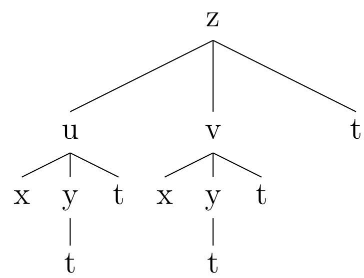
MATH 2130 Problem Workshop 4

1. The tree diagram for the chain rule is



Hence the chain rule is

$$\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}\frac{dy}{dt}+\frac{\partial z}{\partial u}\frac{\partial u}{\partial t}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\frac{dy}{dt}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial t}+\frac{\partial z}{\partial t}.$$

2.

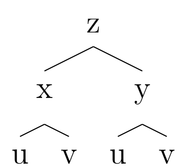
$$\nabla f(x^2-y^2)=\langle f'(x^2-y^2)(2x), f'(x^2-y^2)(-2y)\rangle$$

$$\nabla g(xy)=\langle g'(xy)(y), g'(xy)(x)\rangle$$

Hence

$$\nabla f(x^2-y^2)\cdot\nabla g(xy)=2xyf'(x^2-y^2)g'(xy)-2xyf'(x^2-y^2)g'(xy)=0.$$

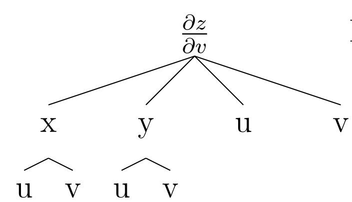
3. First we find $\frac{\partial z}{\partial v}$. The tree is



Hence

$$\frac{\partial z}{\partial v}=\frac{\partial z}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial z}{\partial y}\frac{\partial y}{\partial v}=2x(-u\sin v)+2y(u\cos v).$$

Next we find $\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)$ which has a tree



Hence

$$\begin{aligned}\left.\frac{\partial^2 z}{\partial v^2}\right)_u &= \frac{\partial\frac{\partial z}{\partial v}}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial\frac{\partial z}{\partial v}}{\partial y}\frac{\partial y}{\partial v}+\left.\frac{\partial\frac{\partial z}{\partial v}}{\partial v}\right)_{x,y,u}\\ &= -2u\sin v(-u\sin v)+(2u\cos v)(u\cos v)+2x(-u\cos v)+2y(-u\sin v)\\ &= 2u^2\sin^2 v+2u^2\cos^2 v-2ux\cos v-2uy\sin v\end{aligned}$$

This would be fine as an answer, however note that since $x = u\cos v$ and $y = u\sin v$ we have that

$$2u^2\sin^2 v + 2u^2\cos^2 v - 2ux\cos v - 2uy\sin v = 2u^2\sin^2 v + 2u^2\cos^2 v - 2u^2\cos^2 v - 2u^2\sin^2 v = 0.$$

4.
$$\frac{\partial u}{\partial x} = 3x^2 f(x/y) + x^3 f'(x/y)(1/y)$$

and

$$\frac{\partial u}{\partial y} = x^3 f'(x/y)(-x/y^2)$$

Hence

$$\begin{aligned} x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} &= 3x^3 f(x/y) + x^4 f'(x/y)(1/y) - x^3 y f'(x/y)(x/y^2) \\ &= 3x^3 f(x/y) + f'(x/y)(x^4/y) - f'(x/y)(x^4/y) \\ &= 3x^3 f(x/y) \\ &= 3u \end{aligned}$$

5. This is an implicit differnetiation question. We are finding $\frac{\partial s}{\partial x}$. We are told that s and t are functions of x and y. Hence the formula for $\frac{\partial s}{\partial x}$ is

$$\frac{\partial s}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,t)}}{\frac{\partial(F,G)}{\partial(s,t)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_s & F_t \\ G_s & G_t \end{vmatrix}}$$

where

$$F = x^2 + y + 3s^2 + s - 2t + 1 \text{ and } G = y^2 - x^4 + 2st + 7 - 6s^2t^2$$

Therefore

$$\frac{\partial s}{\partial x} = -\frac{\begin{vmatrix} 2x & -2 \\ -4x^3 & 2s - 12s^2t \end{vmatrix}}{\begin{vmatrix} 6s+1 & -2 \\ 2t - 12st^2 & 2s - 12s^2t \end{vmatrix}} = -\frac{2x(2s - 12s^2t) - (-2)(-4x^3)}{(6s+1)(2s - 12s^2t) - (-2)(2t - 12st^2)}.$$

Note that this is a function of s, t and x while we have only been given values for s, t. Hence we need to find x when $s = 0, t = 1$. Plugging these into the original equations yield.

$$x^2 + y + -1 = 0 \text{ and } y^2 - x^4 + 7 = 0$$

Hence plugging $y = 1 - x^2$ into the second equation gives us

$$(1-x^2)^2 - x^4 + 7 = 0 \Rightarrow 1 - 2x^2 + x^4 + x^4 + 7 = 0 \Rightarrow 2x^2 = 8 \Rightarrow x = \pm 2.$$

Since we have specified that $x > 0$ we can plug in $s = 0, t = 1, x = 2$ into the derivative to get

$$\frac{\partial s}{\partial x} = -\frac{2(2)(2(0) - 12(0)^2(1)) - (-2)(-4(2)^3)}{(6(0)+1)(2(0) - 12(0)^2(1)) - (-2)(2(1) - 12(0)(1)^2)} = -\frac{0 - 64}{0 + 4} = 16.$$

6. This is also an implicit differnetiation question. We are finding $\frac{\partial \phi}{\partial y}$. We are told that r, ϕ and θ are functions of x, y and z. Hence the formula for $\frac{\partial \phi}{\partial y}$ is

$$\frac{\partial \phi}{\partial y} = -\frac{\dfrac{\partial(F,G,H)}{\partial(r,y,\theta)}}{\dfrac{\partial(F,G,H)}{\partial(r,\phi,\theta)}} = -\frac{\begin{vmatrix} F_r & F_y & F_\theta \\ G_r & G_y & G_\theta \\ H_r & H_y & H_\theta \end{vmatrix}}{\begin{vmatrix} F_r & F_\phi & F_\theta \\ G_r & G_\phi & G_\theta \\ H_r & H_\phi & H_\theta \end{vmatrix}}$$

where

$$F = r\sin\phi\cos\theta - x, G = r\sin\phi\sin\theta - y \text{ and } H = r\cos\phi.$$

Therefore

$$\frac{\partial \phi}{\partial y} = -\frac{\begin{vmatrix} \sin\phi\cos\theta & 0 & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & -1 & r\sin\phi\cos\theta \\ \cos\phi & 0 & 0 \end{vmatrix}}{\begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix}}$$

The determinant in the numerator is

$$\begin{vmatrix} \sin\phi\cos\theta & 0 & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & -1 & r\sin\phi\cos\theta \\ \cos\phi & 0 & 0 \end{vmatrix} = \cos\phi \begin{vmatrix} 0 & -r\sin\phi\sin\theta \\ -1 & r\sin\phi\cos\theta \end{vmatrix} = \cos\phi(-r\sin\phi\sin\theta)$$

The determinant in the numerator is

$$
\begin{aligned}
&\begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} \\
&= \cos\phi \begin{vmatrix} r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ r\cos\phi\sin\theta & r\sin\phi\cos\theta \end{vmatrix} - (-r\sin\phi) \begin{vmatrix} \sin\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\sin\phi\cos\theta \end{vmatrix} \\
&= \cos\phi(r\cos\phi)(r\sin\phi) \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} + r\sin\phi(\sin\phi)(r\sin\phi) \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix} \\
&= r^2\cos^2\phi\sin\phi(\cos^2\theta + \sin^2\theta) + r^2\sin^3\phi(\cos^2\theta + \sin^2\theta) \\
&= r^2\cos^2\phi\sin\phi + r^2\sin^3\phi \\
&= r^2\sin\phi(\cos^2\phi + \sin^2\phi) \\
&= r^2\sin\phi.
\end{aligned}
$$

Hence

$$\frac{\partial\phi}{\partial y} = -\frac{\cos\phi(-r\sin\phi\sin\theta)}{r^2\sin\phi} = \frac{\cos\phi\sin\theta}{r}.$$

7. We know the directional derivative is $\nabla f \cdot \hat{\mathbf{v}}$ where the direction vector is the normal to the surface at $(2, 0, 3)$. The normal to the surface $xz^2 - x^2z = 6$ is

$$\nabla(xz^2 - x^2z - 6) = \langle z^2 - 2xz, 0, 2xz - x^2\rangle.$$

At $(2, 0, 3)$ we get $\langle -3, 0, 8\rangle$. Hence the unit vector is

$$\hat{\mathbf{v}} = \frac{1}{\sqrt{73}}\langle -3, 0, 8\rangle.$$

Note that this is in the upward direction because the z-component of $\hat{\mathbf{v}}$ is positive. $\nabla f = \langle y\cos(xy), x\cos(xy), -3z^2\rangle$ so $\nabla f(2, 0, 3) = \langle 0, 2, -27\rangle$ Hence the directional derivative is

$$D_{\hat{\mathbf{v}}}f = \frac{1}{\sqrt{73}}\langle 0, 2, -27\rangle \cdot \langle -3, 0, 8\rangle = \frac{-216}{\sqrt{73}}.$$

8. This can be done either by finding a parametrization, and then finding the tangent vector, or by noting that the tangent line is perpendicular to the normals of both planes at $(1, -1, 3)$. Since the parametrization here would be difficult to find, we go with the second way. Let $f = xyz + z^3 - 24$ and $g = x^3y^2z + y^3 - 4x + 2$.

The normal to the first curve is

$$\mathbf{n_1} = \nabla f(1, -1, 3) = \langle yz, xz, xy + 3z^2\rangle|_{(1,-1,3)} = \langle -3, 3, 26\rangle.$$

The normal to the second curve is

$$\mathbf{n_2} = \nabla g(1, -1, 3) = \langle 3x^2y^2z - 4, 2x^3yz + 3y^2, x^3y^2\rangle|_{(1,-1,3)} = \langle 5, -3, 1\rangle.$$

Hence the tangent to both curves is

$$\mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ -3 & 3 & 26 \\ 5 & -3 & 1 \end{vmatrix} = 81\hat{\mathbf{i}} + 133\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$$

Hence the tangent line is

$$\langle 1, -1, 3\rangle + t\mathbf{v} = \langle 1 + 81t, -1 + 133t, 3 - 6t\rangle.$$

9. The normal to the plane at $(2, -1, -1)$ is $\nabla f(2, -1, -1)$ where $f = x^2y + y^2z + z^2x + 3$. Hence

$$\mathbf{n} = \nabla f(2, -1, -1) = \langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz\rangle|_{(2,-1,-1)} = \langle -3, 6, -3\rangle.$$

Hence the equation of the tangent plane is

$$\mathbf{n} \cdot (\langle x, y, z\rangle - \langle 2, -1, -1\rangle = 0 \Rightarrow -3x + 6y - 3z = -9 \Rightarrow x - 2y + z = 3.$$

10. We need to find f_x, f_y to see where they are both 0 or where at least one of them is undefined.

$$f_x = 3x^2y^3 - 2xy^2 \text{ and } f_y = 3x^3y^2 - 2x^2y.$$

These are both defined everywhere, so setting them both equal to 0 yields

$$3x^2y^3 - 2xy^2 = 0 \Rightarrow xy^2(3xy - 2) = 0$$

and

$$3x^3y^2 - 2x^2y = 0 \Rightarrow x^2y(3xy - 2) = 0$$

Hence either $x = 0, y = 0$ or $y = 2/(3x)$.

11. We need to find f_x, f_y to see where they are both 0 or where at least one of them is undefined.

$$f_x = 3x^2y^2 - y \text{ and } f_y = 2x^3y - x + 3.$$

These are both defined everywhere, so setting them both equal to 0 yields

$$3x^2y^2 - y = 0$$

and

$$2x^3y - x + 3 = 0$$

Rearranging the second equation yields $y = (x-3)/(2x^3)$ ($x = 0$ will not make the second equation 0) so inserting this into the first equation gives

$$0 = \frac{3x^2(x-3)^2}{4x^6} - \frac{x-3}{2x^3} = \frac{3x^2(x^2-6x+9) - 2x^3(x-3)}{4x^6} = \frac{x^4 - 12x^3 + 27x^2}{4x^6} = \frac{x^2 - 12x + 27}{4x^4}$$

Therefore $0 = x^2 - 12x + 27 = (x-9)(x-3)$ implying $x = 3, 9$. Inserting these into our expression for y yields the points $(3, 0)$ and $(9, 1/243)$.

12. Find and classify all critical points of the function as giving relative minima, maxima, saddle points or neither.

   (a) To find the critical points, we do the same as the previous two questions

$$f_x = 3x^2 + y \text{ and } f_y = x + 3y^2$$

   which are always defined. Hence setting these equal to 0 gives $y = -3x^2$ from which the second equation gives $0 = x + 27x^4$. Hence $x = 0$ or $1 = -27x^3 \Rightarrow x = -1/3$. Plugging into $y = -3x^2$ gives the points $(0, 0)$ and $(-1/3, -1/3)$.

   Classifying these points requires $f_{xx} = 6x$, $f_{xy} = 1$ and $f_{yy} = 6y$.

   For the point $(0, 0)$ we get $B^2 - AC = 1^2 - 0^2 = 1 > 0$ hence $(0, 0)$ yields a saddle point.

   For the point $(-1/3, -1/3)$ we get $B^2 - AC = 1^2 - (2)^2 = -3 < 0$ and $A = -2 < 0$ hence $(-1/3, -1/3)$ yields a relative maximum.

   (b)

$$f_x = 3x^2 - y^2 + 3y \text{ and } f_y = -2xy + 3x$$

   which are always defined. Hence setting the second equation equal to 0 gives $x = 0$ or $y = 3/2$. Plugging $x = 0$ into the first equation gives

$$3y - y^2 = 0 \Rightarrow y = 0, 3$$

   Plugging $y = 3/2$ into the first equation yields

$$3x^2 - \frac{9}{4} + \frac{9}{2} = 0 \Rightarrow x^2 = -\frac{3}{4}$$

   which has no solution.

   Therefore the critical points are $(0, 0)$ and $(0, 3)$.

   Classifying these points requires $f_{xx} = 6x$, $f_{xy} = -2y + 3$ and $f_{yy} = -2x$.

   For the point $(0, 0)$ we get $B^2 - AC = 3^2 - 0(0) = 9 > 0$ hence $(0, 0)$ yields a saddle point.

   For the point $(0, 3)$ we get $B^2 - AC = (-3)^2 - (0)(0) = 9 < 0$ hence $(0, 3)$ yields a saddle point.

(c)

$$f_x = 4x^3 - 6xy^2 \text{ and } f_y = 4y^3 - 6x^2y$$

which are always defined. Hence setting the equations equal to 0 yields

$$2x(2x^2 - 3y^2) = 0 \text{ and } 2y(2y^2 - 3x^2) = 0$$

Hence either $x = 0, y = 0$ or both $2x^2 - 3y^2, 2y^2 - 3x^2$ are 0.

If $x = 0$, then in the second equation we get $4y^3 = 0 \Rightarrow y = 0$. If $y = 0$, then in the first equation we get $4x^3 = 0 \Rightarrow x = 0$. In the latter case we get $2x^2 - 3y^2 = 0 = 2y^2 - 3x^2$. Hence $5y^2 = 5x^2$, so $y = x$ or $y = -x$. Either way we get $-x^2 = 0$ so $x = 0$ leading back to the previous cases.

Therefore the only critical point is $(0, 0)$.

Classifying these point requires $f_{xx} = 12x^2 - 6y^2, f_{xy} = -12xy$ and $f_{yy} = 12y^2 - 6x^2$. However $B^2 - AC = 0$ at $(0, 0)$. Hence we need another way to classify the point $(0, 0)$.

Looking back to the original equation $f(x, y) = x^4 - 3x^2y^2 + y^4$. If we look along $y = 0$, we would see that $f(x, 0) = x^4$, meaning we would get a relative minimum. However along $y = x$ we would see that $f(x, x) = -x^4$, meaning that $(0, 0)$ meaning we would get a relative maximum. Hence $(0, 0)$ is a saddle point.

(d) $f(x, y) = y^2 + |x - 1|$ $f_y = 2y$, and $f_x = \frac{\partial}{\partial x}(|x - 1|)$ which is never 0 and is undefined at $x = 1$. Since it is never 0. we can never get a critical point because both partials are zero. Hence the only critical point comes from where f_x is undefined which happens when $x = 1$ and y is any real number.

Since f has a minimum of 0 when $x = 1, y = 0$ that point is a relative minimum. For any other value of y, y^2 is either strictly increasing or decreasing at $(1, y)$ and is therefore neither a minimum nor a maximum.

Hence $(1, 0)$ is a relative minimum and $(1, y)$ is neither a min/max or saddle point when $y \neq 0$.