MATH 2132 Problem Workshop 5

1. Find general solutions for the following differential equations

   (a) $3y''' + 8y'' + 19y' + 10y = 0$

   Solution:
   The auxiliary equation of this homogeneous D.E. with constant coefficients is

   $$3m^3 + 8m^2 + 19m + 10 = 0$$

   Solving for m can use techniques from classical algebra (Rational root theorem etc.) We can test and see that $m = -2/3$ is a solution. Factoring yields

   $$(3m + 2)(m^2 + 2m + 5) = 0$$

   which has solutions $m = -2/3,\ -1 \pm 2i$.
   Thus the general solution is

   $$y = c_1 e^{-2x/3} + e^{-x}\big(c_2 \cos 2x + c_3 \sin 2x\big).$$

   (b) $6y'''' + y'' - y = 0$

   Solution:
   The auxiliary equation of this homogeneous D.E. with constant coefficients is

   $$6m^4 + m^2 - 1 = 0$$

   Factoring yields

   $$(3m^2 - 1)(2m^2 + 1) = 0$$

   which has solutions $m = \pm 1/\sqrt{3},\ \pm\dfrac{i}{\sqrt{2}}$.
   Thus the general solution is

   $$y = c_1 e^{x/\sqrt{3}} + c_2 e^{-x/\sqrt{3}} + c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2}).$$

   (c) $y'' + 3y' + ay = 0$, where $a > 3$ is a constant.

Solution:
The auxiliary equation of this homogeneous D.E. with constant coefficients is

$$m^2 + 3m + a = 0$$

which has solutions

$$m = \frac{-3 \pm \sqrt{9-4a}}{2}.$$

Since $a > 3$ we get two complex solutions $\dfrac{-3}{2} \pm i\dfrac{\sqrt{4a-9}}{2}$
Factoring yields

$$(3m^2 - 1)(2m^2 + 1) = 0$$

which has solutions $m = \pm 1/\sqrt{3}, \pm\dfrac{i}{\sqrt{2}}$.
Thus the general solution is

$$e^{-3x/2}\left(c_1 \cos \frac{\sqrt{4a-9}}{2}x + c_2 \sin \frac{\sqrt{4a-9}}{2}x\right)$$

(d) $y'' + 2y' - 3y = 4e^{5x} - x$

Solution:
First solving the homogeneous system of equations which has auxiliary equation

$$m^2 + 2m - 3 = 0 \Rightarrow (m+3)(m-1) = 0 \Rightarrow m = 1, -3.$$

Thus the homogeneous system has solution $y_h = c_1 e^x + c_2 e^{-3x}$.
Since the $Q(x) = 4e^{5x} - x$ we are looking for a particular solution of the form $y_p = Ae^{5x} + Bx + C$.
$y_p' = 5Ae^{5x} + B$ and $y_p'' = 25Ae^{25x}$. Plugging into the equation

$$4e^{5x} - x = 25Ae^{25x} + 2(5Ae^{5x} + B) - 3(Ae^{5x} + Bx + C) = 32Ae^{5x} - 3Bx + (2B - 3C)$$

Hence $4 = 32A \Rightarrow A = 1/8$. $-3B = -1 \Rightarrow B = 1/3$ and $2B - 3C = 0 \Rightarrow C = 2/9$.
Hence $y_p = e^{5x}/8 + x/3 + 2/9$.

Therefore the general solution is

$$c_1e^x + c_2e^{-3x} + \frac{1}{8}e^{5x} + \frac{1}{3}x + \frac{2}{9}.$$

(e) $y''' + 3y'' - 4y' = 2e^x + \cos 4x$

Solution:
First solving the homogeneous system of equations which has auxiliary equation

$$m^3 + 3m^2 - 4m = 0 \Rightarrow m(m+4)(m-1) = 0 \Rightarrow m = 0,\ 1,\ -4.$$

Thus the homogeneous system has solution $y_h = c_1 + c_2e^x + c_3e^{-4x}$.
Since the $Q(x) = 2e^x + \cos 4x$ we are looking for a particular solution of the form $y_p = Axe^x + B\cos 4x + C\sin 4x$. Note that we need an xe^x term instead of an e^x term because e^x was part of the homogeneous solution.
$y_p' = A(xe^x + e^x) - 4B\sin 4x + 4C\cos 4x$,
$y_p'' = A(xe^x + 2e^x) - 16B\cos 4x - 16C\sin 4x$
and $y_p''' = A(xe^x + 3e^x) + 64B\sin 4x - 64C\cos 4x$.
Plugging into the equation

$$\begin{aligned} 2e^x + \cos 4x &= A(xe^x + 3e^x) + 64B\sin 4x - 64C\cos 4x + 3(A(xe^x + 2e^x) \\ &\quad - 16B\cos 4x - 16C\sin 4x) - 4(A(xe^x + e^x) - 4B\sin 4x + 4C\cos 4x) \\ &= A(5e^x)) + (-80C - 48B)\cos 4x + (80B - 48C)\sin 4x \end{aligned}$$

Hence $2 = 5A \Rightarrow A = 2/5$.

$$1 = -80C - 48B,\ 0 = 80B - 48C \Rightarrow B = -3/544 \Rightarrow C = -5/544.$$

Hence $y_p = 2xe^x/5 - 3(\cos 4x)/544 - 5(\sin 4x)/544$.
Therefore the general solution is

$$c_1 + c_2e^x + c_3e^{-4x} + \frac{2}{5}xe^x - \frac{3}{544}\cos 4x - \frac{5}{544}\sin 4x.$$

(f) $y'' + 6y' - 2y = 3x + \sin x$

Solution:
First solving the homogeneous system of equations which has auxiliary equation

$$m^2 + 6m - 2 = 0 \Rightarrow m = \frac{-6 \pm \sqrt{36+8}}{2} = -3 \pm \sqrt{11}.$$

Thus the homogeneous system has solution $y_h = c_1 e^{(-3+\sqrt{11})x} + c_2 e^{(-3-\sqrt{11})x}$.
Since the $Q(x) = 3x + \sin x$ we are looking for a particular solution of the form $y_p = Ax + B + C\sin x + D\cos x$.
$y_p' = A + C\cos x - D\sin x$ and $y_p'' = -C\sin x - D\cos x$. Plugging into the equation

$$\begin{aligned} 3x + \sin x &= -C\sin x - D\cos x + 6(A + C\cos x - D\sin x) \\ &\quad - 2(Ax + B + C\sin x + D\cos x) \\ &= -2Ax + (6A - 2B) + (6C - 3D)\cos x + (-3C - 6D)\sin x \end{aligned}$$

Hence $-2A = 3 \Rightarrow A = -3/2$.
$6A - 2B \Longrightarrow B = 3A = -9/2$.

$$1 = -3C - 6D,\ 0 = 6C - 3D \Rightarrow C = -1/15 \Rightarrow D = -2/15.$$

Hence $y_p = -3x/2 - 9/2 - (\sin x)/15 - 2(\cos x)/15$.
Therefore the general solution is

$$c_1 e^{(-3+\sqrt{11})x} + c_2 e^{(-3-\sqrt{11})x} + \frac{3}{2}x - \frac{9}{2} - \frac{1}{15}\sin x - \frac{2}{15}\cos x.$$

(g) $4y''' - 3y'' + 7y' + 2y = e^{-x/4}$

Solution:
First solving the homogeneous system of equations which has auxiliary equation

$$4m^3 - 3m^2 + 7m + 2 = 0 \Rightarrow (4m+1)(m^2 - m + 2) = 0 \Rightarrow m = -1/4,\ \frac{1 \pm i\sqrt{7}}{2}.$$

Thus the homogeneous system has solution
$y_h = c_1 e^{-x/4} + e^{x/2}\big(c_2\cos(\sqrt{7}x/2) + c_3\sin(\sqrt{7}x/2)\big)$.

Since the $Q(x) = e^{-x/4}$ we are looking for a particular solution of the form $y_p = Axe^{-x/4}$.
The extra factor of x is due to $e^{-x/4}$ begin part of the homogeneous solution.
$y_p' = A(e^{-x/4} - xe^{-x/4}/4)$,
$y_p'' = A(-e^{-x/4}/2 + xe^{-x/4}/16)$
and $y_p''' = A(3e^{-x/4}/16 - xe^{-x/4}/64)$. Plugging into the equation

$$\begin{aligned} e^{-x/4} &= 4(A(3e^{-x/4}/16 - xe^{-x/4}/64)) - 3(A(-e^{-x/4}/2 + xe^{-x/4}/16)) \\ &\quad + 7(A(e^{-x/4} - xe^{-x/4}/4)) + 2(Axe^{-x/4}) \\ &= 37Ae^{-x/4}/4 \end{aligned}$$

Hence $1 = 37A/4 \Rightarrow A = 4/37$.
Hence $y_p = xe^{-x/4}/25$.
Therefore the general solution is

$$c_1e^{-x/4} + e^{x/2}\big(c_2\cos(\sqrt{7}x/2) + c_3\sin(\sqrt{7}x/2)\big) + \frac{4}{37}e^{-x/4}.$$

2. The roots of an auxiliary equation $\phi(m) = 0$ associated with the differential equation $\phi(D)y = 0$ are

$$2 \pm \sqrt{3}i,\ 2 \pm \sqrt{3}i,\ \pm 4,\ \pm 4,\ \pm 4,\ 2,\ -1 \pm \sqrt{6}$$

What is the general solution of the differential equation?

Solution:

$$\begin{aligned} y_h = e^{2x}&\big(c_1\sin\sqrt{3}x/2 + c_2\cos\sqrt{3}x/2 + c_3x\sin\sqrt{3}x/2 + c_4x\cos\sqrt{3}x/2\big) \\ &+ c_4e^{4x} + c_5xe^{4x} + c_6x^2e^{4x} \\ &\quad + c_7e^{-4x} + c_8xe^{-4x} + c_9x^2e^{-4x} \\ &\qquad + c_{10}e^{2x} + c_{11}e^{-1+\sqrt{6}x} + c_{12}e^{-1-\sqrt{6}x} \end{aligned}$$

3. The roots of an auxiliary equation $\phi(m) = 0$ associated with the differential equation

$$\phi(D)y = 13x^2e^{3x} + e^x\cos 2x + 4x^3 - 5$$

are

$$3,\ 3,\ 3,\ -1 \pm 2i,\ 1 \pm 2i,\ 3 \pm \sqrt{6},\ 0,\ 0,\ -14$$

What is the form of the particular solutions as predicting by the method of undtermined coefficients?

Solution:

$$y_p = Ax^3e^{3x} + Bx^4e^{3x} + Cx^5e^{3x} + De^x \cos 2x + Ee^x \sin 2x + Fx^2 + Gx^3 + Hx^4 + Ix^5$$

4. (a) A 100 gram mass is suspended from a spring with constant 60 N/m. It is lifted 15 cm. above its equilibrium position and given velocity 2 metres per second upward. During its motion, it is acted on by air resistance that is equal, in Newtons, to 5 times the velocity of the mass. Find the position of the mass as a function of time.

Solution:
The differential equation for a mass-spring system is

$$m\frac{d^2y}{dt^2} + \beta\frac{dy}{dt} + ky = F(t)$$

where m is the mass of the object, β is any damping of the system, k is the spring constant and $F(t)$ is any external forces acting on the system.
Thus $m = 0.1$ kg. $k = 60$ N/m and $\beta = 5$ as the air restistance is 5 time the velocity. Further there are initial conditions $y(0) = 0.15$ m and $y'(0) = 2$ from the initial position and velocity. Thus the differential equation is

$$0.1\frac{d^2y}{dt^2} + 5\beta\frac{dy}{dt} + 60y = 0,\ y(0) = 0.15,\ y'(0) = 2.$$

Solving

$$\frac{d^2y}{dt^2} + 50\beta\frac{dy}{dt} + 600y = 0$$

The auxiliary equation is

$$m^2 + 50m + 600 = 0 \Rightarrow (m+20)(m+30) = 0 \Rightarrow m = -20, -30.$$

Hence

$$y = c_1e^{-20t} + c_2e^{-30t} \Rightarrow y' = -20c_1e^{-20t} - 30c_2e^{-30t}$$

From the initial conditions we get that

$$c_1 + c_2 = 0.15 \text{ and } -20c_1 - 30c_2 = 2 \Rightarrow c_2 = -1/2,\ c_1 = 13/20$$

Thus the position is

$$y = \frac{13}{20}e^{-20t} - \frac{1}{2}e^{-30t}$$

(b) What is the maximum displacement from equilibrium experienced by the mass?

Solution:
Solving for where $y'(t) = 0$ (as exponentials are always differentiable) we get

$$0 = y'(t) = -13e^{-20t} + 15e^{-30t} \Rightarrow e^{-30t}(-13e^{10t} + 15) = 0 \Rightarrow e^{10t} = \frac{15}{13}$$

Thus $t = \dfrac{1}{10}\ln\dfrac{15}{13}$.
Plugging in for y we get

$$\begin{aligned}
y &= \frac{13}{20}e^{-20(0.1)\ln(15/13)} - \frac{1}{2}e^{-30(0.1)\ln(15/13)} \\
&= \frac{13}{20}e^{-2\ln(15/13)} - \frac{1}{2}e^{-3\ln(15/13)} \\
&= \frac{13}{20}\left(\frac{15}{13}\right)^{-2} - \frac{1}{2}\left(\frac{15}{13}\right)^{-3} \\
&= \left(\frac{13}{15}\right)^{2}\left(\frac{13}{20} - \frac{13}{30}\right) \\
&= \frac{13}{60}\left(\frac{13}{15}\right)^{2}.
\end{aligned}$$

Hence the maximum displacement is $\dfrac{13}{60}\left(\dfrac{13}{15}\right)^{2}$ m.

Note that we have a maximum since the derivative changes from positive to negative at that point (check) and there is only one critical number.

(c) When, if at all, does the mass pass through its equilibrium position?

Solution:
Solving where $y = 0$ we get

$$0 = y(t) = \frac{13}{20}e^{-20t} - \frac{1}{2}e^{-30t} \Rightarrow e^{-30t}(13e^{10t}/20 - 1/2) = 0 \Rightarrow e^{10t} = \frac{10}{13}$$

Thus $t = \dfrac{1}{10} \ln \dfrac{10}{13}$.
However this t is negative which makes no sense in the context of the word problem. Thus the mass passes through it's equilibrium position (although as t goes to infinity, it does get close.)

5. Recall Newton's Law is that the rate that the temperature changes is proportional to $T_a - T$ where T is the temperature of the object and T_a is the ambient temperature.

   (a) Suppose a potato with initial temperature $20°C$ is placed in an oven with temperature $200°C$. Find the temperature of the potato as a function of time.

Solution:
$T_a = 200$ $T(0) = 20$
The initial value problem is

$$\frac{dT}{dt} = k(200 - T),\ T(0) = 20$$

This is separable, hence

$$\begin{aligned} \int \frac{1}{200 - T}\, dT &= \int k\, dt \\ \Rightarrow -\ln(200 - T) &= kt + C \quad \text{Note that } T < 200 \text{ so the absolute value is not necessary} \\ \Rightarrow 200 - T &= e^{-kt-C} = Ke^{-kt} \quad \text{where } K = e^{-C} > 0 \end{aligned}$$

$\Rightarrow T = 200 - Ke^{-kt}$

Using $T(0) = 20$ we get

$$20 = 200 - K \Rightarrow K = 180.$$

Thus $T(t) = 200 - 180e^{-kt}$.

(b) Suppose a potato with initial temperature $20°C$ is placed in an oven with temperature starting at $20°C$ but rises (linearly) to $200°C$ in 5 minutes. Find the temperature of the potato as a function of time.

Solution:

For the first 5 minutes, $T_a = 20 + 36t$ using that at $t = 0 \to T_a = 20$ and $t = 5 \to T_a = 200$. Hence for the first 5 minutes

$$\frac{dT}{dt} = k(20 + 36t - T) \Rightarrow \frac{dT}{dt} + kT = k(20 + 36t),\ T(0) = 20$$

The integrating factor is
$\mu(t) = e^{\int k\,dt} = e^{kt}$ and thus

$$(e^{kt}T)' = k(20 + 36t)e^{kt}$$

Using integration by parts with $u = k(20 + 36t)$ and $dv = e^{kt}\,dt$ we get

$$\begin{aligned} e^{kt}T &= \int k(20 + 36t)e^{kt}\,dt \\ \int (20 + 36t)e^{kt} &- \int 36e^{kt}\,dt \\ \int (20 + 36t)e^{kt} &- \frac{36}{k}e^{kt} + C \end{aligned}$$

Thus $T = 20 + 36t - \dfrac{36}{k} + Ce^{-kt}$.
Using

$$T(0) = 20 \Rightarrow 20 = 20 - \frac{36}{k} + C \Rightarrow C = \frac{36}{k}$$

Hence for the first 5 minutes, the temperature is

$$T(t) = 20 + 36t - \frac{36}{k} + \frac{36}{k}e^{-kt}$$

After the first 5 minutes, the differential equation changes to $T_a = 200$. The new "initial value" occurs at $t = 5$ which matches the temperature of the previous answer. Namely

$$T(5) = 200 - \frac{36}{k}(1 - e^{-5k})$$

The differential equation is $\frac{dT}{dt} = k(200 - T)$ which has solution $T = 200 - Ke^{-kt}$ from part (a) the difference is the value of K

$$\begin{aligned}
&200 - \frac{36}{k}(1 - e^{-5k}) = T(5) = 200 - Ke^{-5k} \\
&\Rightarrow Ke^{-5k} = \frac{36}{k}(1 - e^{-5k}) \\
&\Rightarrow K = \frac{36}{k}(e^{5k} - 1)
\end{aligned}$$

Hence for $t \geq 5$ we have $T(t) = 200 - \dfrac{36}{k}(e^{5k} - 1)e^{-kt}$.
Hence the temperature is

$$T(t) = \begin{cases} 20 + 36t - \dfrac{36}{k} + \dfrac{36}{k}e^{-kt} & 0 \leq t \leq 5 \\ 200 - \dfrac{36}{k}(e^{5k} - 1)e^{-kt} & 5 \leq t \end{cases}$$