Draft

Review Notes on Sequences (by Dawit)

* Sequences : (representation) $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or $a_1, a_2, \ldots, a_n, \ldots$

always starts @1.

* $\lim_{n\to\infty} a_n = \begin{cases} L \Rightarrow \text{Sequence converges to } L \\ \pm\infty \Rightarrow \text{〃 diverges to } \pm\infty \\ DNE \Rightarrow \text{〃 〃} \end{cases}$

>> Common Sequences (: define a_n as the n^{th}-term or general-term of the Sequence)

1. Arithmetic sequence : (Common difference d) $\Rightarrow a_n = a_1 + d(n-1)$
2. Geometric Sequence : (Common ratio r) $\Rightarrow a_n = a_1 r^{n-1}$
3. SignAlternator : $a_n = \begin{cases} (-1)^n \to 1^{st} \text{ term is -ve} \\ (-1)^{n+1} \to \text{〃 〃 〃 +ve} \end{cases}$
4. Binary Sequence : $a_n = 2^{n-1} \Rightarrow 1, 2, 4, 8, \ldots, 2^{n-1}, \ldots$
5. power Sequences : $a_n = n^2 \Rightarrow 1, 4, 9, 16, \ldots, n^2, \ldots$
   $a_n = n^3 \Rightarrow 1, 8, 27, 64, \ldots, n^3, \ldots$
   $\vdots$
6. Factorial Sequence : $a_n = n! \Rightarrow 1, 2, 6, 24, 120, \ldots, n!, \ldots$
7. product Sequences : $a_n = 2^n n! \Rightarrow 2, 2\cdot4, 2\cdot4\cdot6, \ldots, 2\cdot4\cdot6\cdots(2n), \ldots$
   $a_n = \dfrac{(2n)!}{2^n \cdot n!} \Rightarrow 1, 1\cdot3, 1\cdot3\cdot5, \ldots, 1\cdot3\cdot5\cdots(2n-1), \ldots$
   $a_n = \dfrac{(2n-1)!}{2^{n-1}(n-1)!} \Rightarrow 1, 1\cdot3, 1\cdot3\cdot5, \ldots, 1\cdot3\cdot5\cdots(2n-1), \ldots$

➢ Hierarchy of sequences as $n \to \infty$ $\quad k>1,\ a>1,\ b\geq 2$

$$\text{constants (or numbers)} \ll \ln(\ln n) \ll \ln n \ll n^{1/k} \ll n^k \ll a^n \ll n! \ll (bn)! \ll n^n \ll \cdots$$

the above statement is useful when evaluating limits of the form,

$$\lim_{n\to} \frac{N(n)}{D(n)} = \left\{\begin{array}{ll} 0 & \text{if } N(n) \text{ is to the left of } D(n) \\ \infty & \text{if } N(n) \text{ is to the right of } D(n) \end{array}\right\} \text{ in the list}$$

➢ Some important limit Laws:

1. If $\lim_{n\to\infty} a_n = L \Rightarrow \lim_{n\to\infty} f(a_n) = f(L)$

2. If $a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$

$\Rightarrow \lim_{n\to\infty} b_n = L$ Squeeze (sandwitch) theorem

3. If $\lim_{n\to\infty} \frac{N(n)}{D(n)} = \left\{\begin{array}{l} 0/0 \\ \text{or} \\ \infty/\infty \end{array}\right. \Rightarrow \lim_{x\to\infty} \frac{N(x)}{D(x)} \overset{\hat{LH}}{=} \lim_{x\to\infty} \frac{N'(x)}{D'(x)} = L$

$\hat{LH}$ (L'Hopital's rule)

➢ The seven forms of inderminates are $\underbrace{0/0, \infty/\infty}$; $\underbrace{\infty-\infty, 0\cdot\infty}$, $\underbrace{1^\infty, 0^0, \infty^0}$

- The first 2 forms are ready for $\hat{LH}$
- The next 2 need algebraic Manipulation $\left\{\begin{array}{l} \text{rationatzation for } \infty-\infty \\ \text{reciprocation for } 0\cdot\infty \end{array}\right\}$

  * the $\infty-\infty$ will not require $\hat{LH}$
- The last 3 have to be converted by taking the ln-function.

Some Special limits of Sequences

$$\lim_{n\to\infty} r^n = \begin{cases} \text{Converges to zero if } |r|<1 \\ 1 \text{ if } r=1 \\ \text{DNE if } r=-1 \\ \text{diverges if } |r|>1 \begin{cases} \text{diverges to } \infty \text{ if } r>1 \\ \text{DNE if } r<-1 \end{cases} \end{cases}$$

$$\lim_{n\to\infty} (-1)^n b_n = \begin{cases} \text{Converges to zero if } \lim_{n\to\infty} b_n = 0 \\ \text{Diverges otherwise} \end{cases}$$

$$\lim_{n\to\infty}\left(1+\frac{\square}{n}\right)^n = e^{\square}, \quad \lim_{n\to\infty}\frac{|Kx|^n}{n!} = 0 \quad \text{for any } x \text{ and Constant } K.$$