MATH 2130 – Tutorial Problem Solutions, Thu Feb 15

Gradients and Directional Derivatives

Example. Let $f(x, y) = 2x\sin(2\pi xy)$. At the point $(x, y) = (1, 1)$, find a direction in which the rate of change of f is 4π. At this point, is there a direction in which the rate of change of f is 8π?

Solution. The gradient of f is

$$\nabla f = (f_x, f_y) = (2\sin(2\pi xy) + 4\pi xy\cos(2\pi xy), 4\pi x^2\cos(2\pi xy)).$$

At $(1, 1)$, we have

$$\nabla f|_{(1,1)} = (4\pi, 4\pi).$$

Given any unit vector $\widehat{\mathbf{v}} = (a, b)$, where $a^2 + b^2 = 1$, the rate of change of f in the direction of $\widehat{\mathbf{v}}$ is $\nabla f \cdot \widehat{\mathbf{v}}$. We want to find $\widehat{\mathbf{v}}$ such that $\nabla f|_{(1,1)} \cdot \widehat{\mathbf{v}} = 4\pi$. That is,

$$(4\pi, 4\pi) \cdot (a, b) = 4\pi,$$

which implies that

$$a + b = 1.$$

Substitute $b = 1 - a$ into the condition $a^2 + b^2 = 1$, and simplify:

$$2a^2 - 2a = 0,$$

which implies that $a = 0$ or $a = 1$. The corresponding b values are $b = 1$ or $b = 0$, respectively. Thus there are two possible answers: $\widehat{\mathbf{v}} = (1, 0)$ or $\widehat{\mathbf{v}} = (0, 1)$.

Notice that $|\nabla f|_{(1,1)}| = |(4\pi, 4\pi)| = 4\sqrt{2}\pi < 8\pi$. Since $|\nabla f|$ is the maximum rate of change of f at a given point, there is no direction in which the rate of change of f is 8π at the point $(1, 1)$.

Example. Let $f(x, y, z) = xyz$. Let $\mathcal{C}$ be the curve with vector representation

$$\mathbf{r}(t) = (t^2 + 1)\widehat{\mathbf{i}} + \cos(\pi t)\widehat{\mathbf{j}} + (t^3 - 2t^2)\widehat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

Find the rate of change of f in the direction of $\mathcal{C}$ at the point $(x, y, z) = (5, 1, 0)$.

Solution. Set $\mathbf{r}(t) = (5, 1, 0)$ to find the value of t:

$$(t^2 + 1, \cos(\pi t), t^3 - 2t^2) = (5, 1, 0).$$

From the x-coordinates, we get $t = \pm 2$. From the z-coordinates, we get $t = 0$ or $t = 2$. The only possible solution is $t = 2$. Verify: $\mathbf{r}(2) = (5, 1, 0)$. Now we calculate

$$\begin{aligned}\mathbf{r}'(t) =& (2t, -\pi\sin(\pi t), 3t^2 - 4t),\\ \mathbf{r}'(2) =& (4, 0, 4).\end{aligned}$$

Thus a vector in the direction of $\mathcal{C}$ at the point $(5, 1, 0)$ is $\mathbf{T} = (4, 0, 4)$, and the corresponding unit vector is $\widehat{\mathbf{T}} = \frac{1}{\sqrt{2}}(1, 0, 1)$.

The gradient of f is

$$\nabla f = (yz, xz, xy).$$

At the point $(5,1,0)$, we get

$$\nabla f|_{(5,1,0)} = (0,0,5).$$

The directional derivative of f in the direction of $\mathbf{T}$ is

$$D_{\mathbf{T}} f = \nabla f \cdot \widehat{\mathbf{T}} = (0,0,5) \cdot \frac{1}{\sqrt{2}}(1,0,1) = \frac{5}{\sqrt{2}}.$$

The rate of change of f in the direction of $\mathcal{C}$ at the given point is $\frac{5}{\sqrt{2}}$.

Tangent Lines and Planes

Example. Let $\mathcal{D}$ be the curve formed by the intersection of the surfaces $z = \sqrt{y^2 - x^2}$ and $x^2 + 3y^2 + z^2 = 4$ in 3D space. Find the unit tangent vector to $\mathcal{D}$ at the point $P = \left(\frac{\sqrt{3}}{2}, -1, \frac{1}{2}\right)$ that points in the direction of increasing x.

Solution. Let $F(x,y,z) = \sqrt{y^2 - x^2} - z$ and $G(x,y,z) = x^2 + 3y^2 + z^2$. Then the two surfaces are described by the conditions $F(x,y,z) = 0$ and $G(x,y,z) = 4$, respectively. We will find $\nabla F|_P$ and $\nabla G|_P$. Then a tangent vector to the curve $\mathcal{D}$ at P is $\nabla F|_P \times \nabla G|_P$.

Calculate

$$\begin{aligned}
\nabla F &= \left(-\frac{x}{\sqrt{y^2 - x^2}}, \frac{y}{\sqrt{y^2 - x^2}}, -1\right), \\
\nabla F|_P &= \left(-\frac{\sqrt{3}/2}{1/2}, -\frac{1}{1/2}, -1\right) = \left(-\sqrt{3}, -2, -1\right), \\
\nabla G &= (2x, 6y, 2z), \\
\nabla G|_P &= \left(\sqrt{3}, -6, 1\right).
\end{aligned}$$

Thus a tangent vector to the curve of intersection $\mathcal{D}$ at the point P is

$$\begin{aligned}
\mathbf{T} = \left(-\sqrt{3}, -2, -1\right) \times \left(\sqrt{3}, -6, 1\right) &= \left(-2 - 6, -\sqrt{3} + \sqrt{3}, 6\sqrt{3} + 2\sqrt{3}\right) \\
&= \left(-8, 0, 8\sqrt{3}\right).
\end{aligned}$$

We were asked for the unit tangent vector to the curve in the direction of increasing x. Notice that the vector we calculated above has a negative x-component. This vector points in the direction of decreasing x. To construct the appropriate tangent vector, we take the negative and rescale to length 1. The result is

$$-\widehat{\mathbf{T}} = \left(\frac{1}{2}, 0, -\frac{\sqrt{3}}{2}\right).$$

Example. Consider the surface $x^2 + y^2 + (z-5)^2 = 1$ in 3D space. Find all points (x,y,z) on this surface such that the line through (x,y,z) and the origin is tangent to the surface.

Solution. The given surface is a sphere with radius 1 and center $(0,0,5)$. A sketch might suggest that the points satisfying the given condition form a circle on the sphere, parallel to the xy-plane.

Let $F(x, y, z) = x^2 + y^2 + (z-5)^2$. Then the surface is described by the equation $F(x, y, z) = 1$. Let $P = (x, y, z)$ be an arbitrary point on the surface. The gradient of F at P is

$$\nabla F|_P = (2x, 2y, 2(z-5)).$$

In order for a line through P to be tangent to the surface, it must be perpendicular to the gradient $\nabla F|_P$.

The line through (x, y, z) and the origin is parallel to the vector $\mathbf{P} = (x, y, z)$. Therefore we require (x, y, z) to be a point on the surface such that

$$\nabla F|_P \cdot \mathbf{P} = 0.$$

This equation becomes

$$(2x, 2y, 2(z-5)) \cdot (x, y, z) = 0,$$

and so

$$x^2 + y^2 + z(z-5) = 0.$$

There are two equations to be satisfied simultaneously:

$$x^2 + y^2 + (z-5)^2 = 1 \quad \text{and} \quad x^2 + y^2 + z(z-5) = 0.$$

Subtracting the second equation from the first yields

$$(z-5)^2 - z(z-5) = 1,$$

which simplifies to

$$z - 5 = -\frac{1}{5},$$

so $z = \frac{24}{5}$. We substitute this in the first equation to obtain

$$x^2 + y^2 = \frac{24}{25}.$$

Thus the solutions are the points of the form $\left(x, y, \frac{24}{5}\right)$ where $x^2 + y^2 = \frac{24}{25}$. These points form a circle parallel to the xy-plane, as expected. A parametrization is $\left(\frac{2\sqrt{6}}{5}\cos t, \frac{2\sqrt{6}}{5}\sin t, \frac{24}{5}\right)$, $0 \le t \le 2\pi$.

Example. Consider the surface

$$\frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1$$

in 3D space. Find all points (x, y, z) on this surface such that the tangent plane to the surface is parallel to the plane $x + y + z = 0$.

Solution. The given surface is an ellipsoid with center $(0, 0, 0)$. A sketch might suggest that there are two points on the ellipsoid with an appropriate tangent plane.

Let $F(x, y, z) = \frac{x^2}{4} + \frac{y^2}{9} + z^2$. Then the surface is $F(x, y, z) = 1$. The gradient of F is

$$\nabla F = \left(\frac{x}{2}, \frac{2y}{9}, 2z\right).$$

At each point on the surface, the normal to the tangent plane is given by the gradient vector ∇F.

A normal vector to the plane $x+y+z=0$ is $(1,1,1)$. In order for a tangent plane to be parallel to this one, the normal to the tangent plane must take the form $t(1,1,1)$ for some nonzero $t \in \mathbb{R}$. We set $\nabla F = (t,t,t)$, and obtain

$$x = 2t, \quad y = \frac{9t}{2}, \quad z = \frac{t}{2}.$$

But the solution must also be a point on the surface, which means it must satisfy $F(x,y,z) = 1$. Upon substitution, we get

$$t^2 + \frac{t^2}{4} + \frac{9t^2}{4} = 1,$$

which reduces to

$$t^2 = \frac{4}{14}.$$

There are two solutions: $t = \frac{2}{\sqrt{14}}$ and $t = -\frac{2}{\sqrt{14}}$, corresponding to the points $\left(\frac{4}{\sqrt{14}}, \frac{9}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right)$ and $\left(-\frac{4}{\sqrt{14}}, -\frac{9}{\sqrt{14}}, -\frac{1}{\sqrt{14}}\right)$.