MAT2130: Engineering Mathematical Analysis 1

Final Exam Practice Problems

1. Evaluate each of the following double integrals. You can consider reversing the order of integration, or converting to polar coordinates, or performing the integration as written.

   (a) $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \sqrt{4 - x^2 - y^2}\, dx\, dy$

   (b) $\displaystyle\int_0^1 \int_{x^2}^{2-x^2} x\sqrt{y}\, dy\, dx$

   (c) $\displaystyle\int_0^1 \int_{2x}^{2} e^{y^2}\, dy\, dx$

   (d) $\displaystyle\int_0^1 \int_0^3 \frac{\sqrt{9 - x^2}}{y^2 + 1}\, dx\, dy$

   (e) $\displaystyle\int_2^9 \int_{\sqrt{y}}^{3} \cos(x^3 - 6x)\, dx\, dy$

   (f) $\displaystyle\int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx + \int_1^2 \int_0^{\sqrt{4-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy\, dx$

2. Set up, but do not evaluate, double iterated integral(s) for the following quantities.

   (a) The first moment of area of the region R about the line $x + 2y = 2$, where R is the disk $x^2 + y^2 \leq 1$.

   (b) The center of mass of the region R bounded by the lines $x - y - 3 = 0$, $x + 2y - 3 = 0$ and $5x + y + 3 = 0$, if the density of R is given by $\rho(x, y) = x^2 + y^2$.

   (c) The moment of inertia of the region R about the line $y = -1$, where R is bounded by $y = -2x^2$ and $y = -1$ and has constant density ρ.

   (d) The volume generated by revolving the region R about the line $x = 0$, where R lies in the half-plane $x \geq 0$ and is bounded by $y = 6 - \frac{1}{4}x^4$, $y = \frac{1}{2}x^2$ and $x = 3$.

   (e) The total force due to water pressure on an ellipse with semi-major axis a and semi-minor axis $b < a$, if it is standing in a fluid of density ρ with its major axis aligned vertically and if its center is at depth $2a$.

   (f) The centroid of the region enclosed by the curve $r = \sin\theta - \cos\theta$.

3. Set up, but do not evaluate, double iterated integral(s) for the area of each surface.

   (a) The portion of the surface $z = 3x^2 + y^2$ lying below the plane $3x + y + z = 1$.

   (b) The portion of the surface $z^2 = x - y^2$ contained within the volume that is bounded by $x = 0$, $y = 0$ and $x + y = 1$.

   (c) The portion of the surface $(x-1)^2+(y-2)^2+z^2 = 2$ lying inside the cylinder $x^2+y^2 = 1$.

   (d) The portion of the surface $x^2 = y^3 - z^3$ lying within the sphere $x^2 + y^2 + z^2 = 1$.

4. Use polar coordinates for each of the following.

   (a) Find the surface area of the portion of the sphere $x^2 + y^2 + z^2 = R^2$ lying above the plane $z = h$, where $0 \leq h \leq R$.

   (b) Find the surface area of the cone $z = a\sqrt{x^2 + y^2}$ lying below the plane $z = h$.

5. Find each volume. It might be useful to consider cylindrical or spherical coordinates.

   (a) In the first octant, bounded by the plane $2x + y + 3z = 6$.

   (b) Between $z = 1 - x^2 - y^2$ and the xy-plane.

   (c) Bounded by $y = 0$, $y = 4$, $x = z^2$ and $x = z^3$.

   (d) Bounded by $z = 2 - x^2 - y^2$ and $z = \sqrt{x^2 + y^2}$.

   (e) In the first octant, bounded by $z = 2 - y$ and $x = y + z$.

   (f) In the first octant, contained within $\frac{1}{2}(x^2 + y^2 + z^2)^{5/2} = xy$.

6. Find the area of an ellipse with axes of length $2a$ and $2b$, $a, b > 0$. Check that your answer makes sense in the case where $a = b$.

7. In the xy-plane, consider the two circles $x^2+y^2 = 1$ and $(x-a)^2+y^2 = 1$, where $1 < a < 2$. Set up a double integral in polar coordinates for the area of overlap of the two circles. **Challenge**: calculate the area of overlap.

8. Convert each of the following triple iterated integrals to cylindrical coordinates. Do not evaluate.

   (a) $$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{2-\sqrt{x^2+y^2}} \quad dy\, dx$$

   (b) $$\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} dz\, dy\, dx$$

   (c) $$\int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} dx\, dy\, dz$$

   (d) $$\int_{0}^{1/\sqrt{2}} \int_{-x}^{x} \int_{-1}^{1} dz\, dy\, dx + \int_{1/\sqrt{2}}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1}^{1} dz\, dy\, dx.$$

9. Convert each of the following triple integrals to spherical coordinates. Do not evaluate.

(a) $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2-y^2}}^{0} dx\,dy\,dz$

(b) $\displaystyle\int_{0}^{1}\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy\,dz$

(c) $\displaystyle\int_{0}^{2}\int_{-\sqrt{1-(z-1)^2}}^{\sqrt{1-(z-1)^2}}\int_{-\sqrt{1-(z-1)^2-y^2}}^{\sqrt{1-(z-1)^2-y^2}} dy\,dz.$

(d) $\displaystyle\int_{0}^{\sqrt{6}}\int_{-z/\sqrt{2}}^{z/\sqrt{2}}\int_{-\sqrt{z^2/2-y^2}}^{\sqrt{z^2/2-y^2}} dx\,dy\,dz + \int_{\sqrt{6}}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_{-\sqrt{9-z^2-y^2}}^{\sqrt{9-z^2-y^2}} dx\,dy\,dz$

10. Evaluate each of the following triple integrals. You can consider evaluating the integral as written, or changing the order of integration, or converting to a different coordinate system.

(a) $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\cos\theta}\int_{0}^{\sin\theta} rz\,dr\,dz\,d\theta$ (given in cylindrical coordinates)

(b) $\displaystyle\int_{0}^{2\pi}\int_{0}^{\pi/6}\int_{0}^{2\sec\phi} r^2\sin\phi\,dr\,d\phi\,d\theta$ (given in spherical coordinates)

(c) $\displaystyle\int_{0}^{1}\int_{0}^{x}\int_{y}^{x} \frac{xy}{\sqrt{z^2+y^2}}\,dz\,dy\,dx$ (given in Cartesian coordinates)

(d) $\displaystyle\int_{0}^{\pi/2}\int_{0}^{\phi}\int_{0}^{\cos\theta} r^2\sin\phi\,dr\,d\theta\,d\phi$ (given in spherical coordinates)

(e) $\displaystyle\int_{0}^{\pi}\int_{0}^{1}\int_{r^2+1}^{2} \frac{r(2z+1)}{(z^2+r^2)^2}\,dz\,dr\,d\theta$ (given in cylindrical coordinates)