Math 1210 Tutorial #10 solutions

1. There are two cases: if $k=0$, then the R.E.F. is $\left(\begin{array}{ccc|c} 1 & -1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$.

If $k \neq 0$, then the R.E.F. is $\left(\begin{array}{ccc|c} 1 & -1 & k^2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$. Therefore, if $k=0$, the answers are:

(a) 3 (b) 2 (c) 2 (d) $3-2=1$ (e) infinitely many.

If $k \neq 0$, then:

(a) 3 (b) 3 (c) 2 (d) 0 because the system is inconsistent (e) 0.

2. (a) The coefficient matrix is $\begin{pmatrix} 1 & 2 & 1 & -1 & 0 \\ 2 & 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$. Reducing to R.R.E.F.:

$$\xrightarrow{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 2 & 1 & -1 & 0 \\ 0 & -3 & -5 & 2 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix} \xrightarrow[R_1 \to R_1 - 2R_2]{R_2 \to -\frac{1}{3}R_2} \begin{pmatrix} 1 & 0 & -\frac{7}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & -\frac{2}{3} & 0 \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$\xrightarrow[R_2 \to R_2 + \frac{2}{3}R_3]{R_1 \to R_1 - \frac{1}{3}R_3} \begin{pmatrix} 1 & 0 & -\frac{7}{3} & 0 & -\frac{7}{3} \\ 0 & 1 & \frac{5}{3} & 0 & \frac{14}{3} \\ 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

Free variables: z, t. Basic solutions: $(x,y,z,w,t) = (\frac{7}{3}, -\frac{5}{3}, 1, 0, 0)$ and $(x,y,z,w,t) = (\frac{7}{3}, -\frac{14}{3}, 0, -7, 1)$.

(b) $(x,y,z,w,t) = (2,0,-1,1,1) + z(\frac{7}{3}, -\frac{5}{3}, 1, 0, 0) + t(\frac{7}{3}, -\frac{14}{3}, 0, -7, 1)$. (Column vectors can also be used to write these solutions.)

3. *Solution 1.* The given points belong to the same line if and only if the vectors (x_1-x_3, y_1-y_3) and (x_2-x_3, y_2-y_3) belong to the same line, which means that they are linearly dependent. This happens if and only if $\det\begin{pmatrix} x_1-x_3 & y_1-y_3 \\ x_2-x_3 & y_2-y_3 \end{pmatrix} = 0$. But this determinant is equal to the

determinant from the condition of the problem:

$$\det\begin{pmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{pmatrix} = \det\begin{pmatrix} x_1 - x_3 & y_1 - y_3 & 0 \\ x_2 - x_3 & y_2 - y_3 & 0 \\ x_3 & y_3 & 1 \end{pmatrix} = \det\begin{pmatrix} x_1 - x_3 & y_1 - y_3 \\ x_2 - x_3 & y_2 - y_3 \end{pmatrix}.$$

Solution 2. The given determinant is zero if and only if the column vectors of the matrix are linearly dependent. This means that for some a, b, c not all equal to zero

$$a\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + b\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

that is $ax_j + by_j + c = 0$, $j = 1, 2, 3$ (one can also think of the corresponding homogeneous system of equation on a, b, c which has a non-trivial solution if and only if the determinant is zero). But this means that all three points belong to the line with equation $ax + by + c = 0$.

4. Since A is invertible, A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$. The product of invertible matrices is invertible, and we have $(A^TB)^{-1} = B^{-1}(A^{-1})^T$, and finally $(A^TBC)^{-1} = C^{-1}(A^TB)^{-1} = C^{-1}B^{-1}(A^{-1})^T$.

5. Indeed, since the matrix product satisfies $(XY)Z = X(YZ)$ (associative), by the definition of inverse

$$A^{-1}(I + AB)A - I = A^{-1}IA + (A^{-1}A)BA - I = A^{-1}A + IBA - I = I + BA - I = BA.$$