MAT2130: Engineering Mathematical Analysis 1

Midterm 2 Practice Problems

1. In each of the following, either evaluate the multivariable limit, or show that it does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(1,2)} \frac{x^2+y^2}{x-y}$

   (b) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+2y}{x^2}$

   (c) $\displaystyle\lim_{(x,y)\to(2,-3)} e^{3x+2y}$

   (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+3xy-10y^2}{x-2y}$

   (e) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2x^2+y^2-4x-2y+3}{x+y-2}$

   (f) $\displaystyle\lim_{(x,y)\to(-1,0)} \frac{y}{\sqrt{x+3}-\sqrt{x+2y+3}}$

   (g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2-2y^2}{x^2+5y^2}$

   (h) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{x^2-xy-2y^2}$

   (i) $\displaystyle\lim_{(x,y)\to(2,1)} \frac{\sin(x-2y)}{x+2y}$

   (j) $\displaystyle\lim_{(x,y)\to(1,-1)} \frac{y^2+x+2y}{x-2y-3}$

2. Let

$$f(x,y) = 2xy^2 + y^3 + 4x + 7, \qquad h(x,y,z) = \frac{2y+3}{x^2+1},$$
$$g(x,y) = \ln\left(x^2+y^4+2\right), \qquad j(x,y,z) = \sin x\cos y + \sin y\cos z + \sin z\cos x.$$

   Calculate the following partial derivatives.

   (a) $\dfrac{\partial f}{\partial x}$

   (b) $\dfrac{\partial}{\partial y}(fg)$

   (c) $\dfrac{\partial j}{\partial z}$

   (d) $\dfrac{\partial}{\partial x}(h^2)$

   (e) $\dfrac{\partial^2 f}{\partial y^2}$

   (f) $\dfrac{\partial^2 g}{\partial x^2}$

   (g) $\dfrac{\partial^2 h}{\partial x \partial y}$

   (h) $\dfrac{\partial^3 j}{\partial x \partial y \partial z}$

3. In each of the following, use the chain rule to calculate the indicated derivative.

   (a) $u = s^2 + st + t^2$, $s = 4x^3y - x^2y^2 + 3xy^2$, $t = \frac{y}{x^2+1}$: $\dfrac{\partial u}{\partial x}$.

   (b) $w = \sin(xy + yz + t)$, $x = 3t^2 + 2t$, $y = e^{t^2}$, $z = 3t + 4$: $\dfrac{dw}{dt}$.

   (c) $w = (x + 2y + 3t)^3$, $x = s^2 - 4st$, $y = t^5 - t^3 + st^2$: $\left.\dfrac{\partial w}{\partial t}\right)_s$.

   (d) $w = \sin x\cos y$, $x = 2uv$, $y = \frac{u^2+1}{v^2+1}$, $u = e^{t-s}$, $v = st + s + t$: $\left.\dfrac{\partial w}{\partial s}\right)_t$.

   (e) u, s and t as given in part (a): $\dfrac{\partial^2 u}{\partial y \partial x}$.

4. In each of the following, use implicit differentiation to calculate the indicated derivative.

   (a) Define x and y as functions of t by
   $$xyt = 1, \quad x^2 + y^2 + t^2 = 4.$$
   Find $\dfrac{dx}{dt}$.

   (b) Define x, y and z as functions of s and t by
   $$tx + sy = 2, \quad xyz + x^2 s + z^2 t = 1, \quad 2sxy - 3tx^2 = 0.$$
   Find $\dfrac{\partial y}{\partial t}$.

   (c) Define u and v as functions of r, s and t by
   $$te^{u^2+v^2} = rs, \quad u^2 - rt^2 = s^2 t + v^2.$$
   Find $\dfrac{\partial u}{\partial s}$.

5. In 2D space, the polar coordinates (r, θ) are defined in terms of the Cartesian coordinates (x, y) by
   $$r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

   (a) Calculate $\dfrac{\partial \theta}{\partial x}$ and $\dfrac{\partial \theta}{\partial y}$.

   (b) Let $z = f(r)$. Show that $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$.

   (c) Let $w = g(\theta)$. Show that $x\dfrac{\partial w}{\partial x} + y\dfrac{\partial w}{\partial y} = 0$.

6. In 3D space, the Cartesian coordinates (x, y, z) are defined in terms of the spherical coordinates (r, ϕ, θ) (and vice versa) by
   $$x = r\cos\theta\sin\phi, \quad y = r\sin\theta\sin\phi, \quad z = r\cos\phi.$$

   (a) Let $v = g(x, y)$. Find $v_\phi^2 + v_\theta^2$ in terms of v_x and v_y.

   (b) Let $w = f(\phi, \theta)$. Find w_x in terms of w_ϕ and w_θ.

7. Recall that three noncoplanar vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in 3D space span a parallelepiped. Let α be the angle between $\mathbf{u}$ and $\mathbf{v}$, and let β be the acute angle between $\mathbf{w}$ and a normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$. Let u be the length of $\mathbf{u}$, v be the length of $\mathbf{v}$, and w be the length of $\mathbf{w}$.

   (a) Write the volume V of the parallelepiped as a function of u, v, w, α and β.

   (b) Suppose that the directions of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ remain fixed, while u increases at a rate of 3 cm/sec, v increases at a rate of 2 cm/sec, and w decreases at a rate of 1 cm/sec. Find the rate of change of V, in cm^3/sec, when $(u, v, w, \alpha, \beta) = \left(10 \text{ cm}, 5 \text{ cm}, 8 \text{ cm}, \tfrac{\pi}{2}, \tfrac{\pi}{4}\right)$.

   (c) Now assume that the lengths of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ remain fixed, while the angle α decreases at a rate of 0.1 rad/sec and β increases at a rate of 0.2 rad/sec. Find the rate of change of V at the same point as part (b).

8. Let $\widehat{\mathbf{u}}$ be a constant unit vector in 3D space, and let $\mathbf{x} = (x, y, z)$. If $w(x, y, z, t)$ is a function of space and time, let

$$\nabla w = w_x \widehat{\mathbf{i}} + w_y \widehat{\mathbf{j}} + w_z \widehat{\mathbf{k}},$$

the gradient with respect to the spatial coordinates. Let $F(s)$ be a differentiable function. Show that the function $w(x, y, z, t) = F(\widehat{\mathbf{u}} \cdot \mathbf{x} - ct)$ satisfies the differential equation

$$\widehat{\mathbf{u}} \cdot \nabla w + \frac{1}{c}\frac{\partial w}{\partial t} = 0.$$

9. Find the rate of change of the function in the direction indicated. Some directions might only be specified up to a sign.

   (a) $f(x, y) = \frac{x+1}{y-1}$, at the point $(2, 2)$, in the direction of the line $\mathbf{r}(t) = (2 + 2t, 2 - 3t)$.

   (b) $f(x, y) = e^{1/(x^2+y^2)}$, at the point $(1, 1)$, tangent to the curve $\mathbf{r}(t) = (\sqrt{2}\sin t, -\sqrt{2}\cos t)$.

   (c) $f(x, y, z) = xyz - x^2 - y^2 + z^2$, at the point $(0, 1, 1)$, perpendicular to the plane $3x - y + 1 = 0$.

   (d) $f(x, y, z) = e^x \sin y \cos z$, at the point $\left(0, \frac{\pi}{2}, \pi\right)$, tangent to the curve formed by the intersection of the surfaces $x^2 + 4y^2 = z^2$ and $3x + 2y - z = 0$.

   (e) $f(x, y, z) = z\ln(x^2 + y^2 + 1)$, at the point $(1, 0, 1)$, normal to the surface $2x^2 + 3y^2 + z^2 = 3$.

10. Let $g(x, y, z) = 3x^2y^3 - 4yz^2$.

    (a) At the point $(1, 1, 1)$, find a direction in which g is not changing.

    (b) At the point $(1, 1, 1)$, is there a direction in which g is changing at a rate of 5? Why or why not?

11. Let $h(x, y, z)$ be a function such that, at the point $(1, 1, 1)$, its rate of change in the direction of $\widehat{\mathbf{i}} + \widehat{\mathbf{k}}$ is 2, its rate of change in the direction of $2\widehat{\mathbf{j}} - \widehat{\mathbf{k}}$ is 4, and its rate of change in the direction of $\widehat{\mathbf{i}} - 2\widehat{\mathbf{j}} + \widehat{\mathbf{k}}$ is $\frac{1}{\sqrt{2}}$. Find ∇h at the point $(1, 1, 1)$.

12. Find the indicated tangent line or plane. All objects are in 3D space.

    (a) The tangent line to the curve $\mathbf{r}(t) = (t\cos(\pi t), t\sin(\pi t), t^2)$ at the point $(1, 0, 1)$.

    (b) The tangent line to the curve given by the intersection of the surfaces $xy - z^2 = 1$ and $x^2 + y^2 - z^2 = 20$, at the point $(2, 5, 3)$.

    (c) The tangent plane to the surface $z = 2x^3 - y^3$ at the point $(1, 1, 1)$.

    (d) The tangent plane to the surface $y = \sqrt{x^2 + 3z^2 + 2}$ at the point $(2, 3, 1)$.

13. Prove that the curve

$$\mathbf{r}(t) = (2\cos(\pi t) + 1, 3\sin(\pi t) - 2, t)$$

is tangent to the surface

$$4x^2 + y^2 + z^2 + 2xz + 4y + 1 = 0$$

at the point $(-1, -2, 1)$. You should first prove that the curve intersects the surface at the given point.

14. In each case, find all of the critical points of the given function.
    (a) $f(x,y) = \sqrt{x^2+2y^2}$
    (b) $f(x,y) = 6x^4 - 3x^2y^2 + \frac{2}{3}y^3 + y^2$
    (c) $f(x,y) = |3x-2y+1| + x^2 - y^2$ (Caution: if you write $f(x,y)$ as a piecewise function and find a critical point for one of the pieces, you have to check that the critical point is actually within the correct domain!)
    (d) $f(x,y) = \sqrt{x^4+2x^2y^2}$
    (e) $f(x,y) = \sin x \cos y$

15. In each case, the given point is a critical point of the function. Determine whether it is a relative minimum, relative maximum, saddle point, or none of these.
    (a) The point $(0,0)$ for the function $f(x,y) = \sin(xy)$.
    (b) The point $(3,4)$ for the function $f(x,y) = |4x-3y|$.
    (c) The point $(0,0)$ for the function $f(x,y) = 2 - \sqrt{x^2+y^2}$.
    (d) The point $(-1,0)$ for the function $f(x,y) = x^2y + 3xy^2 + xy$.
    (e) The point $(1,2)$ for the function $f(x,y) = x^4 - 4x^3 + 7x^2 + y^2 - 6x - 4y + 6$.
    (f) The point $(1,-1)$ for the function $f(x,y) = \sqrt{(x-1)^2+(y+1)^2} - 2y(x-1)$.

16. Find the absolute maximum and absolute minimum for the given function over the specified region R.
    (a) $f(x,y) = \frac{x+y}{x-y}$, where R is the square with vertices at $(1,0)$, $(2,0)$, $(1,-1)$ and $(2,-1)$.
    (b) $f(x,y) = 2x + 1 - x^2 - y^2$, where R is the disk $x^2+y^2 \leq 4$.
    (c) $f(x,y) = xy$, where R is the disk $x^2+y^2 \leq 1$.
    (d) $f(x,y) = (x-3y)^{1/3}$, where R is the disk $x^2+y^2 \leq 1$.

17. Consider the lines $\mathbf{r}_1(s) = (3+s, -s, 1-2s)$ and $\mathbf{r}_2(t) = (1+4t, 2+3t, 4-t)$, where $s, t \in \mathbb{R}$.
    (a) Let $D(s,t)$ be the square of the distance between the points $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$. Find the critical point(s) of D.
    (b) Find the smallest value that D takes at a critical point. (This is actually the absolute minimum of D over the entire st-plane – can you explain why?)
    (c) Find the points $\mathbf{r}_1(s)$ and $\mathbf{r}_2(t)$ that correspond to the critical point (s,t) from part (b). Verify that the vector $\mathbf{r}_1(s) - \mathbf{r}_2(t)$ is perpendicular to both lines.