MATH 1210 Assignment 2 Fall 2013 **Due date: October 18 in class**

1. Let $P(x) = x^4 + x^3 + x^2 + x + 1$; find all values of a for which $a^2 P(1) + 5aP(i) = 10P(0)$.

Solution:

$P(1) = 1^4+1^3+1^2+1+1 = 5$, $P(0) = 0^4+0^3+0^2+0+1 = 1$, $P(i) = i^4+i^3+i^2+i+1 = 1-i-1+i+1 = 1$

Substituting in $a^2 P(1) + 5aP(i) = 10P(0)$ we get :

$$5a^2 + 5a = 10 \Rightarrow a^2 + a - 2 = 0 \Rightarrow (a-1)(a+2) = 0 \Rightarrow a = 1, \quad a = -2.$$

2. Find the remainder when the polynomial $P(x) = -2x^5 - 2ix^4 - ix^3 + x^2 + 5$ is divided by $(1+i)x - 1 + i$.

Solution: First note that

$$(1+i)x - 1 + i = 0 \Rightarrow x = \frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{1+i^2-2i}{1-i^2} = \frac{-2i}{2} = -i$$

Now substituting x by $-i$ in $P(x)$ we get

$P(-i) = -2(-i)^5 - 2i(-i)^4 - i(-i)^3 + (-i)^2 + 5 = -2(-i) - 2i(1) - i(i) + i^2 + 5 = 2i - 2i - i^2 + i^2 + 5 = 5$.

Hence the remainder of division of $P(x)$ when is divided by $(1+i)x - 1 + i$ is 5.

3. For each of the following, if it is true prove it and if it is not true give a counter example.

(a) If r_1 is a zero of polynomial $P_1(x)$ and r_2 is a zero of polynomial $P_2(x)$, then $r_1 + r_2$ is a zero of polynomial $P_1(x) + P_2(x)$.

(b) If r is a zero of polynomial $P(x)$, then $2r$ is a zero of polynomial $P(2x)$.

(c) If r is a zero of polynomial $P(x)$ of multiplicity k, then r is a zero of polynomial $(P(x))^n$ of multiplicity nk where $n \geq 1$ is an integer.

Solution:

(a) It is not true in general. For example $r_1 = 1$ is a root of $P_1(x) = x - 1$ and $r_2 = 2$ is a root of $P_2(x) = x - 2$ but $r_1 + r_2 = 1 + 2 = 3$ is not a root of $P_1(x) + P_2(x) = 2x - 3$ because $2(3) - 3 = 3 \neq 0$.

(b) It is not true in general. For example $r = 3$ is a root of $P(x) = x - 3$ but $2r = 2(3) = 6$ is not a root of $P(2x) = 2x - 3$ because $2(6) - 3 = 9 \neq 0$.

(c) It is true and we can prove it either directly or by induction on n.
Direct proof:
Since r is a root of $P(x)$ with multiplicity k, $P(x) = (x-r)^k * Q(x)$ where $Q(r)$ is not 0. Therefore $(P(x))^n = (x-r)^{kn}(Q(x))^n$ where $(Q(r))^n$ is not 0. Hence r is a zero of polynomial $(P(x))^n$ of multiplicity nk.
Proof by induction on n:
If $n = 1$ then it is true in a trivial way. Assuming the statement is true for $n = \ell$,

that is "if r is a zero of polynomial $P(x)$ of multiplicity k, then r is a zero of polynomial $(P(x))^\ell$ of multiplicity ℓk". We need to prove that it is also true for $n = \ell + 1$ that is we need to prove that "if r is a zero of polynomial $P(x)$ of multiplicity k, then r is a zero of polynomial $(P(x))^{\ell+1}$ of multiplicity $(\ell+1)k$". But if r is a zero of polynomial $P(x)$ of multiplicity k, then since

$$(P(x))^{\ell+1} = (P(x))^\ell (P(x)),$$

so by induction assumption r is a zero of polynomial $(P(x))^\ell$ of multiplicity ℓk and also r is a zero of polynomial $P(x)$ of multiplicity k; therefore r is a zero of polynomial $(P(x))^{\ell+1}$ with multiplicity $(\ell)k + k = (\ell+1)k$. Thus by the Principle of Mathematical Induction the statement is true for all $n \geq 1$.

4. Consider the polynomial $P(x) = 7x^6 - 33x^5 + 53x^4 - 3x^3 - 62x^2 + 54x + 20$.

(a) Use Rational Root Theorem to find all possible rational zeros of $P(x)$.

(b) Use Descartes' Rules of Signs to determine how many positive or negative real zeros $P(x)$ may have.

(c) Use Bounds Theorem to determine how large the absolute value of a root of $P(x)$ may be.

(d) Use your answers in part (c) to improve your list of all possible rational zeros of $P(x)$ in part (a).

(e) If $2 + i$ is a complex root of $P(x)$, find an irreducible real quadratic factor of $P(x)$.

Solution:

(a) If $\frac{p}{q}$ is a rational root of $P(x)$ then by Rational Root Theorem p divides 20 and q divides 7 so $p \in \{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20\}$ and $q \in \{\pm1, \pm7\}$ and therefore $\frac{p}{q} \in \{\pm1, \pm2, \pm4, \pm5, \pm10, \pm20, \pm\frac{1}{7}, \pm\frac{2}{7}, \pm\frac{4}{7}, \pm\frac{5}{7}, \pm\frac{10}{7}, \pm\frac{20}{7}\}$.

(b) Sign of coefficients in $P(x)$ is changing 4 times so Descartes' Rules of Signs the number of positive real zeros of $P(x)$ is either 4 or 2 or 0. Now

$$\begin{aligned} P(-x) &= 7(-x)^6 - 33(-x)^5 + 53(-x)^4 - 3(-x)^3 - 62(-x)^2 + 54(-x) + 20 \\ &= 7x^6 + 33x^5 + 53x^4 + 3x^3 - 62x^2 - 54x + 20 \end{aligned}$$

Since sign of coefficients in $P(-x)$ is changing 2 times so by Descartes' Rules of Signs the number of negative real zeros of $P(x)$ is either 2 or 0.

(c) $M = max\{|-33|, |53|, |-3|, |-62|, |54|, |20|\} = 62$, so by Bonds Theorem

$$|x| < \frac{62}{7} + 1 < \frac{63}{7} + 1 = 9 + 1 = 10,$$

that is $|x| < 10$.

(d) By part (c) absolute value of any real root (which includes rational roots) of $P(x)$ is less than 10. Therefore ±10 and ±20 can be omitted from the list of possible rational roots obtained in part (a) and the new improved list of possible rational roots of $P(x)$ is $\{\pm1, \pm2, \pm4, \pm5, \pm\frac{1}{7}, \pm\frac{2}{7}, \pm\frac{4}{7}, \pm\frac{5}{7}, \pm\frac{10}{7}, \pm\frac{20}{7}\}$.

(e) **Since $2+i$ is root and all coefficients of $P(x)$ are real numbers, so $\overline{2+i}=2-i$ is also a root and**

$$\big(x-(2+i)\big)\big(x-(2-i)\big)=x^2-4x+5$$

is an irreducible real quadratic factor of $P(x)$.

5. **Find all the roots of the polynomial $P(x)=2x^4-3x^3-7x^2-5x-3$.**

Solution: If $\dfrac{p}{q}$ is a rational root of $P(x)$ then by Rational Root Theorem p divides -3 and q divides 2 so $p\in\{\pm1,\pm3\}$ and $q\in\{\pm1,\pm2\}$ and therefore $\dfrac{p}{q}\in\{\pm1,\pm3,\pm\frac{1}{2},\pm\frac{3}{2}\}$. Now since

$$P(-1)=2(-1)^4-3(-1)^3-7(-1)^2-5(-1)-3=2+3-7+5-3=0,\ \text{ and}$$
$$P(3)=2(3)^4-3(3)^3-7(3)^2-5(3)-3=162-81-63-15-3=0\,,$$

So -1 and 3 are roots of $P(x)$ and therefore $P(x)$ is divisible by $(x+1)(x-3)=x^2-2x-3$. Using long division we get $P(x)=(x^2-2x-3)(2x^2+x+1)$ and $2x^2+x+1=0$ gives $x=-\dfrac{1}{4}\pm\dfrac{\sqrt{7}}{4}i$. Hence all the roots are -1 ,3 , $-\dfrac{1}{4}+\dfrac{\sqrt{7}}{4}i$ and $-\dfrac{1}{4}-\dfrac{\sqrt{7}}{4}i$.

6. **Given that $2i$ is a root of $P(x)=x^6-x^5+x^4-16x^2+16x-16$, find all roots of $P(x)$ and write $P(x)$ as a product of real linear and irreducible real quadratic factors.**

Solution: Since $2i$ is a root and all coefficients of $P(x)$ are real numbers, so $\overline{2i}=-2i$ is also a root and

$$\big(x-(2i)\big)\big(x-(-2i)\big)=x^2+4$$

is an irreducible real quadratic factor of $P(x)$. Using long division we get $P(x)=(x^2-2x-3)(x^4-x^3-3x^2+4x-4)$. Now let $Q(x)=x^4-x^3-3x^2+4x-4$; if $\dfrac{p}{q}$ is a rational root of $Q(x)$ then by Rational Root Theorem p divides -4 and q divides 1 so $p\in\{\pm1,\pm2,\pm4\}$ and $q\in\{\pm1\}$ and therefore $\dfrac{p}{q}\in\{\pm1,\pm2,\pm4\}$. Now since

$$Q(2)=(2)^4-(2)^3-3(2)^2+4(2)-4=16-8-12+8-4=0,\ \text{ and}$$
$$Q(-2)=(-2)^4-(-2)^3-3(-2)^2+4(-2)-4=16+8-12-8-4=0\,,$$

So 2 and -2 are roots of $Q(x)$ and therefore $Q(x)$ is divisible by $(x+2)(x-2)=x^2-4$. Using long division we get $Q(x)=(x^2-4)(x^2-x+1)$ and $x^2-x+1=0$ gives $x=\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$. Hence all the roots of $P(x)$ are ±2 ,$\pm2i$, and $\dfrac{1}{2}\pm\dfrac{\sqrt{3}}{2}i$. Thus

$$P(x)=(x-2)(x+2)(x^2+4)(x^2-x+1)\,,$$

where both x^2+4 and x^2-x+1 are irreducible real quadratic factors.

7. **Let $P(x)=-2x^{17}-x^{15}+x^2-20$; prove each of the following statements or explain why it is not correct.**

(a) $P(x)$ has at least one zero in the interval $[11, 20]$.

(b) It is impossible for $P(x)$ to have 2 negative real zeros.

Solution:

(a) It is not true because if x is a zero of $P(x)$, then using Bounds Theorem, $M = max\{|-1|, |1|, |-20|\} = 20$, and

$$|x| < \frac{20}{|-2|} + 1 = 10 + 1 = 11\,,$$

that is $|x| < 11$, which means $P(x)$ has no zero in the interval $[11, 20]$ at all.

(b) It is true because

$$P(-x) = -2(-x)^{17} - (-x)^{15} + (-x)^2 - 20 = 2x^{17} + x^{15} + x^2 - 20\,;$$

since sign of coefficients is changing only one time so the number of negative real zeros of $P(x)$ is 1 and therefore it is impossible for $P(x)$ to have 2 negative real zeros.

8. Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 5 \\ -1 & 1 \\ 2 & 6 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$. Evaluate each of the following expressions or explain why it is not defined.

(a) $(3B + 2D)(6C - A^T)$

(b) $BD^T + AC$

Solution:

(a) Since $3B + 2D$ is of size 2×2 and $6C - A^T$ is of size 3×2 and $2 \neq 3$ so $(3B + 2D)(6C - A^T)$ is not defined.

(b) Since BD^T is of size 2×2 and AC is of size 2×2 so $BD^T + AC$ is defined and is of size 2×2. In fact

$$\begin{aligned} BD^T + AC &= \begin{bmatrix} 4 & 0 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 & -3 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -1 & 1 \\ 2 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 8 & -4 \\ 1 & -3 \end{bmatrix} + \begin{bmatrix} -7 & -11 \\ 0 & 16 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -15 \\ 1 & 13 \end{bmatrix}. \end{aligned}$$

9. Let $A = \begin{bmatrix} 1 & a \\ -a & 2 \end{bmatrix}$; find all values of a for which $A^2 - 3A = 3aI_2$ where I_2 is the 2×2 identity matrix.

Solution:

$$A^2-3A=3aI_2 \Rightarrow \begin{bmatrix} 1 & a \\ -a & 2 \end{bmatrix}\begin{bmatrix} 1 & a \\ -a & 2 \end{bmatrix}-3\begin{bmatrix} 1 & a \\ -a & 2 \end{bmatrix}=3a\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\Rightarrow$$

$$\begin{bmatrix} 1-a^2 & 3a \\ -3a & 4-a^2 \end{bmatrix}-\begin{bmatrix} 3 & 3a \\ -3a & 6 \end{bmatrix}=\begin{bmatrix} 3a & 0 \\ 0 & 3a \end{bmatrix}\Rightarrow\begin{bmatrix} -a^2-2 & 0 \\ 0 & -a^2-2 \end{bmatrix}=\begin{bmatrix} 3a & 0 \\ 0 & 3a \end{bmatrix}.$$

Therefore must $-a^2-2=3a$ **which implies** $a^2+3a+2=0$ **that is** $(a+1)(a+2)=0$ **which means** $a=-1$ **or** $a=-2$**.**