MATH 2130 – Tutorial Problem Solutions

Double Integrals

Example. Let $f(x, y) = x(2 - y)^{1/3}$, and let R be the region in the xy-plane that is bounded by $y = x^2$ and $y = 2 - x$. Evaluate

$$\iint_R f(x, y)\, dA.$$

Solution. The line $y = 2 - x$ intersects the parabola $y = x^2$ at the points $(-2, 4)$ and $(1, 1)$. A sketch of R in the xy-plane shows that the double iterated integral requires two pieces if we put y on the outside, but only one if we put x on the outside. We make the latter choice. The region lies within $-2 \le x \le 1$. At each value of x, $x^2 \le y \le 2 - x$. Thus the given integral becomes

$$\begin{aligned}
\int_{-2}^{1}\int_{x^2}^{2-x} x(2-y)^{1/3}\, dy\, dx &= \int_{-2}^{1} x\left[-\frac{3}{4}(2-y)^{4/3}\right]_{y=x^2}^{2-x} dx \\
&= -\frac{3}{4}\int_{-2}^{1} x\left[x^{4/3} - (2-x^2)^{4/3}\right] dx \\
&= -\frac{3}{4}\int_{-2}^{1} \left(x^{7/3} - x(2-x^2)^{4/3}\right) dx \\
&= -\frac{3}{4}\left[\frac{3}{10}x^{10/3} + \frac{1}{2}\cdot\frac{3}{7}(2-x^2)^{7/3}\right]_{x=-2}^{1} \\
&= -\frac{3}{4}\left(\frac{3}{10} + \frac{3}{14} - \frac{3}{10}(-2)^{10/3} - \frac{3}{14}(-2)^{7/3}\right).
\end{aligned}$$

Using $(-2)^{10/3} = -8(-2)^{1/3}$ and $(-2)^{7/3} = 4(-2)^{1/3}$, this eventually reduces to

$$-\frac{27}{70} - \frac{81}{70}(-2)^{1/3}.$$

Alternatively: Suppose we put y on the outside. Then the integral requires two pieces.

- When $0 < y < 1$, $-\sqrt{y} < x < \sqrt{y}$.
- When $1 < y < 4$, $-\sqrt{y} < x < 2 - y$.

We get

$$\iint_R f(x, y)\, dA = \int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} x(2-y)^{1/3}\, dx\, dy + \int_1^4 \int_{-\sqrt{y}}^{2-y} x(2-y)^{1/3}\, dx\, dy.$$

Consider the innermost integral in the first term:

$$\int_{-\sqrt{y}}^{\sqrt{y}} x\, dx.$$

At each fixed value of y, the range of integration is symmetric under $x \mapsto -x$, and the integrand x is an odd function. Thus, by symmetry, this integral is 0. We get no contribution from the first term.

The second term becomes

$$\begin{aligned}\int_1^4 \int_{-\sqrt{y}}^{2-y} x(2-y)^{1/3}\,dx\,dy &= \int_1^4 (2-y)^{1/3}\left[\frac{1}{2}x^2\right]_{x=-\sqrt{y}}^{2-y} dy\\ &= \frac{1}{2}\int_1^4 (2-y)^{1/3}\left[(2-y)^2 - y\right]\,dy\\ &= \frac{1}{2}\int_1^4 (2-y)^{7/3}\,dy - \frac{1}{2}\int_1^4 y(2-y)^{1/3}\,dy.\end{aligned}$$

For the first of these terms, we get

$$\begin{aligned}\frac{1}{2}\int_1^4 (2-y)^{7/3}\,dy =& \frac{1}{2}\left[-\frac{3}{10}(2-y)^{10/3}\right]_{y=1}^4\\ =& -\frac{3}{20}(-2)^{10/3} + \frac{3}{20} = \frac{6}{5}(-2)^{1/3} + \frac{3}{20}.\end{aligned}$$

For the second, let $t = 2 - y$. Then $y = 2 - t$ and $dy = -dt$. When $y = 1$, $t = 1$; and when $y = 4$, $t = -2$. The integral becomes

$$\begin{aligned}-\frac{1}{2}\int_1^{-2} (2-t)t^{1/3}(-1)\,dt &= -\frac{1}{2}\int_{-2}^1 (2t^{1/3} - t^{4/3})\,dt\\ &= -\frac{1}{2}\left[\frac{3}{2}t^{4/3} - \frac{3}{7}t^{3/7}\right]_{t=-2}^1\\ &= -\frac{1}{2}\left(\frac{3}{2} - \frac{3}{7} - \frac{3}{2}(-2)^{4/3} + \frac{3}{7}(-2)^{7/3}\right) = -\frac{15}{28} - \frac{33}{14}(-2)^{1/3}.\end{aligned}$$

The answer is the sum of these two numbers:

$$\frac{6}{5}(-2)^{1/3} + \frac{3}{20} - \frac{15}{28} - \frac{33}{14}(-2)^{1/3} = -\frac{27}{70} - \frac{81}{70}(-2)^{1/3}.$$

We obtain, eventually, the same answer using either method.

Example. Evaluate the integral

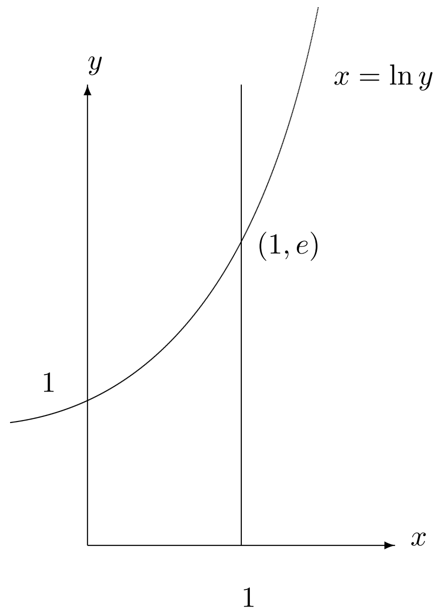
$$\int_0^1 \int_0^1 \sin(e^x)\,dx\,dy + \int_1^e \int_{\ln y}^1 \sin(e^x)\,dx\,dy$$

by first reversing the order of integration.

Solution. The first term in the sum represents the integral over the square with $0 \le x \le 1$ and $0 \le y \le 1$. The second term represents the integral over the region bounded by $y = 1$, $x = 1$ and $x = \ln y$.

We see that these two regions together are bounded by the curves $y = 0$, $x = 0$, $x = 1$ and $x = \ln y$, or equivalently, $y = e^x$. The region lies within $0 \le x \le 1$. At each value of x, we have $0 \le y \le e^x$. Thus the integral becomes

$$\begin{aligned}\int_0^1 \int_0^{e^x} \sin(e^x)\,dy\,dx &= \int_0^1 \Big[y\sin(e^x)\Big]_{y=0}^{e^x} dx\\ &= \int_0^1 e^x \sin(e^x)\,dx\\ &= \Big[-\cos(e^x)\Big]_{x=0}^1\\ &= \cos(1) - \cos(e).\end{aligned}$$



Example. Let $f(x, y) = x^3 y^2 \sin(xy)$, and let R be the disk $(x-2)^2 + y^2 \leq 4$. Evaluate

$$\iint_R f(x, y)\, dA.$$

Solution. The region R is bounded by the circle $(x-2)^2 + y^2 = 4$, which is the circle with center $(2, 0)$ and radius 2.

R lies within the range $0 \leq x \leq 4$, and at each value of x, we have $-\sqrt{4-(x-2)^2} \leq y \leq \sqrt{4-(x-2)^2}$. Alternatively, the disk lies within the range $-2 \leq y \leq 2$, and at each value of y, we have $-\sqrt{4-y^2} \leq x - 2 \leq \sqrt{4-y^2}$, which implies that $2 - \sqrt{4-y^2} \leq x \leq 2 + \sqrt{4-y^2}$. Thus two possible expressions for this integral are

$$\begin{aligned}\iint_R f(x, y)\, dA &= \int_0^4 \int_{-\sqrt{4-(x-2)^2}}^{\sqrt{4-(x-2)^2}} x^3 y^2 \sin(xy)\, dy\, dx \\ &= \int_{-2}^{2} \int_{2-\sqrt{4-y^2}}^{2+\sqrt{4-y^2}} x^3 y^2 \sin(xy)\, dx\, dy,\end{aligned}$$

which both look terrible.

The region of integration is symmetric under the transformation $y \mapsto -y$. Further, y^2 is even in y, and $\sin(xy)$ is odd in y, so the integrand $x^3 y^2 \sin(xy)$ is odd in y, for each value of x. In other words, at every value of x, we are integrating an odd function in y over an interval in y that is symmetric about $y = 0$, so the answer must be 0.

Applications of Double Integrals

Example. Let R be the region in the xy-plane that is bounded by $y = x^2$ and $y = 2 - x$. Find the volume of revolution obtained by revolving R about the line $y = 9 - 3x$.

Solution. Let (x, y) be an arbitrary point within R. We need the perpendicular distance from (x, y) to the line $y = 9 - 3x$.

The equation of the line can be written as $3x + y = 9$. From this form, we see that a normal vector to the line is $(3, 1)$. A point on the line is $(0, 9)$, so a vector from the line to (x, y) is $(x, y-9)$. The desired distance is the absolute value of the component of $(x, y - 9)$ in the direction of $(3, 1)$: that is,

$$\left|(x, y-9) \cdot \frac{1}{\sqrt{10}}(3,1)\right| = \frac{1}{\sqrt{10}} \left|3x + y - 9\right|.$$

The volume of revolution is then

$$V = \iint_R \frac{2\pi}{\sqrt{10}} \left|3x + y - 9\right| \, dA.$$

If we test any point in R – for example, the vertex $(1, 1)$ – we find that $3x + y - 9$ is negative on R. Thus the volume becomes

$$\begin{aligned}
V &= \iint_R \frac{2\pi}{\sqrt{10}} (9 - 3x - y) \, dA = \frac{2\pi}{\sqrt{10}} \int_{-2}^{1} \int_{x^2}^{2-x} (9 - 3x - y) \, dy \, dx \\
&= \frac{2\pi}{\sqrt{10}} \int_{-2}^{1} \left[9y - 3xy - \frac{1}{2}y^2 \right]_{y=x^2}^{2-x} dx = \frac{2\pi}{\sqrt{10}} \int_{-2}^{1} \left(16 - 13x - \frac{13}{2}x^2 + 3x^3 + \frac{1}{2}x^4 \right) dx \\
&= \frac{2\pi}{\sqrt{10}} \left[16x - \frac{13}{2}x^2 - \frac{13}{6}x^3 + \frac{3}{4}x^4 + \frac{1}{10}x^5 \right]_{x=-2}^{1} \\
&= \frac{2\pi}{\sqrt{10}} \left(16 - \frac{13}{2} - \frac{13}{6} + \frac{3}{4} + \frac{1}{10} + 32 + 26 - \frac{52}{3} - 12 + \frac{32}{10} \right).
\end{aligned}$$