Optimization and related rates problems with inverse trig functions

1. Straight line segments are drawn from the fixed points $P(-1,3)$ and $Q(2,7)$ to the movable point R with coordinates $(x,0)$.

(a) Assuming that $-1 \leq x \leq 2$, show that the angle θ between the two line segments PR and RQ is given by

$$\theta = \pi - \mathrm{Cot}^{-1}\left(\frac{x+1}{3}\right) - \mathrm{Cot}^{-1}\left(\frac{2-x}{7}\right).$$

(b) Determine the value of x which makes the angle θ as large as possible.

2. A steel pipe is being carried down a hallway 3 meters wide. At the end of the hall there is a right-angled turn into a narrower hallway 2 meters wide. What is the length of the longest pipe that can be carried horizontally around the corner of the hallway?

3. Find the maximum area of a rectangle that can be circumscribed about a given rectangle with dimensions a and b (each vertex of the smaller rectangle is on a side of the bigger rectangle). Hint: express the area in terms of an angle between the sides of the smaller and the bigger rectangles.

4. An observer stands at a point P, one meter away from a track. Two runners start at the point S of the track (segment SP is perpendicular to the track) and run along the track, one runner is three times as fast as the other.

(a) Find the maximum value of the observer's angle of sight θ between the runners.

(b) If the speed of the slower runner is 2 meters per second, how is the angle of sight changing 3 seconds after they start running?

5. A ferris wheel is 25 meters in diameter and rotates at a constant rate of one revolution every two minutes. If the axle of the wheel is 15 meters above ground level, how fast is a passenger on the wheel rising at the instant when (s)he is 21.25 meters above ground level?

6. A man, who is 2 meters tall, walks along the edge of a straight level road, which is 10 meters wide. On the opposite side of the road is a street light, which is 8 meters tall. If the man walks at a constant rate of 1.5 meters per second, how fast is his shadow lengthening, when he is 10 meters past the point on the edge of the road which lies directly opposite the street light?

7. A plane flies horizontally at an altitude of 5 kilometers and passes directly over a tracking telescope on the ground. When the angle of elevation of the plane is $\pi/3$, this angle is decreasing at a rate of $\pi/6$ radians per minute. How fast is the plane traveling at that time?