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EXAMINATION: APPLIED CALCULUS I

FINAL EXAMINATION
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TIME: 2 hours
EXAMINER: Various

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[12] 1. Compute each limit, if it exists.

(a) $\lim_{x\to 1/3} \dfrac{(2-6x)^2}{(3x-1)(9x^2-1)}$

(b) $\lim_{x\to\infty} \left(\sqrt{x^2+x}-\sqrt{x^2-3x}\right)$

(c) $\lim_{x\to 0} \dfrac{x^2+2x-\sin 3x}{2x}$

(a) $\lim_{x\to\frac{1}{3}} \dfrac{4(3x-1)^2}{(3x-1)(3x-1)(3x+1)} = \lim_{x\to\frac{1}{3}} \dfrac{4}{3x+1} = \dfrac{4}{3\cdot\frac{1}{3}+1} = 2$

(b) $\lim_{x\to\infty} \dfrac{(x^2+x)-(x^2-3x)}{\sqrt{x^2+x}+\sqrt{x^2-3x}} = \lim_{x\to\infty} \dfrac{4x}{\sqrt{x^2+x}+\sqrt{x^2-3x}}$

$= \lim_{x\to\infty} \dfrac{4}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{3}{x}}} = \dfrac{4}{\sqrt{1}+\sqrt{1}} = 2$

(c) $\lim_{x\to 0} \dfrac{x^2+2x-\sin 3x}{2x} = \lim_{x\to 0}\left(\dfrac{x}{2}+1-\dfrac{\sin 3x}{3x}\cdot\dfrac{3}{2}\right)$

$= \left\{ \text{as } \lim_{t\to 0} \dfrac{\sin t}{t} = 1 \right\} = \cancel{\lim}\ \dfrac{0}{2}+1-\dfrac{3}{2} = -\dfrac{1}{2}$

[8] 2. Find $f'(x)$ for each of the following functions. Do NOT simplify your answers.

(a) $f(x) = \sqrt{5 - 2\sin^3(7x)}$

(b) $f(x) = \dfrac{\ln(x^3 + x)}{e^{4\cos(x)}}$

(a) $f'(x) = \dfrac{1}{2\sqrt{5-2\sin^3(7x)}} \cdot (-2) \cdot 3\sin^2(7x) \cdot \cos 7x \cdot 7$

(b) $f'(x) = \dfrac{\frac{1}{x^3+x} \cdot (3x^2+1) \cdot e^{4\cos x} - \ln(x^3+x) \cdot e^{4\cos x} \cdot 4 \cdot (-\sin x)}{e^{8\cos x}}$

[10] 3. Find the equation of the tangent line to the curve $3x^5y^5 - y = 2x$ at the point $(1, 1)$.

$15x^4y^5 + 15x^5y^4 \cdot y' - y' = 2$

$y' = \dfrac{2 - 15x^4y^5}{15x^5y^4 - 1}$ $\qquad y'\Big|_{(x,y)=(1,1)} = \dfrac{2-15}{15-1} = -\underline{\dfrac{13}{14}}$

slope

<u>Ans</u>: $y - 1 = -\dfrac{13}{14}(x-1)$

[20] 4. (a) Draw the graph of the function $f(x) = x^2 e^{-x}$. (You may use the fact that $\lim_{x\to\infty} x^2 e^{-x} = 0$ without proving it.)

(b) Find all points of inflection on this graph, and determine all intervals where the graph of the function is concave upward.

(c) Find the absolute maximum and minimum values of the function $f(x) = x^2 e^{-x}$ on the interval $-1 \le x \le 3$.

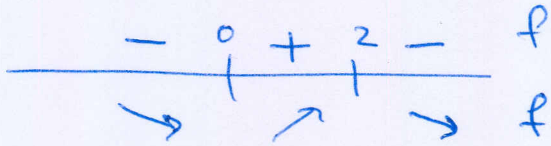
$\lim_{x\to -\infty} x^2 e^{-x} = \infty$

no vertical asymptotes

$x=0$: $y=0$ – only intercept

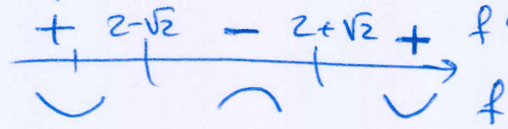
$f'(x) = 2xe^{-x} - x^2 e^{-x} = (2x - x^2)e^{-x}$

$f'=0$: $x=2$, $x=0$

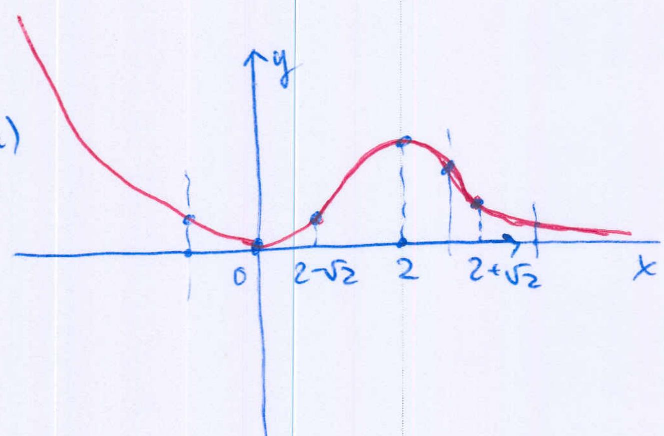


$f''(x) = (2-2x)e^{-x} - (2x - x^2)e^{-x}$

$= (x^2 - 4x + 2)e^{-x} = (x - (2+\sqrt{2}))(x - (2-\sqrt{2}))e^{-x}$



(a)



$f(2+\sqrt{2}) = (2+\sqrt{2})^2 \cdot e^{-2-\sqrt{2}} = (6+4\sqrt{2})e^{-2-\sqrt{2}}$

$f(2-\sqrt{2}) = (2-\sqrt{2})^2 \cdot e^{-2+\sqrt{2}} = (6-4\sqrt{2})e^{-2+\sqrt{2}}$

(b) inflection points:

$(2+\sqrt{2}, (6+4\sqrt{2})e^{-2-\sqrt{2}})$

$(2-\sqrt{2}, (6-4\sqrt{2})e^{-2+\sqrt{2}})$

intervals where concave up:

$(-\infty, 2-\sqrt{2})$, $(2+\sqrt{2}, \infty)$

(c) both 0, 2 are inside

$f(-1) = e$

$f(0) = 0$

$f(2) = 4\cdot e^{-2}$

$f(3) = 9\cdot e^{-3}$

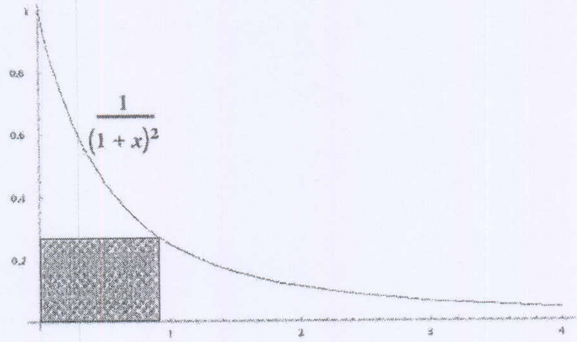
all positive except 0, so <u>0 is abs. min. value</u>

$f(2) > f(3)$ because decreasing on $(2, \infty)$

$f(-1) > f(2)$ because $e > 1 > \left(\frac{2}{e}\right)^2$

so <u>e is abs. max. value</u>

[12] 5. Find the area of the largest rectangle that lies entirely in the first quadrant, has one side on the x-axis, another side on the y-axis, and a vertex on the curve $y = \dfrac{1}{(1+x)^2}$.



$$A = xy = \frac{x}{(1+x)^2}, \qquad x > 0$$

$$A'(x) = \frac{1\cdot(1+x)^2 - x\cdot 2(1+x)}{(1+x)^4} = \frac{1+x-2x}{(1+x)^3} = \frac{1-x}{(1+x)^3}$$

+   − A'

−1   0   1

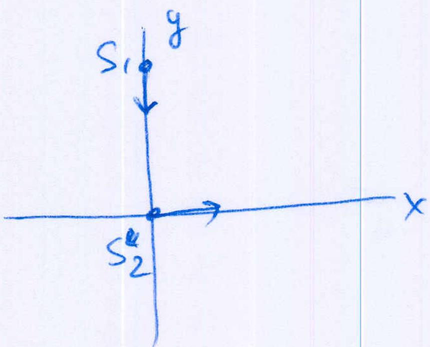
↗   ↘ A

$\underline{x=1}$ − crit. point
loc. max. by
1st der. test
⇓
must be global max.

$$A(1) = \frac{1}{(1+1)^2} = \underline{\frac{1}{4}}$$

$\underline{\text{Ans}}$: $\frac{1}{4}$

[10] 6. At noon a ship S_1 is 20 km north of ship S_2. If S_1 sails south at 6 km/h and S_2 sails east at 8 km/h. How fast are the ships separating at 4 : 00 pm?



S_1 goes along y axis in negative direction, y - coordinate of S_1 in km. $t=0$ - noon, t in hrs.
S_2 goes along x axis in positive direction, x - coordinate of S_2. in km.

$y|_{t=0} = 20$ $\quad x|_{t=0} = 0$ $\quad \ell = \sqrt{x^2+y^2}$ - distance between them

$\frac{dy}{dt} = -6$ $\quad \frac{dx}{dt} = 8$

$$\left.\frac{d\ell}{dt}\right|_{t=4} = ?$$

$$\frac{d\ell}{dt} = \frac{1}{2\sqrt{x^2+y^2}} \cdot \left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$

when $t=4$: $\quad x = 8 \cdot 4 = 32$

$\quad y = 20 - 6 \cdot 4 = -4$

$$\left.\frac{d\ell}{dt}\right|_{t=4} = \frac{32 \cdot 8 + (-4)\cdot(-6)}{\sqrt{32^2+4^2}} = \frac{32 \cdot 2 + 6}{\sqrt{8^2+1}} = \frac{70}{\sqrt{65}}$$

<u>Ans</u>: at $\frac{70}{\sqrt{65}}$ km/h.

[20] 7. Find the following indefinite and definite integrals:

(a) $\displaystyle\int t^2 \cos(t^3)\, dt$

(b) $\displaystyle\int \frac{\sqrt{x} - x^2}{x}\, dx$

(c) $\displaystyle\int_{\pi/2}^{\pi} \left(\sin x - \frac{1}{x}\right) dx$. Simplify as much as possible.

(d) $\displaystyle\int_0^1 x^5 \sqrt{x^3 + 1}\, dx$

(a) $$\int t^2 \cos(t^3)\, dt = \left\{ \begin{array}{l} t^3 = x \\ 3t^2 = \frac{dx}{dt} \end{array} \right\} = \int \cos x \cdot \frac{dx}{3} = \frac{1}{3} \sin x + c$$

$$= \frac{1}{3} \sin(t^3) + C, \ C \in \mathbb{R}$$

(b) $$\int \frac{\sqrt{x} - x^2}{x}\, dx = \int \left(x^{-\frac{1}{2}} - x\right) dx = 2x^{\frac{1}{2}} - \frac{x^2}{2} + c, \quad c \in \mathbb{R}$$

(c) $$\int_{\frac{\pi}{2}}^{\pi} \left(\sin x - \frac{1}{x}\right) dx = \left(-\cos x - \ln|x|\right)\Big|_{\frac{\pi}{2}}^{\pi} = -\cos\pi - \ln\pi$$

$$+ \cos\frac{\pi}{2} + \ln\frac{\pi}{2} = 1 - \ln\pi + 0 + \ln\frac{\pi}{2} = 1 + \ln\frac{\pi}{2\pi}$$

$$= 1 + \ln\frac{1}{2} = 1 - \ln 2$$

(d) $$\int_0^1 x^5 \sqrt{x^3+1}\, dx = \left\{ \begin{array}{l} x^3 + 1 = t \\ 3x^2 = \frac{dt}{dx} \\ x = 0 : t = 1 \\ x = 1 : t = 2 \end{array} \right\}$$

$$\left\{ \begin{array}{l} x^5 \sqrt{x^3+1}\, dx = x^3 \sqrt{x^3+1}\, x^2 d \\ = (t-1)\sqrt{t} \cdot \frac{dt}{3} \end{array} \right\}$$

$$= \int_1^2 \left(t^{3/2} - t^{1/2}\right) \frac{dt}{3} = \frac{1}{3} \cdot \left(\frac{2}{5} t^{5/2} - \frac{2}{3} t^{3/2}\right)\Big|_1^2$$

$$= \frac{8\sqrt{2}}{15} - \frac{4\sqrt{2}}{9} - \frac{2}{15} + \frac{2}{9} = \frac{4}{45}\sqrt{2} + \frac{4}{45}$$

[8] 8. Find $f(x)$ if $f''(x) = 36x^2 + 12x$ with $f'(1) = 19$ and $f(1) = 7$.

$f'(x) = \int (36x^2 + 12x)\,dx = 12x^3 + 6x^2 + c$

$19 = f'(1) = 12 + 6 + c \qquad c = 1$

$f'(x) = 12x^3 + 6x^2 + 1$

$f(x) = \int (12x^3 + 6x^2 + 1) = 3x^4 + 2x^3 + x + k$

$7 = f(1) = 3 + 2 + 1 + k \qquad k = 1$

$\underline{f(x) = 3x^4 + 2x^3 + x + 1}$

[8] Bonus. Find the points on the curve $y^2 - x^2 + 2x = 10$ closest to the point $(5, 0)$.

distance squared $\ell^2 = (x-5)^2 + y^2 = (x-5)^2 + x^2 - 2x + 10$.

$= x^2 - 10x + 25 + x^2 - 2x + 10 = 2x^2 - 12x + 35$

parabola – minimum at the vertex $x = \frac{12}{2 \cdot 2} = 3$

(or, derivative: $(2x^2 - 12x + 35)' = 4x - 12 \Rightarrow \underset{\text{minim.}}{x = 3}$)

$y^2 - 3^2 + 2 \cdot 3 = 10$

$y^2 = 10 + 9 - 6 = 13$

$y = \pm\sqrt{13}$

Ans : $(3, \sqrt{13})$, $(3, -\sqrt{13})$