MATH 1210 ASSIGNMENT #5 SOLUTIONS

① (a) $\begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 1 & 4 & 3 \end{vmatrix} = 3(-1)^{1+2}\begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} + 4(-1)^{3+2}\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix}$

$= -3(6-5) - 4(5+2) = -3 - 28 = -31$

OR

$\begin{vmatrix} 1 & 3 & -1 \\ 2 & 0 & 5 \\ 1 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & -1 \\ 0 & -6 & 7 \\ 0 & 1 & 4 \end{vmatrix} \begin{matrix} \\ [2]-2[1] \\ [3]-[1] \end{matrix}$

$= -\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & -6 & 7 \end{vmatrix} [2]\Leftrightarrow[3] \quad = -\begin{vmatrix} 1 & 3 & -1 \\ 0 & 1 & 4 \\ 0 & 0 & 31 \end{vmatrix} [3]+6[2]$

$= -(1)(1)(31) = -31$

(b) $\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix} = (-1)(-1)^{2+2}\begin{vmatrix} 1 & 2 & 1 \\ 1 & 0 & 3 \\ -1 & 2 & 1 \end{vmatrix}$

$= (-1)\left[2(-1)^{1+2}\begin{vmatrix} 1 & 3 \\ -1 & 1 \end{vmatrix} + 2(-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \right]$

$= (-1)\left[-2(1+3) - 2(3-1) \right] = 12$

OR

$$\begin{vmatrix} 1 & 0 & 2 & 1 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 0 & 3 \\ -1 & 0 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & -1 & -3 & -2 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 4 & 2 \end{vmatrix} \begin{matrix} \\ [2]-2[1] \\ [3]-[1] \\ [4]+[1] \end{matrix}$$

$$= (-1)(-2)(2) \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 1 \end{vmatrix} \begin{matrix} \\ (-1)[2] \\ (-\frac{1}{2})[3] \\ (\frac{1}{2})[4] \end{matrix}$$

$$= 4 \begin{vmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 3 \end{vmatrix} [4]-2[3] \qquad = 4(1)(1)(1)(3) = 12$$

(c) $$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} = 1(-1)^{1+1} \begin{vmatrix} 1 & 2 & 3 \\ -2 & 3 & 4 \\ -6 & 5 & 1 \end{vmatrix} + (-1)(-1)^{1+2} \begin{vmatrix} -1 & 2 & 3 \\ 2 & 3 & 4 \\ 6 & 5 & 1 \end{vmatrix}$$

$$+ 2(-1)^{1+4} \begin{vmatrix} -1 & 1 & 2 \\ 2 & -2 & 3 \\ 6 & -6 & 5 \end{vmatrix}$$

$$= (1)\left[(1)(-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} -2 & 4 \\ -6 & 1 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} -2 & 3 \\ -6 & 5 \end{vmatrix}\right]$$

$$+ (1)\left[(-1)(-1)^{1+1} \begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix} + 2(-1)^{1+2} \begin{vmatrix} 2 & 4 \\ 6 & 1 \end{vmatrix} + 3(-1)^{1+3} \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix}\right]$$

$$+ (-2)\left[(-1)(-1)^{1+1} \begin{vmatrix} -2 & 3 \\ -6 & 5 \end{vmatrix} + (1)(-1)^{1+2} \begin{vmatrix} 2 & 3 \\ 6 & 5 \end{vmatrix} + 2(-1)^{1+3} \begin{vmatrix} 2 & -2 \\ 6 & -6 \end{vmatrix}\right]$$

$$= (1)\left[(1)(3-20) - 2(-2+24) + 3(-16+18)\right]$$

$$+(1)\left[(-1)(3-20) - 2(2-24) + 3(16-18)\right]$$

$$+(-2)\left[(-1)(-10+18) + (-1)(10-18) + 2(\underbrace{-12+12}_{=0})\right]$$

[note: terms cancel in pairs]

$$= 0.$$

OR

$$\begin{vmatrix} 1 & -1 & 0 & 2 \\ -1 & 1 & 2 & 3 \\ 2 & -2 & 3 & 4 \\ 6 & -6 & 5 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 2 & 6 \\ -1 & 1 & -2 & -6 \\ 0 & 2 & 3 & 5 \\ 2 & 3 & 4 & 1 \end{vmatrix}^T$$

$$= \begin{vmatrix} 1 & -1 & 2 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 5 \\ 2 & 3 & 4 & 1 \end{vmatrix}^T \quad [2]+[1] \qquad = 0.$$

(d) $$\begin{vmatrix} x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{vmatrix} = x(-1)^{1+1} \begin{vmatrix} 2x & 3x^2 & 4x^3 \\ 2 & 6x & 12x^2 \\ 0 & 6 & 24x \end{vmatrix}$$

$$+ 1(-1)^{2+1} \begin{vmatrix} x^2 & x^3 & x^4 \\ 2 & 6x & 12x^2 \\ 0 & 6 & 24x \end{vmatrix}$$

$$= x\left[2x(-1)^{1+1}\begin{vmatrix} 6x & 12x^2 \\ 6 & 24x \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} 3x^2 & 4x^3 \\ 6 & 24x \end{vmatrix}\right] +$$

$$+(-1)\left[x^2(-1)^{1+1}\begin{vmatrix} 6x & 12x^2 \\ 6 & 24x \end{vmatrix} + 2(-1)^{2+1}\begin{vmatrix} x^3 & x^4 \\ 6 & 24x \end{vmatrix}\right]$$

$$= x\left[2x(144x^2 - 72x^2) - 2(72x^3 - 24x^3)\right]$$

$$-\left[x^2(144x^2 - 72x^2) - 2(24x^4 - 6x^4)\right] = 12x^4$$

OK

$$\begin{vmatrix} x & x^2 & x^3 & x^4 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 2 & 6x & 12x^2 \\ 0 & 0 & 6 & 24x \end{vmatrix} = x(2)(6)\begin{vmatrix} 1 & x & x^2 & x^3 \\ 1 & 2x & 3x^2 & 4x^3 \\ 0 & 1 & 3x & 6x^2 \\ 0 & 0 & 1 & 4x \end{vmatrix}$$

[using $\frac{1}{x}[1]$, $\frac{1}{2}[3]$, $\frac{1}{6}[4]$]

$$= 12x\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & x & 2x^2 & 3x^3 \\ 0 & 1 & 3x & 6x^2 \\ 0 & 0 & 1 & 4x \end{vmatrix} \quad [2]-[1] \qquad = 12x^2\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 1 & 3x & 6x^2 \\ 0 & 0 & 1 & 4x \end{vmatrix}$$

[using $\frac{1}{x}[2]$]

$$= 12x^2\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & x & 3x^2 \\ 0 & 0 & 1 & 4x \end{vmatrix} \quad [3]-[2]$$

using $[4]-[3]$

$$= 12x^3\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 1 & 3x \\ 0 & 0 & 1 & 4x \end{vmatrix} \quad \frac{1}{x}[3] \qquad = 12x^3\begin{vmatrix} 1 & x & x^2 & x^3 \\ 0 & 1 & 2x & 3x^2 \\ 0 & 0 & 1 & 3x \\ 0 & 0 & 0 & x \end{vmatrix}$$

$$= 12x^3(1)(1)(1)(x) = 12x^4$$

② (a) $\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & 2 \end{vmatrix} \begin{matrix} \\ [2]-2[1] \\ [3]-[1] \end{matrix}$

$= - \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 2 \end{vmatrix} (-1)[2] = - \begin{vmatrix} 1 & 3 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{vmatrix} [3]+[2]$

$= -(1)(1)(3) = -3 \neq 0 \Rightarrow$ system possesses a unique solution.

$$a = \frac{\begin{vmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{vmatrix}} = \frac{(-2)(15-2)+3(-1)(-15-6)+1(-10-30)}{-3}$$

$$= \frac{-3}{-3} = 1$$

$$b = \frac{\begin{vmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{vmatrix}}{-3} = \frac{1(-15-6)+(-2)(-1)(6-1)+1(12+5)}{-3}$$

$$= 6/(-3) = -2$$

$$c = \frac{\begin{vmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{vmatrix}}{-3} = \frac{1(30+10)+3(-1)(12+5)+(-2)(4-5)}{-3}$$

$$= \frac{-9}{-3} = 3$$

(b) $\begin{vmatrix} 5 & 6 & 4 \\ 7 & 8 & 6 \\ 6 & 7 & 5 \end{vmatrix} = 5(40-42) + 6(-1)(35-36) + 4(49-48)$

$= -10+6+4 = 0$

$\therefore$ This system has either no solutions or an infinite # of solutions

$$\begin{pmatrix} 5 & 6 & 4 & 3 \\ 7 & 8 & 6 & 1 \\ 6 & 7 & 5 & 0 \end{pmatrix} \xrightarrow{[1]-[3]} \begin{pmatrix} -1 & -1 & -1 & +3 \\ 7 & 8 & 6 & 1 \\ 6 & 7 & 5 & 0 \end{pmatrix}$$

$$\xrightarrow{(-1)[1]} \begin{pmatrix} 1 & 1 & 1 & -3 \\ 7 & 8 & 6 & 1 \\ 6 & 7 & 5 & 0 \end{pmatrix}$$

$$\xrightarrow[{[3]-6[1]}]{[2]-7[1]} \begin{pmatrix} 1 & 1 & 1 & -3 \\ 0 & 1 & -1 & 22 \\ 0 & 1 & -1 & 18 \end{pmatrix}$$

$$\xrightarrow[{[3]-[2]}]{[1]-[2]} \begin{pmatrix} 1 & 0 & 2 & -25 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

$$\xrightarrow[\frac{1}{4}[4]]{} \begin{pmatrix} 1 & 0 & 2 & -25 \\ 0 & 1 & -1 & 22 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (R.R.E.F)$$

This system is inconsistent since the rank of the augmented matrix is 4 while the rank of the coefficent matrix is 3.

This system has <u>no</u> solutions.

(c) $\begin{vmatrix} 3 & -3 & 3 \\ 2 & -1 & 4 \\ 3 & -5 & -1 \end{vmatrix} = 3\begin{vmatrix} -1 & 4 \\ -5 & -1 \end{vmatrix} + (-3)(-1)\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} + 3\begin{vmatrix} 2 & -1 \\ 3 & -5 \end{vmatrix} = 3(1+20)+3(-2-12) + 3(-10+3)$

$= 3(21) + 3(-14) + 3(-7) = 0$

Since the system is homogeneous, it has an infinite # of solutions.

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 2 & -1 & 4 & 0 \\ 3 & -5 & -1 & 0 \end{array}\right) \xrightarrow{\frac{1}{3}[1]} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 2 & -1 & 4 & 0 \\ 3 & -5 & -1 & 0 \end{array}\right)$$

$$\xrightarrow[{[3]-3[1]}]{[2]-2[1]} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & 0 \end{array}\right) \xrightarrow[{[3]+2[2]}]{[1]+[2]} \left(\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

(R.R.E.F.)

equivalent system: $x_1 - 3x_3 = 0$
$x_2 + 2x_3 = 0$
x_3 - "free"

let $x_3 = t$, then $x_1 = 3t$ & $x_2 = -2t$

(infinite # of solutions)

(3) (a) $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} = 4-6 = -2 \neq 0 \Rightarrow$ linearly independent

remember, vectors go in as columns

(b) 3 2-vectors are always linearly dependent.

Consider $\alpha_1 \vec{u_1} + \alpha_2 \vec{u_2} + \alpha_3 \vec{u_3} = \vec{0}$ —— (*)

augmented matrix: $\begin{pmatrix} 1 & 3 & 5 & 0 \\ 2 & 4 & 6 & 0 \end{pmatrix}$

$$\xrightarrow{[2]-2[1]} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & -2 & -4 & 0 \end{pmatrix}$$

$$\xrightarrow{(-\frac{1}{2})[2]} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \end{pmatrix} \quad \text{RREF}$$

equivalent system: $\alpha_1 - \alpha_3 = 0$

$\alpha_2 + 2\alpha_3 = 0$

let $\alpha_3 = t$, so $\alpha_1 = t$ & $\alpha_2 = -2t$

(*) becomes $t[\vec{u_1} - 2\vec{u_2} + \vec{u_3}] = 0$

for any choice of t

In particular: $t = 1 \Rightarrow \vec{u_1} - 2\vec{u_2} + \vec{u_3} = \vec{0}$

Note: $S = \{\vec{u_1}, \vec{u_2}\}$ is a linearly independent subset, by virtue of part (a).

(c) $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 0 \end{vmatrix} = 3\begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix} + 6(-1)\begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix}$

$= 3(32-35) - 6(8-14) = -9 + 42 = 33 \neq 0$

$\therefore$ this set of vectors is linearly independent

(3) (d) $\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow$ these vectors are linearly dependent.

Note in particular that
$\vec{w_1} = \vec{w_2}$
i.e, $\vec{w_1} - \vec{w_2} + 0\vec{w_3} = \vec{0}$.

$S = \{\vec{w_1}, \vec{w_3}\}$ is a linearly independent subset since $\vec{w_3} \neq k\vec{w_1}$.

(e) 4 3-vectors are naturally linearly dependent
In addition we observe that $\vec{r_2} = 2\vec{r_1}$
i.e, $2\vec{r_1} - \vec{r_2} + 0\vec{r_3} + 0\vec{r_4} = \vec{0}$

$S = \{\vec{r_1}, \vec{r_3}, \vec{r_4}\}$ is a linearly independent subset
since $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 5 & 3 \end{vmatrix} = 5(-1)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = -5(2-1) = -5 \neq 0.$

(f) This set of vectors is linearly dependent since it contains $\vec{0}$.
The subset $S = \{\vec{a}, \vec{b}, \vec{d}\}$ is linearly independent since

$$\begin{vmatrix} 2 & 1 & 5 \\ 0 & 1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 1 & 3 \end{vmatrix}$$
$$= 2(2+3) + (3-5) = 10-2 = 8 \neq 0.$$

(g) $$\begin{vmatrix} 2 & 6 & 1 & 5 \\ 1 & 3 & 0 & 2 \\ 3 & 9 & 0 & 6 \\ 1 & 3 & 1 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 & 1 \\ 6 & 3 & 9 & 3 \\ 1 & 0 & 0 & 1 \\ 5 & 2 & 6 & 3 \end{vmatrix}^T$$

$$= \begin{vmatrix} 2 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 5 & 2 & 6 & 3 \end{vmatrix}^{T} \quad [2]-3[1] \qquad = 0.$$

This set of vectors is linearly dependent.
In particular, we note that $\vec{v_2} = 3\vec{v_1}$

i.e, $3\vec{v_1} - \vec{v_2} + 0\vec{v_3} + 0\vec{v_4} = \vec{0}$

Consider the subset $S_1 = \{\vec{v_1}, \vec{v_3}, \vec{v_4}\}$. To check for linear dependence / independence, we consider

$$\alpha_1 \vec{v_1} + \alpha_2 \vec{v_3} + \alpha_3 \vec{v_4} = \vec{0} \quad \text{———} \quad ✓$$

augmented matrix :

$$\begin{pmatrix} 2 & 1 & 5 & 0 \\ 1 & 0 & 2 & 0 \\ 3 & 0 & 6 & 0 \\ 1 & 1 & 3 & 0 \end{pmatrix} \xrightarrow{[1]\leftrightarrow[2]} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 2 & 1 & 5 & 0 \\ 3 & 0 & 6 & 0 \\ 1 & 1 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow[\substack{[3]-3[1] \\ [4]-[1]}]{[2]-2[1]} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \xrightarrow[{[3]\leftrightarrow[4]}]{} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow[{[3]-[2]}]{} \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

equivalent system : $\alpha_1 + 2\alpha_3 = 0$
$\alpha_2 + \alpha_3 = 0$

$\rightarrow$ let $\alpha_3 = t$, $\alpha_1 = -2t$, $\alpha_2 = -t$

$\therefore$ (✓) becomes $t[-\vec{v_1} - 2\vec{v_2} + \vec{v_3}] = \vec{0}$
for any choice of t.

The set S_1 of vectors is still linearly dependent.
But $S_2 = \{\vec{v_1}, \vec{v_3}\}$ is linearly independent
since $\vec{v_3} \neq k\vec{v_1}$.

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4 (a) $\underset{\sim}{A} = \begin{pmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{pmatrix}$

$\det \underset{\sim}{A} = 1 \begin{vmatrix} 4 & -1 \\ 2 & 5 \end{vmatrix} - 6 \begin{vmatrix} 2 & -1 \\ -1 & 5 \end{vmatrix} + 4 \begin{vmatrix} 2 & 4 \\ -1 & 2 \end{vmatrix}$
$= 1(22) - 6(9) + 4(8) = 0$

$\underset{\sim}{A}$ is not invertible.

4 (c) $\underset{\sim}{B} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}$ $\det \underset{\sim}{B} = 2 \cdot 2 \cdot 2 \cdot 2 = 16 \neq 0$

$\underset{\sim}{B}$ is invertible.

(i) $C_{11} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8$ $C_{12} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = -4$

$C_{13} = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 2$ $C_{14} = -\begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = -1$

$$C_{21} = -\begin{vmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0 \qquad C_{22} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8$$

$$C_{23} = -\begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{vmatrix} = -4 \qquad C_{24} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 2$$

$$C_{31} = \begin{vmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0 \qquad C_{32} = -\begin{vmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

$$C_{33} = +\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 8 \qquad C_{34} = -\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -4$$

$$C_{41} = -\begin{vmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} = 0 \qquad C_{42} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 0$$

$$C_{43} = -\begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 0 \qquad C_{44} = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 8$$

$$\operatorname{adj} \underset{\sim}{B} = \begin{pmatrix} 8 & 0 & 0 & 0 \\ -4 & 8 & 0 & 0 \\ 2 & -4 & 8 & 0 \\ -1 & 2 & -4 & 8 \end{pmatrix}$$

$$\underset{\sim}{B}^{-1} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ -1/4 & 1/2 & 0 & 0 \\ 1/8 & -1/4 & 1/2 & 0 \\ -1/16 & 1/8 & -1/4 & 1/2 \end{pmatrix}$$

4)(c)(ii) by the direct method

with(linalg) :
A := matrix([[2, 0, 0, 0, 1, 0, 0, 0], [1, 2, 0, 0, 0, 1, 0, 0], [0, 1, 2, 0, 0, 0, 1, 0], [0, 0, 1, 2, 0, 0, 0, 1]]);

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

A := swaprow(A, 1, 2);

$$[1] \leftrightarrow [2] \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

A := addrow(A, 1, 2, -2);

$$[2] - 2[1] \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -4 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$

A := swaprow(A, 2, 3);

$$[2] \leftrightarrow [3] \qquad \begin{bmatrix} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -4 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{4}$$

A := addrow(A, 2, 1, -2) : A := addrow(A, 2, 3, 4);

$$\begin{matrix} [1] - 2[2] \\ [3] + 4[2] \end{matrix} \qquad \begin{bmatrix} 1 & 0 & -4 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 & 1 & -2 & 4 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

A := swaprow(A, 3, 4);

$$[3] \leftrightarrow [4] \qquad \begin{bmatrix} 1 & 0 & -4 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 8 & 0 & 1 & -2 & 4 & 0 \end{bmatrix} \tag{6}$$

A := addrow(A, 3, 1, 4) : A := addrow(A, 3, 2, -2) : A := addrow(A, 3, 4, -8);

[1]+4[3]
[2]-2[3]
[4]-8[3]

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -4 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -16 & 1 & -2 & 4 & -8 \end{bmatrix} \tag{7}$$

$A := mulrow\left(A, 4, -\frac{1}{16}\right);$

$-\frac{1}{16}[4]$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -4 & 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -\frac{1}{16} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{8}$$

$A := addrow(A, 4, 1, -8) : A := addrow(A, 4, 2, 4) : A := addrow(A, 4, 3, -2);$

[1]-8[4]
[2]+4[4]
[3]-2[4]

$$\begin{bmatrix} 1 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 & -\frac{1}{16} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{9}$$

$B := matrix([[2, 0, 0, 0], [1, 2, 0, 0], [0, 1, 2, 0], [0, 0, 1, 2]]);$

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \tag{10}$$

$inverse(B);$

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{4} & \frac{1}{2} & 0 & 0 \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} & 0 \\ -\frac{1}{16} & \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \tag{11}$$

4 (b) $\underset{\sim}{C} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$

$$\det \underset{\sim}{C} = 1 \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + 8 \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix}$$
$$= (6-15) + 8(5-4) = -9+8 = -1 \neq 0$$

$\underset{\sim}{C}$ is invertible

matrix of cofactors

$$C_{11} = \begin{vmatrix} 5 & 3 \\ 0 & 8 \end{vmatrix} = 40, \quad C_{12} = -\begin{vmatrix} 2 & 3 \\ 1 & 8 \end{vmatrix} = -13$$
$$C_{13} = \begin{vmatrix} 2 & 5 \\ 1 & 0 \end{vmatrix} = -5$$
$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 0 & 8 \end{vmatrix} = -16, \quad C_{22} = \begin{vmatrix} 1 & 3 \\ 1 & 8 \end{vmatrix} = 5$$
$$C_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 0 \end{vmatrix} = 2$$
$$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} = 6-15 = -9, \quad C_{32} = -\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3$$
$$C_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 1$$

$$\operatorname{adj}(\underset{\sim}{C}) = \begin{pmatrix} 40 & -13 & -5 \\ -16 & 5 & 2 \\ -9 & 3 & 1 \end{pmatrix}^T = \begin{pmatrix} 40 & -16 & -9 \\ -13 & 5 & 3 \\ -5 & 2 & 1 \end{pmatrix}$$

$$\underset{\sim}{C}^{-1} = \begin{pmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{pmatrix}$$

7(b)(ii) by the direct method.

```
with(linalg):
A := matrix([[1, 2, 3, 1, 0, 0], [2, 5, 3, 0, 1, 0], [1, 0, 8, 0, 0, 1]]);
```

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

```
A := addrow(A, 1, 2, -2) : A := addrow(A, 1, 3, -1);
```

$$\begin{bmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{bmatrix} \tag{2}$$

```
A := addrow(A, 2, 1, -2) : A := addrow(A, 2, 3, 2);
```

$$\begin{bmatrix} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{bmatrix} \tag{3}$$

```
A := mulrow(A, 3, -1);
```

$$\begin{bmatrix} 1 & 0 & 9 & 5 & -2 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{bmatrix} \tag{4}$$

```
A := addrow(A, 3, 1, -9) : A := addrow(A, 3, 2, 3);
```

$$\begin{bmatrix} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{bmatrix} \tag{5}$$

```
C := matrix([[1, 2, 3], [2, 5, 3], [1, 0, 8]]);
```

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix} \tag{6}$$

```
inverse(C);
```

$$\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix} \tag{7}$$

(4) (d) $\underset{\sim}{D} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

$\det \underset{\sim}{D} = \cos^2\theta + \sin^2\theta = 1$

$C_{11} = +\cos\theta \qquad C_{12} = -(-\sin\theta)$

$C_{21} = -\sin\theta \qquad C_{22} = \cos\theta$

$\operatorname{adj} \underset{\sim}{D} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

$\underset{\sim}{D}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

<u>Alt.</u> $\begin{pmatrix} \cos\theta & \sin\theta & 1 & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{pmatrix}$

$[1]\cdot\frac{1}{\cos\theta} \quad \begin{pmatrix} 1 & \sin\theta/\cos\theta & 1/\cos\theta & 0 \\ -\sin\theta & \cos\theta & 0 & 1 \end{pmatrix}$

$[2]+[1]\sin\theta \quad \begin{pmatrix} 1 & \sin\theta/\cos\theta & 1/\cos\theta & 0 \\ 0 & 1/\cos\theta & \sin\theta/\cos\theta & 1 \end{pmatrix}$

$[2]\cos\theta \quad \begin{pmatrix} 1 & \sin\theta/\cos\theta & 1/\cos\theta & 0 \\ 0 & 1 & \sin\theta & \cos\theta \end{pmatrix}$

$[1]-\frac{\sin\theta}{\cos\theta}[2] \quad \begin{pmatrix} 1 & 0 & \cos\theta & -\sin\theta \\ 0 & 1 & \sin\theta & \cos\theta \end{pmatrix}$

$\underset{\sim}{D}^{-1} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

5

$$\left.\begin{aligned}(5-\lambda)x_1 + 4x_2 + 2x_3 &= 0\\ 4x_1 + (5-\lambda)x_2 + 2x_3 &= 0\\ 2x_1 + 2x_2 + (2-\lambda)x_3 &= 0\end{aligned}\right\} \quad (X)$$

For this homogeneous system to have non-trivial solutions we must require that

$$D = \begin{vmatrix} (5-\lambda) & 4 & 2 \\ 4 & (5-\lambda) & 2 \\ 2 & 2 & (2-\lambda) \end{vmatrix} = 0$$ (note: this is a polynomial equation in)

But $D = (5-\lambda)\begin{vmatrix} (5-\lambda) & 2 \\ 2 & (2-\lambda) \end{vmatrix} - 4\begin{vmatrix} 4 & 2 \\ 2 & (2-\lambda) \end{vmatrix} + 2\begin{vmatrix} 4 & (5-\lambda) \\ 2 & 2 \end{vmatrix}$

$$\begin{aligned}
&= (5-\lambda)\left[\lambda^2 - 7\lambda + 6\right] - 4\left[4 - 4\lambda\right] + 2\left[-2 + 2\lambda\right]\\
&= (5-\lambda)(\lambda-1)(\lambda-6) + 16(\lambda-1) + 4(\lambda-1)\\
&= (\lambda-1)\left[(5-\lambda)(\lambda-6) + 20\right]\\
&= (\lambda-1)\left[-\lambda^2 + 11\lambda - 10\right] = -(\lambda-1)\left[\lambda^2 - 11\lambda + 10\right]\\
&\qquad = -(\lambda-1)(\lambda-1)(\lambda-10)\\
&\qquad = -(\lambda-1)^2(\lambda-10)
\end{aligned}$$

$$\therefore\ D = 0 \implies \underbrace{\lambda = 1 \text{ or } \lambda = 1}_{\text{multiplicity } 2} \text{ or } \lambda = 2$$

If $\lambda = 1$, (X) becomes

$$\begin{aligned}4x_1 + 4x_2 + 2x_3 &= 0\\ 4x_1 + 4x_2 + 2x_3 &= 0\\ 2x_1 + 2x_2 + x_3 &= 0\end{aligned}$$

& there is effectively only one equation here, namely
$2x_1 + 2x_2 + x_3 = 0$.

let $\underbrace{x_2 = s \text{ \& } x_3 = t}_{\text{"free" variables}} \implies x_1 = -s - t/2$

an infinite # of solutions

$\therefore$ when $\lambda=1$, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -s-t/2 \\ s \\ t \end{pmatrix}$

$= s\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$

In this case the solution is always a linear combination of the vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ & $\begin{pmatrix} -1/2 \\ 0 \\ 1 \end{pmatrix}$

When $\lambda=10$, (X) becomes

$$-5x_1 + 4x_2 + 2x_3 = 0$$
$$4x_1 - 5x_2 + 2x_3 = 0$$
$$2x_1 + 2x_2 - 8x_3 = 0$$

augmented matrix : $\begin{pmatrix} -5 & 4 & 2 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{pmatrix} \longrightarrow$

$\xrightarrow{[1]+[2]} \begin{pmatrix} -1 & -1 & 4 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{pmatrix}$

$\xrightarrow{(-1)[1]} \begin{pmatrix} 1 & 1 & -4 & 0 \\ 4 & -5 & 2 & 0 \\ 2 & 2 & -8 & 0 \end{pmatrix}$

$$\xrightarrow[\,[3]-2[1]\,]{[2]-4[1]} \begin{pmatrix} 1 & 1 & -4 & 0 \\ 0 & -9 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{-\frac{1}{9}[2]} \begin{pmatrix} 1 & 1 & -4 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{[1]-[2]} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

equivalent system: $x_1 - 2x_3 = 0$
$x_2 - 2x_3 = 0$

let $\underbrace{x_3}_{\text{"free" variable}} = r \implies x_1 = 2r,\ x_2 = 2r.$

infinite # of solutions

When $\lambda = 10$, $\underset{\sim}{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2r \\ 2r \\ r \end{pmatrix} = r \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$

In this case every solutions is a multiple of the vector $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$