Solutions to MATH 2130 Test 1 2010

Values

9 **1.** (a) Show that the lines

$$\frac{x-2}{3} = \frac{y+4}{6} = \frac{2z-1}{5} \qquad \text{and} \qquad x+2y+3z=33, \quad x-y+z=6$$

are not parallel.

(b) Assuming that the lines intersect at some point, find the equation of the plane containing the lines. Simplify your equation as much as possible.

(a) A vector along the first line is $3\hat{\mathbf{i}}+6\hat{\mathbf{j}}+(5/2)\hat{\mathbf{k}}$, and so also is $6\hat{\mathbf{i}}+12\hat{\mathbf{j}}+5\hat{\mathbf{k}}$. A vector along the second line is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = 5\hat{\mathbf{i}}+2\hat{\mathbf{j}}-3\hat{\mathbf{k}}.$$

Since these vectors are not multiples of one another, they are not parallel. The lines are not therefore parallel.

(b) A vector normal to the plane is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 5 & 2 & -3 \\ 6 & 12 & 5 \end{vmatrix} = 46\hat{\mathbf{i}}-43\hat{\mathbf{j}}+48\hat{\mathbf{k}}.$$

The equation of the required plane is

$$46(x-2)-43(y+4)+48(z-1/2)=0 \qquad \Longrightarrow \qquad 46x-43y+48z=288.$$

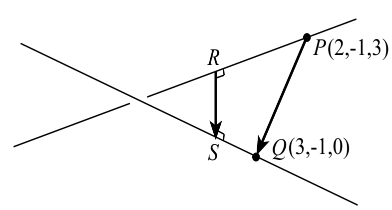
8 **2.** Find the distance between the lines $x=2+t$, $y=-1+t$, $z=3-2t$ and $2x-2y+3z=8$, $x+y-z=2$.

Vectors along the lines are $\hat{\mathbf{i}}+\hat{\mathbf{j}}-2\hat{\mathbf{k}}$ and

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & -2 & 3 \\ 1 & 1 & -1 \end{vmatrix} = -\hat{\mathbf{i}}+5\hat{\mathbf{j}}+4\hat{\mathbf{k}}.$$

The lines are therefore not parallel. Points on the lines are $P(2,-1,3)$ and $Q(3,-1,0)$. A vector normal to both lines is

$$\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & -2 \\ -1 & 5 & 4 \end{vmatrix} = 14\hat{\mathbf{i}}-2\hat{\mathbf{j}}+6\hat{\mathbf{k}}.$$



The required distance is the component of $\mathbf{PQ}$ along $\mathbf{RS}$,

$$|\mathbf{PQ}\cdot\hat{\mathbf{RS}}| = \left|(1,0,-3)\cdot\frac{(14,-2,6)}{\sqrt{196+4+36}}\right| = \left|\frac{-4}{\sqrt{236}}\right| = \frac{2}{\sqrt{59}}.$$

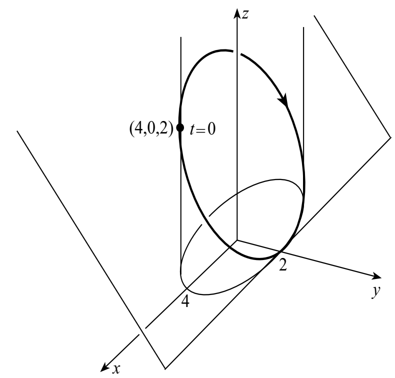
9 **3.** (a) Find parametric equations for the curve

$$x^2 + 4y^2 = 16, \quad y + z = 2.$$

Assume that the curve is directed counterclockwise as viewed from the origin.

(b) Find a unit tangent vector to the curve in part (a) at the point $(0, 2, 0)$.

(c) Set up, but do **NOT** evaluate, a definite integral for the length of the curve in part (a).



(a) Parametric equations for the curve (without regard to direction) are

$x = 4\cos t,$

$y = 2\sin t,$

$z = 2 - 2\sin t,$

where $0 \le t \le 2\pi$. Since $t = 0$ gives the point $(4, 0, 2)$, and when t is slightly larger than zero, y is positive, these equations give the wrong direction along the curve. Correct parametric equations are

$$x = 4\cos t, \quad y = -2\sin t, \quad z = 2 + 2\sin t, \quad 0 \le t \le 2\pi.$$

(b) A tangent vector at any point on the curve is

$$\mathbf{T} = (-4\sin t)\hat{\mathbf{i}} - (2\cos t)\hat{\mathbf{j}} + (2\cos t)\hat{\mathbf{k}}.$$

Since $t = 3\pi/2$ gives the required point

$$\mathbf{T}(3\pi/2) = 4\hat{\mathbf{i}}.$$

A unit tangent vector is $\hat{\mathbf{i}}$.

(c) The length of the curve is

$$L = \int_0^{2\pi} \sqrt{(-4\sin t)^2 + (-2\cos t)^2 + (2\cos t)^2}\, dt.$$

9 **4.** Determine whether the following limit exists,

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - x^2y + 2y^4}{2x^4 - 5x^2y + 4y^4}.$$

If the limit does not exist, explain why not.

If we approach $(0, 0)$ along the parabolas $y = ax^2$, the limit becomes

$$\lim_{x\to 0} \frac{x^4 - x^2(ax^2) + 2(ax^2)^4}{2x^4 - 5x^2(ax^2) + 4(ax^2)^4} = \lim_{x\to 0} \frac{1 - a + 2a^4x^4}{2 - 5a + 4a^4x^4} = \frac{1 - a}{2 - 5a}.$$

Since this value depends on which parabola is used, it follows that the original limit does not exist.

5 **5.** Find all points where the curve $y^2 = z^2 + 1$, $z = x^2$ intersects the surface $x^2 + y^2 + z^2 = 2$.

If we substitute $x^2 = z$ and $y^2 = z^2 + 1$ into the equation of the surface,

$$z + z^2 + 1 + z^2 = 2 \qquad \Longrightarrow \qquad 0 = 2z^2 + z - 1 = (2z - 1)(z + 1).$$

Thus, $z = 1/2$ or $z = -1$. The second of these is impossible. With $z = 1/2$, we find that $x = \pm 1/\sqrt{2}$ and $y = \pm\sqrt{5}/2$. The points of intersection are

$$\left(\frac{1}{\sqrt{2}}, \frac{\pm\sqrt{5}}{2}, \frac{1}{2}\right) \quad \text{and} \quad \left(-\frac{1}{\sqrt{2}}, \frac{\pm\sqrt{5}}{2}, \frac{1}{2}\right).$$