MATH 1210 A01 Summer 2013 Problem Workshop 5 Solutions

1. (a) To use the rational root theorem, we need to make the coefficients integers. So change the polynomial to

$$Q(x) = 2x^3 + 3x^2 + 9x + 4$$

which has no sign changes. Hence there are no positive roots. Letting x be $-x$ we get

$$Q(-x) = -2x^3 + 3x^2 - 9x + 4$$

which has three sign changes. Hence there are 3 or 1 negative roots.

For the bounds theorem, we have $M = 9, |a_n| = 2$, hence

$$|x| < \frac{9}{2} + 1 = \frac{11}{2}.$$

The rational root theorem says any rational root p/q with p dividing 4 and q dividing 2. This leaves us with

$$x = \pm 1, \pm 2, \pm 4, \pm 1/2.$$

The bounds theorem won't eliminate any of these and Descartes Rule of Signs eliminates the positive ones. This leaves us with

$$x = -1, -2, -4, -1/2.$$

Testing these values leads us to

$$Q\left(-\frac{1}{2}\right) = 2\left(-\frac{1}{2}\right)^3 + 3\left(-\frac{1}{2}\right)^2 + 9\left(-\frac{1}{2}\right) + 4 = -\frac{1}{4} + \frac{3}{4} - \frac{9}{2} + 4 = 0.$$

Hence $(2x+1)$ is a factor of $Q(x)$. Division leaves us with

$$Q(x) = (2x+1)(x^2 + x + 4).$$

Solutions to the resulting quadratic is

$$x = \frac{-1 \pm \sqrt{1-16}}{2} = \frac{-1 \pm \sqrt{-15}}{2} = \frac{-1 \pm i\sqrt{15}}{2}.$$

Hence the solutions are

$$-\frac{1}{2}, \frac{-1 \pm i\sqrt{15}}{2}.$$

(b) The polynomial

$$P(x) = 2x^3 - x^2 + 3x - 9$$

has 3 sign changes. Hence there are 1 or 3 positive roots. Letting x be $-x$ we get

$$P(-x) = -2x^3 - x^2 - 3x - 9$$

which has no sign changes. Hence there are 0 negative roots.
For the bounds theorem, we have $M = 9, |a_n| = 2$, hence

$$|x| < \frac{9}{2} + 1 = \frac{11}{2}.$$

The rational root theorem says any rational root p/q with p dividing 9 and q dividing 2. This leaves us with

$$x = \pm 1, \pm 3, \pm 9, \pm 1/2, \pm 3/2, \pm 9/2.$$

The bounds theorem eliminates ± 9 and Descartes Rule of Signs eliminates the negative ones. This leaves us with

$$x = 1, 3, 1/2, 3/2, 9/2$$

Testing these values leads us to

$$P\left(\frac{3}{2}\right) = 2\left(\frac{3}{2}\right)^3 - \left(\frac{3}{2}\right)^2 + 3\left(\frac{3}{2}\right) - 9 = \frac{27}{4} - \frac{9}{4} + \frac{9}{2} - 9 = 0.$$

Hence $(2x - 3)$ is a factor of $P(x)$. Division leaves us with

$$P(x) = (2x - 3)(x^2 + x + 3).$$

Solutions to the resulting quadratic are

$$x = \frac{-1 \pm \sqrt{1 - 12}}{2} = \frac{-1 \pm \sqrt{-11}}{2} = \frac{-1 \pm i\sqrt{11}}{2}.$$

Hence the solutions are

$$\frac{3}{2}, \frac{-1 \pm i\sqrt{11}}{2}.$$

(c) The polynomial

$$P(x) = 45x^3 + 54x^2 + 19x + 2$$

has no sign changes. Hence there are 0 positive roots. Letting x be $-x$ we get

$$P(-x) = -45x^3 + 54x^2 - 19x + 2$$

which has three sign changes. Hence there are 1 or 3 negative roots.

For the bounds theorem, we have $M = 54, |a_n| = 45$, hence

$$|x| < \frac{54}{45} + 1 = \frac{11}{5}.$$

The rational root theorem says any rational root p/q with p dividing 2 and q dividing 45. This leaves us with

$$x = \pm 1, \pm 1/3, \pm 1/5, \pm 1/9, \pm 1/15, \pm 1/45, \pm 2, \pm 2/3, \pm 2/5, \pm 2/9, \pm 2/15, \pm 2/45.$$

The bounds theorem doesn't eliminate any of them and Descartes Rule of Signs eliminates the positive ones. This leaves us with

$$x = -1, -1/3, -1/5, -1/9, -1/15, -1/45, -2, -2/3, -2/5, -2/9, -2/15, -2/45$$

Testing these values leads us to

$$P\left(-\frac{1}{3}\right) = 45\left(-\frac{1}{3}\right)^3 + 54\left(-\frac{1}{3}\right)^2 + 19\left(-\frac{1}{3}\right) + 2 = -\frac{5}{3} + 6 - \frac{19}{3} + 2 = 0.$$

Hence $(3x + 1)$ is a factor of $P(x)$. Division leaves us with

$$P(x) = (3x + 1)(15x^2 + 13x + 2).$$

Solutions to the resulting quadratic can be found from the quadratic formula or by noticing that

$$15x^2 + 13x + 2 = (3x + 2)(5x + 1)$$

Hence the solutions are

$$-\frac{1}{3}, -\frac{2}{3}, -\frac{1}{5}.$$

(d) The polynomial

$$P(x) = 15x^3 - 13x^2 + 2x + 30$$

has two sign changes. Hence there are 2 or 0 positive roots. Letting x be $-x$ we get

$$P(-x) = -15x^3 - 13x^2 - 2x + 30$$

which has one sign change. Hence there is 1 negative roots.

For the bounds theorem, we have $M = 30, |a_n| = 15$, hence

$$|x| < \frac{30}{15} + 1 = 3.$$

The rational root theorem says any rational root p/q with p dividing 30 and q dividing 15. This leaves us with

$$x = \pm 1, \pm 1/3, \pm 1/5, \pm 1/15, \pm 2, \pm 2/3, \pm 2/5, \pm 2/15, \pm 3, \pm 3/5, \pm 5, \pm 5/3, \pm 6, \pm 6/5, \pm 15, \pm 30$$

The bounds theorem eliminates $\pm 3, \pm 5, \pm 6, \pm 15$ and ± 30 and Descartes Rule of Signs doesn't eliminate any. This leaves us with

$$x = \pm 1, \pm 1/3, \pm 1/5, \pm 1/15, \pm 2, \pm 2/3, \pm 2/5, \pm 2/15, \pm 3/5, \pm 5/3, \pm 6/5$$

Testing these values leads us to

$$P(-1) = 15(-1)^3 - 13(-1)^2 + 2(-1) + 30 = -15 - 13 - 2 + 30 = 0.$$

Hence $(x + 1)$ is a factor of $P(x)$. Division leaves us with

$$P(x) = (x + 1)(15x^2 - 28x + 30).$$

Solutions to the resulting quadratic are

$$x = \frac{28 \pm \sqrt{784 - 1800}}{30} = \frac{28 \pm \sqrt{-1016}}{30} = \frac{14 \pm i\sqrt{254}}{15}.$$

Hence the solutions are

$$-1, \frac{14 \pm i\sqrt{254}}{15}.$$

(e) First we note that we can factor x out of $P(x)$ and so $P(x) = xQ(x)$ where $Q(x) = 6x^4 - 13x^3 - 15x^2 + 23x - 6$. The polynomial $Q(x)$ has three sign changes. Hence there are 1 or 3 positive roots. Letting x be $-x$ we get

$$Q(-x) = 6x^4 + 13x^3 - 15x^2 - 23x - 6$$

which has one sign change. Hence there is 1 negative root.
For the bounds theorem, we have $M = 23$, $|a_n| = 6$, hence

$$|x| < \frac{23}{6} + 1 = \frac{29}{6}.$$

The rational root theorem says any rational root p/q with p dividing 6 and q dividing 6. This leaves us with

$$x = \pm 1, \pm 1/2, \pm 1/3, \pm 1/6, \pm 2, \pm 2/3, \pm 3, \pm 3/2, \pm 6$$

The bounds theorem eliminates ± 6 and Descartes Rule of Signs doesn't eliminate any. This leaves us with

$$x = \pm 1, \pm 1/2, \pm 1/3, \pm 1/6, \pm 2, \pm 2/3, \pm 3, \pm 3/2.$$

Testing these values leads us to

$$
\begin{aligned}
Q\left(\frac{2}{3}\right) &= 6\left(\frac{2}{3}\right)^4 - 13\left(\frac{2}{3}\right)^3 - 15\left(\frac{2}{3}\right)^2 + 23\left(\frac{2}{3}\right) - 6 \\
&= \frac{32}{27} - \frac{104}{27} - \frac{20}{3} + \frac{46}{3} - 6 \\
&= 0.
\end{aligned}
$$

Hence $(3x-2)$ is a factor of $Q(x)$. Division leaves us with

$$P(x) = x(3x-2)(2x^3 - 3x^2 - 7x + 3).$$

Let $H(x) = 2x^3 - 3x^2 - 7x + 3$. We test some more values to try and get it to a quadratic (note that we wouln't have to test rational numbers previously checked, and also for the new polynomial, the rational root theorem eliminates $\pm 1/3, \pm 1/6$ and $\pm 2/3$.
Testing values leads us to

$$
\begin{aligned}
H\left(-\frac{3}{2}\right) &= 2\left(-\frac{3}{2}\right)^3 - 3\left(-\frac{3}{2}\right)^2 - 7\left(-\frac{3}{2}\right) + 3 \\
&= -\frac{27}{4} - \frac{27}{4} + \frac{21}{2} + 3 \\
&= 0.
\end{aligned}
$$

Hence $(2x+3)$ is a factor of $H(x)$. Therefore

$$P(x) = x(3x-2)(2x+3)(x^2 - 3x + 1).$$

Solutions to the resulting quadratic are

$$x = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

Hence the solutions are

$$0, \frac{2}{3}, -\frac{3}{2}, \frac{3 \pm \sqrt{5}}{2}.$$