MATH 1210 Problem Workshop 3 Solutions

1. Simplify each of the following expressions to Cartesian form:

   (a) Here we repeatedly use $i^2 = -1$.

$$\begin{aligned}
\frac{(1+2i^3)^2\overline{(3-i)}}{4+i} &= \frac{(1+2i^2 i)^2(3+i)}{4+i} \\
&= \frac{(1-2i)^2(3+i)}{4+i} \\
&= \frac{(1-4i+4i^2)(3+i)}{4+i} \\
&= \frac{(-4i-3)(3+i)}{4+i} \\
&= \frac{-12i-9-4i^2-3i}{4+i} \\
&= \frac{-5-15i}{4+i} \\
&= \frac{(-5-15i)(4-i)}{(4+i)(4-i)} \\
&= \frac{-20+5i-60i+15i^2}{16-i^2} \\
&= \frac{-35-55i}{17} \\
&= -\frac{35}{17} - \frac{55}{17}i
\end{aligned}$$

   (b) First we convert to exponential form.
   For the modulus,

$$\left|\sqrt{3}-i\right| = \sqrt{(\sqrt{3})^2 + (-1)^2} = \sqrt{4} = 2.$$

   For the argument,

$$\tan\theta = \frac{y}{x} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = -\frac{\pi}{6}.$$

   Hence

$$\sqrt{3} - i = 2e^{-\pi i/6}.$$

   From here we get

$$\begin{aligned}
(\sqrt{3}-i)^{14} &= (2e^{-\pi i/6})^{14} \\
&= 2^{14}e^{-14\pi i/6} \\
&= 2^{14}\left(\cos\left(-\frac{14\pi}{6}\right)+i\sin\left(-\frac{14\pi}{6}\right)\right) \\
&= 2^{14}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \\
&= 2^{13}-2^{13}\sqrt{3}i
\end{aligned}$$

2. Express each of the following in exponential form. Your final answer should have the principal argument.

   (a) First we convert to exponential form.
   For the modulus,

   $$\left|3+3\sqrt{3}i\right| = \sqrt{3^2+(3\sqrt{3})^2} = \sqrt{36} = 6.$$

   For the argument,

   $$\tan\theta = \frac{y}{x} = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}.$$

   Hence

   $$3+3\sqrt{3}i = 6e^{\pi i/3}.$$

   From here we get

   $$\begin{aligned}
   (3+3\sqrt{3}i)^7e^{5\pi i/6} &= (6e^{\pi i/3})^7e^{5\pi i/6} \\
   &= 6^7e^{7\pi i/3}e^{5\pi i/6} \\
   &= 6^7e^{19\pi i/6} \\
   &= 6^7e^{-5\pi i/6}
   \end{aligned}$$

   (b) First we convert to exponential form.
   For the modulus,

   $$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}.$$

   For the argument,

   $$\tan\theta = \frac{y}{x} = 1 \Rightarrow \theta = \frac{\pi}{4}.$$

   Hence

   $$1+i = \sqrt{2}e^{\pi i/4}.$$

   From here we get

$$
\begin{aligned}
\frac{\sqrt{2}e^{\pi i/4}e^{3\pi i/4}}{3e^{-\pi i/3}} &= (6e^{\pi i/3})^7 e^{5\pi i/6} \\
&= \frac{\sqrt{2}}{3}e^{4\pi i/3} \\
&= \frac{\sqrt{2}}{3}e^{-2\pi i/3}
\end{aligned}
$$

3. Find exact values for all solutions of the following equations. Express final answers in Cartesian form.

   (a) Let $w = x^2$, then the equation turns into $2w^2 + 3w - 1 = 0$. Using the quadratic formula we get

   $$w = \frac{-3 \pm \sqrt{9+8}}{4} = \frac{-3 \pm \sqrt{17}}{4}$$

   Hence we have that

   $$x^2 = \frac{-3 \pm \sqrt{17}}{4}.$$

   For $x^2 = \frac{-3+\sqrt{17}}{4} > 0$, we have that

   $$x = \pm\sqrt{\frac{-3+\sqrt{17}}{4}}$$

   For $x^2 = \frac{-3-\sqrt{17}}{4} < 0$, we have that

   $$x = \pm\sqrt{\frac{-3-\sqrt{17}}{4}} = \pm\sqrt{\frac{3+\sqrt{17}}{4}}i.$$

   (b) We first convert $-4i$ to exponential form to get $4e^{3\pi i/2}$. Since we may get multiple solutions, we note that

   $$-4i = 4e^{3\pi i/2 + 2k\pi i}$$

   for any integer k.
   Hence

   $$
   \begin{aligned}
   z = (-4i)^{1/4} &= \left(4e^{3\pi i/2+2k\pi i}\right)^{1/4} \\
   &= 4^{1/4}e^{3\pi i/8 + k\pi i/2}
   \end{aligned}
   $$

   Using $k = 0, 1, 2$ and 3 we get the 4 solutions

   $$z = \sqrt{2}e^{3\pi i/8}, \sqrt{2}e^{7\pi i/8}, \sqrt{2}e^{11\pi i/8}, \sqrt{2}e^{15\pi i/8}.$$

In cartesian form we get $\sqrt{2}\cos\left(\frac{3\pi}{8}\right)+\sqrt{2}\sin\left(\frac{3\pi}{8}\right)i$, $\sqrt{2}\cos\left(\frac{7\pi}{8}\right)+\sqrt{2}\sin\left(\frac{7\pi}{8}\right)i$, $\sqrt{2}\cos\left(\frac{11\pi}{8}\right)+\sqrt{2}\sin\left(\frac{11\pi}{8}\right)i$, $\sqrt{2}\cos\left(\frac{15\pi}{8}\right)+\sqrt{2}\sin\left(\frac{15\pi}{8}\right)i$

4. Find the square roots of $5+12i$ by:

(a) Let $z = x + yi$, then we are finding when $(x+yi)^2 = 5+12i$. Therefore

$$5+12i = (x^2-y^2)+2xyi \Rightarrow 2xy = 12, x^2-y^2 = 5$$

where x and y are real.

$$\begin{aligned}
xy = 6 \Rightarrow y &= \frac{6}{x} \\
&\Rightarrow x^2 - \left(\frac{6}{x}\right)^2 = 5 \\
&\Rightarrow x^4 - 5x^2 - 36 = 0 \\
&\Rightarrow (x^2-9)(x^2+4) = 0 \\
&\Rightarrow x^2 = 9, x^2 = -4 \\
&\Rightarrow x = \pm 3 \\
&\Rightarrow y = \pm 2
\end{aligned}$$

since x must be real.

Therefore the roots are $3+2i$ and $-3-2i$.

(b) Converting to exponential form yields

$$z^2 = 13e^{\pi\theta+2\pi ki}$$

where $\cos\theta = \frac{5}{13}$.

Hence

$$z = \sqrt{13}e^{\pi(\theta/2)+\pi ki}.$$

Hence if $k=0$ we get

$$z = \sqrt{13}\left(\cos\left(\frac{\theta}{2}\right)+i\sin\left(\frac{\theta}{2}\right)\right)$$

Using the half angle indentities

$$\cos\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\cos\theta}{2}} = \sqrt{\frac{1+\frac{5}{13}}{2}} = \sqrt{\frac{9}{13}} = \frac{3}{\sqrt{13}}.$$

$$\sin\left(\frac{\theta}{2}\right) = \sqrt{\frac{1-\cos\theta}{2}} = \sqrt{\frac{1-\frac{5}{13}}{2}} = \sqrt{\frac{4}{13}} = \frac{2}{\sqrt{13}}.$$

Therefore we get that

$$z = \sqrt{13}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right) = 3+2i.$$

From there we can get the other root is the negative.

5. We first note that

$$\begin{aligned}\cos 3\theta + i\sin 3\theta &= e^{i(3\theta)} \\ &= (e^{i\theta})^3 \\ &= (\cos\theta + i\sin\theta)^3 \\ &= (\cos^3\theta - 3\cos\theta\sin^2\theta) + i(3\cos^2\theta\sin\theta - \sin^3\theta).\end{aligned}$$

Hence

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta \quad \text{and} \quad \sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

6. We use that $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$ to get

$$e^{-i\theta} = \cos(-\theta) + i\sin(-\theta) = \cos\theta - i\sin\theta$$

Therefore

$$\begin{aligned}\frac{e^{i\theta}+e^{-i\theta}}{2} &= \frac{\cos\theta + i\sin\theta + \cos\theta - i\sin\theta}{2} \\ &= \frac{2\cos\theta}{2} \\ &= \cos\theta\end{aligned}$$

and

$$\begin{aligned}\frac{e^{i\theta}-e^{-i\theta}}{2i} &= \frac{\cos\theta + i\sin\theta - (\cos\theta - i\sin\theta)}{2i} \\ &= \frac{2i\sin\theta}{2i} \\ &= \sin\theta\end{aligned}$$

7. First we turn $-2-2i$ into exponential form

$$|-2-2i| = \sqrt{8} = 2^{3/2} \text{ and } \tan\theta = \frac{-2}{-2} = 1 \Rightarrow \theta = \frac{5\pi}{4}.$$

Now to find the fifth roots of $-2-2i$ we need to find solutions to $z^5 = 2^{3/2}e^{i(5\pi/4+2k\pi)}$ where k can be any integer.

We can use $k = 0,1,2,3,4$ to get the 5 different answers (note that using $k = 5$ will yield the same solution as $k = 0$.

using $z = re^{i\theta}$ we get that $z^5 = r^5e^{i(5\theta)}$ and hence we are solving

$$r^5 = 2^{3/2} \Rightarrow r = 2^{3/10}$$

and

$$5\theta = \frac{5\pi}{4} + 2k\pi \Rightarrow \theta = \frac{\pi}{4} + \frac{2k\pi}{5}$$

$\underline{k=0}$

$\theta = \pi/4$ and hence

$$z = 2^{3/10}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) = 2^{3/10}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = 2^{-1/5} + 2^{-1/5}i$$

$\underline{k=1}$

$\theta = 13\pi/20$ and hence

$$z = 2^{3/10}\left(\cos\left(\frac{13\pi}{20}\right) + i\sin\left(\frac{13\pi}{20}\right)\right) = 2^{3/10}\cos\left(\frac{13\pi}{20}\right) + 2^{3/10}\sin\left(\frac{13\pi}{20}\right)i$$

(It's fine to leave it like this for non nice values.)

$\underline{k=2}$

$\theta = 21\pi/20$ and hence

$$z = 2^{3/10}\left(\cos\left(\frac{21\pi}{20}\right) + i\sin\left(\frac{21\pi}{20}\right)\right) = 2^{3/10}\cos\left(\frac{21\pi}{20}\right) + 2^{3/10}\sin\left(\frac{21\pi}{20}\right)i$$

$\underline{k=3}$

$\theta = 29\pi/20$ and hence

$$z = 2^{3/10}\left(\cos\left(\frac{29\pi}{20}\right) + i\sin\left(\frac{29\pi}{20}\right)\right) = 2^{3/10}\cos\left(\frac{29\pi}{20}\right) + 2^{3/10}\sin\left(\frac{29\pi}{20}\right)i$$

and

$\underline{k=4}$

$\theta = 37\pi/20$ and hence

$$z = 2^{3/10}\left(\cos\left(\frac{37\pi}{20}\right) + i\sin\left(\frac{37\pi}{20}\right)\right) = 2^{3/10}\cos\left(\frac{37\pi}{20}\right) + 2^{3/10}\sin\left(\frac{37\pi}{20}\right)i$$