**THE UNIVERSITY OF MANITOBA**

**DATE**: December 17, 2011 **FINAL EXAMINATION**

**DEPARTMENT & NO**: MATH2132 **TIME**: 3 hours

**EXAMINATION**: Engineering Mathematical Analysis 2 **EXAMINER**: M. Despic, D. Trim



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10 **1.** For what value of the constant $a$ will the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^2+1}{a^n}(x-1)^{2n}$$

be equal to 5?

If we set $y=(x-1)^2$, the series becomes $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+1}{a^n} y^n$. Its radius of convergence is

$$R_y = \lim \left| \frac{\dfrac{n^2+1}{a^n}}{\dfrac{(n+1)^2+1}{a^{n+1}}} \right| = |a|.$$

Hence, $R_x = \sqrt{|a|}$. For this to be equal to 5, we must have $\sqrt{|a|}=5$, which implies that $a=\pm 25$.

13 **2.** Find the Taylor series about $x=3$ for the function

$$f(x) = (x-3)\ln(x+1).$$

Use a method that guarantees that the series converges to $f(x)$. Express your answer in sigma notation, simplified as much as possible. Determine the open interval of convergence for the series.

$\boxed{x-3}$

$$\frac{1}{x+1} = \frac{1}{(x-3)+4} = \frac{1}{4\left[1+\left(\dfrac{x-3}{4}\right)\right]} = \frac{1}{4}\sum_{n=0}^{\infty}\left[-\left(\frac{x-3}{4}\right)\right]^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^{n+1}}(x-3)^n,$$

and this is valid for

$$\left|-\left(\frac{x-3}{4}\right)\right| < 1 \quad\Longrightarrow\quad |x-3|<4 \quad\Longrightarrow\quad -4<x-3<4 \quad\Longrightarrow\quad -1<x<7.$$

Since the radius of convergence of the series is positive, we may integrate to get

$$\ln|x+1| = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)4^{n+1}}(x-3)^{n+1} + C.$$

When we set $x = 3$, we get $\ln 4 = C$, and therefore

$$\ln|x+1| = \ln 4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}}(x-3)^{n+1}.$$

Because $x > -1$, we may drop the absolute values, and

$$\begin{aligned}(x-3)\ln(x+1) &= (\ln 4)(x-3) + \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)4^{n+1}}(x-3)^{n+2} \\ &= (\ln 4)(x-3) + \sum_{n=2}^{\infty} \frac{(-1)^n}{(n-1)4^{n-1}}(x-3)^{n}.\end{aligned}$$

The open interval of convergence is $-1 < x < 7$.

7 **3.** Find a maximum possible error for all $x$ in the interval $1 \le x \le 3$ if the series

$$\sum_{n=1}^{\infty} \frac{n(-1)^{n+1}}{4^n}(x-1)^n$$

is truncated after its $10^{\text{th}}$ term. Justify your answer. (Note: the question was changed during the exam.)

Since the series is alternating when $1 \le x \le 3$, with absolute values of terms decreasing and approaching zero, the maximum error when the series is truncated after the $10^{\text{th}}$ term is the absolute value of the $11^{\text{th}}$ term,

$$\left|\frac{11(-1)^{12}}{4^{11}}(x-1)^{11}\right| \le \frac{11(3-1)^{11}}{4^{11}} = \frac{11}{2^{11}}.$$

6 **4.** You are given that the roots of the auxiliary equation associated with the linear, differential equation

$$\phi(D)y = 3 - 5x + 4\cos x + x^3 e^{-2x}$$

are $m = 0,\ 3 \pm i,\ 3 \pm i,\ -2,\ -2,\ -2,\ \pm\sqrt{5}$. Write down the form of a particular solution of the differential equation as predicted by the method of undetermined coefficients. You must use the minimum number of terms possible. Do **NOT** find the coefficients, just the form of the particular solution.

$$y_h(x) = C_1 + e^{3x}[(C_2 + C_3 x)\cos x + (C_4 + C_5 x)\sin x] + (C_6 + C_7 x + C_8 x^2)e^{-2x} + C_9 e^{\sqrt{5}x} + C_{10} e^{\sqrt{5}x}$$

$$y_p(x) = Ax^2 + Bx + D\cos x + E\sin x + Fx^6 e^{-2x} + Gx^5 e^{-2x} + Hx^4 e^{-2x} + Ix^3 e^{-2x}$$

9 **5.** Find a one-parameter family of solution for the differential equation

$$y\frac{dy}{dx} = (y+2)(\sin 3x - x).$$

Are there any singular solutions of your family?

The equation is separable

$$\frac{y}{y+2}dy = (\sin 3x - x)dx, \qquad \text{provided } y \neq -2.$$

A one-parameter family of solutions is defined implicitly by

$$\int \frac{y}{y+2}dy = \int (\sin 3x - x)\,dx + C$$

$$\int \left(1 - \frac{2}{y+2}\right) dy = -\frac{1}{3}\cos 3x - \frac{x^2}{2} + C$$

$$y - 2\ln|y+2| = -\frac{1}{3}\cos 3x - \frac{x^2}{2} + C.$$

Since $y = -2$ is a solution of the differential equation, and it is not in the family, it is a singular solution of the family.

15 **6.** (a) A mass of 2 kilograms is suspended from a spring with constant 50 newtons per metre. At time $t = 0$, it is lifted 10 centimetres above its equilibrium position and given velocity 4 metres per second downward. During its subsequent motion, it is also subjected to a damping force that (in newtons) is equal to 20 times its velocity (in metres per second). Determine the maximum distance from its equilibrium position that the mass ever achieves.
(b) If damping is removed, and an additional force $4\cos\omega t$ acts on the mass, what value of $\omega$ causes resonance?

(a) The initial-value problem for displacement $x(t)$ from equilibrium is

$$2\frac{d^2x}{dt^2} + 20\frac{dx}{dt} + 50x = 0, \quad x(0) = \frac{1}{10}, \quad x'(0) = -4.$$

The auxiliary equation is $0 = 2m^2 + 20m + 50 = 2(m+5)^2$ with roots $m = -5, -5$. Hence,

$$x(t) = (C_1 + C_2 t)e^{-5t}.$$

The initial conditions require

$$\frac{1}{10} = C_1, \quad -4 = -5C_1 + C_2 \qquad \Longrightarrow \qquad C_2 = -\frac{7}{2}.$$

The displacement of the mass is

$$x(t) = \left(\frac{1}{10} - \frac{7t}{2}\right)e^{-5t} \text{ m}.$$

(b) To find maximum displacement, we find when velocity is zero,

$$0 = x'(t) = -\frac{7}{2}e^{-5t} - 5\left(\frac{1}{10} - \frac{7t}{2}\right)e^{-5t} = \left(-4 + \frac{35t}{2}\right)e^{-5t}.$$

This implies that $t = 8/35$, and therefore maximum displacement is

$$x(8/35) = \left[\frac{1}{10} - \frac{7}{2}\left(\frac{8}{35}\right)\right]e^{-5(8/35)} = -\frac{7}{10}e^{-8/7} \text{ m.}$$

(b) When damping is removed, the differential equation becomes

$$2\frac{d^2x}{dt^2} + 50x = 0.$$

The auxiliary equation is $0 = 2m^2 + 50 = 2(m^2 + 25)$ with roots $m = \pm 5i$. The solution of the differential equation is

$$x(t) = C_1 \cos 5t + C_2 \sin 5t.$$

Resonance occurs when $\omega = 5$.

13 **7.** (a) Find the Laplace transform for the function in the figure below. Do not simplify your answer.

(b) Find the inverse Laplace transform for

$$F(s) = \frac{se^{-2s}}{s^2 + 4s + 7}.$$

(a) Since the function has period 2,

$$\begin{aligned}
F(s) &= \frac{1}{1 - e^{-2s}} \int_0^2 f(t)e^{-st}\,dt \\
&= \frac{1}{1 - e^{-2s}} \mathcal{L}\{2t^2[h(t) - h(t-1)] + 2[h(t-1) - h(t-2)]\} \\
&= \frac{1}{1 - e^{-2s}} \mathcal{L}\{2t^2 + (2 - 2t^2)h(t-1) - 2h(t-2)\} \\
&= \frac{1}{1 - e^{-2s}} \left[\frac{4}{s^3} + e^{-s}\mathcal{L}\{2 - 2(t+1)^2\} - \frac{2}{s}e^{-2s}\right] \\
&= \frac{1}{1 - e^{-2s}} \left[\frac{4}{s^3} + e^{-s}\mathcal{L}\{-4t - 2t^2\} - \frac{2}{s}e^{-2s}\right] \\
&= \frac{1}{1 - e^{-2s}} \left[\frac{4}{s^3} - e^{-s}\left(\frac{4}{s^2} + \frac{4}{s^3}\right) - \frac{2}{s}e^{-2s}\right].
\end{aligned}$$

(b) Consider

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+4s+7}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+2)-2}{(s+2)^2+3}\right\} = e^{-2t}\mathcal{L}^{-1}\left\{\frac{s-2}{s^2+3}\right\}$$
$$= e^{-2t}\left(\cos\sqrt{3}t - \frac{2}{\sqrt{3}}\sin\sqrt{3}t\right).$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{se^{-2s}}{s^2+4s+7}\right\} = e^{-2(t-2)}\left[\cos\sqrt{3}(t-2) - \frac{2}{\sqrt{3}}\sin\sqrt{3}(t-2)\right]h(t-2).$$

11 **8.** Find a general solution of the following initial value problem where $f(t)$ is some unspecified function of time,

$$y'' + 4y' - 5y = f(t), \quad y(0) = 1, \quad y'(0) = 0.$$

When we take Laplace transforms of both sides of the differential equation, we obtain

$$(s^2Y - s) + 4(sY - 1) - 5Y = F(s),$$

from which

$$Y(s) = \frac{s+4+F(s)}{s^2+4s-5} = \frac{s+4+F(s)}{(s+5)(s-1)} = \left(\frac{1/6}{s+5} + \frac{5/6}{s-1}\right) + \left(\frac{-1/6}{s+5} + \frac{1/6}{s-1}\right)F(s).$$

Since $\mathcal{L}^{-1}\left\{\frac{-1/6}{s+5} + \frac{1/6}{s-1}\right\} = -\frac{1}{6}e^{-5t} + \frac{1}{6}e^t$, we can use convolutions on the last term to write

$$y(t) = \frac{1}{6}e^{-5t} + \frac{5}{6}e^t + \int_0^t \left(-\frac{1}{6}e^{-5u} + \frac{1}{6}e^u\right) f(t-u)\,du.$$

16 **9.** A mass of 1 kilogram is suspended from a spring with constant 6 newtons per metre. At time $t = 0$, it is released from 5 centimetres below its equilibrium position. During its subsequent motion, it is subjected to a constant force of 4 newtons upward, and at time $t = 5$ seconds, it is struck upward with an instantaneous force of 3 newtons. Find its position as a function of time.

The initial-value problem for displacement $x(t)$ of the mass from equilibrium is

$$\frac{d^2x}{dt^2} + 6x = 4 + 3\delta(t-5), \quad x(0) = -\frac{1}{20}, \quad x'(0) = 0.$$

When we take Laplace transforms,

$$s^2X - \frac{s}{20} + 6X = \frac{4}{s} + 3e^{-5s},$$

from which

$$X(s) = \frac{\frac{s}{20} + \frac{4}{s} + 3e^{-5s}}{s^2+6} = \frac{s}{20(s^2+6)} + \frac{4}{s(s^2+6)} + \frac{3e^{-5s}}{s^2+6}$$
$$= \frac{s}{20(s^2+6)} + 4\left(\frac{1/6}{s} - \frac{s/6}{s^2+6}\right) + \frac{3e^{-5s}}{s^2+6}.$$

Inverse transforms give

$$x(t) = \frac{1}{20}\cos\sqrt{6}t + \frac{2}{3}\left(1 - \cos\sqrt{6}t\right) + \frac{3}{\sqrt{6}}\sin\sqrt{6}(t-5)h(t-5) \text{ m}.$$

**3** **10.** Use the definition of the Laplace transform to prove that when $f(t)$ has a Laplace transform $F(s)$, then

$$\mathcal{L}\{t\,f(t)\} = -F'(s).$$

The definition of the Laplace transform of $f(t)$ is

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

If we differentiate both sides with respect to $s$, we get

$$F'(s) = \int_0^\infty f(t)(-te^{-st})\,dt = -\int_0^\infty tf(t)e^{-st}\,dt = -\mathcal{L}\{tf(t)\}.$$

The following table of Laplace transforms may be used without proof.

| $f(t)$ | | $F(s) = \mathcal{L}\{f(t)\}$ |
|---|---|---|
| $t^n \quad (n = 0, 1, 2, \ldots)$ | $\leftrightarrow$ | $\dfrac{n!}{s^{n+1}}$ |
| $e^{at}$ | $\leftrightarrow$ | $\dfrac{1}{s-a}$ |
| $\sin at$ | $\leftrightarrow$ | $\dfrac{a}{s^2+a^2}$ |
| $\cos at$ | $\leftrightarrow$ | $\dfrac{s}{s^2+a^2}$ |
| $h(t-a)$ | $\leftrightarrow$ | $\dfrac{e^{-as}}{s}$ |
| $\delta(t-a)$ | $\leftrightarrow$ | $e^{-as}$ |
| $e^{at}f(t)$ | $\leftrightarrow$ | $F(s-a)$ |
| $f(t)h(t-a)$ | $\rightarrow$ | $e^{-as}\mathcal{L}\{f(t+a)\}$ |
| $f(t-a)h(t-a)$ | $\leftarrow$ | $e^{-as}F(s)$ |
| $p-$periodic $f(t)$ | $\rightarrow$ | $\dfrac{1}{1-e^{-ps}}\displaystyle\int_0^p e^{-st}f(t)\,dt$ |
| $\displaystyle\int_0^t f(u)g(t-u)\,du$ | $\leftarrow$ | $F(s)G(s)$ |
| $f'(t)$ | $\rightarrow$ | $sF(s)-f(0)$ |
| $f''(t)$ | $\rightarrow$ | $s^2F(s)-sf(0)-f'(0)$ |