MATH 1210
Assignment 3

Due November 15, in class.

1. Given the points $A : (2, 3, 1)$, $B : (3, 5, -2)$, and $C : (-2, 9, -1)$, find the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Solution:
$\overrightarrow{AB} = \langle 3-2\,,\, 5-3\,,\, -2-1\rangle = \langle 1\,,\, 2\,,\, -3\rangle$ and
$\overrightarrow{AC} = \langle -2-2\,,\, 9-3\,,\, -1-1\rangle = \langle -4\,,\, 6\,,\, -2\rangle$.

So $\cos\theta = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}$, where θ is the desired angle.

Now

$$\begin{aligned}\cos\theta =& \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}|\,|\overrightarrow{AC}|}\\ =& \frac{(1)(-4)+(2)(6)+(-3)(-2)}{\sqrt{(1)^2+(2)^2+(-3)^2}\sqrt{(-4)^2+(6)^2+(-2)^2}}\\ =& \frac{-4+12+6}{\sqrt{1+4+9}\sqrt{16+36+4}}\\ =& \frac{14}{\sqrt{14}\sqrt{56}} = \frac{14}{\sqrt{14}(2\sqrt{14})} = \frac{1}{2}\end{aligned}$$

Hence $\theta = \dfrac{\pi}{3}$.

2. For vectors $\overrightarrow{u}$ and $\overrightarrow{v}$ in 3-space, prove that:

$$|\overrightarrow{u}\times\overrightarrow{v}|^2 = |\overrightarrow{u}|^2|\overrightarrow{v}|^2 - (\overrightarrow{u}\cdot\overrightarrow{v})^2$$

Solution:
Let $\overrightarrow{u} = \langle u_x, u_y, u_z\rangle$ and $\overrightarrow{v} = \langle v_x, v_y, v_z\rangle$.

Then

$$
\begin{aligned}
|\overrightarrow{u} \times \overrightarrow{v}|^2 =& |\langle u_y v_z - u_z v_y, u_z v_x - u_x v_z, u_x v_y - u_y v_x \rangle|^2 \\
=& \sqrt{(u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2}^2 \\
=& (u_y v_z - u_z v_y)^2 + (u_z v_x - u_x v_z)^2 + (u_x v_y - u_y v_x)^2 \\
=& u_y^2 v_z^2 - 2u_y u_z v_y v_z + u_z^2 v_y^2 + u_z^2 v_x^2 - 2u_x u_z v_x v_z + u_z^2 v_x^2 + u_x^2 v_y^2 - 2u_x u_y v_x v_y + u_y^2 v_x^2 \\
=& u_y^2 v_z^2 + u_z^2 v_y^2 + u_z^2 v_x^2 + u_z^2 v_x^2 + u_x^2 v_y^2 + u_y^2 v_x^2 + (u_x^2 v_x^2 + u_y^2 v_y^2 + u_z^2 v_z^2) \\
& - 2u_x u_y v_x v_y - 2u_x u_z v_x v_z - 2u_y u_z v_y v_z - (u_x^2 v_x^2 + u_y^2 v_y^2 + u_z^2 v_z^2) \\
=& (u_x^2 + u_y^2 + u_z^2)(v_x^2 + v_y^2 + v_z^2) - (u_x v_x + u_y v_y + u_z v_z)^2 \\
=& |\overrightarrow{u}|^2 |\overrightarrow{v}|^2 - (\overrightarrow{u} \cdot \overrightarrow{v})^2 .
\end{aligned}
$$

Alternately, we could do the following:

$$
\begin{aligned}
\sin^2\theta + \cos^2\theta &= 1 \\
|\overrightarrow{u}|^2|\overrightarrow{v}|^2 \sin^2\theta + |\overrightarrow{u}|^2|\overrightarrow{v}|^2 \cos^2\theta &= |\overrightarrow{u}|^2|\overrightarrow{v}|^2 \\
(|\overrightarrow{u}||\overrightarrow{v}|\sin\theta)^2 + (|\overrightarrow{u}||\overrightarrow{v}|\cos\theta)^2 &= |\overrightarrow{u}|^2|\overrightarrow{v}|^2 \\
|\overrightarrow{u} \times \overrightarrow{v}|^2 + (\overrightarrow{u} \cdot \overrightarrow{v})^2 &= |\overrightarrow{u}|^2|\overrightarrow{v}|^2 \\
|\overrightarrow{u} \times \overrightarrow{v}|^2 &= |\overrightarrow{u}|^2|\overrightarrow{v}|^2 - (\overrightarrow{u} \cdot \overrightarrow{v})^2
\end{aligned}
$$

3. Find the point of intersection of the line $\dfrac{x-1}{-2} = \dfrac{y+4}{3} = z-2$ and the plane that passes through the point $(1,1,1)$ and is parallel to the lines $x = 3+t\,;\ y = -2-2t\,;\ z = 4-t$ and $x = 4-t\,;\ y = 1+6t\,;\ z = 3-t$.

Solution:

A vector in the direction of the first line parallel to the plane is $v_1 = \langle 1, -2, -1 \rangle$, and this vector is perpendicular to the plane normal. Similarly, the vector in the direction of the second line, $v_2 = \langle -1, 6, -1\rangle$, is also perpendicular to the plane normal.

Hence the plane normal is $v_1 \times v_2$:

$$v_1 \times v_2 = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & -2 & -1 \\ -1 & 6 & -1 \end{vmatrix} = (2-(-6))\widehat{i} - (-1-1)\widehat{j} + (6-2)\widehat{k}$$
$$= \langle 8, 2, 4\rangle = 2\langle 4, 1, 2\rangle$$

Since for a plane only direction matters, we can use $\langle 4, 1, 2\rangle$ for the plane normal.

So the plane equation is :

$$\begin{aligned} \langle 4,1,2\rangle \cdot \langle x-1, y-1, z-1\rangle &= 0 \\ 4(x-1) + (y-1) + 2(z-1) &= 0 \\ 4x + y + 2z &= 7 \end{aligned}$$

Using the parametric form of the line which we want intersecting with the plane: $x = 1-2t\,;\ y = -4+3t\,;\ z = 2+t$, we substitute into the plane to get

$$\begin{aligned} 4(1-2t) + (-4+3t) + 2(2+t) &= 7 \\ 4 - 8t - 4 + 3t + 4 + 2t &= 7 \\ -3t &= 3 \\ t &= -1 \end{aligned}$$

We substitute this value back into the equation of the line to find the point $(3, -7, 1)$.

4. Find all values of a and b such that the following system of equations:

$$\begin{array}{rcrcrl} x & - & y & + & 2z = & 4 \\ 3x & - & 2y & + & 9z = & 14 \\ 2x & - & 4y & + & az = & b \end{array}$$

Solution: We start by putting the system in an augmented matrix, then we reduce as much as possible:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 3 & -2 & 9 & 14 \\ 2 & -4 & a & b \end{array}\right) \Longrightarrow \begin{array}{lcl} R_2 & \to & R_2 - 3R_1 \\ R_3 & \to & R_3 - 2R_1. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 1 & 3 & 2 \\ 0 & -2 & a-4 & b-8 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \to & R_1 + R_2 \\ R_3 & \to & R_3 + 2R_2. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 6 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & a+2 & b-4 \end{array}\right)$$

If we were to try to reduce this further, we would look for a leading one in the last row, which contains unknowns, so we stop here.

(a) has no solutions.

Solution: This will have no solutions if $a+2=0$ and $b-4 \neq 0$. Hence we get $a=-2$ and $b \neq 4$.

(b) has exactly one solution.

Solution: This will have exactly one solution if $a+2 \neq 0$ (regardless of what b is). Hence we get $a \neq -2$ (and b and real number).

(c) has exactly three solutions.

Solution: This will never have exactly three solutions.

(d) has infinitely many solutions.

Solution: This will infinitely many solutions if $a+2=0$ and $b-4=0$. Hence we get $a=-2$ and $b=4$.

5. Solve the following systems of equations using the Gauss Jordan method:

(a) $$\begin{array}{rcrcrcrcr} x_1 & - & x_2 & + & 2x_3 & + & 8x_4 & = & 1 \\ -x_1 & + & x_2 & - & x_3 & - & 5x_4 & = & -2 \\ 3x_1 & - & 3x_2 & + & 4x_3 & + & 18x_4 & = & 5 \end{array}$$

Solution:

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 8 & 1 \\ -1 & 1 & -1 & -5 & -2 \\ 3 & -3 & 4 & 18 & 5 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_2 & \to & R_2 + R_1 \\ R_3 & \to & R_3 - 3R_1. \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 2 & 8 & 1 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & -2 & -6 & 2 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \to & R_1 - 2R_2 \\ R_3 & \to & R_3 + 2R_2. \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & -1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_2 = s$ and $x_4 = t$ where $s, t \in \mathbb{R}$, then

$$\begin{array}{lcl} x_1 & = & 3 - 2t + s \\ x_2 & = & s \\ x_3 & = & -1 - 3t \\ x_4 & = & t \end{array}$$

(b)
$$\begin{array}{rcrcrl} x & - & y & - & 2z = & 3 \\ 3x & - & 2y & - & 4z = & 9 \\ -2x & + & 2y & + & 5z = & -7 \\ 4x & - & 3y & - & 6z = & 12 \end{array}$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 3 & -2 & -4 & 9 \\ -2 & 2 & 5 & -7 \\ 4 & -3 & -6 & 12 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_2 & \to & R_2 - 3R_1 \\ R_3 & \to & R_3 + 2R_1 \\ R_4 & \to & R_4 - 4R_1. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & -2 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 2 & 0 \end{array}\right) \Longrightarrow \begin{array}{lcl} R_1 & \to & R_1 + R_2 \\ R_4 & \to & R_4 - R_2. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \Longrightarrow R_2 \;\to\; R_2 - 2R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

So the unique solution is: $\begin{array}{rcl} x &=& 3 \\ y &=& 2 \\ z &=& -1 \end{array}$

(c) $\begin{array}{rcrcrcr} x_1 &-& x_2 &+& 2\,x_3 &=& 0 \\ 4\,x_1 &-& x_2 &+& 6\,x_3 &=& 0 \\ -3\,x_1 &-& 3\,x_2 &-& 2\,x_3 &=& 0 \end{array}$

Solution:

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 4 & -1 & 6 & 0 \\ -3 & -3 & -2 & 0 \end{array}\right) \Longrightarrow \begin{array}{rcl} R_2 & \to & R_2 - 4R_1 \\ R_3 & \to & R_3 + 3R_1. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 3 & -2 & 0 \\ 0 & -6 & 4 & 0 \end{array}\right) \Longrightarrow R_2 \to \tfrac{1}{3}R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & -6 & 4 & 0 \end{array}\right) \Longrightarrow \begin{array}{rcl} R_1 & \to & R_1 + R_2 \\ R_3 & \to & R_3 + 6R_2. \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{4}{3} & 0 \\ 0 & 1 & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Let $x_3 = t$ where $t \in \mathbb{R}$, then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3}t \\ \frac{2}{3}t \\ t \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} \\ \frac{2}{3} \\ 1 \end{pmatrix} t$$

6. Let $A = \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & x & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 4 & -1 & 2 \end{pmatrix}$, and $B = \begin{pmatrix} 4 & 1 & 2 \\ 3 & x & -1 \\ 2 & 2 & 5 \end{pmatrix}$.

Find all values of x such that $\det A = \det B$.

Solution:

$$\begin{aligned}
\det A &= \begin{vmatrix} x & 0 & 0 & 0 \\ 0 & x & 1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 4 & -1 & 2 \end{vmatrix} \text{(expand along first row (or column))} \\
&= x \begin{vmatrix} x & 1 & 1 \\ 2 & 1 & 3 \\ 4 & -1 & 2 \end{vmatrix} \text{(expand along first column)} \\
&= x \left(x \begin{vmatrix} 1 & 3 \\ -1 & 2 \end{vmatrix} + 2(-1) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} + 4 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} \right) \\
&= x\left(x(2-(-3)) - 2(2-(-1)) + 4(3-1)\right) \\
&= x(5x+2) \\
&= 5x^2 + 2x
\end{aligned}$$

And

$$\begin{aligned}
\det B &= \begin{vmatrix} 4 & 1 & 2 \\ 3 & x & -1 \\ 2 & 2 & 5 \end{vmatrix} \text{(expand along first column)} \\
&= 4 \begin{vmatrix} x & -1 \\ 2 & 5 \end{vmatrix} + 3(-1) \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ x & -1 \end{vmatrix} \\
&= 4(5x-(-2)) - 3(5-4) + 2(-1-2x) \\
&= 20x + 8 - 3 - 2 + 4x \\
&= 16x + 3
\end{aligned}$$

Now setting $\det A = \det B$ we get:

$$\begin{aligned}
5x^2 + 2x &= 16x + 3 \\
5x^2 - 14x - 3 &= 0 \\
(5x+1)(x-3) &= 0
\end{aligned}$$

Hence the values of x are $x = \frac{-1}{5}$, and $x = 3$.

7. Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$. Suppose $\det A = 4$, find the determinant of the following

matrices:

Solution: For each of the following, we want to perform a series of row operations, keeping track of the impact on the determinant, attempting to form the matrix A.

(a) $B_1 = \begin{pmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g+3a & h+3b & i+3c \end{pmatrix}$.

Solution:

$$\begin{aligned}
&\begin{vmatrix} a & b & c \\ 2d+a & 2e+b & 2f+c \\ g+3a & h+3b & i+3c \end{vmatrix} \begin{pmatrix} R_2 & \to & R_2 - R_1 \\ R_3 & \to & R_3 - 3R_1 \end{pmatrix} \\
=&\begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \begin{pmatrix} R_2 & \to & \frac{1}{2}R_2 \end{pmatrix} \\
=&(2)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}
\end{aligned}$$

Hence $\det B_1 = (2) \det A = 8$.

(b) $B_2 = \begin{pmatrix} a+g & b+h & c+i \\ d & e & f \\ a+d+g & b+e+h & c+e+i \end{pmatrix}$.

Solution:

$$\begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ a+d+g & b+e+h & c+e+i \end{vmatrix} \left(R_3 \;\to\; R_3 - R_1 \right)$$
$$= \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ d & e & f \end{vmatrix} \left(R_3 \;\to\; R_3 - R_2 \right)$$
$$= \begin{vmatrix} a+g & b+h & c+i \\ d & e & f \\ 0 & 0 & 0 \end{vmatrix}$$

Since this last matrix has a row of zeros, $\det B_2 = 0$.

(c) $B_3 = \begin{pmatrix} g & h & i \\ d+2a & e+2b & f+2c \\ a+g & b+h & c+i \end{pmatrix}$.

Solution:

$$\begin{vmatrix} g & h & i \\ d+2a & e+2b & f+2c \\ a+g & b+h & c+i \end{vmatrix} \left(R_3 \;\to\; R_3 - R_1 \right)$$
$$= \begin{vmatrix} g & h & i \\ d+2a & e+2b & f+2c \\ a & b & c \end{vmatrix} \left(R_2 \;\to\; R_2 - 2R_3 \right)$$
$$= \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \left(R_1 \;\leftrightarrow\; R_3 \right)$$
$$= (-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Hence $\det B_3 = (-1) \det A = -4$.

(d) $B_4 = \begin{pmatrix} 3d & 3e & 3f \\ 2a+3d & 2b+3e & 2c+3f \\ a+\frac{1}{12}g & b+\frac{1}{12}h & c+\frac{1}{12}i \end{pmatrix}$.

Solution:

$$\begin{vmatrix} 3d & 3e & 3f \\ 2a+3d & 2b+3e & 2c+3f \\ a+\frac{1}{12}g & b+\frac{1}{12}h & c+\frac{1}{12}i \end{vmatrix} \left(R_2 \to R_2 - R_1 \right)$$

$$= \begin{vmatrix} 3d & 3e & 3f \\ 2a & 2b & 2c \\ a+\frac{1}{12}g & b+\frac{1}{12}h & c+\frac{1}{12}i \end{vmatrix} \begin{pmatrix} R_1 & \to & \frac{1}{3}R_1 \\ R_2 & \to & \frac{1}{2}R_2 \end{pmatrix}$$

$$=(2)(3) \begin{vmatrix} d & e & f \\ a & b & c \\ a+\frac{1}{12}g & b+\frac{1}{12}h & c+\frac{1}{12}i \end{vmatrix} \left(R_3 \to R_3 - R_2 \right)$$

$$=(2)(3) \begin{vmatrix} d & e & f \\ a & b & c \\ \frac{1}{12}g & \frac{1}{12}h & \frac{1}{12}i \end{vmatrix} \left(R_3 \to 12R_3 \right)$$

$$=(2)(3)(\frac{1}{12}) \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} \left(R_2 \leftrightarrow R_1 \right)$$

$$=(-1)(2)(3)(\frac{1}{12}) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Hence $\det B_4 = -\frac{6}{12} \det A = -2$.