Complete Answers

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U Manitoba-Laplace Transform Final Exam Questions 2001-2004

(note: $h(t)$ and $U(t)$ are heavyside Unit Step functions.)

Values

5 1. Find the Laplace transform for $f(t) = t^3e^{2t} - h(t-2\pi)\sin 3t$, where $h(t-2\pi)$ is the Heaviside unit step function.

Ans) $F(s) = \frac{6}{(s-2)^4} - e^{-2\pi s}\left(\frac{3}{s^2+9}\right)$

4 1. Find the Laplace transform for

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \\ 3, & t > 2. \end{cases}$$

Ans) $\frac{1}{s^2} + e^{-s}\left(-\frac{2}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

Values

6 1. Find the Laplace transform of the function

$$f(t) = \begin{cases} \sin t, & 0 < t < 2\pi \\ 0, & 2\pi < t < 4\pi \end{cases} \qquad f(t+4\pi) = f(t).$$

Simplify the transform as much as possible.

ans) $\frac{1}{\left(1+e^{-2\pi s}\right)\left(s^2+1\right)}$

7 2. Find the Laplace transform of the function defined as follows:

$$f(t) = 1-t, \quad 0 < t < 2 \qquad f(t+2) = f(t)$$

Ans) $\frac{1}{1-e^{-2s}}\left[\frac{1}{s} - \frac{1}{s^2} + e^{-2s}\left(\frac{1}{s} + \frac{1}{s^2}\right)\right]$

10 4. Find the inverse Laplace transform of

$$Y(s) = \frac{6s^2-3s+5}{(s-5)(s^2+3)(1+e^{-2s})}.$$

Ans) $\sum_{n=0}^{\infty}(-1)^n\left[5e^{5(t-2n)} + \cos\sqrt{3}(t-2n) + \frac{2}{\sqrt{3}}\sin\sqrt{3}(t-2n)\right]U(t-2n)$

8 3. Find the inverse Laplace transform for $F(s) = \frac{e^{-2s}}{s^3+4s}$.

Ans) $\frac{1}{4}\left[U(t-2) - \cos 2(t-2)U(t-2)\right]$

8 2. Find the inverse Laplace transform for

$$F(s) = \frac{3s+4}{(s+1)(s^2+2s+2)}$$

ans) $e^{-t} + e^{-t}\left(-\cos t + 3\sin t\right)$

8 3. Solve the following initial value problem using Laplace transforms

$$\frac{d^2y}{dt^2} + 4y = 3 + 2\delta(t-3), \quad y(0) = 1, \quad y'(0) = 0.$$

Ans) $\frac{3}{4} + \frac{1}{4}\cos 2t + \sin 2(t-3)\,U(t-3)$

10 3. (a) Show that the Laplace transform of the function $y(t)$ satisfying the initial-value problem

$$y'' + 4y' + 8y = 3\delta(t-2), \quad y(0) = 1, \quad y'(0) = -1,$$

is

$$Y(s) = \frac{s+3+3e^{-2s}}{s^2+4s+8}.$$

(b) Find $y(t)$.

Ans) $y(t) = \left[\frac{3}{2}e^{-2(t-2)}\sin 2(t-2)\right]U(t-2) + e^{-2t}\left[\cos 2t + \frac{1}{2}\sin 2t\right]$

14 2. Solve the following initial value problem using Laplace transforms

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t\,h(t-1), \quad y(0) = 0, \quad y'(0) = 2.$$

Ans) $Y(s) = \frac{2}{(s+1)^2} + e^{-s}\left(-\frac{1}{s} + \frac{1}{s^2} + \frac{1}{s+1}\right)$

$y(t) = 2te^{-t} + \left[(t-2) + e^{1-t}\right]U(t-1).$

$= 2te^{-t} + \left[-1 + (t-1) + e^{-(t-1)}\right]h(t-1)$

1. Let $f(t) = \begin{cases} 0 & 0 \le t < \pi/4; \\ e^{(t-\frac{\pi}{4})}\cos(t-\frac{\pi}{4}), & t \ge \pi/4. \end{cases}$ (7 pts)

(a) Find the compact form (representation in terms of Heaviside (unit step) function(s)) of f.

ans) $e^{(t-\pi/4)}\cos(t-\pi/4)\,h(t-\pi/4)$

(b) Find the Laplace transform of f.

ans) $e^{-\pi/4 s}\dfrac{(s-1)}{(s-1)^2+1}$

10] 1 Find $\mathcal{L}\{f(t)+g(t)\}$ where $f(t)=\begin{cases} t-1, & 0\le t<1 \\ t^2, & t\ge 1 \end{cases}$

and $g(t)=\left(1+e^{2t}\right)^2$.

ans) $e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}+\frac{2}{s^3}\right)+\frac{1}{s^2}+\frac{2}{s-2}+\frac{1}{s-4}$

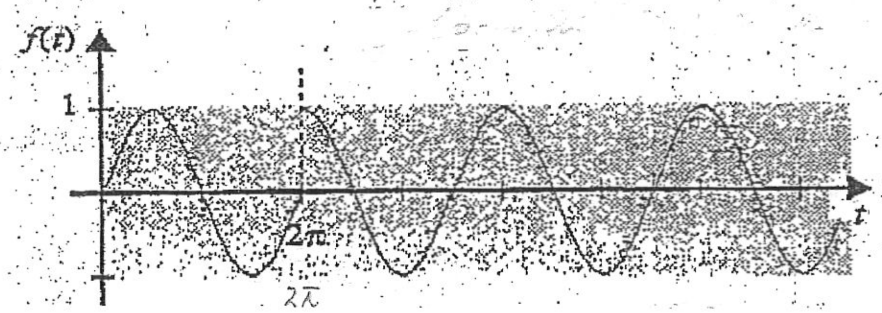
1. (9 marks) Using the Unit Step function, find the Laplace transform of $f(t) = \begin{cases} t & 0 \le t < \pi \\ \sin(2t) & t \ge \pi. \end{cases}$

ans) $\frac{1}{s^2} + e^{-\pi s}\left(\frac{2}{s^2+4} - \frac{1}{s^2} - \frac{\pi}{s}\right)$

1. (8 Marks) Given the piecewise-defined function,

$$f(t) = \begin{cases} \sin t, & 0 \le t < 2\pi \\ \cos t, & t \ge 2\pi \end{cases}$$

as illustrated below:



a) Write $f(t)$ in terms of the unit step function.

b) Find the Laplace transform of $f(t)$.

ans) a) $[1 - h(t-2\pi)]\sin t + \cos t \, h(t-2\pi)$

b) $\frac{1}{s^2+1}\left[1 + (s-1)e^{-2\pi s}\right]$

4 1. Find the Laplace transform for

t

$$f(t) = \begin{cases} t, & 0 < t < 1 \\ 2-t, & 1 < t < 2 \\ 3, & t > 2. \end{cases}$$

t

ans) $\frac{1}{s^2} - \frac{2}{s^2}e^{-s} + e^{-2s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

3. (12 marks) Find the inverse Laplace transform of $F(s) = \frac{11}{(s^2+2s+3)(s-2)}$.

ans) $e^{2t} - e^{-t}\left(\cos\sqrt{2}t + \frac{3}{\sqrt{2}}\sin\sqrt{2}t\right)$

6 1. Find the inverse Laplace transform for

$$F(s) = \frac{2s}{s^2+2s+7}$$

ans) $2e^{-t}\left(\cos\sqrt{6}t - \frac{1}{\sqrt{6}}\sin\sqrt{6}t\right)$

5. (12 Marks) Find the inverse Laplace transform of $F(s) = \dfrac{11}{(s+1)(s^2-4s+6)}$

ans) $e^{-t} + e^{2t}\left(\frac{3}{\sqrt{2}} \sin\sqrt{2}t - \cos\sqrt{2}t\right)$

8 **2.** Find the inverse Laplace transform for

$$F(s) = \frac{3s+4}{(s+1)(s^2+2s+2)}.$$

ans) $e^{-t}(1 - \cos t + 3\sin t)$

8 **3.** Find the inverse Laplace transform for $F(s) = \dfrac{e^{-2s}}{s^3+4s}$.

ans) $\frac{1}{4}\left[1 - \cos 2(t-2)\right] h(t-2)$

6. (14 points) Solve the initial value problem:

$$\begin{cases} y'' - 2y' + y = t^3 e^t \\ y(0) = 0,\ y'(0) = 1. \end{cases}$$

ans) $\frac{te^t}{20}(t^4 + 20)$

5. (14 points) Solve the initial value problem:

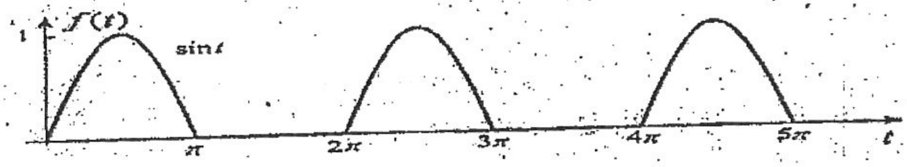
$$\begin{cases} y'' + y = f(t) \\ y(0) = 0, \; y'(0) = 1, \end{cases}$$

where

$$f(t) = \begin{cases} \sin t, & \text{if } 0 \le t < \frac{\pi}{2} \\ \cos 2t, & \text{if } \frac{\pi}{2} \le t. \end{cases}$$

ans) $\frac{3}{2}\sin t - \frac{1}{2}t\cos t + \left[\frac{1}{2}(t-\frac{\pi}{2}) - \frac{1}{3}(\sin t + \cos 2t)\right] h(t-\frac{\pi}{2})$

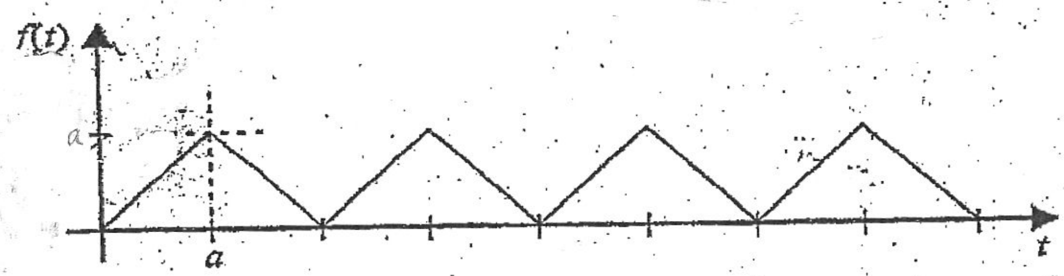
7 1. Find the Laplace transform for the periodic function shown below.



ans) $\frac{1}{(1-e^{-\pi s})(s^2+1)}$

find the Laplace with period

2. (12 Marks[redacted]ace transform of the periodic function, $f(t)$[redacted]iod $2a$:



ans) $\frac{1-e^{-as}}{s^2(1+e^{-as})}$

9 2. Solve the following initial-value problem using Laplace transforms

$$\frac{d^2y}{dt^2} + 4y = 3 + 2\delta(t-3), \quad y(0) = 1, \quad y'(0) = 0.$$

ans) $\frac{3}{4} + \frac{1}{4}\cos 2t + \sin 2(t-3)\, h(t-3)$

10 3. Solve the following initial-value problem, simplifying your answer as much as possible

$$\frac{d^2y}{dt^2} + 9y = 4\delta(t-2\pi), \qquad y(0) = 1, \quad y'(0) = -2$$

ans) $\frac{4}{3}\sin 3t\, h(t-2\pi) + \cos 3t - \frac{2}{3}\sin 3t$

or

$$\begin{cases} \cos 3t - \frac{2}{3}\sin 3t, & 0 \le t \le 2\pi \\ \cos 3t + \frac{2}{3}\sin 3t & t \ge 2\pi \end{cases}$$

4. (10 marks) A cantilever beam of length L is supported at both ends. A large thin mass is placed in the center of the beam resulting in an impulse $w(x) = 5\delta(x - L/2)$ at $x = L/2$. The deflection of the beam due to the load can be found by solving the fourth order boundary-value problem:

$$EI\frac{d^4y}{dx^4} = w(x), \quad y(0) = y''(0) = y(L) = y''(L) = 0.$$

Given that $EI = 1$, and letting $A = y'(0)$ and $B = y'''(0)$, solve the differential equation.

<u>DO NOT</u> calculate the values of A and B.

ans) $\frac{1}{EI}\left[A x + \frac{B}{6}x^3 + \frac{5}{6}(x - L/2)^3 h(x - L/2) \right]$

7. (14 points) Use the Laplace transform to solve the integral equation:

$$f(t) + \int_0^t f(\tau)\, d\tau = 2t.$$

ans) $2(1 - e^{-t})$