16.107 Final Examination (April 23, 1999) - Solutions

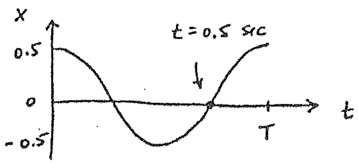
Sforge 28/11/99.

A1 $x(t) = 0.5 \cos(3\pi t)$ metres $= 0.5 \cos(2\pi \frac{t}{T})$

at $t = 0.5$ sec, $x(0.5) = 0.5 \cos(\frac{3\pi}{2}) = 0$

the period $T = \frac{2}{3}$ sec; at $t = 0.5$ sec, it has gone through

$(\frac{t}{T}) = \frac{1}{2} / \frac{2}{3} = 3/4$ of a cycle:

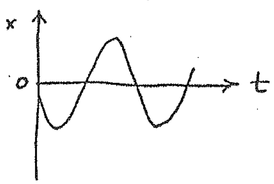


$\therefore$ distance travelled is $3(0.5)$

$\boxed{= 1.5\text{ m c)}}$

A2 $x = x_m \cos(\omega t + \phi)$ at $t = 0$, $x = x_m \cos\phi$

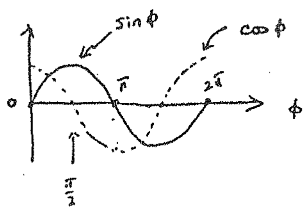
from the graph, $x < 0$ at $t = 0$ $\therefore$ $\cos\phi$ is negative



Next check the slope at $t = 0$

$\frac{dx}{dt}$ is negative $= -x_m \sin\phi$ @ $t = 0$

so $\sin\phi$ is positive



so $\boxed{\frac{\pi}{2} \le \phi \le \pi \text{ d)}}$

A3 Max length of spring = equilibrium length + amplitude of oscillation

$L_{max} = 8.0\text{ cm} + x_0$

let $x = x_0 \cos(\omega t + \phi)$ for SHM.

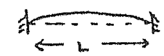
Use conservation of energy; at $x = 0$, $E = \frac{1}{2}mv^2$, all kin.
at $x = x_0$, $E = \frac{1}{2}kx_0^2$, all pot.

$\therefore \frac{1}{2}mv^2 = \frac{1}{2}kx_0^2$ $\therefore$ $x_0 = \sqrt{\frac{m}{k}}\, v = 2.1\text{ cm}$

$L_{max} = 8.0 + 2.1\text{ cm} = 10.1\text{ cm}$, round off to $\boxed{10\text{ cm b)}}$

A4 Analyze the fundamental modes of the wire and the tube:

(i) wire:

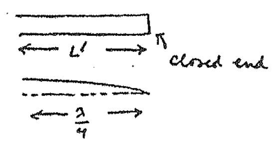


$L = \frac{\lambda}{2}$

Speed of wave: $\lambda f = v = \sqrt{\frac{T}{\mu}}$ where T = tension
$\mu = \frac{M}{L}$ = mass density

solve for $T = \mu v^2 = \mu \lambda^2 f^2 = \mu (2L)^2 f^2 = 4MLf^2$ ~ ①

(ii) tube



$L' = \frac{\lambda}{4}$ for fundamental mode

$\lambda f = v = 340$ m/s

$\therefore f = \frac{v}{\lambda} = \frac{v}{4L'}$ ~ ②

plug into ① and solve for T:

$T = 4MLf^2 = \frac{4ML\, v^2}{16(L')^2} = \boxed{360\text{ N e)}}$

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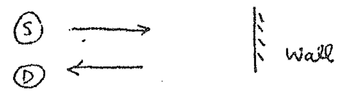
A5 The bat is both the source and detector of the sound!



$v = 340$ m/s speed of sound

Doppler effect: $f' = f\dfrac{(v \pm v_D)}{(v \pm v_S)}$; $f' > f$ when S/D move towards each other

$f = 80$ kHz. Sound emitted by moving source.

$$f' = 80 \text{ kHz}\left(\frac{(340)}{(340-12)}\right)$$

Reflected at this frequency but detected by the moving bat:

$$f'' = f'\frac{(340+12)}{340} = 80 \text{ kHz}\frac{(340+12)}{(340-12)} = 85.85 \text{ kHz}$$

Period $T = \dfrac{1}{f''} = 1.2 \times 10^{-5}$ sec $= \boxed{0.012 \text{ ms (a)}}$

A6

travelling wave $y = y_m \cos(kx - \omega t + \phi)$

speed of travel: $\dfrac{\omega}{k} = 12.4$ m/s ← ①

transverse velocity: $\dfrac{dy}{dt} = +\omega y_m \sin(kx - \omega t + \phi)$

∴ max value is $\omega y_m = 9.4$ m/s $\rightarrow \omega = \dfrac{9.4}{y_m}$ ← ②

find $\lambda = \dfrac{2\pi}{k} = 2\pi\cdot\dfrac{(12.4)}{\omega} = 2\pi\dfrac{(12.4)\cdot(4.5 \text{ cm})}{(9.4)}$

∴ $\boxed{\lambda = 37 \text{ cm b)}}$

A7 Two light waves: in air, $\lambda f = c \Rightarrow \lambda = \dfrac{c}{f} = 620$ nm

in the media, $c \rightarrow \dfrac{c}{n}$ ∴ $\lambda_1 = \dfrac{\lambda}{n_1}$ and $\lambda_2 = \dfrac{\lambda}{n_2}$

wave oscillates as $\sin(kx - \omega t + \phi)$ for each one;

for path length L, the phase change of each wave is

$\Delta\phi = kL = \dfrac{2\pi L}{\lambda}$ after passing through the med

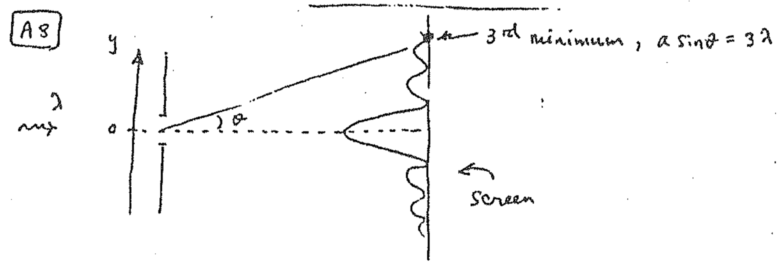
we want $|\Delta\phi_2 - \Delta\phi_1| = \pi$ to bring them in phase again

∴ $2\pi L\left(\dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1}\right) = \pi$ or $\boxed{\dfrac{2\pi L}{\lambda}(n_2 - n_1) = \pi}$

∴ $L = \dfrac{\lambda}{2(n_2 - n_1)} = \dfrac{620}{2(0.2)} = 1550$ nm $= \boxed{1.55 \mu m}$

(This is the smallest L that will work - larger L's that will do the job arise from setting $\dfrac{2\pi L}{\lambda}(n_2 - n_1) = m\pi$ where $m = 1, 3, 5$

A8



for slit width a, we have $a \sin\theta = m\lambda$ for $m = 1, 2, 3 \ldots$

So for the 3rd min, $a \sin\theta = 3\lambda$

** $(a \sin\theta)$ is the path length difference for rays from the two edges of the slit at $y = \pm\frac{a}{2}$

path length difference from midpoint to top ($y = 0$ to $y = \frac{a}{2}$) is just half this much. $\dfrac{a \sin\theta}{2} = \dfrac{3\lambda}{2} = \boxed{900 \text{ nm c)}}$

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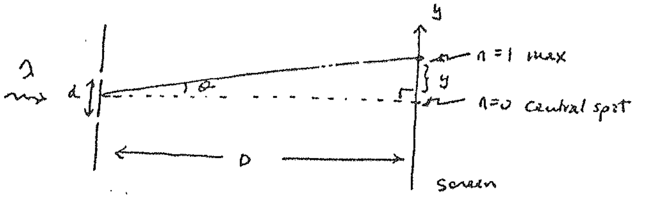
A9 Two slit diffraction : maxima given by $n\lambda = d\sin\theta$ where d is the slit spacing



$\lambda = 633$ nm

given $n=1$ max is at $y = 0.82$ m, $D = 12$ m we find

$\theta = \tan^{-1}\left(\frac{y}{D}\right)$, and $\lambda = d\sin\theta$. $\left\{ \underline{\text{small angle}} \cdot \theta \approx \tan\theta \approx \sin\theta \right\}$

$$\therefore d = \frac{\lambda}{\sin\theta} = \frac{\lambda D}{y} = \boxed{9.3\,\mu m \ \ a)}$$

A10 diffraction grating $n = 5$ for 5^{th} order maxima

slit spacing $d = \frac{1}{4000}$ cm

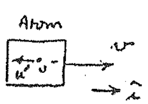
$5\lambda = d\sin\theta$; max λ is when $\theta = 90°$

then $5\lambda = d$, $\lambda = \frac{d}{5} = 5\times10^{-7}$ m $= \boxed{500\text{ nm} \ \ e)}$

A11 Velocity transformation Atom = moving ref. frame, $v = 0.8c$

e^- is emitted backwards in that frame

e $\vec{u}' = -0.5c\hat{\imath}$



$u = \frac{u' + v}{1 + \frac{u'v}{c^2}}$ = e^- velocity in the lab frame (fixed)

$$\therefore u = \left[\frac{-0.5 + 0.8}{1 + (-0.5)(0.8)}\right]c = \frac{0.3}{0.6}c = 0.50c$$

to the right

$\therefore$ in the lab frame, e^- moves in the $\hat{\imath}$ direction at speed $\boxed{0.50c \ \ c)}$

A12 relativistic energy : $E = 2mc^2$

find momentum :

$$E^2 = (mc^2)^2 + c^2p^2 = (2mc^2)^2$$

$$\therefore c^2p^2 = 3m^2c^4$$

$$p^2 = 3m^2c^2$$

$$\boxed{p = \sqrt{3}\,mc \ \ d)}$$

A13 ship moves at $\frac{v}{c} = 0.95$ $\therefore \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = 3.20$

contracted length of space ship is $\frac{L_0}{\gamma} = 62$ m

$\therefore L_0 = 199$ m

Measured in the frame of the ship, $L_0 = c\Delta t$ for light travelling from tail to nose

$$\therefore \Delta t = \frac{199}{c} = \boxed{0.66\,\mu s \ \ a)}$$

A14 flashes of light in frame S : $\Delta x = 2400$ m, $\Delta t = 5.0\,\mu s$

but in frame S' : $\Delta x' = \gamma(\Delta x - v\Delta t)$

$\Delta t' = \gamma\left(\Delta t - \frac{v\Delta x}{c^2}\right) = 0$, simultaneous

$\therefore \Delta t = \frac{v\Delta x}{c^2}$ $\qquad \frac{v}{c} = \frac{c\Delta t}{\Delta x} = \boxed{0.625 \ \ b)}$

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A15 Wave function $\psi(x) = \sqrt{\frac{2}{a}}\, e^{-x/a}$ for $x > 0$

$= 0$ for $x < 0$

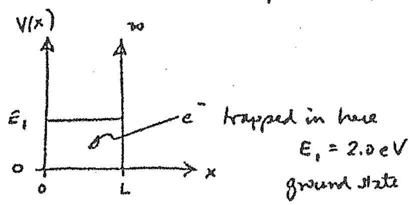
Probability density $\psi^*(x)\psi(x) = \frac{2}{a} e^{-2x/a}$

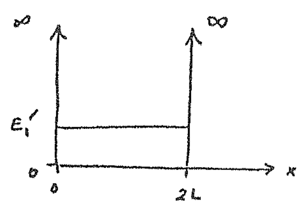
probability between 0 and a is $\int_0^a \psi^*(x)\psi(x)\,dx = \frac{2}{a}\int_0^a e^{-2x/a}\,dx$

$$= \frac{2}{a}\left(-\frac{a}{2}\right) e^{-\frac{2x}{a}}\Big]_0^a = (-1)\left[e^{-2} - e^{0}\right]$$

$$= 1 - e^{-2} = \boxed{0.86 \quad c)}$$

A16 one-dimensional potential problem: compare 2 infinite wells





twice as wide $L' = 2L$

energy formula: $E_1 = \dfrac{h^2}{8mL^2} = 2.0$ eV

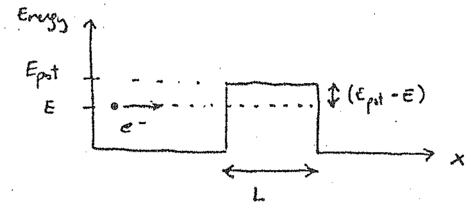
$$E_1' = \frac{h^2}{8m(2L)^2} = \frac{1}{4}\cdot E_1 = \boxed{0.5 \text{ eV} \quad a)}$$

A17 Photon emission corresponds to downwards arrow, $\Delta E = \frac{hc}{\lambda}$

∴ shortest λ for photon corresponds to largest ΔE

∴ $\boxed{\text{Transition III is the shortest } \lambda \quad c)}$

A18 Barrier tunnelling



tunnelling probability

$$T \sim e^{-\sqrt{(E_{pot}-E)}\cdot L}$$

when does T increase?

a) no change to $(E_{pot} - E)$ ∴ no

b) $L \downarrow$ ∴ $e^{-L} \uparrow$ yes

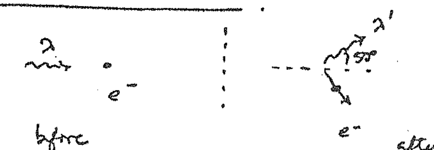
c) $E \downarrow$ ∴ $T \downarrow$ no

d) no change to $(E_{pot} - E)$ ∴ no

e) $E_{pot} \uparrow$, $T \downarrow$ no

$\boxed{\therefore \text{ only b) is correct}}$

A19 Compton scattering



$\lambda = 5.70 \times 10^{-12}$ m ; $\Delta\lambda = \frac{h}{mc}(1 - \cos 50°) = 0.87 \times 10^{-12}$ m

$$\lambda' = \lambda + \Delta\lambda = \boxed{6.57 \times 10^{-12}\text{ m} \quad e)}$$

A20 Single slit diffraction: minima occur when $m\lambda = a\sin\theta$

a = slit width, $m = 1, 2, 3$.

Spacing of minima for small angle: $\sin\theta \approx \theta$;

$a\,\Delta\sin\theta = a\,\Delta\theta = \lambda$ $(\Delta m = 1)$

$\Delta\theta$ increases if λ increases and decreases if it decreases

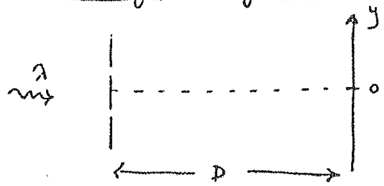
$E\uparrow$, $\lambda\downarrow$ for electrons ∴ $\boxed{\text{Central max. narrows} \quad b)}$

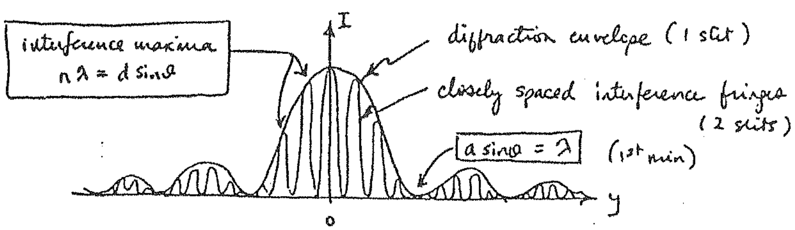
Part B - Long Answer Questions

B1 Double slit. $a = 9.0\ \mu m$ width; $d = 50\ \mu m$ spacing

$\lambda = 632.8$ nm light shines on the slits.

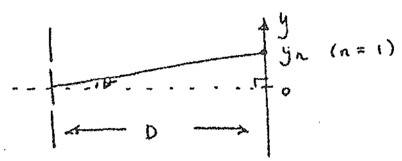
a) sketch of intensity pattern: $I(y)$ for coordinates as in the sketch





(b) interference fringes $n\lambda = d\sin\theta$ $n = 0, 1, 2, 3 \ldots$

occur on the screen at certain values of $y = y_n$



assume θ small

$$\frac{y_n}{D} = \tan\theta_n \approx \theta_n$$

$n\lambda = d\theta_n = d\dfrac{y_n}{D}$; spacing given by $\Delta n = 1$

$$\Delta n \cdot \lambda = \left(\frac{d}{D}\right)\Delta y_n = 1 \cdot \lambda \quad \therefore \quad \Delta y_n = \frac{\lambda D}{d} = \text{constant}$$

Calculate: $$\Delta y_n = \frac{(632.8 \times 10^{-9}\ m)(2.00\ m)}{(50 \times 10^{-6}\ m)} = 2.5 \times 10^{-2}\ m = 2.5\ cm$$

c) intensity drops because of the rapid falloff of the main diffraction envelope

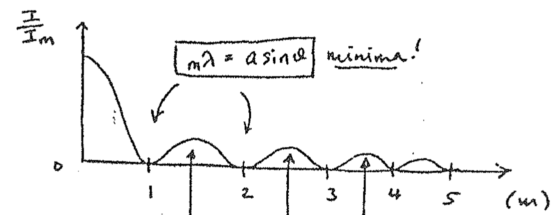
$$\left(\frac{I}{I_m}\right) = \cos^2\beta \left(\frac{\sin\alpha}{\alpha}\right)^2 \qquad \begin{cases} \beta = \dfrac{\pi d}{\lambda}\sin\theta \\ \boxed{\alpha = \dfrac{\pi a}{\lambda}\sin\theta} \end{cases}$$

$\cos^2\beta$ term describes the more rapid interference oscillations.

→ sit at maximum of the interference pattern so that $\cos\beta = 1$;

then $\left(\dfrac{I}{I_m}\right) = \left(\dfrac{\sin\alpha}{\alpha}\right)^2 = 0.01$ limits what can be seen

diffraction envelope:



Maxima: $m = 1.5, 2.5, 3.5 \ldots$

Maxima occur when $(1.5, 2.5, 3.5 \ldots)\lambda = a\sin\theta$

they will be visible if $\left(\dfrac{\sin\alpha}{\alpha}\right)^2 \geq 0.01$ or $\left(\dfrac{\sin\alpha}{\alpha}\right) > \dfrac{1}{10}$

Now $\alpha = \dfrac{\pi}{\lambda}(a\sin\theta) = (1.5, 2.5, 3.5 \ldots)\pi$ for all these

As $\sin\alpha = \pm 1$ for all the maxima

$$\left|\frac{\sin\alpha}{\alpha}\right| = 1, \frac{1}{1.5\pi}, \frac{1}{2.5\pi}, \frac{1}{3.5\pi} \ldots = 1, 0.21, 0.13, 0.09.$$

so, the maximum at $m = 3.5$ is too faint to be detected.

$\therefore$ Observer can see the central peak plus two oscillations of the main diffraction envelope on either side of the central peak.

d) Possibility to extend the angular range of the measurements:

$\longrightarrow$ put a photographic film on the screen and wait a long time; exposure = intensity × time, so weaker features would be observable

note that the intensity goes as $\left(\frac{\sin\alpha}{\alpha}\right)^2$, and for a given maximum of the diffraction pattern, $\boxed{(m+0.5)\lambda = a\sin\theta}$

$\therefore$ increasing the wavelength will increase the angle at which a given maximum occurs but will not allow more oscillations to be detected. Same is true for decreasing the slit width a; increasing the intensity of the light source won't help because the detection limit was specified as $\left(\frac{I}{I_m}\right) = 1\%$ and this is independent of the input intensity!

B2 Photoelectric Effect

a) The photoelectric effect is a phenomenon wherein light can eject electrons from a metal surface provided that the frequency of the light exceeds some threshold frequency that is a characteristic of the material. The current of electrons ejected is proportional to the intensity of the light.

b) Einstein assumed that light was quantized in little bundles called photons, which behave like particles in some respects. The energy of a photon is $E = hf$ where h is Planck's constant and f is the frequency. Einstein assumed that a photon could be absorbed, transferring its energy to an electron and allowing it to escape from the metal surface, after overcoming a surface binding energy Φ which is referred to as the work function of the material. Thus, $K_e = hf - \Phi$ where K_e is the kinetic energy of the electron that escapes

c) Stopping potential: $K_e = eV_{stop}$ when a retarding potential is applied to the surface of just the right magnitude, then electrons can no longer be released via the photoelectric effect, and the photoelectron current drops to zero even though the surface is still illuminated. (Actually, electrons are released with a spread of kinetic energies depending on how far below the surface they are released from; $K_{e,\underline{max}} = eV_{stop}$ for the most energetic electrons)

$$eV_{stop} = hf - \Phi \quad \text{or} \quad \boxed{V_{stop} = \left(\frac{h}{e}\right)f - \frac{\Phi}{e}}$$

This is plotted in the figure.

The slope is $\dfrac{\Delta V}{\Delta f} = \dfrac{(2.65\,V - 0\,V)}{(12\times10^{14} - 5.5\times10^{14}\ Hz)} = 4.1\times10^{-15}$ Volt-sec

This gives an estimate for (h/e).

notice the units: 1 Volt = 1 $\frac{\text{Joule}}{\text{Coulomb}}$

$\therefore \frac{h}{e} = 4.1\times10^{-15} \frac{\text{Joule-sec}}{\text{Coul}} \longrightarrow h = 4.1\times10^{-15} \frac{J.s}{C} \times 1.6\times10^{-19} \frac{C}{\text{electron}}$

$\therefore h = 6.6\times10^{-34}$ J.s
from the graph
(Accepted value: $h = 6.63\times10^{-34}$ J.s ✓)

d) Work function

$\boxed{V_{stop} = \left(\frac{h}{e}\right)f - \frac{\Phi}{e}}$; $V_{stop} = 0$ when $\frac{h}{e}f = \frac{\Phi}{e}$

from the graph, this occurs when $f = 5.5\times10^{14}$ Hz

$\therefore \frac{\Phi}{e} = (4.1\times10^{-15})(5.5\times10^{14}) = 2.25$ Volts.

$\therefore \Phi = 2.25$ electron-volts (eV)

e) Maximum speed of electrons at $f = 2.0\times10^{15}$ Hz?

$V_{stop} = (4.1\times10^{-15})(2.0\times10^{15}) - 2.25 = 6.0$ Volts

$K_{e,max} = eV_{stop} = 6.0\text{ eV} \times 1.6\times10^{-19} \frac{J}{eV} = 9.6\times10^{-19}$ J

$K_e = \frac{1}{2}mv^2 \longrightarrow v = \sqrt{\frac{2(9.6\times10^{-19})}{9.1\times10^{-31}}} = 1.5\times10^{6}$ m/s