TIME: 70 minutes

DATE: February 11, 2016
COURSE: MATH 2130

EXAMINER: G.I. Moghaddam

NAME: Test 1, Math 2130

STUDENT # : ____________________

Q1 (11)	Q2 (10)	Q3 (11)	Q4 (8)	Q5 (10)	Total (50)

1. Let S be the surface $x^2 + 4y^2 - 4z^2 - 16y + 16z = 0$..

[7] (a) Identify and sketch the surface S. Mark the important points.

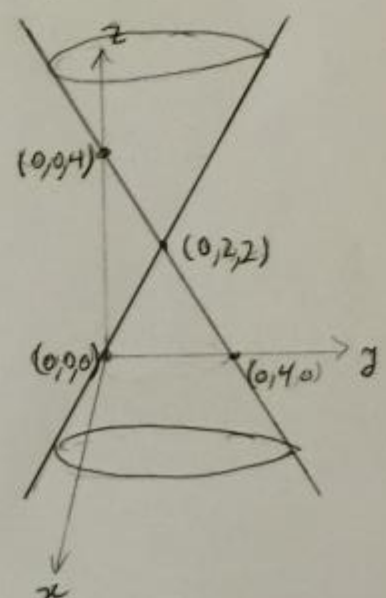
$x^2 + 4(y^2 - 4) - 4(z^2 - 4) = 0$

$x^2 + 4(y-2)^2 - 16 - 4(z-2)^2 + 16 = 0$

$(z-2)^2 = \frac{x^2}{4} + \frac{(y-2)^2}{1}$

Elliptic Cone with centre $(0, 2, 2)$

with $a = 2$, $b = 1$



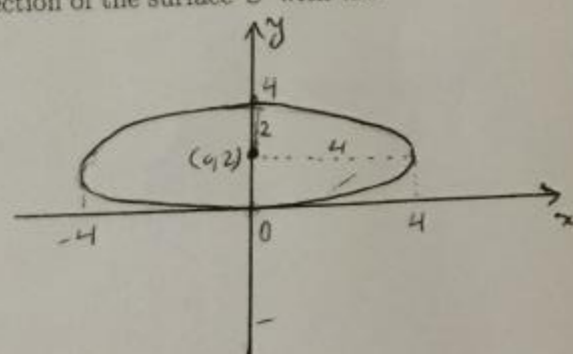
[4] (b) Identify and sketch the cross section of the surface S with the xy–plane.

$z = 0 \Rightarrow 4 = \frac{x^2}{4} + \frac{(y-2)^2}{1}$

$\frac{x^2}{16} + \frac{(y-2)^2}{4} = 1$

Ellipse with Centre $(0,2)$

$a = 4$, $b = 2$



[10] 2. Let P be the point of intersection of the line $\ell: \quad x = 1+t\,,\ y = 2-t\,,\ z = -1+2t$ and the plane $\Pi: 2x + y - z = 4$. Find the distance between the point P and the line $\ell_1: \quad x = 3-2r\,,\ y = r\,,\ z = 4+r$.

$$2(1+t)+(2-t)-(-1+2t)=4$$

$$5-t=4 \Rightarrow t=1 \Rightarrow P(2,1,1)$$

$$\vec{v}=(-2,1,1)\,,\quad Q(3,0,4)$$

$$D=\frac{\|\vec{u}\times\vec{v}\|}{\|\vec{v}\|} \quad \text{where } \vec{u}=\vec{PQ}=(1,-1,3)$$

$$\text{but } \vec{u}\times\vec{v}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 3 \\ -2 & 1 & 1 \end{vmatrix}=(-4,-7,-1)$$

$$\|\vec{u}\times\vec{v}\|=\sqrt{(-4)^2+(-7)^2+(-1)^2}=\sqrt{66}$$

$$\|\vec{v}\|=\sqrt{4+1+1}=\sqrt{6}$$

$$D=\frac{\sqrt{66}}{\sqrt{6}}=\sqrt{11}$$

OR use $D=|\vec{PQ}\times\hat{PR}|$ where $\hat{PR}=\dfrac{(\vec{v}\times\vec{PQ})\times\vec{v}}{\|(\vec{v}\times\vec{PQ})\times\vec{v}\|}$

$$\text{but } \vec{v}\times\vec{PQ}=\begin{vmatrix} \hat{i} & \hat{j} & k \\ -2 & 1 & 1 \\ 1 & -1 & 3 \end{vmatrix}=(4,7,1)$$

$$(\vec{v}\times\vec{PQ})\times\vec{v}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 7 & 1 \\ -2 & 1 & 1 \end{vmatrix}=(6,-6,18)=6(1,-1,3)$$

$$\|(\vec{v}\times\vec{PQ})\times\vec{v}\|=6\sqrt{1^2+1^2+9}=6\sqrt{11}$$

$$\text{so } \hat{PR}=\frac{6(1,-1,3)}{6\sqrt{11}}=\frac{1}{\sqrt{11}}(1,-1,3)$$

$$\text{so } D=\left|(1,-1,3)\cdot\frac{1}{\sqrt{11}}(1,-1,3)\right|=\left|\frac{1}{\sqrt{11}}(1+1+9)\right|=\frac{11}{\sqrt{11}}=\sqrt{11}$$

3. Let the curve C be the intersection of the surfaces

$$2x + 2xy + z = 0\,, \qquad z = (x-y)^2\,.$$

[7] (a) Find a parametric representation for the curve C, directed so that x decreases when y is positive.

$z = -2x - 2xy \Rightarrow -2x - 2xy = (x-y)^2 \Rightarrow -2x - \cancel{2xy} = x^2 + y^2 - \cancel{2xy} \Rightarrow$

$x^2 + 2x + y^2 = 0 \Rightarrow (x+1)^2 - 1 + y^2 = 0 \Rightarrow (x+1)^2 + y^2 = 1$

Let $x+1 = \cos t$ (so $x = -1 + \cos t$) then $\cos^2 t + y^2 = 1 \Rightarrow y^2 = 1 - \cos^2 t = \sin^2 t$

so either $y = \sin t$ or $y = -\sin t$. But $y = \sin t$ is the right choice

[because $y \geq 0$ means $\sin t \geq 0$ so $0 \leq t \leq \pi$, then

$t = 0 \Rightarrow x = -1 + \cos 0 = -1 + 1 = 0$ and $y = \sin 0 = 0$

$t = \pi/2 \Rightarrow x = -1 + \cos \pi/2 = -1 + 0 = -1$ and $y = \sin \pi/2 = 1$

that is x decreases from 0 to -1]

Now $z = -2x - 2xy$ so $z = -2(-1+\cos t) - 2(-1+\cos t)\sin t = 2 - 2\cos t + 2\sin t - 2\sin t\cos t$

so $z = 2 - 2\cos t + 2\sin t - \sin(2t)$

Therefore the answer is $x = -1 + \cos t$, $y = \sin t$, $z = 2 - 2\cos t + 2\sin t - \sin(2t)$

[4] (b) Find the unit tangent vector to the curve C at the point $(-1, 1, 4)$.

$\vec{r}(t) = (-1 + \cos t,\ \sin t,\ 2 - 2\cos t + 2\sin t - \sin(2t))$

at $(-1, 1, 4)$, $t = \pi/2$ and

$\vec{T}(t) = \vec{r}\,'(t) = (-\sin t,\ \cos t,\ 2\sin t + 2\cos t - 2\cos(2t))$

$\vec{T}(\pi/2) = (-1,\ 0,\ 2(1) + 2(0) - 2(-1)) = (-1, 0, 4)$

$\| \vec{T}(\pi/2) \| = \sqrt{(-1)^2 + 0^2 + 4^2} = \sqrt{17}$

so a unit tangent vector at the point $(-1, 1, 4)$ is

$\hat{T}(\pi/2) = \frac{1}{\sqrt{17}}(-1, 0, 4) = \left(\frac{-1}{\sqrt{17}}, 0, \frac{4}{\sqrt{17}}\right)$

[8] 4. Let C be the curve with vector representation

$$\mathbf{r}(t) = (\frac{1}{3}t^3)\,\hat{\mathbf{i}} + (\frac{1}{3}(2t)^{\frac{3}{2}})\,\hat{\mathbf{j}} + (\ln t)\,\hat{\mathbf{k}}.$$

Find the arc length of the curve C from the point $A(\frac{1}{3}e^3, \frac{2\sqrt{2}}{3}e^{\frac{3}{2}}, 1)$ to the point $B(\frac{1}{3}e^6, \frac{2\sqrt{2}}{3}e^3, 2)$.

$$\vec{r}\,'(t) = (t^2,\ \tfrac{1}{3}(\tfrac{3}{2})(2)(2t)^{\frac{1}{2}},\ \tfrac{1}{t}) = (t^2, \sqrt{2t}, \tfrac{1}{t})$$

$$\|\vec{r}\,'(t)\| = \sqrt{(t^2)^2 + (\sqrt{2t})^2 + (\tfrac{1}{t})^2}$$

$$= \sqrt{t^4 + 2t + \frac{1}{t^2}}$$

$$= \sqrt{(t^2 + \tfrac{1}{t})^2}$$

$$= t^2 + \tfrac{1}{t}$$

at A, $t = e$ and at B, $t = e^2$

$$L = \int_e^{e^2} \|\vec{r}\,'(t)\|\,dt = \int_e^{e^2} (t^2 + \tfrac{1}{t})\,dt$$

$$= (\tfrac{1}{3}t^3 + \ln t)\Big|_e^{e^2}$$

$$= (\tfrac{1}{3}(e^2)^3 + \ln e^2) - (\tfrac{1}{3}e^3 + \ln e)$$

$$= (\tfrac{1}{3}e^6 + 2) - (\tfrac{1}{3}e^3 + 1)$$

$$= \tfrac{1}{3}(e^6 - e^3) + 1$$

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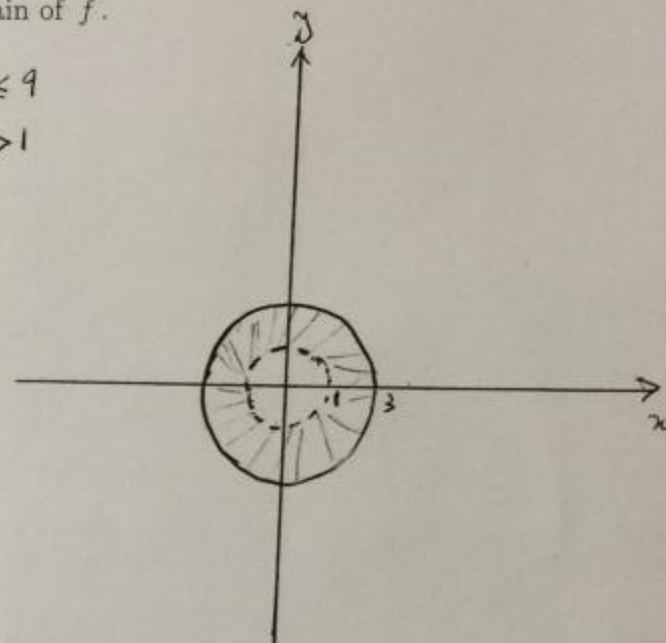
5. Let $f(x,y) = \dfrac{\sqrt{9-x^2-y^2}}{\sqrt{x^2+y^2-1}}$.

[5] (a) Find and sketch the domain of f.

must $9-x^2-y^2 \geq 0 \Rightarrow x^2+y^2 \leq 9$

Also $x^2+y^2-1 > 0 \Rightarrow x^2+y^2 > 1$

Domain $= \{(x,y) \mid 1 < x^2+y^2 \leq 9\}$



[5] (b) Identify all level curves of the function f and **draw only two** of the level curves.

Range of f is $[0,\infty)$ so let $k \geq 0$, then

$$\frac{\sqrt{9-x^2-y^2}}{\sqrt{x^2+y^2-1}} = k \Rightarrow 9-x^2-y^2 = k^2(x^2+y^2-1) \Rightarrow x^2+y^2 = \frac{k^2+9}{k^2+1}$$

that is level curves are a family of circles with centre at $(0,0)$ and radius $\sqrt{\dfrac{k^2+9}{k^2+1}}$

If $k=0$, then $x^2+y^2=9$

If $k=1$, then $x^2+y^2=5$