1. Base of induction $n=1$:

$$LHS = \frac{1}{1\cdot 5} = \frac{1}{5}$$

$$RHS = \frac{1}{4\cdot 1+1} = \frac{1}{5}$$

$LHS = RHS$

Hypothesis: Assume for some $n = k \geq 1$

$$\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \ldots + \frac{1}{(4k-3)(4k+1)} = \frac{k}{4k+1}$$

We need to prove that for $n=k+1$ our equality is valid:

$$\underbrace{\frac{1}{1\cdot 5} + \frac{1}{5\cdot 9} + \ldots + \frac{1}{(4k-3)(4k+1)}} + \frac{1}{(4k+1)(4k+5)} = \frac{k+1}{4k+5}.$$

Using hypothesis:

$$LHS = \frac{k}{4k+1} + \frac{1}{(4k+1)(4k+5)} = \frac{k(4k+5)+1}{(4k+1)(4k+5)}$$

$$= \frac{4k^2+5k+1}{(4k+1)(4k+5)} *$$

$$RHS = \frac{k+1}{4k+5} = \frac{(4k+1)(k+1)}{(4k+1)(4k+5)} = \frac{4k^2+5k+1}{(4k+1)(4k+5)}$$

So, $LHS = RHS$ which proves the statement for $n = k+1$.

So, by PMI, the identity is verified for all $n \geq 1$.

2. Base $n=1$:

$$7\cdot 5^2 + 1 = 7\cdot 25 + 1 = 176 = 8\cdot 22 \text{ - divisible by } 8.$$

Hypthesis: Assume for some $n = k \geq 1$

8 divides $7\cdot 5^{2k} + 1$.

We need to prove our statement for $n=k+1$, i.e. that

$$8 \text{ divides } 7\cdot 5^{2(k+1)}+1$$

We transform the last expression to factor $7\cdot 5^{2k}+1$:

$$7\cdot 5^{2(k+1)}+1 = 7\cdot 5^{2k}\cdot 25+1 = 7\cdot 5^{2k}\cdot 25+25-25+1$$

$$= 25\underbrace{\left(7\cdot 5^{2k}+1\right)}_{\substack{\text{divisible by 8}\\ \text{by hypothesis}}} - \underbrace{24}_{\text{divisible by 8}}$$

So, 8 divides $7\cdot 5^{2(k+1)}+1$, and by PMI 8 divides $7\cdot 5^{2n}+1$ for any $n\geq 1$.

3. Since $2=\frac{2}{2^0}$, we have $2+\frac{3}{2^1}+\ldots+\frac{22}{2^{20}}=\sum_{n=1}^{21}\frac{n+1}{2^{n-1}}$.

Alternatively, the following are also correct:

$$\sum_{n=0}^{20}\frac{n+2}{2^n} \quad \text{or} \quad \sum_{n=2}^{22}\frac{n}{2^{n-2}}$$

(and other letters can be used instead of n)

4. Since $\sum_{k=6}^{19}a_k=\sum_{k=1}^{19}a_k-\sum_{k=1}^{5}a_k$, we have

$$\sum_{k=6}^{19}(2k-3k^2+k^3)=2\sum_{k=6}^{19}k-3\sum_{k=6}^{19}k^2+\sum_{k=6}^{19}k^3$$

$$=2\left(\frac{19(19+1)}{2}-\frac{5(5+1)}{2}\right)-3\left(\frac{19(19+1)(2\cdot 19+1)}{6}-\frac{5(5+1)(2\cdot 5+1)}{6}\right)$$

$$+\left(\frac{19^2(19+1)^2}{4} - \frac{5^2(5+1)^2}{4}\right) = 19\cdot 20 - 5\cdot 6 - 19\cdot 10\cdot 38$$

$$+\,5\cdot 3\cdot 11 + 19^2\cdot 10^2 - 5^2\cdot 3^2 \quad \text{(no need to compute further)}$$

5. Base $n=1, n=2$:

$$3^1 = 3 > 1 = 1^2 \quad \text{- true}$$
$$3^2 = 9 > 4 = 2^2 \quad \text{- true}$$

Hypothesis: Assume that for $n = k \geq 2$ we know that

(1) $$3^k > k^2.$$

We need to prove our inequality for $n = k+1$, i.e.

(2). $$3^{k+1} > (k+1)^2.$$

To obtain (2) from (1), we need to multiply the inequality (1) by the inequality

$$3 > \left(\frac{k+1}{k}\right)^2.$$

The last inequality is true since

$$\sqrt{3} > \frac{3}{2} \geq 1 + \frac{1}{k} = \frac{k+1}{k} \text{ for } k \geq 2.$$

This proves (2), and by PMI $3^n > n^2$ for all $n \geq 1$.

__NOTE__: Inductive step uses $k \geq 2$, so we have to check both $n=1$ and $n=2$ in the base of induction.