Math 1210 Assignment 5 - Solutions Nov 24, 2008

1. $$\begin{bmatrix} 1 & 4 & 3 & 1 & -6 & 3 \\ 2 & 4 & 4 & 0 & -8 & 5 \\ 1 & 6 & 4 & 1 & -9 & 5 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{matrix} \begin{bmatrix} 1 & 4 & 3 & 1 & -6 & 3 \\ 0 & -4 & -2 & -2 & 4 & -1 \\ 0 & 2 & 1 & 0 & -3 & 2 \end{bmatrix} R_3 \leftrightarrow R_2$$

$$\begin{bmatrix} 1 & 4 & 3 & 1 & -6 & 3 \\ 0 & 2 & 1 & 0 & -3 & 2 \\ 0 & -4 & -2 & -2 & 4 & -1 \end{bmatrix} \begin{matrix} R_1 \to R_1 - 2R_2 \\ \\ R_3 \to R_3 + 2R_2 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 2 & 1 & 0 & -3 & 2 \\ 0 & 0 & 0 & -2 & -2 & 3 \end{bmatrix} \begin{matrix} \\ R_2 \to \frac{1}{2} R_2 \\ R_3 \to -\frac{1}{2} R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 1/2 & 0 & -3/2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3/2 \end{bmatrix} R_1 \to R_1 - R_3 \begin{bmatrix} 1 & 0 & 1 & 0 & -1 & 1/2 \\ 0 & 1 & 1/2 & 0 & -3/2 & 1 \\ 0 & 0 & 0 & 1 & 1 & -3/2 \end{bmatrix}$$

Solutions. $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -r + s - \frac{1}{2}t \\ -\frac{1}{2}r + \frac{3}{2}s - t \\ r \\ -s + \frac{3}{2}t \\ s \\ t \end{bmatrix}$$

The basic solutions are
$$\begin{bmatrix} -1 \\ -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3/2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ -1 \\ 0 \\ 3/2 \\ 0 \\ 1 \end{bmatrix}$$

2a) $\det A = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$

$$= 2(-1)^{2+1}\begin{vmatrix} 4 & -6 \\ 6 & 4 \end{vmatrix} + 8(-1)^{2+2}\begin{vmatrix} 1 & -6 \\ 1 & 4 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 1 & 6 \end{vmatrix}$$

$$= -2(16+36) - 8(4+6) - 5(6-4)$$
$$= -2(52) - 8(10) - 5(2)$$
$$= -194$$

2b) $\det A = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$

$$= (2)(-1)^{1+3}\begin{vmatrix} 1 & 2 \\ -1 & -3 \end{vmatrix} + (1)(-1)^{2+3}\begin{vmatrix} 3 & -1 \\ -1 & -3 \end{vmatrix} + (-4)(-1)^{3+3}\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}$$

$$= 2(-3+2) - 1(-9-1) - 4(6+1)$$

$$= 2(-1) - (-10) - 4(7)$$

$$= -20$$

2c) $$\begin{vmatrix} 0 & 3 & 9 \\ 5 & 10 & 25 \\ 0 & 2 & -5 \end{vmatrix} \overset{R_1 \leftrightarrow R_2}{=} (-1)\begin{vmatrix} 5 & 10 & 25 \\ 0 & 3 & 9 \\ 0 & 2 & -5 \end{vmatrix} \overset{R_2 \to \frac{1}{3}R_2}{=} (-1)(3)\begin{vmatrix} 5 & 10 & 25 \\ 0 & 1 & 3 \\ 0 & 2 & -5 \end{vmatrix}$$

$$\overset{R_3 \to R_3 - 2R_2}{=} (-1)(3)\begin{vmatrix} 5 & 10 & 25 \\ 0 & 1 & 3 \\ 0 & 0 & -11 \end{vmatrix} = (-1)(3)(5)(1)(-11) = 165.$$

3. This system is $AX = B$ where $A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & -3 & 5 \\ 4 & 7 & 8 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ $B = \begin{bmatrix} 6 \\ -10 \\ 22 \end{bmatrix}$

$$\det A = 4\begin{vmatrix} -3 & 5 \\ 7 & 8 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 4 & 8 \end{vmatrix} + 2\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix} = -172$$

So Cramer's Rule applies

$$A_{x_1} = \begin{bmatrix} 6 & 3 & 2 \\ -10 & -3 & 5 \\ 22 & 7 & 8 \end{bmatrix} \quad A_{x_2} = \begin{bmatrix} 4 & 6 & 2 \\ 2 & -10 & 5 \\ 4 & 22 & 8 \end{bmatrix} \quad A_{x_3} = \begin{bmatrix} 4 & 3 & 6 \\ 2 & -3 & -10 \\ 4 & 7 & 22 \end{bmatrix}$$

(Q3 - continued)

$$\det A_{x_1} = 6\begin{vmatrix} -3 & 5 \\ 7 & 8 \end{vmatrix} - 3\begin{vmatrix} -10 & 5 \\ 22 & 8 \end{vmatrix} + 2\begin{vmatrix} -10 & -3 \\ 22 & 7 \end{vmatrix} = 208$$

$$\det A_{x_2} = 4\begin{vmatrix} -10 & 5 \\ 22 & 8 \end{vmatrix} - 6\begin{vmatrix} 2 & 5 \\ 4 & 8 \end{vmatrix} + 2\begin{vmatrix} 2 & -10 \\ 4 & 22 \end{vmatrix} = -568$$

$$\det A_{x_3} = 4\begin{vmatrix} -3 & -10 \\ 7 & 22 \end{vmatrix} - 3\begin{vmatrix} 2 & -10 \\ 4 & 22 \end{vmatrix} + 6\begin{vmatrix} 2 & -3 \\ 4 & 7 \end{vmatrix} = -80$$

So $x_1 = \dfrac{\det A_{x_1}}{\det A} = \dfrac{208}{-172} = -\dfrac{52}{43}$

$x_2 = \dfrac{\det A_{x_2}}{\det A} = \dfrac{-568}{-172} = \dfrac{142}{43}$

$x_3 = \dfrac{\det A_{x_3}}{\det A} = \dfrac{-80}{-172} = \dfrac{20}{43}$

Q4 a) These vectors are linearly dependant since there are more vectors (3) than coordinates (2)

b) Since $\det\begin{bmatrix} 11 & 4 \\ 22 & 44 \end{bmatrix} = 396$ ($\neq 0$) these vectors are linearly independent.

c) $\det\begin{bmatrix} 1 & 2 & 4 \\ -1 & 5 & 1 \\ 1 & 3 & -6 \end{bmatrix} = \begin{vmatrix} 5 & 1 \\ 3 & -6 \end{vmatrix} - 2\begin{vmatrix} -1 & 1 \\ 1 & -6 \end{vmatrix} + 4\begin{vmatrix} -1 & 5 \\ 1 & 3 \end{vmatrix}$

$= -75$

Hence the vectors are linearly independent

d) $\det\begin{bmatrix} 3 & 1 & 6 \\ 1 & 2 & 2 \\ 3 & 4 & 6 \end{bmatrix} = 3\begin{vmatrix} 2 & 2 \\ 4 & 6 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix} + 6\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0$

(or you can note that two columns are one a scalar multiple of the other or $[6,2,6] = 2[3,1,3]$)

Hence the vectors are linearly dependent.

e) $C_1[1,-1,1,-1] + C_2[2,3,-1,2] + C_3[3,2,-1,1] = \vec{0}$

$$\begin{bmatrix} 1 & 2 & 3 \\ -1 & 3 & 2 \\ 1 & -1 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{matrix} \\ R_2 \to R_2 + R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 + R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 5 \\ 0 & -3 & -4 \\ 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ R_2 \to \frac{1}{5}R_2 \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & -3 & -4 \\ 0 & 4 & 4 \end{bmatrix} \begin{matrix} \\ \\ R_3 \to R_3 + 3R_2 \\ R_4 \to R_4 - 4R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \to -R_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The vectors are linearly independent. *

* Since the rank of the coefficient matrix of the homogeneous system is 3 and there are 3 variables, the above equation only has the trivial solution.