1. Find the interval of convergence for the power series

$$\sum_{n=2}^{\infty} \frac{2^n}{n}(x+1)^{2n+3}.$$

If we set $y=(x+1)^2$, the series becomes

$$(x+1)^3 \sum_{n=2}^{\infty} \frac{2^n}{n} y^n. \quad \sim 1$$

$$R_y = \lim_{n\to\infty} \left| \frac{\frac{2^n}{n}}{\frac{2^{n+1}}{n+1}} \right| = \frac{1}{2}. \sim 2 \quad \text{Thus, } R_x = \frac{1}{\sqrt{2}}. \sim 1$$

The open interval of convergence is $-\frac{1}{\sqrt{2}} < x+1 < \frac{1}{\sqrt{2}} \Rightarrow -1-\frac{1}{\sqrt{2}} < x < -1$ ~ 1

At $x = -1-\frac{1}{\sqrt{2}}$, the series becomes

$$\sum_{n=2}^{\infty} \frac{2^n}{n}\left(-1-\frac{1}{\sqrt{2}}+1\right)^{2n+3} = -\frac{1}{2\sqrt{2}} \sum_{n=2}^{\infty} \frac{1}{n}. \sim 1$$

Since this is the harmonic series, less its first term, it diverges. ~ 1

At $x=-1+\frac{1}{\sqrt{2}}$, the series becomes

$$\sum_{n=2}^{\infty} \frac{2^n}{n}\left(-1+\frac{1}{\sqrt{2}}+1\right)^{2n+3} = \frac{1}{2\sqrt{2}} \sum_{n=2}^{\infty} \frac{1}{n}. \sim 1$$

Once again this is the harmonic series, less its first term, and therefore diverges. ~ 1

The interval of convergence is

2. Find the Taylor series about $x = -2$ for the function $\frac{1}{\sqrt{2x+7}}$. Express your answer in sigma notation simplified as much as possible. Include its open interval of convergence. You must use a method that guarantees that the series converges to the function.

$\boxed{x+2}$

$$\frac{1}{\sqrt{2x+7}} = \frac{1}{\sqrt{2(x+2)+3}} = \frac{1}{\sqrt{3}}\left[1+\frac{2}{3}(x+2)\right]^{-1/2} \sim 2$$

$$= \frac{1}{\sqrt{3}}\left\{1 - \frac{1}{2}\left[\frac{2}{3}(x+2)\right] + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left[\frac{2}{3}(x+2)\right]^2 + \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}\left[\frac{2}{3}(x+2)\right]^3 \dots\right.$$

$$= \frac{1}{\sqrt{3}}\left\{1 - \underbrace{\frac{\cancel{2}}{\cancel{2}\cdot 3}(x+2)}_{n=1} + \underbrace{\frac{\cancel{2^2}(1\cdot 3)}{\cancel{2^2}\cdot 2!\, 3^2}(x+2)^2}_{n=2} - \underbrace{\frac{(1\cdot 3\cdot 5)\cancel{2^3}}{\cancel{2^3}\cdot 3!\, 3^3}(x+2)^3}_{n=3} + \dots\right\}$$

$$= \frac{1}{\sqrt{3}}\left\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 3\cdot 5\cdots(2n-1)]}{n!\,3^n}(x+2)^n\right\} \sim 2$$

$$= \frac{1}{\sqrt{3}}\left\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n[1\cdot 2\cdot 3\cdot 4\cdots(2n)]}{n!\,3^n[2\cdot 4\cdots(2n)]}(x+2)^n\right\}$$

$$= \frac{1}{\sqrt{3}}\left\{1 + \sum_{n=1}^{\infty}\frac{(-1)^n(2n)!}{(n!)^2 2^n 3^n}(x+2)^n\right\}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{(n!)^2 2^n 3^{n+1/2}}(x+2)^n \sim 2$$

This is valid for $\left|\frac{2}{3}(x+2)\right| < 1 \sim 1 \implies |x+2| < \frac{3}{2}$

$$\implies -\frac{3}{2} < x+2 < \frac{3}{2}$$

3. (a) Find a series of constants that represents the value of the integral

$$\int_0^{1/2} \frac{x - \sin x}{x^3}\,dx.$$

Express your answer in sigma notation.

(b) Explain how you would obtain an approximation to the integral correct to 6 decimal places. Justify your reasoning.

(a) $$\int_0^{1/2} \frac{x-\sin x}{x^3}\,dx = \int_0^{1/2} \frac{1}{x^3}\left[\cancel{x} - \left(\cancel{x} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)\right] dx \quad \sim 1$$

$$= \int_0^{1/2} \left[\frac{1}{3!} - \frac{x^2}{5!} + \frac{x^4}{7!} - \cdots\right] dx \quad \sim 1$$

$$= \left\{\frac{x}{3!} - \frac{x^3}{3\cdot 5!} + \frac{x^5}{5\cdot 7!} - \cdots\right\}_0^{1/2} \quad \sim 1$$

$$= \overset{n=1}{\frac{1}{2\cdot 3!}} - \overset{n=2}{\frac{1}{3\cdot 2^3\cdot 5!}} + \overset{n=3}{\frac{1}{5\cdot 2^5\cdot 7!}} - \cdots \quad \sim 1$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)2^{2n-1}(2n+1)!} \quad \sim 1$$

(b) The series is alternating and the sequence $\left\{\frac{1}{(2n-1)2^{2n-1}(2n+1)}\right\}$ is decreasing, and has limit zero. The series therefore converges by the alternating series test. To find an approximation correct to 6 decimal places, take partial sums of this series until two successive partial sums 6 decimal places.

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4. Find, in explicit form, a one-parameter family of solutions of the differential equation

$$x\frac{dy}{dx} = \frac{\sqrt{x-2}}{x} - 3y.$$

Is your solution a general solution? Explain.

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{\sqrt{x-2}}{x^2}$$

An integrating factor is $e^{\int \frac{3}{x}dx} = e^{3\ln|x|} = |x|^3$, since $x>2$

~~For either x<0 or x>0, we obtain~~

$$x^3 \frac{dy}{dx} + 3x^2 y = x\sqrt{x-2}.$$

$$\frac{d}{dx}(x^3 y) = x\sqrt{x-2}$$

$$x^3 y = \int x\sqrt{x-2}\,dx \qquad u = x-2$$

$$= \int (u+2)\sqrt{u}\,du$$

$$= \frac{2}{5}u^{5/2} + \frac{4}{3}u^{3/2} + C$$

$$= \frac{2}{5}(x-2)^{5/2} + \frac{4}{3}(x-2)^{3/2} + C$$

$$\therefore y(x) = \frac{2}{5x^3}(x-2)^{5/2} + \frac{4}{3x^3}(x-2)^{3/2} + \frac{C}{x^3}$$

Since the differential equation is linear, the soluti mixture containing 2 kilograms of sugar for each 100 litres of solution is being added to the tank at a rate of 10 millilitres per second. At the same time, 20 millilitres of well-stirred mixture is being withdrawn from the tank. Set up an initial-value problem for the number of grams of sugar in the tank as a function of time. Make **NO ATTEMPT** to solve the problem, but determine how long the differential equation is in effect.

Let $S(t)$ represent the number of grams of sugar in the tank. Then

$$\frac{dS}{dt} = (\text{rate sugar enters}) - (\text{rate sugar leaves})$$

$$= \frac{2000}{10^5} 10 - \frac{S}{10^6 - 10t}(20) \quad 2$$

$$= \frac{1}{5} - \frac{20S}{10^6 - 10t}, \quad S(0) = 5000 \quad 1$$

The equation is valid for

$$10^6 - 10t\,20 \Rightarrow t \leq 10^5 s. \quad 1$$

6. You are given that the roots of the auxiliary equation associated with the differential equation

$$\phi(D)y = 3x^2 + 4\cos 2x - 3xe^{3x},$$

are $m = 0, 0, 1 \pm 2i, 1 \pm 2i, 3, -2$. What is the form of a particular solution to the differential equation as predicted by undetermined coefficients? Your answer should contain the minimum number of terms possible. Make **NO** attempt to find the coefficients.

$$y_h(x) = C_1 + C_2 x + e^x[(C_3 + C_4 x)\cos 2x + (C_5 + C_6 x)\sin 2x] + C_7 e^{3x} + C_8 e^{-2x} \quad 2$$

$$y_p(x) = Ax^4 + Bx^3 + Cx^2 + D\cos 2x + E\sin 2x + Fx^2 e^{3x} + Gxe^{3x}$$

Find a general solution for the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = \sin 2x - x.$$

The auxiliary equation is 1

$$m^2 + 2m + 4 = 0 \Rightarrow m = \frac{-2 \pm \sqrt{4 - 16}}{2} = -1 \pm \sqrt{3}$$

$$y_h(x) = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right). \quad 1$$

$$y_p(x) = A\sin 2x + B\cos 2x + Cx + D. \quad 2$$

Substituting into the differential equation,

$$(-4A\sin 2x - 4B\cos 2x) + 2(2A\cos 2x - 2B\sin 2x + C)$$
$$+ 4(A\sin 2x + B\cos 2x + Cx + D) = \sin 2x - x$$

When we equate coefficients:

$\sin 2x$: $-4A - 4B + 4A = 1 \Rightarrow B = -1/4$

$\cos 2x$: $-4B + 4A + 4B = 0 \Rightarrow A = 0$

x : $4C = -1 \Rightarrow C = -1/4$

1 : $2C + 4D = 0 \Rightarrow D = 1/8$

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$$\therefore y_p(x) = -\frac{1}{4}\cos 2x - \frac{x}{4} + \frac{1}{8}$$

$$y(x) = e^{-x}\left(C_1 \cos\sqrt{3}x + C_2 \sin\sqrt{3}x\right) - \frac{1}{4}\cos 2x - \frac{x}{4} + \frac{1}{8}$$

(a) $f(t) = 3t^2 - 1,\ 0 < t < 3, \quad f(t+3) = f(t)$ (b) $f(t) = \begin{cases} 0, & 0 \le t < \pi \\ e^{2t}\sin t, & t > \pi \end{cases}$

(a) $$F(s) = \frac{1}{1-e^{-3s}} \int_0^3 (3t^2-1)e^{-st}\,dt \sim 1 = \frac{1}{1-e^{-3s}} \mathcal{L}\{(3t^2-1)[1-h(t-3)]\}$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{6}{s^3} - \frac{1}{s} - e^{-3s}\mathcal{L}\{3(t+3)^2-1\}\right] \sim 1$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{6}{s^3} - \frac{1}{s} - e^{-3s}\mathcal{L}\{3t^2+18t+26\}\right] \sim 1$$

$$= \frac{1}{1-e^{-3s}}\left[\frac{6}{s^3} - \frac{1}{s} - e^{-3s}\left(\frac{6}{s^3} + \frac{18}{s^2} + \frac{26}{s}\right)\right] \sim 1$$

(b) $$F(s) = \mathcal{L}\{e^{2t}\sin t\, h(t-\pi)\} \sim 1 = e^{-\pi s}\mathcal{L}\{e^{2(t+\pi)}\sin(t+\pi)\} \sim 1$$

$$= e^{-\pi s} e^{2\pi}\mathcal{L}\{-e^{2t}\sin t\} \sim 1 = -e^{\pi(2-s)}\mathcal{L}\{\sin t\}|_{s \to s-2}$$

$$= -e^{\pi(2-s)} \frac{1}{(s-2)^2+1} \sim 1$$

9. Find inverse Laplace transforms for the following functions:

(a) $F(s) = \dfrac{6s^2 + 11s - 1}{s^3 + 3s^2 + s + 3}$ (b) $F(s) = \dfrac{se^{-s}}{s^2 + 2s + 4}$

(a) $$f(t) = \mathcal{L}^{-1}\left\{\frac{6s^2+11s-1}{(s+3)(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{2}{s+3} + \frac{4s-1}{s^2+1}\right\} \sim 4$$

$$= 2e^{-3t} + 4\cos t - \sin t \sim 2$$

(b) $$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+4}\right\} = \mathcal{L}^{-1}\left\{\frac{(s+1)-1}{(s+1)^2+3}\right\} \sim 2 = e^{-t}\mathcal{L}^{-1}\left\{\frac{s-1}{s^2+3}\right\}$$

$$= e^{-t}\left(\cos\sqrt{3}t - \frac{1}{\sqrt{3}}\sin\sqrt{3}t\right) \sim 2$$

$$\therefore \cancel{f}\, f(t) = e^{-(t-1)}\left[\cos\sqrt{3}(t-1) - \frac{1}{\sqrt{3}}\sin\sqrt{3}(t-1)\right]h(t-1) \sim 2$$

metre. The other end of the spring is attached to a wall. The mass is set into horizontal motion (directly away from the wall) by pulling it 10 centimetres to the right of its equilibrium position and releasing it. During its subsequent motion, there is friction between the mass and the table with coefficient of kinetic friction equal to $\mu = 0.1$, but motion is free of any damping force proportional to velocity. After 1/2 second, the mass is hit to the left with an impulse force of 2 Newtons.

(a) Set up an initial-value problem for the displacement $x(t)$ of the mass from its equilibrium position.

(b) Show that, as long as the mass moves to the left, the Laplace transform of $x(t)$ is

$$X(s) = \frac{\frac{g}{5s} - 2e^{-s/2} + \frac{s}{5}}{2s^2 + 40},$$

where $g = 9.81$.

(c) Find the position of the mass as a function of time until it comes to rest for the first time.

(a) $2\frac{d^2x}{dt^2} + 40x = \frac{1}{10}(2)g - 2\delta(t-\frac{1}{2})$ -2, $x(0) = \frac{1}{10}$, $x'(0) = 0$

(b) When we take Laplace transforms,

$$2\left[s^2X - \frac{s}{10}\right] + 40X = \frac{g}{5s} - 2e^{-s/2}$$

$$X(s) = \frac{\frac{g}{5s} - 2e^{-s/2} + \frac{s}{5}}{2s^2+40} \quad -2$$

8 (c) $x(t) = \mathcal{L}^{-1}\left\{\frac{g}{10(s^2+20)s} - \frac{e^{-s/2}}{s^2+20} + \frac{s}{10(s^2+20)}\right\}$

$$= \frac{g}{10}\mathcal{L}^{-1}\left\{\frac{1/20}{s} + \frac{-s/20}{s^2+20}\right\} - \frac{1}{2\sqrt{5}}\sin 2\sqrt{5}(t-\tfrac{1}{2})h(t-\tfrac{1}{2}) \quad -2$$

$$+ \frac{1}{10}\cos 2\sqrt{5}t \quad -1$$

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$$= \frac{g}{200}\left[1 - \cos 2\sqrt{5}t\right] + \frac{1}{10}\cos 2\sqrt{5}t - \frac{1}{2\sqrt{5}}\sin 2\sqrt{5}(t-\tfrac{1}{2})h(t-\tfrac{1}{2})$$

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