MATH 2130 Problem Workshop 2 Solutions

1. Let $P = (3, -1, 5)$. There are a few ways to do this question. All of which require the vector on the line which is $\mathbf{v} = \langle 3, 2, 1\rangle$ and a point on the line. Using $t = 0$ we get the point is $Q = (2, -1, 4)$. (You could use other values of t if you'd like.) Let R be the point on the line which is closest to P. We want to find the length of PR.

   <u>Method 1: In class</u> $|PR| = |PQ| \sin\theta$ where θ is the angle between $\mathbf{PQ}$ and $\mathbf{QR}$. Since the unit vector in the direction of $\mathbf{QR}$ has length 1 since is a unit vector we get $|PR| = \left|\hat{QR}\right| |PQ| \sin\theta = \left|\hat{\mathbf{QR}} \times \mathbf{PQ}\right|$. Since $\mathbf{QR}$ is along the line, we get the unit vector is $\frac{1}{\sqrt{14}}\langle 3, 2, 1\rangle$.

   $$\left|\hat{\mathbf{QR}} \times \mathbf{PQ}\right| = \frac{1}{\sqrt{14}} \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & 1 \\ 3 & 2 & 1 \end{vmatrix} = \frac{1}{14}(-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

   Hence $|PR| = \left|\hat{\mathbf{QR}} \times \mathbf{PQ}\right| = \frac{1}{\sqrt{14}}\sqrt{4+4+4} = \sqrt{6/7}$.

   <u>Method 2: In textbook</u> $|PR|$ is the component of $\mathbf{PQ}$ onto $\mathbf{PR}$. The vector $\mathbf{PQ}$ is $\langle 1, 0, 1\rangle$. To find a vector in the direction of $\mathbf{PR}$ we need two vectors which are perpendicular to $\mathbf{PR}$. One of them is $\mathbf{v}$. The other can be found by taking $\mathbf{v} \times \mathbf{PQ}$.

   $$\mathbf{v} \times \mathbf{PQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & 1 \\ 1 & 0 & 1 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

   Hence $\mathbf{PR}$ is parallel to

   $$\mathbf{v} \times \mathbf{PQ} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 3 & 2 & 1 \\ 2 & -2 & -2 \end{vmatrix} = -2\hat{\mathbf{i}} - 8\hat{\mathbf{j}} - 10\hat{\mathbf{k}}.$$

   Therefore

   $$|PR| = \left|\mathbf{PQ} \cdot \hat{\mathbf{PR}}\right| = \frac{|\langle 1, 0, 1\rangle \cdot \langle -2, -8, -10\rangle|}{\sqrt{4 + 64 + 100}} = \frac{12}{\sqrt{168}} = \frac{6}{\sqrt{42}}.$$

2. First thing we need is to find the parametric equation of the intersection of the two planes. Putting the first plane is standard form, we get $2x - y + 3z = 4$ Hence the normal vectors to the two planes are $\langle 2, -1, 3\rangle$ and $\langle 3, 1, -2\rangle$. Hence the vector along the line of intersection is perpendicular to both normal lines and is therefore

   $$\langle 2, -1, 3\rangle \times \langle 3, 1, -2\rangle = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -1 & 3 \\ 3 & 1 & -2 \end{vmatrix} = -\hat{\mathbf{i}} + 13\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$$

We also need a point on the line, and hence a point on both planes. Setting $z = 0$, we get $2x - y = 4$ and $3x + y = 6$. Adding the equations yields $5x = 10$ or $x = 2$. Inserting into either equations gives $y = 0$ and so the point is $(2, 0, 0)$. Therefore the two lines we are finding the distance between are

$$x = 2 - s, y = 13s, z = 5s \text{ and } x = 2 + t, y = 3 - 2t, z = 1 + t.$$

The vectors parallel to these lines are $\langle -1, 13, 5\rangle$ and $\langle 1, -2, 1\rangle$ which are clearly not parallel. Also if the lines intersect, then $2 - s = 2 + t \Rightarrow t = -s$. Inserting these into y and z gives $13s = 3 + 2s$ and $5s = 1 - s$. The first says $s = 3/11$ and the second yields $s = 1/6$. Hence the lines don't intersect and therefore they are skew.

To find the distance between the lines, we want to find the length of the vector $\mathbf{PQ}$ where $\mathbf{PQ}$ is perpendicular to both lines. To find this we take any point R on the first line and S on the second line. Then $|PQ| = \left|\hat{\mathbf{PQ}} \cdot \mathbf{RS}\right|$. Let $R = (2, 0, 0)$ and $S = (2, 3, 1)$, hence $\mathbf{RS} = \langle 0, 3, 1\rangle$. As for $\hat{\mathbf{PQ}}$, we know that it is perpendicular to both $\langle -1, 13, 5\rangle$ and $\langle 1, -2, 1\rangle$, which is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -1 & 13 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 23\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 11\hat{\mathbf{k}}$$

Hence

$$|PQ| = \left|\mathbf{RS} \cdot \hat{\mathbf{PQ}}\right| = \frac{|\langle 0, 3, 1\rangle \cdot \langle 23, 6, -11\rangle|}{\sqrt{529 + 36 + 121}} = \frac{7}{\sqrt{686}} = \frac{1}{\sqrt{14}}.$$

3. First we need to find the 3 points that give the triangle. Let P be the intersection of the first two lines, Q be the intersection of the first and third lines and R be the intersection of the last two lines. For P we get

$$-11 + 5s = 1 + 2u, s = 1 - u, -2 + 2s = -2 - 4u$$

The last equation implies $s = -2u$. Inserting this into the second yields $-2u = 1 - u \Rightarrow u = -1, s = 2$. This yields the point $P = (-1, 2, 2)$. For Q we get

$$-11 + 5s = -2 + 3t, s = -1 + 2t, -2 + 2s = -8 + 6t$$

The last equation implies $s = -3 + 3t$. Inserting this into the second yields $-3 + 3t = -1 + 2t \Rightarrow t = 2, s = 3$. This yields the point $Q = (4, 3, 4)$. For R we get

$$-2 + 3t = 1 + 2u, -1 + 2t = 1 - u, -8 + 6t = -2 - 4u$$

The middle equation implies $u = 2 - 2t$. Inserting this into the first yields $-2 + 3t = 1 + 4 - 4t \Rightarrow 7t = 7 \Rightarrow t = 1, u = 0$. This yields the point $R = (1, 1, -2)$.

The area is $\frac{1}{2}\left|\mathbf{PQ}\times\mathbf{PR}\right|$. Using the points we calculated, we get that $\mathbf{PQ} = \langle 5,1,2\rangle$ and $\mathbf{PR} = \langle 2,-1,-4\rangle$.

$$\mathbf{PQ}\times\mathbf{PR} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 1 & 2 \\ 2 & -1 & -4 \end{vmatrix} = -2\hat{\mathbf{i}} + 24\hat{\mathbf{j}} - 7\hat{\mathbf{k}}.$$

Hence the area is

$$\frac{1}{2}\sqrt{4+576+49} = \frac{1}{2}\sqrt{629}.$$

4. We can do this from any point to the midpoint of the opposite side. For example use $P = (-1,2,2)$ and the midpoint of QR is $S = (5/2,2,1)$. The point two thirds from P to S is therefore $(-1+7/3, 2, 2-2/3) = (4/3, 2, 4/3)$.

5. $\mathbf{v}'(t) = 2t\hat{\mathbf{i}} + \frac{1/4}{\sqrt{1-t^2/16}}\hat{\mathbf{j}} + \frac{2}{2t+1}\hat{\mathbf{k}} = 2t\hat{\mathbf{i}} + \frac{1}{\sqrt{16-t^2}}\hat{\mathbf{j}} + \frac{2}{2t+1}\hat{\mathbf{k}}$. Hence $\mathbf{v}'(3) = 6\hat{\mathbf{i}} + \frac{1}{\sqrt{16-9}}\hat{\mathbf{j}} + \frac{2}{7}\hat{\mathbf{k}} = 6\hat{\mathbf{i}} + \frac{1}{\sqrt{7}}\hat{\mathbf{j}} + \frac{2}{7}\hat{\mathbf{k}}$.

6.

$$\begin{aligned} f(t)\mathbf{v}(t) &= (t^2+1)(e^t\hat{\mathbf{i}} + \left[t/(t^2+1)^3\right]\hat{\mathbf{j}} - t\sqrt{t^2+1}\hat{\mathbf{k}}) \\ &= e^t(t^2+1)\hat{\mathbf{i}} + \left[t/(t^2+1)^2\right]\hat{\mathbf{j}} - t(t^2+1)^{3/2}\hat{\mathbf{k}} \end{aligned}$$

Taking an antiderivative of each component individually using substitution ($u = t^2+1$) and in the first case, integration by parts, we get

$$\begin{aligned} \int e^t(t^2+1)dt &= e^t(t^2+1) - \int e^t(2t)dt \\ &= e^t(t^2+1) - 2te^t + \int e^t(2)dt \\ &= e^t(t^2+1) - 2te^t + 2e^t \\ \int \frac{t}{(t^2+1)^2}dt &= \frac{1}{2}\int u^{-2}du = -\frac{1}{2}u^{-1} = -\frac{1}{2(t^2+1)} \\ -\int \frac{t}{(t^2+1)^{3/2}}dt &= -\frac{1}{2}\int u^{3/2}du = -\frac{1}{5}u^{5/2} = -\frac{1}{5}(t^2+1)^{5/2} \end{aligned}$$

Hence the antiderivative of $f(t)\mathbf{v}(t)$ is $(t^2-2t+3)e^t\hat{\mathbf{i}} + \left[1/2(t^2+1)\right]\hat{\mathbf{j}} - \frac{1}{5}(t^2+1)^{5/2}\hat{\mathbf{k}} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.

7. (a) Squaring both sides of the first equation yields $z^2 = 4(x^2+y^2)$ where z is positive. Inserting the second equation gives $z^2 = 4(3-z) \Rightarrow z^2 + 4z - 12 \Rightarrow z = 2, -6$ Since z is positive or using the points given, we know $z = 2$. From this we get $x^2+y^2 = 1$. We generally would like to use x, y being $\sin t$ and $\cos t$ in some order, however we only have half the circle where (x, y) goes from $(1, 0)$ to $(-1, 0)$ where y is negative. Hence $x = \cos t$ but $y = -\sin t$ where $t = 0$ to π. Hence the solution is

$$x = \cos t, y = -\sin t, z = 2, 0 \le t \le \pi.$$

   (b) Using $x^2 + z^2 = 4$ we want x, z being $2\sin t$ and $2\cos t$ in some order. Since x, z are both positive, we know $0 \le t \le \pi/2$. On that interval we know that $\sin t$ increases, so let $z = 2\sin t$ and $x = 2\cos t$. From $x + y = 1$ we get $y = 1 - 2\cos t$. Since we are in the first octant, we need $1 - 2\cos t \ge 0 \Rightarrow \cos t \le 1/2$. Hence $\pi/3 \le t \le \pi/2$. Hence our parameterization is

$$x = 2\cos t, y = 1 - 2\cos t, z = 2\sin t, \pi/3 \le t \le \pi/2.$$

   (c) Using the second equation, we can rearrange it to be $x^2 + (y-2)^2 = 4$. Again we'd like $x, y-2$ being $2\sin t$ and $2\cos t$. Since we want it to be clockwise, one option is $x = 2\cos t, y - 2 = -2\sin t$ where we need the negative on y to ensure the clockwise direction. Then

$$z = x^2 + y^2 = 4\cos^2 t + 4\sin^2 t - 8\sin t + 4 = 8(1 - \sin t).$$

   Hence we get

$$x = 2\cos t, y = 2 - 2\sin t, z = 8(1 - \sin t), 0 \le t \le 2\pi.$$

8. We first need to find a parameterization. However we did this in 7b and got $x = 2\cos t, y = 1 - 2\cos t, z = 2\sin t$. (The point we are dealing with is not in the first quadrant, but the parameterization is fine.) The point occurs when $\cos t = \sin t = 1/\sqrt{2}$. Hence $t = \pi/4$.

   $\mathbf{T}(t) = \langle -2\sin t, 2\sin t, 2\cos t\rangle$ and so $\mathbf{T}(\pi/4) = \langle -2\sin \pi/4, 2\sin \pi/4, 2\cos \pi/4\rangle = \langle -\sqrt{2}, \sqrt{2}, \sqrt{2}\rangle$. The length of $\mathbf{T}(\pi/4)$ is $\sqrt{6}$ and so the unit tangents are $\pm\frac{1}{\sqrt{3}}\langle -1, 1, 1\rangle$. (the $\pm$ is because we could have used a different parameterization.)

9. Using the existing parameterization, we have $t = 0$ at the origin. Hence $\mathbf{T}(t) = \langle 2t, 6t^2, 6t\rangle$ which is the zero vector at $t = 0$. This is a problem since the zero vector is not a valid tangent vector. Therefore we need to change the parameterization. Let $u = t^2$, then the parameterization becomes $x = u, y = 2u^{3/2}, z = 3u$ and we are still finding it when $u = 0$. Hence $\mathbf{T}(u) = \langle 1, 3u^{1/2}, 3\rangle$. At $u = 0$ we get $\mathbf{T}(0) = \langle 1, 0, 3\rangle$. Therefore the unit tangent vector is $\frac{1}{\sqrt{10}}\langle 1, 0, 3\rangle$.

10. We need to find a parameterization. For the first, let $x = t$, then $y = 5/2 - t/2$. Hence $z = t^2 - t/2 - 3/2$. For the second, let $y = u$, then $x = 5 - u^2$, $z = \frac{1}{4}(4 - 2(5 - u^2) - 3u) = \frac{1}{4}(2u^2 - 3u - 6)$.

    Next we need to find when the lines intersect. From the x component we get $t = 5 - u^2$ and from y we get $u = 5/2 - t/2 \Rightarrow t = 5 - 2u$. Equating these yields

$$5 - 2u = 5 - u^2 \Rightarrow u^2 - 2u \Rightarrow u = 0, 2 \Rightarrow (u, t) = (0, 5), (2, 1)$$

    Looking at the z component, we get that $(0, 5)$ doesn't work and $(2, 1)$ does work. Hence $u = 2, t = 1$ and $(x, y, z) = (1, 2, -1)$.

    For the first curve, $\mathbf{T_1}(t) = \langle 1, -1/2, 2t - 1/2\rangle \Rightarrow \mathbf{T_1}(1) = \langle 1, -1/2, 3/2\rangle$. For the second curve, $\mathbf{T_2}(t) = \langle -2u, 1, u - 3/4\rangle \Rightarrow \mathbf{T_2}(2) = \langle -4, 1, 5/4\rangle$.

    Now the angle between vectors $\mathbf{u}$ and $\mathbf{v}$ satifies $\cos\theta = \mathbf{u} \cdot \mathbf{v} / |\mathbf{u}|\,|\mathbf{v}|$. We can rescale the vectors to avoid fractions, so we are find the angle between $\mathbf{u} = \langle 2, -1, 3\rangle$ and $\mathbf{v} = \langle -16, 4, 5\rangle$. $\mathbf{u} \cdot \mathbf{v} = -32 - 4 + 15 = -21$, $|\mathbf{u}| = \sqrt{4 + 1 + 9} = \sqrt{14}$ and $|\mathbf{v}| = \sqrt{256 + 16 + 25} = \sqrt{297}$. Hence the solution is $\arccos\left(\dfrac{-21}{\sqrt{14}\sqrt{297}}\right)$. Note that it is possible with different parameterizations to get the supplementary angle $\arccos\left(\dfrac{21}{\sqrt{14}\sqrt{297}}\right)$.

11. Let $\mathbf{r}(t) = \langle t + 1, 2t^{3/2} - 3, 4t - 2\rangle$. The points $(2, -1, 2)$ and $(1, -3, -2)$ occur at $t = 1$ and $t = 0$ respectively. Hence the arc length is $\int_0^1 |\mathbf{r}'(t)|\,dt$.

    $\mathbf{r}'(t) = \langle 1, 3t^{1/2}, 4\rangle$, and so $|\mathbf{r}'(t)| = \sqrt{17 + 9t}$ Therefore using the substitution $w = 17 + 9t$, the arc length is

$$\int_0^1 (17 + 9t)^{1/2} dt = \frac{1}{9}\int_1 7^2 6 w^{1/2} dt = \frac{2}{27} w^{3/2}]_{17}^{26} = \frac{2}{27}(26^{3/2} - 17^{3/2})$$

12. The first step is to find a parameterization. Let $x = t$, then $y = 4 - t$. Solving for z gives

$$z^2 = 4 + t^2 + t^2 - 8t + 16 = 2t^2 - 8t + 20.$$

    Since the points we are dealing with have $z > 0$, we can take the positive square root $z = \sqrt{2t^2 - 8t + 20}$. The values of t go from 2 to 4.

$\mathbf{r}'(t) = \langle 1, -1, (2t^2 - 8t + 20)^{-1/2}(2t - 4)\rangle$, and so

$$\begin{aligned}
|\mathbf{r}'(t)| &= \sqrt{1 + 1 + \frac{4t^2 - 16t + 16}{2t^2 - 8t + 20}} \\
&= \sqrt{1 + 1 + \frac{2t^2 - 8t + 8}{t^2 - 4t + 10}} \\
&= \sqrt{\frac{4t^2 - 16t + 28}{t^2 - 4t + 10}} \\
&= 2\sqrt{\frac{t^2 - 4t + 7}{t^2 - 4t + 10}}
\end{aligned}$$

Hence the arc length is given by

$$2\int_2^4 \sqrt{\frac{t^2 - 4t + 7}{t^2 - 4t + 10}}dt$$