MATH 1210 ASSIGNMENT #1 SOLUTIONS

1. $P_n: \sum_{l=1}^{n} (3l-2) = \frac{n(3n-1)}{2}$ for $n \geq 1$.

[note $\sum_{l=1}^{n} (3l-2) = 1 + 4 + \cdots + (3n-2)$]

(A) Is P_1 true? $LHS = \sum_{l=1}^{1} (3l-2) = 1$

$RHS = \frac{1(2)}{2} = 1$

$\therefore P_1$ is true.

(B) If P_k is true $\left[\sum_{l=1}^{k} (3l-2) = \frac{k(3k-1)}{2} \right]$

is P_{k+1} also true $\left[\sum_{l=1}^{k+1} (3l-2) = \frac{(k+1)(3(k+1)-1)}{2} = \frac{(k+1)(3k+2)}{2} \right]$?

$$\sum_{l=1}^{k+1} (3l-2) = \sum_{l=1}^{k} (3l-2) + (3(k+1)-2)$$

$$= \frac{k(3k-1)}{2} + (3k+1) \quad [\text{if } P_k \text{ is true}]$$

$$= \frac{3k^2 - k + 6k + 2}{2} = \frac{3k^2 + 5k + 2}{2}$$

$$= \frac{(k+1)(3k+2)}{2}$$

$\therefore$ if P_k is true, so is P_{k+1}!

(C) by P.M.I., P_n is true for all $n \geq 1$.

i.e, $\sum_{l=1}^{n} (3l-2) = \frac{n(3n-1)}{2}$ for $n \geq 1$.

(2) P_n: $6^n - 1$ is div by 5 for $n \geq 1$

(A) Is P_1 true: $n=1 \Rightarrow 6^n - 1 = 6^1 - 1 = 6-1 = 5$ which is div. by $5 \Rightarrow P_1$ is true

(B) If P_k is true [$6^k - 1$ is div. by 5]
is P_{k+1} also true [$6^{k+1} - 1$ is div. by 5]?

$$6^{k+1} - 1 = 6^k \cdot 6 - 1 = (6^k - 1)\cdot 6 + 6 - 1$$
$$= \underbrace{(6^k - 1)}_{\substack{\text{div. by 5} \\ \text{if } P_k \text{ is} \\ \text{true}}} + \underbrace{5}_{\text{div. by 5}}$$

$\therefore$ if P_k is true, P_{k+1} is also true.

(C) $\therefore$ by P.M.I., P_n is true for all $n \geq 1$
i.e, $6^n - 1$ is div. by 5 for $n \geq 1$.

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(3) P_n: $\displaystyle\sum_{\ell=1}^{2n} \ell^2 = \frac{n(2n+1)(4n+1)}{3}$ for $n \geq 1$.

(A) Is P_1 true? $LHS = \displaystyle\sum_{\ell=1}^{2} \ell^2 = 1^2 + 2^2 = 1+4 = 5$

$RHS = \dfrac{1(3)(5)}{3} = 5$

$\therefore P_1$ is true!

(B) If P_k is true $\left[\displaystyle\sum_{\ell=1}^{2k} \ell^2 = \frac{k(2k+1)(4k+1)}{3} \right]$

is P_{k+1} also true $\left[\displaystyle\sum_{\ell=1}^{2k+2} \ell^2 = \frac{(k+1)(2k+3)(4k+5)}{3} \right]$?

$$\sum_{\ell=1}^{2k+2} \ell^2 = \sum_{\ell=1}^{2k} \ell^2 + (2k+1)^2 + (2k+2)^2$$

$$= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \quad [\text{If } P_k \text{ is true}]$$

$$= \frac{k(2k+1)(4k+1) + 3(2k+1)^2 + 3(2k+2)^2}{3}$$

$$= \frac{8k^3 + 30k^2 + 37k + 15}{3}$$

But $(k+1)(2k+3)(4k+5) = 8k^3 + 30k^2 + 37k + 15$

Therefore, if P_k is true, so is P_{k+1}.

(C) By P.M.I., P_n is true for all $n \geq 1$.

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(4) Given : $x_1 = 1$

$x_{n+1} = \sqrt{1 + 2x_n}$ for $n \geq 1$

Claim : $x_n < 4$ for $n \geq 1$.

(A) Is $x_1 < 4$? Yes, since $x_1 = 1$.

(B) If $x_k < 4$, is $x_{k+1} < 4$ also?

$x_k < 4$

$\Rightarrow 2x_k < 8$

$\Rightarrow 1 + 2x_k < 9$

$\Rightarrow \sqrt{1 + 2x_k} < \sqrt{9} = 3$

But $x_{k+1} = \sqrt{1 + 2x_k}$ so $x_{k+1} < 4$ if $x_k < 4$.

(C) by P.M.I, $x_n < 4$ for all $n \geq 1$.

5 $n! > 3^n$ for $n \geq 7$. [note: $6! = 720$, $3^6 = 729$, so $6! < 3^6$]

(A) Is $7! > 3^7$? $\left.\begin{array}{l} 7! = 5040 \\ 3^7 = 2187 \end{array}\right\}$ $\therefore 7! > 3^7$.

(B) If $k! > 3^k$, is $(k+1)! > 3^{k+1}$?

$$\begin{aligned} (k+1)! = (k+1)\,k! &> (k+1)\,3^k && [\text{if } k! > 3^k] \\ &> 3 \cdot 3^k && [\text{since } k \geq 7] \\ &= 3^{k+1} \end{aligned}$$

Therefore, if $k! > 3^k$, then $(k+1)! > 3^{k+1}$.

(C) by P.M.I., $n! > 3^n$ for $n \geq 7$.

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6 $p_0 = 1$

$p_1 = \cos\theta$

$p_{n+1} = 2p_1 p_n - p_{n-1}$ for $n \geq 1$

Claim: $p_n = \cos(n\theta)$ for $n \geq 0$.

(A) Is $p_0 = \cos(0)$? yes, since $p_0 = 1$ & $\cos(0) = 1$.
Is $p_1 = \cos\theta$? yes, by definition.

(B) If $p_k = \cos(k\theta)$ & $p_{k+1} = \cos((k+1)\theta)$
is $p_{k+2} = \cos((k+2)\theta)$?

$$p_{k+2} = 2p_1 p_{k+1} - p_k = 2\cos\theta\cos((k+1)\theta) - \cos(k\theta)$$

But $\cos((k+2)\theta) = \cos(((k+1)+1)\theta) = \cos((k+1)\theta + \theta)$
$= \cos((k+1)\theta)\cos\theta - \sin((k+1)\theta)\sin\theta$ (i)

& $\cos(k\theta) = \cos((k+1-1)\theta) = \cos((k+1)\theta - \theta)$
$= \cos((k+1)\theta)\cos\theta + \sin((k+1)\theta)\sin\theta$ (ii)

$\therefore$ if we add (i) & (ii) together, we have

$$\cos((k+2)\theta) + \cos(k\theta) = 2\cos\theta\cos((k+1)\theta)$$

$\therefore p_{k+2} = 2\cos\theta\cos((k+1)\theta) - \cos(k\theta) = \cos((k+2)\theta)$.

Thus, if $p_k = \cos(k\theta)$ & $p_{k+1} = \cos((k+1)\theta)$

then $p_{k+2} = \cos((k+2)\theta)$

(c) by extended P.M.I., $p_n = \cos(n\theta)$ for $n \geq 0$.

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(7) Fibonacci Sequence : $x_1 = 1$, $x_2 = 1$

$x_n = x_{n-1} + x_{n-2}$ for $n \geq 3$.

Claim : P_n : $x_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ for $n \geq 1$

Proof. (A) Are P_1 & P_2 true ?

$x_1 = 1$, $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^1 - \left(\frac{1-\sqrt{5}}{2}\right)^1\right] = \frac{1}{\sqrt{5}}\left(\frac{2\sqrt{5}}{2}\right)$

$= 1$

$\therefore P_1$ is true

$x_2 = 1$, $\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{1+2\sqrt{5}+5}{4} - \left(\frac{1-2\sqrt{5}+5}{4}\right)\right]$

$= \frac{1}{\sqrt{5}}\left[\frac{4\sqrt{5}}{4}\right] = 1$

$\therefore P_2$ is true

(B) if P_k & P_{k+1} are true

$\left[x_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]\right.$

& $x_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$

is P_{k+2} also true $\left[x_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]\right]$?

$$x_{k+2} = x_{k+1} + x_k$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

$$+ \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k} - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k}\left[\frac{1+\sqrt{5}}{2}+1\right] - \left(\frac{1-\sqrt{5}}{2}\right)^{k}\left[\frac{1-\sqrt{5}}{2}+1\right]\right]$$

But $\left[\frac{1+\sqrt{5}}{2}+1\right] = \frac{3+\sqrt{5}}{2} = \frac{6+2\sqrt{5}}{4} = \left(\frac{1+\sqrt{5}}{2}\right)^2$

and $\left[\frac{1-\sqrt{5}}{2}+1\right] = \frac{3-\sqrt{5}}{2} = \frac{6-2\sqrt{5}}{4} = \left(\frac{1-\sqrt{5}}{2}\right)^2$

Therefore $x_{k+2} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+2}\right]$

i.e., if P_k & P_{k+1} are true, so is P_{k+2}.

© by extended P.M.I., P_n is true for all $n \geq 1$.