UNIVERSITY OF MANITOBA

DATE: February 8, 2011	Midterm #1
COURSE: MATH 1210	TITLE PAGE
EXAMINATION: Techniques of Classical & Linear Algebra	TIME: 45 minutes

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FAMILY NAME: (Print in ink, capitals) $\boxed{\text{Solutions}}$

GIVEN NAME(S): (Print in ink, capitals) ______________________

STUDENT NUMBER: ______________

LAB SECTION: ________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offence. I have read the instructions below twice.)

- ❑ A01 R. Thomas
- ❑ A02 T. Mohammed

INSTRUCTIONS TO STUDENTS:

This is a 45 minute exam. **Please show your work clearly.**

No calculators, texts, notes, or other aids are permitted. No cellphones or electronic translators, or other electronic devices able to receive or transmit a signal are permitted.

This exam has a title page and 3 pages of questions. Please check that you have all the pages.

The value of each question is indicated in the lefthand margin beside the statement of the question. The total value of all questions is 35 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the previous page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	12	
2	5	
3	7	
4	11	
Total:	35	

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[12] 1. Give in the form $a + bi$, with a and b real, the following, simplified as far as practical:

$$(2 + i2\sqrt{3})^{27}, \qquad \left(\frac{2+i}{1-i}\right) e^{2+3i}.$$

Solution: $(2 + i2\sqrt{3})^{27}$
Let $z = 2 + i\,2\sqrt{3} = 4\,e^{\frac{\pi}{3}\,i}$, where $|\,z\,| = \sqrt{x^2 + y^2} = 2\sqrt{4} = 4$ and $arg\,z = \frac{\pi}{3}$

$z^{27} = 4^{27}\,e^{27\,(\frac{\pi}{3})i} = 4^{27}\,e^{9\,\pi\,i} = 4^{27}\,e^{\pi\,i} = 4^{27}[\cos\pi + i\,\sin\pi] = 4^{27}\,(-1) + 0\,i = -2^{54} + 0\,i$,
(where $9\pi = \pi + 4(2\,\pi)$)

$$\left(\frac{2+i}{1-i}\right) e^{2+3i} = \left(\frac{2+i}{1-i}\cdot\frac{1+i}{1+i}\right) e^2\,e^{3i} = e^2\,\frac{(2-1)+(2+1)\,i}{2}\,(\cos 3 + i\,\sin 3)$$

$$= \frac{e^2}{2}\,(1+3i)(\cos 3 + i\,\sin 3) = \underbrace{\frac{e^2}{2}(\cos 3 - 3\,\sin 3)}_{a} + \underbrace{\frac{e^2}{2}(\sin 3 + 3\cos 3)}_{b}\,i$$

[5] 2. Rewrite the following using sigma notation with index beginning at 1:

$$1(10) - 2(9) + 3(8) - \cdots - 10(1).$$

Solution:
When $i = 1 \;\Rightarrow\; 1(10) = 1(11-1)$ and when $i = 2 \;\Rightarrow\; 2(9) = 2(11-2)$ $\cdots$

when $i = 10 \;\Rightarrow\; 10(1) = 10(11-10)$.

Therefore the above sum can be written as

$$\sum_{i=1}^{10} (-1)^{i+1}\,i\,(11-i)\,.$$

[7] 3. (a) Rewrite the sum $8 + 11 + 14 + \cdots + 38$ using sigma notation with index beginning at 1.

Solution:

when $k = 1 \quad \Rightarrow \quad 8 = 3(1) + 5$, and when $k = 2 \quad \Rightarrow \quad 11 = 3(2) + 5 \quad \cdots$
The upper limit can be found as following:
when $k = n$ then $38 = 3(n) + 5 \quad \Rightarrow \quad n = \frac{38-5}{3} = 11$

$$\sum_{k=1}^{11} (3k + 5)$$

(b) Using $\displaystyle\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$, evaluate the sum given in part (a).

Solution:

$$\sum_{k=1}^{11} (3k + 5) = \sum_{k=1}^{11} 3\,k + \sum_{k=1}^{11} 5 = 3 \sum_{k=1}^{11} k + 11(5) = 3\,\frac{11(12)}{2} + 55 = 253$$

[11] 4. Show by mathematical induction that 6 divides $19^n - 13^n$ for all values of n greater than or equal to 1.

Solution:

Let $P_n: \quad 6 \mid 19^n - 13^n, \quad n \geq 1$

1) When $n = 1$, $19 - 13 = 6$ and it is clearly divisible by 6. That is $P_1: \quad 6 \,|\, 19 - 13$, is true.

2) Assume $P_k: 6 \mid 19^k - 13^k$, to prove $P_{k+1}: 6 \mid 19^{k+1} - 13^{k+1}$.

Proof:

$$\begin{aligned} 19^{k+1} - 13^{k+1} &= 19^k\,(19) - 13^k\,(13) + 19(13^k) - 19\,(13^k) \\ &= 19(19^k - 13^k) + 13\,(19 - 13) \\ &= 19 \quad \underbrace{(19^k - 13^k)}_{\text{by } P_k \text{ it is divisible by 6}} \quad +13\,(6) = 6(g(k)\,) \end{aligned}$$

Where $g(k)$ is a number depending on k. So, P_{k+1} is true.

3) By PMI, P_n is true for all $n \geq 1$.