MATH 1700 Problem Workshop 11 Solutions

1. (a) Solving for y yields
$$y = \frac{x^3}{24} + \frac{2}{x}$$
which has derivative
$$\frac{dy}{dx} = \frac{x^2}{8} - \frac{2}{x^2}.$$
Hence the length is
$$\begin{aligned}
L &= \int_2^3 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{x^2}{8} - \frac{2}{x^2}\right)^2}\, dx \\
&= \int_2^3 \sqrt{1 + \left(\frac{x^4}{64} - \frac{1}{2} + \frac{4}{x^4}\right)}\, dx \\
&= \int_2^3 \sqrt{\frac{x^4}{64} + \frac{1}{2} + \frac{4}{x^4}}\, dx \\
&= \int_2^3 \left(\frac{x^2}{8} + \frac{2}{x^2}\right) dx \\
&= \frac{x^3}{24} - \frac{2}{x}\bigg|_2^3 \\
&= \left(\frac{3^3}{24} - \frac{2}{3}\right) - \left(\frac{2^3}{24} - \frac{2}{2}\right) \\
&= \left(\frac{9}{8} - \frac{2}{3}\right) - \left(\frac{1}{3} - 1\right) \\
&= \frac{9}{8}.
\end{aligned}$$

(b)
$$\frac{dy}{dx} = \frac{-\sin x}{\cos x} = -\tan x.$$
Hence the length is

$$
\begin{aligned}
L &= \int_0^{\pi/4} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_0^{\pi/4} \sqrt{1 + (-\tan x)^2}\, dx \\
&= \int_0^{\pi/4} \sec x\, dx \\
&= \ln|\sec x + \tan x|\,\Big|_0^{\pi/4} \\
&= \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \ln|\sec 0 + \tan 0| \\
&= \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| \\
&= \ln(\sqrt{2} + 1)
\end{aligned}
$$

2. (a)

$$
\frac{dx}{dt} = 2t\sin t + t^2\cos t, \quad \frac{dy}{dt} = 2t\cos t - t^2\sin t
$$

Hence

$$
\begin{aligned}
\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= (2t\sin t + t^2\cos t)^2 + (2t\cos t - t^2\sin t)^2 \\
&= (4t^2\sin^2 t + 4t^3\sin t\cos t + t^4\cos^2 t) + (4t^2\cos^2 t - 4t^3\sin t\cos t + t^4\sin^2 t) \\
&= 4t^2\sin^2 t + 4t^2\cos^2 t + t^4\sin^2 t + t^4\cos^2 t \\
&= 4t^2 + t^4
\end{aligned}
$$

Hence the length is

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{4t^2 + t^4}\, dt \\
&= \int_0^{2\pi} t\sqrt{t^2 + 4}\, dt \\
&= \int_4^{4\pi^2+4} \frac{1}{2}w^{1/2}\, dw \\
&= \frac{1}{3}w^{3/2}\Big|_4^{4\pi^2+4} \\
&= \frac{1}{3}(4\pi^2 + 4)^{3/2} - \frac{1}{3}(4)^{3/2} \\
&= \frac{1}{3}(4\pi^2 + 4)^{3/2} - \frac{8}{3}
\end{aligned}
$$

(b)

$$\frac{dx}{dt} = 3t^2, \quad \frac{dy}{dt} = -2t$$

Hence

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (3t^2)^2 + (-2t)^2 = 9t^4 + 4t^2$$

Hence the length is

$$\begin{aligned}
L &= \int_{-1}^{2} \sqrt{9t^4 + 4t^2}\, dt \\
&= \int_{-1}^{2} |t| \sqrt{9t^2 + 4}\, dt \\
&= -\int_{-1}^{0} t\sqrt{9t^2 + 4}\, dt + \int_{0}^{2} t\sqrt{9t^2 + 4}\, dt \\
&= -\int_{13}^{4} \frac{1}{18} w^{1/2}\, dw + \int_{4}^{40} \frac{1}{18} w^{1/2}\, dw \\
&= -\frac{1}{27} w^{3/2}\Big|_{13}^{4} + \frac{1}{27} w^{3/2}\Big|_{4}^{40} \\
&= -\left(\frac{1}{27}(4)^{3/2} - \frac{1}{27}(13)^{3/2}\right) + \left(\frac{1}{27}(40)^{3/2} - \frac{1}{27}(4)^{3/2}\right) \\
&= \frac{1}{27}(40)^{3/2} + \frac{1}{27}(13)^{3/2} - \frac{16}{27}
\end{aligned}$$

3. (a) The cardioid goes from $\theta = 0$ to 2π. Hence

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(-\sin\theta)^2 + (1+\sin\theta)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{\sin^2\theta + 1 + 2\cos\theta + \cos^2\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{2 + 2\cos\theta}\, d\theta \\
&= \int_0^{2\pi} \sqrt{4\cos^2(\theta/2)}\, d\theta \\
&= \int_0^{2\pi} |2\cos(\theta/2)|\ d\theta \\
&= \int_0^{\pi} 2\cos(\theta/2)\, d\theta - \int_{\pi}^{2\pi} 2\cos(\theta/2)\, d\theta \\
&= 4\sin(\theta/2)|_0^{\pi} - 4\sin(\theta/2)|_{\pi}^{2\pi} \\
&= (4\sin(\pi/2) - 4\sin(0)) - (4\sin(\pi) - 4\sin(\pi/2)) \\
&= (4-0) - (0-4) \\
&= 8.
\end{aligned}
$$

(b)

$$
\begin{aligned}
L &= \int_0^{2\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\, d\theta \\
&= \int_0^{\pi} \sqrt{(2\theta)^2 + (\theta^2)^2}\, d\theta \\
&= \int_0^{\pi} \sqrt{4\theta^2 + \theta^4}\, d\theta \\
&= \int_0^{\pi} \theta\sqrt{\theta^2 + 4}\, d\theta \\
&= \int_4^{\pi^2+4} \frac{1}{2}\sqrt{w}\, dw \\
&= \frac{1}{3} w^{3/2}\Big|_4^{\pi^2+4} \\
&= \frac{1}{3}(\pi^2+4)^{3/2} - \frac{1}{3}(4)^{3/2} \\
&= \frac{1}{3}(\pi^2+4)^{3/2} - \frac{8}{3}
\end{aligned}
$$

4. (a) We can use symmetry to find the length of

$$b^2x^2 + a^2y^2 = a^2b^2 \Rightarrow y = \sqrt{\frac{a^2b^2 - b^2x^2}{a^2}} = b\left(1 - \frac{x^2}{a^2}\right)^{1/2}.$$

$$\frac{dy}{dx} = \frac{b}{2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right).$$

Hence

$$\begin{aligned} 1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(\frac{b}{2}\left(\frac{a^2 - x^2}{a^2}\right)^{-1/2}\left(-\frac{2x}{a^2}\right)\right)^2 \\ &= 1 + \left(\frac{4a^2b^2x^2}{4a^4(a^2 - x^2)}\right) \\ &= 1 + \left(\frac{b^2x^2}{a^2(a^2 - x^2)}\right). \end{aligned}$$

Hence

$$L = 4\int_0^a \sqrt{1 + \left(\frac{b^2x^2}{a^2(a^2 - x^2)}\right)}\,dx$$

(b) We can use a parameterization

$$x = a\cos t, \quad y = b\sin t, \quad 0 \leq t \leq 2\pi.$$

Hence

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = a^2\sin^2 t + b^2\cos^2 t.$$

Hence

$$L = \int_0^{2\pi} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt. \text{ or}$$
$$4\int_0^{\pi/2} \sqrt{a^2\sin^2 t + b^2\cos^2 t}\,dt.$$