## MATH 1210 (WINTER 2014): SOLUTIONS TO ASSIGNMENT TWO

*Q1. Consider the polynomial*

$$p(x) = x^3 + k\,x + (3 - 2\,i),$$

*where $k$ is an unknown complex number. It is given to you that if $p(x)$ is divided by $2\,x - (4 - i)$, then the remainder is $5 + i$. Find the value of $k$.*

**Solution:** By the Remainder Theorem (page 25 of the textbook), we have the equality

$$p(\frac{4-i}{2}) = 5 + i,$$

that is to say,

$$(\frac{4-i}{2})^3 + k\,(\frac{4-i}{2}) + 3 - 2\,i = 5 + i.$$

After expanding the left hand side, we get

$$\frac{13}{2} - \frac{47}{8}\,i + k\,(\frac{4-i}{2}) + 3 - 2\,i = \frac{19}{2} - \frac{63}{8}\,i + k\,(\frac{4-i}{2}) = 5 + i.$$

Hence

$$k\,(\frac{4-i}{2}) = -\frac{9}{2} + \frac{71}{8}\,i.$$

Now solving for $k$, we get

$$k = \frac{2(-\frac{9}{2} + \frac{71}{8}\,i)}{4 - i} = \frac{(-9 + \frac{71}{4}\,i)\,(4 + i)}{(4 - i)\,(4 + i)} = \frac{(-\frac{215}{4} + 62\,i)}{17} = \frac{1}{68}\,(-215 + 248\,i).$$

Hence the answer is

$$k = \frac{-215 + 248\,i}{68}.$$

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*Q2. Find all values of $k$ (which may be complex numbers) such that $k\,x - (k + 1)$ is a factor of the polynomial*

$$p(x) = x^2 + (2 + 3\,i)\,x - (17 + i)\,.$$

**Solution:** By the Factor Theorem (page 26), if we substitute $x = \frac{k+1}{k}$ into $p(x)$, the result should be zero; that is,

$$p(\frac{k+1}{k}) = 0.$$

This becomes

$$(\frac{k+1}{k})^2 + \frac{(2 + 3\,i)(k + 1)}{k} - (17 + i) = 0.$$

Now multiply by $k^2$ to clear the denominators, and expand. After collecting the powers of $k$, we get

$$(-14+2\,i)\,k^2+(4+3\,i)\,k+1=0.$$

We use the quadratic formula to solve this equation for $k$, which gives

$$k=\frac{-(4+3\,i)\pm\sqrt{(4+3\,i)^2-4\,(-14+2\,i)}}{2\,(-14+2\,i)}.$$

After simplifying the expression under the square-root sign, this becomes

$$k=\frac{-(4+3\,i)\pm\sqrt{63+16\,i}}{-28+4\,i}. \tag{1}$$

In order to find the square roots explicitly, we will use the technique of Questions 44-45 on page 18 of the textbook. Let us set

$$\sqrt{63+16\,i}=a+b\,i,$$

where $a, b$ are some real numbers. Then

$$(a+b\,i)^2=a^2-b^2+2\,a\,b\,i=63+16\,i.$$

Comparing the real and imaginary parts, we get two equations

$$a^2-b^2=63, \qquad \text{and} \qquad 2\,a\,b=16,\implies a\,b=8.$$

Substituting $b=8/a$ in the first equation, we get

$$a^2-\frac{64}{a^2}=63.$$

Now let $a^2=c$, so that we have $c-\frac{64}{c}=63$. Multiply by $c$ to clear the denominator, when we have $c^2-63\,c-64=0$. This factors as $(c-64)(c+1)=0$, hence $c=64$ or $-1$. But remember that $c=a^2$ for some real number $a$, hence $c$ must be positive. Thus $c=64$, which gives $a=\pm 8$. This in turn gives $b=\pm 1$. Hence we have

$$\sqrt{63+16\,i}=\pm\,(8+i).$$

Now substitute this into (1) to get:

$$k=\frac{-(4+3\,i)+(8+i)}{-28+4\,i}=\frac{4-2\,i}{-28+4\,i}=\frac{(4-2\,i)\,(-28-4\,i)}{(-28+4\,i)(-28-4\,i)}=\frac{-120+40\,i}{800}=\frac{-3+i}{20},$$

or

$$k=\frac{-(4+3\,i)-(8+i)}{-28+4\,i}=\frac{-12-4\,i}{-28+4\,i}=\frac{(-12-4\,i)\,(-28-4\,i)}{800}=\frac{320+160\,i}{800}=\frac{2+i}{5}.$$

Hence, the answers are

$$k=\frac{-3+i}{20} \quad \text{and} \quad \frac{2+i}{5}.$$

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*Q3. Let*

$$p(x) = x^5 + a\,x^4 + b\,x^3 + c\,x^2 + d\,x + e$$

*where $a, b, c, d, e$ are unknown real numbers. It is given that*

$$2 + i\,\sqrt{3}, \quad 1 - i\,\sqrt{5}, \quad -3$$

*are three of the roots of $p(x)$. Find the remaining two roots, and then find the values of $a, b, c, d, e$.*

**Solution:** Note that $p(x)$ is a polynomial with real coefficients, hence Theorem 3.5 on page 28 is applicable. If $2 + i\,\sqrt{3}$ is a root, then $2 - i\,\sqrt{3}$ must also be a root. By the same reasoning, $1 + i\,\sqrt{5}$ must also be a root. Hence we know all the five roots of $p(x)$, namely

$$2 + i\,\sqrt{3}, \quad 2 - i\,\sqrt{3}, \quad 1 + i\,\sqrt{5}, \quad 1 - i\,\sqrt{5}, \quad -3.$$

Note that if $z$ is any root of $p(x)$, then $x - z$ is a factor $p(x)$. Hence we know all the linear factors of $p(x)$, that is to say

$$p(x) = (x - (2 + i\,\sqrt{3}))\,(x - (2 - i\,\sqrt{3}))\,(x - (1 + i\,\sqrt{5}))\,(x - (1 - i\,\sqrt{5}))\,(x + 3).$$

After expansion, this is

$$p(x) = (x^2 - 4\,x + 7)\,(x^2 - 2\,x + 6)\,(x + 3) = x^5 - 3\,x^4 + 3\,x^3 + 25\,x^2 - 72\,x + 126.$$

Hence

$$a = -3, \quad b = 3, \quad c = 25, \quad d = -72, \quad e = 126.$$

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*Q4. Consider the polynomial*

$$p(x) = 3\,x^4 - 2\,x^3 - 22\,x^2 + 23\,x + 10.$$

(a) *Use the Rational Root Theorem to list all possible rational roots of $p(x)$.*
(b) *Use the Bounds Theorem to eliminate some of the possibilities from your list in (a).*
(c) *Use direct substitution to check whether any of the remaining possibilities are in fact roots.*
(d) *Use your results from (c) to find all the roots of $p(x)$.*

**Solution:**
(a) Let $p/q$ denote a possible rational root. (As usual, we assume that $p, q$ are integers without any common factors.) Then $p$ must divide $10$, and $q$ must divide $3$. Hence

$$p = \pm 1, \pm 2, \pm 5, \pm 10, \quad \text{and} \quad q = \pm 1, \pm 3.$$

Hence the possible rational roots are

$$\frac{p}{q} = \pm 1,\ \pm 2,\ \pm 5,\ \pm 10,\ \pm 1/3,\ \pm 2/3,\ \pm 5/3,\ \pm 10/3.$$

(b) In the notation of the Bounds Theorem, $|a_4| = 3, M = 23$. Hence if $z$ is any root, then

$$|z| < \frac{23}{3} + 1 = \frac{26}{3} \simeq 8.66.$$

This implies that $\pm 10$ are not possible roots.

(c) After checking the remaining possibilities by substitution, we see that $2$ and $-\frac{1}{3}$ are the only rational roots.

(d) Now it follows that $x-2$ and $3\,x+1$ are factors of $p(x)$, hence $(x-2)\,(3\,x+1) = 3\,x^2 - 5\,x - 2$ must divide $p(x)$. After carrying out the long division, we get

$$p(x) = (x-2)\,(3\,x+1)\,(x^2 + x - 5).$$

Applying the quadratic formula to the last factor, we get $x = \frac{1}{2}\,(-1 \pm \sqrt{21})$. Hence the roots are

$$2, \quad -\frac{1}{3}, \quad \frac{1}{2}\,(-1 \pm \sqrt{21}).$$

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*Q5. Consider the polynomial*

$$p(x) = x^7 - 3\,x^5 + m\,x - 2,$$

*where $m$ is some integer. Find all possible values of $m$ such that $p(x)$ has a rational root.*

**Solution:** If $p/q$ denotes a rational root, then by the Rational Root Theorem, we know that $p$ divides $2$ and $q$ divides $1$. Hence $p = \pm 1, \pm 2$ and $q = \pm 1$. Hence the possible rational roots are

$$\frac{p}{q} = \pm 1, \pm 2.$$

Now substitute each of them into $p(x)$ and solve for $m$. For example, if $1$ is to be a root of $p(x)$, then

$$p(1) = 0 \implies 1 - 3 + m - 2 = 0,$$

which gives $m = 4$. Similarly, after substituting $x = -1, 2, -2$, we get $m = 0, -15, -17$. Hence the possible values of $m$ are

$$m = 4,\ 0,\ -15,\ -17.$$

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*Q6. Consider the polynomial*

$$p(x) = x^4 + (-1 + 2\,i)\,x^3 - 3\,x^2 - (2 + i)\,x + 1 - 3\,i\ .$$

*It is given that $x^2 + i$ is a factor of $p(x)$. Use long division followed by the quadratic formula to find all the roots of $p(x)$.*

**Solution:** After long division of $p(x)$ by $x^2 + i$, the quotient turns out to be

$$Q(x) = x^2 + (-1 + 2\,i)\,x - (3 + i).$$

That is to say,

$$p(x) = (x^2 + i)\, Q(x).$$

In order to find the roots of $p(x)$, we have to treat the two factors separately. First we solve for $x^2 + i = 0$, i.e., $x^2 = -i$. Using the exponential form, we can write

$$-i = e^{3\,i\,\pi/2} = e^{7\,i\,\pi/2}.$$

Hence the roots are

$$x = e^{3\,i\,\pi/4} = \frac{-1+i}{\sqrt{2}}, \qquad \text{and} \qquad x = e^{7\,i\,\pi/4} = \frac{1-i}{\sqrt{2}}.$$

Now we have to find the roots of $Q(x)$. Using the quadratic formula, these are

$$\frac{-(-1+2\,i) \pm \sqrt{(-1+2\,i)^2 + 4\,(3+i)}}{2} = \frac{1 - 2\,i \pm \sqrt{9}}{2},$$

that is,

$$\frac{1 - 2\,i + 3}{2} = 2 - i, \quad \text{and} \quad \frac{1 - 2\,i - 3}{2} = -1 - i.$$

In conclusion, the roots of $p(x)$ are

$$\pm \frac{1}{\sqrt{2}}\,(1 - i), \quad 2 - i, \quad -1 - i.$$

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*Q7. Consider the polynomial*

$$p(x) = 1 + \sum_{k=1}^{13} \frac{(-1)^k}{k^2}\, x^k\,.$$

(a) *Show that* $p(x)$ *must have at least one positive real root.*
(b) *Show that* $p(x)$ *has no negative real roots.*
(c) *Show that if* $z$ *is any root of* $p(x)$*, then* $|z| < 170$*.*

**Solution:**
(a) Notice that the coefficient $\frac{(-1)^k}{k^2}$ is positive if $k$ is even, and negative if $k$ is odd. Hence the coefficients of $p(x)$ follow the alternating pattern:

$$+ \quad - \quad + \quad - \quad + \quad \dots \quad -$$

There are altogether $13$ sign changes, hence Descartes's Rule of Signs implies that the number of positive roots is

$$13 \quad \text{or} \quad 11 \quad \text{or} \quad 9 \quad \text{or} \dots \quad 1.$$

This implies that that there must be at least one positive root.

(2) The coefficients in $p(-x)$ are all positive, i.e., there are no sign changes. Hence there is no negative root.

(3) In the notation of the Bounds Theorem, we get $a_{13} = -\frac{1}{13^2}$ and $M = 1$. (This is so, because all the coefficients have absolute value $\leqslant 1$, and the constant term is $1$.) Hence for any root $z$, we have

$$|z| < \frac{1}{1/13^2} + 1 = 13^2 + 1 = 170.$$

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