MATH 2130 – Tutorial Problem Solutions, Thu Mar 29

Applications of Double Integrals

Example. Use polar coordinates to find the second moment of area of a disk of radius R about a line tangent to the disk.

Solution. Choose xy-axes so that the center of the disk is at the origin. Then the disk is given by $x^2 + y^2 \le R^2$. By rotating about the origin if necessary, assume that the equation of the tangent line is $x = -R$.

The perpendicular distance between the line $x = -R$ and any point (x, y) within the disk is $x + R$. Therefore the second moment of area about the line is

$$I = \iint_{disk} (x+R)^2 \, dA.$$

Now we convert to polar coordinates. The disk lies within $0 \le \theta \le 2\pi$ and $0 \le r \le R$. In polar coordinates, $x = r\cos\theta$, so

$$(x+R)^2 = (r\cos\theta + R)^2 = r^2\cos^2\theta + 2Rr\cos\theta + R^2.$$

Thus

$$\begin{aligned} I &= \int_0^{2\pi}\int_0^R (r^2\cos^2\theta + 2Rr\cos\theta + R^2) r \, dr \, d\theta \\ &= \int_0^{2\pi}\int_0^R r^3\cos^2\theta \, dr \, d\theta + 2R\int_0^{2\pi}\int_0^R r^2\cos\theta \, dr \, d\theta \\ &\quad + R^2\int_0^{2\pi}\int_0^R r \, dr \, d\theta. \end{aligned}$$

The third term in the sum is

$$R^2\int_0^{2\pi}\int_0^R r \, dr \, d\theta = R^2\int_0^{2\pi}\frac{1}{2}R^2 \, d\theta = \pi R^2.$$

The second term is

$$2R\int_0^{2\pi}\int_0^R r^2\cos\theta \, dr \, d\theta = 2R\int_0^R r^2\int_0^{2\pi}\cos\theta \, d\theta \, dr = 0,$$

since $\int_0^{2\pi}\cos\theta \, d\theta = 0$. Lastly, the first term is

$$\begin{aligned} \int_0^{2\pi}\int_0^R r^3\cos^2\theta \, dr \, d\theta &= \int_0^{2\pi}\frac{1}{4}R^4\cos^2\theta \, d\theta \\ &= \frac{1}{4}R^4\int_0^{2\pi}\frac{\cos 2\theta + 1}{2} \, d\theta \\ &= \frac{\pi}{4}R^4. \end{aligned}$$

Thus the second moment of area is

$$I = \pi R^4 + \frac{\pi}{4}R^4 = \frac{5\pi}{4}R^4.$$

This integral can also be performed in Cartesian coordinates, but one of the terms requires a trig substitution.

Example. Find the area enclosed by the curve $r = 1 + \cos\theta$. Then find its centroid.

Solution. If $r = 1 + \cos\theta$, then r takes values between 0 and 2.

- The positive x-axis corresponds to $\theta = 0$, in which case $r = 2$. Therefore the point $(x = 2, y = 0)$ is on the curve.
- The positive y-axis corresponds to $\theta = \frac{\pi}{2}$, in which case $r = 1$. Similarly, the negative y-axis corresponds to $\theta = \frac{3\pi}{2}$, in which case $r = 1$. Therefore the points $(x = 0, y = \pm 1)$ are on the curve.
- The negative x-axis corresponds to $\theta = \pi$, in which case $r = 0$. Therefore the point $(0, 0)$ is on the curve.

When these points are connected, they form a closed curve that bends inward to create a cusp at the origin. This is the region R whose area we are asked to find.

The region lies within $0 \le \theta \le 2\pi$. At each value of θ, $0 \le r \le 1 + \cos\theta$. Thus the area of the region is

$$\begin{aligned}
A =& \iint_R dA \\
=& \int_0^{2\pi} \int_0^{1+\cos\theta} r\, dr\, d\theta \\
=& \int_0^{2\pi} \frac{1}{2}(1 + \cos\theta)^2\, d\theta \\
=& \frac{1}{2}\int_0^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta \\
=& \frac{1}{2}\left(2\pi + 0 + \pi\right) \\
=& \frac{3\pi}{2}.
\end{aligned}$$

We could also say, by symmetry, that the total area is twice the area lying within $0 \le \theta \le \pi$.

The centroid is the center of mass in the special case where the density of the region is 1 everywhere. Therefore the centroid is the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{1}{A}\iint_R x\, dA, \quad \overline{y} = \frac{1}{A}\iint_R y\, dA.$$

Since the region is symmetric about the x-axis, its centroid must lie on the x-axis: that is, $\overline{y} = 0$.

To find $\overline{x}$, we must evaluate

$$\begin{aligned}
\iint_R x\, dA =& \int_0^{2\pi} \int_0^{1+\cos\theta} r\cos\theta r\, dr\, d\theta \\
=& \int_0^{2\pi} \frac{1}{3}(1 + \cos\theta)^3 \cos\theta\, d\theta \\
=& \frac{1}{3}\int_0^{2\pi} \left(\cos\theta + 3\cos^2\theta + 3\cos^3\theta + \cos^4\theta\right)\, d\theta.
\end{aligned}$$

The terms containing an odd power of $\cos\theta$ must vanish. The remaining terms are

$$\frac{1}{3}\int_0^{2\pi} 3\cos^2\theta\, d\theta = \pi,$$

and

$$\begin{aligned}\frac{1}{3}\int_0^{2\pi} \cos^4\theta\, d\theta =& \frac{1}{3}\int_0^{2\pi}\left(\frac{\cos 2\theta + 1}{2}\right)^2 d\theta \\ =& \frac{1}{12}\int_0^{2\pi}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) d\theta \\ =& \frac{\pi}{6} + \frac{1}{12}\int_0^{2\pi}\left(\frac{1+\cos 4\theta}{2}\right) d\theta \\ =& \frac{\pi}{6} + \frac{\pi}{12} = \frac{\pi}{4}.\end{aligned}$$

Therefore

$$\overline{x} = \frac{1}{A}\left(\pi + \frac{\pi}{4}\right) = \frac{2}{3\pi}\frac{5\pi}{4} = \frac{5}{6}.$$

Triple Integrals

Example. Set up a triple iterated integral for the integral of a continuous function $f(x, y, z)$ over the volume bounded by the surfaces $2z = x^2 + y^2$ and $2x = y^2 + z^2$.

Solution. (This will not make much sense unless you sketch the diagrams.) The surface $2z = x^2+y^2$ is a paraboloid opening in the $+z$-direction, and the surface $2x = y^2 + z^2$ is a paraboloid opening in the $+x$-direction. The volume they enclose has $x \geq 0$ and $z \geq 0$, and is symmetric under $y \mapsto -y$.

Let's try different values of y. If $y = 0$, then in the xz-plane we have the parabolas $2z = x^2$ (absolute minimum $z = 0$) and $2x = z^2$ (absolute minimum $x = 0$). They intersect at the points $(0, 0)$ and $(2, 2)$.

If $y > 0$, the parabolas in the xz-plane become $2z = x^2 + y^2$ (absolute minimum $z = \frac{1}{2}y^2$) and $2x = z^2 + y^2$ (absolute minimum $x = \frac{1}{2}y^2$). That is, as y increases, the parabolas move further apart. Eventually, when y is large enough, they will not intersect at all. That value of y is a boundary for the volume.

To find the maximum value of y, we solve the system $2z = x^2 + y^2$, $2x = z^2 + y^2$ for x and z, viewing y as a constant. Subtracting one equation from the other and rearranging yields

$$z^2 + 2z = x^2 + 2x.$$

Adding 1 to both sides lets us complete the square:

$$(z + 1)^2 = (x + 1)^2,$$

which implies that $z + 1 = x + 1$ or $z + 1 = -(x + 1)$. That is, $z = x$ or $z = -x - 2$.

Substitute $z = x$ into the first equation:

$$x^2 - 2x + y^2 = 0,$$

which has roots

$$x = \frac{2 \pm \sqrt{4 - 4y^2}}{2} = 1 \pm \sqrt{1 - y^2}.$$

A root exists as long as $y^2 \le 1$. Thus $-1 \le y \le 1$.

Now substitute $z = -x - 2$ into the first equation:

$$x^2 + 2x + y^2 + 4 = 0,$$

which has roots

$$x = \frac{-2 \pm \sqrt{4 - 4(y^2 + 4)}}{2} = -1 \pm \sqrt{-y^2 - 3}.$$

These are not real numbers for any values of y.

Conclusion: when $-1 \le y \le 1$, the points of intersection between the parabolas occur at $x = z = 1 + \sqrt{1 - y^2}$ and $x = z = 1 - \sqrt{1 - y^2}$.

Fix a value of y, $-1 \le y \le 1$. Then $1 - \sqrt{1 - y^2} \le x \le 1 + \sqrt{1 - y^2}$. At each value of x, z is bounded below by the upward-opening parabola, and bounded above by the rightward-opening parabola. That is, $\frac{1}{2}(x^2 + y^2) \le z \le \sqrt{2x - y^2}$. Finally, a triple iterated integral over this volume is

$$\int_{-1}^{1} \int_{1-\sqrt{1-y^2}}^{1+\sqrt{1-y^2}} \int_{(x^2+y^2)/2}^{\sqrt{2x-y^2}} f(x, y, z)\, dz\, dx\, dy.$$