MATH 2130 Problem Workshop 5 Solutions

1. The maximum and minimum can occur either at a critical point or along the boundry.

$$f_x = 2x, f_y = -2y$$

which are always defined and are equal to 0 when $x = y = 0$.

Along the boundry $x^2+y^2 = 1$, the function becomes $g(x) = f(x, \pm\sqrt{1-x^2}) = 2x^2 - 1$ and we are finding extrema on the interval $[-1, 1]$. Hence we wish to find critical points of g.

$$0 = g'(x) = 4x = 0 \Rightarrow x = 0.$$

Therefore the maximum along the boundry occurs when $x = -1, 0$ or 1. $g(-1) = 1, g(0) = -1, g(1) = 1$ and the critical point yields $f(0, 0) = 0$. Therefore the maximum is 1 which occurs $(\pm 1, 0)$, and the minimum is -1 which occurs $(0, \pm 1)$.

2. The maximum can occur either at a critical point or along the boundry.

$$f_x = y(3-x-2y) - xy = y(3-2x-2y), f_y = x(3-x-2y) - 2xy = x(3-x-4y)$$

which are always defined. Either $x = 0, y = 0$ or both $3 - 2x - 2y$ and $3 - x - 4y$ are both zero.

If $x = 0$, then the first equation becomes $y(3 - 2y)$ and so $y = 0, 3/2$.

If $y = 0$, then the second equation becomes $x(3 - x)$ and so $y = 0, 3$. Only the critical point $(0, 0)$ is in the region R.

If $3 - 2x - 2y = 0, 3 - x - 4y = 0$, then the first equation minus two times the second yields

$$-3 + 6y = 0 \Rightarrow y = 1/2 \Rightarrow x = 1$$

which is also outside the interval. Hence the only critical point in R is $(0, 0)$.

The boundry is composed of $x = 0$ where $0 \le y \le 1$, $y = 0$ where $0 \le x \le 1$ and $y = 1 - x$ where $0 \le x \le 1$.

If $x = 0$, $g(y) = f(0, y) = 0$ which has a maximum of 0 everywhere.

If $y = 0$, $g(x) = f(x, 0) = 0$ which has a maximum of 0 everywhere.

If $y = 1-x$, $g(x) = f(x, 1-x) = x(1-x)(3-x-2(1-x)) = x(1-x)(x+1) = x - x^3$. Finding the derivative

$g'(x) = 1 - 3x^2 = 0$ when $x = 1/\sqrt{3}$. Hence we test $x = 0, 1/\sqrt{3}, 1$ giving

$$g(0) = 0, g(1) = 0 \text{ and } g\left(\frac{1}{\sqrt{3}}\right) = \left(\frac{1}{\sqrt{3}}\right) - \left(\frac{1}{\sqrt{3}}\right)^3 = \frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}.$$

Since $(0, 0)$ yields $f(0, 0) = 0$ we get a maximum of $\frac{2}{3\sqrt{3}}$ at the point $\left(\frac{1}{\sqrt{3}}, 1 - \frac{1}{\sqrt{3}}\right)$.

3. The maximum can occur either at a critical point or along the boundry.

$$f_x = 2x + 2, f_y = -2y + 9/2$$

which are always defined and are equal to 0 when $x = -1, y = 9/4$ which is outside the region.

The boundry is composed of $x = 0$ where $-1 \le y \le 1$ and $x = 1-y^2$ where $-1 \le y \le 1$. If $x = 0$, $g(y) = f(0, y) = -y^2 + \frac{9y}{2}$. Finding a critical point of g gives

$$g'(y) = -2y + \frac{9}{2} = 0 \Rightarrow y = \frac{9}{4}$$

which is outside the interval. Hence the maximum of g occurs at an endpoint. $g(-1) = -1 - 9/2 = -11/2, g(1) = -1 + 9/2 = 7/2$.

If $x = 1 - y^2$, $h(y) = f(1 - y^2, y) = (1 - y^2)^2 - y^2 + 2(1 - y^2) + \frac{9y}{2} = y^4 - 5y^2 + \frac{9y}{2} + 3$ Finding a critical point of h gives

$$g'(y) = 4y^3 - 10y + \frac{9}{2} = 0 \Rightarrow 0 = 8y^3 - 20y + 9 = (2y - 1)(4y^2 + 2y - 9)$$

Hence

$$y = \frac{1}{2}, y = \frac{-2 \pm \sqrt{4 + 144}}{8} = \frac{-1 \pm \sqrt{37}}{4}.$$

The latter two solutions are outside the interval $[-1, 1]$. Hence we test $-1, 1/2, 1$ in h.

$$h(-1) = 1-5-\frac{9}{2}+3 = -\frac{11}{2}, h\left(\frac{1}{2}\right) = \frac{1}{16} - \frac{5}{4} + \frac{9}{4} + 3 = \frac{65}{16}, h(1) = 1-5+\frac{9}{2}+3 = \frac{7}{2}.$$

Hence the maximum value is $\frac{65}{16}$ at $(3/4, 1/2)$.

4. The anti-derivative of $\sqrt{y - x}$ with respect to y is $\frac{2}{3}(y - x)^{3/2}$ Hence

$$\begin{aligned}
\int_{-2}^{0}\int_{0}^{-x} \sqrt{y - x}dydx &= \int_{-2}^{0} \frac{2}{3}(y - x)^{3/2}\Big|_{0}^{-x} dx \\
&= \frac{2}{3}\int_{-2}^{0} \left((-2x)^{3/2} - (-x)^{3/2}\right)dx \\
&= \frac{2}{3}\left(\frac{(-2x)^{5/2}}{5} - \frac{(-x)^{5/2}}{5/2}\right)\Big|_{-2}^{0} \\
&= \frac{2}{3}\left(\frac{(-2(0))^{5/2}}{5} - \frac{(-0)^{5/2}}{5/2}\right) - \frac{2}{3}\left(\frac{(-2(-2))^{5/2}}{5} - \frac{(-(-2))^{5/2}}{5/2}\right) \\
&= -\frac{2}{3}\left(\frac{4^{5/2}}{5} - \frac{2^{5/2}}{5/2}\right) \\
&= \frac{16(4 - \sqrt{2})}{15}.
\end{aligned}$$

5. The region is bounded by $x^2-1 \le y \le -x^2, -1/\sqrt{2} \le x \le 1/\sqrt{2}$. Hence the integral is

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2-1}^{-x^2} (x^3y^3 - 3xy^2 + y) dy dx.$$

We could do this directly, but it would be a mess. One way to simplify this is by noting that the region is symmetric about the x-axis and the function $x^3y^3 - 3xy^2$ is an odd function (with respect to x). Hence

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2-1}^{-x^2} (x^3y^3 - 3xy^2) dy dx = 0$$

This leaves us with

$$\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2-1}^{-x^2} y dy dx$$

which is much simpler.

$$\begin{aligned}
\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \int_{x^2-1}^{-x^2} y dy dx &= \frac{1}{2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} y^2 \Big|_{x^2-1}^{-x^2} dx \\
&= \frac{1}{2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \big(x^4 - (x^4 - 2x^2 + 1)\big) dx \\
&= \frac{1}{2} \int_{-1/\sqrt{2}}^{1/\sqrt{2}} (2x^2 - 1) dx \\
&= \frac{1}{2} \left(\frac{2x^3}{3} - x \right) \Big|_{-1/\sqrt{2}}^{1/\sqrt{2}} \\
&= \frac{1}{2} \left(\left(\frac{2(1/\sqrt{2})^3}{3} - (1/\sqrt{2}) \right) - \left(\frac{2(-1/\sqrt{2})^3}{3} - (-1/\sqrt{2}) \right) \right) \\
&= \frac{1}{2} \left(\left(\frac{1}{3\sqrt{2}} - \frac{1}{\sqrt{2}} \right) - \left(-\frac{1}{3\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right) \\
&= \frac{1}{2} \left(-\frac{4}{3\sqrt{2}} \right) \\
&= -\frac{2}{3\sqrt{2}}
\end{aligned}$$

6. We may try to integrate e^{y^2} with respect to y, but that is impossible. Hence we need to switch the order of integration to allow us to integrate in term of x first. The region R is bounded by $-3x \le y \le 6, -2 \le x \le 0$. By noting the line $y = -3x$ can be

rearranged to be $x = -\frac{1}{3}y$, the region can be changed to $-\frac{1}{3}y \leq x \leq 0, 0 \leq y \leq 6$. Therefore the integral becomes

$$\begin{aligned}\int_0^6 \int_{-y/3}^0 e^{y^2} dxdy &= \int_0^6 e^{y^2} x|_{-y/3}^0 dy \\ &= \int_0^6 \frac{ye^{y^2}}{3} dy \\ &= \frac{e^{y^2}}{6}|_0^6 \\ &= \frac{e^{6^2}}{6} - \frac{e^{0^2}}{6} \\ &= \frac{e^{36}-1}{6}\end{aligned}$$

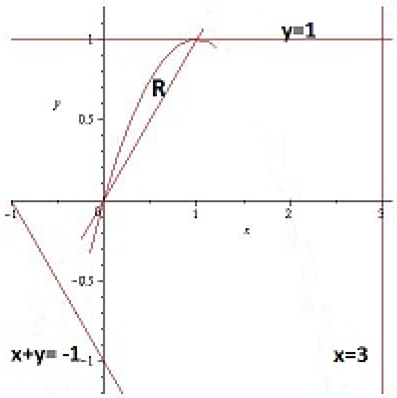
7. The region is bounded by $x \leq y \leq 2x, 2 \leq x \leq 3$. Hence the integral becomes

$$\begin{aligned}\int_2^3 \int_x^{2x} \frac{1}{y-1} dydx &= \int_2^3 \ln|y-1|\,|_x^{2x} dx \\ &= \int_2^3 \ln|2x-1| - \ln|x-1|\, dx \\ &= \int_2^3 \big(\ln(2x-1) - \ln(x-1)\big) dx\end{aligned}$$

Recall using integration by parts with $u = \ln x, dv = dx$ that $\int \ln x dx = x\ln - \int 1dx = x\ln x - x + C$. Hence the integral above becomes

$$\begin{aligned}\int_2^3 \int_x^{2x} \frac{1}{y-1} dydx &= \int_2^3 \big(\ln(2x-1) - \ln(x-1)\big) dx \\ &= \left(\frac{(2x-1)\ln(2x-1) - (2x-1)}{2} - \big((x-1)\ln(x-1) - (x-1)\big)\right)\Bigg|_2^3 \\ &= \left(\frac{5\ln 5 - 5}{2} - \big(2\ln 2 - 2\big)\right) - \left(\frac{3\ln 3 - 3}{2} - \big(1\ln 1 - 1\big)\right) \\ &= \frac{5\ln 5}{2} - \frac{3\ln 3}{2} - 2\ln 2\end{aligned}$$

8. The volumes of revolutions are $\iint_R 2\pi d dA$ where R is the region and d is the distance from the point (x, y) to whatever line we are rotating about. A graph of the region along with the 3 lines being rotated is given below



The region R is bounded above by $y = 2x - x^2$ and below by $y = x$. To see where they intersect, we set them equal to each other

$$2x - x^2 = x \Rightarrow x - x^2 = 0 \Rightarrow x(1-x) = 0 \Rightarrow x = 0, 1.$$

Therefore the region R is

$$x \leq y \leq 2x - x^2, \qquad 0 \leq x \leq 1.$$

(a) For the first line $x = 3$, the distance is $d = 3 - x$. Therefore the volume is

$$\begin{aligned}
\iint_R 2\pi d dA &= \int_0^1 \int_x^{2x-x^2} 2\pi(3-x)dydx \\
&= 2\pi \int_0^1 (3-x)y\Big]_x^{2x-x^2} dx \\
&= 2\pi \int_0^1 (3-x)\big[(2x-x^2) - x\big]dx \\
&= 2\pi \int_0^1 (3x - 4x^2 + x^3)dx \\
&= 2\pi \left(\frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{1}{4}x^4\right]_0^1 \\
&= 2\pi\left(\frac{3}{2} - \frac{4}{3} + \frac{1}{4}\right) - 0 \\
&= 2\pi\left(\frac{5}{12}\right) \\
&= \frac{5\pi}{6}.
\end{aligned}$$

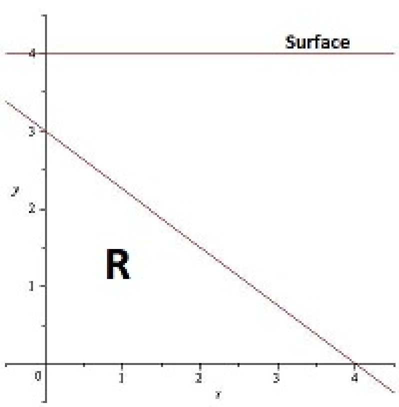
(b) For the first line $y = 1$, the distance is $d = 1 - y$. Therefore the volume is

$$\begin{aligned}
\iint_R 2\pi d dA &= \int_0^1 \int_x^{2x-x^2} 2\pi(1-y)dydx \\
&= 2\pi \int_0^1 \left(y - \frac{y^2}{2} \right]_x^{2x-x^2} dx \\
&= 2\pi \int_0^1 \left[\left(2x - x^2 - \frac{(2x-x^2)^2}{2} \right) - \left(x - \frac{x^2}{2} \right) \right] dx \\
&= 2\pi \int_0^1 \left(x - \frac{5}{2}x^2 + 2x^3 - \frac{1}{2}x^4 \right) dx \\
&= 2\pi \left(\frac{1}{2}x^2 - \frac{5}{6}x^3 + \frac{1}{2}x^4 - \frac{1}{10}x^5 \right]_0^1 \\
&= 2\pi \left(\frac{1}{2} - \frac{5}{6} + \frac{1}{2} - \frac{1}{10} \right) - 0 \\
&= 2\pi \left(\frac{1}{15} \right) \\
&= \frac{2\pi}{15}.
\end{aligned}$$

(c) For the first line $x + y + 1 = 0$, the distance is $d = \frac{|x+y+1|}{\sqrt{2}}$. The region is above the line, so that means that $x + y + 1 \geq 0$ meaning we can just drop the absolute value. Another way to see this is to test points in the interval, or to note that both $x, y \geq 0$ so $x + y + 1 \geq 0$. Therefore the volume is

$$
\begin{aligned}
\iint_R 2\pi d dA &= \int_0^1 \int_x^{2x-x^2} 2\pi\left(\frac{x+y+1}{\sqrt{2}}\right) dydx \\
&= \sqrt{2}\pi \int_0^1 \int_x^{2x-x^2} (x+y+1) dydx \\
&= \sqrt{2}\pi \int_0^1 \left(xy + \frac{y^2}{2} + y\right]_x^{2x-x^2} dx \\
&= \sqrt{2}\pi \int_0^1 \left[\left(x(2x-x^2) + \frac{(2x-x^2)^2}{2} + (2x-x^2)\right) - \left(x^2 + \frac{x^2}{2} + x\right)\right] dx \\
&= \sqrt{2}\pi \int_0^1 \left(x + \frac{3}{2}x^2 - 3x^3 + \frac{1}{2}x^4\right) dx \\
&= \sqrt{2}\pi \left(\frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{3}{4}x^4 + \frac{1}{10}x^5\right]_0^1 \\
&= \sqrt{2}\pi \left(\frac{1}{2} + \frac{1}{2} - \frac{3}{4} + \frac{1}{10}\right) - 0 \\
&= \sqrt{2}\pi \left(\frac{7}{20}\right) \\
&= \frac{7\sqrt{2}\pi}{20}.
\end{aligned}
$$

9. A picture of the triangle is the region below. Note that we could set up the coordinates differently.



The hypoteneuse of the triangle has the equation $y = 3 - \frac{3}{4}x$ where $0 \leq x \leq 4$. The formula for finding the fluid pressure is

$$\iint_R 9.81 \rho d dA$$

where ρ is the density of the liquid and d is the distance to the surface. Here $\rho = 950$ and $d = 4 - y$. Hence the region is

$$
\begin{aligned}
\iint_R 9.81\rho d\, dA &= \int_0^4 \int_0^{3-3x/4} (9.81)(950)(4-y)dydx \\
&= (9.81)(950) \int_0^4 \left(4y - \frac{y^2}{2}\right]_0^{3-3x/4} dx \\
&= (9.81)(950) \int_0^4 \left[\left(4\left(3 - \frac{3x}{4}\right) - \frac{(3-3x/4)^2}{2}\right) - 0\right] dx \\
&= (9.81)(950) \int_0^4 \left[\left(\frac{15}{2} - \frac{3}{4}x - \frac{9}{32}x^2\right)\right] dx \\
&= (9.81)(950)\left(\frac{15}{2}x - \frac{3}{8}x^2 - \frac{3}{32}x^3\right]_0^4 \\
&= (9.81)(950)\left(\frac{15}{2}(4) - \frac{3}{8}(16) - \frac{3}{32}(64)\right) - 0 \\
&= (9.81)(950)(30-6-6) \\
&= (9.81)(950)(18) \\
&= 167751N \approx 1.68 \times 10^5.
\end{aligned}
$$