In Exercises 1-4 express the complex numbers in simplified Cartesian form.

1. $\left(\frac{3}{5}+\frac{4}{5}i\right)^{39}\left(\frac{3}{5}i+\frac{4}{5}\right)^{39}$

Solution: There are several ways to simplify the given expression. Here is one:

$$\left(\frac{3}{5}+\frac{4}{5}i\right)^{39}\left(\frac{3}{5}i+\frac{4}{5}\right)^{39}=\left(\frac{3}{5}+\frac{4}{5}i\right)^{39}\left[\left(\frac{3}{5}-\frac{4}{5}i\right)i\right]^{39}=\left[\left(\frac{3}{5}+\frac{4}{5}i\right)\left(\frac{3}{5}-\frac{4}{5}i\right)\right]^{39}i^{39}$$
$$=\left[\left(\frac{3}{5}\right)^2+\left(\frac{4}{5}\right)^2\right]^{39}\left(i^{36}\cdot i^3\right)=1^{39}\cdot\left(i^4\right)^9\cdot(-i)=-i$$

2. $\dfrac{\left(9e^{\frac{\pi i}{9}}\right)\left(8e^{-\frac{5\pi i}{18}}\right)}{\left(6e^{-\frac{\pi i}{3}}\right)^2}$

Solution: $$\frac{\left(9e^{\frac{\pi i}{9}}\right)\left(8e^{-\frac{5\pi i}{18}}\right)}{\left(6e^{-\frac{\pi i}{3}}\right)^2}=\frac{72e^{\frac{\pi i}{9}-\frac{5\pi i}{18}}}{36e^{-\frac{2\pi i}{3}}}=2e^{\frac{-\pi i}{6}+\frac{2\pi i}{3}}=2e^{\frac{\pi i}{2}}=2i$$

3. $\left[2\left(\cos\frac{3\pi}{7}\right)+2\left(\sin\frac{3\pi}{7}\right)i\right]^7$

Solution: $$=\left[2\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)\right]^7=2^7\left(\cos\frac{3\pi}{7}+i\sin\frac{3\pi}{7}\right)^7$$
$$=128\left(\cos 3\pi+i\sin 3\pi\right)=-128$$

4. $\left(i-\sqrt{3}\right)^{10}$

Solution: Since $\left|i-\sqrt{3}\right|=\sqrt{\left(-\sqrt{3}\right)^2+1^2}=2$, the exponential form of $i-\sqrt{3}$ is

$$-\sqrt{3}+i=2\left(-\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)=2e^{\frac{5\pi i}{6}}.$$

Therefore

$$\left(i-\sqrt{3}\right)^{10}=\left(2e^{\frac{5\pi i}{6}}\right)^{10}=2^{10}e^{\frac{25\pi i}{3}}=2^{10}\left(\cos\frac{25\pi}{3}+i\sin\frac{25\pi}{3}\right)=2^{10}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=512+512\sqrt{3}i$$

In Exercises 5-6 find all solutions of the equation.

5. $z^4+4z^2+16=0$

Solution: Set $y=z^2$. Then $y^2+4y+16=0$. Hence

$$y=\frac{-4\pm\sqrt{4^2-4\cdot 16}}{2}=\frac{-4\pm\sqrt{-48}}{2}=-2\pm 2\sqrt{3}i$$

If $y=-2+2\sqrt{3}i$, then

$$z^2=-2+2\sqrt{3}i=4\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=4e^{\frac{2\pi i}{3}}.$$

This leads to two solutions:

$$z=\pm 2e^{\frac{\pi i}{3}}=\pm 2\left(\cos\frac{\pi i}{3}+i\sin\frac{\pi i}{3}\right)=\pm 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\pm\left(1+i\sqrt{3}\right).$$

If $y = -2 - 2\sqrt{3}i$, then

$$z^2 = -2 - 2\sqrt{3}i = 4\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 4e^{-\frac{2\pi i}{3}}.$$

This leads to two more solutions:

$$z = \pm 2e^{\frac{-\pi i}{3}} = \pm 2\left(\cos\frac{-\pi i}{3} + i\sin\frac{-\pi i}{3}\right) = \pm 2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \pm\left(1 - i\sqrt{3}\right).$$

In total, there are four solutions: $z = \pm 1 \pm i\sqrt{3}$.

6. $z^6 - 7z^3 = 8$

Solution: The given equation is equivalent to $z^6 - 7z^3 - 8 = 0$, or

$$(z^3 - 8)(z^3 + 1) = 0.$$

Hence $z^3 = 8$ or $z^3 = -1$.

If $z^3 = 8$, then z can be any of the three cube roots of 8:

$$z = \left(8e^{2k\pi i}\right)^{1/3} = 2e^{\frac{2k\pi i}{3}} \qquad (k = 0, 1, 2).$$

This leads to three solutions: $z = 2$, $z = -1 + i\sqrt{3}$, and $z = -1 - i\sqrt{3}$.

If $z^3 = -1$, then z can be any of the three cube roots of -1 $(= e^{\pi i})$:

$$z = \left(e^{\pi i}e^{2k\pi i}\right)^{1/3} = e^{\frac{(1+2k)\pi i}{3}} \qquad (k = 0, 1, 2).$$

This leads to three more solutions: $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $z = -1$, and $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i$.

In total, there are six solutions.