EXERCISES FOR CHAPTER 2: Riemann Sums

1. (a) By finding a partition of the interval [0,1], using the function $f(x)=e^x$ and a suitable Riemann sum show that $\lim_{n\to\infty}\frac{1}{n}\left(1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}\right)=e-1$. (b) Rederive this limit by explicitly finding the sum $1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}$ and using L'Hôpital's rule.

Solution

(a) A Riemann sum for this function on this interval is (choosing $c_k=\frac{k-1}{n}$)

$$\begin{aligned}\sum_{k=1}^{k=n} f(c_k)\Delta x_k &= \sum_{k=1}^{k=n} e^{c_k}\,\frac{1}{n}\\ &= \sum_{k=1}^{k=n} e^{\frac{k-1}{n}}\,\frac{1}{n}\\ &= \frac{1}{n}\left(1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}\right)\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left(1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}\right) &= \int_0^1 e^x\,dx\\ &= e-1\end{aligned}$$

(b)

$$\begin{aligned}1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}} &= \frac{(1-e^{\frac{n}{n}})}{1-e^{\frac{1}{n}}} && \text{summed a geometric series}\\ &= \frac{1-e}{1-e^{\varepsilon}} && \text{we let } \frac{1}{n}=\varepsilon\end{aligned}$$

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left(1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}\right) &= \lim_{\varepsilon\to 0}\frac{\varepsilon(1-e)}{1-e^{\varepsilon}}\\ &= (1-e)\frac{1}{-1} && \text{by L'Hôpital's rule}\\ &= e-1\end{aligned}$$

2. Show that $\lim_{n\to\infty}\frac{1}{n}\left(\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\ln(1+\frac{3}{n})+\cdots+\ln(1+\frac{n}{n})\right)=2\ln 2-1.$

Solution

Use the function $f(x)=\ln x$ on the interval $[1,2]$ with $c_k = 1+\dfrac{k}{n}$. Then

$$\begin{aligned}\sum_{k=1}^{k=n} f(c_k)\Delta x_k &= \sum_{k=1}^{k=n} \ln c_k \,\frac{1}{n}\\ &= \sum_{k=1}^{k=n} \ln\left(1+\frac{k}{n}\right)\frac{1}{n}\\ &= \frac{1}{n}\left(\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\ln(1+\frac{3}{n})+\cdots+\ln(1+\frac{n}{n})\right)\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\left(\ln(1+\frac{1}{n})+\ln(1+\frac{2}{n})+\ln(1+\frac{3}{n})+\cdots+\ln(1+\frac{n}{n})\right) &= \int_1^2 \ln x\, dx\\ &= x\ln x - x\Big|_1^2\\ &= (2\ln 2-2)-(-1)\\ &= 2\ln 2 - 1\end{aligned}$$

3. Show that $\lim\limits_{n\to\infty}\dfrac{1}{n}\left(\sin\dfrac{\pi}{n}+\sin\dfrac{2\pi}{n}+\sin\dfrac{3\pi}{n}+\cdots+\sin\dfrac{(n-1)\pi}{n}\right)=\dfrac{2}{\pi}$.

Solution

Use the function $f(x)=\sin x$ with $c_k=\dfrac{k-1}{n}\pi$ on the interval $[0,\pi]$. A Riemann sum is then

$$\begin{aligned}\sum_{k=1}^{k=n} f(c_k)\Delta x_k &= \sum_{k=1}^{k=n}\sin(c_k)\,\frac{\pi}{n}\\ &= \sum_{k=1}^{k=n}\sin\left(\frac{k-1}{n}\pi\right)\frac{\pi}{n}\\ &= \frac{\pi}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)\end{aligned}$$

Thus

$$\pi\lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)=\int_0^{\pi}\sin x\,dx = -\cos x\Big|_0^{\pi}=2$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{n}\left(\sin\frac{\pi}{n}+\sin\frac{2\pi}{n}+\sin\frac{3\pi}{n}+\cdots+\sin\frac{(n-1)\pi}{n}\right)=\frac{2}{\pi}$$

4. Show that $\lim\limits_{n\to\infty}\dfrac{1}{n^{k+1}}\left(1^k+2^k+3^k+\cdots+n^k\right)=\dfrac{1}{k+1}$.

Solution

Use the function $f(x)=x^k$ with $c_k=\dfrac{k}{n}$ on the interval $[0,1]$. A Riemann sum is then

$$\begin{aligned}\sum_{k=1}^{k=n} f(c_k)\Delta x_k &= \sum_{k=1}^{k=n} f(c_k)\,\frac{1}{n}\\ &= \sum_{k=1}^{k=n}\left(\frac{k}{n}\right)^k \frac{1}{n}\\ &= \frac{1}{n}\left((\frac{1}{n})^k + (\frac{2}{n})^k + (\frac{3}{n})^k + \cdots + (\frac{n}{n})^k\right)\\ &= \frac{1}{n^{k+1}}\left(1^k + 2^k + \cdots + n^k\right)\end{aligned}$$

Thus

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n^{k+1}}\left(1^k + 2^k + \cdots + n^k\right) &= \int_0^1 x^k\, dx\\ &= \left.\frac{x^{k+1}}{k+1}\right|_0^1\\ &= \frac{1}{k+1}\end{aligned}$$

5. Show that $\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1+n}}+\frac{1}{\sqrt{2+n}}+\frac{1}{\sqrt{3+n}}+\cdots+\frac{1}{\sqrt{2n}}\right)=2(\sqrt{2}-1)$.

Solution

Use the function $f(x)=\frac{1}{\sqrt{1+x}}$ with $c_k=\frac{k}{n}$ on the interval $[0,1]$. A Riemann sum is then

$$\begin{aligned}\sum_{k=1}^{k=n} f(c_k)\Delta x_k &= \sum_{k=1}^{k=n} f(c_k)\,\frac{1}{n}\\ &= \sum_{k=1}^{k=n}\frac{1}{\sqrt{1+\frac{k}{n}}}\,\frac{1}{n}\\ &= \sum_{k=1}^{k=n}\frac{\sqrt{n}}{\sqrt{n+k}}\,\frac{1}{n}\\ &= \sum_{k=1}^{k=n}\frac{1}{\sqrt{n+k}}\,\frac{1}{\sqrt{n}}\\ &= \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1+n}}+\frac{1}{\sqrt{2+n}}+\frac{1}{\sqrt{3+n}}+\cdots+\frac{1}{\sqrt{2n}}\right)\end{aligned}$$

Thus

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{1+n}}+\frac{1}{\sqrt{2+n}}+\frac{1}{\sqrt{3+n}}+\cdots+\frac{1}{\sqrt{2n}}\right)=\int_0^1\frac{1}{\sqrt{x+1}}\,dx$$

$$=\left.\frac{\sqrt{x+1}}{1/2}\right|_0^1$$

$$=2\sqrt{2}-2\sqrt{1}$$

$$=2(\sqrt{2}-1)$$

6. Evaluate $\int_0^\infty \frac{dx}{x^2+a^2}$.

Solution

$$\int_0^\infty \frac{dx}{x^2+a^2}=\lim_{\Lambda\to\infty}\int_0^\Lambda \frac{dx}{x^2+a^2}=\frac{1}{a}\lim_{\Lambda\to\infty}\arctan\left.\frac{x}{a}\right|_0^\Lambda=\frac{1}{a}\lim_{\Lambda\to\infty}\arctan\frac{\Lambda}{a}=\frac{\pi}{2a}$$

7. Evaluate $\int_0^\infty e^{-3x}\,dx$.

Solution

$$\int_0^\infty e^{-3x}\,dx=\lim_{\Lambda\to\infty}\int_0^\Lambda e^{-3x}\,dx=\lim_{\Lambda\to\infty}\left.\frac{e^{-3x}}{-3}\right|_0^\Lambda=\lim_{\Lambda\to\infty}\frac{e^{-3\Lambda}}{-3}+\frac{1}{3}=\frac{1}{3}.$$

8. Evaluate $\int_0^\infty xe^{-5x}\,dx$.

Solution

$$\int_0^\infty xe^{-5x}\,dx=\lim_{\Lambda\to\infty}\int_0^\Lambda xe^{-5x}\,dx=\lim_{\Lambda\to\infty}\left.\frac{xe^{-5x}}{-5}\right|_0^\Lambda+\frac{1}{5}\lim_{\Lambda\to\infty}\int_0^\Lambda e^{-5x}\,dx$$

$$=0-\lim_{\Lambda\to\infty}\left.\frac{e^{-5x}}{25}\right|_0^\Lambda=\frac{1}{25}$$

9. Evaluate $\int_1^\infty \frac{1}{x^4}\,dx$.

Solution

$$\int_1^\infty \frac{1}{x^4}\,dx=\lim_{\Lambda\to\infty}\int_1^\Lambda \frac{1}{x^4}\,dx=\lim_{\Lambda\to\infty}\left.\frac{1}{-3x^3}\right|_1^\Lambda=\lim_{\Lambda\to\infty}\frac{1}{-3\Lambda^3}+\frac{1}{3}=\frac{1}{3}.$$

10. Evaluate $\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution

Integrate by parts to get

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{\Lambda\to\infty}\int_1^\Lambda \frac{\ln x}{x^2}\,dx = \lim_{\Lambda\to\infty}\left(-\frac{1}{x\ln x}\right)\Bigg|_1^\Lambda + \lim_{\Lambda\to\infty}\int_1^\Lambda \frac{1}{x^2}\,dx$$

$$= \lim_{\Lambda\to\infty}\left(-\frac{\ln x}{x}\right)\Bigg|_1^\Lambda + \lim_{\Lambda\to\infty}\left(-\frac{1}{x}\right)\Bigg|_1^\Lambda = \lim_{\Lambda\to\infty} -\frac{\ln\Lambda}{\Lambda} + \lim_{\Lambda\to\infty} -\frac{1}{\Lambda} + 1$$

$$= 1$$

since $\lim_{\Lambda\to\infty}\frac{\ln\Lambda}{\Lambda} = \lim_{\Lambda\to\infty}\frac{\frac{1}{\Lambda}}{1} = 0$ by L' Hôpital's rule.

11. Determine whether the improper integral $\int_2^\infty \frac{1}{x}\,dx$ exists.

Solution

$\int_2^\infty \frac{1}{x}\,dx = \lim_{\Lambda\to\infty}\int_2^\Lambda \frac{1}{x}\,dx = \lim_{\Lambda\to\infty} \ln x\big|_2^\Lambda = \lim_{\Lambda\to\infty} \ln\Lambda - \ln 2 \to \infty$, integral does not exist.

12. Determine whether the improper integral $\int_0^\infty \sin^2 x\,dx$ exists.

Solution

$\int_0^\infty \sin^2 x\,dx = \lim_{\Lambda\to\infty}\int_0^\Lambda \sin^2 x\,dx = \lim_{\Lambda\to\infty}\int_0^\Lambda \frac{1-\cos 2x}{2}\,dx = \lim_{\Lambda\to\infty}\left(\frac{x}{2} - \frac{\sin 2x}{4}\right)_0^\Lambda \to \infty$ and so the integral does not exist.

13. Evaluate $\int_1^\infty \frac{x}{2^x}\,dx$.

Solution

$$\int_1^\infty \frac{x}{2^x}\,dx = \lim_{\Lambda\to\infty}\int_1^\Lambda x2^{-x}\,dx = \lim_{\Lambda\to\infty} -\frac{x2^{-x}}{\ln 2}\Bigg|_0^\Lambda + \frac{1}{\ln 2}\lim_{\Lambda\to\infty}\int_1^\Lambda 2^{-x}\,dx$$

$$= -\frac{1}{\ln 2}\left(\lim_{\Lambda\to\infty}\frac{\Lambda}{2^\Lambda} - \frac{1}{2}\right) - \frac{1}{(\ln 2)^2}\left(\lim_{\Lambda\to\infty}\frac{1}{2^\Lambda} - \frac{1}{2}\right)$$

$$= -\frac{1}{\ln 2}\lim_{\Lambda\to\infty}\frac{1}{2^\Lambda \ln 2} + \frac{1}{2\ln 2} + \frac{1}{2(\ln 2)^2}$$

$$= \frac{1}{2\ln 2} + \frac{1}{2(\ln 2)^2}$$

$$= \frac{\ln 2 + 1}{2(\ln 2)^2}$$

14. Consider the improper integral $\Gamma(n)=\int_0^\infty x^{n-1}e^{-x}\,dx$ that defines a function called the gamma function. Here n is a positive integer. (a) By using integration by parts show that $\Gamma(n)=(n-1)\Gamma(n-1)$. (b) Show that $\Gamma(1)=\Gamma(2)=1$. (c) Hence or otherwise establish that $\Gamma(n)=(n-1)!$.

Solution

(a)

$$\int_0^\infty x^{n-1}e^{-x}\,dx=\lim_{\Lambda\to\infty}\int_0^\Lambda x^{n-1}e^{-x}\,dx=\lim_{\Lambda\to\infty}\left(-x^{n-1}e^{-x}\right)_0^\Lambda+(n-1)\lim_{\Lambda\to\infty}\int_0^\Lambda x^{n-2}e^{-x}\,dx$$
$$=\lim_{\Lambda\to\infty}\left(-\Lambda^{n-1}e^{-\Lambda}\right)+(n-1)\Gamma(n-1)$$
$$=(n-1)\Gamma(n-1)$$

because $\lim_{\Lambda\to\infty}\left(\Lambda^{n-1}e^{-\Lambda}\right)=\lim_{\Lambda\to\infty}\left(\frac{\Lambda^{n-1}}{e^\Lambda}\right)=0$ by repeated application of L' Hôpital's rule.

(b)

$$\Gamma(2)=(2-1)\Gamma(1)=\int_0^\infty e^{-x}\,dx=\lim_{\Lambda\to\infty}\int_0^\Lambda e^{-x}\,dx=\lim_{\Lambda\to\infty}\left(-e^{-x}\right)_0^\Lambda=1.$$

(c) $\Gamma(n)=(n-1)\Gamma(n-1)=(n-1)(n-2)\Gamma(n-2)=\cdots=(n-1)!\Gamma(1)=(n-1)!$

15. (a) By careful diagrams establish that when $f(n)$ is a decreasing function $\int_1^\infty f(n)dn<\sum_{n=1}^\infty f(n)<f(1)+\int_1^\infty f(n)dn$. Hence show that

$$\frac{2}{e}<\sum_{n=1}^\infty ne^{-n}<\frac{3}{e}$$

(b) Consider $F(\alpha)=\sum_{n=1}^\infty e^{-\alpha n}$. Show that $F(\alpha)=\frac{1}{e^\alpha-1}$ and that $-\left.\frac{dF(\alpha)}{d\alpha}\right|_{\alpha=1}=\sum_{n=1}^\infty ne^{-n}$.

Hence find the exact value of $\sum_{n=1}^\infty ne^{-n}$. (c) How many terms must we keep in the sum so that the error is less than 5×10^{-6}?

Solution

(a) The discussion of the inequalities is given in the text. We must do the integral

$$\int_1^\infty xe^{-x}\,dx=\lim_{\Lambda\to\infty}\int_1^\Lambda xe^{-x}\,dx=\lim_{\Lambda\to\infty}\left(-xe^{-x}\Big|_1^\Lambda+\int_1^\Lambda e^{-x}\,dx\right)=\lim_{\Lambda\to\infty}-(x+1)e^{-x}\Big|_1^\Lambda=2e^{-1}$$

and so $2e^{-1}<\sum_{n=1}^\infty ne^{-n}<e^{-1}+2e^{-1}$, i.e. $\frac{2}{e}<\sum_{n=1}^\infty ne^{-n}<\frac{3}{e}$ as required.

(b) $F(\alpha)=e^{-\alpha}+e^{-2\alpha}+\cdots=\dfrac{e^{-\alpha}}{1-e^{-\alpha}}=\dfrac{1}{e^{\alpha}-1}$ since we have a geometric series. $\dfrac{dF(\alpha)}{d\alpha}=\dfrac{d}{d\alpha}\sum_{n=1}^{\infty}e^{-\alpha n}=-\sum_{n=1}^{\infty}ne^{-\alpha n}$, which gives our series when $\alpha=1$.

Hence $\sum_{n=1}^{\infty}ne^{-\alpha n}=-\dfrac{d}{d\alpha}\dfrac{1}{e^{\alpha}-1}\bigg|_{\alpha=1}=\dfrac{e^{\alpha}}{(e^{\alpha}-1)^2}\bigg|_{\alpha=1}=\dfrac{e}{(e-1)^2}\approx 0.9206735942$. (This is within the bounds of the inequality in (a), i.e. $\dfrac{2}{e}=0.736<0.921<1.104=\dfrac{3}{e}$.)

(c) In keeping N terms the error made is at most $R_N=\int_N^{\infty}xe^{-x}\,dx=e^{-N}(N+1)$. We demand that $e^{-N}(N+1)<5\times10^{-6}$ and so $N\geq 15$. With 15 terms the sum is approximately 0.9206706422 with an error therefore of 2.952×10^{-6}, i.e. less than 5×10^{-6} as expected.

16. By considering the function $f(n)=\dfrac{1}{n^2+3}$ show that $\dfrac{\pi\sqrt{3}}{9}<\sum_{n=1}^{\infty}\dfrac{1}{n^2+3}<\dfrac{9+4\pi\sqrt{3}}{36}$.

Solution

The required integral is

$$\int_1^{\infty}\frac{dn}{n^2+3}=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{dn}{n^2+3}=\frac{1}{\sqrt{3}}\left(\arctan\frac{\Lambda}{\sqrt{3}}-\arctan\frac{1}{\sqrt{3}}\right)=\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{\pi\sqrt{3}}{9}$$

Hence

$$\frac{\pi\sqrt{3}}{9}<\sum_{n=1}^{\infty}\frac{1}{n^2+3}<\frac{1}{4}+\frac{\pi\sqrt{3}}{9}$$

$$\frac{\pi\sqrt{3}}{9}<\sum_{n=1}^{\infty}\frac{1}{n^2+3}<\frac{9+4\pi\sqrt{3}}{36}$$

17. Estimate the sum of the infinite series $\sum_{n=1}^{\infty}\dfrac{1}{1+n^2}$ by keeping just the first 10 terms. Give an estimate of the error involved. (The exact sum is 1.07667.)

Solution

$\sum_{n=1}^{10}\dfrac{1}{1+n^2}\approx 0.981793$. The error is less than

$$R_{10}=\int_{10}^{\infty}\frac{1}{x^2+1}dx=\arctan\infty-\arctan 10\approx 0.0996687$$

The actual error is $1.07667-0.981793=0.0948812$ which is less than our estimate of the error as expected.

18. Show that if $f(x)$ is an *increasing* function $f(1)+\int_1^n f(x)dx < \sum_{k=1}^{n} f(k) < \int_1^{n+1} f(x)dx$.

Hence find out how many terms must be kept so that the sum $\sum_{n=1} n2^n$ equals approximately 4.1 million.

Solution

The proof of the inequalities is a straightforward application of the rectangle trick. For the function $f(x)=x2^x$, the integral to be performed is

$$\int x2^x\,dx = x\frac{2^x}{\ln 2}-\frac{1}{\ln 2}\int 2^x\,dx = x\frac{2^x}{\ln 2}-\frac{2^x}{(\ln 2)^2}=\frac{2^x}{(\ln 2)^2}(x\ln 2-1)$$

and so we have the inequalities

$$2+\frac{2^N}{(\ln 2)^2}\left(N\ln 2-1\right)-\frac{2}{(\ln 2)^2}(\ln 2-1)<4.1\times 10^6$$

$$4.1\times 10^6<\frac{2^{N+1}}{(\ln 2)^2}\left((N+1)\ln 2-1)\right)-\frac{2}{(\ln 2)^2}(\ln 2-1)$$

The first inequality states that

$2^N\left(N\ln 2-1\right)<(4.1\times 10^6-2)(\ln 2)^2+2(\ln 2-1)=2.0\times 10^6$ i.e. $N\le 17$ (by trial and error with the GDC).

The second inequality is

$2^{N+1}\left((N+1)\ln 2-1)\right)>(4.1\times 10^6)(\ln 2)^2+2(\ln 2-1)=2.0\times 10^6$ i.e. $N\ge 17$.

We must thus choose 17 terms. The computer gives the sum of the first 17 terms as 4.19 million. (The sum of the first 16 is about 1.97 million and that of the first 18 about 8.9 million.)

19. Apply the inequalities of the previous problem on the increasing function $f(x)=\ln x$ to show that $\frac{n^n}{e^{n-1}}<n!<\frac{(n+1)^{n+1}}{e^n}$. Deduce that $\lim_{n\to\infty}\frac{n!}{n^n}=0$ and that $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$.

Solution

From $f(1)+\int_1^n f(x)dx<\sum_{k=1}^{n} f(k)<\int_1^{n+1} f(x)dx$ we have that

$$\begin{aligned}
&\ln 1 + (x\ln x - x)\Big|_1^n < \ln 1 + \ln 2 + \cdots + \ln n < (x\ln x - x)\Big|_1^{n+1} \\
&n\ln n - n + 1 < \ln(1\times 2\times 3\times\cdots\times n) < (n+1)\ln(n+1) - n \\
&\ln n^n - (n-1) < \ln(n!) < \ln(n+1)^{(n+1)} - n \\
&\ln(n^n e^{-(n-1)}) < \ln(n!) < \ln((n+1)^{(n+1)} e^{-n})
\end{aligned}$$

Hence

$$\frac{n^n}{e^{(n-1)}} < n! < \frac{(n+1)^{(n+1)}}{e^n}$$

as required.

Thus,

$$\begin{aligned}
&\frac{1}{e^{(n-1)}} < \frac{n!}{n^n} < \frac{(n+1)^{(n+1)}}{n^n e^n} \\
&\frac{1}{e^{(n-1)}} < \frac{n!}{n^n} < \frac{(n+1)^{(n+1)}}{n^{n+1} e^n} n \\
&\frac{1}{e^{(n-1)}} < \frac{n!}{n^n} < \left(\frac{n+1}{n}\right)^{n+1} \frac{n}{e^n}
\end{aligned}$$

and so $\lim\limits_{n\to\infty} \frac{1}{e^{(n-1)}} \le \lim\limits_{n\to\infty} \frac{n!}{n^n} \le \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^{n+1} \frac{n}{e^n}$. Because $\lim\limits_{n\to\infty} \frac{1}{e^{(n-1)}} = 0$ and $\lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^{n+1} \frac{n}{e^n} = e \times 0 = 0$ we have that $0 \le \lim\limits_{n\to\infty} \frac{n!}{n^n} \le 0$ and so by the squeeze theorem,

$$\lim_{n\to\infty} \frac{n!}{n^n} = 0 .$$

Further, taking the *n*th root of $\frac{1}{e^{(n-1)}} < \frac{n!}{n^n} < \left(\frac{n+1}{n}\right)^{n+1} \frac{n}{e^n}$ we have that

$$\frac{1}{e^{(n-1)/n}} < \frac{\sqrt[n]{n!}}{n} < \left(\frac{n+1}{n}\right)^{(n+1)/n} \frac{\sqrt[n]{n}}{e}$$

and taking limits $\lim\limits_{n\to\infty} \frac{1}{e^{(n-1)/n}} \le \lim\limits_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \le \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^{(n+1)/n} \frac{\sqrt[n]{n}}{e}$. Because

$$\lim_{n\to\infty} \frac{1}{e^{(n-1)/n}} = \lim_{n\to\infty} \frac{1}{ee^{-1/n}} = \frac{1}{e}$$

and

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{(n+1)/n} \frac{\sqrt[n]{n}}{e} = \frac{1}{e} \times \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{1+\frac{1}{n}} \times \lim_{n\to\infty} \sqrt[n]{n} = \frac{1}{e} \times 1 \times 1 = \frac{1}{e}$$

we have by the squeeze theorem that

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},$$

as required.

20. The value of $\sum_{k=1}^{\infty}\frac{1}{k^2}$ is to be approximated by keeping just the first n terms. What should n be so that the error is less than 5×10^{-5}, i.e. so that we have agreement to 4 decimal places? The exact answer for the infinite sum is $\frac{\pi^2}{6}$.

Solution

The error involved is $\sum_{k=n+1}^{\infty}\frac{1}{k^2}<\int_n^{\infty}\frac{1}{x^2}\,dx=-\frac{1}{x}\Big|_n^{\infty}=\frac{1}{n}$. So we need

$$\frac{1}{n}<5\times10^{-5}\Rightarrow n>\frac{1}{5\times10^{-5}}=2\times10^4$$

The sum of the first $2\times10^4+1$ terms is $1.64488\approx1.6449$ to 4 decimal places. The exact sum is $1.64493\approx1.6449$ giving agreement to 4 decimal places.

21. If the functions $f(x)$ and $g(x)$ satisfy $0\le f(x)\le g(x)$ then $\int_a^b f(x)\,dx\le\int_a^b g(x)\,dx$.

Use this fact to determine whether the improper integrals (a) $\int_1^{\infty}\frac{dx}{x^4+1}$ (b) $\int_1^{\infty}\frac{x^3}{x^5+x^3+x+1}\,dx$ and (c) $\int_1^{\infty}\frac{\sin x}{x}\,dx$ exist.

Solution

(a) We may compare,

$$\int_1^{\infty}\frac{dx}{x^4+1}<\int_1^{\infty}\frac{dx}{x^4}\quad\text{and}\quad\int_1^{\infty}\frac{dx}{x^4}=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{dx}{x^4}=-\lim_{\Lambda\to\infty}\frac{1}{3x^3}\Big|_1^{\Lambda}=\frac{1}{3}$$

so the original integral exists.

(b) We may compare,

$$\int_1^{\infty}\frac{x^3}{x^5+x^3+x+1}\,dx\le\int_1^{\infty}\frac{x^3}{x^5}\,dx=\int_1^{\infty}\frac{dx}{x^2}=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{dx}{x^2}=-\lim_{\Lambda\to\infty}\frac{1}{x}\Big|_1^{\Lambda}=1$$

so the original integral exists.

(c) By integration by parts,

$$\int_1^{\infty}\frac{\sin x}{x}\,dx=-\frac{\cos x}{x}\Big|_1^{\infty}-\int_1^{\infty}\frac{\cos x}{x^2}\,dx$$

$$=-\lim_{\Lambda\to\infty}\frac{\cos x}{x}\Big|_1^{\Lambda}-\int_1^{\infty}\frac{\cos x}{x^2}\,dx$$

The first term is $\cos 1$ and the integral is finite since $\int_1^\infty \frac{\cos x}{x^2}\,dx \le \int_1^\infty \left|\frac{\cos x}{x^2}\right| dx \le \int_1^\infty \frac{1}{x^2}\,dx = 1$. Hence $\int_1^\infty \frac{\sin x}{x}\,dx$ exists. (In fact also $\int_0^\infty \frac{\sin x}{x}\,dx$ exists and equals $\frac{\pi}{2}$.)