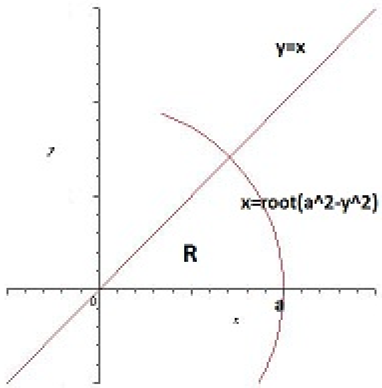
MATH 2130 Problem Workshop 6 Solutions

1. The region is question is given below



(a) The formula for mass is

$$\iint_R \rho dA.$$

Since density is constant, this is the same as

$$\rho \iint_R dA = \rho(\text{area of } R).$$

To find the area of R, we can note that it is just one-eighth of the area of the circle of radius a. Hence the mass is

$$\rho\frac{\pi a^2}{8}.$$

Failing that we can use polar coordinates to re-write the region as

$$0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}.$$

(Note that we can get $\pi/4$ since when $y = x$, $r\cos\theta = r\sin\theta \Rightarrow \tan\theta = 1 \Rightarrow \theta = \pi/4$)

Therefore

$$M = \rho\iint_R dA = \rho\int_0^{\pi/4}\int_0^a r dr d\theta = \rho\int_0^{\pi/4}\left.\frac{r^2}{2}\right]_0^a d\theta = \rho\int_0^{\pi/4}\frac{a^2}{2}d\theta = \rho\frac{\pi}{4}\frac{a^2}{2} = \rho\frac{\pi a^2}{8}.$$

(b) To find the moment about the x-axis, we use the formula for first moment

$$\iint_R \rho d dA$$

where d is the distance to whatever line we are find the first moment about. For the x-axis, $d = y$, and so we get

$$M_x = \iint_R y dA.$$

From here we have a couple different options. The region is part of a circle, so polar coordinates might be useful. I'll do it both ways <u>Solution 1:</u> Cartesian

The region is awful if y is in terms of x, so we'll do it the other way. Setting the curves equal to each other to find bounds on y gives

$$y = \sqrt{a^2 - y^2} \Rightarrow y^2 = a^2 - y^2 \Rightarrow y^2 = \frac{a^2}{2} \Rightarrow y = \frac{a}{\sqrt{2}}$$

using $a, y \geq 0$. Hence

$$y \leq x \leq \sqrt{a^2 - y^a}, \qquad 0 \leq y \leq \frac{a}{\sqrt{2}}.$$

Therefore

$$\begin{aligned}
M_x &= \rho \int_0^{a/\sqrt{2}} \int_y^{\sqrt{a^2-y^2}} y dx dy \\
&= \rho \int_0^{a/\sqrt{2}} xy\Big]_y^{\sqrt{a^2-y^2}} dy \\
&= \rho \int_0^{a/\sqrt{2}} (y\sqrt{a^2 - y^2} - y^2) dy \\
&= \rho \Big(-\frac{1}{3}(a^2 - y^2)^{3/2} - \frac{y^3}{3} \Big]_0^{a/\sqrt{2}} \\
&= \rho \left[\left(-\frac{1}{3}(a^2 - (a/\sqrt{2})^2)^{3/2} - \frac{(a/\sqrt{2})^3}{3} \right) - \left(-\frac{1}{3}(a^2 - 0^2)^{3/2} - \frac{0^3}{3} \right) \right] \\
&= \rho \left[\left(-\frac{1}{3}\left(\frac{a^2}{2}\right)^{3/2} - \frac{a^3}{6\sqrt{2}} \right) - \left(-\frac{a^3}{3} \right) \right] \\
&= \rho \left(-\frac{a^3}{6\sqrt{2}} - \frac{a^3}{6\sqrt{2}} + \frac{a^3}{3} \right) \\
&= \rho \frac{(\sqrt{2} - 1)a^3}{3\sqrt{2}}
\end{aligned}$$

<u>Solution 2:</u> Polar

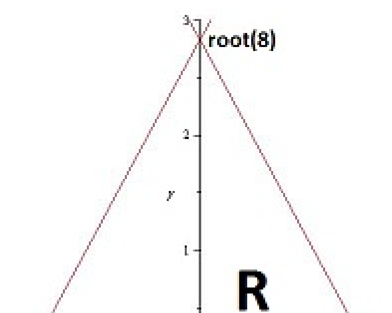
The region is as in part (a). The only change is we have to replace $y = r \sin\theta$ in the integrand. Hence we get

$$
\begin{aligned}
M_x &= \rho \iint_R y dA \\
&= \rho \int_0^{\pi/4} \int_0^a r^2 \sin\theta dr d\theta \\
&= \rho \int_0^{\pi/4} \frac{r^3}{3} \sin\theta \Big]_0^a d\theta \\
&= \rho \int_0^{\pi/4} \frac{a^3 \sin\theta}{3} d\theta \\
&= -\frac{\rho a^3 \cos\theta}{3} \Big]_0^{\pi/4} \\
&= -\frac{\rho a^3 \cos(\pi/4)}{3} + \frac{\rho a^3 \cos 0}{3} \\
&= \frac{\rho a^3(-1/\sqrt{2}+1)}{3} \\
&= \frac{\rho a^3(\sqrt{2}-1)}{3\sqrt{2}}
\end{aligned}
$$

(c)

$$
M\overline{y} = M_x \Rightarrow \overline{y} = \frac{M_x}{M} = \frac{\rho \frac{(\sqrt{2}-1)a^3}{3\sqrt{2}}}{\rho \frac{\pi a^2}{8}} = \frac{8(\sqrt{2}-1)a}{3\pi\sqrt{2}}
$$

2. We take the isosceles triangle and put the base on the x-axis (symmetrically). Using the pythagorean theorem, we get that the triangle has height $\sqrt{8}$. Hence we get the region



from which we'll find I_x the moment of inertia about the x-axis. Since the distance to the x-axis is $d = y$, we get the moment of inertia is

$$
\iint_R \rho d^2 dA = \rho \iint_R y^2 dA.
$$

Using some symmetry, since the region is symmetric and y^2 is even, we get

$$\begin{aligned}
I &= \rho \iint_R y^2 dA \\
&= 2\rho \int_0^1 \int_0^{\sqrt{8}-\sqrt{8}x} y^2 dy dx \\
&= 2\rho \int_0^1 \frac{y^3}{3}\Big]_0^{\sqrt{8}-\sqrt{8}x} dx \\
&= 2\rho \int_0^1 \left(\frac{(\sqrt{8}-\sqrt{8}x)^3}{3} - 0\right) dx \\
&= 2(\sqrt{8})^3\rho \int_0^1 \left(\frac{(1-x)^3}{3}\right) dx \\
&= 32\sqrt{2}\rho \bigg(-\frac{(1-x)^4}{12}\bigg]_0^1 \\
&= 32\sqrt{2}\rho \left[0 - \left(-\frac{1}{12}\right)\right] \\
&= \frac{8\sqrt{2}\rho}{3}
\end{aligned}$$

3. We are finding surface area which has the formula

$$S = \iint_R \sqrt{1 + z_x^2 + z_y^2} dA.$$

The region R, is the circle $x^2 + y^2 = a^2$ in the xy-plane. The integrand is

$$\sqrt{1 + z_x^2 + z_y^2} = \sqrt{1 + y^2 + x^2}.$$

Therefore we get

$$S = \iint_R \sqrt{1 + y^2 + x^2} dA.$$

Since we have a circle and $x^2 + y^2$ in the integrand, this is a clear case for polar coordinates. The region has $0 \leq r \leq a, 0 \leq \theta \leq 2\pi$. Hence the surface area is

$$\begin{aligned} S &= \int_0^{2\pi} \int_0^a \sqrt{1+r^2} r dr d\theta \\ &= \frac{1}{3} \int_0^{2\pi} (1+r^2)^{3/2} \Big]_0^a d\theta \\ &= \frac{1}{3} \int_0^{2\pi} \big((1+a^2)^{3/2} - 1\big) d\theta \\ &= \frac{2\pi}{3}\big((1+a^2)^{3/2} - 1\big) \end{aligned}$$

4. The integrand is

$$\sqrt{1+z_x^2+z_y^2} = \sqrt{1+(4x)^2+(2y)^2} = \sqrt{1+16x^2+4y^2}.$$

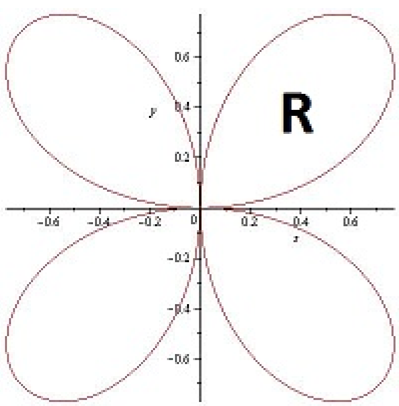
The region R is the triangle bounded by the lines $y=0, x=0, y=1-x$. Hence we get

$$S = \iint_R \sqrt{1+z_x^2+z_y^2} dA = \int_0^1 \int_0^{1-x} \sqrt{1+16x^2+4y^2} dy dx.$$

5. Determining what this region looks like, we note that polar is helpful since we see x^2+y^2. Hence we get

$$(r^2)^3 = 4a^2(r^4 \cos^2\theta \sin^2\theta) \Rightarrow r^2 = 4a^2\cos^2\theta\sin^2\theta \Rightarrow r = \pm 2a \sin\theta\cos\theta.$$

The picture of which (with $a=1$) is



Symmetry allows us to take 4 times the leaf in the first quadrant, hence using R as the region in the first quadrant, we get the area is

$$
\begin{aligned}
A &= 4\iint_R dA \\
&= 4\int_0^{\pi/2}\int_0^{2a\sin\theta\cos\theta} r\,dr\,d\theta \\
&= 4\int_0^{\pi/2} \frac{r^2}{2}\Big]_0^{2a\sin\theta\cos\theta} d\theta \\
&= 4\int_0^{\pi/2}\left(\frac{(2a\sin\theta\cos\theta)^2}{2} - 0\right)d\theta \\
&= 8a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta
\end{aligned}
$$

Using $\sin^2\theta = \frac{1-\cos 2\theta}{2}$ and $\cos^2\theta = \frac{1+\cos 2\theta}{2}$, we get the area is

$$
\begin{aligned}
A &= 8a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta \\
&= 8a^2\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)\left(\frac{1+\cos 2\theta}{2}\right)d\theta \\
&= 2a^2\int_0^{\pi/2}(1-\cos^2 2\theta)\,d\theta \\
&= 2a^2\int_0^{\pi/2}\left(1-\frac{1+\cos 4\theta}{2}\right)d\theta \\
&= a^2\int_0^{\pi/2}(1-\cos 4\theta)\,d\theta \\
&= a^2\left(\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} \\
&= a^2\left(\frac{\pi}{2} - \frac{\sin 2\pi}{4} - (0-0)\right) \\
&= \frac{\pi a^2}{2}.
\end{aligned}
$$

6. The region strongly suggests that we use polar coordinates, even though the function itself doesn't. We are basically forced into polar by the region

$$R = \{(r,\theta) | 1 \le r \le 2, 0 \le \theta \le \pi/2\}.$$

Hence we get

$$
\begin{aligned}
I &= \iint_R xy(x+y)dA \\
&= \int_0^{\pi/2}\int_1^2 (r\cos\theta)(r\sin\theta)(r\cos\theta + r\sin\theta)rdrd\theta \\
&= \int_0^{\pi/2}\int_1^2 r^4(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta)drd\theta \\
&= \int_0^{\pi/2} \frac{r^5}{5}(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta)]_1^2 d\theta \\
&= \int_0^{\pi/2} \left(\frac{32}{5} - \frac{1}{5}\right)(\cos^2\theta\sin\theta + \cos\theta\sin^2\theta)d\theta \\
&= \frac{31}{5}\int_0^{\pi/2} (\cos^2\theta\sin\theta + \cos\theta\sin^2\theta)d\theta \\
&= \frac{31}{5}\Big(- \frac{\cos^3\theta}{3} + \frac{\sin^3\theta}{3}\Big]_0^{\pi/2} \\
&= \frac{31}{15}\big[(-\cos^3(\pi/2) + \sin^3(\pi/2)) - (-\cos^3(0) + \sin^3(0)\big] \\
&= \frac{31}{15}\big[(0+1) - (-1+0)\big] \\
&= \frac{62}{15}.
\end{aligned}
$$

7. The region is above the xy-plane and below the plane $z = 6 - 2x - 3y$. Hence we get the bounds on z to be

$$0 \le z \le 6 - 2x - 3y.$$

Projecting the region into the xy-plane makes the triangle bounded by $x = 0, y = 0$ and $2x + 3y = 6$. Hence we get bounds on y, x to be

$$0 \le y \le 2 - \frac{2}{3}x, \qquad 0 \le x \le 3.$$

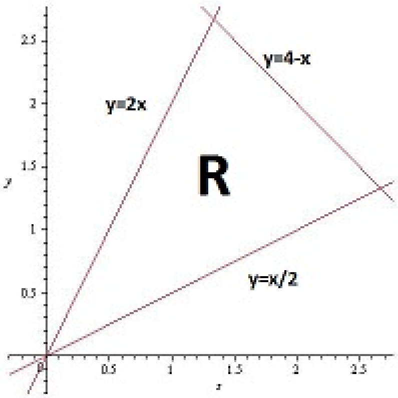
Putting this together leads to the integral

$$
\begin{aligned}
I &= \int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} x\,dz\,dy\,dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} xz\Big]_0^{6-2x-3y} dy\,dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} x(6-2x-3y-0)\,dy\,dx \\
&= \int_0^3 \int_0^{2-\frac{2}{3}x} (6x-2x^2-3xy)\,dy\,dx \\
&= \int_0^3 \left(6xy - 2x^2y - \frac{3xy^2}{2}\right]_0^{2-\frac{2}{3}x} dx \\
&= \int_0^3 \left(6x\big(2-\frac{2}{3}x\big) - 2x^2\big(2-\frac{2}{3}x\big) - \frac{3x\big(2-\frac{2}{3}x\big)^2}{2} - 0\right) dx \\
&= \int_0^3 \left(12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 - \frac{3x\big(4-\frac{8}{3}x+\frac{4}{9}x^2\big)}{2}\right) dx \\
&= \int_0^3 \left(12x - 4x^2 - 4x^2 + \frac{4}{3}x^3 - \big(6x - 4x^2 + \frac{2}{3}x^3\big)\right) dx \\
&= \int_0^3 \left(6x - 4x^2 + \frac{2}{3}x^3\right) dx \\
&= \left(3x^2 - \frac{4x^3}{3} + \frac{1}{6}x^4\right]_0^3 \\
&= \left(3(3)^2 - \frac{4(3)^3}{3} + \frac{1}{6}(3)^4\right) - 0 \\
&= 27 - 36 + \frac{27}{2} \\
&= \frac{9}{2}.
\end{aligned}
$$

8. The volume is

$$\iiint_V dV.$$

We need to figure out bounds for our region. Clearly z is bounded below by $z=0$ and is bounded above by $z = 16-4x-4y$. This leaves us with the region in the xy-plane bounded by $4x+4y=16, y=x/2, y=2x$ graphed below.



The intersection of $y = x/2$ and $y = 2x$ is clearly $(0, 0)$. The intersection of $y = 2x$ and $4x + 4y = 16$ is

$$4x + 8x = 16 \Rightarrow x = \frac{16}{12} = \frac{4}{3} \Rightarrow \left(\frac{4}{3}, \frac{8}{3}\right).$$

The intersection of $y = x/2$ and $4x + 4y = 16$ is

$$4x + 2x = 16 \Rightarrow x = \frac{16}{6} = \frac{8}{3} \Rightarrow \left(\frac{8}{3}, \frac{4}{3}\right).$$

Since the upper function changes from $0 \leq x \leq 4/3$ to $4/3 \leq x \leq 8/3$ we will need to split the function up into two integrals.

$$I_1 = \int_0^{4/3} \int_{x/2}^{2x} \int_0^{16-4x-4y} dzdydx, \qquad I_2 = \int_{4/3}^{8/3} \int_{x/2}^{4-x} \int_0^{16-4x-4y} dzdydx$$

$$
\begin{aligned}
I_1 &= \int_0^{4/3} \int_{x/2}^{2x} \int_0^{16-4x-4y} dzdydx \\
&= \int_0^{4/3} \int_{x/2}^{2x} (16-4x-4y)dydx \\
&= \int_0^{4/3} 16y - 4xy - 2y^2]_{x/2}^{2x} dx \\
&= \int_0^{4/3} \left[(32x - 8x^2 - 8x^2) - \left(8x - 2x^2 - \frac{1}{2}x^2\right)\right] dx \\
&= \int_0^{4/3} \left(24x - \frac{27}{2}x^2\right) dx \\
&= \left(12x^2 - \frac{9}{2}x^3\right]_0^{4/3} \\
&= \left(12(4/3)^2 - \frac{9}{2}(4/3)^3\right) - \left(12(0)^2 - \frac{9}{2}(0)^3\right) \\
&= \left(\frac{64}{3} - \frac{32}{3}\right) - 0 \\
&= \frac{32}{3}.
\end{aligned}
$$

$$
\begin{aligned}
I_2 &= \int_{4/3}^{8/3} \int_{x/2}^{4-x} \int_0^{16-4x-4y} dzdydx \\
&= \int_{4/3}^{8/3} \int_{x/2}^{4-x} (16-4x-4y)dydx \\
&= \int_{4/3}^{8/3} 16y - 4xy - 2y^2]_{x/2}^{4-x} dx \\
&= \int_{4/3}^{8/3} \left[(64 - 16x - 4x(4-x) - 2(16 - 8x + x^2)) - \left(8x - 2x^2 - \frac{1}{2}x^2\right)\right] dx \\
&= \int_{4/3}^{8/3} \left[(64 - 16x - (16x - 4x^2) - (32 - 16x + 2x^2) - \left(8x - 2x^2 - \frac{1}{2}x^2\right)\right] dx \\
&= \int_{4/3}^{8/3} \left(32 - 24x + \frac{9}{2}x^2\right) dx \\
&= \left(32x - 12x^2 - \frac{3}{2}x^3\right]_{4/3}^{8/3} \\
&= \left(32(8/3) - 12(8/3)^2 + \frac{3}{2}(8/3)^3\right) - \left(32(4/3) - 12(4/3)^2 + \frac{3}{2}(4/3)^3\right) \\
&= \left(\frac{256}{3} - \frac{256}{3} + \frac{256}{9}\right) - \left(\frac{128}{3} - \frac{64}{3} + \frac{32}{9}\right) \\
&= \left(\frac{256}{9}\right) - \left(\frac{224}{9}\right) \\
&= \frac{32}{9}
\end{aligned}
$$

Hence the total volume is

$$
\frac{32}{3} + \frac{32}{9} = \frac{128}{9}
$$