EXERCISES FOR CHAPTER 5: Power Series

Find the interval of convergence of the power series below. For each state the radius of convergence.

1. (a) $\sum_{n=0}^{\infty} x^n$ (b) $\sum_{n=0}^{\infty} (-1)^{n+1} x^n$

Solution

(a) $\rho = \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{n+1}}{(-1)^{n+1} x^n} \right| = |x|, \quad -1 < x < 1 \Rightarrow \textit{converges}$. At the end points, $x = 1$ the series is $\sum_{n=0}^{\infty} (-1)^{n+1}$ and diverges by the divergence test and at $x = -1$ the series is $\sum_{n=0}^{\infty} (-1)^n$ and diverges by the divergence test. Thus $R = 1$.

(b) $\rho = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = |x|, \quad -1 < x < 1 \Rightarrow \textit{converges}$. At the end points, $x = 1$ the series is $\sum_{n=0}^{\infty} (-1)^{n+1}$ and diverges by the divergence test and at $x = -1$ the series is $\sum_{n=0}^{\infty} (-1)^n$ and diverges by the divergence test. Thus $R = 1$.

2. (a) $\sum_{n=0}^{\infty} \frac{x^n}{n+2}$ (b) $\sum_{n=0}^{\infty} \frac{(-x)^n}{n+1}$

Solution

(a) $\rho = \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+3}}{\frac{x^n}{n+2}} \right| = |x| \lim_{n\to\infty} \left| \frac{n+2}{n+3} \right| = |x|, \quad -1 < x < 1 \Rightarrow \textit{converges}$. At the end points, $x = 1$ the series is $\sum_{n=0}^{\infty} \frac{1}{n+2}$ and diverges by the integral test and at $x = -1$ the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ and converges by the alternating series test. Thus $R = 1$.

(b) $\rho = \lim_{n\to\infty} \left| \frac{\frac{(-x)^{n+1}}{n+2}}{\frac{(-x)^n}{n+1}} \right| = |x| \lim_{n\to\infty} \left| \frac{n+1}{n+2} \right| = |x|, \quad -1 < x < 1 \Rightarrow \textit{converges}$. At the end points, $x = 1$ the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ and converges by the alternating series test and at $x = -1$ the series is $\sum_{n=0}^{\infty} \frac{1}{n+1}$ and diverges by the integral test. Thus $R = 1$.

3. (a) $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$ (b) $\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{x^{2n+2}}{(2n+2)!}}{(-1)^n\dfrac{x^{2n}}{(2n)!}}\right| = x^2\lim_{n\to\infty}\frac{(2n)!}{(2n+2)!} = x^2\lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0$$ for all x and so series converges for all x. Thus $R=\infty$.

(b)

$$\rho = \lim_{n\to\infty}\left|\frac{\dfrac{x^{2n+3}}{(2n+3)!}}{\dfrac{x^{2n+1}}{(2n+1)!}}\right| = x^2\lim_{n\to\infty}\frac{(2n+1)!}{(2n+3)!} = x^2\lim_{n\to\infty}\frac{1}{(2n+3)(2n+2)} = 0$$ for all x and so series converges for all x. Thus $R=\infty$.

4. $\frac{x}{1\times 2}-\frac{x^2}{2\times 3}+\frac{x^3}{3\times 4}-\cdots$ (b) $1+2x+3x^2+4x^3+\cdots$

Solution

(a) The general term is $(-1)^{n+1}\frac{x^n}{n(n+1)}$. Hence:

$$\rho = \lim_{n\to\infty}\left|\frac{\dfrac{x^{n+1}}{(n+1)(n+2)}}{\dfrac{x^n}{n(n+1)}}\right| = |x|\lim_{n\to\infty}\frac{n(n+1)}{(n+1)(n+2)} = |x|$$. The series thus converges for $-1<x<1$. At $x=-1$, the series becomes $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(-1)^n}{n(n+1)} = -\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ that converges (apply for example the direct comparison test) and at $x=1$ it is $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n(n+1)}$ that converges absolutely. Hence the series converges for $-1\le x\le 1$. Thus $R=1$.

(b) The general term is nx^{n-1}. Hence, $\rho = \lim_{n\to\infty}\left|\frac{(n+1)x^n}{nx^{n-1}}\right| = |x|\lim_{n\to\infty}\frac{(n+1)}{n} = |x|$. The series converges for $-1<x<1$. At $x=-1$ the series becomes $\sum_{n=1}^{\infty}n(-1)^n$ that diverges by the divergence test and at $x=1$ it is $\sum_{n=1}^{\infty}n$ that diverges by the divergence test. Hence series converges for $-1<x<1$. Thus $R=1$.

5. (a) $\sum_{n=1}^{\infty} 5^n \frac{(x-2)^n}{n}$ (b) $\sum_{n=1}^{\infty} 2^n \frac{(x+3)^n}{n+1}$

Solution

(a) $\rho = \lim_{n\to\infty}\left|\frac{\frac{5^{n+1}(x-2)^{n+1}}{n+1}}{\frac{5^n(x-2)^n}{n}}\right| = 5|x-2|\lim_{n\to\infty}\frac{n}{n+1} = 5|x-2|$. The series converges for $-1 < 5(x-2) < 1$, $-\frac{1}{5} < x-2 < \frac{1}{5}$, $\frac{9}{5} < x < \frac{11}{5}$. At the endpoints we have: at $x = \frac{9}{5}$ the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ that converges by the alternating series test. At $x = \frac{11}{5}$ the series is $\sum_{n=1}^{\infty}\frac{1}{n}$ that diverges being the harmonic series.

The series thus converges for $\frac{9}{5} \le x < \frac{11}{5}$ and $R = \frac{1}{5}$.

(b) $\rho = \lim_{n\to\infty}\left|\frac{\frac{2^{n+1}(x+3)^{n+1}}{n+2}}{\frac{2^n(x+3)^n}{n+1}}\right| = 2|x+3|\lim_{n\to\infty}\frac{n+1}{n+2} = 2|x+3|$. The series thus converges for $-1 < 2(x+3) < 1$, $-\frac{1}{2} < x+3 < \frac{1}{2}$, $-\frac{7}{2} < x < -\frac{5}{2}$. At the endpoints we have: at $x = -\frac{7}{2}$ the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n+1}$ that converges by the alternating series test. At $x = -\frac{5}{2}$ the series is $\sum_{n=1}^{\infty}\frac{1}{n+1}$ that diverges by the integral test.

The series thus converges for $-\frac{7}{2} \le x < -\frac{5}{2}$ and $R = \frac{1}{2}$.

6. (a) $\sum_{n=1}^{\infty}\frac{(x-1)^n}{n}$ (b) $\sum_{n=1}^{\infty}\frac{(x-2)^n}{n^2}$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(x-1)^{n+1}}{n+1}}{\frac{(x-1)^n}{n}}\right| = |x-1|\lim_{n\to\infty}\frac{n}{n+1} = |x-1|, \quad -1 < x-1 < 1,$$

$0 < x < 2 \Rightarrow \textit{converges}$

For $x=0$ series becomes $\sum_1^\infty \frac{(-1)^n}{n} \Rightarrow$ *converges* by the alternating series test.
For $x=2$ series becomes $\sum_1^\infty \frac{1}{n} \Rightarrow$ *diverges* being a p=1 series.

Series converges for $0 \le x < 2$. Thus $R=1$.
(b)
$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}}\right| = |x-2|\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^2 = |x-2|, \quad -1 < x-2 < 1,$$
$1 < x < 3 \Rightarrow$ *converges*

For $x=1$ series becomes $\sum_1^\infty \frac{(-1)^n}{n^2}$ that converges absolutely.
For $x=3$ series becomes $\sum_1^\infty \frac{1}{n^2}$ that converges being a p =2 series.

Series converges for $1 \le x \le 3$. Thus $R=1$.

7. (a) $\sum_1^\infty \frac{x^n}{n}$ (b) $\sum_{n=1}^\infty \frac{x^n}{n+3}$

Solution
(a)
$\rho = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}}\right| = |x|\lim_{n\to\infty}\frac{n}{n+1} = |x|, \quad -1 < x < 1 \Rightarrow$ *converges*. At the endpoints the series becomes $\sum_1^\infty \frac{(-1)^n}{n}$ that converges by the alternating series test and $\sum_1^\infty \frac{1}{n}$ that is the harmonic series and diverges. Hence the series converges for $-1 \le x < 1$. Thus $R=1$.

(b)
$\rho = \lim_{n\to\infty}\left|\frac{\frac{x^{n+1}}{n+4}}{\frac{x^n}{n+3}}\right| = |x|\lim_{n\to\infty}\frac{n+3}{n+4} = |x|, \quad -1 < x < 1 \Rightarrow$ *converges*. At the endpoints the series becomes $\sum_1^\infty \frac{(-1)^n}{n+3}$ that converges by the alternating series test and $\sum_1^\infty \frac{1}{n+3}$ that diverges by the integral test. Hence the series converges for $-1 \le x < 1$. Thus $R=1$.

8. (a) $\sum_{n=2}^\infty \frac{\ln n}{n} x^n$ (b) $\sum_{n=2}^\infty \frac{\ln n}{n^2} x^n$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{\dfrac{\ln(n+1)\,x^{n+1}}{n+1}}{\dfrac{\ln n\;x^{n}}{n}}\right| = |x|\lim_{n\to\infty}\frac{n}{n+1}\frac{\ln(n+1)}{\ln n} = |x|,\quad -1<x<1 \Rightarrow \textit{converges}$$. At the endpoints, $x=-1$ the series is $\sum_{n=2}^{\infty}\frac{\ln n}{n}(-1)^n$ and converges by the alternating series test and at $x=1$ the series is $\sum_{n=2}^{\infty}\frac{\ln n}{n}$ and diverges by the direct comparison test with $\sum_{n=2}^{\infty}\frac{1}{n}$. Hence series converges for $-1\le x<1$. Thus $R=1$.

(b)

$$\rho = \lim_{n\to\infty}\left|\frac{\dfrac{\ln(n+1)\,x^{n+1}}{(n+1)^2}}{\dfrac{\ln n\;x^{n}}{n^2}}\right| = |x|\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\frac{\ln(n+1)}{\ln n} = |x|,\quad -1<x<1 \Rightarrow \textit{converges}$$. At the endpoints, $x=-1$ the series is $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}(-1)^n$ and converges by the alternating series test and at $x=1$ the series is $\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ and converges by the integral test. Hence series converges for $-1\le x\le 1$. Thus $R=1$.

9. (a) $\sum_{1}^{\infty}\left(\frac{x}{4}\right)^n$ (b) $\sum_{1}^{\infty}(-1)^{n+1}\frac{n!}{n^2}x^n$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{\left(\dfrac{x}{4}\right)^{n+1}}{\left(\dfrac{x}{4}\right)^{n}}\right| = \left|\frac{x}{4}\right|,\quad -4<x<4 \Rightarrow \textit{converges}$$. At the endpoints the series is $\sum_{1}^{\infty}(-1)^n$ and $\sum_{1}^{\infty}1$, both diverge. Hence series converges for $-4<x<4$. Thus $R=4$.

(b)

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+2}\dfrac{(n+1)!x^{n+1}}{(n+1)^2}}{(-1)^{n+1}\dfrac{n!x^{n}}{n^2}}\right| = |x|\lim_{n\to\infty}\frac{n^2}{(n+1)^2}\frac{(n+1)!}{n!} = |x|\lim_{n\to\infty}(n+1)=\infty \text{ unless } x=0 .$$

Thus $R=0$.

10. (a) $\sum_{n=1}^{\infty}\frac{(-1)^n n}{2^{2n}}(x-2)^n$ (b) $\sum_{1}^{\infty}\frac{(2x-1)^n}{n}$

Solution

(a)

$$\rho=\lim_{n\to\infty}\left|\frac{(n+1)\dfrac{(x-2)^{n+1}}{2^{2n+2}}}{n\dfrac{(x-2)^n}{2^{2n}}}\right|=\frac{|x-2|}{4}\lim_{n\to\infty}\frac{n}{n+1}=\frac{|x-2|}{4},\quad -4<x-2<4,$$

$-2<x<6 \Rightarrow \textit{converges}$

For $x=-2$ series becomes $\sum_{1}^{\infty}\frac{n4^n}{2^{2n}}=\sum_{1}^{\infty}n \Rightarrow \textit{diverges}$ by the divergence test.

For $x=6$ series becomes $\sum_{1}^{\infty}\frac{(-1)^n n4^n}{2^{2n}}=\sum_{1}^{\infty}(-1)^n n \Rightarrow \textit{diverges}$ by the alternating series test.

Series converges for $-2<x<6$. Thus $R=4$.

(b)

$$\rho=\lim_{n\to\infty}\left|\frac{\dfrac{(2x-1)^{n+1}}{n+1}}{\dfrac{(2x-1)^n}{n}}\right|=|2x-1|\lim_{n\to\infty}\frac{n}{n+1}=|2x-1|,\quad -1<2x-1<1,$$

$0<x<1 \Rightarrow \textit{converges}$

For $x=0$ series becomes $\sum_{1}^{\infty}\frac{(-1)^n}{n} \Rightarrow \textit{converges}$ by the alternating series test.

For $x=1$ series becomes $\sum_{1}^{\infty}\frac{1}{n} \Rightarrow \textit{diverges}$ being a *p* series with *p*=1.

Series converges for $0\le x<1$. Thus $R=\frac{1}{2}$.

11. (a) $\sum_{1}^{\infty}2^n x^n$ (b) $\sum_{1}^{\infty}\frac{x^n}{2^n}$

Solution

(a) $\rho=\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}}{2^n x^n}\right|=2|x|,\quad -1<2x<1,\quad -\frac{1}{2}<x<\frac{1}{2}\Rightarrow \textit{converges}$.

For $x=\pm\frac{1}{2}$ series becomes $\sum_{1}^{\infty}1 \Rightarrow \textit{diverges}$ by divergence test and

$\sum_{1}^{\infty}(-1)^n \Rightarrow \textit{diverges}$ by divergence test.

Series converges for $-\frac{1}{2}<x<\frac{1}{2}$. Thus $R=\frac{1}{2}$.

(b) $\rho = \lim_{n\to\infty}\left|\dfrac{\dfrac{x^{n+1}}{2^{n+1}}}{\dfrac{x^n}{2^n}}\right| = \dfrac{|x|}{2}, \quad -1 < \dfrac{x}{2} < 1, \quad -2 < x < 2 \Rightarrow \textit{converges}$.

For $x = \pm 2$ series becomes $\sum_1^\infty 1 \Rightarrow$ *diverges* by divergence test and $\sum_1^\infty (-1)^n \Rightarrow$ *diverges* by divergence test.

Series converges for $-2 < x < 2$. Thus $R = 2$.

12. (a) $\sum_1^\infty n^2(x-2)^n$ (b) $\sum_1^\infty n^n x^n$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{(n+1)^2(x-2)^{n+2}}{n^2(x-2)^n}\right| = |x-2|\lim_{n\to\infty}\frac{(n+1)^2}{n^2} = |x-2|, \quad -1 < x-2 < 1$$

$1 < x < 3 \Rightarrow \textit{converges}$

At the endpoints we have: $x = 1$ and the series becomes $\sum_1^\infty n^2(-1)^n$ that diverges by the divergence test. At $x = 3$ and the series becomes $\sum_1^\infty n^2$ that diverges by the divergence test. Hence the series converges for $1 < x < 3$. Thus $R = 1$.

(b)

$\rho = \lim_{n\to\infty}\left|\dfrac{(n+1)^{n+1}x^{n+1}}{n^n x^n}\right| = |x|\lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^{n+1} n = |x|\lim_{n\to\infty}\left(1+\dfrac{1}{n}\right)^{n+1} n = |x|e\lim_{n\to\infty} n = \infty$ unless $x = 0$. Thus $R = 0$.

13. (a) $\sum_1^\infty \dfrac{n!x^n}{(2n)!}$ (b) $\sum_1^\infty \dfrac{n^4 x^n}{n!}$

Solution

(a)

$\rho = \lim_{n\to\infty}\left|\dfrac{\dfrac{(n+1)!x^{n+1}}{(2n+2)!}}{\dfrac{n!x^n}{(2n)!}}\right| = |x|\lim_{n\to\infty}\dfrac{n+1}{(2n+2)(2n+1)} = 0$ for all values of x. Thus $R = \infty$.

(b)

$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(n+1)^4 x^{n+1}}{(n+1)!}}{\frac{n^4 x^n}{n!}}\right| = |x| \lim_{n\to\infty}\frac{1}{(n+1)}\left(\frac{n+1}{n}\right)^4 = 0 \text{ for all values of } x\text{. Thus } R = \infty .$$

14. (a) $\sum_1^\infty (-1)^{n+1}\frac{(2x-1)^n}{n+1}$ (b) $\sum_1^\infty (2^n+3^n)x^n$

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{\frac{(2x-1)^{n+1}}{n+1}}{\frac{(2x-1)^n}{n}}\right| = |2x-1|, \quad -1 < 2x-1 < 1, \quad 0 < x < 1 \Rightarrow \textit{converges}$$

For $x=0$ series becomes $-\sum_1^\infty \frac{1}{n+1} \Rightarrow$ *diverges* by the integral test.

For $x=1$ series becomes $\sum_1^\infty \frac{(-1)^{n+1}}{n+1} \Rightarrow$ *converges* by the alternating series test.

Series converges for $0 < x \le 1$. Thus $R = \frac{1}{2}$.

(b)

$$\rho = \lim_{n\to\infty}\left|\frac{(2^{n+1}+3^{n+1})x^{n+1}}{(2^n+3^n)x^n}\right| = |x| \lim_{n\to\infty}\left|\frac{(\left(\frac{2}{3}\right)^{n+1}+1)}{(\frac{1}{3}\left(\frac{2}{3}\right)^n + \frac{1}{3}\left(\frac{3}{3}\right)^n)}\right| = 3|x|$$

$$-1 < 3x < 1, \quad -\frac{1}{3} < x < \frac{1}{3} \Rightarrow \textit{converges}$$

For $x = \pm\frac{1}{3}$ series becomes $\sum_1^\infty (\left(\frac{2}{3}\right)^n + 1)$ and $\sum_1^\infty (\left(\frac{2}{3}\right)^n + 1)(-1)^n$ and both diverge by divergence test and alternating series test.

Series converges for $-\frac{1}{3} < x < \frac{1}{3}$. Thus $R = \frac{1}{3}$.

15. The Bessel function $J_0(x)$ may be defined by the power series $\sum_{k=0}^\infty (-1)^k \frac{x^{2k}}{2^{2k}(k!)^2}$. (a) Find the radius of convergence of this series. (b) The Bessel function $J_1(x)$ may be defined through $J_1(x) = -\frac{d}{dx}J_0(x)$. Find the power series for $J_1(x)$.

Solution

(a)

$$\rho = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\dfrac{x^{2n+2}}{2^{2n+2}((n+1)!)^2}}{(-1)^n\dfrac{x^{2n}}{2^{2n}(n!)^2}}\right| = \frac{x^2}{4}\lim_{n\to\infty}\frac{(n!)^2}{((n+1)!)^2} = \frac{x^2}{4}\lim_{n\to\infty}\frac{1}{(n+1)^2} = 0$$ for all x. Hence

$R = \infty$.

(b)

$$\begin{aligned}
J_1(x) &= -\frac{d}{dx}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}(k!)^2}\\
&= -\frac{d}{dx}\sum_{k=1}^{\infty}(-1)^k\frac{x^{2k}}{2^{2k}(k!)^2}\\
&= -\sum_{k=0}^{\infty}(-1)^k\frac{(2k)x^{2k-1}}{2^{2k}(k!)^2}\\
&= \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-1}}{2^{2k-1}(k-1)!k!}\\
&= \sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{2^{2k+1}(k+1)!k!}
\end{aligned}$$

16. Find the sum of the power series $\sum_{n=1}^{\infty} nx^{2n-1}$ by using another power series of known sum.

Solution

Since, $\sum_{n=0}^{\infty} x^{2n} = 1 + x^2 + x^4 + \ldots = \frac{1}{1-x^2}$ (for $-1 < x < 1$) we have by differentiation that

$$\begin{aligned}
&\frac{d}{dx}\sum_{n=0}^{\infty}x^{2n} = \sum_{n=0}^{\infty}2nx^{2n-1} = \sum_{n=1}^{\infty}2nx^{2n-1} = \frac{d}{dx}\frac{1}{1-x^2} = \frac{2x}{(1-x^2)^2}\\
&\Rightarrow \sum_{n=1}^{\infty}nx^{2n-1} = \frac{x}{(1-x^2)^2}
\end{aligned}$$

17. Consider the differential equation $\frac{dy}{dx} = xy$ with initial condition $y = 1$ when $x = 0$. (a) This is a separable differential equation. Solve this equation. (b) Now try to solve the equation again by assuming that the solution can be written as a power series, $y = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n$. Calculate $\frac{dy}{dx}$ and then xy and equate the coefficients of x^n on both sides to find the value of a_n. Hence find the solution of the differential equation as a power series. (c) Find the interval of convergence of the power series you got in (b).

Solution

(a) $\frac{dy}{dx} = xy \Rightarrow \frac{dy}{y} = xdx \Rightarrow \ln y = \frac{x^2}{2} + \ln C \Rightarrow y = Ce^{\frac{x^2}{2}}$. The initial condition means that $C = 1$ and so $y = e^{\frac{x^2}{2}}$.

(b)

$$\begin{aligned} &y = a_0 + a_1 x + a_2 x^2 + \cdots \\ &\Rightarrow yx = a_0 x + a_1 x^2 + a_2 x^3 + \cdots \\ &\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \end{aligned}$$

and so

$$a_1 + 2a_2 x + 3a_3 x^2 + \cdots = a_0 x + a_1 x^2 + a_2 x^3 + \cdots$$

Now, the initial condition dictates that $a_0 = 1$. Since there is no constant term on the right side we must have $a_1 = 0$. Matching coefficients of equal powers of *x* we have that

$$\begin{aligned} &2a_2 = a_0 = 1 \Rightarrow a_2 = \frac{1}{2} \\ &3a_3 = a_1 = 0 \Rightarrow a_3 = 0 \\ &4a_4 = a_2 = \frac{1}{2} \Rightarrow a_4 = \frac{1}{2} \times \frac{1}{4} \end{aligned}$$

and in general

$$\begin{aligned} &a_{2n+1} = 0 \\ &a_{2n} = \frac{1}{2n} \times \frac{1}{2n-2} \times \frac{1}{2n-4} \times \cdots \times \frac{1}{2} = \frac{1}{2^n n!} \end{aligned}$$

so that the solution is $y = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$. We may then deduce that $e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!}$.

(c) Applying the power series ratio test gives

$$\rho = \lim_{n\to\infty} \left| \frac{\frac{x^{2n+2}}{2^{n+1}(n+1)!}}{\frac{x^{2n}}{2^n n!}} \right| = \frac{1}{2}\left|x^2\right| \lim_{n\to\infty} \left| \frac{n!}{(n+1)!} \right| = \frac{1}{2}\left|x^2\right| \lim_{n\to\infty} \frac{1}{n+1} = 0$$, for all values of *x*.