MATH 1210 Assignment #3 Solutions

Due: February 22, 2016; At the start of class

Reminder: all assignments *must* be accompanied by a signed copy of the honesty declaration available on the course website.

1. Consider the polynomial
   $P(x) = \sum_{k=0}^{2015} \frac{(-1)^k}{k+1} x^k$.
   (a) Show that $P(x)$ must have at least one positive real root.
   (b) Show that $P(x)$ has no negative real roots.
   (c) Show that if z is any root of $P(x)$, then $|z| < 2020$.

Solution:
a) One can rewrite the polynomial as $P(x) = -\frac{1}{2016}x^{2015} + \frac{1}{2015}x^{2014} - \frac{1}{2014}x^{2013} + \cdots - \frac{1}{2}x + 1$. Each following coefficient has a sign opposite to the previous one, therefore the number of sign changes in the sequence of coefficients is 2015. By Descarte's rule of signs, $P(x)$ must have an odd number (and not greater than 2015) of positive real roots, so this number cannot be equal to 0.

b)$P(-x) = \frac{1}{2016}x^{2015} + \frac{1}{2015}x^{2014} + \frac{1}{2014}x^{2013} + \cdots + \frac{1}{2}x + 1$.
There are no sign changes in the sequence of coefficients, so By Descarte's rule of signs, $P(x)$ must have 0 negative real roots.

c) By the Bounds Theorem, if z is any root of $P(x)$, then $|z| < \frac{M}{|a_{2015}|} + 1$, where $M = max\{|-\frac{1}{2016}|, |\frac{1}{2015}|, \ldots, |-\frac{1}{2}|, |1|\} = 1$. Therefore $|z| < \frac{1}{\frac{1}{2016}} + 1 = 2016 + 1 = 2017 < 2020$.

2. Consider the polynomial $P(x) = x^3 + 4x^2 + k^3x + 3$, where k is some integer. Find all possible values of k such that $P(x)$ has a rational root. (Clearly explain why there are no other values of k that work.)

Solution:
By the Rational Root theorem, if $\frac{p}{q}$ is a rational root (in lowest terms) of $P(x)$, then p divides 3 and q divides 1. So the only possible rational roots are $1, 3, -1$ and -3.
If 1 is a root, then $0 = P(1) = 1 + 4 + k^3 + 3 = k^3 + 8$, so $k^3 = -8$ and since k must be an integer, $k = -2$.
If 3 is a root, then $0 = P(3) = 27 + 4 \cdot 9 + 3k^3 + 3 = 3k^3 + 66$, so $k^3 = -22$ and there

are no integers that satisfy this equation.
If -1 is a root, then $0 = P(-1) = -1 + 4 - k^3 + 3 = -k^3 + 6$, so $k^3 = 6$ and there are no integers that satisfy this equation.
If -3 is a root, then $0 = P(-3) = -27 + 4 \cdot 9 + -3k^3 + 3 = -3k^3 + 12$, so $k^3 = 4$ and there are no integers that satisfy this equation.
Therefore, the only k such that $P(x)$ has a rational root is $k = -2$.

3. In each part of this question: (i) use Descartes rules of signs to state the number of possible positive and negative zeros of the polynomial; (ii) use the bounds theorem to find bounds for zeros of the polynomial; (iii) use the rational root theorem to list all possible rational zeros of the polynomial; (iv) use this information to find all the zeros of the polynomial.

   (a) $6x^5 + 7x^4 - 13x^3 - 85x^2 - 50x$
   (b) $x^9 + 3x^8 + 3x^7 + 3x^6 + 6x^5 + 6x^4 + 4x^3 + 6x^2 + 6x + 2$

Solution:
(a)Let $P(x) = 6x^5 + 7x^4 - 13x^3 - 85x^2 - 50x$.
(i)There is one sign change in the sequence of coefficients, so $P(x)$ has 1 positive root.
There are 3 sign changes in the sequence of coefficients of $P(-x) = -6x^5 + 7x^4 + 13x^3 - 85x^2 + 50x$, so $P(x)$ has 3 or 1 negative root.

(ii)If x is a root of $P(x)$, then $|x| < \frac{85}{6} + 1 = 15\frac{1}{6}$.

(iii) We can't use the Rational Root theorem right away, because the last coefficient is 0. Notice that 0 is a root of $P(x)$, and $P(x) = x(6x^4 + 7x^3 - 13x^2 - 85x - 50)$.
Then we can use the Rational Root theorem for $Q(x) = 6x^4 + 7x^3 - 13x^2 - 85x - 50$.
If $\frac{p}{q}$ is a root of $Q(x)$, then p divides 50 and q divides 6, so
$\frac{p}{q} \in \pm\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{2}{6}, \frac{5}{1}, \frac{5}{2}, \frac{5}{3}, \frac{5}{6}, \frac{10}{1}, \frac{10}{2}, \frac{10}{3}, \frac{10}{6}, \frac{25}{1}, \frac{25}{2}, \frac{25}{3}, \frac{25}{6}, \frac{50}{1}, \frac{50}{2}, \frac{50}{3}, \frac{50}{6}\} =$
$= \pm\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 2, \frac{2}{3}, 5, \frac{5}{2}, \frac{5}{3}, \frac{5}{6}, 10, \frac{10}{3}, 25, \frac{25}{2}, \frac{25}{3}, \frac{25}{6}, 50, \frac{50}{3}\}$

(iv) Using the Bounds Theorem, we can limit the possible candidates for rational roots to $\pm\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 2, \frac{2}{3}, 5, \frac{5}{2}, \frac{5}{3}, \frac{5}{6}, 10, \frac{10}{3}, \frac{25}{2}, \frac{25}{3}, \frac{25}{6}\}$
By plugging different values in $Q(x)$, we eventually get that $Q(\frac{5}{2}) = 0$, so $Q(x)$ can be divided by $2x - 5$.
$6x^4 + 7x^3 - 13x^2 - 85x - 50 = (2x - 5)(3x^3 + 11x^2 + 21x + 10)$.
$3x^3 + 11x^2 + 21x + 10$ can have rational roots from the set $\pm\{1, 2, 5, 10, \frac{1}{3}, \frac{2}{3}, \frac{5}{3}, \frac{10}{3}\}$.
Since $Q(x)$ has only one positive root (by Descartes'), which is $\frac{5}{2}$, we can try only negative roots.
By plugging different values in $3x^3+11x^2+21x+10$, we eventually get that $Q(-\frac{2}{3}) = 0$, so $Q(x)$ can be divided by $3x + 2$.

$3x^3+11x^2+21x+10=(3x+2)(x^2+3x+5)$, and x^2+3x+5 has roots $\frac{-3\pm\sqrt{9-4\cdot 5}}{2} = -\frac{3}{2}\pm\frac{\sqrt{11}}{2}i$.
To summarize, all zeros of $P(x)$ are $0, \frac{5}{2}, -\frac{2}{3}, -\frac{3}{2}+\frac{\sqrt{11}}{2}i, -\frac{3}{2}-\frac{\sqrt{11}}{2}i$.

(b) Let $P(x) = x^9+3x^8+3x^7+3x^6+6x^5+6x^4+4x^3+6x^2+6x+2$

(i) There are no sign changes in the sequence of coefficients, so $P(x)$ has no positive roots.
There are 9 sign changes in the sequence of coefficients of $P(-x) = -x^9+3x^8-3x^7+3x^6-6x^5+6x^4-4x^3+6x^2-6x+2$, so $P(x)$ has $9, 7, 5, 3$ or 1 negative roots.

(ii) If x is a root of $P(x)$, then $|x| < \frac{6}{1}+1 = 7$.

(iii) If $\frac{p}{q}$ is a root of $P(x)$, then p divides 2 and q divides 1, so
$\frac{p}{q} \in \pm\{\frac{1}{1}, \frac{2}{1}\} = \pm\{1, 2\}$

(iv) Since $P(x)$ has no positive roots, the only possible rational roots are -1 and -2.
$P(-1) = 0$, and $P(x) = (x+1)(x^8+2x^7+x^6+2x^5+4x^4+2x^3+2x^2+4x+2)$.

$(-1)^8+(-1)^7+(-1)^6+2(-1)^5+4(-1)^4+2(-1)^3+2(-1)^2+4(-1)+2 = 0$, and
$x^8+2x^7+x^6+2x^5+4x^4+2x^3+2x^2+4x+2 = (x+1)(x^7+x^6+2x^4+2x^3+2x+2)$.

$(-1)^7+(-1)^6+2(-1)^4+2(-1)^3+2(-1)+2 = 0$, and
$x^7+x^6+2x^4+2x^3+2x+2 = (x+1)(x^6+2x^3+2)$.

So, $P(x) = (x+1)^3(x^6+2x^3+2)$ and x^6+2x^3+2 has no rational roots.

To find roots of x^6+2x^3+2, we can make a substitution $y = x^3$. Then $y^2+2y+2 = 0$ and $y = \frac{-2\pm\sqrt{4-4\cdot 2}}{2} = -1\pm i$.

If $x^3 = -1+i = \sqrt{2}e^{\frac{3\pi}{4}}$, then $x = \sqrt[6]{2}e^{\frac{\frac{3\pi}{4}+2k\pi}{3}}, k = 0, 1, 2$.
In this case we have 3 roots $x = \sqrt[6]{2}e^{\frac{\pi}{4}}$, $x = \sqrt[6]{2}e^{\frac{11\pi}{12}}$, $x = \sqrt[6]{2}e^{\frac{19\pi}{12}} = \sqrt[6]{2}e^{-\frac{5\pi}{12}}$.

If $x^3 = -1-i = \sqrt{2}e^{\frac{5\pi}{4}}$, then $x = \sqrt[6]{2}e^{\frac{\frac{5\pi}{4}+2k\pi}{3}}, k = 0, 1, 2$.
In this case we have 3 roots $x = \sqrt[6]{2}e^{\frac{5\pi}{12}}$, $x = \sqrt[6]{2}e^{\frac{13\pi}{12}} = \sqrt[6]{2}e^{-\frac{11\pi}{12}}$, $x = \sqrt[6]{2}e^{\frac{21\pi}{12}} = \sqrt[6]{2}e^{-\frac{\pi}{4}}$.

To summarize, the roots of $P(x)$ are:
-1 (with multiplicity 3), $\sqrt[6]{2}e^{\frac{\pi}{4}}, \sqrt[6]{2}e^{-\frac{\pi}{4}}, \sqrt[6]{2}e^{\frac{5\pi}{12}}, \sqrt[6]{2}e^{-\frac{5\pi}{12}}, \sqrt[6]{2}e^{\frac{11\pi}{12}}, \sqrt[6]{2}e^{-\frac{11\pi}{12}}$.

4. Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 2 & 0 & 2 \\ 2 & 0 & 1 & 6 \end{bmatrix}$; $B = (b_{ij})_{3\times 4}$, $b_{ij} = i - j$.

   Find a matrix X such that $3(X^T + I) = 2(B^T A)^T$, or explain why such X does not exist.

   Solution:
   After taking transpose of both sides of the equation, we get $3X + 3I = 2A^T(B^T)^T = 2A^T B$, so

   $$X = \tfrac{1}{3}(2A^T B - 3I) = \tfrac{2}{3}A^T B - I = \tfrac{2}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 6 \end{bmatrix}\begin{bmatrix} 0 & -1 & -2 & -3 \\ 1 & 0 & -1 & -2 \\ 2 & 1 & 0 & -1 \end{bmatrix} - I =$$

   $$= \tfrac{2}{3}\begin{bmatrix} 6 & 1 & -4 & -9 \\ 2 & -2 & -6 & -10 \\ 2 & 0 & -2 & -4 \\ 14 & 6 & -2 & -10 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & \frac{2}{3} & -\frac{8}{3} & -6 \\ \frac{4}{3} & -\frac{7}{3} & -4 & -\frac{20}{3} \\ \frac{4}{3} & 0 & -\frac{7}{3} & -\frac{8}{3} \\ \frac{28}{3} & 4 & -\frac{4}{3} & -\frac{23}{3} \end{bmatrix}.$$

5. Let x and y be real numbers; $A = \begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}$.

   Prove that for any integer $n \geq 0$, $A^{2n+1} = \begin{bmatrix} x^{2n+1} & x^{2n}y \\ 0 & -x^{2n+1} \end{bmatrix}$.

   Solution:
   Since $A^2 = \begin{bmatrix} x & y \\ 0 & -x \end{bmatrix}\begin{bmatrix} x & y \\ 0 & -x \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix}$,

   and multiplication for diagonal matrices is the same is multiplication of their corresponding entries,

   $A^{2n} = (A^2)^n = (\begin{bmatrix} x^2 & 0 \\ 0 & x^2 \end{bmatrix})^n = \begin{bmatrix} x^{2n} & 0 \\ 0 & x^{2n} \end{bmatrix}$.

   Therefore $A^{2n+1} = A^{2n}A = \begin{bmatrix} x^{2n} & 0 \\ 0 & x^{2n} \end{bmatrix}\begin{bmatrix} x & y \\ 0 & -x \end{bmatrix} = \begin{bmatrix} x^{2n+1} & x^{2n}y \\ 0 & -x^{2n+1} \end{bmatrix}$.

   Note: it is also possible to prove the statement using mathematical induction by n.

6. Let $\mathbf{u}$ be a vector from point $(1, -4, 0)$ to point $(-2, 3, 5)$; $\mathbf{v}$ be the vector with length 5 in the opposite direction to $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$.

   (a) Find $\mathbf{2u} \times \mathbf{v} + (\mathbf{u} \cdot \mathbf{v})|\mathbf{v}|\hat{\mathbf{u}}$, where $\hat{\mathbf{u}}$ is the unit vector in the direction of $\mathbf{u}$.
   (b) Find a vector of length 8 perpendicular to both **3u+v** and **u-2v**.

Solution:
$\mathbf{u} = \langle -2-1, 3+4, 5-0\rangle = \langle -3, 7, 5\rangle$.

$\mathbf{v} = -\frac{5\langle 1,2,-2\rangle}{|\langle 1,2,-2\rangle|} = -\frac{5\langle 1,2,-2\rangle}{\sqrt{1^2+2^2+(-2)^2}} = \langle -\frac{5}{3}, -\frac{10}{3}, \frac{10}{3}\rangle$

(a)

$\mathbf{u}\times\mathbf{v} = \langle -3, 7, 5\rangle \times \langle -\frac{5}{3}, -\frac{10}{3}, \frac{10}{3}\rangle =$
$= \langle 7\cdot\frac{10}{3} + 5\cdot\frac{10}{3}, 5(-\frac{5}{3}) + 3\cdot\frac{10}{3}, (-3)(-\frac{10}{3}) - 7(-\frac{5}{3})\rangle = \langle 40, \frac{5}{3}, \frac{65}{3}\rangle$.

$\mathbf{u}\cdot\mathbf{v} = \langle -3, 7, 5\rangle \cdot \langle -\frac{5}{3}, -\frac{10}{3}, \frac{10}{3}\rangle = -3\cdot(-\frac{5}{3}) + 7\cdot(-\frac{10}{3}) + 5\cdot\frac{10}{3} = -\frac{5}{3}$

$\hat{\mathbf{u}} = \frac{\mathbf{u}}{|\mathbf{u}|} = \frac{\langle -3,7,5\rangle}{\sqrt{3^2+7^2+5^2}} = \frac{1}{\sqrt{83}}\langle -3, 7, 5\rangle$;
and $|\mathbf{v}| = 5$ is given is the question.

So, $\mathbf{2u}\times\mathbf{v} + (\mathbf{u}\cdot\mathbf{v})|\mathbf{v}|\hat{\mathbf{u}} = 2\langle 40, \frac{5}{3}, \frac{65}{3}\rangle - \frac{5}{3}\cdot 5\cdot\frac{1}{\sqrt{83}}\langle -3, 7, 5\rangle =$
$= \langle 80 - \frac{25}{\sqrt{83}}, \frac{10}{3} - \frac{175}{3\sqrt{83}}, \frac{130}{3} - \frac{125}{3\sqrt{83}}\rangle = \langle \frac{80\sqrt{83}-25}{\sqrt{83}}, \frac{10\sqrt{83}-175}{3\sqrt{83}}, \frac{130\sqrt{83}-125}{3\sqrt{83}}\rangle$.

(b)
Using properties of the cross product, we can write
$(\mathbf{3u{+}v})\times(\mathbf{u\text{-}2v}) = 3\mathbf{u}\times\mathbf{u} - 2\mathbf{u}\times\mathbf{v} + \mathbf{v}\times\mathbf{u} - 2\mathbf{v}\times\mathbf{v} = 3\cdot\mathbf{0} - 2\mathbf{u}\times\mathbf{v} - \mathbf{u}\times\mathbf{v} - \mathbf{0} =$
$-3\mathbf{u}\times\mathbf{v} = -3\langle 40, \frac{5}{3}, \frac{65}{3}\rangle = \langle -120, -5, -65\rangle$.

Since $(\mathbf{3u{+}v})\times(\mathbf{u\text{-}2v})$ is perpendicular to both $\mathbf{3u{+}v}$ and $\mathbf{u\text{-}2v}$, so is $-\frac{1}{5}(\mathbf{3u{+}v})\times(\mathbf{u\text{-}2v}) = \langle 24, 1, 13\rangle$.

A vector of length 8 parallel to the last one will be $\frac{8\langle 24,1,13\rangle}{|\langle 24,1,13\rangle|} = \frac{8}{\sqrt{24^2+1^2+13^2}}\langle 24, 1, 13\rangle =$
$\frac{8}{\sqrt{746}}\langle 24, 1, 13\rangle = \langle \frac{192}{\sqrt{746}}, \frac{8}{\sqrt{746}}, \frac{104}{\sqrt{746}}\rangle$.

7. Let $\mathbf{u}$ and $\mathbf{v}$ be two unit vectors such that $\mathbf{u}\cdot\mathbf{v} = \frac{1}{32}$.
(a) Prove that vectors **u-v** and **3u+3v** are perpendicular.
(b) Find the angle between vectors **2u+6v** and **3u-v**.
Hint: Consider how dot product of a vector with itself is related to its length.

Solution:
(a) We will use the fact that $\mathbf{u}\cdot\mathbf{u} = |\mathbf{u}|^2 = 1^2 = 1$ (and $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2 = 1^2 = 1$).

$(\mathbf{u}\text{-}\mathbf{v})\cdot(\mathbf{3u}+\mathbf{3v}) = 3\mathbf{u}\cdot\mathbf{u} + 3\mathbf{u}\cdot\mathbf{v} - 3\mathbf{v}\cdot\mathbf{u} - 3\mathbf{v}\cdot\mathbf{v} = 3\cdot 1 + 3\mathbf{u}\cdot\mathbf{v} - 3\mathbf{u}\cdot\mathbf{v} - 3\cdot 1 = 0$, therefore vectors **u-v** and **3u+3v** are perpendicular.

(b) $(\mathbf{2u}+\mathbf{6v})\cdot(\mathbf{3u}\text{-}\mathbf{v}) = 6\mathbf{u}\cdot\mathbf{u} - 2\mathbf{u}\cdot\mathbf{v} + 18\mathbf{v}\cdot\mathbf{u} - 6\mathbf{v}\cdot\mathbf{v} = 6 + 16\cdot\frac{1}{32} - 6 = \frac{1}{2}$

$|\mathbf{2u}+\mathbf{6v}|^2 = (\mathbf{2u}+\mathbf{6v})\cdot(\mathbf{2u}+\mathbf{6v}) = 2\mathbf{u}\cdot\mathbf{u} + 24\mathbf{u}\cdot\mathbf{v} + 36\mathbf{v}\cdot\mathbf{v} = 2 + \frac{24}{32} + 36 = \frac{116}{3}$, so $|\mathbf{2u}+\mathbf{6v}| = \sqrt{\frac{116}{3}}$.

$|\mathbf{3u}\text{-}\mathbf{v}|^2 = (\mathbf{3u}\text{-}\mathbf{v})\cdot(\mathbf{3u}\text{-}\mathbf{v}) = 9\mathbf{u}\cdot\mathbf{u} - 6\mathbf{u}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{v} = 9 - \frac{6}{32} + 1 = \frac{157}{16}$, so $|\mathbf{3u}\text{-}\mathbf{v}| = \frac{\sqrt{157}}{4}$.

Then cosine of the angle between **2u+6v** and **3u-v** is equal to $\frac{(\mathbf{2u}+\mathbf{6v})\cdot(\mathbf{3u}\text{-}\mathbf{v})}{|\mathbf{2u}+\mathbf{6v}||\mathbf{3u}\text{-}\mathbf{v}|} = \frac{\frac{1}{2}}{\sqrt{\frac{116}{3}}\frac{\sqrt{157}}{4}} = \frac{2\sqrt{3}}{\sqrt{18212}}$, and the angle is $\cos^{-1}(\frac{2\sqrt{3}}{\sqrt{18212}})$.