MATH 1210 Tutorial # 11

Solutions

1. Find all values of c, if any, for which the matrix

$$A = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix}$$

is invertible. Find A^{-1} for those values of c.

<u>*Solution*</u>: The cofactors of the entries of A are $C_{11} = c^2 - 1,\ C_{12} = -c,$ $C_{13} = -c,\ C_{21} = -c,\ C_{22} = c^2,\ C_{23} = -c,\ C_{31} = 1,\ C_{32} = -c,\ C_{33} = c^2 - 1.$ Thus,

$$\det(A)I_3 = A{\cdot}\mathrm{adj}(A) = \begin{pmatrix} c & 1 & 0 \\ 1 & c & 1 \\ 0 & 1 & c \end{pmatrix} \cdot \begin{pmatrix} c^2-1 & -c & 1 \\ -c & c^2 & -c \\ 1 & -c & c^2-1 \end{pmatrix} = c(c^2{-}2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

so A is invertible if and only if $\det(A) = c(c^2 - 1) \neq 0$, if and only if $c \neq 0, \pm\sqrt{2}$, and for all those values of c we obtain

$$A^{-1} = \frac{1}{\det(A)}\mathrm{adj(A)} = \frac{1}{c(c^2-2)} \begin{pmatrix} c^2-1 & -c & 1 \\ -c & c^2 & -c \\ 1 & -c & c^2-1 \end{pmatrix}.$$

2. Find $\det(\mathrm{adj}(A))$ if A is a 7×7 matrix such that $\det(A) = 3$. Does the answer depend on the choice of A? Why or why not?"

Hint: use $A^{-1} = [1/\det(A)]\mathrm{adj}(A)$.

<u>*Solution*</u>: By using the hint and $\det(A^{-1}) = \frac{1}{\det(A)}$ we get

$$\frac{1}{3} = \det(A^{-1}) = \det\left[\frac{1}{\det(A)}\mathrm{adj}(A)\right] = \left(\frac{1}{\det(A)}\right)^7 \det(\mathrm{adj})(A) = \left(\frac{1}{3}\right)^7 \det(\mathrm{adj}(A)),$$

so $\det(\operatorname{adj}(A)) = (\frac{1}{3})(\frac{1}{3})^{-7}) = (\frac{1}{3})^{-6}) = 3^6 = 729$. Clearly, the answer does not depend on the choice of A; it depends only on the dimensions of A and on the value of its determinant.

3. Given the matrix

$$A = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find $\operatorname{adj}(A)$.
(b) Find $\det(A)$ and determine for which values of θ the matrix A is invertible.
(c) Find A^{-1} by using
  (i) the adjoint matrix method for inversion of a matrix;
  (ii) the direct method.

Solution: (a) The cofactors of the entries of A are $C_{11} = \cos\theta, C_{12} = \sin\theta, C_{13} = 0, C_{21} = -\sin\theta, C_{22} = \cos\theta, C_{23} = 0, C_{31} = 0, C_{32} = 0, C_{33} = 1$. Thus

$$\operatorname{adj}(A) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(b) Expanding along the third row (or column) we get

$$\det(A) = 1(-1)^{3+3}(\cos^2\theta - (-\sin^2\theta)) = \cos^2\theta + \sin^2\theta = 1,$$

regardless of the value of θ. Therefore A is invertible for all values of θ.

(c)(i) We get

$$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{1}\operatorname{adj}(A) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c)(ii) Assume first that $\cos\theta \neq 0$. Then

$$A \mid I_3 = \left(\begin{array}{ccc|ccc} \cos\theta & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 \to (\cos\theta)R_1} \left(\begin{array}{ccc|ccc} \cos^2\theta & \sin\theta\cos\theta & 0 & \cos\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + (-\sin\theta)R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2\to R_2+(\sin\theta)R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta\cos\theta & 1-\sin^2\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$=\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \cos\theta & 0 & \sin\theta\cos\theta & \cos^2\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)\xrightarrow{R_2\to\frac{1}{\cos\theta}R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \cos\theta & -\sin\theta & 0 \\ 0 & 1 & 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right).$$

If $\cos\theta=0$, then $\sin^2\theta=1$, and we get

$$A\mid I_3=\left(\begin{array}{ccc|ccc} 0 & \sin\theta & 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)\xrightarrow{R_1\leftrightarrow R_2}\left(\begin{array}{ccc|ccc} -\sin\theta & 0 & 0 & 0 & 1 & 0 \\ 0 & \sin\theta & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_2\to R_2(\sin\theta)]{R_1\to R_1(-\sin\theta)}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & \sin\theta & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Therefore, in both cases we get that $A^{-1}=\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which coincides with the answer to (c)(i).

4. Determine whether or not the system of linear equations

$$\begin{array}{rcrcrcrcl} x_1 & + & 3x_2 & + & x_3 & + & x_4 & = & 1 \\ 2x_1 & + & 5x_2 & + & 2x_3 & + & 2x_4 & = & 1 \\ x_1 & + & 3x_2 & + & 8x_3 & + & 9x_4 & = & 1 \\ x_1 & + & 3x_2 & + & 2x_3 & + & x_4 & = & 1 \end{array}$$

has a unique solution. If "yes", find the solution by the inverse matrix method.

Solution: Denoting the coefficient matrix of the system by A, we get

$$A\mid I_4=\left(\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 2 & 5 & 2 & 2 & 0 & 1 & 0 & 0 \\ 1 & 3 & 8 & 9 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right)\xrightarrow[R_2\to R_2-2R_1;R_3\to R_3-R_1]{R_4\to R_4-R_1}$$

$$\left(\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \end{array}\right)\xrightarrow[R_2\to(-1)R_2]{R_3\leftrightarrow R_4}\left(\begin{array}{cccc|cccc} 1 & 3 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 7 & 8 & -1 & 0 & 1 & 0 \end{array}\right)\xrightarrow[R_4\to R_4-7R_3]{R_1\to R_1-3R_2-R_3}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 8 & 6 & 0 & 1 & -7 \end{array}\right) \xrightarrow{R_4 \to \frac{1}{8}R_4} \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & -4 & 3 & 0 & -1 \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & 0 & \frac{1}{8} & -\frac{7}{8} \end{array}\right) \xrightarrow{R_1 \to R_1 - R_4}$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -\frac{19}{4} & 3 & -\frac{1}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & 0 & 2 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{4} & 0 & \frac{1}{8} & -\frac{7}{8} \end{array}\right)$$

It follows that the rank of the 4×4-matrix A is 4, so the system has a unique solution

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = A^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{19}{4} & 3 & -\frac{1}{8} & -\frac{1}{8} \\ 2 & -1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ \frac{3}{4} & 0 & \frac{1}{8} & -\frac{7}{8} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$