MATH1210 Test #1 | 12 February 2009 | Time : 60 minute

Instructor: (check one) | [ ] Berry (A01) | [ ] Borgersen (A02)

NAME: ______________________________ **ID#:** _______________

|10 marks| Use the Principle of Mathematical Induction to show that the statement

$P_n: \sum_{\ell=n}^{2n} \ell = \frac{3n(n+1)}{2}$ is true for n any positive integer.

(SHOW ALL YOUR WORK!)

(A) Consider P_1 (Is P_1 true?) $LHS = \sum_{\ell=1}^{2} \ell = 1+2=3$

$RHS = \frac{3(1)(2)}{2} = 3$

$\therefore P_1$ is true.

(B) If P_k is true $\left[\sum_{\ell=k}^{2k} \ell = \frac{3k(k+1)}{2} \right]$

is P_{k+1} also true $\left[\sum_{\ell=k+1}^{2(k+1)} \ell = \frac{3(k+1)(k+2)}{2} \right]$?

But $\sum_{\ell=k+1}^{2(k+1)} \ell = \sum_{\ell=k+1}^{2k+2} \ell = \sum_{\ell=k}^{2k} \ell + (2k+1)+(2k+2)-k$

$= \frac{3k(k+1)}{2} + (3k+3)$ [if P_k is true]

$= 3(k+1)\left[\frac{k}{2}+1\right] = 3(k+1)\left(\frac{k+2}{2}\right)$

$\therefore$ If P_k is true, so is P_{k+1}

(C) Therefore by PMI P_n is true for all $n \geq 1$.

[10 marks] Consider the complex number $z = \left(1 + i - \dfrac{1}{1-i}\right)$.

(a) Express z in **Cartesian form**, simplifying your answer as far as possible.

$$z = \frac{(1+i)(1-i)-1}{(1-i)} = \frac{1-i^2-1}{(1-i)} = \frac{1}{1-i}$$

$$= \frac{(1+i)}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1+i}{2} = \frac{1}{2} + \frac{1}{2}i$$

(b) Express z in **exponential form**, indication clearly its modulus and the **principal value** of its argument.

$$z = \frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1+i) = \frac{1}{2}\sqrt{2}\, e^{i(\pi/4)}$$

$$= \frac{1}{\sqrt{2}} e^{i(\pi/4)}$$

MODULUS $= \dfrac{1}{\sqrt{2}}$, P.V. argument $= \pi/4$

(c) Express $\bar{z}$ in **exponential form**.

$$\bar{z} = \frac{1}{\sqrt{2}} e^{-i(\pi/4)} = \frac{1}{\sqrt{2}} e^{i(-\pi/4)}$$

(d) Use the above results to compute $\bar{z}^3\left(\dfrac{1}{z}\right)$, expressing your answer in **Cartesian form**.

$$\left.\begin{aligned} \bar{z}^3 &= \frac{1}{2\sqrt{2}} e^{i(-3\pi/4)} \\ \frac{1}{z} &= \sqrt{2}\, e^{i(-\pi/4)} \end{aligned}\right\} \quad \bar{z}^3\left(\frac{1}{z}\right) = \frac{1}{2} e^{i(-\pi)}$$

$$= \frac{1}{2}(-1)$$

$$= -1/2.$$

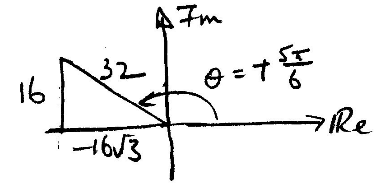
[10 marks] Consider the complex number $z = 16(-\sqrt{3}+i)$.

(a) Express z in exponential form.

$r = \sqrt{(16\sqrt{3})^2 + (16)^2} = 32$

$\theta = 5\pi/6$

$z = 32 e^{i(5\pi/6)}$



(b) Find the **modulus** and **principal value of the argument** of each of the **fifth** roots of $z = 16(-\sqrt{3}+i)$.

$z = 32 e^{i\left(\frac{5\pi}{6} + 2k\pi\right)}$

let $x = Re^{i\phi}$ so $x^5 = R^5 e^{i5\phi} = z$
$= 32 e^{i\left(\frac{5\pi}{6} + 2k\pi\right)}$

$\therefore R^5 = 32 \Longrightarrow R = 2$

$\& \ 5\phi = \frac{5\pi}{6} + 2k\pi \Longrightarrow \phi = \frac{\pi}{6} + \frac{2k\pi}{5}$ for $k = 0,1,2,3,4$

$\therefore R = 2 \ \& \ \phi = \frac{\pi}{6}, \frac{17\pi}{30}, \frac{29\pi}{30}, \underbrace{\frac{41\pi}{30}, \frac{53\pi}{30}}_{\text{not in p.v range}}$

$R = 2 \ \& \ \phi = \frac{\pi}{6}, \frac{17\pi}{30}, \frac{29\pi}{30}, -\frac{19\pi}{30}, -\frac{7\pi}{30}$

± 1 $\quad$ $\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20$

[10 marks] Consider the real polynomial $P(x) = x^3 - 7x^2 + 17x - 20$.

(a) Find the remainder when $P(x)$ is divided $(x+2i)$. $\quad x = -2i$

$$P(-2i) = (-2i)^3 - 7(-2i)^2 + 17(-2i) - 20$$
$$= -8i^3 - 28i^2 - 34i - 20$$
$$= 8i + 28 - 34i - 20 = 8 - 26i$$

(b) Use Descarte's Rule of signs to determine the maximum number of negative real zeros of $P(x)$.

$$P(-x) = -x^3 - 7x^2 - 17x - 20 \longrightarrow \text{no sign changes}$$
$\therefore 0$

(c) Use the rational roots theorem to list all the possible rational roots of $P(x) = 0$.

$1, 2, 4, 5, 10, 20$ $\quad$ (since there are no negative real roots)

(d) Show that $P(x)$ may be written in the form $P(x) = (x-4)\,Q(x)$ in which $Q(x)$ is an **irreducible real quadratic** factor.

$$x^3 - 7x^2 + 17x - 20 = (x-4)\left[\underset{1}{\_}\,x^2 + \underset{-3}{\_}\,x + \underset{5}{\_}\right]$$

$\therefore Q(x) = x^2 - 3x + 5$ — quadratic

irreducible since $b^2 - 4ac = 9 - 4(1)(5) = -11 < 0$.

(e) Express $P(x)$ as the product of **linear factors** only.

$$x^2 - 3x + 5 = 0 \longrightarrow x = \frac{3 \pm \sqrt{-11}}{2} = \frac{3 \pm \sqrt{11}\,i}{2}$$

$$P(x) = (x-4)\left[x - \left(\frac{3+\sqrt{11}\,i}{2}\right)\right]\left[x - \left(\frac{3-\sqrt{11}\,i}{2}\right)\right]$$

Problem	1	2	4	4	Total
MARK					
Possible	10	10	10	10	40