Derivatives of trigonometric functions

3.9.1

(a)

$$\frac{dy}{dx} = \sec x \cdot \tan x \cdot \cos x^3 + \sec x \cdot (-\sin x^3) \cdot (3x^2)$$

(b)

Implicit differentiation: $\frac{d}{dx}(y \cdot \tan x^2) = \frac{d}{dx}(x \cdot \tan y^2)$

$$\frac{dy}{dx} \cdot \tan x^2 + y \cdot \sec^2 x^2 \cdot (2x) = \tan y^2 + x \cdot \sec^2 y^2 \cdot 2y \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\tan y^2 - 2xy \cdot \sec^2 x^2}{\tan x^2 - 2xy \cdot \sec^2 y^2}$$

3.9.2

Differentiate the equation for $f(x)$ w.r.t. x:

$$(f(x) + x \cdot \cos(f(x)))' = (x^2)' = 2x$$

$$f'(x) + \cos(f(x)) + x \cdot (-\sin(f(x))) \cdot f'(x) = 2x \Big|_{x=1}$$

$$f'(1) + \cos(f(1)) + 1 \cdot (-\sin(f(1))) \cdot f'(1) = 2 \cdot 1$$

Because $f(1)=0$, $\cos(f(1)) = 1$ and $\sin(f(1)) = 0$, so

$$f'(1) + 1 = 2 \Rightarrow \boxed{f'(1) = 1}$$

3.9.3

$$f(x) = \sin(2x)$$

$$f'(x) = \cos(2x) \cdot 2$$

$$f''(x) = -\sin(2x) \cdot 2^2$$

$$f'''(x) = -\cos(2x) \cdot 2^3$$

$$f^{IV}(x) = \sin(2x) \cdot 2^4$$

. . .

From the pattern, $\boxed{f^{(21)}(x) = \cos(2x) \cdot 2^{21}.}$

3.9.4

(a)

$$\lim_{x\to 0} \frac{\cancel{\tan}\sin 6x}{(\cos 6x)\sin(2x)} = \lim_{x\to 0} \frac{\frac{\sin 6x}{2x}}{\cos(6x)\frac{\sin 2x}{2x}} = \lim_{x\to 0} \frac{\sin 6x}{2x\cos(6x)} = \lim_{x\to 0} \frac{3\left(\frac{\sin 6x}{6x}\right)}{\cos 6x} = \frac{3\,(1)}{1} = 3$$

(b)

$$\lim_{x\to 1} \frac{(x-1)(x+2)}{\sin(x-1)} = \lim_{x\to 1} \frac{x-1}{\sin(x-1)} \lim_{x\to 1}(x+2) = 1\,(1+2) = 3$$