MATH 1210 - ASSIGNMENT #2 SOLUTIONS

(1)

(a) $(1+i)-(7+2i) = -6-i$

(b) $i^{57} = (i^2)^{28} i = (-1)^{28} i = (1)i = i$

(c) $i^2(-1+3i) = (-1)(-1+3i) = 1-3i$

(d) $\dfrac{2-3i}{4+2i} = \dfrac{(2-3i)}{(4+2i)}\dfrac{(4-2i)}{(4-2i)} = \dfrac{8-4i-12i+6i^2}{16-8i+8i-4i^2} = \dfrac{2-16i}{20}$

$= \dfrac{1-8i}{10}$

(e) $\overline{(7-2\bar{i})^2} = \overline{(7+2i)^2}$

$= \overline{49+14i+14i+2i^2}$

$= 47-28i$

(f) $(1-\sqrt{3}i)^4$

$= (1-\sqrt{3}i)^2(1-\sqrt{3}i)^2$

$= (1-2\sqrt{3}i+3i^2)(1-2\sqrt{3}i+3i^2)$

$= (-2(1+\sqrt{3}i))(-2(1+\sqrt{3}i))$

$= 4(1+2\sqrt{3}i+3i^2)$

$= 4(-2+2\sqrt{3}i) = 8(-1+\sqrt{3}i) = -8+8\sqrt{3}i$

(g) $(4-2i)^6 = 2^6(2-i)^6$

$= 2^6[1(2)^6(-i)^0 + 6(2)^5(-i)^1 + 15(2)^4(-i)^2 + 20(2)^3(-i)^3 + 15(2)^2(-i)^4 + 6(2)^1(-i)^5 + 1(2)^0(-i)^6]$

[using the BINOMIAL THEOREM]

$= 2^6[2^6 - 6(2)^5 i - 15(2)^4 + 20(2)^3 i + 15(2)^2 - 6(2)i - 1]$

since $(-i)^2 = i^2 = -1$

$(-i)^3 = -(i^3) = -(-i) = i$

$(-i)^4 = i(-i) = -i^2 = 1$

$(-i)^5 = 1(-i) = -i$

$(-i)^6 = (-i)(-i) = i^2 = -1$

$\therefore (4-2i)^6 = 2^6[(2^6 - 15(2)^4 + 15(2)^2 - 1) + i(-6(2)^5 + 20(2)^3 - 6(2))]$

$$= 64\Big[\,(64 - 15(16) + 15(4) - 1)$$
$$+ i\,(-6(32) + 20(8) - 12)\Big]$$
$$= 64[-117 - 44i]$$
$$= -64[117 + 44i]$$

(h) $\left(2 + \overline{(-1+i)}\right)^2 = (2 + (-1-i))^2 = (1-i)^2 = 1 - 2i + i^2$
$$= -2i$$

(i) $$\frac{(2+i)(4-6i)}{(3-i)i^3} = \frac{8 - 12i + 4i - 6i^2}{(3-i)(-i)} = \frac{14-8i}{-3i - i^2}$$
$$= \frac{14-8i}{1-3i} = \frac{(14-8i)(1+3i)}{(1-3i)(1+3i)} = \frac{14 + 42i - 8i - 24i^2}{1 + 3i - 3i - 9i^2}$$
$$= \frac{38 - 44i}{10} = \frac{1}{5}(19 - 22i)$$

(j) $$1 + \frac{1}{1-i} = \frac{1-i+1}{1-i} = \frac{2-i}{1-i} = \frac{(2-i)(1+i)}{(1-i)(1+i)} = \frac{2 + 2i - i - i^2}{1 + i - i - i^2}$$
$$= \frac{3+i}{2}$$

$$\therefore \frac{2+i}{1 + \frac{1}{1-i}} = \frac{2+i}{(3+i)/2} = \frac{2(2+i)}{(3+i)} = \frac{2(2+i)(3-i)}{(3+i)(3-i)}$$

$$= \frac{2(6 - 2i + 3i - i^2)}{9 - 3i + 3i - i^2} = \frac{2(7+i)}{10} = \frac{1}{5}(7+i)$$

② (a) $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\dfrac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}$

But $\dfrac{\cos\theta_1 + i\sin\theta_1}{\cos\theta_2 + i\sin\theta_2} = \dfrac{(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$

$$= \frac{\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= (\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)$$

since $\cos^2\theta_2 + \sin^2\theta_2 = 1$

$$= \cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

$\therefore \dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$

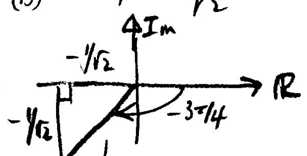
i.e., $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$

& $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg(z_1) - \arg(z_2)$

as required

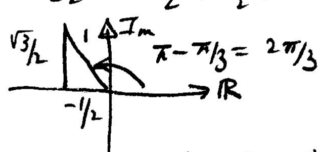
(b) $z_1 = -\dfrac{1}{\sqrt{2}}(1+i)$ $\qquad |z_1| = \sqrt{\left(-\tfrac{1}{\sqrt{2}}\right)^2 + \left(-\tfrac{1}{\sqrt{2}}\right)^2} = 1$

$\arg(z_1) = -3\pi/4$



$z_2 = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$ $\qquad |z_2| = \sqrt{\left(-\tfrac{1}{2}\right)^2 + \left(\tfrac{\sqrt{3}}{2}\right)^2} = \sqrt{\tfrac{1}{4} + \tfrac{3}{4}} = 1$

$\arg z_2 = \dfrac{2\pi}{3}$



$$\frac{z_1}{z_2} = \frac{-1/\sqrt{2}\,(1+i)}{1/2\,(-1+\sqrt{3}i)} = -\sqrt{2}\left(\frac{1+i}{-1+\sqrt{3}i}\right) = -\sqrt{2}\left(\frac{1+i}{-1+\sqrt{3}i}\right)\left(\frac{-1-\sqrt{3}i}{-1-\sqrt{3}i}\right)$$

$$= -\sqrt{2}\left[\frac{-1-\sqrt{3}i - i + \sqrt{3}}{1+3}\right] = -\sqrt{2}\frac{\left((\sqrt{3}-1) - i(\sqrt{3}+1)\right)}{4}$$

$$= \frac{(\sqrt{3}-1) - i(\sqrt{3}+1)}{-2\sqrt{2}} = \frac{1}{2\sqrt{2}}\left[(1-\sqrt{3}) + i(\sqrt{3}+1)\right]$$

$$\therefore \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{1}{2\sqrt{2}}(1-\sqrt{3})\right)^2 + \left(\frac{1}{2\sqrt{2}}(\sqrt{3}+1)\right)^2}$$
$$= \sqrt{\frac{1}{8}(1-2\sqrt{3}+3) + \frac{1}{8}(3+2\sqrt{3}+1)}$$
$$= \sqrt{\frac{1}{2}+\frac{1}{2}} = 1 \qquad \text{ie, } \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

as expected

To find $\arg\left(\frac{z_1}{z_2}\right) = \phi$

we need to use the equations

$$\left.\begin{aligned} \cos\phi &= \frac{1-\sqrt{3}}{2\sqrt{2}} < 0 \\ \sin\phi &= \frac{\sqrt{3}+1}{2\sqrt{2}} > 0 \end{aligned}\right\} \Rightarrow \phi \text{ must lie in the 2nd quadrant}$$

$$\& \therefore \phi = \cos^{-1}\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) \pm 2k\pi$$
$$= \cos^{-1}(-0.26) \pm 2k\pi$$
$$= (1.83 \pm 2k\pi) \text{ radians}$$

On the other hand $\arg(z_1) - \arg(z_2)$

$$= -\frac{3\pi}{4} - \frac{2\pi}{3} = -\frac{9\pi - 8\pi}{12} = -\frac{17\pi}{12}$$
$$= -4.45 \text{ radians}$$

or equivalently

$$\arg(z_1) - \arg(z_2) = -4.45 + 2\pi \text{ radians}$$
$$= 1.83 \text{ radians}$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2\pi$$

as expected (provided we include the 2π correction term)

(c) note: since $\arg(z_1)$ & $\arg(z_2)$ lie between $-\pi$ & π.

$$p.v.(\arg(z_1)) = \arg(z_1) = -3\pi/4$$
$$p.v.(\arg(z_2)) = \arg(z_2) = \frac{2\pi}{3}$$
$$p.v. \arg(z_1) - p.v.(\arg(z_2)) = -\frac{17\pi}{3} \quad \text{(see above)}$$

& this lies outside the interval $(-\pi, \pi]$ so cannot be p.v. $\left(\frac{z_1}{z_2}\right)$, because it must lie in the interval $(-\pi, \pi]$

3 (a) $(a+ib)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} (ib)^k$.

$$(a+ib)^2 = 1(a)^2(ib)^0 + 2(a)(ib) + 1(a)^0(ib)^2$$
$$= (a^2-b^2) + i(2ab)$$

$$(a+ib)^3 = 1(a)^3(ib)^0 + 3(a)^2(ib)^1 + 3(a)^1(ib)^2 + 1(a)^0(ib)^3$$
$$= (a^3 - 3ab^2) + i(3a^2b - b^3)$$

$$(a+ib)^4 = (a)^4 + 4(a)^3(ib) + 6(a)^2(ib)^2 + 4(a)(ib)^3 + (ib)^4$$
$$= (a^4 - 6a^2b^2 + b^4) + i(4a^3b - 4ab^2)$$

$$(a+ib)^5 = a^5 + 5a^4(ib) + 10a^3(ib)^2 + 10a^2(ib)^3 + 5a(ib)^4 + (ib)^5$$
$$= (a^5 - 10a^3b^2 + 5ab^4) + i(5a^4b - 10a^2b^3 + b^5)$$

(b) If $a = \cos\theta$ & $b = \sin\theta$

$a+ib = \cos\theta + i\sin\theta = e^{i\theta}$ [Euler's identity]

$\therefore (a+ib)^n = (e^{i\theta}) = e^{i(n\theta)}$ by the theorem of de Moivre

$= \cos(n\theta) + i\sin(n\theta)$

$\therefore$ for $n=2$ we obtain $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

$\sin(2\theta) = 2\sin\theta\cos\theta$

for $n=3$: $\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$

$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^2\theta$

for $n=4$: $\cos(4\theta) = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

$\sin(4\theta) = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

& for $n=5$: $\cos(5\theta) =$ as given on sheet

$\sin(5\theta) =$ as given on sheet.

4 (a) $x^6 = -1$. Let $x = Re^{i\phi}$

$-1 = e^{i(\pi + 2k\pi)}$

$\therefore R^6 e^{i6\phi} = e^{i(\pi+2k\pi)} \Rightarrow R^6 = 1 \Rightarrow R = 1$ (remember R must be real & +ve)

$6\phi = \pi + 2k\pi$

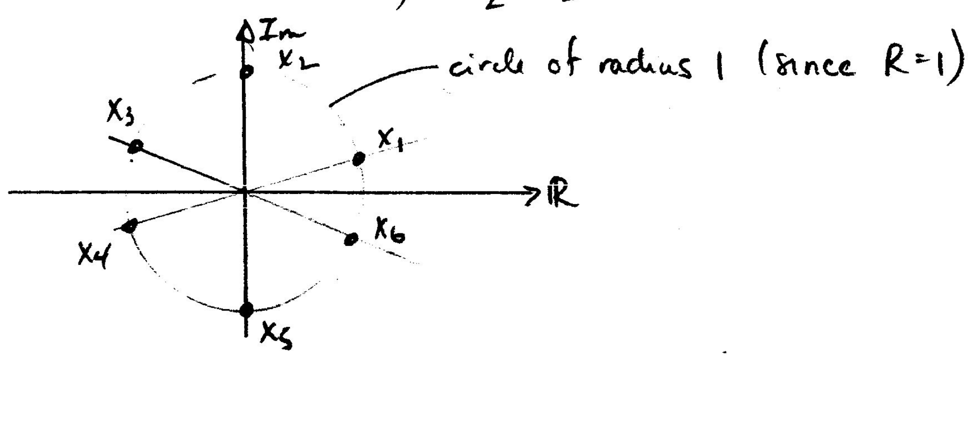
$\therefore \phi = \frac{\pi}{6} + \frac{k\pi}{3}$

for $k=0$: $\phi = \phi_1 = \pi/6$
for $k=1$: $\phi = \phi_2 = \pi/6 + \frac{\pi}{3} = \pi/2$
for $k=2$: $\phi = \phi_3 = \pi/6 + 2\pi/3 = 5\pi/6$
for $k=3$: $\phi = \phi_4 = \pi/6 + \pi = 7\pi/6$
for $k=4$: $\phi = \phi_5 = \pi/6 + \frac{4\pi}{3} = 3\pi/2$
for $k=5$: $\phi = \phi_6 = \pi/6 + \frac{5\pi}{3} = 11\pi/6$

Thus the sixth roots of -1 are

$$
\begin{aligned}
x_1 &= e^{i(\pi/6)} = \cos(\pi/6) + i\sin(\pi/6) = \frac{\sqrt{3}}{2} + i\frac{1}{2} \\
x_2 &= e^{i(\pi/2)} = i \\
x_3 &= e^{i(5\pi/6)} = \cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + i\frac{1}{2} \\
x_4 &= e^{i(7\pi/6)} = \cos(7\pi/6) + i\sin\left(\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{2} - i\frac{1}{2} \\
x_5 &= e^{i(3\pi/2)} = -i \\
x_6 &= e^{i(11\pi/6)} = \cos\left(\frac{11\pi}{6}\right) + i\sin\left(\frac{11\pi}{6}\right) = \frac{\sqrt{3}}{2} - i\frac{1}{2}
\end{aligned}
$$

i.e., the sixth roots of -1 are
$\pm i, \pm\frac{\sqrt{3}}{2} \pm \frac{1}{2}i$



(b) $x^2 = -i$ with $-i = e^{i(-\frac{\pi}{2} + 2k\pi)}$

let $x = Re^{i\phi}$ so $R^2 e^{i2\phi} = e^{i(-\frac{\pi}{2} + 2k\pi)}$

$\rightarrow R^2 = 1 \Rightarrow R = 1$

& $2\phi = -\frac{\pi}{2} + 2k\pi \Rightarrow \phi = -\frac{\pi}{4} + k\pi$

cases: $k = 0$: $\phi = -\pi/4$

$k = 1$: $\phi = -\frac{\pi}{4} + \pi = \frac{3\pi}{4}$

$\therefore$ the square roots of $-i$ are

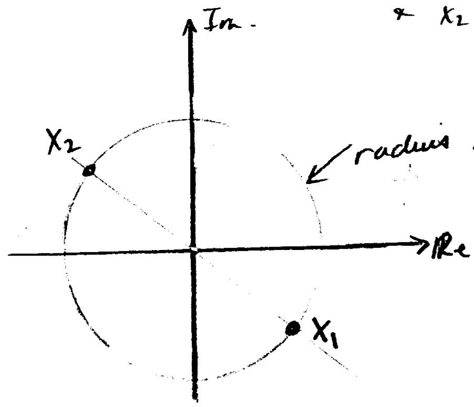
$x_1 = e^{i(-\pi/4)} = \cos(-\frac{\pi}{4}) + i\sin(-\pi/4)$

$= +\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} i = \frac{1}{\sqrt{2}}(1-i)$

& $x_2 = e^{i(\frac{5\pi}{4})} = \cos(\frac{5\pi}{4}) + i\sin(5\pi/4)$

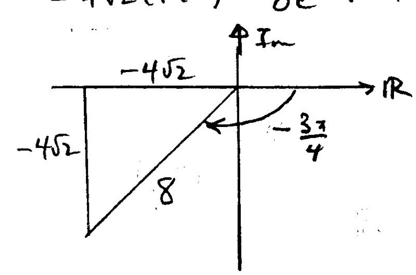
$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = \frac{1}{\sqrt{2}}(-1+i)$

$= -\frac{1}{\sqrt{2}}(1-i)$



(c) $x^3 = -4\sqrt{2}(1+i)$

$-4\sqrt{2}(1+i) = 8e^{i(-\frac{3\pi}{4} + 2k\pi)}$



let $x = Re^{i\phi}$

$R^3 e^{i3\phi} = 8\, e^{i(-\frac{3\pi}{4} + 2k\pi)}$

$\rightarrow R^3 = 8$ & $3\phi = -3\pi/4 + 2k\pi$

$(R = 2)$ ie, $\phi = -\frac{\pi}{4} + \frac{2k\pi}{3}$

$k=0 \Rightarrow \phi = \phi_1 = -\pi/4$

$k=1 \Rightarrow \phi = \phi_2 = -\pi/4 + \frac{2\pi}{3} = \frac{5\pi}{12}$

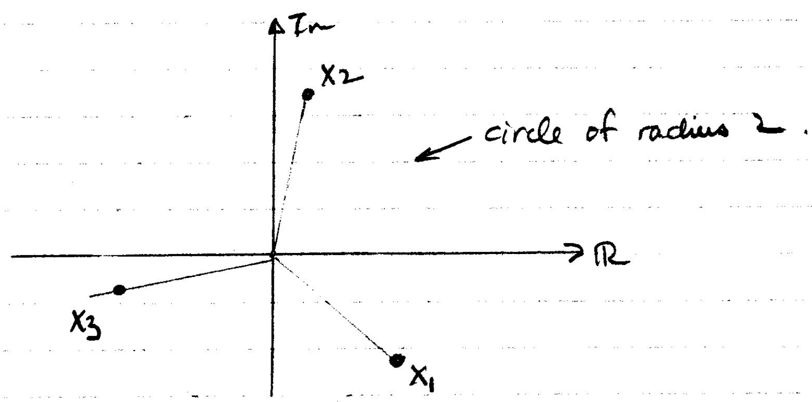
$k=2 \Rightarrow \phi = \phi_3 = -\pi/4 + 4\pi/3 = 13\pi/12$

The cube roots of $-4\sqrt{2}\,(1+i)$ are

$$x_1 = 2e^{i(-\pi/4)} = 2\left(\cos(-\pi/4) + i\sin(-\pi/4)\right)$$
$$= 2\left(1/\sqrt{2} - \tfrac{1}{\sqrt{2}}i\right) = \sqrt{2}\,(1-i),$$

$$x_2 = 2e^{i(5\pi/12)} = 2\left(\cos\left(\tfrac{5\pi}{12}\right) + i\sin(5\pi/12)\right)$$
$$\simeq 2\,(0.25882 + i\,0.96593)$$

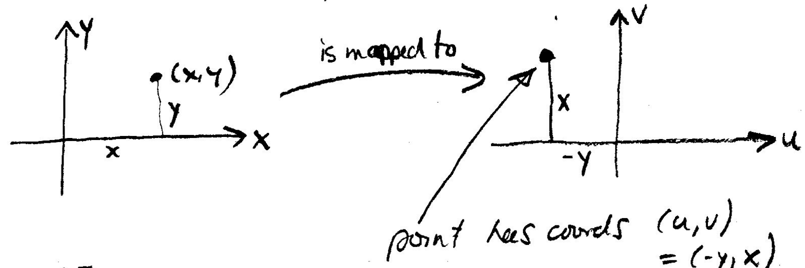
$$x_3 = 2e^{i(13\pi/12)} = 2\left(\cos\left(\tfrac{13\pi}{12}\right) + i\sin\left(\tfrac{13\pi}{12}\right)\right)$$
$$\simeq 2\,(-0.96593 - i\,0.25882)$$



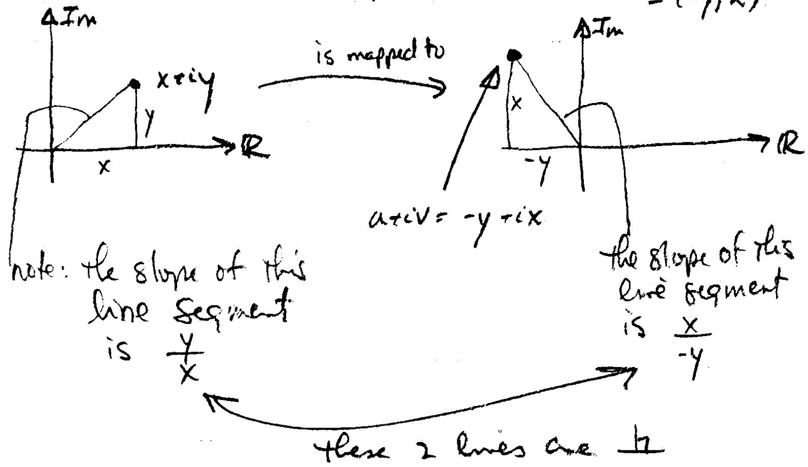
(5) (a) $w = iz \Rightarrow u + iv = i(x+iy) = ix - y$

$\Rightarrow u = -y$ & $v = x$. as required

(b)



(c)



∴ the transformation $w = iz$ corresponds to a rotation about O through an angle of $\pi/2$ radians

(d) if $x+iy = re^{i\theta}$

& $w = u+iv = Re^{i\phi}$

then $Re^{i\phi} = ire^{i\theta}$

$= re^{i(\theta+\pi/2)}$ since $i = e^{i(\pi/2)}$

$\Rightarrow R = r$

& $\phi = \theta + \pi/2 \Rightarrow \arg(w) = \arg(z) + \pi/2$

i.e, this mapping corresponds to a rotation about O through $\frac{\pi}{2}$ radians