MATH 2132 Problem Workshop 1

1. Evaluate the limit if it converges. If the limit tends to ∞, $-\infty$ indicate it as such.

   (a) $\displaystyle\lim_{n\to\infty} \frac{n+2}{3n^2+5}$

   Solution:

   $$\lim_{n\to\infty} \frac{n+2}{3n^2+5} = \lim_{n\to\infty} \frac{1/n+2/n^2}{3+5/n^2} = \frac{0}{3} = 0$$

   (b) $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n+2}{3n^2+5}$

   Solution:
   Let $c_n = \dfrac{n+2}{3n^2+5}$. Since $\displaystyle\lim_{n\to\infty} c_n = 0$, $\displaystyle\lim_{n\to\infty} (-1)^n c_n = 0$.

   (c) $\displaystyle\lim_{n\to\infty} \frac{n^2+2}{3n^2+5}$

   Solution:

   $$\lim_{n\to\infty} \frac{n^2+2}{3n^2+5} = \lim_{n\to\infty} \frac{1+2/n^2}{3+5/n^2} = \frac{1}{3}$$

   (d) $\displaystyle\lim_{n\to\infty} (-1)^n \frac{n^2+2}{3n^2+5}$

   Solution:
   Let $c_n = \dfrac{n^2+2}{3n^2+5}$. Since $\displaystyle\lim_{n\to\infty} c_n \neq 0$, $\displaystyle\lim_{n\to\infty} (-1)^n c_n$ does not exist.

   (e) $\displaystyle\lim_{n\to\infty} \frac{n^3+2}{3n^2+5}$

   Solution:

   $$\lim_{n\to\infty} \frac{n^3+2}{3n^2+5} = \lim_{n\to\infty} \frac{n(1+2/n^3)}{3+5/n^2} = \infty.$$

(f) $\lim_{n\to\infty} (-1)^n \frac{n^3+2}{3n^2+5}$

Solution:
Let $c_n = \frac{n^3+2}{3n^2+5}$. Since $\lim_{n\to\infty} c_n \neq 0$, $\lim_{n\to\infty} (-1)^n c_n$ does not exist.

(g) $\lim_{n\to\infty} \left(\sqrt{n^2+3n-4} - \sqrt{n^2+6n+5}\right)$

Solution:

$$\begin{aligned}
&\lim_{n\to\infty} \left(\sqrt{n^2+3n-4} - \sqrt{n^2+6n+5}\right) \\
&= \lim_{n\to\infty} \frac{\left(\sqrt{n^2+3n-4} - \sqrt{n^2+6n+5}\right)\left(\sqrt{n^2+3n-4} + \sqrt{n^2+6n+5}\right)}{\left(\sqrt{n^2+3n-4} + \sqrt{n^2+6n+5}\right)} \\
&= \lim_{n\to\infty} \frac{\left(n^2+3n-4\right) - \left(n^2+6n+5\right)}{\left(\sqrt{n^2+3n-4} + \sqrt{n^2+6n+5}\right)} \\
&= \lim_{n\to\infty} \frac{-3n-9}{\left(\sqrt{n^2+3n-4} + \sqrt{n^2+6n+5}\right)} \\
&= \lim_{n\to\infty} \frac{-3-9/n}{\left(\sqrt{1+3/n-4/n^2} + \sqrt{1+6/n+5/n^2}\right)} \\
&= \frac{-3}{\left(\sqrt{1}+\sqrt{1}\right)} \\
&= -\frac{3}{2}
\end{aligned}$$

(h) $\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{2n}$

Solution:
Letting x be a continuous variable
$\lim_{n\to\infty} \left(1+\frac{3}{n}\right)^{2n} = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x}$
This is a 1^∞ form so we want to use L'Hoptial's Rule
Let $L = \lim_{x\to\infty} \left(1+\frac{3}{x}\right)^{2x}$.

$$\begin{aligned}
\ln L &= \lim_{x\to\infty} \ln\left(\left(1+\frac{3}{x}\right)^{2x}\right)\\
&= \lim_{x\to\infty} 2x\ln\left(1+\frac{3}{x}\right)\\
&= \lim_{x\to\infty} \frac{2\ln\left(1+\frac{3}{x}\right)}{x^{-1}} \text{ which is a } \frac{0}{0} \text{ form}\\
&= \lim_{x\to\infty} \frac{-6x^{-2}/(1+3/x)}{-x^{-2}}\\
&= \lim_{x\to\infty} \frac{6}{1+\frac{3}{x}}\\
&= 6
\end{aligned}$$

Therefore $L = e^6$.

(i) $\displaystyle\lim_{n\to\infty}\left(\frac{3n+2}{2-n}\right)\cot^{-1}\left(\frac{3-\sqrt{3}n^3}{2+3n+n^3}\right)$

Solution:

$$\lim_{n\to\infty}\frac{3n+2}{2-n} = \lim_{n\to\infty}\frac{3+2/n}{2/n-1} = -3.$$

$$\lim_{n\to\infty}\frac{3-\sqrt{3}n^3}{2+3n+n^3} = \lim_{n\to\infty}\frac{3/n^3-\sqrt{3}}{2/n^3+3/n^2+1} = -\sqrt{3}.$$

Therefore since $\cot^{-1}$ is a continuous function on its domain,

$$\begin{aligned}
&\lim_{n\to\infty}\left(\frac{3n+2}{2-n}\right)\cot^{-1}\left(\frac{3-\sqrt{3}n^3}{2+3n+n^3}\right)\\
&= \lim_{n\to\infty}\left(\frac{3n+2}{2-n}\right)\lim_{n\to\infty}\cot^{-1}\left(\frac{3-\sqrt{3}n^3}{2+3n+n^3}\right)\\
&= \lim_{n\to\infty}\left(\frac{3n+2}{2-n}\right)\cot^{-1}\left(\lim_{n\to\infty}\frac{3-\sqrt{3}n^3}{2+3n+n^3}\right)\\
&= -3\cot^{-1}(-\sqrt{3})\\
&= -3(5\pi/6)\\
&= -\frac{5\pi}{2}.
\end{aligned}$$

(j) $\lim_{n\to\infty} \left(\frac{3}{n}\right)^{2n}$

Solution: The base goes to 0 and the exponent goes to ∞ and thus the limit goes to 0.

(k) $\lim_{n\to\infty} \frac{\sin n}{n}$

Solution:
By the squeeze theorem.
Since $-1 < \sin n < 1$ and $n > 0$ we have $-\frac{1}{n} < \frac{\sin n}{n} < \frac{1}{n}$.
Since both $\lim_{n\to\infty} -\frac{1}{n}$ and $\lim_{n\to\infty} \frac{1}{n}$ are both 0, by the squeeze theorem $\lim_{n\to\infty} \frac{\sin n}{n} = 0$.

(l) $\lim_{n\to\infty} \left(\tan^{-1}(1/n)\right)^{1/n}$

Solution:
This is a 0^0 form so we want to use L'Hoptial's Rule
Letting x be a continuous variable
$\lim_{n\to\infty} \left(\tan^{-1}(1/n)\right)^{1/n} = \lim_{x\to\infty} \left(\tan^{-1}(1/x)\right)^{1/x}$
While it can be done directly, for simplicity, we can let $h = 1/x$ and thus
$L = \lim_{x\to\infty} \left(\tan^{-1}(1/x)\right)^{1/x} = \lim_{h\to 0^+} (\tan^{-1} h)^h$

$$\begin{aligned}
\ln L &= \lim_{h\to 0^+} \ln(\tan^{-1} h)^h \\
&= \lim_{h\to 0^+} h \ln(\tan^{-1} h) \\
&= \lim_{h\to 0^+} \frac{\ln(\tan^{-1} h)}{h^{-1}} \text{ which is a } \frac{-\infty}{\infty} \text{ form} \\
&= \lim_{h\to 0^+} \frac{\frac{1}{(1+h^2)\tan^{-1} h}}{-h^{-2}} \\
&= \lim_{h\to 0^+} \frac{h^2}{(1+h^2)\tan^{-1} h}
\end{aligned}$$

Using L'Hopital's Rule again as it is a 0/0 form

$$\ln L = \lim_{h\to 0^+} \frac{2h}{\frac{1+h^2}{1+h^2} + 2h\tan^{-1} h}$$
$$= \frac{0}{1}$$
$$= 0.$$

Therefore $L = e^0 = 1$.

2. Find the general term of the sequence $8, \frac{11}{7}, \frac{14}{25}, \frac{17}{79}, \ldots$.

Solution: Thinking of 8 as $\frac{8}{1}$ we can see the patterm of the numerators is adding 3 each time. Hence the pattern is $3n + k$ for some number k. Since $n = 1$ makes $3n + k = 8$, we get that $8 = 3 + k \Rightarrow k = 5$ thus the numerator is $3n + 5$.

The denominator is trickier, however each term is 1, 7, 25, 79 which is 2 less than 3, 9, 27, 81 which are powers of 3. Thus the denominator is $3^n - 2$.

Hence the general term is $c_n = \dfrac{3n+5}{3^n-2}$.

3. Find the general term of the sequence $1, -\frac{6}{5}, \frac{12}{10}, -\frac{20}{17}, \frac{30}{26}, \ldots$.

Solution:

The alternating sign means that we need a factor of $(-1)^n$ or $(-1)^{n+1}$. Since $n = 1$ gives a positive, we use $(-1)^{n+1}$.

The denominator is similar to the last example since 5, 10, 17, 26 is one more than 4, 9, 16, 25 and hence the denominator is $n^2 + 1$. However this doesn't seem to match the first term. A quick fix is to think of 1 as $\frac{2}{2}$ and then the patterns match.

Therefore the terms in the numerator are 2, 6, 12, 20, 30 which can be written as $1 \cdot 2$, $2 \cdot 3$, , $3 \cdot 4$, $4 \cdot 5$, $5 \cdot 6$ Hence the pattern is $n(n + 1)$.

Hence the general term is $c_n = (-1)^{n+1}\dfrac{n(n+1)}{n^2+1}$.

4. It can be proven that if $\lim_{n\to\infty} c_n = C$ and $\lim_{n\to\infty} d_n = d$, then $\lim_{n\to\infty} c_n d_n = CD$. Use this to prove the following result. Suppose that $\lim_{n\to\infty} c_n = C \neq 0$ and $c_n \neq 0$ for all n. Suppose further that $\lim_{n\to\infty} d_n$ does not exist. Show $\lim_{n\to\infty} c_n d_n$ does not exist.

Solution: Suppose the limit of $\lim_{n\to\infty} c_n d_n$ does exist and equals L. Since $c_n \neq 0$ and has non-zero limit C, from the given result

$$\lim_{n\to\infty} d_n = \lim_{n\to\infty} \frac{c_n d_n}{c_n} = \lim_{n\to\infty} c_n d_n \lim_{n\to\infty} \frac{1}{c_n} = \frac{L}{C}.$$

However $\lim_{n\to\infty} d_n$ did not exist and therefore cannot equal L/C. Hence $\lim_{n\to\infty} c_n d_n$ could not have existed in the first place.

5. Determine to which function, if it exists, the sequence of functions $\{f_n(x)\}$ converges for x in the given interval.

(a) $f_n(x) = \dfrac{n^2x^2 + 3nx}{2n^2x + 5}$, $(-\infty, \infty)$

Solution: For any $x \neq 0$

$$\lim_{n\to\infty} \frac{n^2x^2 + 3nx}{2n^2x + 5} = \lim_{n\to\infty} \frac{x^2 + 2x/n}{3x + 5/n^2} = \frac{x^2}{3x} = \frac{x}{3}$$

If $x = 0$,

$$\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} \frac{0}{0+5} = 0$$

which matches $\dfrac{x}{3}$.

Thus the sequence converges to $\dfrac{x}{3}$.

(b) $f_n(x) = \dfrac{\sin nx}{nx}$, $(0\,\infty)$

Solution:

By the squeeze theorem.

Since $-1 < \sin nx < 1$ and $n, x > 0$ we have $-\dfrac{1}{nx} < \dfrac{\sin nx}{nx} < \dfrac{1}{nx}$.

Since both $\lim_{n\to\infty} -\dfrac{1}{nx}$ and $\lim_{n\to\infty} \dfrac{1}{nxx}$ are both 0, by the squeeze theorem $\lim_{n\to\infty} \dfrac{\sin nx}{nx} = 0$.

(c) $f_n(x) = \dfrac{n\,\sin(x/n)}{x}$, $(0, \infty)$

Solution:
Using the limit rule $\lim_{h\to 0} \frac{\sin h}{h} = 1$,

$$\lim_{n\to\infty} \frac{n\,\sin(x/n)}{x} = \lim_{n\to\infty} \frac{\sin(x/n)}{x/n} = 1$$

since for any $x > 0$, $x/n \to 0$ as $n \to \infty$.

(d) $f_n(x) = \big(\ln(x^{n+1})\big)^{1/n}$, $(1, \infty)$

Solution:
There are two way of thinking of this question. Either way uses L'Hopital's Rule.
Solution 1:
For $x > 1$, $(\ln x^{n+1})^{1/n}$ is an ∞^0 form.
Letting t be a continuous variable

$$L = \lim_{n\to\infty} (\ln x^{n+1})^{1/n} = \lim_{n\to\infty} (\ln x^{t+1})^{1/t}$$

$$\begin{aligned}
\ln L &= \lim_{t\to\infty} \ln(\ln x^{t+1})^{1/t} \\
&= \lim_{t\to\infty} \frac{\ln\ln x^{t+1}}{t} \text{ which is an } \frac{\infty}{\infty} \text{ form} \\
&= \lim_{t\to\infty} \frac{\frac{1}{\ln x^{t+1}} \cdot \frac{x^{t+1}\ln x}{x^{t+1}}}{1} \\
&= \lim_{t\to\infty} \frac{\ln x}{\ln x^{t+1}} \\
&= 0
\end{aligned}$$

Thus $L = e^0 = 1$.
Solution 2:
The function can be rearranged to be

$$\big(\ln(x^{n+1})\big)^{1/n} = \big((n+1)\ln x\big)^{1/n} = (n+1)^{1/n}(\ln x)^{1/n}.$$

For any x, $(\ln x)^{1/x} \to 1$ as $n \to \infty$.
For $(n+1)^{1/n}$ we have an ∞^0 form. Letting t be a continuous variable

$$M = \lim_{n\to\infty}(n+1)^{1/n} = \lim_{t\to\infty}(t+1)^{1/t}$$

$$\begin{aligned}
\ln M &= \lim_{t\to\infty} \ln(t+1)^{1/t} \\
&= \lim_{t\to\infty} \frac{\ln(t+1)}{t} \text{ which is an } \frac{\infty}{\infty} \text{ form} \\
&= \lim_{t\to\infty} \frac{\frac{1}{t+1}}{1} \\
&= 0
\end{aligned}$$

Thus $M = e^0 = 1$ and thus
$(n+1)^{1/n}(\ln x)^{1/n} \to (1)(1) = 1$ as $n \to \infty$.