UNIVERSITY OF MANITOBA

DATE: October 24, 2016 MIDTERM

EXAMINATION: Techniques of Classical and Linear Algebra TIME: 75 minutes

COURSE: MATH 1210 EXAMINER: Grafton, Moghaddam, Szestopalow

[8] 1. (a) Use mathematical induction on integer $n \geq 1$ to prove that

$$1(3) + 2(4) + 3(5) + \ldots + n(n+2) = \frac{1}{6}n(n+1)(2n+7).$$

Let $P(n)$ be the above statement.

$P(1)$ is true because L.H.S $= 1(3) = 3$, R.H.S $= \frac{1}{6}(1)(2)(9) = 3$ ✓

Assume that for $n=k$, $P(k)$ is true, that is

$$1(3)+2(4)+\cdots+k(k+2) = \frac{1}{6}k(k+1)(2k+7) \quad (1)$$

we need to prove that for $n=k+1$, $P(k+1)$ is true, that is

$$1(3)+2(4)+\cdots+(k+1)(k+3) = \frac{1}{6}(k+1)(k+2)(2k+9)$$

But $1(3)+2(4)+\cdots+(k+1)(k+3) = [1(3)+2(4)+\cdots+k(k+2)]+(k+1)(k+3)$

$$\overset{\text{by }(1)}{=} \frac{1}{6}k(k+1)(2k+7)+(k+1)(k+3)$$

$$= \frac{1}{6}(k+1)[k(2k+7)+6(k+3)]$$

$$= \frac{1}{6}(k+1)(2k^2+13k+18)$$

$$= \frac{1}{6}(k+1)(k+2)(2k+9)$$

$= R.H.S.$

So by the principle of mathematical induction, $P(n)$ is true for all $n \geq 1$.

[3] (b) Write $1(3) + 2(4) + 3(5) + \ldots + n(n+2)$ in sigma notation such that the index starts from 2.

$$1(3)+2(4)+\cdots+n(n+2) = \sum_{\ell=1}^{n} \ell(\ell+2)$$

$$= \sum_{\ell=2}^{n+1} (\ell-1)(\ell+1)$$

$$= \sum_{\ell=2}^{n+1} (\ell^2-1)$$

[10] 2. Find the Cartesian form of $(i^5+1)^{10} + (i^5-1)^{10}$. Simplify as much as possible.

$i^5 = i^4 i = (1)i = i$ so $i^5+1 = 1+i$ and $i^5-1 = -1+i$.

For $z_1 = 1+i$, $r_1 = \sqrt{1^2+1^2} = \sqrt{2}$, $\tan\theta_1 = \frac{1}{1} = 1 \Rightarrow \theta_1 = \pi/4$

and for $z_2 = -1+i$, $r_2 = \sqrt{(-1)^2+1^2} = \sqrt{2}$, $\tan\theta_2 = \frac{1}{-1} = -1$

$\Rightarrow \theta_2 = 3\pi/4$.

So $z_1 = 1+i = \sqrt{2}\, e^{\pi/4 i}$ and $z_2 = -1+i = \sqrt{2}\, e^{3\pi/4 i}$

Now $(i^5+1)^{10} + (i^5-1)^{10} = z_1^{10} + z_2^{10}$

$$= (\sqrt{2}\, e^{\pi/4 i})^{10} + (\sqrt{2}\, e^{3\pi/4 i})^{10}$$

$$= (2^5) e^{10\pi/4 i} + (2^5) e^{30\pi/4 i}$$

$$= 2^5 \left(e^{5\pi/2 i} + e^{15\pi/2 i} \right)$$

$$= 2^5 \left(e^{(2\pi + \pi/2) i} + e^{(8\pi - \pi/2) i} \right)$$

$$= 2^5 \left(e^{\pi/2 i} + e^{-\pi/2 i} \right)$$

$$= 2^5 \left[(\cos \pi/2 + i \sin \pi/2) + (\cos(-\pi/2) + i \sin(-\pi/2)) \right]$$

$$= 2^5 (0 + i(1) + 0 + i(-1))$$

$$= 2^5 (0)$$

$$= 0.$$

[7] 3. Find all third roots of $-\sqrt{2}+\sqrt{6}\,i$. Leave your answer in exponential form but simplify it.

$$r = \sqrt{(-\sqrt{2})^2 + (\sqrt{6})^2} = \sqrt{2+6} = \sqrt{8}$$

$$\tan\theta = \frac{\sqrt{6}}{-\sqrt{2}} = -\frac{\sqrt{2}\sqrt{3}}{\sqrt{2}} = -\sqrt{3} \Rightarrow \theta = 2\pi/3$$

$$z = \sqrt{8}\, e^{\frac{2\pi}{3} i} = \sqrt{2^3}\, e^{(\frac{2\pi}{3}+2k\pi)i}$$

so $z_k = (\sqrt{2^3})^{1/3}\, e^{\frac{(2\pi/3 + 2k\pi)}{3} i}$

so $z_k = \sqrt{2}\, e^{\frac{2\pi(3k+1)}{9} i}$, $k = 0, 1, 2$

If $k=0$, then $z_0 = \sqrt{2}\, e^{\frac{2\pi}{9} i}$

If $k=1$, then $z_1 = \sqrt{2}\, e^{\frac{8\pi}{9} i}$

If $k=2$, then $z_2 = \sqrt{2}\, e^{\frac{14\pi}{9} i}$

[16] 4. Consider the polynomial equation of $P(x) = 0$ where

$$P(x) = 3x^4 - 6x^3 + 7x^2 - 8x + 4$$

(a) What are the possible rational zeros of $P(x)$?

If $r = \frac{p}{q}$ is a rational root, then by the rational root theorem

$p \mid 4$ and $q \mid 3$ so p is $\pm 1, \pm 2, \pm 4$ and q is $\pm 1, \pm 3$

Therefore possible rational zeros of $P(x)$ are

$\pm 1, \pm 2, \pm 4, \pm \frac{1}{3}, \pm \frac{2}{3}, \pm \frac{4}{3}$

(b) Show that $P(x)$ has no zero in the interval $\overset{-3,\,-1}{[\cancel{-1}, -3]}$.

$P(-x) = 3x^4 + 6x^3 + 7x^2 + 8x + 4$

since there is no sign change, so there is no negative real zero for $P(x)$. Hence $P(x)$ has no zero in the interval $[-3, -1]$.

(c) Use Bounds Theorem to find a bound on the zeros of $P(x)$.

$M = \text{maximum}\{|-6|, |7|, |-8|, |4|\} = 8$

For any zero x of $P(x)$,

$|x| < \frac{M}{|a_n|} + 1 = \frac{8}{3} + 1 = \frac{11}{3}$

that is $-\frac{11}{3} < x < \frac{11}{3}$ if x is a real zero.

[*Recall that* $P(x) = 3x^4 - 6x^3 + 7x^2 - 8x + 4$]

(d) Update the list from part (a) using the information from parts (b) and (c).

$1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{4}{3}$

(e) Find all the zeros of $P(x)$.

since $P(1) = 3-6+7-8+4=0$ so $x-1$ is a factor

So by long division

$P(x) = (x-1)\underbrace{(3x^3-3x^2+4x-4)}_{Q(x)}$

since $Q(1) = 3-3+4-4=0$ so $x-1$ is a factor of $Q(x)$ as well. Long division gives (or factoring)

$P(x) = (x-1)(x-1)(3x^2+4) = (x-1)^2(3x^2+4)$

So $\boxed{x=1}$, $3x^2+4=0 \Rightarrow x^2 = -\frac{4}{3} \Rightarrow \boxed{x = \pm\frac{2}{\sqrt{3}}i}$

[8] 5. Let $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}$. Find all values of a and b for which $A^2 + AB^T - 6I = 0$.

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -5 \end{pmatrix}^T - 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} + \begin{pmatrix} a & 0 \\ 0 & -5b \end{pmatrix} - \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} a^2 + a - 6 & 0 \\ 0 & b^2 - 5b - 6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So must $a^2 + a - 6 = 0 \Rightarrow (a+3)(a-2) = 0 \Rightarrow a = -3, a = 2$

and $b^2 - 5b - 6 = 0 \Rightarrow (b-6)(b+1) = 0 \Rightarrow b = 6, b = -1$

6. Let $\mathbf{u} = \langle 1, 3, 3 \rangle$, $\mathbf{v} = \langle 1, 2, 2 \rangle$ and $\mathbf{w} = \langle 1, 1, -\frac{1}{2} \rangle$.

[4] (a) Find the angle between $\mathbf{u} \times \mathbf{v}$ and $2\mathbf{w}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 3 \\ 1 & 2 & 2 \end{vmatrix} = \langle 0, 1, -1 \rangle$$

$$2\vec{w} = 2\langle 1, 1, -\tfrac{1}{2} \rangle = \langle 2, 2, -1 \rangle$$

$$\cos\theta = \frac{(\vec{u} \times \vec{v}) \cdot (2\vec{w})}{|\vec{u} \times \vec{v}|\,|2\vec{w}|}, \quad \text{but } |\vec{u} \times \vec{v}| = \sqrt{0+1+1} = \sqrt{2}$$

$$|2\vec{w}| = \sqrt{4+4+1} = 3$$

$$\text{so } \cos\theta = \frac{\langle 0,1,-1 \rangle \cdot \langle 2,2,-1 \rangle}{(\sqrt{2})(3)} = \frac{0+2+1}{3\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

[4] (b) Let $\mathbf{r} = \langle 1, 2a - 12, a^2 \rangle$. Find all values of "$a$" for which $\mathbf{r}$ is perpendicular to $\mathbf{u} + \mathbf{v} - 2\mathbf{w}$.

must $(\vec{u} + \vec{v} - 2\vec{w}) \cdot \vec{r} = 0$ but

$$\vec{u} + \vec{v} - 2\vec{w} = \langle 1,3,3 \rangle + \langle 1,2,2 \rangle - \langle 2,2,-1 \rangle = \langle 0, 3, 6 \rangle$$

$$\text{so } \langle 0, 3, 6 \rangle \cdot \langle 1, 2a-12, a^2 \rangle = 0$$

$$0 + 3(2a-12) + 6(a^2) = 0$$

$$6a^2 + 6a - 36 = 0$$

$$a^2 + a - 6 = 0$$

$$(a+3)(a-2) = 0$$

$$a = -3, \quad a = 2$$