MATH 2130 – Midterm 1 Solutions

1. Let $\mathbf{r}(t) = (1, \cos(t^3), \sin(t^3))$. Calculate the indefinite integral

$$\int \mathbf{r}(t) \times \mathbf{r}'(t)\, dt.$$

Solution. We have

$$\begin{aligned}
\mathbf{r}(t) =&(1, \cos(t^3), \sin(t^3)),\\
\mathbf{r}'(t) =&(0, -3t^2\sin(t^3), 3t^2\cos(t^3)),\\
\mathbf{r}(t) \times \mathbf{r}'(t) =&(1, \cos(t^3), \sin(t^3)) \times (0, -3t^2\sin(t^3), 3t^2\cos(t^3))\\
=&(3t^2\cos^2(t^3) + 3t^2\sin^2(t^3), -3t^2\cos(t^3), -3t^2\sin(t^3))\\
=&(3t^2, -3t^2\cos(t^3), -3t^2\sin(t^3)),\\
\int \mathbf{r}(t) \times \mathbf{r}'(t)\, dt =& \int (3t^2, -3t^2\cos(t^3), -3t^2\sin(t^3))\, dt\\
=& \left(\int 3t^2\, dt, \int (-3t^2\cos(t^3))\, dt, \int (-3t^2\sin(t^3))\, dt\right)\\
=& \left(t^3, -\sin(t^3), \cos(t^3)\right) + \mathbf{C},
\end{aligned}$$

where $\mathbf{C} \in \mathbb{R}^3$.

2. Let Π be the plane

$$3x - 2y + 6z - 6 = 0,$$

and let ℓ be the line with parametric representation

$$x = 4 + t, \quad y = 6 + 4t, \quad z = 1 + 2t, \quad t \in \mathbb{R}.$$

Let P be the point $(2, 1, -2)$.

(a) Find the distance between ℓ and Π.

Solution. A normal vector for Π is $(3, -2, 6)$. A direction vector for ℓ is $(1, 4, 2)$. Observe that

$$(3, -2, 6) \cdot (1, 4, 2) = 3 - 8 + 12 = 7 \neq 0.$$

This show that ℓ and Π are not parallel, which implies that they intersect. The distance between them is 0.

(b) Find the distance between P and Π.

Solution. A normal vector for Π is $\mathbf{v} = (3, -2, 6)$, and the corresponding unit vector is $\widehat{\mathbf{v}} = \frac{1}{\sqrt{9+4+36}}(3, -2, 6) = \frac{1}{7}(3, -2, 6)$. By inspection, a point on Π is $Q = (2, 0, 0)$. The distance from Π to P is then

$$|\mathbf{PQ} \cdot \widehat{\mathbf{v}}| = \left|(0, -1, 2) \cdot \frac{1}{7}(3, -2, 6)\right| = \frac{0 + 2 + 12}{7} = 2.$$

(c) Find the equation of the plane containing P and ℓ.

Solution. A direction vector for ℓ is $\mathbf{w} = (1, 4, 2)$. The desired plane contains ℓ, so it is parallel to $\mathbf{w}$.

A point on ℓ is $R = (4, 6, 1)$. The desired plane contains ℓ and P, which means it is also parallel to $\mathbf{PR} = (2, 5, 3)$. Thus a normal vector for this plane is

$$\mathbf{w} \times \mathbf{PR} = (1, 4, 2) \times (2, 5, 3) = (12 - 10, 4 - 3, 5 - 8) = (2, 1, -3).$$

Using this normal vector and the point $P = (2, 1, -2)$, the equation for the desired plane is

$$2(x - 2) + (y - 1) - 3(z + 2) = 0.$$

3. On a large, clearly labeled diagram, sketch the surface

$$2x^2 - 4x - 3y^2 - 6y + z^2 - 6z + 2 = 0$$

Mark at least one important point. Give either the name of the surface, or the names of two different cross sections.

Solution. We complete the square:

$$\begin{aligned} 2(x^2 - 2x + 1 - 1) - 3(y^2 + 2y + 1 - 1) + (z^2 - 6z + 9 - 9) + 2 &= 0 \\ \implies \quad 2(x - 1)^2 - 3(y + 1)^2 + (z - 3)^2 - 2 + 3 - 9 + 2 &= 0 \\ \implies \quad 2(x - 1)^2 + (z - 3)^2 = 6 + 3(y + 1)^2. \end{aligned}$$

This surface is a translation by $(1, -1, 3)$ of the surface

$$2x^2 + z^2 = 6 + 3y^2.$$

The following observations can be made about the surface $2x^2 + z^2 = 6 + 3y^2$:

- It is an elliptic hyperboloid of one sheet, opening in the $\pm y$-directions.
- When $y = 0$, the cross section is the ellipse $2x^2 + z^2 = 6$. More generally, a slice at a constant value of y is an ellipse, centered on the y-axis, whose principal axes increase as $|y|$ increases.
- The slice at $x = 0$ is the hyperbola $z^2 - 3y^2 = 6$, and the slice at $z = 0$ is the hyperbola $2x^2 - 3y^2 = 6$. Thus the surface consists of ellipses, bounded by hyperbolas.

Sketch the simpler surface, then relabel the origin as $(1, -1, 3)$.

4. Find a parametric representation for the intersection of the surfaces

$$x^3 + y + z = 0, \quad x^2 y - z = 0,$$

having the property that y increases when x is positive.

Solution. There are no obvious constraints on any of the variables, in either equation. We are asked to relate the sign of x to the rate of change of y. This suggests we should let $x = t$, $t \in \mathbb{R}$. The two equations become

$$t^3 + y + z = 0 \ (1), \quad t^2 y - z = 0 \ (2).$$

Adding (1) and (2) yields

$$t^3 + y + t^2 y = 0,$$

which rearranges to

$$y = -\frac{t^3}{1+t^2}.$$

With this substitution in (2), we get

$$z = t^2 y = -\frac{t^5}{1+t^2}.$$

Thus a parametrization of the curve is

$$x = t, \quad y = -\frac{t^3}{1+t^2}, \quad z = -\frac{t^5}{1+t^2}, \quad t \in \mathbb{R}.$$

Now we check the constraint. We calculate

$$\frac{dy}{dt} = \frac{-3t^2(1+t^2) + t^3(2t)}{(1+t^2)^2} = -\frac{3t^2 + t^4}{(1+t^2)^2},$$

which is negative for all $t > 0$. Thus y is decreasing when x is positive. We must reverse direction. We substitute $t = -s$ in the parametrization above. The result is

$$x = -s, \quad y = \frac{s^3}{1+s^2}, \quad z = \frac{s^5}{1+s^2}, \quad s \in \mathbb{R}.$$

5. Let $\mathcal{C}$ be the curve with vector representation

$$\mathbf{r}(t) = \left(\frac{2}{\sqrt{3}}t^3 - \frac{1}{\sqrt{3}}\right)\hat{\mathbf{i}} + \left(\frac{3}{2}t^2\right)\hat{\mathbf{j}} + \left(2 - \frac{2}{3}t^3\right)\hat{\mathbf{k}}, \quad t \in \mathbb{R}.$$

(a) Find a unit tangent vector $\widehat{\mathbf{T}}$ to $\mathcal{C}$ at the point $(\frac{1}{\sqrt{3}}, \frac{3}{2}, \frac{4}{3})$, in the direction of decreasing t.

Solution. We set $\mathbf{r}(t) = (\frac{1}{\sqrt{3}}, \frac{3}{2}, \frac{4}{3})$, and find $t = 1$. Now,

$$\begin{aligned}
\mathbf{r}(t) &= \left(\frac{2}{\sqrt{3}}t^3 - \frac{1}{\sqrt{3}}, \frac{3}{2}t^2, 2 - \frac{2}{3}t^3\right), \\
\mathbf{r}'(t) &= \left(2\sqrt{3}t^2, 3t, -2t^2\right), \\
\mathbf{r}'(1) &= \left(2\sqrt{3}, 3, -2\right).
\end{aligned}$$

By construction, $\mathbf{r}'(1)$ points in the direction of increasing t. We therefore take $\mathbf{T} = -\mathbf{r}'(1) = (-2\sqrt{3}, -3, 2)$. The corresponding unit vector is

$$\widehat{\mathbf{T}} = \frac{1}{\sqrt{12+9+4}}(-2\sqrt{3}, -3, 2) = \frac{1}{5}(-2\sqrt{3}, -3, 2).$$

(b) Find the arc length of $\mathcal{C}$ between the points $(-\frac{1}{\sqrt{3}}, 0, 2)$ and $(\frac{1}{\sqrt{3}}, \frac{3}{2}, \frac{4}{3})$.

Solution. In part (a), we found that the point $(\frac{1}{\sqrt{3}}, \frac{3}{2}, \frac{4}{3})$ corresponds to $t = 1$. We set $\mathbf{r}(t) = (-\frac{1}{\sqrt{3}}, 0, 2)$, and find $t = 0$.

In part (a), we computed

$$\mathbf{r}'(t) = (2\sqrt{3}t^2, 3t, -2t^2).$$

From this, we get

$$|\mathbf{r}'(t)| = \sqrt{(2\sqrt{3}t^2)^2 + (3t)^2 + (-2t^2)^2} = \sqrt{12t^4 + 9t^2 + 4t^4} = \sqrt{16t^4 + 9t^2} = t\sqrt{16t^2 + 9}.$$

The arc length of the curve is

$$\begin{aligned}\int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 t\sqrt{16t^2+9}\, dt &= \left[\frac{1}{32} \cdot \frac{2}{3}(16t^2+9)^{3/2}\right]_{t=0}^1 \\ &= \frac{1}{48}\left[25^{3/2} - 9^{3/2}\right] = \frac{49}{24}.\end{aligned}$$