MAT2130: Engineering Mathematical Analysis 1

Midterm 2 Practice Problems – Selected Solutions

1. General suggestion: try substitution first. If the denominator is nonzero, then the limit is defined. If the numerator is nonzero but the denominator is zero, then the limit is undefined. The ambiguous case is $\frac{0}{0}$.

   Answers:
   (a) -5, (b) DNE, (c) 1, (d) 0, (f) $-\sqrt{2}$, (g) DNE, (h) $\frac{1}{3}$, (i) 0, (j) DNE

   I *meant* for (e) to DNE, but it turns out to be 0.

   Solutions:

(d) Notice that the numerator factors as

$$x^2 + 3xy - 10y^2 = (x - 2y)(x + 5y).$$

Therefore the limit becomes

$$\lim_{(x,y)\to(0,0)} \frac{(x-2y)(x+5y)}{x-2y} = \lim_{(x,y)\to(0,0)} (x+5y) = 0.$$

(e) When we complete the square in the numerator and rearrange the denominator, we get

$$\lim_{(x,y)\to(1,1)} \frac{2(x-1)^2 + (y-1)^2}{(x-1)+(y-1)}.$$

The usual choices $x = 1$ or $y = 1$ or even $(y - 1) = m(x - 1)$ all yield a limit of 0. This is a strong hint – but *not* a proof – that the limit exists. In fact, the limit exists and equals 0, but the proof is outside the scope of the course.

(f) When we rationalize the denominator, we get

$$\begin{aligned}
\lim_{(x,y)\to(-1,0)} \frac{y\left(\sqrt{x+3}+\sqrt{x+2y+3}\right)}{x+3-x-2y-3} &= \lim_{(x,y)\to(-1,0)} \frac{y\left(\sqrt{x+3}+\sqrt{x+2y+3}\right)}{-2y} \\
&= \lim_{(x,y)\to(-1,0)} \frac{\sqrt{x+3}+\sqrt{x+2y+3}}{-2} \\
&= -\sqrt{2}.
\end{aligned}$$

(g) Try the paths $x = 0$ and $y = 0$.

(h) We factor the denominator to obtain

$$\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{(x-2y)(x+y)}.$$

Notice that

$$\lim_{(x,y)\to(2,1)} \frac{1}{x+y} = \frac{1}{3},$$

and that

$$\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{x-2y} = \lim_{u\to 0} \frac{\sin u}{u} = \lim_{u\to 0} \frac{\cos u}{1} = 1.$$

Since both limits exist, we can split up the product:

$$\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{(x-2y)(x+y)} = \left(\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{(x-2y)}\right)\left(\lim_{(x,y)\to(2,1)} \frac{1}{x+y}\right) = (1)\left(\frac{1}{3}\right) = \frac{1}{3}.$$

Key observation: in the expression $\frac{\sin(x-2y)}{x-2y)}$, (x, y) only appeared in the form $x - 2y$. This let us reduce from a 2D to a 1D limit, in which we used L'Hôpital's rule.

Also note that we cannot not distribute the limit over the product until we establish that the limits of both factors exist.

(i) Direct substitution.

(j) Complete the square in the numerator and rearrange the denominator to obtain

$$\lim_{(x,y)\to(1,-1)} \frac{(y+1)^2 + (x-1)}{(x-1) - 2(y+1)},$$

then try the paths $y = -1$ and $x = 1$.

2. These are all routine derivatives. The key is to keep track of which variables are being held constant.

Answers:

(a) $2y^2 + 4$,

(b) $\frac{\partial}{\partial y}(fg) = \frac{\partial f}{\partial y} g + f \frac{\partial g}{\partial y} = \left(4xy + 3y^2\right) \ln\left(x^2 + y^4 + 2\right) + \left(2xy^2 + y^3 + 4x + 7\right) \frac{4y^3}{x^2 + y^4 + 2}$,

(c) $-\sin y \sin z + \cos z \cos x$,

(d) $\frac{\partial}{\partial x}(h^2) = 2h \frac{\partial h}{\partial x} = 2\frac{2y+3}{x^2+1}\left[-\frac{2y+3}{(x^2+1)^2}(2x)\right]$,

(e) $4x + 6y$,

(f) $\frac{2}{x^2 + y^4 + 2} - \frac{2x}{(x^2 + y^4 + 2)^2}(2x)$,

(g) $-\frac{2}{(x^2+1)^2}(2x)$

(h) 0

3. Chain rules:

(a) $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}$,

(b) $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} + \left.\frac{\partial w}{\partial t}\right)_{x,y,z}$,

(c) $\left.\frac{\partial w}{\partial t}\right)_s = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \left.\frac{\partial w}{\partial t}\right)_{x,y}$,

(d) $\left.\frac{\partial w}{\partial s}\right)_t = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial x}\frac{\partial x}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial v}\frac{\partial v}{\partial s}$,

(e) Let $w = \frac{\partial u}{\partial x}$. Then w depends on (s, t, x, y), so $\frac{\partial^2 u}{\partial y \partial x} = \left.\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} + \left.\frac{\partial w}{\partial y}\right)_{s,t,x}$.

Solutions:

(a) We have the dependences $u = f(s,t)$, $s = g(x,y)$ and $t = h(x,y)$, so the relevant chain rule is

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x}.$$

From the equations given, we calculate

$$\frac{\partial u}{\partial s} = 2s + t, \qquad \frac{\partial u}{\partial t} = s + 2t,$$
$$\frac{\partial s}{\partial x} = 12x^2 y - 2xy^2 + 3y^2, \qquad \frac{\partial t}{\partial x} = -\frac{2xy}{(x^2+1)^2},$$

so

$$\frac{\partial u}{\partial x} = (2s+t)\left(12x^2y - 2xy^2 + 3y^2\right) + (s+2t)\left(-\frac{2xy}{(x^2+1)^2}\right).$$

(c) We have $w = f(x,y,t)$, $x = g(s,t)$ and $y = h(s,t)$, so the relevant chain rule is

$$\left.\frac{\partial w}{\partial t}\right)_s = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \left.\frac{\partial w}{\partial t}\right)_{x,y}.$$

From the given equations, we calculate

$$\frac{\partial w}{\partial x} = 3(x+2y+3t)^2, \qquad \frac{\partial w}{\partial y} = 6(x+2y+3t)^2,$$
$$\left.\frac{\partial w}{\partial t}\right)_{x,y} = 9(x+2y+3t)^2,$$
$$\frac{\partial x}{\partial t} = -4s, \qquad \frac{\partial y}{\partial t} = 5t^4 - 3t^2 + 2st.$$

Therefore

$$\left.\frac{\partial w}{\partial t}\right)_s = 3(x+2y+3t)^2(-4s) + 6(x+2y+3t)^2\left(5t^4 - 3t^2 + 2st\right) + 9(x+2y+3t)^2.$$

(e) Let $w = \frac{\partial u}{\partial x}$ as calculated in part (a). Then $w = j(s,t,x,y)$, where $s = g(x,y)$ and $t = h(x,y)$, so the relevant chain rule is

$$\frac{\partial^2 u}{\partial y \partial x} = \left.\frac{\partial w}{\partial y}\right)_x = \frac{\partial w}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial w}{\partial t}\frac{\partial t}{\partial y} + \left.\frac{\partial w}{\partial y}\right)_{s,t,x}.$$

We calculate

$$\begin{aligned}\frac{\partial w}{\partial s} =& 2\left(12x^2y - 2xy^2 + 3y^2\right) - \frac{2xy}{(x^2+1)^2},\\ \frac{\partial w}{\partial t} =& 12x^2y - 2xy^2 + 3y^2 - 2\frac{2xy}{(x^2+1)^2},\\ \left.\frac{\partial w}{\partial y}\right)_{s,t,x} =& (2s+t)\left(12x^2 - 4xy + 6y\right) + (s+2t)\left(-\frac{2x}{(x^2+1)^2}\right),\\ \frac{\partial s}{\partial y} =& 4x^3 - 2x^2y + 6xy,\\ \frac{\partial t}{\partial y} =& \frac{1}{x^2+1}.\end{aligned}$$

We can combine these expressions to form the (lengthy) desired partial derivative.

4. Implicit differentiation rules:

(a) Let the two constraints be $G(x,y,t) = 1$ and $H(x,y,t) = 4$. Then $\frac{dx}{dt} = -\frac{\frac{\partial(G,H)}{\partial(t,y)}}{\frac{\partial(G,H)}{\partial(x,y)}}$.

(b) Let the three constraint functions be $F(x,y,z,s,t) = 2$, $G(x,y,z,s,t) = 1$ and $H(x,y,z,s,t) = 0$. Then $\frac{\partial y}{\partial t} = -\frac{\frac{\partial(F,G,H)}{\partial(x,t,z)}}{\frac{\partial(F,G,H)}{\partial(x,y,z)}}$.

(c) Rewrite the two constraints in the form $F(u,v,r,s,t) = 0$, $G(u,v,r,s,t) = 0$. Then $\frac{\partial u}{\partial s} = -\frac{\frac{\partial(F,G)}{\partial(s,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$.

Solutions:

(b) Let

$$F(x,y,z,s,t) = tx + sy, \quad G(x,y,z,s,t) = xyz + x^2s + z^2t, \quad H(x,y,z,s,t) = 2sxy - 3tx^2.$$

Then the variables x, y and z are defined as functions of s and t by the equations

$$F(x,y,z,s,t) = 2, \quad G(x,y,z,s,t) = 1, \quad H(x,y,z,s,t) = 0.$$

From our procedure for calculating implicit derivatives, we have

$$\frac{\partial y}{\partial t} = -\frac{\frac{\partial(F,G,H)}{\partial(x,t,z)}}{\frac{\partial(F,G,H)}{\partial(x,y,z)}}.$$

We calculate

$$\begin{aligned}\frac{\partial(F,G,H)}{\partial(x,t,z)} =& \begin{vmatrix} t & x & 0 \\ yz + 2xs & z^2 & xy + 2zt \\ 2sy - 6tx & -3x^2 & 0 \end{vmatrix}\\ =& -(xy+2zt)\begin{vmatrix} t & x \\ 2sy - 6tx & -3x^2 \end{vmatrix}\\ =& -(xy+2zt)\left[-3x^2t - x(2sy - 6tx)\right]\end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial(F,G,H)}{\partial(x,y,z)} &= \begin{vmatrix} t & s & 0 \\ yz+2xs & xz & xy+2zt \\ 2sy-6tx & 2sx & 0 \end{vmatrix} \\
&= -(xy+2zt)\begin{vmatrix} t & s \\ 2sy-6tx & 2sx \end{vmatrix} \\
&= -(xy+2zt)\left[2stx - s(2sy-6tx)\right].
\end{aligned}
$$

Therefore

$$
\frac{\partial y}{\partial t} = -\frac{-(xy+2zt)\left[-3x^2t - x(2sy-6tx)\right]}{-(xy+2zt)\left[2stx - s(2sy-6tx)\right]} = \frac{3x^2t + x(2sy-6tx)}{2stx - s(2sy-6tx)}.
$$

5. We view r and θ as functions of x and y.

(a) We could use the implicit differentiation process, but in this case it is simpler to differentiate the equation $\tan\theta = \frac{y}{x}$ with respect to x and with respect to y. The derivative with respect to x yields

$$
\begin{aligned}
\sec^2\theta \frac{\partial\theta}{\partial x} &= -\frac{y}{x^2} \\
\implies \quad \frac{\partial\theta}{\partial x} &= -\frac{y}{x^2}\cos^2\theta,
\end{aligned}
$$

while the derivative with respect to y yields

$$
\begin{aligned}
\sec^2\theta \frac{\partial\theta}{\partial y} &= \frac{1}{x} \\
\implies \quad \frac{\partial\theta}{\partial y} &= \frac{1}{x}\cos^2\theta.
\end{aligned}
$$

(b) Let $z = f(r)$. Then

$$
\frac{\partial z}{\partial x} = \frac{df}{dr}\frac{\partial r}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{df}{dr}\frac{\partial r}{\partial y}.
$$

By differentiating the equation $r = \sqrt{x^2+y^2}$, we find that

$$
\frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}.
$$

Therefore

$$
\frac{\partial z}{\partial x} = \frac{df}{dr}\frac{x}{r}, \quad \frac{\partial z}{\partial y} = \frac{df}{dr}\frac{y}{r},
$$

from which it follows that

$$
y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y\frac{df}{dr}\frac{x}{r} - x\frac{df}{dr}\frac{y}{r} = 0,
$$

as desired.

6. (b) We view r, ϕ and θ as functions of x, y and z. Then

$$w_x = w_\phi \frac{\partial \phi}{\partial x} + w_\theta \frac{\partial \theta}{\partial x}.$$

Let $F(r,\phi,\theta,x,y,z) = r\cos\theta\sin\phi - x$, $G(r,\phi,\theta,x,y,z) = r\sin\theta\sin\phi - y$, and $H(r,\phi,\theta,x,y,z) = r\cos\phi - z$. Then

$$\frac{\partial \phi}{\partial x} = -\frac{\frac{\partial(F,G,H)}{\partial(r,x,\theta)}}{\frac{\partial(F,G,H)}{\partial(r,\phi,\theta)}}, \quad \frac{\partial \theta}{\partial x} = -\frac{\frac{\partial(F,G,H)}{\partial(r,\phi,x)}}{\frac{\partial(F,G,H)}{\partial(r,\phi,\theta)}}.$$

We calculate

$$\begin{aligned}
\frac{\partial(F,G,H)}{\partial(r,\phi,\theta)} =& \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & r\cos\theta\sin\phi \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} \\
=& - r\sin\theta\sin\phi \begin{vmatrix} \sin\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & -r\sin\phi \end{vmatrix} \\
& - r\cos\theta\sin\phi \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi \\ \cos\phi & -r\sin\phi \end{vmatrix} \\
=& - r\sin\theta\sin\phi \left(-r\sin\theta\sin^2\phi - r\sin\theta\cos^2\phi\right) \\
& - r\cos\theta\sin\phi \left(-r\cos\theta\sin^2\phi - r\cos\theta\cos^2\phi\right) \\
=& r^2\sin^2\theta\sin\phi + r^2\cos^2\theta\sin\phi \\
=& r^2\sin\phi, \\
\frac{\partial(F,G,H)}{\partial(r,x,\theta)} =& \begin{vmatrix} \cos\theta\sin\phi & -1 & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & 0 & r\cos\theta\sin\phi \\ \cos\phi & 0 & 0 \end{vmatrix} \\
=& \begin{vmatrix} \sin\theta\sin\phi & r\cos\theta\sin\phi \\ \cos\phi & 0 \end{vmatrix} \\
=& - r\cos\theta\sin\phi\cos\phi, \\
\frac{\partial(F,G,H)}{\partial(r,\phi,x)} =& \begin{vmatrix} \cos\theta\sin\phi & r\cos\theta\cos\phi & -1 \\ \sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} \\
=& - \begin{vmatrix} \sin\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & -r\sin\phi \end{vmatrix} \\
=& - \left(-r\sin\theta\sin^2\phi - r\sin\theta\cos^2\phi\right) \\
=& r\sin\theta.
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{\partial \phi}{\partial x} &= -\frac{-r\cos\theta\sin\phi\cos\phi}{r^2\sin\phi} = \frac{\cos\theta\cos\phi}{r}, \\
\frac{\partial \theta}{\partial x} &= -\frac{r\sin\theta}{r^2\sin\phi} = -\frac{\sin\theta}{r\sin\phi}.
\end{aligned}$$

With these substitutions, we find

$$w_x = w_\phi \frac{\cos\theta\cos\phi}{r} - w_\theta \frac{\sin\theta}{r\sin\phi}.$$

7. The relevant formula is

$$V(u,v,w,\alpha,\beta) = uvw\sin\alpha\cos\beta.$$

Answers:

(b) $\dfrac{dV}{dt} = \dfrac{1}{\sqrt{2}}\left(u'vw + uv'w + uvw'\right) = \dfrac{230}{\sqrt{2}}\ \text{cm}^3/\text{sec}.$

(c) $\dfrac{dV}{dt} = \left(400\ \text{cm}^3\right)\left(\alpha'\cos\alpha\cos\beta - \beta'\sin\alpha\sin\beta\right) = -\dfrac{80}{\sqrt{2}}\ \text{cm}^3/\text{sec}.$

8. Let $\widehat{\mathbf{u}} = (u_1, u_2, u_3)$. Since $\widehat{\mathbf{u}}$ is a unit vector, $u_1^2 + u_2^2 + u_3^2 = 1$. Further let $s = \widehat{\mathbf{u}}\cdot\mathbf{x} - ct = u_1x + u_2y + u_3z - ct$. From the chain rule, we have

$$\begin{aligned}\nabla w =& w_x\widehat{\mathbf{i}} + w_y\widehat{\mathbf{j}} + w_k\widehat{\mathbf{k}}\\ =& \frac{dF}{ds}\frac{\partial s}{\partial x}\widehat{\mathbf{i}} + \frac{dF}{ds}\frac{\partial s}{\partial y}\widehat{\mathbf{j}} + \frac{dF}{ds}\frac{\partial s}{\partial z}\widehat{\mathbf{k}}\\ =& \frac{dF}{ds}\left(u_1\widehat{\mathbf{i}} + u_2\widehat{\mathbf{j}} + u_3\widehat{\mathbf{k}}\right)\\ =& \frac{dF}{ds}\widehat{\mathbf{u}}.\end{aligned}$$

Therefore

$$\widehat{\mathbf{u}}\cdot\nabla w = \frac{dF}{ds}\widehat{\mathbf{u}}\cdot\widehat{\mathbf{u}} = \frac{dF}{ds}.$$

Further,

$$\frac{\partial w}{\partial t} = \frac{dF}{ds}\frac{\partial s}{\partial t} = -c,$$

so

$$\frac{1}{c}\frac{\partial w}{\partial t} = -\frac{dF}{ds}.$$

It follows that

$$\widehat{\mathbf{u}}\cdot\nabla w + \frac{1}{c}\frac{\partial w}{\partial t} = 0,$$

as desired.

9. In each case, the formula for a directional derivative is $D_{\mathbf{v}}f = \nabla f\cdot\widehat{\mathbf{v}}$.

Answers:

(a) $\dfrac{11}{\sqrt{13}}$,

(b) 0

(c) $\dfrac{5}{\sqrt{10}}$ (up to a sign),

(d) 0,

(e) $\dfrac{2+\ln 2}{\sqrt{5}}$ (up to a sign).

Solution:

(d) The gradient of f is

$$\nabla f = (e^x \sin y \cos z, e^x \cos y \cos z, -e^x \sin y \sin z)\,.$$

At the point $\left(0, \frac{\pi}{2}, \pi\right)$, this becomes

$$\nabla f|_{\left(0,\frac{\pi}{2},\pi\right)} = (-1, 0, 0)\,.$$

Let $G(x, y, z) = x^2 + 4y^2 - z^2$. Then one of the surfaces is described by $G(x, y, z) = 0$. The gradient of G is

$$\nabla G = (2x, 8y, -2z).$$

At the point $\left(0, \frac{\pi}{2}, \pi\right)$, which is on the surface, we get

$$\nabla G|_{\left(0,\frac{\pi}{2},\pi\right)} = (0, 4\pi, -2\pi)\,.$$

This vector is normal to the surface at the given point. Since we are only interested in the direction of the normal vector, we simplify this to $(0, 2, -1)$. The other surface is a plane, and we see immediately that its normal vector is $(3, 2, -1)$. Thus a tangent vector to the curve formed by the intersection of these two surfaces is

$$\mathbf{v} = (0, 2, -1) \times (3, 2, -1) = (0, -3, -6).$$

The corresponding unit vector is

$$\widehat{\mathbf{v}} = \frac{1}{\sqrt{5}}\,(0, -1, -2)\,.$$

Thus the directional derivative is

$$D_{\mathbf{v}} f = \nabla f \cdot \widehat{\mathbf{v}} = (-1, 0, 0) \cdot \frac{1}{\sqrt{5}}(0, -1, -2) = 0.$$

10. (a) The gradient of g is

$$\nabla g = (6xy^3, 9x^2y^2 - 4z^2, -8yz).$$

At the point $(1, 1, 1)$, this becomes

$$\nabla g|_{(1,1,1)} = (6, 5, -8).$$

Then g is not changing in the direction of any vector $\mathbf{v}$ such that $(6, 5, -8) \cdot \mathbf{v} = 0$. By inspection, one such vector is

$$\mathbf{v} = (4, 0, 3).$$

(b) At the point $(1, 1, 1)$, we have

$$|\nabla g| = \sqrt{36 + 25 + 64} = \sqrt{125} = 5\sqrt{5}.$$

Thus the maximum rate of change of g at this point is $5\sqrt{5}$, and it occurs in the direction $(6, 5, -8)$. Since $5 < 5\sqrt{5}$, it is possible to find a vector $\mathbf{v}$ such that the projection of ∇g in the direction of $\mathbf{v}$ is 5. Thus there is a direction (in fact, a cone of directions) in which g changes at a rate of 5.

11. For short, let $\mathbf{v} = \nabla h|_{(1,1,1)}$, and write its components as $\mathbf{v} = (v_1, v_2, v_3)$. Then, from the

conditions given, we have

$$\begin{aligned}
(v_1, v_2, v_3) \cdot \frac{1}{\sqrt{2}}(1,0,1) =& \frac{v_1 + v_3}{\sqrt{2}} = 2, \\
(v_1, v_2, v_3) \cdot \frac{1}{\sqrt{5}}(0,2,-1) =& \frac{2v_2 - v_3}{\sqrt{5}} = 4, \\
(v_1, v_2, v_3) \cdot \frac{1}{\sqrt{6}}(1,-2,1) =& \frac{v_1 - 2v_2 + v_3}{\sqrt{6}} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

This is a system of three linear equations in three unknowns, which can be solved by standard techniques.

Answer: $\nabla h|_{(1,1,1)} = \left(4\sqrt{5} + \sqrt{3}, \sqrt{2} - \frac{\sqrt{3}}{2}, 2\sqrt{2} - 4\sqrt{5} - \sqrt{3}\right).$

12. Answers:

(a) The point $(1,0,1)$ corresponds to $t = -1$. Tangent line: $(1,0,1) + s\,(-1, \pi, -2)$, $s \in \mathbb{R}$.
(b) $(2,5,3) + t(8,1,7)$, $t \in \mathbb{R}$.
(c) $6(x-1) - 3(y-1) - (z-1) = 0$.
(d) $2(x-1) - 3(y-3) + 9(z-1) = 0$.

Solutions:

(b) Let $F(x,y,z) = xy - z^2$ and $G(x,y,z) = x^2 + y^2 - z^2$. Then the two surfaces are described by $F(x,y,z) = 1$ and $G(x,y,z) = 20$. We take the gradient of each function, and evaluate at the point $(2,5,3)$:

$$\begin{aligned}
\nabla F =& (y, x, -2z), \\
\nabla F|_{(2,5,3)} =& (5, 2, -6), \\
\nabla G =& (2x, 2y, -2z), \\
\nabla G|_{(2,5,3)} =& (4, 10, -6).
\end{aligned}$$

Therefore a tangent vector to the curve at the point $(2,5,3)$ is perpendicular to the vectors $(5,2,-6)$ and $(4,10,-6)$. Since we are only interested in the direction, not the length, we reduce the second vector to $(2,5,-3)$. Then a tangent vector is

$$(5,2,-6) \times (2,5,-3) = (24, 3, 21).$$

Again, only the direction matters, so we choose $(8,1,7)$.

The desired tangent line passes through $(2,5,3)$ in the direction of $(8,1,7)$. Therefore the vector equation for the tangent line is

$$\mathbf{r}(t) = (2,5,3) + t(8,1,7), \quad t \in \mathbb{R}.$$

(d) Let $F(x,y,z) = \sqrt{x^2 + 3z^2 + 2} - y$. Then the surface is described by $F(x,y,z) = 0$. At the point $(2,3,1)$, a vector perpendicular to the surface is $\nabla F|_{(2,3,1)}$. We calculate

$$\begin{aligned}\nabla F &= \left(\frac{x}{\sqrt{x^2+3z^2+2}}, -1, \frac{3z}{\sqrt{x^2+3z^2+2}}\right), \\ \nabla F|_{(2,3,1)} &= \left(\frac{2}{3}, -1, 3\right).\end{aligned}$$

This is a normal vector to the tangent plane at the given point. For convenience, we rescale by 3 to obtain the normal vector $(2,-1,9)$. The tangent plane passes through the point $(2,3,1)$ and has normal vector $(2,-3,9)$. Therefore the equation of the plane is

$$2(x-2)-3(y-3)+9(z-1)=0.$$

13. Let $G(x,y,z)=4x^2+y^2+z^2+2xz+4y+1$, so that the surface is described by $G(x,y,z)=0$. We verify that

$$G(-1,-2,1)=4+4+1-2-8+1=0,$$

so the point $(-1,-2,1)$ is on the surface.

The vector representation of the curve is

$$\mathbf{r}(t)=(2\cos(\pi t)+1, 3\sin(\pi t)-2, t).$$

If we set $t=1$, we get

$$\mathbf{r}(1)=(-1,-2,1),$$

which proves that $(-1,-2,1)$ is on the curve and corresponds to $t=1$.

Now we need to show that the curve is tangent to the plane. The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t)=(-2\pi\sin(\pi t), 3\pi\cos(\pi t), 1).$$

In particular, at $t=1$, we have

$$\mathbf{r}'(1)=(0,-3\pi,1).$$

This is a tangent vector to the curve at $(-1,-2,1)$.

The vector $\nabla G|_{(-1,-2,1)}$ is perpendicular to the surface at $(-1,-2,1)$. We calculate

$$\begin{aligned}\nabla G =& (8x+2z, 2y+4, 2z+2x), \\ \nabla G|_{(-1,-2,1)} =& (-6,0,0).\end{aligned}$$

To show that the curve is tangent to the surface, it suffices to show that the gradient vector above is perpendicular to the curve. We see immediately that

$$(0,-3\pi,1)\cdot(-6,0,0)=0,$$

as needed.

14. Critical points are points where $\nabla f=\mathbf{0}$, or point where some component of f does not exist.

Answers:

(a) $(0,0)$,
(b) $(0,0)$, $(0,-1)$, $(1,2)$, $(-1,2)$, $\left(\frac{1}{3},-\frac{2}{3}\right)$, $\left(-\frac{1}{3},-\frac{2}{3}\right)$,

(c) all points on the line $3x - 2y + 1 = 0$,
(d) all points on the line $x = 0$,
(e) all points of the form $\left(\left(n + \frac{1}{2}\right)\pi, m\pi\right)$ or $\left(n\pi, \left(m + \frac{1}{2}\right)\pi\right)$ where n and m are integers.

Solutions:

(c) Given $f(x, y) = |3x - 2y + 1| + x^2 - y^2$, we know that f is not differentiable when $3x - 2y + 1 = 0$. Therefore every point on the line $3x - 2y + 1 = 0$ is a critical point.

Away from that line, we write the function as

$$f(x, y) = \begin{cases} 3x - 2y + 1 + x^2 - y^2, & 3x - 2y + 1 > 0, \\ 2y - 3x - 1 + x^2 - y^2, & 3x - 2y + 1 < 0. \end{cases}$$

Then the gradient is

$$\nabla f = \begin{cases} (2x + 3, -2y - 2), & 3x - 2y + 1 > 0, \\ (2x - 3, -2y + 2), & 3x - 2y + 1 < 0. \end{cases}$$

It looks like there are two more critical points, $\left(-\frac{3}{2}, -1\right)$ and $\left(\frac{3}{2}, 1\right)$. However, observe that

$$3\left(-\frac{3}{2}\right) - 2(-1) + 1 = -\frac{3}{2} < 0.$$

That is, the point $\left(-\frac{3}{2}, -1\right)$ is not within the range where $3x - 2y + 1 > 0$, so the gradient is not given by $(2x + 3, -2y - 2)$ at this point. Similarly,

$$3\left(\frac{3}{2}\right) - 2(1) + 1 = \frac{7}{2} > 0,$$

so the gradient is not given by $(2x - 3, -2y + 2)$ at the point $\left(\frac{3}{2}, 1\right)$. Thus $\nabla f \neq \mathbf{0}$ everywhere that it is defined, and the only critical points lie on the line $3x - 2y + 1 = 0$.

(e) The gradient is

$$\nabla f = (\cos x \cos y, -\sin x \sin y).$$

Clearly this is defined for all (x, y), so the only critical points are the points where $\nabla f = \mathbf{0}$. Such points must satisfy

$$[\cos x = 0 \ \text{ OR } \ \cos y = 0] \quad \text{AND} \quad [\sin x = 0 \ \text{ OR } \ \sin y = 0].$$

We proceed by cases on the first pair of conditions.

Suppose $\cos x = 0$. Then $x = \left(n + \frac{1}{2}\right)\pi$ for some integer n. We need $\sin x = 0$ or $\sin y = 0$, but $\sin x \neq 0$ for any such x, so the only option is $\sin y = 0$. Therefore $y = m\pi$ for some integer m. The critical points take the form $\left(\left(n + \frac{1}{2}\right)\pi, m\pi\right)$ for some integers n and m.

Now suppose $\cos y = 0$. Then $y = \left(m + \frac{1}{2}\right)$ for some integer m. We must have $\sin x = 0$ or $\sin y = 0$. The only possibility is $\sin x = 0$, in which case $x = n\pi$ for some integer n. The critical points take the form $\left(n\pi, \left(m + \frac{1}{2}\right)\pi\right)$ for integers n and m. This exhausts the possibilities.

15. Answers:

(a) Saddle, (b) relative minimum, (c) relative maximum, (d) saddle, (e) relative minimum, (f) none of the above.

Solutions:

(b) Note that $f(3,4) = 0$, which is the smallest value that f can ever take on the xy-plane. Therefore $(3,4)$ must be a relative minimum (and in fact an absolute minimum).

(c) Note that the expression $\sqrt{x^2+y^2} \geq 0$ for all (x,y). Therefore $2-\sqrt{x^2+y^2} \geq 2$ for all (x,y). At $(0,0)$, we have $f(x,y) = 2$, which is the largest value it can ever take. Thus $(0,0)$ is a relative maximum.

(e) Nothing obvious arises by inspection, so we proceed with the second derivative test. We compute

$$\nabla f = (f_x, f_y) = \left(4x^3 - 12x^2 + 14x - 6, 2y - 4\right),$$

and

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} 12x^2 - 24x + 14 & 0 \\ 0 & 2 \end{pmatrix}.$$

We take the determinant of this matrix at the point $(1,2)$:

$$\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4 > 0.$$

Thus the point $(1,2)$ is a relative extreme. In particular, since $f_{xx} = 2 > 0$, the function is concave up near this point, so it must be a relative minimum.

(f) Note that

$$\nabla f = \left(\frac{x-1}{\sqrt{(x-1)^2+(y+1)^2}} - 2y, \frac{y+1}{\sqrt{(x-1)^2+(y+1)^2}} - 2(x-1) \right),$$

which is not defined at $(1,-1)$. This point can still be a relative maximum or minimum, but it cannot be a saddle point.

Fix $y = -1$, and consider the path $(x,-1)$ through $(1,-1)$. On this path, the function reduces to

$$f(x,-1) = \sqrt{(x-1)^2} + 2(x-1) = |x-1| + 2(x-1).$$

At $(1,-1)$, the value of the function is 0. If $x < 1$, then

$$|x-1| + 2(x-1) = -(x-1) + 2(x-1) = x - 1 < 0.$$

If $x > 1$, then

$$|x-1| + 2(x-1) = x - 1 + 2(x-1) = 3(x-1) > 0.$$

Therefore $f(x,-1) > f(1,-1)$ on one side and $f(x,-1) < f(1,-1)$ on the other. This shows that $(1,-1)$ is not a relative maximum or minimum. It cannot be a saddle either, so this critical point is uncategorized.

16. Answers:

(a) Absolute maximum is 1, absolute minimum is 0,
(b) absolute maximum is 2, absolute minimum is -7,
(c) absolute maximum is $\frac{1}{2}$, absolute minimum is $-\frac{1}{2}$,

(d) absolute maximum is $10^{1/6}$, absolute minimum is $-10^{1/6}$.

Solutions:

(a) The gradient of f is

$$\nabla f = \left(\frac{1}{x-y} - \frac{x+y}{(x-y)^2}, \frac{1}{x-y} + \frac{x+y}{(x-y)^2}\right) = \left(-\frac{2y}{(x-y)^2}, \frac{2x}{(x-y)^2}\right).$$

This is undefined whenever $x = y$, but the line $x = y$ does not intersect R. Therefore there are no critical points within the region R. It remains to check the boundaries.

We have four boundary lines: $y = 0$ for $1 \le x \le 2$, $y = -1$ for $1 \le x \le 2$, $x = 1$ for $-1 \le y \le 0$, and $x = 2$ for $-1 \le y \le 0$.

On $y = 0$, $1 \le x \le 2$, the function becomes

$$f(x,0) = \frac{x}{x} = 1.$$

On $y = -1$, $1 \le x \le 2$, the function becomes

$$f(x,-1) = \frac{x-1}{x+1}.$$

To determine if there are critical points on the boundary, we calculate the derivative:

$$\frac{d}{dx} f(x,-1) = \frac{1}{x+1} - \frac{x-1}{(x+1)^2} = \frac{2}{(x+1)^2}.$$

This is undefined when $x = -1$, but we are only considering the range $1 \le x \le 2$. Thus there are no relevant critical points, and we only need to evaluate at $x = 1$ and $x = 2$. The values are

$$f(1,-1) = 0, \quad f(2,-1) = \frac{1}{3}.$$

On $x = 1$, $-1 \le y \le 0$, the function becomes

$$f(1,y) = \frac{1+y}{1-y}.$$

The derivative is

$$\frac{d}{dy} f(1,y) = \frac{1}{1-y} + \frac{1+y}{(1-y)^2} = \frac{2}{(1-y)^2}.$$

This is undefined at $y = 1$, which is outside the range of interest. We only need to check the boundary points, but we already have $f(1,-1)$ from the line $y = -1$, and we already have $f(1,0)$ from the line $y = 0$.

Lastly, on $x = 2$, $-1 \le y \le 0$, the function becomes

$$f(2,y) = \frac{2+y}{2-y}.$$

The derivative is

$$\frac{d}{dy} f(2,y) = \frac{2}{(2-y)^2}.$$

This is undefined at $y = 2$, which is outside the range of interest. We have already calculated both boundary points.

Comparing all of the values above, we find that the absolute maximum is 1 and the absolute minimum is 0.

(c) The gradient of f is

$$\nabla f = (y, x).$$

The only critical point is $(0,0)$, which is within the region R. The value of the function there is

$$f(0,0) = 0.$$

It remains to check the boundary, which is the unit circle $x^2 + y^2 = 1$. A parametrization of the unit circle is $(x, y) = (\cos t, \sin t)$, $0 \le t \le 2\pi$. With this substitution, we get

$$f(t) = \cos t \sin t = \frac{1}{2}\sin(2t).$$

The derivative is

$$f'(t) = \cos(2t),$$

which equals 0 when $t = \pi/4$, $3\pi/4$, $5\pi/4$ or $7\pi/4$. We must also check the endpoints $t = 0$ and $t = 2\pi$. The various function values are

$$f(0) = 0, \quad f\left(\frac{\pi}{4}\right) = \frac{1}{2}, \quad f\left(\frac{3\pi}{4}\right) = -\frac{1}{2},$$

and the remaining three are repetitions of these. Thus the absolute maximum is $\frac{1}{2}$ and the absolute minimum is $-\frac{1}{2}$.

(d) The gradient of f is

$$\nabla f = \left(\frac{1}{3(x-3y)^{2/3}}, -\frac{1}{(x-3y)^{2/3}}\right).$$

This is never $\mathbf{0}$, and it is undefined whenever $x = 3y$. At any point such that $x = 3y$,

$$f(3y, y) = 0.$$

It remains to check the boundary, which is the unit circle $x^2 + y^2 = 1$. A parametrization is $(x, y) = (\cos t, \sin t)$ for $0 \le t \le 2\pi$. With this substitution, the function becomes

$$f(t) = (\cos t - 3\sin t)^{1/3}.$$

The derivative is

$$f'(t) = \frac{1}{3(\cos t - 3\sin t)^{2/3}}(-\sin t - 3\cos t).$$

Thus $f'(t)$ is undefined whenever $\cos t = 3\sin t$, and it is 0 whenever $\sin t = -3\cos t$. If we rewrite these conditions in terms of x and y, we have critical points at $x = 3y$ and at $y = -3x$. If $x = 3y$, we already know that the function value is 0. The points on the unit circle where $y = -3x$ must satisfy

$$x^2 + 9x^2 = 1,$$

which occurs when $x = \pm\frac{1}{\sqrt{10}}$. The corresponding points are $\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right)$ and $\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right)$. We evaluate the function here:

$$f\left(\frac{1}{\sqrt{10}}, -\frac{3}{\sqrt{10}}\right) = \left(\sqrt{10}\right)^{1/3} = 10^{1/6},$$

$$f\left(-\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}}\right) = \left(-\sqrt{10}\right)^{1/3} = -\left(\sqrt{10}\right)^{1/3} = -10^{1/6}.$$

Thus the absolute maximum is $10^{1/6}$ and the absolute minimum is $-10^{1/6}$.

The function $D(s,t)$ is

$$\begin{aligned} D(s,t) &= (3+s-1-4t)^2 + (-s-2-3t)^2 + (1-2s-4+t)^2 \\ &= (s-4t+2)^2 + (s+3t+2)^2 + (t-2s-3)^2. \end{aligned}$$

Its gradient is

$$\begin{aligned} \nabla D = &\Big(2(s-4t+2) + 2(s+3t+2) - 4(t-2s-3), \\ &\quad -8(s-4t+2) + 6(s+3t+2) + 2(t-2s-3)\Big) \\ =&(12s-6t+20, -6s+52t-10). \end{aligned}$$

The gradient equals $\mathbf{0}$ when

$$12s - 6t + 20 = 0 \quad \text{and} \quad -6s + 52t - 10 = 0.$$

When we solve this system of linear equations, we find $(s,t) = \left(-\frac{5}{3}, 0\right)$. This is the only critical point of D.

(b) We have

$$D\left(-\frac{5}{3}, 0\right) = \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \frac{1}{3}.$$

Since there is only one critical point, this is the smallest (and, for that matter, largest) value that D takes at a critical point.

To find the absolute minimum of D, we should be considering a finite region in the st-plane. Roughly speaking, as s and t both get larger, the two lines $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$ should get further and further apart. So we can take our finite region to be a disk in the st-plane with a very large radius – sufficiently large that the lines satisfy $D(s,t) > \frac{1}{3}$ everywhere on the boundary and outside it. Then $D = \frac{1}{3}$ becomes the absolute minimum of D on the disk, and in fact over all (s,t).

(c) The corresponding points on the two lines are

$$\mathbf{R} = \mathbf{r}_1\left(-\frac{5}{3}\right) = \left(\frac{4}{3}, \frac{5}{3}, \frac{13}{3}\right) \quad \text{and} \quad \mathbf{S} = \mathbf{r}_2(0) = (1,2,4).$$

The vector from the first point to the second is

$$(1,2,4) - \left(\frac{4}{3}, \frac{5}{3}, \frac{13}{3}\right) = \left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right).$$

A vector parallel to the line $\mathbf{r}_1(s)$ is $(1,-1,-2)$, and a vector parallel to $\mathbf{r}_2(t)$ is $(4,3,-1)$. Taking the dot product of $\left(-\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right)$ with each of these direction vectors yields 0.

Thus our 2D minimization process has correctly identified the points R and S on the two lines such that the vector $\mathbf{RS}$ is perpendicular to both lines. This proves that the distance $|\mathbf{RS}|$ does in fact minimize the distance between the two lines.