MATH 1210 Fall 2013 Assignment 4 Solutions

Attempt all questions and show all your work. The assignment is due Friday, November 29.

1. *Decide if the following expressions can be determined from the given information. If they can, determine them, otherwise state that they can be determined.*

   Let A, B, C and D be 4×4 invertible matrices such that $|A| = 4, \quad |B| = -2, \quad |C| = 3$. Determine

   (a) $\left|A^2B^TC^{-1}\right|$

$$\begin{aligned}\left|A^2B^TC^{-1}\right| &= \left|A^2\right|\left|B^T\right|\left|C^{-1}\right| \\ &= |A|^2\,|B|\,\frac{1}{|C|} \\ &= 4^2(-2)\left(\frac{1}{3}\right) \\ &= -\frac{32}{3}.\end{aligned}$$

   (b) $\left|AB + C^TA\right|$

   This cannot be determined from the given information since there is no theorem which allows us to split up sum in the determinant. More specifically we can come up with different matrices which yield different answers

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

   Then

$$\left|AB + C^TA\right| = \begin{vmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -4$$

   However, change C to

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and we get

$$\left|AB + C^T A\right| = \begin{vmatrix} 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{vmatrix} = -8.$$

(c) $\left|ADBC^T D^{-1}\right|$

$$\begin{aligned} \left|ADBC^T D^{-1}\right| &= |A|\,|D|\,|B|\,\left|C^T\right|\,\left|D^{-1}\right| \\ &= |A|\,|D|\,|B|\,|C|\,\frac{1}{|D|} \\ &= |A|\,|B|\,|C| \\ &= 4(-2)(3) \\ &= -24. \end{aligned}$$

2. *Let* $A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$, *and* $B = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}$.

(a) *Use row reduction to find* A^{-1} *if it exists.*

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 1 & -2 & 3 & 0 & 0 & 1 \end{array}\right] \Rightarrow^{R_3 \to R_3 - R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & -3 & 5 & -1 & 0 & 1 \end{array}\right] \Rightarrow^{\substack{R_1 \to R_1 - R_2 \\ R_3 \to R_3 + 3R_2}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 3 & 1 \end{array}\right] \Rightarrow^{R_3 \to -R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{array}\right] \Rightarrow^{R_2 \to R_2 + 2R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 2 & -5 & -2 \\ 0 & 0 & 1 & 1 & -3 & -1 \end{array}\right].$$

Hence $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -5 & -2 \\ 1 & -3 & -1 \end{bmatrix}$.

(b) *Solve the system* $AX = B$.

If $AX = B$, then

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & -5 & -2 \\ 1 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -2 \\ -21 \\ -12 \end{bmatrix} \end{aligned}$$

(c) *Solve the system* $A^T X = B$. $X = (A^T)^{-1}B$. So we need to find $(A^T)^{-1}$. However, since $(A^T)^{-1} = (A^{-1})^T$ we have that

$$\begin{aligned} X &= (A^T)^{-1}B \\ &= \begin{bmatrix} 1 & -1 & 0 \\ 2 & -5 & -2 \\ 1 & -3 & -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 & 1 \\ -1 & -5 & -3 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 11 \\ -28 \\ -10 \end{bmatrix} \end{aligned}$$

(d) *Solve the system* $A^{-1}X = B$.
Multiplying both sides by A yields

$$\begin{aligned} X &= AB \\ &= \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} -4 \\ -5 \\ 7 \end{bmatrix} \end{aligned}$$

3. *Let* $A = \begin{bmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(a) *Find* $|A|$

This can be done a variety of ways, however row reduction is likely the easiest.

$$|A| = \begin{vmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 1 & -2 & 3 & 2 \\ 0 & 0 & 1 & 2 \end{vmatrix} \Rightarrow^{R_3 \to R_3 - R_1}$$

$$= \begin{vmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & -3 & 5 & 0 \\ 0 & 0 & 1 & 2 \end{vmatrix} \Rightarrow^{R_3 \to R_3 + 3R_2}$$

$$= \begin{vmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 1 & 2 \end{vmatrix} \Rightarrow^{R_4 \to R_4 + R_3}$$

$$= \begin{vmatrix} 1 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 6 \\ 0 & 0 & 0 & 8 \end{vmatrix} = 1(1)(-1)(8) = -8.$$

(b) Use Cramer's Rule to solve for x_3 only in $AX = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

$x_3 = \dfrac{|A_3|}{|A|}$ where A_3 is the matrix A with the 3rd column replaced by B. Hence

$$|A_3| = \begin{vmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 1 & -2 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix}.$$

Expanding along row 4 yields

$$|A_3| = -0 + 0 - 0 + 2 \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{vmatrix}$$

Expanding along column 3 yields

$$|A_3| = 2(1) \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 2(1)(1) = 2.$$

Hence $x_3 = \dfrac{2}{-8} = -\dfrac{1}{4}$.

(c) *If* $adj(A) = \begin{bmatrix} -8 & 8 & 0 & 0 \\ -6 & 10 & 6 & -10 \\ a & 6 & 2 & -6 \\ b & -3 & -1 & -1 \end{bmatrix}$, *find* a *and* b.

Since $adj(A) = C^T$ where C is the cofactor matrix, we have that $a = c_{13}$ and $b = c_{14}$. Hence

$$a = (-1)^4 \begin{vmatrix} 0 & 1 & 2 \\ 1 & -2 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = 2(-1) = -2$$

by expanding along the 3rd row.

$$b = (-1)^5 \begin{vmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ 0 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} 0 & 1 \\ 1 & -2 \end{vmatrix} = -1(-1) = 1$$

by expanding along the 3rd row.

(d) *Find* A^{-1}.

From part (c) we know that

$$adj(A) = \begin{bmatrix} -8 & 8 & 0 & 0 \\ -6 & 10 & 6 & -10 \\ -2 & 6 & 2 & -6 \\ 1 & -3 & -1 & -1 \end{bmatrix}$$

and from part (a) we know that $|A| = -8$. Hence

$$A^{-1} = \frac{adj(A)}{|A|} = -\frac{1}{8} \begin{bmatrix} -8 & 8 & 0 & 0 \\ -6 & 10 & 6 & -10 \\ -2 & 6 & 2 & -6 \\ 1 & -3 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3/4 & -5/4 & -3/4 & 5/4 \\ 1/4 & -3/4 & -1/4 & 3/4 \\ -1/8 & 3/8 & 1/8 & 1/8 \end{bmatrix}$$

4. *For the matrix* $A = \begin{bmatrix} x & 2 & -2 \\ 0 & 1 & -2 \\ 1 & -2 & x \end{bmatrix}$, *find* x *such that* A *is not invertible.*

A is not invertible when $|A| = 0$. Hence we need to find $|A|$. Using cofactor expansion along row 2 we get the determinant is

$$-0 + 1 \begin{vmatrix} x & -2 \\ 1 & x \end{vmatrix} - (-2) \begin{vmatrix} x & 2 \\ 1 & -2 \end{vmatrix} = (x^2 + 2) + 2(-2x - 2) = x^2 - 4x - 2.$$

Setting this equal to 0 and solving using the quadratic formula yields

$$x = \frac{4 \pm \sqrt{16+8}}{2} = \frac{4 \pm \sqrt{24}}{2} = 2 \pm \sqrt{6}.$$

5. *Determine whether the following vectors are linearly independent or dependent.*

$$\mathbf{u_1} = \langle 1, -2, 1, 4 \rangle, \ \mathbf{u_2} = \langle 2, 6, -3, 3 \rangle, \ \mathbf{u_3} = \langle 1, 2, -1, 2 \rangle.$$

If they are linearly dependent, write one vector as a linear combination of the others.

We are trying to solve the equation

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} = \mathbf{0}$$

which leads to

$$c_1\langle 1, -2, 1, 4 \rangle + c_2\langle 2, 6, -3, 3 \rangle + c_3\langle 1, 2, -1, 2 \rangle = \langle 0, 0, 0, 0 \rangle$$

or

$$\langle c_1 + 2c_2 + c_3, -2c_1 + 6c_2 + 2c_3, c_1 - 3c_2 - c_3, 4c_1 + 3c_2 + 2c_3 \rangle = \langle 0, 0, 0, 0 \rangle$$

leading to the system of equation

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ -2c_1 + 6c_2 + 2c_3 &= 0 \\ c_1 - 3c_2 - c_3 &= 0 \\ 4c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

Putting these into an augmented matrix yields

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -2 & 6 & 2 & 0 \\ 1 & -3 & -1 & 0 \\ 4 & 3 & 2 & 0 \end{array}\right] \Rightarrow^{R_2 \to R_2 + 2R_1,\ R_3 \to R_3 - R_1}_{R_4 \to R_4 - 4R_1}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 10 & 4 & 0 \\ 0 & -5 & -2 & 0 \\ 0 & -5 & -2 & 0 \end{array}\right] \Rightarrow^{R_2 \to R_2/10}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & -5 & -2 & 0 \\ 0 & -5 & -2 & 0 \end{array}\right] \Rightarrow^{R_1 \to R_1 - 2R_2,\ R_3 \to R_3 + R_2}_{R_4 \to R_4 - +5R_2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1/5 & 0 \\ 0 & 1 & 2/5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Sinve there is no leading one in column 3, we get non-trivial solutions and therefore the vectors are linearly dependent. As for the solution, we have that c_3 is arbitrary, $c_1 = -c_3/5$ and $c_2 = -2c_3/5$. Letting c_3 be anything non-zero (say 5) we get

$$c_1 = -1,\ c_2 = -2,\ c_3 = 5$$

Makes

$$-\mathbf{u_1} - 2\mathbf{u_2} + 5\mathbf{u_3} = \mathbf{0}$$

leading to one of the following

$$\begin{aligned} \mathbf{u_1} &= -2\mathbf{u_2} + 5\mathbf{u_3} \\ \mathbf{u_2} &= -\frac{1}{2}\mathbf{u_1} + \frac{5}{2}\mathbf{u_3} \\ \mathbf{u_3} &= \frac{1}{5}\mathbf{u_1} + \frac{2}{5}\mathbf{u_2} \end{aligned}$$

6. *Determine whether each of the following sets of vectors are linearly independent or dependent. Explain your answers.*

   (a) $\{\langle 1,2,3\rangle,\ \langle 2,3,4\rangle,\ \langle 1,-1,5\rangle\}$
   We are trying to solve the equation

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} = \mathbf{0}$$

   which leads to

$$c_1\langle 1,2,3\rangle + c_2\langle 2,3,4\rangle + c_3\langle 1,-1,5\rangle = \langle 0,0,0\rangle$$

   or

$$\langle c_1 + 2c_2 + c_3, 2c_1 + 3c_2 - c_3, 3c_1 + 4c_2 + 5c_3\rangle = \langle 0,0,0\rangle$$

   leading to the system of equation

$$\begin{aligned} c_1 + 2c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 - c_3 &= 0 \\ 3c_1 + 4c_2 + 5c_3 &= 0 \end{aligned}$$

   The coefficient matrix is square, so we can determine if we get only the trivial solution by finding its determinant.

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & -1 \\ 3 & 4 & 5 \end{vmatrix} = 1 \begin{vmatrix} 3 & -1 \\ 4 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & -1 \\ 3 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 1(19) - 2(13) + (-1) = -8.$$

Since $|A| \neq 0$ the system has only the trivial solution and hence the vectors are linearly independent.

(b) $\{\langle 1, 2, 3, 4, 5, 6\rangle,\ \langle -2, -4, -6, -8, -10, -6\rangle\}$
Since there are only two vectors which are not muliples of each other, we know the vectors are linearly independent.

(c) $\{\langle 1, 2, 3\rangle,\ \langle 2, 3, 4\rangle,\ \langle 1, -1, 5\rangle,\ \langle 4, -10, 3\rangle\}$
Since there are more vectors than components in the set, we know the vectors are linearly dependent.

(d) $\{\langle 1, -4, 3, 5\rangle,\ \langle 2, 3, 4, -1\rangle,\ \langle -2, 8, -6, -10\rangle\}$
Since the third vector is a multiple of the first, we know the vectors are linearly dependent.

7. *Show that every vector with 3 components can be written as a linear combination of $\langle 1, 2, 3\rangle$, $\langle 2, 3, 4\rangle$ and $\langle 1, -1, 5\rangle$.*

We need to show that

$$c_1\mathbf{u_1} + c_2\mathbf{u_2} + c_3\mathbf{u_3} = \langle v_x, v_y, v_z\rangle$$

has a solution for all v_x, v_y and v_z which leads to

$$c_1\langle 1, 2, 3\rangle + c_2\langle 2, 3, 4\rangle + c_3\langle 1, -1, 5\rangle = \langle v_x, v_y, v_z\rangle$$

or

$$\langle c_1 + 2c_2 + c_3, 2c_1 + 3c_2 - c_3, 3c_1 + 4c_2 + 5c_3\rangle = \langle v_x, v_y, v_z\rangle$$

leading to the system of equation

$$\begin{aligned} c_1 + 2c_2 + c_3 &= v_x \\ 2c_1 + 3c_2 - c_3 &= v_y \\ 3c_1 + 4c_2 + 5c_3 &= v_z \end{aligned}$$

Since the coeffecient matrix is square, this has a (unique) solution for all v_x, v_y and v_z if the determinant is non-zero. From question 6(a) we know $|A| = -8 \neq 0$ and therefore the system has a solution for all v_x, v_y and v_z.