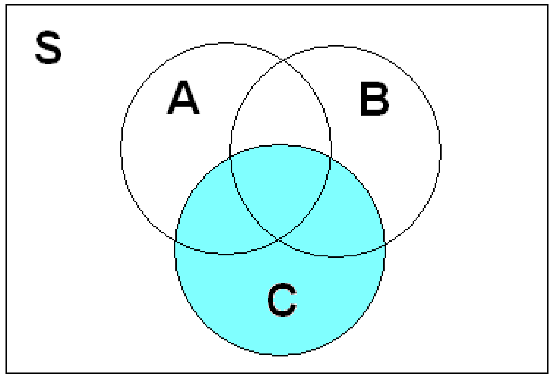
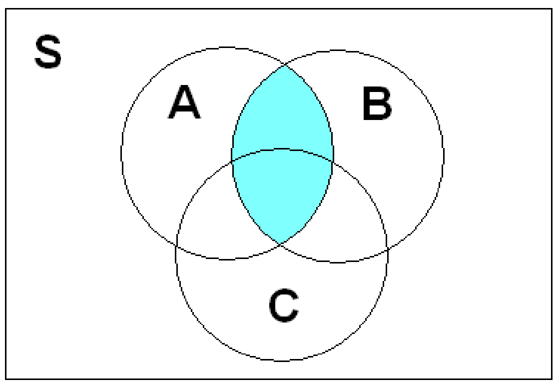
Unit 5 Assignment Solutions

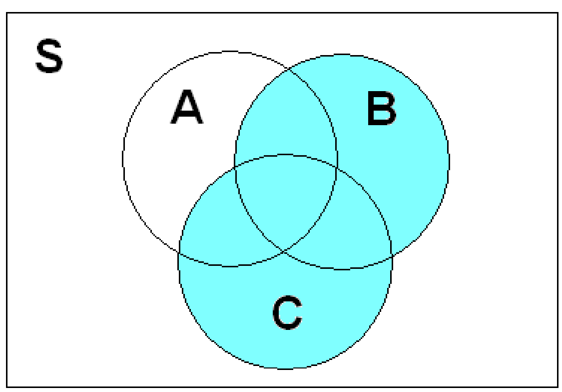
1. (a) $P(C) = 0.10 + 0.05 + 0.08 + 0.12 + 0.06 = 0.41$.



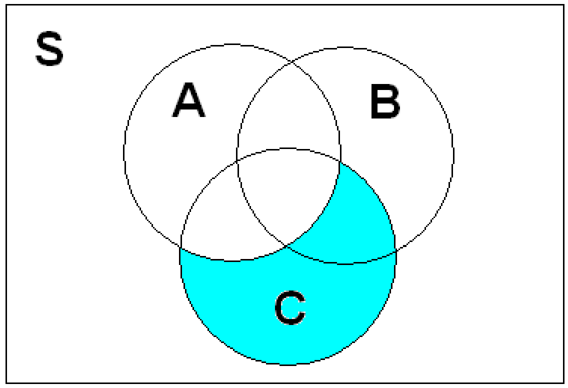
(b) $P(A \cap B) = 0.09 + 0.05 = 0.14$.



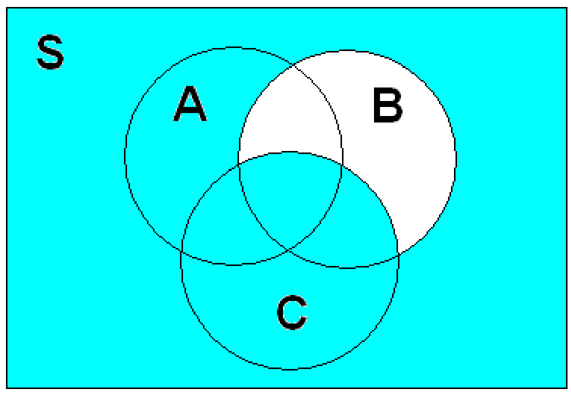
(c) $P(B \cup C) = 0.04 + 0.07 + 0.09 + 0.10 + 0.05 + 0.08 + 0.12 + 0.06 = 0.61$.



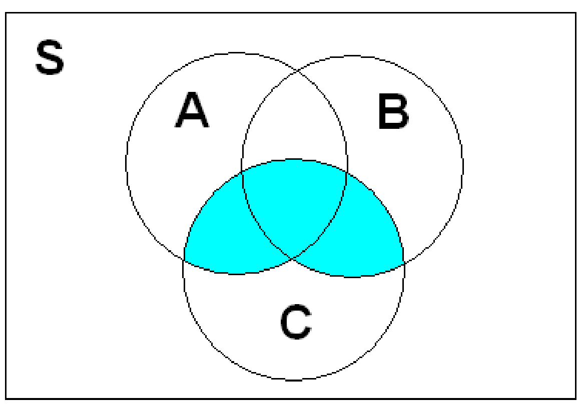
(d) $P(A' \cap C) = 0.12 + 0.06 + 0.08 = 0.26$.



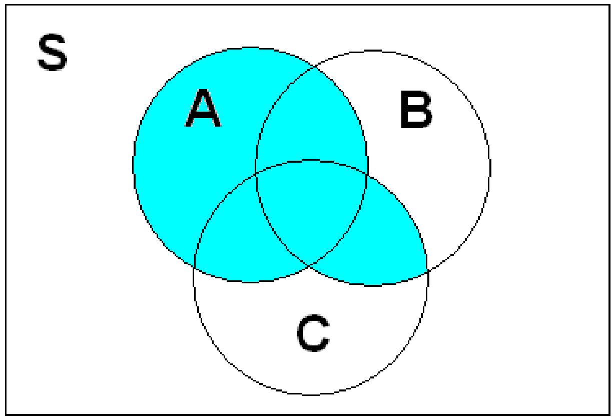
(e) $P(B' \cup C) = 0.05 + 0.09 + 0.10 + 0.05 + 0.08 + 0.06 + 0.12 + 0.11 + 0.14 = 0.80$.



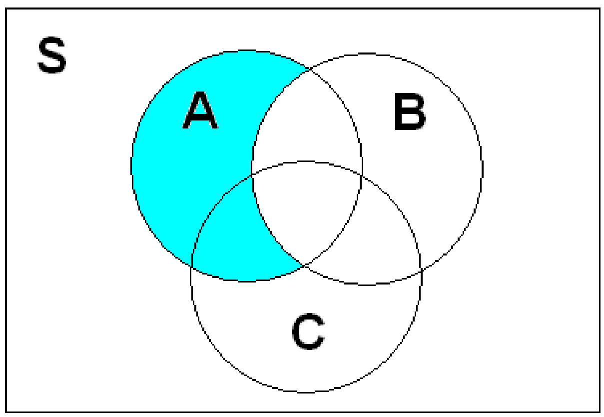
(f) $P((A \cup B) \cap C) = 0.10 + 0.05 + 0.08 = 0.23$.



(g) $P(A \cup (B \cap C)) = 0.05 + 0.09 + 0.09 + 0.05 + 0.10 + 0.08 = 0.46$.

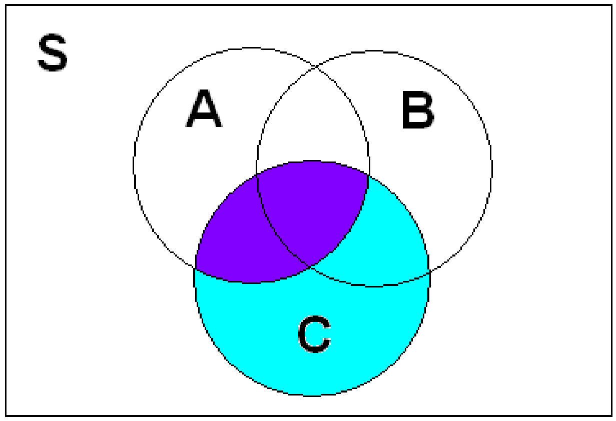


(h) $P(A' \cup B)' = 0.05 + 0.09 + 0.10 = 0.24$.



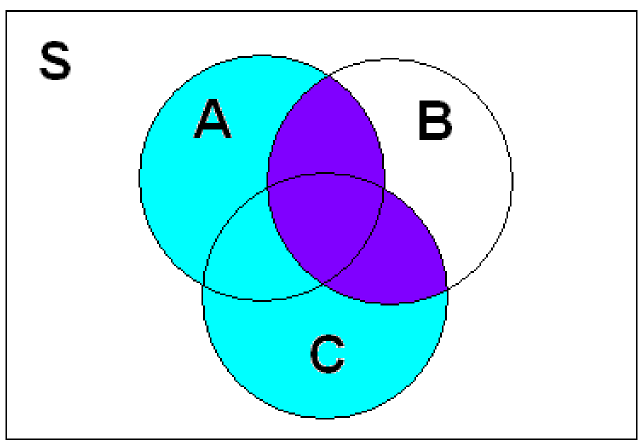
*For all Venn diagrams shown for conditional probabilities, the probability of the event in the numerator is the area shaded in purple and the probability of the event in the denominator is the area shaded in purple plus the area shaded in blue.

(i) $$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{0.10 + 0.05}{0.10 + 0.05 + 0.08 + 0.12 + 0.06} = \frac{0.15}{0.41} = 0.3659.$$

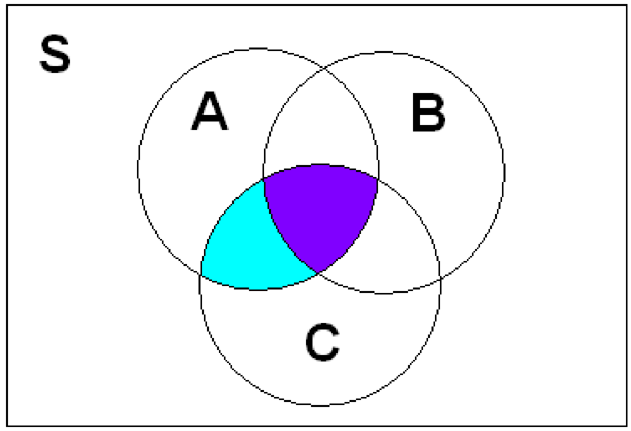


(j) $$P(B \mid A \cup C) = \frac{P(B \cap (A \cup C))}{P(A \cup C)}$$

$$= \frac{0.09 + 0.05 + 0.08}{0.09 + 0.05 + 0.08 + 0.05 + 0.09 + 0.10 + 0.12 + 0.06} = \frac{0.22}{0.64} = 0.3438.$$



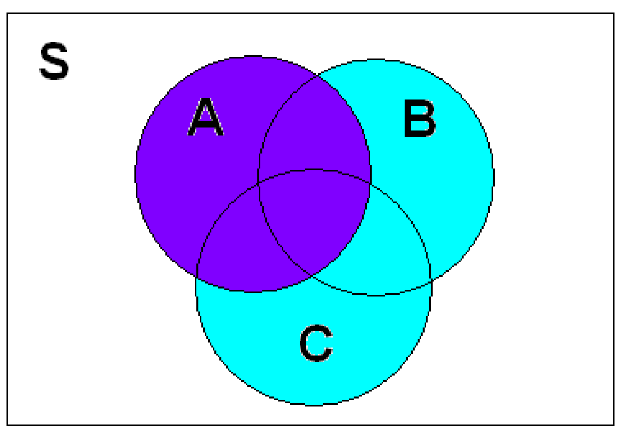
(k) $$P(B \mid A \cap C) = \frac{P(B \cap A \cap C)}{P(A \cap C)} = \frac{0.05}{0.05 + 0.10} = \frac{0.05}{0.15} = 0.3333.$$



(l) $$P(A \mid A \cup B \cup C) = \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A)}{P(A \cup B \cup C)}$$

$$= \frac{0.05 + 0.09 + 0.09 + 0.05 + 0.10}{0.05 + 0.09 + 0.09 + 0.05 + 0.10 + 0.04 + 0.07 + 0.08 + 0.12 + 0.06} = \frac{0.38}{0.75} = 0.5067.$$

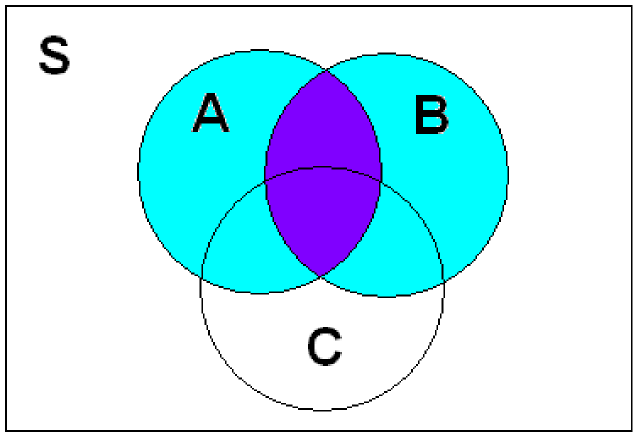
*Note that $A \subset (A \cup B \cup C)$, i.e. A is completely contained in $(A \cup B \cup C)$, so we must have that $P(A \cap (A \cup B \cup C)) = P(A)$.



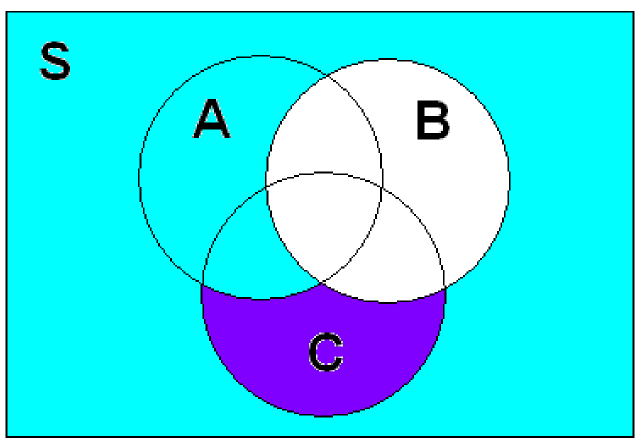
(m) $$P(A \cap B \mid A \cup B) = \frac{P((A \cap B) \cap (A \cup B))}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)}$$

$$= \frac{0.09 + 0.05}{0.09 + 0.05 + 0.05 + 0.09 + 0.10 + 0.04 + 0.07 + 0.08} = \frac{0.14}{0.57} = 0.2456$$

*Note that $(A \cap B) \subset (A \cup B)$, so we must have that $P((A \cap B) \cap (A \cup B)) = P(A \cap B)$.



(n) $$P(A' \cap C \mid B') = \frac{P((A' \cap C) \cap B')}{P(B')} = \frac{0.12 + 0.06}{0.12 + 0.06 + 0.05 + 0.09 + 0.10 + 0.11 + 0.14} = \frac{0.18}{0.67} = 0.2687.$$



2. (a) P(your friend wins) = P(friend gets T before you get T)
= P(T on friend's first toss)
+ P(H on friend's first toss, H on your first toss, T on friend's second toss)
+ P(H on friend's first toss, H on your first toss, H on friend's second toss, H on your second toss, T on friend's third toss)
$+\cdots$

The flips are independent, so this is equal to

$(0.5) + (0.5)(0.4)(0.5) + (0.5)(0.4)(0.5)(0.4)(0.5) + \cdots = a + ar + ar^2 + \cdots$

This is an infinite geometric series with first term a = 0.5 and multiplicative term (0.5)(0.4) = 0.2. It follows that

$$\text{P(your friend wins)} = \sum_{i=0}^{\infty} (0.5)[(0.2)]^i = \frac{0.5}{1-0.2} = \frac{0.5}{0.8} = 0.625.$$

(b) It is clear from the above solution that, if p is the probability of flipping tails on your coin, then the probability that your friend wins is

$$\text{P(your friend wins)} = \sum_{i=0}^{\infty} 0.5[0.5(1-p)]^i = \frac{0.5}{1-0.5(1-p)} = \frac{0.5}{0.5+p}$$

For this to be a fair game, the probability your friend wins must be 0.5. Clearly then, we must have that $p = 1$.

3. Let G be the event that the player scores a goal. Let P be the event that the player gets a penalty. Let W be the event that the player's team wins.

(a) $P(G \cup P) = P(G) + P(P) - P(G \cap P) = 0.53 + 0.40 - 0.20 = 0.73.$

(b) $P(G \cup W) = P(G) + P(W) - P(G \cap W) \Rightarrow P(W) = P(G \cup W) - P(G) + P(G \cap W)$
$= 0.74 - 0.53 + 0.39 = 0.60\ (60\%).$

(c) $P(W \mid G') = \dfrac{P(W \cap G')}{P(G')} = \dfrac{0.21}{0.47} = 0.4468.$

Note that, we know P(G) = 0.53, so the probability he doesn't score a goal is $P(G') = 1 - P(G) = 0.47$. The event that his team wins can be divided into two mutually exclusive events, i.e. {Team wins and he scores a goal} and {Team wins and he doesn't score a goal}. We know his team wins 60% of their games, and in 39% of their games, his team wins and he scores a goal. Therefore, his team wins and he doesn't score a goal in 60% – 39% = 21% of his games, i.e. $P(W \cap G') = 0.21$.

(d) $P(G)P(P) = (0.53)(0.40) = 0.212 \neq P(G \cap P) = 0.20$, so G and P are not independent.
$P(G)P(W) = (0.53)(0.60) = 0.318 \neq P(G \cap W) = 0.39$, so G and W are not independent.
$P(P)P(W) = (0.40)(0.60) = 0.24 = P(P \cap W)$, so P and W are independent.

(e) Let $D = \{B \cup C\}$. We have

$$
\begin{aligned}
&P(A \cup B \cup C) = P(A \cup D) = P(A) + P(D) - P(A \cap D) = P(A) + P(B \cup C) - P(A \cap (B \cup C)) \\
&= P(A) + P(B \cup C) - P((A \cap B) \cup (A \cap C)) \\
&= P(A) + \left[P(B) + P(C) - P(B \cap C)\right] - \left[P(A \cap B) + P(A \cap C) - P(A \cap B \cap C\right] \\
&= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)
\end{aligned}
$$

(f) $P(G' \cap P' \cap W') = P(G \cup P \cup W)' = 1 - P(G \cup P \cup W)$
$= 1 - [P(G) + P(P) + P(W) - P(G \cap P) - P(G \cap W) - P(P \cap W) + P(G \cap P \cap W)$
$1 - [0.53 + 0.40 + 0.60 - 0.20 - 0.39 - 0.24 + 0.12] = 1 - 0.82 = 0.18.$

4. (a) S = {FFF, RFFF, FRFF, FFRF, RRFFF, RFRFF, RFFRF, FRRFF, FRFRF, FFRRF,
RRR, FRRR, RFRR, RRFR, FFRRR, FRFRR, FRRFR, RFFRR, RFRFR, RRFFR}

* Note that there is no outcome such as FFFRR, because if Federer won the first three sets, the fourth and fifth set would not have been played.

(b) Let A be the event that the match lasts five sets. Then

A = {RRFFF, RFRFF, RFFRF, FRRFF, FRFRF, FFRRF,
FFRRR, FRFRR, FRRFR, RFFRR, RFRFR, RRFFR}

The probability that the match lasts five sets is equal to the sum of the probabilities of these twelve outcomes. Since outcomes are independent, we can find these probabilities using multiplication:

P(RRFFF) = P(RFRFF) = P(RFFRF) = P(FRRFF) = P(FRFRF) = P(FFRRF) = $(0.4)^2(0.6)^3 = 0.03456$.
P(FFRRR) = P(FRFRR) = P(FRRFR) = P(RFFRR) = P(RFRFR) = P(RRFFR) = $(0.4)^3(0.6)^2 = 0.02304$.

The probability that the match lasts five sets is 6(0.03456) + 6(0.02304) = 0.3456.

(c) Let B be the event that Roddick wins the match. Then

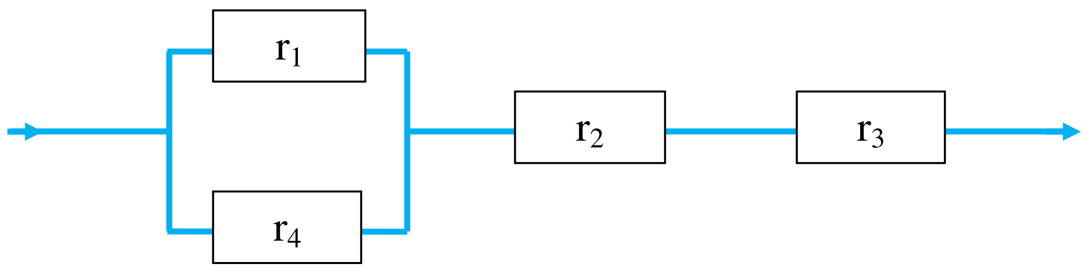
B = {RRR, FRRR, RFRR, RRFR, FFRRR, FRFRR, FRRFR, RFFRR, RFRFR, RRFFR}

The probability that Roddick wins the series is equal to the sum of the probabilities of these ten outcomes. Since sets are independent, we can find these probabilities using multiplication:

Outcome	P(Outcome)
RRR	(0.4)(0.4)(0.4) = 0.06400
FRRR	(0.6)(0.4)(0.4)(0.4) = 0.03840
RFRR	(0.4)(0.6)(0.4)(0.4) = 0.03840
RRFR	(0.4)(0.4)(0.6)(0.4) = 0.03840
FFRRR	(0.6)(0.6)(0.4)(0.4)(0.4) = 0.02304
FRFRR	(0.6)(0.4)(0.6)(0.4)(0.4) = 0.02304
FRRFR	(0.6)(0.4)(0.4)(0.6)(0.4) = 0.02304
RFFRR	(0.4)(0.6)(0.6)(0.4)(0.4) = 0.02304
RFRFR	(0.4)(0.6)(0.4)(0.6)(0.4) = 0.02304
RRFFR	(0.4)(0.4)(0.6)(0.6)(0.4) = 0.02304
	P(Roddick wins) = 0.31774

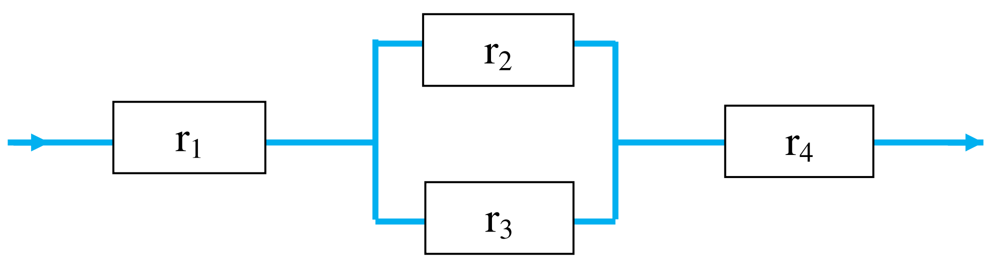
5. (a) If the fourth component is placed in parallel with the first component, then

$$
\begin{aligned}
r_s &= P((C_1 \cup C_4) \cap C_2 \cap C_3) = [1-(1-r_1)(1-r_4)]\, r_2\, r_3 \\
&= [1-(1-r_1-r_4+r_1 r_4)]\, r_2\, r_3 = [r_1+r_4-r_1 r_4]\, r_2\, r_3 = r_1\, r_2\, r_3 + r_2\, r_3\, r_4 - r_1\, r_2\, r_3 r_4
\end{aligned}
$$



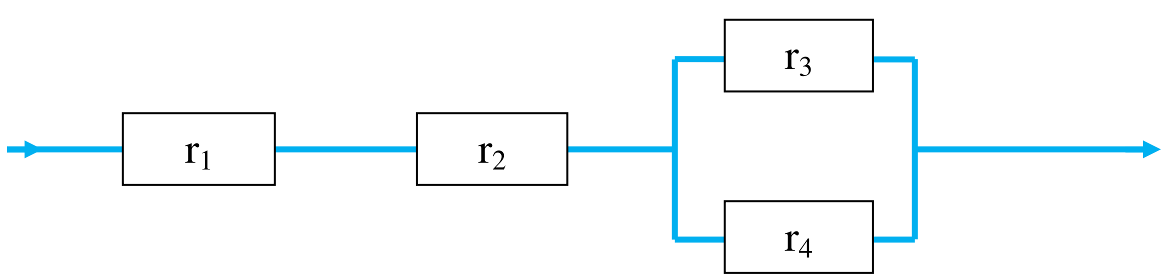
If the fourth component is placed in parallel with the second component, then

$$
\begin{aligned}
r_s &= P(C_1 \cap (C_2 \cup C_4) \cap C_3) = r_1\, [1-(1-r_2)(1-r_4)]\, r_3 \\
&= r_1\, [1-(1-r_2-r_4+r_2 r_4)]\, r_3 = r_1 [r_2+r_4-r_2 r_4]\, r_3 = r_1\, r_2\, r_3 + r_1\, r_3\, r_4 - r_1\, r_2\, r_3 r_4
\end{aligned}
$$



If the fourth component is placed in parallel with the third component, then

$$
\begin{aligned}
r_s &= P(C_1 \cap C_2 \cap (C_3 \cup C_4)) = r_1\, r_2\, [1-(1-r_3)(1-r_4)] \\
&= r_1\, r_2\, [1-(1-r_3-r_4+r_3 r_4)] = r_1\, r_2 [r_3+r_4-r_3 r_4] = r_1\, r_2\, r_3 + r_1\, r_2\, r_4 - r_1\, r_2\, r_3 r_4
\end{aligned}
$$

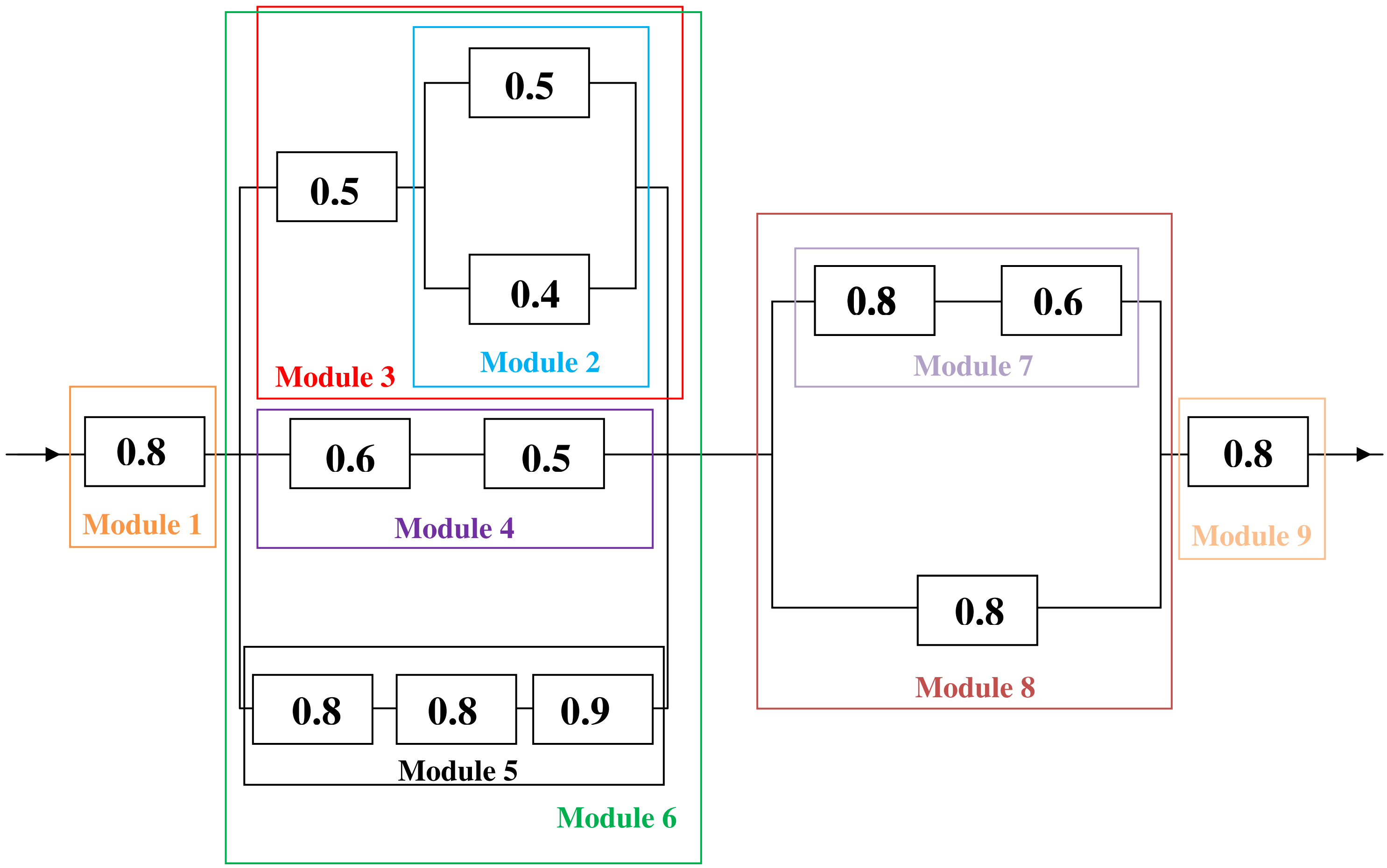


The reliabilities are summarized below:

C_4 placed in parallel with	Reliability r_s
C_1	$r_1 r_2 r_3 - r_1 r_2 r_3 r_4 + r_2 r_3 r_4$
C_2	$r_1 r_2 r_3 - r_1 r_2 r_3 r_4 + r_1 r_3 r_4$
C_3	$r_1 r_2 r_3 - r_1 r_2 r_3 r_4 + r_1 r_2 r_4$

Note that the green parts are common to all three reliabilities. The system reliabilities differ **only** by the terms in red. Notice that, when C_4 is placed in parallel with component C_i ($i = 1, 2$ or 3), this term in red is the product of the reliabilities of C_4 and the **other two** components. Therefore, this product will be greatest when C_4 is placed in parallel with the component with the **lowest** reliability among C_1, C_2 and C_3 (i.e. component 1). This is true **regardless** of the values of r_1, r_2, r_3, and r_4.

(b) The system is divided into modules as shown below:



The reliability of the system is $r_s = P(M_1)P(M_6)P(M_8)P(M_9)$

We know that $P(M_1) = 0.8$ and $P(M_9) = 0.8$.

Now, $P(M_6) = P(M_3 \cup M_4 \cup M_5)$

$P(M_3) = 0.5P(M_2) = 0.5(1 - (1 - 0.5)(1 - 0.4)) = 0.5(1 - 0.3) = 0.5(0.7) = 0.35$.

$P(M_4) = 0.6(0.5) = 0.30$.

$P(M_5) = 0.8(0.8)(0.9) = 0.576$.

So $P(M_6) = 1 - (1 - 0.35)(1 - 0.30)(1 - 0.576) = 1 - 0.19292 = 0.807$.

$P(M_7) = 0.8(0.6) = 0.48$

So $P(M_8) = 1 - (1 - 0.48)(1 - 0.8) = 1 - 0.104 = 0.896$.

The system reliability is therefore $r_s = P(M_1)P(M_6)P(M_8)P(M_9)= 0.8(0.807)(0.896)(0.8) = 0.4628$.