MATH 1210 Summer 2015 Quiz 1

Surname: ______________________________

Given Name: ______________________________

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[9] 1. Prove the following using induction. For all $n \geq 1$,

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}.$$

Solution:

Let $P(n)$ stand for the statement $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{n(n+1)}=1-\frac{1}{n+1}$.

<u>Part A</u>

Show $P(1)$ is true.

$$LHS=\frac{1}{1\cdot 2}=\frac{1}{2}$$
$$RHS=1-\frac{1}{1+1}=1-\frac{1}{2}=\frac{1}{2}$$

Since LHS=RHS we know $P(1)$ is true.

<u>Part B</u> Suppose $P(k)$ is true for an integer k. that is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{k(k+1)}=1-\frac{1}{k+1}.$$

We want to show $P(k+1)$ is true, that is

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(k+1)(k+2)}=1-\frac{1}{k+2}.$$

$$
\begin{aligned}
LHS &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(k+1)(k+2)} \\
&= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{k+2}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)} \\
&= 1 - \frac{k+1}{(k+1)(k+2)} \\
&= 1 - \frac{1}{k+2}.
\end{aligned}
$$

Therefore $P(k+1)$ is true.

Therefore by the principle of mathematical induction, $P(n)$ is true for $n \geq 1$.

[5] 2. Turn the following sum into sigma notation

$$\frac{2\cdot 3}{1\cdot 4} + \frac{6\cdot 7}{5\cdot 8} + \frac{10\cdot 11}{9\cdot 12} + \cdots + \frac{414\cdot 415}{413\cdot 416}$$

Solution:

If we start with $i = 1$, we can notice that the second term in the denominator is 4 times i. Then since the other terms are $4i$ minus either 1,2 or 3. Hence the general term is

$$\frac{(4i-2)(4i-1)}{(4i-3)(4i)}.$$

The first term is when $i = 1$ and the last term would be when $4i = 416 \Rightarrow i = 104$.

Therefore in sigma notation, the sum is

$$\sum_{i=1}^{104} \frac{(4i-2)(4i-1)}{(4i-3)(4i)}.$$

[6] 3. Possibly using the sums $\sum_{i=1}^{n} i = \dfrac{n(n+1)}{2}$ and/or $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$, find the sum of

$$\sum_{i=25}^{132} (2i+1)(i-2).$$

Leave your answer as an unsimplified numerical expression. There can not be any sigma notation remaining in your answer.

Solution:

Solution 1:

$$\begin{aligned}
S &= \sum_{i=25}^{132} (2i+1)(i-2) \\
&= \sum_{i=25}^{132} (2i^2 - 3i - 2) \\
&= \sum_{i=1}^{132} (2i^2 - 3i - 2) - \sum_{i=1}^{24} (2i^2 - 3i - 2) \\
&= \left(2\sum_{i=1}^{132} i^2 - 3\sum_{i=1}^{132} i - \sum_{i=1}^{132} 2\right) - \left(2\sum_{i=1}^{24} i^2 - 3\sum_{i=1}^{24} i - \sum_{i=1}^{24} 2\right) \\
&= \left(2\left(\frac{132 \cdot 133 \cdot 263}{6}\right) - 3\left(\frac{132 \cdot 133}{2}\right) - 2(132)\right) \\
&\qquad - \left(2\left(\frac{24 \cdot 25 \cdot 49}{6}\right) - 3\left(\frac{24 \cdot 25}{2}\right) - 2(24)\right)
\end{aligned}$$

Solution 2:

Using a substitution $i = j - 24$ we can change the indices of summation from $i = 25$ to $j = 1$ and $i = 132$ to $j = 108$. This allows us to use the given formula. Rewriting the substitution yields $j = i + 24$ and therefore the summation becomes

$$
\begin{aligned}
S &= \sum_{i=25}^{132} (2i+1)(i-2) \\
&= \sum_{j=1}^{108} \big[(2(j+24)+1)((j+24)-2)\big] \\
&= \sum_{j=1}^{108} (2j+49)(j+22)\big] \\
&= \sum_{j=1}^{108} (2j^2+93j+1078)\big] \\
&= 2\sum_{j=1}^{108} j^2 + 93\sum_{j=1}^{108} j + \sum_{j=1}^{108} 1078 \\
&= 2\left(\frac{108\cdot 109\cdot 217}{6}\right) + 93\left(\frac{108\cdot 109}{2}\right) + 1078(108)
\end{aligned}
$$