sp

10 **1.** Find the interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}(x-1)^n.$$

$$R = \lim_{n\to\infty} \left| \frac{\frac{(-1)^n}{n2^n}}{\frac{(-1)^{n+1}}{(n+1)2^{n+1}}} \right| = 2$$ 2 1

The open interval of convergence is $|x-1| < 2 \Rightarrow -1 < x < 3$.

At $x=-1$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}(-2)^n = \sum_{n=1}^{\infty} \frac{1}{n}.$$ 1

This is the harmonic series, which diverges. 1

At $x=3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n2^n}2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}.$$ 1

This is the negative of the alternating harmonic series, which converges. 1

The interval of convergence is therefore $-1 < x \le 3$. 1

8 **2.** Find the Taylor series about $x = 4$ for the function

$$f(x) = \frac{1}{(x+1)^2}.$$

Use a method that guarantees that the series converges to $f(x)$. Express your answer in sigma notation, simplified as much as possible. Determine the open interval of convergence for the series.

$$\frac{1}{x+1} = \frac{1}{(x-4)+5} = \frac{1}{5\left(1 + \frac{x-4}{5}\right)} \;\sim 1\; = \frac{1}{5}\sum_{n=0}^{\infty} \frac{(-1)^n (x-4)^n}{5^n}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} (x-4)^n \;\sim 1, \quad \text{valid for } \left|-\frac{x-4}{5}\right| < 1 \;\sim 1$$

$$\Rightarrow |x-4| < 5 \Rightarrow -5 < x-4 < 5$$

$$\Rightarrow -1 < x < 9. \;\sim 1$$

If we differentiate,

$$-\frac{1}{(x+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{5^{n+1}} n (x-4)^{n-1} \;\sim 1$$

$$\therefore \frac{1}{(x+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} n}{5^{n+1}} (x-4)^{n-1} \;\sim 1$$

$$= \sum_{n=-1}^{\infty} \frac{(-1)^n (n+1)}{5^{n+2}} (x-4)^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{5^{n+2}} (x-4)^n. \;\sim 1$$

Since differentiation preserves ~1 open intervals of convergence, $-1 < x < 9$.

12 **3.** (a) Find the Maclaurin series for the function

$$f(x) = x^4 \ln(x+2).$$

Express your answer in sigma notation simplified as much as possible. Include its radius of convergence.

(b) Use the series in part (a) to find $f^{(10)}(0)$.

(a) $$\frac{1}{x+2} = \frac{1}{2\left(1+\frac{x}{2}\right)} \sim 1 \quad = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}x^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^n \sim 2,$$ valid for $\left|-\frac{x}{2}\right| < 1 \Rightarrow |x| < 2.$ ~1

Integration gives

$$\ln|x+2| = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}x^{n+1} + C \sim 2$$

If we set $x = 0$, then $\ln 2 = C$ ~1. Since $|x| < 2$, we may drop absolute values and write

$$\ln(x+2) = \ln 2 + \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}x^{n+1}$$

$$\therefore\ x^4\ln(x+2) = (\ln 2)x^4 + \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}(n+1)}x^{n+5} \sim 1$$

$$= (\ln 2)x^4 + \sum_{n=5}^{\infty}\frac{(-1)^{n+1}}{2^{n-4}(n-4)}x^n \sim 1$$

Since $|x| < 2$, $R = 2$ ~1

(b) The coefficient of x^{10} in the series is $\frac{f^{(10)}(0)}{10!}$ ~1. Thus,

$$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^{11}}{2^6(6)} \Rightarrow f^{(10)}(0) = -\frac{10!}{2^6(6)}. \sim 1$$

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12 **4.** Find a general solution for the differential equation

$$y'''' + 4y'' + 4y = x - 3\sin 3x.$$

The auxiliary equation is

$$0 = m^4 + 4m^2 + 4 \overset{\sim 1}{=} (m^2+2)^2 \Rightarrow m = \pm\sqrt{2}i, \pm\sqrt{2}i. \quad \sim 2$$

Thus,

$$y_h(x) = (C_1 + C_2 x)\cos\sqrt{2}x + (C_3 + C_4 x)\sin\sqrt{2}x. \quad \sim 2$$

A particular solution is of the form

$$y_p(x) = Ax + B + C\sin 3x + D\cos 3x. \quad \sim 2$$

Substitution into the DE gives

$$(81C\sin 3x + 81D\cos 3x) + 4(-9C\sin 3x - 9D\cos 3x) + 4(Ax + B + C\sin 3x + D\cos 3x) = x - 3\sin 3x \quad \sim 2$$

When we equate coefficients:

x: $4A = 1 \Rightarrow A = 1/4$

1: $4B = 0$

$\sin 3x$: $81C - 36C + 4C = -3 \Rightarrow C = -3/49$

$\cos 3x$: $81D - 36D + 4D = 0 \Rightarrow D = 0$.

Thus,

$$y_p(x) = \frac{x}{4} - \frac{3}{49}\sin 3x \quad \sim 2$$

$$y(x) = (C_1 + C_2 x)\cos\sqrt{2}x + (C_3 + C_4 x)\sin\sqrt{2}x + \frac{x}{4} - \frac{3}{49}\sin 3x. \quad \sim 1$$

6 **5.** You are given that the roots of the auxiliary equation associated with the linear, differential equation

$$\phi(D)y = x^3 - x + 2\sin 5x + e^{3x}$$

are $m = 0, 0, \pm 5i, 3 \pm 4i, 6$. Write down the form of a particular solution of the differential equation as predicted by the method of undetermined coefficients. Do **NOT** find the coefficients, just the form of the particular solution.

$$y_h(x) = C_1 + C_2 x + C_3 \cos 5x + C_4 \sin 5x + e^{3x}(C_5 \cos 4x + C_6 \sin 4x) + C_7 e^{6x}$$

$\sim \cancel{2}\ 1$

$$y_p(x) = \underbrace{Ax^5 + Bx^4 + Cx^3 + Dx^2}_{2} + \underbrace{Ex\sin 5x + Fx\cos 5x}_{2} + \underbrace{Ge^{3x}}_{1}$$

5 **6.** Find an integrating factor for the differential equation

$$(x+1)\frac{dy}{dx} + xy = \cos 2x, \quad x > 0.$$

$$\frac{dy}{dx} + \frac{x}{x+1}y = \frac{\cos 2x}{x+1} \quad -1$$

Simplify your result as much as possible. Do **NOT** solve the differential equation.

$$\underbrace{e^{\int \frac{x}{x+1}dx}}_{1} = \underbrace{e^{\int\left(1 - \frac{1}{x+1}\right)dx}}_{1} = \underbrace{e^{x - \ln|x+1|}}_{1} = e^x e^{\ln(x+1)^{-1}}$$

$$= \underbrace{\frac{e^x}{x+1}}_{1}$$

7 **7.** Two substances A and B react to form a third substance C in such a way that 2 grams of A react with 3 grams of B to produce 5 grams of C. The rate at which C is formed is proportional to the product of the amounts of A and B still present in the mixture. Set up an initial-value problem (differential equation plus initial condition) for the amount $C(t)$ of C present in the mixture as a function of time t when the original amounts of A and B brought together at time $t = 0$ are 20 grams and 10 grams, respectively.

$$\frac{dC}{dt} = k\left(20 - \frac{2C}{5}\right)\left(10 - \frac{3C}{5}\right), \quad C(0) = 0.$$

8 **8.** Find the Laplace transform for the function

$$e^{-3t}\sin 2t\, h(t-\pi).$$

$$F(s) = e^{-\pi s}\mathcal{L}\{e^{-3(t+\pi)}\sin 2(t+\pi)\} \sim 2$$

$$= e^{-\pi s}e^{-3\pi}\mathcal{L}\{e^{-3t}\sin 2t\} \sim 2$$

$$= e^{-\pi(s+3)}\mathcal{L}\{\sin 2t\}_{|s+3} \sim 2$$

$$= e^{-\pi(s+3)}\left[\frac{2}{(s+3)^2+4}\right] \sim 2$$

4 **9.** Can the function

$$F(s) = \frac{s^2e^s}{(s^2+1)(e^s+1)}$$

be the Laplace transform of a piecewise continuous function of exponential order? Explain.

No. since $\lim_{s\to\infty} F(s) = 1$, which is not 0.

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8 **10.** Find the inverse Laplace transform for the function

$$F(s) = \frac{e^{-2s}(1-e^{s})}{s^3+2s}.$$

$$\frac{1}{s^3+2s} = \frac{1}{s(s^2+2)} = \frac{1/2}{s} + \frac{-s/2}{s^2+2} \quad \sim 3$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^3+2s}\right\} = \frac{1}{2} - \frac{1}{2}\cdot \cancel{\sin}\cos\sqrt{2}\,t \quad \sim 2$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{e^{-2s}-e^{-s}}{s^3+2s}\right\} \quad \sim 1$$

$$= \left[\frac{1}{2} - \frac{1}{2}\cos\sqrt{2}(t-2)\right]h(t-2) \quad \sim 1$$

$$- \left[\frac{1}{2} - \frac{1}{2}\cos\sqrt{2}(t-1)\right]h(t-1) \quad \sim 1$$

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8 **11.** A mass of 1 kilogram is suspended from a spring with constant 50 newtons per metre. At time $t = 0$, it is at its equilibrium position and is given velocity 2 metres per second downward. During its subsequent motion, it is also subjected to air resistance that (in newtons) is equal to 3/2 times its velocity (in metres per second). Use Laplace transforms to find the position of the mass as a function of time.

$$1\frac{d^2x}{dt^2} + \frac{3}{2}\frac{dx}{dt} + 50x = 0 \quad \checkmark 1, \quad x(0) = 0 \quad \checkmark 1/2, \quad x'(0) = -2. \quad \checkmark 1/2$$

When we take Laplace transforms

$$(s^2X + 2) + \frac{3}{2}(sX) + 50X = 0. \quad \checkmark 2$$

$$X(s) = \frac{-2}{s^2 + \frac{3s}{2} + 50} = \frac{-2}{\left(s+\frac{3}{4}\right)^2 + \frac{791}{16}} \quad \checkmark 2$$

$$x(t) = -2e^{-3t/4}\mathcal{L}^{-1}\left\{\frac{1}{s^2 + \frac{791}{16}}\right\} \quad \checkmark 1$$

$$= -2e^{-3t/4}\left(\frac{4}{\sqrt{791}}\right)\sin\frac{\sqrt{791}\,t}{4} \text{ m.} \quad \checkmark 1$$

12 **12.** Find an integral representation for the solution of the initial-value problem

$$y'' - 2y' - 3y = f(t), \qquad y(0) = 1, \quad y'(0) = 0,$$

where $f(t)$ is some unspecified function.

when we take Laplace transforms,

$$(s^2 Y - s) - 2(sY - 1) - 3Y = F(s) \quad \sim 2$$

$$\therefore\ Y(s) = \frac{F(s)}{s^2-2s-3} + \frac{s-2}{s^2-2s-3} \quad \sim 1$$

$$= \left(\frac{1/4}{s-3} + \frac{-1/4}{s+1}\right)F(s) \sim 2 + \frac{1/4}{s-3} + \frac{3/4}{s+1} \quad \sim 2$$

Using convolutions,

$$y(t) = \int_0^t \left[\frac{1}{4}e^{3(t-u)} - \frac{1}{4}e^{-(t-u)}\right] f(u)\,du \quad \sim 3$$

$$+ \frac{1}{4}e^{3t} \sim 1 + \frac{3}{4}e^{-t} \sim 1$$