Summer 2014 Final — Brent's draft solution (1/6)

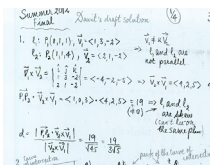
1. ℓ_1: $P_1(1,1,1)$, $\vec{v}_1 = \langle 1,3,-2\rangle$ $\Rightarrow$ $\vec{v}_1 \neq k\vec{v}_2$ ℓ_1 and ℓ_2 are not parallel.
   ℓ_2: $P_2(1,1,4)$, $\vec{v}_2 = \langle 2,1,-2\rangle$

$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle -4,-2,-5\rangle \Rightarrow \vec{v}_1 \times \vec{v}_2 = \langle 4,2,5\rangle$

$\vec{P_1P_2} \cdot \vec{v}_1 \times \vec{v}_1 = \langle 1,1,3\rangle \cdot \langle 4,2,5\rangle = 19 \neq 0$ $\Rightarrow$ ℓ_1 and ℓ_2 are skew (can't lie on the same plane)

$d = \dfrac{|\vec{P_1P_2} \cdot \vec{v}_1 \times \vec{v}_2|}{|\vec{v}_1 \times \vec{v}_2|} = \dfrac{19}{\sqrt{45}} = \dfrac{19}{3\sqrt{5}}$

2. Intersection



projection onto the xy plane

parametric equations: $x = -2\cos t$, $y = 2\cos t$, $z = -2\sin t$ ✓ vs. $x = -2\sin t$, $y = 2\sin t$, $z = 2\cos t$

$0 \le t \le 2\pi$

b) $P(\sqrt{2}, -\sqrt{2}, \sqrt{2})$ $\xrightarrow{\text{So } t = 3\pi/4}$ $t = \pi/4$

$\vec{r}'_P = \langle 2\sin t, -2\cos t, 2\sin t\rangle = \langle \sqrt{2}, +\sqrt{2}, \sqrt{2}\rangle$ ($t = 3\pi/4$)

$x = \sqrt{2} - \sqrt{2}u$, $y = -\sqrt{2} + \sqrt{2}u$, $z = \sqrt{2} + \sqrt{2}u$ or

$x = \sqrt{2} + \sqrt{2}u$, $y = -\sqrt{2} - \sqrt{2}u$, $z = \sqrt{2} - \sqrt{2}u$

c. $\int_{-3\pi/4}^{-\pi/4} \sqrt{8\sin^2 t + 8\cos^2 t}\, dt$

d) $\nabla f|_P = \left\langle \dfrac{2xy}{x^2+y^2}, z^2 \sin\sqrt{xy} - \ln(z\sqrt{x^2+y^2}), xy\sin\sqrt{xy} + \dfrac{z^2 x y}{\sqrt{x^2+y^2}} \right\rangle$

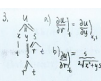
$= \langle -1, \ln 4, -1\rangle$ length be positive

x increasing along C $\Rightarrow$ $\vec{T} = \langle \sqrt{2}, -\sqrt{2}, -\sqrt{2}\rangle \parallel$ $\vec{T} = \langle 1,-1,-1\rangle$

$D_{\vec{T}} f = \nabla f|_P \cdot \vec{T} = \langle -1, \ln 4, -1\rangle \cdot \langle 1,-1,-1\rangle / \sqrt{3}$

$= \dfrac{(-1 - \ln 4 + 1)}{\sqrt{3}} = -\dfrac{\ln 4}{\sqrt{3}}$

3. (2/6)



a) $\left.\dfrac{\partial z}{\partial r}\right|_t = \left.\dfrac{\partial u}{\partial v}\right|_{t,s}$, $\left.\dfrac{\partial z}{\partial s}\right|_t = \left.\dfrac{\partial z}{\partial t}\right|_{s,r} + \left.\dfrac{\partial u}{\partial s}\right|_{t,r} \left.\dfrac{\partial z}{\partial r}\right|_{s,t}$

b) $\left.\dfrac{\partial u}{\partial r}\right|_t = \dfrac{r}{\sqrt{r^2+s^2}} \left(= \Delta u(\langle r,s\rangle) \cdot \left\langle \frac{r}{s}\right\rangle\right) \cdot \dfrac{1}{\sqrt{r^2+s^2}} \left\langle \frac{r}{s}\right\rangle$

4. $f_x = y(5-y-2x^2) - 2x^2y = y(5-y-3x^2) = 0$

$f_y = x(5-y-x^2) - xy = x(5-2y-x^2) = 0$

$x=0$, $y=0 \to (0,0)$

$x=0$, $5-y=0 \to y=5 \to (0,5)$

$y=0$, $5-x^2=0 \to x=\pm\sqrt{5} \to (\pm\sqrt{5}, 0)$

$\left.\begin{array}{l} 5-y-3x^2=0 \\ 5-2y-x^2=0 \end{array}\right\}$ $\begin{array}{l} 5-y-3(5-2y) = -10+5y=0 \to y=2 \\ x^2 = 5-2y \end{array}$ $\xrightarrow{\text{So}}$ $x^2=1 \to x=\pm 1$

$(\pm 1, 2)$

CPs: $(0,0), (0,5), (\pm\sqrt{5}, 0), (\pm 1, 2)$

$f_{xx} = -6xy$; $f_{xy} = 5-2y-3x^2$, $f_{yy} = -2x$

for $(0,0)$: $A=0$, $B=5$, $C=0$ $\to B^2-AC = 25 > 0$ $\to$ $(0,0)$ yields SP.

for $(1,2)$: $A=-12$, $B=-2$, $C=-2$, $\to B^2-AC = 4-24 = -20$ $\to$ $(1,2)$ yields rel-max

c) the only CP inside R is $(1,2)$

$f(1,2) = (1)(2)[5-2-1^2] = \boxed{12}$

on C_1: $x=0 \to f(x,y) = \boxed{0}$ $0 \le y \le 5$

on C_2: $y=0 \to f(x,y) = \boxed{0}$ $0 \le x \le \sqrt{5}$

on C_3: $y = 5-x^2 \to f(x,5-x^2) = x(5-x^2)[5-(5-x^2)-x^2] = \boxed{0}$ $0 \le x \le \sqrt{5}$

Abs-max $= 12$ @ $(1,2)$, Abs-min 0 on the boundary of R

5. $y \le x \le 1$, $0 \le y \le 1$

$I = \int_0^1 \int_y^1 6y^2 \cos x^2\, dx\, dy$ (1) reversing order of integration (2) method of strips

$= \int_0^1 \cos x^2 \left(2y^3\right)\Big|_0^x dx = \int_0^1 2x^3 \cos x^2\, dx$

let $u = x^2$, $\frac{1}{2}du = x\,dx$ $\int_0^1 u \cos u\, du$

$= \int_0^1 u \cos u\, du$, $v = u$, $dw = \cos u$, $\frac{1}{2}dv = du$, $w = \sin u$

Integration by parts