MATH 2130 – Tutorial Problem Solutions, Thu Jan 8

Multivariable Limits

Example. Evaluate the limit

$$\lim_{(x,y)\to(1,2)} \frac{x^2-2x+2y^2-8y+9}{3x^2-6x-y^2+4y-1},$$

or show that it does not exist.

Solution. The substitution $(x,y)=(1,2)$ yields 0 in the numerator and denominator.
We complete the square everywhere:

$$\begin{aligned}\frac{x^2-2x+2y^2-8y+9}{3x^2-6x-y^2+4y-1} &= \frac{(x-1)^2-1+2(y-2)^2-8+9}{3(x-1)^2-3-(y-2)^2+4-1}\\ &= \frac{(x-1)^2+2(y-2)^2}{3(x-1)^2-(y-2)^2}.\end{aligned}$$

Consider the two paths $(1,y)$ as $y\to 2$ and $(x,2)$ as $x\to 1$. Using the first path, we get

$$\lim_{y\to 2}\frac{2(y-2)^2}{-(y-2)^2} = -2,$$

and using the second path, we get

$$\lim_{x\to 1}\frac{(x-1)^2}{3(x-1)^2} = \frac{1}{3}.$$

Since the two paths yield different limits, we conclude that the limit does not exist.

Example. Evaluate the limit

$$\lim_{(x,y)\to(1,-1)} \frac{x^2-y^2-2x-2y}{\sqrt{x-2y+1}-\sqrt{2x-y+1}},$$

or show that it does not exist.

Solution. Upon substitution, we get 0 in the numerator and denominator.
Rationalize the denominator:

$$\begin{aligned}\frac{x^2-y^2-2x-2y}{\sqrt{x-2y+1}-\sqrt{2x-y+1}} &= \frac{x^2-y^2-2x-2y}{\sqrt{x-2y+1}-\sqrt{2x-y+1}}\,\frac{\sqrt{x-2y+1}+\sqrt{2x-y+1}}{\sqrt{x-2y+1}+\sqrt{2x-y+1}}\\ &= \frac{x^2-y^2-2x-2y}{x-2y+1-2x+y-1}\left(\sqrt{x-2y+1}+\sqrt{2x-y+1}\right)\\ &= \frac{x^2-y^2-2x-2y}{-x-y}\left(\sqrt{x-2y+1}+\sqrt{2x-y+1}\right)\\ &= \frac{(x+y)(x-y)-2(x+y)}{-(x+y)}\left(\sqrt{x-2y+1}+\sqrt{2x-y+1}\right)\\ &= (y-x+2)\left(\sqrt{x-2y+1}+\sqrt{2x-y+1}\right).\end{aligned}$$

Therefore

$$\begin{aligned}
& \lim_{(x,y)\to(1,-1)} \frac{x^2-y^2-2x-2y}{\sqrt{x-2y+1}-\sqrt{2x-y+1}} \\
= & \lim_{(x,y)\to(1,-1)} (y-x+2)\left(\sqrt{x-2y+1}+\sqrt{2x-y+1}\right) \\
= & \ 0.
\end{aligned}$$

Example. (a) Evaluate

$$\lim_{(x,y)\to(2,-1)} \frac{\sin(x+2y)}{x+2y}.$$

Solution. Recall the one-dimensional limit

$$\lim_{z\to 0} \frac{\sin z}{z} = 1.$$

Let $F(z) = \dfrac{\sin z}{z}$, and observe that $\dfrac{\sin(x+2y)}{(x+2y)} = F(x+2y)$. Since $x+2y \to 0$ as $(x,y) \to (2,-1)$, we get

$$\lim_{(x,y)\to(2,-1)} \frac{\sin(x+2y)}{x+2y} = \lim_{z\to 0} \frac{\sin z}{z} = 1.$$

(b) Consider the limit

$$\lim_{(x,y)\to(0,0)} \frac{\cos(x+2y)-1}{x+y}.$$

Verify that all paths of the form $y = mx$ yield the same limit. Do you think that this limit exists?

Solution. Recall the one-dimensional limit

$$\lim_{z\to 0} \frac{\cos z - 1}{z} = 0.$$

Let $y = mx$ for some $m \in \mathbb{R}$, $m \neq -1$ (since the line $y = -x$ is not in the domain of the function). Then we get

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(x+2mx)-1}{x+mx} &= \lim_{x\to 0} \frac{\cos[(2m+1)x]-1}{(m+1)x} \\
&= \left(\frac{2m+1}{m+1}\right) \lim_{x\to 0} \frac{\cos[(2m+1)x]-1}{(2m+1)x} \\
&= \frac{2m+1}{m+1} \cdot 0 = 0.
\end{aligned}$$

All linear paths agree.

The limit does not exist, but finding a path to prove it takes some work. Try $y = x^2 - x$. Then $x + y = x^2$ and $x + 2y = 2x^2 - x$. We get

$$\begin{aligned}
\lim_{x\to 0} \frac{\cos(2x^2-x)-1}{x^2} &= \lim_{x\to 0} \frac{-(4x-1)\sin(2x^2-x)}{2x} \quad \text{by l'Hôpital's rule} \\
&= \lim_{x\to 0} \frac{-4\sin(2x^2-x)-(4x-1)^2\cos(2x^2-x)}{2} \quad \text{by l'Hôpital's rule again} \\
&= -\frac{1}{2}.
\end{aligned}$$

We have found a path that does not yield a limit of 0. Therefore the multivariable limit does not exist, even though we couldn't see this using linear paths.

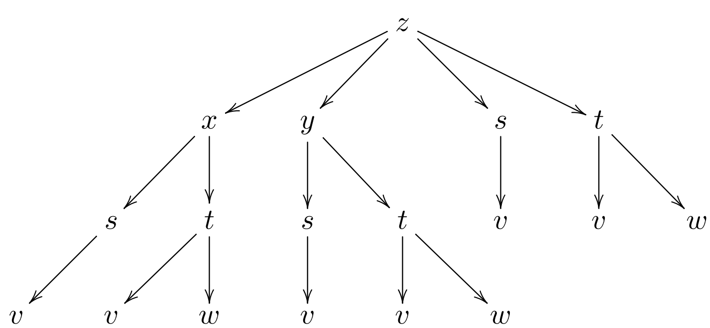
Strategy for limits. To show that a limit as $(x, y) \to (a, b)$ does not exist:

- Try the simplest possible paths: (x, b) as $x \to a$ and (a, y) as $y \to b$.
- If that doesn't work, try an arbitrary line through (a, b) with slope m: $y - b = m(x - a)$.
- If that doesn't work, *maybe* try a quadratic through (a, b): $y - b = m(x - a)^2$, but also reconsider the possibility that the limit exists.

Chain Rules

Example. Let $z = f(x, y, s, t)$, where $x = g(s, t)$, $y = h(s, t)$, $s = k(v)$ and $t = m(v, w)$. Find $\left.\dfrac{\partial z}{\partial v}\right)_w$.

Solution. A schematic is

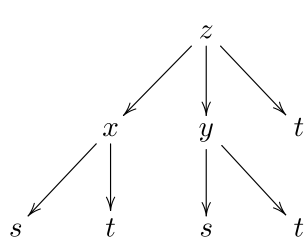


The chain rule we get is

$$\left.\frac{\partial z}{\partial v}\right)_w = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial s}\frac{ds}{dv} + \frac{\partial x}{\partial t}\frac{\partial t}{\partial v}\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial s}\frac{ds}{dv} + \frac{\partial y}{\partial t}\frac{\partial t}{\partial v}\right) + \left.\frac{\partial z}{\partial s}\right)_{x,y,t}\frac{ds}{dv} + \left.\frac{\partial z}{\partial t}\right)_{x,y,s}\frac{\partial t}{\partial v}.$$

Example. Let $z = x^2y^2 + yt^3$, $x = t^2 + s^3$, $y = 1 + st + s^2t^2$. Find $\left.\dfrac{\partial^2 z}{\partial t^2}\right)_s$.

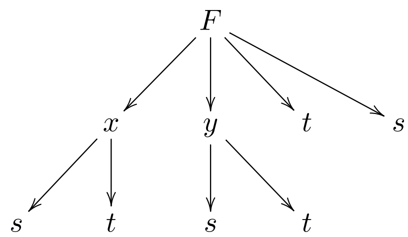
Solution. First, we find $\left.\dfrac{\partial z}{\partial t}\right)_s$. A schematic is



From the chain rule, we get

$$\begin{aligned}\left.\frac{\partial z}{\partial t}\right)_s =& \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} + \left.\frac{\partial z}{\partial t}\right)_{x,y} \\ =& \left(2xy^2\right)(2t) + \left(2x^2y + t^3\right)\left(s + 2s^2t\right) + 3yt^2.\end{aligned}$$

Let $F(x,y,s,t) = \left.\frac{\partial z}{\partial t}\right)_s$. We need to calculate $\left.\frac{\partial F}{\partial t}\right)_s$. A schematic is



The chain rule is

$$\left.\frac{\partial F}{\partial t}\right)_s = \frac{\partial F}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial t} + \left.\frac{\partial F}{\partial t}\right)_{x,y,s}.$$

From the expression for F above, we get

$$\begin{aligned}\frac{\partial F}{\partial x} &= \left(2y^2\right)(2t) + (4xy)\left(s + 2s^2t\right), \\ \frac{\partial F}{\partial y} &= (4xy)(2t) + \left(2x^2\right)\left(s + 2s^2t\right) + 3t^2, \\ \left.\frac{\partial F}{\partial t}\right)_{x,y,s} &= \left(2xy^2\right)(2) + \left(3t^2\right)\left(s + 2s^2t\right) + \left(2x^2y + t^3\right)\left(2s^2\right) + 6yt.\end{aligned}$$

Therefore

$$\begin{aligned}\left.\frac{\partial^2 z}{\partial t^2}\right)_s = \left.\frac{\partial F}{\partial t}\right)_s =& \left[\left(2y^2\right)(2t) + (4xy)\left(s + 2s^2t\right)\right](2t) \\ &+ \left[(4xy)(2t) + \left(2x^2\right)\left(s + 2s^2t\right) + 3t^2\right]\left(s + 2s^2t\right) \\ &+ \left(2xy^2\right)(2) + \left(3t^2\right)\left(s + 2s^2t\right) + \left(2x^2y + t^3\right)\left(2s^2\right) + 6yt.\end{aligned}$$

Implicit Differentiation

Example. Let the equations

$$\begin{aligned}F(x,y,z,s,t) =& x\sin(ys) + z\cos(yt) = 0, \\ G(x,y,z,s,t) =& x^2 + y^2 + z^2 - s^2 - t^2 = 0, \\ H(x,y,z,s,t) =& ye^{s+t} + xz^3 = 0\end{aligned}$$

define x, y, z implicitly as functions of s, t. Find $\frac{\partial x}{\partial s}$ and $\frac{\partial x}{\partial t}$ at the point $(x,y,z,s,t) = (1,0,0,0,1)$.

Solution. The implicit differentiation procedure yields

$$\frac{\partial x}{\partial s} = -\frac{\frac{\partial(F,G,H)}{\partial(s,y,z)}}{\frac{\partial(F,G,H)}{\partial(x,y,z)}}, \qquad \frac{\partial x}{\partial t} = -\frac{\frac{\partial(F,G,H)}{\partial(t,y,z)}}{\frac{\partial(F,G,H)}{\partial(x,y,z)}}.$$

The Jacobian determinants we need are

$$\frac{\partial(F,G,H)}{\partial(x,y,z)} = \begin{vmatrix} F_x & F_y & F_z \\ G_x & G_y & G_z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} \sin(ys) & xs\cos(ys) - zt\sin(yt) & \cos(yt) \\ 2x & 2y & 2z \\ z^3 & e^{s+t} & 3xz^2 \end{vmatrix},$$

$$\frac{\partial(F,G,H)}{\partial(s,y,z)} = \begin{vmatrix} F_s & F_y & F_z \\ G_s & G_y & G_z \\ H_s & H_y & H_z \end{vmatrix} = \begin{vmatrix} xy\cos(ys) & xs\cos(ys) - zt\sin(yt) & \cos(yt) \\ -2s & 2y & 2z \\ ye^{s+t} & e^{s+t} & 3xz^2 \end{vmatrix},$$

$$\frac{\partial(F,G,H)}{\partial(t,y,z)} = \begin{vmatrix} F_t & F_y & F_z \\ G_t & G_y & G_z \\ H_t & H_y & H_z \end{vmatrix} = \begin{vmatrix} -yz\sin(yt) & xs\cos(ys) - zt\sin(yt) & \cos(yt) \\ -2t & 2y & 2z \\ ye^{s+t} & e^{s+t} & 3xz^2 \end{vmatrix}.$$

Evaluate at the point $(x, y, z, s, t) = (1, 0, 0, 0, 1)$:

$$\left.\frac{\partial(F,G,H)}{\partial(x,y,z)}\right|_{(1,0,0,0,1)} = \begin{vmatrix} 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & e & 0 \end{vmatrix} = 2e,$$

$$\left.\frac{\partial(F,G,H)}{\partial(s,y,z)}\right|_{(1,0,0,0,1)} = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & e & 0 \end{vmatrix} = 0,$$

$$\left.\frac{\partial(F,G,H)}{\partial(t,y,z)}\right|_{(1,0,0,0,1)} = \begin{vmatrix} 0 & 0 & 1 \\ -2 & 0 & 0 \\ 0 & e & 0 \end{vmatrix} = -2e.$$

Therefore

$$\frac{\partial x}{\partial s} = -\frac{0}{2e} = 0, \qquad \frac{\partial x}{\partial t} = -\frac{-2e}{2e} = 1.$$

Example. Let $z = \ln(x^2 + y^2 + 1)$, where x, y are defined implicitly as functions of t by

$$\begin{aligned} F(x,y,t) =& x^3 - yt^3 = 0, \\ G(x,y,t) =& xe^{yt-1} - t = 0. \end{aligned}$$

Find $\frac{dz}{dt}$ when $(x, y, t) = (1, 1, 1)$.

Solution. We have $z = f(x, y)$, where $x = g(t)$ and $y = h(t)$. Using partial differentiation, we find

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

From the given expression for z in terms of x, y, we find

$$\frac{\partial z}{\partial x} = \frac{2x}{x^2 + y^2 + 1}, \quad \frac{\partial z}{\partial y} = \frac{2y}{x^2 + y^2 + 1}.$$

When $(x, y) = (1, 1)$, these are

$$\left.\frac{\partial z}{\partial x}\right|_{(1,1)} = \frac{2}{3}, \quad \left.\frac{\partial z}{\partial y}\right|_{(1,1)} = \frac{2}{3}.$$

Next, using implicit differentiation, we have

$$\frac{dx}{dt} = -\frac{\frac{\partial(F,G)}{\partial(t,y)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_t & F_y \\ G_t & G_y \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} -3yt^2 & -t^3 \\ xye^{yt-1} - 1 & xte^{yt-1} \end{vmatrix}}{\begin{vmatrix} 3x^2 & -t^3 \\ e^{yt-1} & xte^{yt-1} \end{vmatrix}},$$

$$\frac{dy}{dt} = -\frac{\frac{\partial(F,G)}{\partial(x,t)}}{\frac{\partial(F,G)}{\partial(x,y)}} = -\frac{\begin{vmatrix} F_x & F_t \\ G_x & G_t \end{vmatrix}}{\begin{vmatrix} F_x & F_y \\ G_x & G_y \end{vmatrix}} = -\frac{\begin{vmatrix} 3x^2 & -3yt^2 \\ e^{yt-1} & xye^{yt-1} - 1 \end{vmatrix}}{\begin{vmatrix} 3x^2 & -t^3 \\ e^{yt-1} & xte^{yt-1} \end{vmatrix}}.$$

At the point $(x, y, t) = (1, 1, 1)$, these become

$$\left.\frac{dx}{dt}\right|_{(1,1,1)} = -\frac{\begin{vmatrix} -3 & -1 \\ 0 & 1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}} = -\frac{-3}{4} = \frac{3}{4},$$

$$\left.\frac{dy}{dt}\right|_{(1,1,1)} = -\frac{\begin{vmatrix} 3 & -3 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}} = -\frac{3}{4}.$$

Therefore

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \frac{2}{3} \cdot \frac{3}{4} - \frac{2}{3} \cdot \frac{3}{4} = 0.$$