MATH 1210 Assign 2 Solutions.

1. Let z be a complex number. Prove that for all $n \geq 1$, $(\overline{z})^n = \overline{z^n}$.

<u>Proof</u>. For all $n \geq 1$, let P_n denote the statement that for any complex number z, $(\overline{z})^n = \overline{z^n}$.

<u>Base Case</u>. The statement P_1 says $\overline{z} = \overline{z^1}$, which is true.

<u>Inductive Step</u>. Fix $k \geq 1$, and assume P_k holds, that is, for any complex number z, $(\overline{z})^k = \overline{z^k}$. It remains to show that P_{k+1} holds, that is, for any complex number z, $(\overline{z})^{k+1} = \overline{z^{k+1}}$.

Let z be a complex number. Then

$$\begin{aligned}
(\overline{z})^{k+1} &= \overline{z}^k \cdot \overline{z}^1 \\
&= \overline{z^k} \cdot \overline{z} \quad (\text{by } P_k) \\
&= \overline{z^k \cdot z} \quad (\text{since } \overline{z_1} \cdot \overline{z_2} = \overline{z_1 z_2} \text{ for any complex numbers } z_1, z_2) \\
&= \overline{z^{k+1}}.
\end{aligned}$$

Thus P_{k+1} holds.

Therefore, by PMI, for all $n \geq 1$, P_n holds.

<u>Note</u>: $\overline{z_1} \cdot \overline{z_2} = \overline{z_1 z_2}$

<u>Proof</u>: Let $z_1 = a+bi$, $z_2 = c+di$.

Then:

$$\overline{z_1} \cdot \overline{z_2} = \overline{(a+bi)} \cdot \overline{(c+di)}$$

$$= (a-bi)(c-di)$$

$$= ac - adi - bci + bdi^2$$

$$= (ac-bd) - (ad+bc)i$$

$$= \overline{(ac-bd) + (ad+bc)i}$$

$$= \overline{(a+bi)(c+di)}$$

$$= \overline{z_1 \cdot z_2}.$$

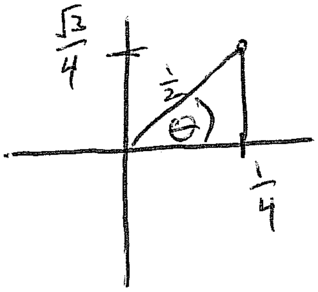
2.

$$z^2 - w = (-2+7i)^2 - (3+4i)$$
$$= (-2+7i)(-2+7i) \; -3-4i$$
$$= 4 - 14i - 14i + 49i^2 - 3 - 4i$$
$$= 1 - 49 - 28i - 4i$$
$$= -48 - 32i.$$

$$z + w = (-2+7i) + (3+4i)$$
$$= 1 + 11i$$

$$\frac{z^2 - w}{z+w} = \frac{-48-32i}{1+11i}\left(\frac{1-11i}{1-11i}\right)$$
$$= \frac{-48-32i+528i+352i^2}{1-121i^2}$$
$$= \frac{-400+496i}{122}$$
$$= -\frac{400}{122} + \frac{496}{122}i$$
$$= -\frac{200}{61} + \frac{248}{61}i$$

3. a) $\frac{1}{4}+\frac{\sqrt{3}}{4}i$



$$r=\sqrt{\left(\tfrac{1}{4}\right)^2+\left(\tfrac{\sqrt{3}}{4}\right)^2}$$
$$=\sqrt{\tfrac{1}{16}+\tfrac{3}{16}}$$
$$=\sqrt{\tfrac{4}{16}}=\tfrac{2}{4}=\tfrac{1}{2}$$

$$\sin\theta=\frac{\sqrt{3}}{4}\Big/\frac{1}{2}$$
$$=\frac{\sqrt{3}}{2}$$
$$\Rightarrow \theta=\frac{\pi}{3}$$

$$\therefore \frac{1}{4}+\frac{\sqrt{3}}{4}i=\frac{1}{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$
$$=\frac{1}{2}e^{\frac{\pi}{3}i}$$

Therefore $\left(\frac{1}{4}+\frac{\sqrt{3}}{4}i\right)^{14}=\left(\frac{1}{2}e^{\frac{\pi}{3}i}\right)^{14}$

$$=2^{-14}e^{\frac{14\pi}{3}i}$$
$$=\frac{1}{2^{14}}e^{\frac{2\pi}{3}i}$$
$$=\frac{1}{2^{14}}\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$$

$$=\frac{1}{2^{14}}\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)=\underline{\frac{-1}{2^{15}}+i\frac{\sqrt{3}}{2^{15}}}$$

$$\left(=\frac{-1}{32768}+\frac{i\sqrt{3}}{32768}\right)$$

3. (b)

$$\left(10e^{(-\pi/6)i}\right)^5 = 10^5 \cdot e^{-5\pi/6\, i}$$

Polar form

$$= 100{,}000\left(\cos\left(\frac{-5\pi}{6}\right) + i\sin\left(\frac{-5\pi}{6}\right)\right)$$

$$= 100{,}000\left(-\cos\left(\frac{\pi}{6}\right) + i\left(-\sin\frac{\pi}{6}\right)\right)$$

$$= 100{,}000\left(-\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right)$$

$$= -50000\sqrt{3} - 50000\,i.$$

Cartesian form.

3. c) $\overline{(5+4i)^2} + (1+4i^3)z = -4i$

$(\overline{5+4i})^2 + (1+4i^2 i)z = -4i$

$(5-4i)(5-4i) + (1-4i)z = -4i$

$(25-16)+(-40)i + (1-4i)z = -4i$

$9-40i + (1-4i)z = -4i$

$(1-4i)z = -4i - 9 + 40i$

$\frac{\cancel{(1-4i)}z}{\cancel{1-4i}} = \frac{-9+36i}{1-4i}\left(\frac{1+4i}{1+4i}\right)$

$z = \frac{(-9-144)+i(-36+36)}{1-16i^2}$

$= \frac{-153}{17} = \underline{\underline{-9.}}$

4. $-32 = 32e^{\pi i} = 32e^{(\pi + 2k\pi)i}$

$$z^5 = -32 \iff z = \left(32e^{(\pi+2k\pi)i}\right)^{1/5}$$

$$z = 32^{1/5} e^{\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)i}$$

$$= 2e^{\left(\frac{\pi}{5} + \frac{2}{5}k\pi\right)i}$$

$k=0$: $z_0 = 2e^{\pi/5 \, i} = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right) = 2\cos\frac{\pi}{5} + 2\sin\frac{\pi}{5}\, i$

$k=1$: $z_1 = 2e^{\frac{3\pi}{5}i} = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right) = 2\cos\frac{3\pi}{5} + 2\sin\frac{3\pi}{5}\, i$

$k=2$: $z_2 = 2e^{\frac{5\pi}{5}i} = 2e^{\pi i} = -2 = -2 + 0i$

$k=3$: $z_3 = 2e^{\frac{7\pi}{5}i} = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right) = 2\cos\frac{7\pi}{5} + 2\cdot\sin\frac{7\pi}{5}\, i$

$k=4$: $z_4 = 2e^{\frac{9\pi}{5}i} = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right) = 2\cos\frac{9\pi}{5} + 2\cdot\sin\frac{9\pi}{5}\, i$

5. $z^4 + 100z^2 + 10000 = 0$

Let $x = z^2$

$\Rightarrow x^2 + 100x + 10000 = 0$

$$X = \frac{-100 \pm \sqrt{100^2 - 4(1)(10000)}}{2(1)}$$

$$= -50 \pm \frac{\sqrt{(-3)10000}}{2}$$

$$= -50 \pm \frac{100\sqrt{3}}{2} i$$

$$= -50 \pm 50\sqrt{3}\, i$$

$$X_1 = -50 + 50\sqrt{3}\, i =$$

$$= 100\left(\frac{-1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$= 100\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

$$= 100e^{\frac{2\pi}{3}i}$$

$$X_2 = -50 - 50\sqrt{3}\, i$$

$$= 100e^{\frac{4\pi}{3}i}$$

$$z = \left(100e^{\left(\frac{2\pi}{3} + 2k\pi\right)i}\right)^{1/2}$$

$$= 10e^{\left(\frac{\pi}{3} + \frac{3k\pi}{3}\right)i}$$

$k=0$: $z_1 = 10e^{\pi/3\, i}$

$k=1$: $z_2 = 10e^{\frac{4\pi}{3}i}$

$$z = \left(100e^{\left(\frac{4\pi}{3} + 2k\pi\right)i}\right)^{1/2}$$

$$= 10e^{\left(\frac{2\pi}{3} + \frac{3k\pi}{3}\right)i}$$

$k=0$: $z_3 = 10e^{\frac{2\pi}{3}i}$

$k=1$: $z_4 = 10e^{\frac{5\pi}{3}i}$

6. (a)

$$
\begin{array}{r}
x^4 - 2x^3 - 15x^2 + 4x + 4 \\
x-3 \overline{\smash{)}\, x^5 - 5x^4 - 9x^3 + 49x^2 - 8x - 60} \\
-\ x^5 - 3x^4 \qquad\qquad\qquad\qquad\qquad\qquad \\
\hline
-2x^4 - 9x^3 \qquad\qquad\qquad\qquad\quad \\
-\ \ -2x^4 + 6x^3 \qquad\qquad\qquad\qquad\quad \\
\hline
-15x^3 + 49x^2 \qquad\qquad\quad \\
-\ \ -15x^3 + 45x^2 \qquad\qquad\quad \\
\hline
4x^2 - 8x \qquad\quad \\
-\ \ 4x^2 - 12x \qquad\quad \\
\hline
4x - 60 \\
-\ \ 4x - 12 \\
\hline
-48.
\end{array}
$$

$\therefore\ x^5 - 5x^4 - 9x^3 + 49x^2 - 8x - 60$

$= (x-3)(x^4 - 2x^3 - 15x^2 + 4x + 4) + -48$

6 (b) $F(x) = (2+i)x^4 + 3x^3 + (2-i)x + 1$

$ix - 2$ $\qquad \frac{2}{i}\left(\frac{i}{i}\right) = \frac{2i}{i^2} = -2i$

$F(2/i) = F(-2i)$

$= (2+i)(-2i)^4 + 3(-2i)^3 + (2-i)(-2i) + 1$

$= (2+i)(16i^4) + 3(-8i^3) - 4i + 2i^2 + 1$

$= 32\underset{1}{\cancel{i^4}} + 16\underset{i}{\cancel{i^5}} - 24\underset{(-i)}{\cancel{i^3}} - 4i + 2\underset{(-1)}{\cancel{i^2}} + 1$

$= 32 + 16i + 24i - 4i - 2 + 1$

$= \underline{31 + 36i}$

6.(c) Need $g(3/2)=0$

$$g(x)=6x^3-49x^2+dx+21$$

$$g\left(\tfrac{3}{2}\right)=6\cdot\left(\tfrac{3}{2}\right)^3-49\left(\tfrac{3}{2}\right)^2+d\left(\tfrac{3}{2}\right)+21=0$$

$$\frac{3^4}{2^2}-\frac{7^2 3^2}{2^2}+\frac{3}{2}d+21=0$$

$$\frac{1}{4}(81-441)+\frac{3}{2}d+21=0$$

$$-90+\frac{3}{2}d+21=0$$

$$\frac{3}{2}d-69=0$$

$$\frac{3}{2}d=69$$

$$\underline{d=46}.$$

6.(d) $F(x) = 3x^4 + hx^3 - 15x^2 - 18x + k$

$F(2) = 3 \cdot 2^4 + h \cdot 2^3 - 15(2)^2 - 18(2) + k = 0$

$48 + 8h - 60 - 36 + k = 0$

$8h + k - 48 = 0$

$k = 48 - 8h.$

$F(-1) = 3(-1)^4 + h(-1)^3 - 15(-1)^2 - 18(-1) + k = 0$

$3 - h - 15 + 18 + k = 0$

$-h + k + 6 = 0$

$-h + (48 - 8h) + 6 = 0$

$54 - 9h = 0$

$9h = 54$

$h = 6$

$k = 48 - 8(6) = 48 - 48 = 0$

$$\boxed{\begin{array}{l} k = 0 \\ h = 6 \end{array}}$$

7. (a) $p(1) = 0$. Thus,

$$2x^4 - 9x^3 + 33x^2 - 56x + 30$$
$$= (x-1)(2x^3 - 7x^2 + 26x - 30)$$

Let $p_2(x) = 2x^3 - 7x^2 + 26x - 30$. Then
$p_2(-x) = -2x^3 - 7x^2 - 26x - 30$ has no sign changes, and thus $p_2(x)$ has no negative zeros.

By rational root theorem, if $\frac{p}{q}$ is a rational root of $p_2(x) = 0$, then $p \mid 30$, $q \mid 2$

$$\Rightarrow \frac{p}{q} \in \left\{1, 2, 3, 5, 6, 10, 15, 30, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{15}{2}\right\}$$

$p_2\left(\frac{3}{2}\right) = 0 \Rightarrow$

$p_2(x) = (2x-3)(x^2 - 2x + 10)$. Let $p_3(x) = x^2 - 2x + 10$

$$p_3(x) = 0 \iff x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(10)}}{2(1)}$$
$$= 1 \pm \frac{1}{2}\sqrt{-9(4)}$$
$$= 1 \pm \frac{1}{2} 6i$$
$$= 1 \pm 3i.$$

$$\therefore p(x) = (x-1)(2x-3)(x-(1+3i))(x-(1-3i))$$

7. b) Since $p(2+i)=0$, $p(2-i)=0$ too.

$$\Rightarrow (x-2-i)(x-2+i) = x^2 \cancel{-2x} \cancel{+ix} -2x \cancel{-ix} +4 \cancel{-2i} \cancel{+2i} -i^2$$

$$= x^2-4x+5 \mid p(x).$$

$$p(x) = (x^2-4x+5)(x^2+10x+25)$$
$$= (x^2-4x+5)(x+5)^2$$

Thus $p(x) = (x-(2+i))(x-(2-i))(x+5)(x+5)$

The roots are

$2+i$ (w/ multiplicity 1)
$2-i$ (w/ multiplicity 1)
-5 (w/ multiplicity 2).

8. Since 2 is a zero w/ multiplicity 6 of $f(x)$,

$$f(x) = (x-2)^6 a(x) \quad (\text{where } a(2) \neq 0).$$

Since 2 is a zero of $g(x)$ w/ multiplicity 4,

$$g(x) = (x-2)^4 b(x) \quad (\text{where } b(2) \neq 0)$$

Thus,

$$h(x) = f(x)g(x) = (x-2)^6 a(x)(x-2)^4 b(x)$$
$$= (x-2)^{10} a(x) b(x).$$

Therefore, $h(2) = 0$, and since $a(2) \neq 0$, $b(2) \neq 0$, the multiplicity of 2 as a zero of $h(x)$ is 10.

9. a) $p(x) = 6x^5 + 2x^4 + 3x^3 - 11x^2 + 10x - 10.$

$p(1) = 0$

$\Rightarrow p(x) = (x-1)(6x^4 + 8x^3 + 11x^2 + 10)$

Let $q(x) = 6x^4 + 8x^3 + 11x^2 + 10$. By Descartes' rule of signs, since $q(x)$ has no sign changes, $q(x)$ has no positive real zeros.

Thus, the only positive (real) zero of $p(x)$ is $x = 1$, and so, no, there are no zeros of $p(x)$ in $(2,4)$.

9. (b) $q(x) = 5x^4 - 14x^3 + 5x - 9.$

Let x_0 be any real zero of $q(x)$. Then, by the bounds theorem, if

$$M = \max\{|-14|, |5|, |-9|\} = 14,$$

then

$$|x| < \frac{M}{5} + 1 = \frac{14}{5} + 1 < \frac{15}{5} + 1 = 4.$$

Thus, no, there can be no zeros of $q(x)$ in the interval $[4, 10]$.