Unit 7 Assignment Solutions

1. (a) Let X be the number of draws required to obtain the first face card. Draws are independent (since we are replacing the card after each draw), there are only two events of interest (face card or "not face card"), and the probability of success is constant, p = 12/52. As such, X has a geometric distribution with parameter p = 0.24. Therefore,

$$P(X = 6) = (40/52)^5 (12/52) = 0.0622.$$

(b) Let X be the number of draws required to obtain the first face card. Since draws are made without replacement,

$$P(X = 6) = (40/52)(39/51)(38/50)(37/49)(36/48)(12/47) = 0.0646$$

(c) Let X be the number of draws required to obtain the fourth Club. Draws are independent (since we are replacing the card after each draw), there are only two events of interest (Club or "not Club"), and the probability of success is constant, p = 1/4. As such, X has a negative binomial distribution with parameters r = 3 and p = 0.25. Therefore,

$$P(X = 3) = \binom{14}{3}(0.75)^{11}(0.25)^4 = 0.0601.$$

(d) Let X be the number of Diamonds obtained in 12 draws. We have a fixed number of draws (n = 12), there are only two events of interest (Diamond or "not Diamond"), and draws are independent with a constant probability of success (p = 1/4). As such, X has a binomial distribution with parameters n = 12 and p = 0.25. Therefore,

$$P(X = 3) = \binom{12}{3}(0.25)^3 (0.75)^9 = 0.2581$$

(e) Let X be the number of Aces obtained in 20 draws. We have a fixed number of draws (n = 20), there are only two events of interest (Ace or “not Ace), and draws are independent with a constant probability of success (p = 1/13). As such, X has a binomial distribution with parameters n = 20 and p = 1/13. Therefore,

$$P(X \geq 2) = 1 - P(X \leq 1) = 1 - [P(X = 0) + P(X = 1)]$$

$$= 1 - \left[\binom{20}{0} (1/13)^0 (12/13)^{20} + \binom{20}{1} (1/13)^1 (12/13)^{19} \right] = 1 - [0.2017 + 0.3362] = 1 - 0.5379 = 0.4621$$

(f) Let X be the number of people who get at least two Aces. We have a fixed number of trials (n = 5), there are only two events of interest (person gets at least two Aces or person doesn’t get at least two Aces), and trials are independent with a constant probability of success (p = 0.4621, as calculated in (e)). As such, X has a binomial distribution with parameters n = 5 and p = 0.4621. Therefore,

$$P(X = 3) = \binom{5}{3} (0.4621)^3 (0.5379)^2 = 0.2855$$

(g) Let X be the number of Spades in 15 draws. Since the cards are no longer being replaced, draws are not independent and the probability of success is no longer constant. It follows that X has a hypergeometric distribution with parameters N = 52 (number of cards in the deck), n = 15 (number of draws) and r = 13 (number of Spades in the deck). Therefore,

$$P(X = 3) = \frac{\binom{13}{3}\binom{39}{12}}{\binom{52}{15}} = 0.2496$$

2. Before we can find the probability of winning at least three of the next ten games, we need to find the probability of winning on any one game. Let X be the number of correct matches. There are a fixed number of draws, there are only two possible outcomes of interest on each draw (match or no match), and since draws are made without replacement, they are not independent and there is no constant probability of success. It follows that X has a hypergeometric distribution with parameters $N = 80$, $n = 20$ and $r = 8$. Therefore,

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - \left[\frac{\binom{8}{0}\binom{72}{20}}{\binom{80}{20}} + \frac{\binom{8}{1}\binom{72}{19}}{\binom{80}{20}} + \frac{\binom{8}{2}\binom{72}{18}}{\binom{80}{20}} \right] = 1 - (0.0883 + 0.2665 + 0.3281)$$

$$= 1 - 0.6829 = 0.3171$$

Now let Y be the number of times the player wins a prize in the next ten games. There are a fixed number of games ($n = 10$), there are only two events of interest for each game (win or lose), and games are independent with constant probability of success ($p = 0.3171$). It follows that Y has a binomial distribution with parameters $n = 10$ and $p = 0.3171$. Therefore,

$$P(Y \geq 3) = 1 - P(Y \leq 2) = 1 - [P(Y = 0) + P(Y = 1) + P(Y = 2)]$$

$$= 1 - \left[\binom{10}{0}(0.3171)^0 (0.6829)^{10} + \binom{10}{1}(0.3171)^1 (0.6829)^9 + \binom{10}{2}(0.3171)^2 (0.6829)^8 \right]$$

$$= 1 - (0.0221 + 0.1024 + 0.2140) = 1 - 0.3385 = 0.6615.$$

3. The probability that Roddick wins any set is. The outcome of each set is independent of any other, and there are only two events of interest (Roddick wins a set or Federrer wins a set). Let X be the number of sets it takes for Rodick to win three sets. Then X has a negative binomial distribution with parameters r = 3 and p = 0.4, and the probability that the Roddick wins the match is

$$\mathrm{P(Roddick\ wins) = P(Roddick\ win\ in\ 3\ sets) + P(Roddick\ wins\ in\ 4\ sets) + P(Roddick\ wins\ in\ 5\ sets)}$$

$$= \binom{2}{2}(0.6)^0(0.4)^3 + \binom{3}{2}(0.6)^1(0.4)^3 + \binom{4}{2}(0.6)^2(0.4)^3 = 0.0640 + 0.1152 + 0.1382 = 0.3174.$$

4. We prove the lack-of-memory property as follows:

$$P(X > s + t \mid X > s) = \frac{P(X > s \ \cap \ X > s + t)}{P(X > s)} = \frac{P(X > s + t)}{P(X > s)} = \frac{\sum_{x=s+t+1}^{\infty} (1-p)^{x-1} p}{\sum_{x=s+1}^{\infty} (1-p)^{x-1} p}$$

For any integer k > 0,

$$P(X > k) = \sum_{x=k+1}^{\infty} (1-p)^{x-1} p$$

This is an infinite geometric series with first term $a = (1 - p)^k p$ and multiplicative term $r = 1 - p$. Therefore,

$$\sum_{x=k+1}^{\infty} (1-p)^{x-1} p = \frac{(1-p)^k p}{1-(1-p)} = \frac{(1-p)^k p}{p} = (1-p)^k$$

It follows that

$$P(X > s + t \mid X > s) = \frac{\sum_{x=s+t+1}^{\infty} (1-p)^{x-1} p}{\sum_{x=s+1}^{\infty} (1-p)^{x-1} p} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t = P(X > t)$$

5. (a) Let X be the number of crickets observed in one square meter of the field. We know that X has a Poisson distribution with parameter $\lambda = 2.7$, so

$$P(X \geq 3) = 1 - P(X \leq 2) = 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$
$$= 1 - \left[\frac{(2.7)^0 e^{-2.7}}{0!} + \frac{(2.7)^1 e^{-2.7}}{1!} + \frac{(2.7)^2 e^{-2.7}}{2!}\right] = 1 - (0.0672 + 0.1815 + 0.2450)$$
$$= 1 - 0.4937 = 0.5063$$

(b) Let X be the number of crickets in five square meters of the field. Then X has a Poisson distribution with parameter $5\lambda = 5(2.7) = 13.5$. Therefore,

$$P(X = 15) = \frac{(13.5)^{15} e^{-13.5}}{15!} = 0.0945$$

(c) Let X be the number of crickets in a circular area of the field. Then X has a Poisson distribution with parameter $\pi r^2 \lambda = 2.7\pi r^2$. Therefore,

$$P(X \geq 1) = 0.99 \quad \Rightarrow \quad P(X = 0) = 0.01 \quad \Rightarrow \quad \frac{(2.7\pi r^2)^0 e^{-2.7\pi r^2}}{0!} = 0.01$$
$$\Rightarrow \quad e^{-2.7\pi r^2} = 0.01 \quad \Rightarrow \quad -2.7\pi r^2 = \ln 0.01 \quad \Rightarrow \quad r^2 = \frac{\ln 0.01}{-2.7\pi}$$
$$\Rightarrow \quad r = \sqrt{\frac{\ln 0.01}{-2.7\pi}} = \sqrt{0.5429} = 0.7368$$

6. (a) The time X until the first student falls asleep (and the time between students falling asleep) follows an exponential distribution with parameter $\lambda = 0.2$. Therefore, the probability that the first student falls asleep after 8:40 (i.e. it takes more than ten minutes from the start of class for the first student to fall asleep) is

$$P(X > 10) = \int_{10}^{\infty} 0.1e^{-0.1x}dx = \left[-e^{-0.1x}\right]_{10}^{\infty} = e^{-0.1(10)} = e^{-1} = 0.3686$$

Note that we could also calculate this using the Poisson distribution, since if it takes longer than 10 minutes for the first student to fall asleep, then zero students fall asleep in the first ten minutes. The expected number of students falling asleep in one minute is 0.1, and so the expected number of students falling asleep in 10 minutes is 10(0.1) = 1. Therefore,

$$P(X = 0) = \frac{(1)^0 e^{-1}}{0!} = e^{-1} = 0.3686$$

(b) Since an exponential random variable possesses the lack of memory property, we know

$$P(X > 30 \mid X > 20) = P(X > 10) = 0.3686$$

(c) We know the c.d.f. of the exponential distribution is

$$F(x) = 1 - e^{-\lambda x} \quad \text{for } x \geq 0$$

The first quartile is the value of x such that F(x) = 0.25 and the third quartile is the value of x such that F(x) = 0.75.

$$F(x) = 0.25 \;\Rightarrow\; 1 - e^{-0.1x} = 0.25 \;\Rightarrow\; e^{-0.1x} = 0.75 \;\Rightarrow\; x = \frac{\ln 0.75}{-0.1} = 2.8768$$

$$F(x) = 0.75 \;\Rightarrow\; 1 - e^{-0.1x} = 0.75 \;\Rightarrow\; e^{-0.1x} = 0.25 \;\Rightarrow\; x = \frac{\ln 0.25}{-0.1} = 13.8629$$

So Q1 = 2.8768 and Q3 = 13.8629, and so the interquartile range is

$$IQR = Q3 - Q1 = 13.8629 - 2.8768 = 10.9861$$

(d) Let X be the number of students that fall asleep in a thirty-minute period. Then X has a Poisson distribution with parameter $30\lambda = 30(0.1) = 3$. Therefore,

P(fourth student falls asleep after 9:00)
= P(less than three students fall asleep in first 30 minutes)

$$P(X < 4) = P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$$
$$\frac{(3)^0 e^{-3}}{0!} + \frac{(3)^1 e^{-3}}{1!} + \frac{(3)^2 e^{-3}}{2!} + \frac{(3)^3 e^{-3}}{3!} = 0.0498 + 0.1494 + 0.2240 + 0.2240 = 0.6472.$$

7. (a) $P(X_1 < 55) = P\left(Z < \frac{55 - 54.2}{0.7}\right) = P(Z < 1.14) = 0.8729.$

(b) $P(60 < X_2 < 62) = P\left(\frac{60 - 60.3}{0.9} < Z < \frac{62 - 60.3}{0.9}\right) = P(-0.33 < Z < 1.89)$
$= P(Z < 1.89) - P(Z < -0.33) = 0.9706 - 0.3707 = 0.5999.$

(c) $P(X_3 > 51) = P\left(Z > \frac{51 - 52.1}{0.6}\right) = P(Z > -1.83) = 1 - P(Z < -1.83) = 1 - 0.0336 = 0.9664.$

(d) We want to find the value x such that $P(X_4 \le x) = 0.8500$. First we find that the value z such that $P(Z \le z) = 0.8500$ is $z = 1.04$. Now we have

$$z = \frac{x - \mu_4}{\sigma_4} \Rightarrow x = \mu_4 + z\sigma_4 = 47.0 + 1.04(0.5) = 47.52.$$

(e) In order to determine in which swimmer did the best relative to his past times, we find each of their z-scores.

$$z_1 = \frac{x_1 - \mu_1}{\sigma_1} = \frac{54.7 - 54.2}{0.7} = 0.71 \qquad z_2 = \frac{x_2 - \mu_2}{\sigma_2} = \frac{59.5 - 60.3}{0.9} = -0.89$$

$$z_3 = \frac{x_3 - \mu_3}{\sigma_3} = \frac{51.6 - 52.1}{0.6} = -0.83 \qquad z_4 = \frac{x_4 - \mu_4}{\sigma_4} = \frac{46.6 - 47.0}{0.5} = -0.80$$

Swimmer 2 did the best relative to her past times, since he had the lowest z-score. (Remember, in swimming, a **low** time is good.)

(f) Let X_{TC} be Team Canada's total time. Then $X_{TC} = X_1 + X_2 + X_3 + X_4$, and

$$X_{TC} \sim N\left(\mu_1 + \mu_2 + \mu_3 + \mu_4,\ \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \sigma_4^2}\right)$$

$$\Rightarrow X_{TC} \sim N\left(54.2 + 60.3 + 52.1 + 47.0,\ \sqrt{(0.7)^2 + (0.9)^2 + (0.6)^2 + (0.5)^2}\right)$$

$$\Rightarrow X_{TC} \sim N(213.6, 1.382)$$

And so

$$P(X_{TC} < 3.5 \text{ min}) = P(X_{TC} < 210 \text{ sec}) = P\left(Z < \frac{210 - 213.6}{1.382}\right) = P(Z < -2.60) = 0.0047.$$

(g) Let X_{TU} be Team U.S.A.'s total time. Then $X_{TU} = X_{1U} + X_{2U} + X_{3U} + X_{4U}$, and

$$X_{TU} \sim N\left(\mu_{1U} + \mu_{2U} + \mu_{3U} + \mu_{4U},\ \sqrt{\sigma_{1U}^2 + \sigma_{2U}^2 + \sigma_{3U}^2 + \sigma_{4U}^2}\right)$$

$$\Rightarrow X_{TU} \sim N\left(53.6 + 60.1 + 50.9 + 47.9,\ \sqrt{(0.8)^2 + (1.0)^2 + (0.7)^2 + (0.4)^2}\right)$$

$$\Rightarrow X_{TC} \sim N(212.5,\ 1.513)$$

Team Canada will beat Team U.S.A. if $X_{TC} < X_{TU}$, i.e. if $X_{TC} - X_{TU} < 0$. We know

$$X_{TC} - X_{TU} \sim N\left(\mu_{TC} - \mu_{TU},\ \sqrt{\sigma_{TC}^2 + \sigma_{TU}^2}\right)$$

$$\Rightarrow X_{TC} - X_{TU} \sim N\left(213.6 - 212.5,\ \sqrt{(1.382)^2 + (1.513)^2}\right)$$

$$\Rightarrow X_{TC} - X_{TU} \sim N(1.1,\ 2.049)$$

And so the probability that Team Canada beats team U.S.A. is

$$P(X_{TC} - X_{TU} < 0) = P\left(Z < \frac{0 - 1.1}{2.049}\right) = P(Z < -0.54) = 0.2946.$$

8. (a) We want to find μ such that $P(X > 6) = 0.0200$. First we find the value z such that $P(Z > z) = 0.0200 \Rightarrow P(Z < z) = 0.9800$ to be $z = 2.05$. We then use the z-score formula to solve for μ as follows:

$$z = \frac{x - \mu}{\sigma} \quad \Rightarrow \quad \mu = x - z\sigma = 6 - 2.05(0.2) = 5.59 \text{ ounces.}$$

(b) $P(X < 5.7) = P\left(Z < \dfrac{5.7 - 5.59}{0.2}\right) = P(Z < 0.55) = 0.7088.$

(c) $P(X > 5.2) = P\left(Z > \dfrac{5.2 - 5.59}{0.2}\right) = P(Z > -1.95) = 1 = P(Z < -1.95) = 1 - 0.0256 = 0.9744.$

(d) $P(5.5 < X \le 5.8) = P\left(\dfrac{5.5 - 5.59}{0.2} < Z < \dfrac{5.8 - 5.59}{0.2}\right) = P(-0.45 < Z < 1.05)$

$= P(Z < 1.05) - P(Z < -0.45) = 0.8531 - 0.3264 = 0.5267.$

(e) We want to find the first and third quartiles of the distribution of X, i.e. the values x_{Q1} and x_{Q3} such that $P(X < x_{Q1}) = 0.2500$ and $P(X < x_{Q3})$. First we find the values z_{Q1} and z_{Q3} such that $P(Z < z_{Q1}) = 0.2500$ and $P(Z < z_{Q3}) = 0.7500$ to be $z_{Q1} = -1.645$ and $z_{Q3} = 1.645$. We now use the z-score formula to solve for x_{Q1} and x_{Q3} as follows:

$$z_{Q1} = \frac{x_{Q1} - \mu}{\sigma} \quad \Rightarrow \quad x_{Q1} = \mu + z_{Q1}\sigma = 5.59 + (-0.67)(0.2) = 5.456 \text{ ounces}$$

$$z_{Q3} = \frac{x_{Q3} - \mu}{\sigma} \quad \Rightarrow \quad x_{Q3} = \mu + z_{Q3}\sigma = 5.59 + (0.67)(0.2) = 5.724 \text{ ounces}$$

And so the interquartile range of fill volumes is

$$IQR = x_{Q3} - x_{Q1} = 5.724 - 5.456 = 0.268 \text{ ounces.}$$