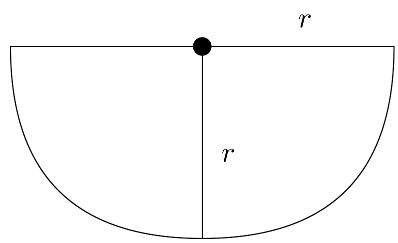
MATH 2130 – Tutorial Problem Solutions, Thu Mar 22

Applications of Double Integrals

Example. A trough has a semicircular cross section with radius r, as shown below.

(a) What is the force due to fluid pressure on one of the semicircular sides of the trough if it is completely filled with a fluid of constant density ρ?

(b) What is the force due to fluid pressure on one of the semicircular sides of the trough if it is filled with the same fluid to an arbitrary depth H, $H < r$?



Solution. Let us choose coordinates so that the origin is at the center of the semicircle. Then the equation of the semicircle is $y = -\sqrt{r^2 - x^2}$.

(a) Assume that the tank is completely full. The point (x, y) within the semicircle (where $y \leq 0$) lies at depth $-y$. Therefore the integral to be evaluated is

$$F = \iint_R g\rho(-y)\, dA,$$

where R is the semicircle.

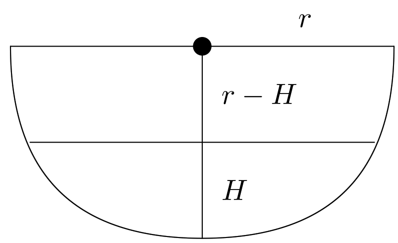
The region R lies within $-r \leq y \leq 0$. At each value of y, we have $-\sqrt{r^2 - y^2} \leq x \leq \sqrt{r^2 - y^2}$. Thus

$$\begin{aligned}
F &= \int_{-r}^{0} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} g\rho(-y)\, dx\, dy \\
&= -g\rho \int_{-r}^{0} \Big[xy \Big]_{x=-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dy \\
&= -2g\rho \int_{-r}^{0} y\sqrt{r^2 - y^2}\, dy \\
&= -2g\rho \left[-\frac{1}{3}(r^2 - y^2)^{3/2} \right]_{y=-r}^{0} \\
&= \frac{2g\rho}{3} r^3.
\end{aligned}$$

Note that we could have also used the symmetry of the circle to write

$$F = 2\int_{-r}^{0} \int_{0}^{\sqrt{r^2-y^2}} g\rho(-y)\, dx\, dy.$$

(b) Now assume that the depth of fluid in the trough is H for some $H < r$.



The region of integration is that portion of the semicircle lying below the line $y = -(r-H) = H - r$. A point (x, y) within this region is at a depth $H - r - y$ below the surface of the fluid. The total force is

$$\begin{aligned} F =& \int_{-r}^{H-r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} g\rho(H-r-y)\,dx\,dy \\ =& 2g\rho \int_{-r}^{H-r} \left[(H-r)\sqrt{r^2-y^2} - y\sqrt{r^2-y^2} \right] dy \\ =& 2g\rho(H-r) \int_{-r}^{H-r} \sqrt{r^2-y^2}\,dy - 2g\rho \int_{-r}^{H-r} y\sqrt{r^2-y^2}\,dy. \end{aligned}$$

The second integral in the above expression becomes

$$\begin{aligned} -2g\rho \int_{-r}^{H-r} y\sqrt{r^2-y^2}\,dy =& -2g\rho \left[-\frac{1}{3}(r^2-y^2)^{3/2} \right]_{y=-r}^{H-r} \\ =& \frac{2\rho g}{3}\left(r^2-(H-r)^2\right)^{3/2}. \end{aligned}$$

The first integral requires a trig substitution. Let $\frac{y}{r} = \sin\theta$. Then $dy = r\cos\theta\,d\theta$, and $\sqrt{r^2-y^2} = r\cos\theta$. When $y = -r$, $\theta = -\frac{\pi}{2}$. When $y = H - r$, $\theta = \sin^{-1}\left(\frac{H-r}{r}\right)$. We get

$$\begin{aligned} 2g\rho(H-r) \int_{-r}^{H-r} \sqrt{r^2-y^2}\,dy =& 2g\rho(H-r)r^2 \int_{-\pi/2}^{\sin^{-1}((H-r)/r)} \cos^2\theta\,d\theta \\ =& 2g\rho(H-r)r^2 \int_{-\pi/2}^{\sin^{-1}((H-r)/r)} \frac{\cos(2\theta)+1}{2}\,d\theta \\ =& g\rho(H-r)r^2 \left[\frac{1}{2}\sin(2\theta) + \theta \right]_{\theta-\pi/2}^{\sin^{-1}((H-r)/r)} \\ =& g\rho(H-r)r^2 \left[\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{H-r}{r}\right)\right) + \sin^{-1}\left(\frac{H-r}{r}\right) + \frac{\pi}{2} \right]. \end{aligned}$$

The total force is then

$$F = \frac{2\rho g}{3}\left(r^2-(H-r)^2\right)^{3/2} + g\rho(H-r)r^2 \left[\frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{H-r}{r}\right)\right) + \sin^{-1}\left(\frac{H-r}{r}\right) + \frac{\pi}{2} \right].$$

Example. Let R be the region in the xy-plane that is bounded by $y = x^2$ and $y = 1$. Assume that R contains a mass described by the density function $\rho(x, y) = x^2 + y$. Find the center of mass of R.

Solution. The points of intersection between $y = x^2$ and $y = 1$ are $(-1, 1)$ and $(1, 1)$. The region R lies within $-1 \le x \le 1$. At each x, $x^2 \le y \le 1$.

First, we need to find the total mass M contained in R. This is

$$\begin{aligned}
M = \iint_R \rho(x, y)\, dA &= \int_{-1}^{1} \int_{x^2}^{1} (x^2 + y)\, dy\, dx \\
&= \int_{-1}^{1} \left[x^2 y + \frac{1}{2} y^2 \right]_{y=x^2}^{1} dx \\
&= \int_{-1}^{1} \left(\frac{1}{2} + x^2 - \frac{3}{2} x^4 \right) dx \\
&= \left[\frac{1}{2} x + \frac{1}{3} x^3 - \frac{3}{10} x^5 \right]_{x=-1}^{1} \\
&= \frac{16}{15}.
\end{aligned}$$

Now, the center of mass is the point $(\overline{x}, \overline{y})$, where

$$\begin{aligned}
\overline{x} =& \frac{1}{M} \iint_R x\rho(x, y)\, dA = \frac{15}{16} \int_{-1}^{1} \int_{x^2}^{1} x(x^2 + y)\, dy\, dx \\
=& \frac{15}{16} \int_{-1}^{1} \left[x^3 y + \frac{1}{2} x y^2 \right]_{y=x^2}^{1} dx \\
=& \frac{15}{16} \int_{-1}^{1} \left(\frac{1}{2} x + x^3 - \frac{3}{2} x^5 \right) dx \\
=& \frac{15}{16} \left[\frac{1}{4} x^2 + \frac{1}{4} x^4 - \frac{1}{4} x^6 \right]_{x=-1}^{1} \\
=& \ 0.
\end{aligned}$$

This is exactly the result we expect, since the region R and the density function $\rho(x, y)$ are both symmetric under $x \mapsto -x$. The y-coordinate is

$$\begin{aligned}
\overline{y} =& \frac{1}{M} \iint_R y\rho(x, y)\, dA = \frac{15}{16} \int_{-1}^{1} \int_{x^2}^{1} y(x^2 + y)\, dy\, dx \\
=& \frac{15}{16} \int_{-1}^{1} \left[\frac{1}{2} x^2 y^2 + \frac{1}{3} y^3 \right]_{y=x^2}^{1} dx \\
=& \frac{15}{16} \int_{-1}^{1} \left(\frac{1}{2} x^2 + \frac{1}{3} - \frac{5}{6} x^6 \right) dx \\
=& \frac{15}{16} \left[\frac{1}{6} x^3 + \frac{1}{3} x - \frac{5}{42} x^7 \right]_{x=-1}^{1} \\
=& \frac{15}{16} \cdot \frac{16}{21} = \frac{5}{7}.
\end{aligned}$$

Thus the center of mass of the given region is $(0, \frac{5}{7})$.

Example. Set up, but don't evaluate, an integral for the area of the surface $z = 4 - (x^2 + y^2)$ that lies above the plane $x + 3y + z = 3$.

Solution. The surface $z = 4 - (x^2 + y^2)$ is a paraboloid, centered on the z-axis, opening in the negative z-direction with its maximum at $z = 4$. The surface $x + 3y + z = 3$ is a plane with intercepts $(3, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 3)$.

For a point on the paraboloid to lie above the plane, we must have $z \geq 3 - x - 3y$. Therefore $4 - x^2 - y^2 \geq 3 - x - 3y$, which rearranges to

$$x^2 - x + y^2 - 3y \leq 1.$$

If we complete the square in x and y, we find

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 \leq 1 + \frac{1}{4} + \frac{9}{4} = \frac{7}{2}.$$

Thus the projection of the desired piece of the paraboloid onto the xy-plane is the circular disk $\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 \leq \frac{7}{2}$. This is the region of integration for the surface area calculation.

The disk lies within $\frac{1}{2} - \sqrt{\frac{7}{2}} \leq x \leq \frac{1}{2} + \sqrt{\frac{7}{2}}$. At each value of x, $\frac{3}{2} - \sqrt{\frac{7}{2} - \left(x - \frac{1}{2}\right)^2} \leq y \leq \frac{3}{2} + \sqrt{\frac{7}{2} - \left(x - \frac{1}{2}\right)^2}$.

It remains to find the integrand. We are treating z as a function of x and y, so the required expression is

$$\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}.$$

From the equation of the paraboloid, we get

$$\frac{\partial z}{\partial x} = -2x, \quad \frac{\partial z}{\partial y} = -2y.$$

Therefore the integrand is

$$\sqrt{1 + 4x^2 + 4y^2}.$$

Thus the surface area of the paraboloid is given by the integral

$$\int_{\frac{1}{2}-\sqrt{\frac{7}{2}}}^{\frac{1}{2}+\sqrt{\frac{7}{2}}} \int_{\frac{3}{2}-\sqrt{\frac{7}{2}-\left(x-\frac{1}{2}\right)^2}}^{\frac{3}{2}+\sqrt{\frac{7}{2}-\left(x-\frac{1}{2}\right)^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx.$$

Example. Set up and evaluate an integral for the area of that part of the surface $z = x^{3/2} + 2y^{3/2}$ that is cut off by the plane $x + 4y = 4$.

Solution. We are given z as a function of x and y. The partial derivatives are

$$\frac{\partial z}{\partial x} = \frac{3}{2} x^{1/2}, \quad \frac{\partial z}{\partial y} = 3y^{1/2},$$

so the integrand is

$$\sqrt{1 + \frac{9}{4}x + 9y}.$$

Notice that the surface is defined if and only if $x \geq 0$ and $y \geq 0$. The region of integration is the triangle in the xy-plane bounded by $x = 0$, $y = 0$ and $x + 4y = 4$. The region lies within $0 \leq y \leq 1$. At each y, $0 \leq x \leq 4 - 4y$.

The surface area is

$$
\begin{aligned}
S &= \int_0^1 \int_0^{4-4y} \sqrt{1 + \frac{9}{4}x + 9y}\, dx\, dy \\
&= \int_0^1 \left[\frac{2}{3} \cdot \frac{4}{9} \left(1 + \frac{9}{4}x + 9y \right)^{3/2} \right]_{x=0}^{4-4y} dx \\
&= \frac{8}{27} \int_0^1 \left[(1 + 9(1-y) + 9y)^{3/2} - (1+9y)^{3/2} \right] dy \\
&= \frac{8}{27} \int_0^1 \left[10^{3/2} - (1+9y)^{3/2} \right] dy \\
&= \frac{8}{27} \left[10^{3/2} y - \frac{2}{5} \cdot \frac{1}{9} (1+9y)^{5/2} \right]_{y=0}^{1} \\
&= \frac{8}{27} \left[10^{3/2} - \frac{2}{45} (19)^{5/2} + \frac{2}{45} \right].
\end{aligned}
$$