - 3 + 4 = 5$ (m/s)

(b) What is the position of the particle at $t = 1\text{s}$?

$x(t) = \int (4t - 3t^2 + 4)\,dt = 2t^2 - t^3 + 4t + k$

$x(0) = 3$ $\quad k = 3$ $\quad x(t) = 2t^2 - t^3 + 4t + 3$

$x(1) = 2 - 1 + 4 + 3 = 8$ (m.)

(c) Is the particle speeding up or slowing down at $t = 1\text{s}$? (Explain!)

$\left.\begin{array}{l} v(1) = 5 \\ a(1) = 4 - 6 = -2 \end{array}\right\} v(1)\cdot a(1) < 0$

Ans: slowing down

(d) Is the particle speeding up or slowing down at $t = 3\text{s}$? (Explain!)

$v(3) = 4\cdot 3 - 3\cdot 3^2 + 4 = 12 - 27 + 4 = -11$

$a(3) = 4 - 6\cdot 3 = 4 - 18 = -14$

$v(3)\cdot a(3) > 0$ $\quad$ Ans: speeding up

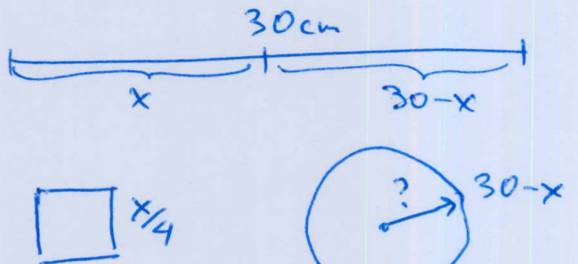
[8] 5. Find the absolute maximum value and the absolute minimum value of $g(t) = t^2 e^{-t}$ on the interval $[-1, 3]$.

See 2005 4 (c)

[8] 6. Consider the following word problem:

> "A jeweler is going to cut a piece of gold wire 30cm long into two pieces. One piece will be bent into a square, the other piece will be bent into a circle. Find the length of the piece that will be bent into a square so that the total area enclosed by the square and the circle is maximized."

DO NOT SOLVE THIS WORD PROBLEM! Just set up the equivalent mathematical question: draw a neat sketch illustrating the situation described; identify the variables involved; set up the equations described by the problem; and find a function of one variable to be maximized. State any restrictions on the variables involved (that is, determine the domain of the function).



x is in cm

x is for the square

$30-x$ – for the circle

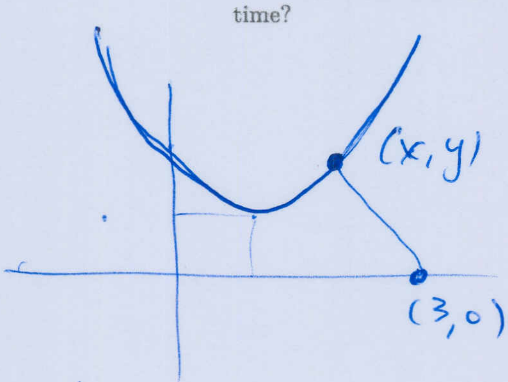
$r = \dfrac{30-x}{2\pi}$ – radius of the circle

$a = \dfrac{x}{4}$ – side of the square

$$A = a^2 + \pi r^2 = \underbrace{\frac{x^2}{16} + \pi\left(\frac{30-x}{2\pi}\right)^2}_{\text{f-n to maximize}}$$

interval: $0 \le x \le 30$ (or $[0, 30]$)

[10] 7. A particle moves on the parabola $y = x^2 - 2x + 2$, x and y measured in metres. When the particle is at the point $(2, 2)$, its x-coordinate is decreasing at $\frac{1}{20}$m/s. How fast is the distance from the particle to the point $(3, 0)$ changing at this time?



$$\left.\frac{dx}{dt}\right|_{(x,y)=(2,2)} = -\frac{1}{20}$$

ℓ - distance from (x, y) to $(3, 0)$

$$\left.\frac{d\ell}{dt}\right|_{(x,y)=(2,2)} = ?$$

$$\ell = \sqrt{(x-3)^2 + y^2}$$

$$\frac{d\ell}{dt} = \frac{1}{2\sqrt{(x-3)^2 + y^2}} \cdot \left(2(x-3)\cdot\frac{dx}{dt} + 2y\cdot\frac{dy}{dt}\right)$$

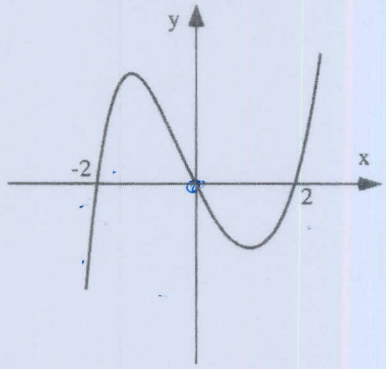
$$\frac{dy}{dt} = (2x-2)\cdot\frac{dx}{dt} \qquad \left.\frac{dy}{dt}\right|_{(x,y)=(2,2)} = (2\cdot 2 - 2)\cdot\frac{-1}{20} = -\frac{1}{10}$$

$$\left.\frac{d\ell}{dt}\right|_{(x,y)=(2,2)} = \frac{1}{\sqrt{1+4}} \cdot \left((-1)\cdot\left(-\frac{1}{20}\right) + 2\cdot\left(-\frac{1}{10}\right)\right)$$

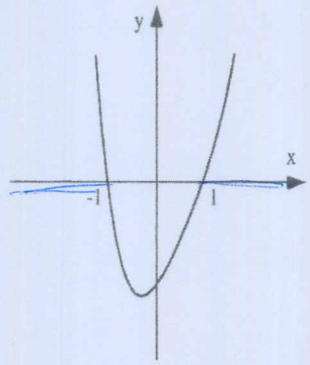
$$= \frac{1}{\sqrt{5}}\left(\frac{1}{20} - \frac{1}{5}\right) = \frac{-3}{20\sqrt{5}}$$

Ans: decreasing at $\frac{3}{20\sqrt{5}}$ m/s.

[17] 8. Consider the following two sketches and table of information about the function $f(x)$, which is defined and continuous on $(-\infty, \infty)$:



$y = f'(x)$



$y = f''(x)$

x	-3	-2	-1	0	1	2	3
$f(x)$	0	-3	-2	0	$-1/2$	-1	0

This information includes EVERYTHING that is "interesting" about the curve. Please note that there are no "tricks" hidden in minor flaws in the sketches!

(a) On what intervals is f increasing? $(-2, 0)$, $(2, \infty)$

(b) On what intervals is f decreasing? $(-\infty, -2)$, $(0, 2)$

(c) Find the coordinates of all the local maxima of f. $(0, 0)$

(d) Find the coordinates of all the local minima of f. $(-2, -3)$, $(2, -1)$

(e) On what intervals is f concave up? $(-\infty, -1)$, $(1, \infty)$

(f) On what intervals is f concave down? $(-1, 1)$

(g) Find the coordinates of all the inflection points of f. $(-1, -2)$, $(1, -\frac{1}{2})$

(h) Give a rough sketch of the graph of $y = f(x)$.