In Exercises 1-3, for the given transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$, do the following:

(a) Find the image of $\vec{v} = (3, -1, 2)$ under T.
(b) Is T a linear transformation? If so, find the matrix associated with T.
(c) Find all vectors $\vec{v}$ whose image under T is $\vec{v'} = (1, 3, 2)$.
(d) Find the inverse transformation, T^{-1}, or give a reason why T^{-1} does not exist.
(e) Is T^{-1} a linear transformation? If so, find the matrix associated with T^{-1}.

1. $T(v_1, v_2, v_3) = (v'_1, v'_2, v'_3)$, where

$$\begin{aligned} v'_1 &= v_3 \\ v'_2 &= v_1 + v_2 \\ v'_3 &= v_1 v_2 \end{aligned}$$

Solution: (a) $T(3, -1, 2) = (2, 2, -3)$.

(b) No, T is not linear. For example, if we set $\vec{v} = (3, -1, 2)$, then $T(2\vec{v}) \neq 2T(\vec{v})$.

(c) If $\vec{v} = (v_1, v_2, v_3)$ and $T(\vec{v}) = (1, 3, 2)$, then

$$\begin{aligned} v_3 &= 1 \\ v_1 + v_2 &= 3 \\ v_1 v_2 &= 2\,. \end{aligned}$$

Solving for v_1, v_2 and v_3 leads to two solutions: $\vec{v} = (1, 2, 1)$ and $\vec{v} = (2, 1, 1)$.

(d) The inverse transformation does not exist since the inverse image of $\vec{v'} = (1, 3, 2)$ is not uniquely defined: it could be either $\vec{v} = (1, 2, 1)$ or $\vec{v} = (2, 1, 1)$.

(e) Not applicable, since T^{-1} does not exist.

2. $T : \mathbb{R}^3 \to \mathbb{R}^3$, $T(v_1, v_2, v_3) = (v'_1, v'_2, v'_3)$, where

$$\begin{aligned} v'_1 &= v_1 + 2v_2 - v_3 \\ v'_2 &= v_2 + v_3 \\ v'_3 &= 2 - v_3 \end{aligned}$$

Solution: (a) $T(3, -1, 2) = (-1, 1, 0)$.

(b) Since $T(0, 0, 0) = (0, 0, 2)$, it follows that $T(\vec{0}) \neq \vec{0}$, so T is not linear.

(c) If $\vec{v} = (v_1, v_2, v_3)$ and $T(\vec{v}) = (1, 3, 2)$, then

$$\begin{aligned} v_1 + 2v_2 - v_3 &= 1 \\ v_2 + v_3 &= 3 \\ 2 - v_3 &= 2\,. \end{aligned}$$

Solving for v_1, v_2 and v_3 (say, by backward substitution) leads to a unique solution: $\vec{v} = (-5, 3, 0)$.

(d) For any vector $\vec{v'}$ in $\mathbb{R}^3$, $T^{-1}(\vec{v'})$ is defined if and only if there is a *unique* vector $\vec{v}$ such that $T(\vec{v}) = \vec{v'}$. Thus we need to determine whether there is exactly one solution of the system:

$$\begin{aligned} v_1 + 2v_2 - v_3 &= v_1' \\ v_2 + v_3 &= v_2' \\ 2 - v_3 &= v_3', \end{aligned}$$

where we treat v_1, v_2, v_3 as the unknowns and v_1', v_2', v_3' are fixed constants. Solving the system (say, by backward substitution) leads to a unique solution:

$$\begin{aligned} v_1 &= v_1' - 2v_2' - 3v_3' + 6 \\ v_2 &= v_2' + v_3' - 2 \\ v_3 &= 2 - v_3' \end{aligned}$$

These equations define the inverse transformation T^{-1}, where $T^{-1}(v_1', v_2', v_3') = (v_1, v_2, v_3)$.

(e) Since $T^{-1}(0,0,0) = (6,-2,2)$, it follows that $T^{-1}(\vec{0}) \neq \vec{0}$, so T^{-1} is not linear.

3. $T : \mathbb{R}^3 \to \mathbb{R}^3,\ T(v_1, v_2, v_3) = (v_1', v_2', v_3')$, where

$$\begin{aligned} v_1' &= v_1 - 5v_3 \\ v_2' &= v_2 + 3v_3 \\ v_3' &= 2v_1 + 3v_2 \end{aligned}$$

Solution: (a) $T(3,-1,2) = (-7,5,3)$.

(b) The transformation is linear since the given equations can be written in the matrix form:

$$\begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix} = \begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The 3×3 matrix occurring here is the matrix associated with the transformation T.

(c) If $\vec{v} = (v_1, v_2, v_3)$ and $T(\vec{v}) = (1,3,2)$, then

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}.$$

Solving for v_1, v_2 and v_3 (say, by Gauss-Jordan elimination) leads to a unique solution: $\vec{v} = (-44, 30, -9)$.

(d) For any vector $\vec{v'}$ in $\mathbb{R}^3$, $T^{-1}(\vec{v'})$ is defined if and only if there is a *unique* vector $\vec{v}$ such that $T(\vec{v}) = \vec{v'}$. Thus we need to determine whether there is exactly one solution of the system:

$$\begin{pmatrix} 1 & 0 & -5 \\ 0 & 1 & 3 \\ 2 & 3 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_1' \\ v_2' \\ v_3' \end{pmatrix}.$$

where we treat v_1, v_2, v_3 as the unknowns and v_1', v_2', v_3' are fixed constants. This is indeed the case since the determinant of the coefficient matrix is not zero:

$$\det\begin{pmatrix}1 & 0 & -5\\ 0 & 1 & 3\\ 2 & 3 & 0\end{pmatrix} = 1\cdot\det\begin{pmatrix}1 & 3\\ 3 & 0\end{pmatrix} + 2\cdot\det\begin{pmatrix}0 & -5\\ 1 & 3\end{pmatrix} = 1$$

In fact, we can solve the system by finding the inverse of the coefficient matrix A:

$$A^{-1} = \frac{1}{\det(A)}\text{adj}(A) = \begin{pmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{pmatrix}^T = \begin{pmatrix}M_{11} & -M_{21} & M_{31}\\ -M_{12} & M_{22} & -M_{32}\\ M_{13} & -M_{23} & M_{33}\end{pmatrix} = \begin{pmatrix}-9 & -15 & 5\\ 6 & 10 & -3\\ -2 & -3 & 1\end{pmatrix}$$

Therefore the unique solution is $\vec{v} = A^{-1}\vec{v'}$, or:

$$\begin{aligned} v_1 &= -9v_1' - 15v_2' + 5v_3' \\ v_2 &= 6v_1' + 10v_2' - 3v_3' \\ v_3 &= -2v_1 - 3v_2 + v_3' \end{aligned}$$

These equations define the inverse transformation T^{-1}, where $T^{-1}(v_1', v_2', v_3') = (v_1, v_2, v_3)$.

(e) The inverse transformation is linear since it can be given in the matrix form:

$$\begin{pmatrix}v_1\\ v_2\\ v_3\end{pmatrix} = \begin{pmatrix}-9 & -15 & 5\\ 6 & 10 & -3\\ -2 & -3 & 1\end{pmatrix}\begin{pmatrix}v_1'\\ v_2'\\ v_3'\end{pmatrix}.$$

The 3×3 matrix occurring here is the matrix associated with T^{-1}. This matrix is the inverse of the matrix associated with T.

4. Find all eigenvalues and corresponding eigenvectors for the matrix

$$A = \begin{pmatrix}7 & -5 & 7 & -1 & 4\\ 0 & 2 & 5 & -4 & 7\\ 0 & 0 & 7 & 1 & 17\\ 0 & 0 & 0 & 7 & 10\\ 0 & 0 & 0 & 0 & 2\end{pmatrix}.$$

Solution: Each eigenvalue λ must be a root of the equation:

$$\det\begin{pmatrix}7-\lambda & -5 & 7 & -1 & 4\\ 0 & 2-\lambda & 5 & -4 & 7\\ 0 & 0 & 7-\lambda & 1 & 17\\ 0 & 0 & 0 & 7-\lambda & 10\\ 0 & 0 & 0 & 0 & 2-\lambda\end{pmatrix} = 0.$$

Since the matrix is upper-triangular, its determinant is the product of the entries on the main diagonal. This leads to the equation:

$$(7-\lambda)^3(2-\lambda)^2 = 0$$

Therefore $\lambda = 7$ and $\lambda = 2$ are the only eigenvalues of the matrix A.
Case 1: $\lambda = 7$

A vector $\vec{v}$ is an eigenvector associated with $\lambda = 7$ if $(A - 7I_5)\vec{v} = \vec{0}$. This leads to a linear system:

$$\begin{pmatrix} 0 & -5 & 7 & -1 & 4 \\ 0 & -5 & 5 & -4 & 7 \\ 0 & 0 & 0 & 1 & 17 \\ 0 & 0 & 0 & 0 & 10 \\ 0 & 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving the system (say, by backward substitution), we get $v_5 = 0$, $v_4 = 0$, $v_3 = 0$, $v_2 = 0$, and v_1 is a free variable. Thus the eigenvectors corresponding to $\lambda = 7$ are of the form:

$$\vec{v} = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = v_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Case 2: $\lambda = 2$

A vector $\vec{v}$ is an eigenvector associated with $\lambda = 2$ if $(A - 2I_5)\vec{v} = \vec{0}$. This leads to a linear system:

$$\begin{pmatrix} 5 & -5 & 7 & -1 & 4 \\ 0 & 0 & 5 & -4 & 7 \\ 0 & 0 & 5 & 1 & 17 \\ 0 & 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Solving the system (say, by backward substitution), we get $v_4 = -2v_5$, $v_3 = -3v_5$, and $v_1 = v_2 + 3v_5$, where v_2 and v_5 are free variables. Thus the eigenvectors corresponding to $\lambda = 2$ are of the form:

$$\vec{v} = \begin{pmatrix} v_2 + 3v_5 \\ v_2 \\ -3v_5 \\ -2v_5 \\ v_5 \end{pmatrix} = v_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_5 \begin{pmatrix} 3 \\ 0 \\ -3 \\ -2 \\ 1 \end{pmatrix}.$$

5. Let A and E be square matrices of the same size. Suppose E is invertible and let $B = E^{-1}AE$. Show that if $(\lambda, \vec{v})$ is an eigenpair of the matrix A, then $(\lambda, E^{-1}\vec{v})$ is an eigenpair of the matrix B.

Solution: The assumption that $(\lambda, \vec{v})$ is an eigenpair of the matrix A means that

$$A\vec{v} = \lambda\vec{v}.$$

Hence

$$\begin{aligned} B(E^{-1}\vec{v}) &= (E^{-1}AE)(E^{-1}\vec{v}) = (E^{-1}A)((EE^{-1})\vec{v}) \\ &= (E^{-1}A)(I\vec{v}) = E^{-1}(A\vec{v}) = E^{-1}(\lambda\vec{v}) = \lambda(E^{-1}\vec{v}). \end{aligned}$$

Thus $(\lambda, E^{-1}\vec{v})$ is an eigenpair of the matrix B.