Math 1710. Homework Problems III (January 9, 2012)

1. $\displaystyle\int \left(\tan^2 x + \tan^4 x\right)\, dx = \int \tan^2 x \left(1 + \tan^2 x\right)\, dx = \int \tan^2 x \sec^2 x \, dx$

   We make substitution $u = \tan x :\quad du = \sec^2 x \, dx$,

   $$\int u^2 \, du = \frac{u^3}{3} + C = \boxed{\frac{\tan x}{3} + C, \quad C \in \mathbb{R}}$$

2. $\displaystyle\int \frac{x^2+1}{\sqrt{x^6 - 7x^4 + x^2}}\, dx = \int \frac{x^2(1+\frac{1}{x^2})}{\sqrt{x^4(x^2 - 7 + \frac{1}{x^2})}}\, dx = \int \frac{1+\frac{1}{x^2}}{\sqrt{x^2 - 7 + \frac{1}{x^2}}}\, dx$

   We make substitution $u = x - \dfrac{1}{x} :\quad du = \left(1 + \dfrac{1}{x^2}\right) dx$,

   $$u^2 = x^2 - 2 + \frac{1}{x^2} \Rightarrow x^2 - 7 + \frac{1}{x^2} = u^2 - 5$$

   $$= \int \frac{du}{\sqrt{u^2-5}}\, du$$

   We make new substitution $u = \sqrt{5}\sec t :\quad du = \sqrt{5}\tan t \sec t \, dt$

   $$= \int \frac{\sqrt{5}\tan t \sec t}{\sqrt{5\sec^2 t - 5}}\, dt = \int \frac{\sqrt{5}\tan t \sec t}{\sqrt{5(\sec^2 t - 1)}}\, dt = \int \frac{\sqrt{5}\tan t \sec t}{\sqrt{5}\tan t}\, dt$$

   $$= \int \sec t \, dt = \ln(\sec t + \tan t) + C = \ln\left(\frac{u}{\sqrt{5}} + \sqrt{\frac{u^2}{5} - 1}\right) + C$$

   $$= \ln\left(\frac{x - 1/x}{\sqrt{5}} + \sqrt{\frac{(x-1/x)^2}{5} - 1}\right) + C = \ln\left(\frac{x^2-1}{\sqrt{5}x} + \sqrt{\frac{x^4 - 2x^2 + 1}{5x^2} - 1}\right) + C$$

   $$= \boxed{\ln\left(\frac{x^2 - 1 + \sqrt{x^4 - 7x^2 + 1}}{\sqrt{5}x}\right) + C, \quad C \in \mathbb{R}}$$