MATH 2132 Problem Workshop 3

1. Find the open interval of convergence for the power series:

(a) $\sum\limits_{n=3}^{\infty} \frac{n3^n}{n^2+1} x^{2n+3}$

Solution:

$$\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \frac{\dfrac{n3^n}{n^2+1}}{\dfrac{(n+1)3^{n+1}}{(n+1)^2+1}} \\
&= \lim_{n\to\infty} \frac{n(n^2+2n+2)}{3(n+1)(n^2+1)} \\
&= \frac{1}{3}
\end{aligned}$$

Therefore the open interval of convergence is $|x^2| < \frac{1}{3} \to -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$.

(b) $\sum\limits_{n=0}^{\infty} (-1)^{n+1} \sqrt{\frac{2n+3}{n+6}} \ln(n+6)(x+2)^n$

Solution:

$$\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1}\sqrt{\dfrac{2n+3}{n+6}}\ln(n+6)}{(-1)^{n+2}\sqrt{\dfrac{2n+5}{n+7}}\ln(n+7)} \right| \\
&= \lim_{n\to\infty} \sqrt{\frac{(2n+3)(n+7)}{(n+6)(2n+5)}}\frac{\ln(n+6)}{\ln(n+7)} \\
&= \lim_{n\to\infty} \sqrt{1}(1) \\
&= 1
\end{aligned}$$

Noting that by L'Hopital's Rule, $\lim_{t\to\infty} \dfrac{\ln(t+6)}{\ln(t+7)} = \lim_{t\to\infty} \dfrac{1/(t+6)}{1/(t+7)} = 1$, using that the limit was an ∞/∞ form.
Therefore the open interval of convergence is $|x+2| < 1 \to -3 < x < -1$.

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{n!}{(3n)!}(3x-1)^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \\
&= \lim_{n\to\infty} \frac{\frac{n!}{(3n)!}}{\frac{(n+1)!}{(3n+3)!}} \\
&= \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)}{n+1)} \\
&= \infty
\end{aligned}
$$

Therefore the open interval of convergence is $|3x-1| < \infty \to -\infty < x < \infty$.

2. Find the interval of convergence for the power series

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(3n)4^n}{n+1}(x+1)^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \lim_{n\to\infty}\frac{\frac{(3n)4^n}{n+1}}{\frac{(3n+3)4^{n+1}}{n+2}} \\
&= \lim_{n\to\infty}\frac{(3n)(n+2)}{4(n+1)(3n+3)} \\
&= \frac{1}{4}.
\end{aligned}
$$

Therefore the open interval of convergence is $|x+1| < \frac{1}{4} \to -\frac{5}{4} < x < -\frac{3}{4}$.

Testing the endpoint, $x = -\frac{5}{4}$ we get the series

$\sum_{n=1}^{\infty}\frac{(-1)^n(3n)}{n+1}$. Since $\lim_{n\to\infty}\frac{3n}{n+1} = 3 \neq 0$, the terms diverge. Therefore the series diverges.

Testing the endpoint, $x = -\frac{3}{4}$ we get the series

$\sum_{n=1}^{\infty}\frac{3n}{n+1}$. Since $\lim_{n\to\infty}\frac{3n}{n+1} = 3 \neq 0$, the series diverges.

Therefore the interval of convergence is $-\frac{5}{4} < x < -\frac{3}{4}$.

(b) $\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{5^n}(2x-3)^n$

Solution:

$$
\begin{aligned}
R &= \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| \\
&= \lim_{n\to\infty}\left|\frac{\frac{(-1)^n 3^n}{5^n}}{\frac{(-1)^{n+1}3^{n+1}}{5^{n+1}}}\right| \\
&= \lim_{n\to\infty}\frac{5}{3} \\
&= \frac{5}{3}.
\end{aligned}
$$

Therefore the open interval of convergence is $|2x-3| < \frac{5}{3} \to \frac{2}{3} < x < \frac{7}{3}$.

Testing the endpoint, $x = \frac{2}{3}$ we get the series

$\sum_{n=1}^{\infty} 1$. Since $\lim_{n\to\infty} 1 = 1 \neq 0$, the series diverges.

Testing the endpoint, $x\frac{7}{3}$ we get the series

$\sum_{n=1}^{\infty} (-1)^n$. Since this is a geometric series with $|r| = 1 \geq 1$, the series diverges.

Therefore the interval of convergence is $\frac{2}{3} < x < \frac{7}{3}$.

3. (a) Find the Maclaurin series for the function $f(x) = x/(4x+1)$. Express your answer in sigma notation, simplifying as much as possible. What is the interval of convergence of the series.

Solution:

First finding the Maclaurin series for $1/(4x+1)$

$$\frac{1}{1+4x} = \frac{1}{1-(-4x)} = \sum_{n=0}^{\infty}(-4x)^n$$

with interval $|-4x| < 1 \Rightarrow -\frac{1}{4} < x < \frac{1}{4}$.

Therefore the Maclaurin series of $f(x)$ is

$$\sum_{n=0}^{\infty}(-1)^n 4^n x^{n+1}$$

with interval of convergence $-\frac{1}{4} < x < \frac{1}{4}$.

(b) Repeat part (a), but find the Taylor series about $x = 1$.

Solution:

Let $y = x-1$, then the function is $f(y) = \frac{y+1}{4y+5}$. First finding the Maclaurin series of $\frac{1}{4y+5}$.

$$\frac{1}{5+4y} = \frac{1}{5(1-(-4y/5))} = \frac{1}{5}\sum_{n=0}^{\infty}(-4y/5)^n$$

with interval $|-4y/5| < 1 \Rightarrow -\frac{5}{4} < y < \frac{5}{4}$.

Hence

$$\begin{aligned}
\frac{y+1}{4y+5} &= \sum_{n=0}^{\infty}(-1)^n \frac{4^n}{5^{n+1}} y^n + \sum_{n=0}^{\infty}(-1)^n \frac{4^n}{5^{n+1}} y^{n+1} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{4^n}{5^{n+1}} y^n + \sum_{n=1}^{\infty}(-1)^{n-1} \frac{4^{n-1}}{5^n} y^n \\
&= \frac{1}{5} + \sum_{n=1}^{\infty}\left[(-1)^n \frac{4^n}{5^{n+1}} + (-1)^{n-1}\frac{4^{n-1}}{5^n}\right] y^n \\
&= \frac{1}{5} + \sum_{n=1}^{\infty}(-1)^{n-1} \frac{4^{n-1}}{5^{n+1}} y^n
\end{aligned}$$

Replacing $y = x - 1$ in the series and interval, we get

$$\frac{1}{5} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{4^{n-1}}{5^{n+1}}(x-1)^n$$

with interval $-\frac{1}{4} < x < \frac{9}{4}$.

4. Find the Maclaurin series of $\sin^2 2x$. What is its interval of convergence?

Solution:

$$\sin^2(2x) = \frac{1-\cos(4x)}{2} = \frac{1}{2} - \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(4x)^{2n} = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{(-1)^{n+1}2^{4n-1}}{(2n)!}x^{2n}$$

with interval $-\infty < 4x < \infty \Rightarrow -\infty < x < \infty$.

5. Find the Taylor series about $x = 5$ for $\ln(3+x)$. What is its open interval of convergence?

Solution:

Let $y = x - 5$, then we are finding the Maclaurin series of $f(y) = \ln(8 + y)$.

$$f'(y) = \frac{1}{8+y} = \frac{1}{8(1+y/8)} = \sum_{n=0}^{\infty} \frac{1}{8}\left(-\frac{y}{8}\right)^n$$

with open interval $|-y/8| < 1 \Rightarrow -8 < y < 8$.

Thus

$$f(y) = C + \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+1}} \frac{y^{n+1}}{n+1}$$

Letting $y = 0$ we get $\ln 8 = C$ and thus

$$f(y) = \ln 8 + \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+1}} \frac{y^{n+1}}{n+1}.$$

Hence the Taylor series is

$$f(y) = \ln 8 + \sum_{n=0}^{\infty} \frac{(-1)^n}{8^{n+1}(n+1)}(x-5)^{n+1}$$

with interval $-3 < x < 13$.

6. Find the Maclaurin series of $f(x) = 1/(4+3x)^2$. What is its interval of convergence?

Solution:

We know

$$\frac{1}{4+3x} = \frac{1}{4(1-(-3x/4)} = \sum_{n=0}^{\infty} \frac{1}{4}\left(-\frac{3x}{4}\right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{4^{n+1}} x^n$$

with open interval $|-3x/4| < 1 \Rightarrow -\dfrac{4}{3} < x < \dfrac{4}{3}$.

Differentiating both sides yields

$$\frac{-3}{(4+3x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^n n 3^n}{4^{n+1}} x^{n-1}$$

and then dividing by -3 yields

$$\frac{1}{(4+3x)^2} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n 3^{n-1}}{4^{n+1}} x^{n-1}$$

with open interval $|-3x/4| < 1 \Rightarrow -\frac{4}{3} < x < \frac{4}{3}$.

7. Find the Maclaurin series for the function $\tan^{-1}(2x^2)$. Express your answer in sigma notation, simplifying as much as possible. What is the open interval of convergence of the series.

Solution:

$$f'(x) = \frac{4x}{1+4x^4} = 4x \sum_{n=0}^{\infty} (-4x^4)^n + \sum_{n=0}^{\infty} 4^{n+1} (-1)^n x^{4n+1}$$

with open interval $|-4x^4| < 1 \Rightarrow -\frac{1}{2} < x < \frac{1}{2}$.

Thus

$$f(x) = C + \sum_{n=0}^{\infty} \frac{4^{n+1}(-1)^n x^{4n+2}}{4n+2} = C + \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^n x^{4n+2}}{2n+1}$$

Letting $x = 0$ we get $0 = C$ and thus

$$f(x) = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^n x^{4n+2}}{2n+1}$$

with interval $-\frac{1}{2} < x < \frac{1}{2}$.

8. Find the Taylor series for the function $1/\sqrt{10-3x}$ about $x = 2$. Express your answer in sigma notation, simplifying as much as possible. What is the radius of convergence of the series.

Solution:

Let $y = x - 2$

$f(y) = 1/\sqrt{10-3x} = 1/\sqrt{4-3y}$ can be re-written as $\frac{1}{2}(1-3y/4)^{-1/2}$. so using the binomial series, we get

$$\frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1/2)(-3/2)(-5/2)\cdots(-1/2-n+1)}{n!}\bigg(-\frac{3}{4}y\bigg)^n$$

with interval of convergence $|3/4| < 1 \Rightarrow R = \dfrac{4}{3}$

Simplifying the series

$$\begin{aligned}
f(x) &= \frac{1}{2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{(-1)^n(1)(3)(5)\cdots(2n-1)}{2^n n!}\bigg((-1)^n\frac{3^n}{2^{2n}}y^n\bigg) \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{3^n(2n)!}{2^{3n+1}n!(2)(4)\cdots(2n)}y^n \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{3^n(2n)!}{2^{4n+1}n!n!}y^n\bigg) \\
&= \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(2n)!3^n}{2^{4n+1}(n!)^2})(x-2)^n
\end{aligned}$$

9. Find the Maclaurin series for the function $(x^2+2)/(x+3)^2$. Express your answer in sigma notation, simplifying as much as possible. What is the open interval of convergence of the series.

Solution:

We know

$$\frac{1}{3+x}=\frac{1}{3(1-(-x/3))}=\sum_{n=0}^{\infty}\frac{1}{3}\bigg(-\frac{x}{3}\bigg)^n=\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}}x^n$$

with open interval $|-x/3| < 1 \Rightarrow -3 < x < 3$.

Differentiating both sides yields

$$\frac{-1}{(x+3)^2}=\sum_{n=1}^{\infty}\frac{(-1)^n n}{3^{n+1}}x^{n-1}$$

and then dividing by -1 yields

$$\frac{1}{(x+3)^2}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n}{3^{n+1}}x^{n-1}$$

with open interval $-3 < x < 3$.

Multiplying by $x^2 + 2$ yields

$$\begin{aligned}
\frac{x^2+2}{(x+3)^2} &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{3^{n+1}} x^{n+1} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{3^{n+1}} x^{n-1} \\
&= \sum_{n=3}^{\infty} \frac{(-1)^{n-1}(n-2)}{3^{n-1}} x^{n-1} + 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{3^{n+1}} x^{n-1} \\
&= \frac{2}{9} - \frac{4}{27}x + \sum_{n=3}^{\infty} (-1)^{n-1} \left[\frac{(n-2)}{3^{n-1}} + \frac{2n}{3^{n+1}} \right] x^{n-1} \\
&= \frac{2}{9} - \frac{4}{27}x + \sum_{n=3}^{\infty} (-1)^{n-1} \left[\frac{11n-18}{3^{n+1}} \right] x^{n-1} \\
&= \frac{2}{9} + \sum_{n=2}^{\infty} (-1)^{n} \left[\frac{11n-18}{3^{n+1}} \right] x^{n-1}
\end{aligned}$$

10. Find the sum of the series. Include the open interval of convergence.

(a) $\sum\limits_{n=0}^{\infty} \frac{1}{n+2} x^n$

Solution:

Let $f(x)$ be the series. First note that if $x = 0$ we get $f(0) = \frac{1}{2}$.

We want to get rid of the $n+2$ term in the denominator. We can do that by differentiating, but we would need an x^{n+2} term to do it.

Hence $f(x) = x^{-2} \sum\limits_{n=0}^{\infty} \frac{1}{n+2} x^{n+2}$

Note that we have a problem when $x = 0$ which is why we did that one separately.

Let $g(x)$ be the second series. We solve for $g(x)$

$g'(x) = \sum\limits_{n=0}^{\infty} x^{n+1} = \frac{x}{1-x}$

Integrating gives

$$g(x) = \int \frac{x}{1-x}\, dx = \int \left(-1 - \frac{1}{1-x} \right) dx = -x - \ln(1-x) + C$$

When $x = 0, g(0) = 0$ and thus $C = 0$. Hence $g(x) = -x - \ln(1-x)$

Thus $f(x) = -\frac{1}{x} - \frac{\ln(1-x)}{x^2}$ for $x \neq 0$. $f(0) = \frac{1}{2}$. Open interval of convergence is $(-1, 1)$.

Alternatively, we could note that the series looks similar to the $\ln x$ series. With some manipulations:

$\ln y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(y-1)^n$ (int: $(0, 2)$)

Replacing x with $y - 1$ we get

$\ln(x+1) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(x)^n$ (int: $(-1, 1)$) and replacing x with $-x$ we get

$\ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(-x)^n = -\sum_{n=1}^{\infty} \frac{1}{n}x^n$

Now this looks close to what we want, with two exceptions. The n in the denominator can be turned into $m + 2$ with a substitution. $n = m + 2$

$\ln(1-x) = -\sum_{m=-1}^{\infty} \frac{1}{m+2}x^{m+2}$

The starting point can be changed to $m = 0$ by splitting the first term.

$\ln(1-x) = -x - \sum_{m=0}^{\infty} \frac{1}{m+2}x^{m+2} = -x - x^2 f(x)$

Thus $f(x) = -\frac{1}{x} - \frac{\ln(1-x)}{x^2}$ for $x \neq 0$. $f(0) = \frac{1}{2}$. Open interval of convergence is $(-1, 1)$.

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n n}{(2n)!}x^{2n}$

Solution:

Let $f(x) = \sum_{n=2}^{\infty} \frac{(-1)^n n}{(2n)!}x^{2n}$ be the series. The issue is the n term. If we integrate the x^{2n} term we get a $2n + 1$ term in the bottom. However we want n term. If we factored out an x then that would leave us with x^{2n-1} which integrate would give a $2n$ in the denominator which can cancel the n.

Hence

$$f(x) = x\sum_{n=2}^{\infty} \frac{(-1)^n n}{(2n)!}x^{2n-1}$$

Call the remaining series $g(x)$.

$$\int g(x)\,dx = \sum_{n=2}^{\infty} \frac{(-1)^n n}{(2n)(2n)!}x^{2n} = \frac{1}{2}\sum_{n=2}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = \frac{1}{2}(\cos x - 1 + x^2/2)$$

Solving for $g(x)$ by differentiating leads to

$$g(x) = \frac{1}{2}(-\sin x + 2x) \Rightarrow f(x) = xg(x) = -\frac{1}{2}x\sin x + \frac{x^2}{2}.$$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{n} x^{2n+1}$

Solution:
Similar to the last example, we want to differentiate to remove the n in the denominator, but we can't because we have x^{2n+1} instead of x^{2n}.
$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{n} x^{2n+1} = xg(x)$ where $g(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}}{n} x^{2n}$.

$$g'(x) = \sum_{n=1}^{\infty} 2(-1)^n 2^{2n} x^{2n-1}$$

which is geometric. Hence

$$g'(x) = \frac{-8x}{1+4x^2}$$

with interval $|-4x^2| < 1 \Rightarrow -1/2 < x < 1/2$.
Integrating using a substitution of $u = 1 + 4x^2$ leads to

$$g(x) = -\ln(1+4x^2) + C$$

Plugging in $x = 0$ leads to $g(0) = 0$, and so $C = 0$.
Hence $f(x) = -x\ln(1+4x^2)$ with interval $-1/2 < x < 1/2$.

(d) $\sum_{n=1}^{\infty} (2n+1)(x-1)^n$

Solution:
We want to integrate. The problem is we need a power of $2n$ to cancel the $2n+1$. Hence we substitute.
Let $y^2 = x - 1$. Then the series becomes
$g(y) = \sum_{n=1}^{\infty} (2n+1) y^{2n}$.
Integrating leads to

$$\int g(y)\, dy = \sum_{n=1}^{\infty} y^{2n+1} = \frac{y^3}{1-y^2}$$

since the series is geometric. The interval is $|y^2| < 1 \Rightarrow -1 < y < 1$.
Differentiating to solve for $g(y)$ leads to

$$g(y) = \frac{3y^2(1-y^2) - y^3(-2y)}{1-y^2)^2} = \frac{3y^2 - y^4)}{1-y^2)^2}$$

Replacing $y^2 = x - 1$ leads to
$f(x) = \dfrac{3(x-1) - (x-1)^2}{1-(x-1)^2} = \dfrac{-x^2+5x-4}{(2-x)^2}$ with interval $0 < x < 2$.

11. Use series to evaluate $\displaystyle\lim_{x\to 0} \frac{\sqrt{3+4x^3} - \sqrt{3}}{x^3}$.

Solution:

Finding the binomial series for the square root.
$\sqrt{3+4x^3}$ can be re-written as $\sqrt{3}(1+4x^3/3)^{1/2}$. so using the binomial series, we get

$$\sqrt{3} + \sqrt{3}\sum_{n=1}^{\infty} \frac{(1/2)(-1/2)(-3/2)(-5/2)\cdots(1/2-n+1)}{n!}\left(\frac{4}{3}x^3\right)^n$$

Thus

$$\frac{\sqrt{3+4x^3} - \sqrt{3}}{x^3} = \sqrt{3}\sum_{n=1}^{\infty} 4^n \frac{(1/2)(-1/2)(-3/2)(-5/2)\cdots(1/2-n+1)}{3^n n!} x^{3n-3}$$

The only term which doesn't go to zero as $x \to 0$ is the first term. Hence the limit is
$\dfrac{\sqrt{3}(4)(1/2)}{3} = \dfrac{2\sqrt{3}}{3}$.