MATH 1210 ASSIGNMENT #3 SOLUTIONS

① (a) $x^2 - 7x + 11 = 0$ reducible since $(-7)^2 - 4(1)(11) = 5 > 0$

roots: $x = \dfrac{7 \pm \sqrt{5}}{2}$

real factored form: $\left(x - \left(\frac{7+\sqrt{5}}{2}\right)\right)\left(x - \left(\frac{7-\sqrt{5}}{2}\right)\right) = 0$

(b) $x^2 - 2x + 9$ irreducible since $(-2)^2 - 4(1)(9) = -32 < 0$

roots: $x = \dfrac{2 \pm \sqrt{-32}}{2} = 1 \pm 2\sqrt{2}\,i$

real factored form: $x^2 - 2x + 9 = 0$

(c) $x^3 + 5x^2 + 6x = 0$

$x(x^2 + 5x + 6) = 0$

$x(x+3)(x+2) = 0$ ← real factored form

roots: $x = 0, -2, -3$.

(d) $(x-1)^2(x^4 - 16) = 0$

$(x-1)^2(x^2 - 4)(x^2 + 4) = 0$

$(x-1)^2(x-2)(x+2)\underbrace{(x^2+4)}_{\text{irreducible since } 0^2 - 4(1)(4) = -16 < 0} = 0$

this is the real factored form of the equation.

roots: $x = \underbrace{1, 1}_{\text{multiplicity } 2}, 2, -2, \pm 2i$

(e) $x^4\underbrace{(x^2+4)}_{\text{irreducible since } 0^2 - 4(1)(4) = -16 < 0}(x^3 - 3x^2 + 3x - 1) = 0$

$\Leftrightarrow x^4(x^2+4)(x-1)^3 = 0$ [using the Binomial Theorem] real factored form : $x^4(x^2+4)(x-1)^3=0$

roots: $x=\underbrace{0,0,0,0}_{\text{multiplicity } 4},\pm 2i,\underbrace{1,1,1}_{\text{multiplicity } 3}.$

(f) $x^6+5x^4+4x^2=0$

$x^2(x^4+5x^2+4)=0$

$x^2(x^2+4)(x^2+1)=0$ ⟸ represents the real factored form

both of these factors are irreducible.

roots: $x=\underbrace{0,0}_{\text{multiplicity } 2},\pm 2i,\pm i$

(g) $p_1(x)=x^4+x^3-x^2+2x-6=0$

$p_1(\sqrt{2}i)=(\sqrt{2}i)^4+(\sqrt{2}i)^3-(\sqrt{2}i)^2+2\sqrt{2}i-6$

$=4-2\sqrt{2}i+2+2\sqrt{2}i-6=0$ (given)

$\therefore (x-\sqrt{2}i)$ is a factor

$\therefore (x+\sqrt{2}i)$ is also a factor, since $p_1(x)$ is a real polynomial

$\therefore (x-\sqrt{2}i)(x+\sqrt{2}i)=x^2+2$ is an irreducible real quadratic factor.

$\therefore p_1(x)=(x^2+2)[\underset{1}{\_\_}x^2+\underset{1}{\_\_}x+\underset{-3}{\_\_}]$ (alternative use long division)

$p_1(x)=(x^2+2)[\underbrace{x^2+x-3}_{\text{reducible since } (1)^2-4(1)(-3)=13>0}]$

roots of $x^2+x-3=0$ are

$x=\dfrac{-1\pm\sqrt{13}}{2}$

$p_1(x)=(x^2+2)\left(x-\left(\frac{-1+\sqrt{13}}{2}\right)\right)\left(x-\left(\frac{-1-\sqrt{13}}{2}\right)\right)=0$

is the real factored form

roots: $\pm\sqrt{2}i, (-1\pm\sqrt{13})/2$.

(h) $p_2(x) = x^3 + 9x^2 + 16x + 14 = 0$

$p_2(-1+i) = (-1+i)^3 + 9(-1+i)^2 + 16(-1+i) + 14$

$= -1 + 3(-1)^2 i + 3(-1)i^2 + i^3$

$+ 9((-1)^2 + 2(-1)i + i^2) - 16 + 16i + 14$

[by the Binomial Theorem]

$= -1 + 3i + 3 - i + 9 - 18i - 9 - 16 + 16i + 14$

$= 0$ (given)

Since $p_2(x)$ is a real polynomial, both $(x-(-1+i))$ & $(x-(-1-i))$ are factors

& $(x+1-i)(x+1+i)$

$= (x^2+2x+2)$ is an irreducible real quadratic factor.

$$p_2(x) = (x^2+2x+2)[\underset{\uparrow \atop 1}{\_\_}\, x + \underset{\uparrow \atop 7}{\_\_}]$$

$= (x^2+2x+2)(x+7) = 0$ is the real factored form.

roots: $x = -1 \pm i, -7$.

(i) $p_3(x) = x^5 - 4x^4 + 5x^3 + x^2 - 4x + 5$

$= x^3(x^2-4x+5) + 1(x^2-4x+5)$

$= (x^3+1)\underbrace{(x^2-4x+5)}_{\text{irreducible since } (-4)^2 - 4(1)(5) = -4 < 0.} = 0$

$p_3(x) = 0 \iff x^3+1=0$ or $x^2-4x+5=0$

$\Downarrow$ $\Downarrow$

$x^3 = -1$ $\quad x = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$

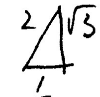
$\Downarrow$

To find cube roots of $-1 = 1e^{i(\pi + 2k\pi)}$

let $x = Re^{i\phi}$ so $R^3 e^{i3\phi} = e^{i(\pi+2k\pi)}$

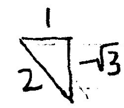
$\Rightarrow R = 1$ (since $R > 0$) & $\phi = \frac{\pi}{3} + \frac{2k\pi}{3}$

∴ The cube roots of -1 are



$$x = 1e^{i(\pi/3)} = \cos\left(\tfrac{\pi}{3}\right) + i\sin\left(\tfrac{\pi}{3}\right) = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i = \tfrac{1}{2}(1+\sqrt{3}\,i)$$

$$x = 1e^{i(\frac{\pi}{3}+\frac{2\pi}{3})} = e^{i\pi} = -1$$



$$x = 1e^{i(\frac{\pi}{3}+\frac{4\pi}{3})} = e^{i(5\pi/3)} = \cos\left(\tfrac{5\pi}{3}\right) + i\sin\left(\tfrac{5\pi}{3}\right) = \cos\left(-\tfrac{\pi}{3}\right) + i\sin\left(-\tfrac{\pi}{3}\right) = \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}i = \tfrac{1}{2}(1-\sqrt{3}\,i)$$

Thus $x^3+1 = \underset{\text{real}}{(x-(-1))}\underbrace{\left(x-\tfrac{1}{2}(1+\sqrt{3}i)\right)\left(x-\tfrac{1}{2}(1-\sqrt{3}i)\right)}_{\text{complex}}$

$= (x+1)(x^2-x+1)$

$P_3(x) = 0 \iff (x+1)\underbrace{(x^2-x+1)(x^2-4x+5)}_{\text{both are irreducible real quadratic factors}} = 0$

so this is the real factorization of $P_3(x)$.

roots of $P_3(x)=0$ are $-1,\ -\frac{1}{2}(1\pm\sqrt{3}i),\ 2\pm i$

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(2) let $P(x) = x^6 + 12x^5 + 42x^4 + 43x^3 - 30x^2 - 60x - 8$.

(a) Since $P(x)$ has only one sign change in its coefficients, it has at most one +ve real root.

(b) Possible rational roots: divisors of 1 are ±1

divisors of -8 are $\pm1, \pm2, \pm4, \pm8$

$\Rightarrow$ possible rational roots are: $\pm1, \pm2, \pm4, \pm8$.

$P(1) = 1+12+42+43-30-60-8 = 0$

$\Rightarrow x=1$ is the only real zero of $P(x)$

$\Rightarrow (x-1)$ is a factor of $P(x)$.

(c)

$$
\begin{array}{r}
x^5 + 13x^4 + 55x^3 + 98x^2 + 68x + 8 \\
(x-1)\overline{\big)\, x^6 + 12x^5 + 42x^4 + 43x^3 - 30x^2 - 60x - 8} \\
\underline{x^6 - x^5} \\
13x^5 \\
\underline{13x^5 - 13x^4} \\
55x^4 \\
\underline{55x^4 - 55x^3} \\
98x^3 \\
\underline{98x^3 - 98x^2} \\
+68x^2 \\
\underline{68x^2 - 68x} \\
8x - 8 \\
\underline{8x - 8} \\
0
\end{array}
$$

$P(x) = (x-1)\,Q(x)$

where $Q(x) = x^5 + 13x^4 + 55x^3 + 98x^2 + 68x + 8$.

(d) Note: since $P(x)$ can have no more real +ve zeros, neither can $Q(x)$, so we need only test $-1, -2, -4, -8$.

$Q(-1) = -1 + 13 - 55 + 98 - 68 + 8 \neq 0$

$Q(-2) = -32 + 13(16) + 55(-8) + 98(4) + 68(-2) + 8 = 0$

$\Rightarrow$ $x = -2$ is a zero of $Q(x)$ & $(x - (-2)) = x+2$ is a factor of $Q(x)$.

$$
\begin{array}{r}
x^4 + 11x^3 + 33x^2 + 32x + 4 \\
(x+2)\overline{\big)\, x^5 + 13x^4 + 55x^3 + 98x^2 + 68x + 8} \\
\underline{x^5 + 2x^4} \\
11x^4 \\
\underline{11x^4 + 22x^3} \\
33x^3 \\
\underline{33x^3 + 66x^2} \\
32x^2 \\
\underline{32x^2 + 64x} \\
4x + 8
\end{array}
$$

c'td

$$
\begin{array}{r}
\underline{4x + 8} \\
0
\end{array}
$$

$\therefore \quad P(x) = (x-1)(x+2)\,R(x)$

where $R(x) = x^4 + 11x^3 + 33x^2 + 32x + 4$.

Continuing the above reasoning $R(x)$ cannot have any +ve real zeros & $R(-1)$ cannot be zero.

But $R(-2) = 16 - 88 + 132 - 64 + 2 = 0$

$\Rightarrow \quad x = -2$ is a zero of $R(x)$

$\Rightarrow x - (-2) = x+2$ is a factor of $R(x)$

$$
\begin{array}{r l}
 & x^3 + 9x^2 + 15x + 2 \\ \hline
x+2 \,\big) & x^4 + 11x^3 + 33x^2 + 32x + 4 \\
 & \underline{x^4 + 2x^3} \\
 & 9x^3 \\
 & \underline{9x^2 + 18x^2} \\
 & 15x^2 \\
 & \underline{15x^2 + 30x} \\
 & 2x \\
 & \underline{2x + 4} \\
 & 0
\end{array}
$$

$\therefore \quad P(x) = (x-1)(x+2)^2\,S(x)$

where $S(x) = x^3 + 9x^2 + 15x + 2$.

Again, using the above reasoning, $S(x)$ has no +ve real zeros & $S(-1) \neq 0$.

But $S(-2) = -8 + 36 - 30 + 2 = 0$

$\Rightarrow \quad x = -2$ is a zero of $S(x)$

$\Rightarrow (x+2)$ is a factor of $S(x)$

$$
\begin{array}{r l}
 & x^2 + 7x + 1 \\ \hline
x+2 \,\big) & x^3 + 9x^2 + 15x + 2 \\
 & \underline{x^3 + 2x^2} \\
 & 7x^2 \\
 & \underline{7x^2 + 14x} \\
 & x + 2 \\
 & \underline{x + 2} \\
 & 0
\end{array}
$$

$\Rightarrow P(x) = (x-1)(x+2)^3(x^2+7x+1)$

reducible since $7^2 - 4(1)(1) = 45 > 0$

thus the roots of $x^2+7x+1=0$ are $x = \dfrac{-7 \pm \sqrt{45}}{2}$

$\Rightarrow x^2+7x+1 = \left(x-\left(\frac{-7+\sqrt{45}}{2}\right)\right)\left(x-\left(\frac{-7-\sqrt{45}}{2}\right)\right)$

$+ \; P(x) = (x-1)(x+2)^3\left(x-\left(-\frac{7+\sqrt{45}}{2}\right)\right)\left(x-\left(-\frac{7-\sqrt{45}}{2}\right)\right)$

(e) $P(x)=0 \Rightarrow x = \underset{\text{multiplicity } 1}{1}, \underbrace{-2,-2,-2}_{\text{multiplicity } 1}, \underbrace{\frac{-7\pm\sqrt{45}}{2}}_{\text{each has mult. } 1}$

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(3) $P(x) = ix^4 + 3x^3 + 8ix + 24$

(a) $P\left(\frac{-3}{i}\right) = P\left(-\frac{3i}{i^2}\right) = P(+3i)$

$= 81i^5 + 81i^3 + 24i^2 + 24$

$= -81i + 81i - 24 + 24 = 0.$

Therefore $(ix+3)$ is a factor of $P(x)$

(b) [Alt: $(x-3i)$ is a factor of $P(x)$]

$$\begin{array}{r|l} & x^3 \qquad + 8 \\ \hline ix+3 & ix^4+3x^3+8ix+24 \\ & ix^4+3x^3 \\ \hline & 0 + 8ix + 24 \\ & 8ix+24 \\ \hline & 0 \end{array}$$

$\Rightarrow P(x) = (ix+3)(x^3+8)$

[Alternatively

$$\begin{array}{r|l} & ix^3 \qquad + 8i \\ \hline x-3i & ix^4+3x^3+8ix+24 \\ & ix^4+3x^3 \\ \hline & 0 + 8ix+24 \\ & +8ix+24 \\ \hline & 0 \end{array}$$

so that we may write.

$$P(x) = (x-3i)(ix^3+8i) = i(x-3i)(x^3+8)$$
$$= (ix+3)(x^3+8)$$
as above.

(c) $P(x) = 0 \iff (ix+3) = 0$ OR $x^3+8=0$

$$\Downarrow \qquad\qquad \Downarrow$$

$$x = -\frac{3}{i} = 3i \qquad x^3 = -8 = 8e^{i(\pi+2k\pi)}$$

let $x = Re^{i\theta}$

then $R^3 e^{i3\theta} = 8e^{i(\pi+2k\pi)}$

$$\Rightarrow R^3 = 8 \Rightarrow \boxed{R=2}$$

& $3\theta = \pi + 2k\pi$

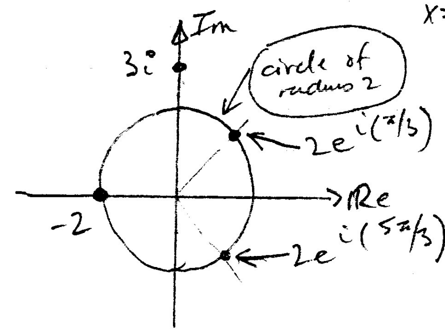
$$\theta = \frac{\pi}{3} + \frac{2k\pi}{3} \quad \text{for } k = 0,1,2.$$

Thus the roots of $x^3+8=0$ are

$$x = 2e^{i(\pi/3)}$$
$$x = 2e^{i\left(\frac{\pi}{3}+\frac{2\pi}{3}\right)} = 2e^{i\pi} = -2$$
$$\& \ x = 2e^{i\left(\frac{\pi}{3}+\frac{4\pi}{3}\right)} = 2e^{i(5\pi/3)}$$

& the roots of $P(x)=0$ are $x = 3i$, $x = 2e^{i(\pi/3)}$, $x = -2$ & $x = 2e^{i\left(\frac{5\pi}{3}\right)}$.

(d)



4 $\quad l_1 : x = 1+s,\ y = 1-s,\ z = 4s$
$\quad l_2 : x = 2-3t,\ y = 6+2t,\ z = 1-t.$

(a) $\vec{v_1} = \hat{i} - \hat{j} + 4\hat{k}$ is a vector along l_1.
[if you like, consider the vector joining the 2 pts on l_1 determined by setting $s=0$ & $s=1$]
$\vec{v_2} = -3\hat{i} + 2\hat{j} - \hat{k}$ is a vector along l_2.

(b) Let $P_1(1+s, 1-s, 4s)$ be any pt on l_1
& $P_2(2-3t, 6+2t, 1-t)$ be any pt on l_2
then
$$\vec{u} = \overrightarrow{P_1P_2} = \big((2-3t)-(1+s)\big)\hat{i} + \big((6+2t)-(1-s)\big)\hat{j} + \big((1-t)-4s\big)\hat{k}.$$
$$= (1-3t-s)\hat{i} + (5+2t+s)\hat{j} + (1-t-4s)\hat{k}$$
is a vector joining the points P_1 & P_2.

(c) $\vec{v_1} \cdot \overrightarrow{P_1P_2} = 0 \implies 1(1-3t-s) - 1(5+2t+s) + 4(1-t-4s) = 0$
i.e, $-9t - 18s = 0$
$\vec{v_1} \cdot \overrightarrow{P_1P_2} = 0 \implies -3(1-3t-s) + 2(5+2t+s) - 1(1-t-4s) = 0$
i.e, $6 + 14t + 9s = 0$

i.e, for $\overrightarrow{P_1P_2}$ to be perpendicular to both $\vec{v_1}$ & $\vec{v_2}$ we must have
$$\left.\begin{array}{l} -9t - 18s = 0 \\ +14t + 9s = -6 \end{array}\right\} \text{2 equations in 2 unknowns}$$

solve for s & t: $\quad t = -2s$
$\implies 14(-2s) + 9s = -6$
$\implies -19s = -6$
$s = +6/19$ & $t = \frac{-12}{19}$

thus since $s = \frac{6}{19}$, P_1 has coords $\left(\frac{25}{19}, \frac{13}{19}, \frac{24}{19}\right)$

& $t = \frac{-12}{19} \Rightarrow P_2$ has coords $\left(\frac{74}{19}, \frac{90}{19}, \frac{31}{19}\right)$

$\therefore \; \overrightarrow{P_1P_2} = \left(\frac{49}{19}\right)\hat{i} + \left(\frac{77}{19}\right)\hat{j} + \left(\frac{7}{19}\right)\hat{k}$

(e) $$d = |P_1P_2| = \sqrt{\left(\frac{49}{19}\right)^2 + \left(\frac{77}{19}\right)^2 + \left(\frac{7}{19}\right)^2}$$
$$= \frac{1}{19}\sqrt{49^2 + 77^2 + 7^2}$$
$$= \frac{1}{19}\sqrt{8379} \simeq 4.82$$

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(5) $\Pi : 2x - y + 3z = 4$

$\ell : x = 2 + t, \; y = 3 + 5t, \; z = 4 + t.$

(a) If ℓ intersects Π, then there must be a value of t for which

$$2(2+t) - (3+5t) + 3(4+t) = 4$$

ie, $13 = 4$ But this is impossible so ℓ cannot intersect Π

(b) (i) let P be a fixed point on ℓ with coords $(2,3,4)$ [set $t = 0$]

(ii) $\vec{N} = 2\hat{i} - \hat{j} + 3\hat{k}$ is a normal vector to Π

(iii) the equation of a line through P in the direction of $\vec{N}$ is given by

$$\vec{r} = \vec{r_0} + s\vec{N} \qquad \text{(parameter } s\text{)} \qquad (*)$$

where $\vec{r_0} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ is the position vector of P.

(*) represents the vector parametric equations of a line l_1 through P in the direction of $\vec{N}$.
(*) may be written in component form as.

$$\left.\begin{aligned} x &= 2+2s \\ y &= 3-s \\ z &= 4+3s \end{aligned}\right\} \quad (**)$$

(iv) l_1 intersects π when $2(2+2s)-(3-s)+3(4+3s)=4$

$$13+14s=4$$
$$14s=-9$$
$$s=-9/14$$

$\therefore$ l_1 intersects π at the point $Q\left(\frac{5}{7}, \frac{61}{14}, \frac{10}{7}\right)$
[put $s=-9/14$ in (**)]

(v) $\vec{PQ} = \left(\frac{5}{7}-2\right)\hat{i} + \left(\frac{61}{14}-3\right)\hat{j} + \left(\frac{10}{7}-4\right)\hat{k}$
$= -\frac{9}{7}\hat{i} + \frac{19}{14}\hat{j} - \frac{18}{7}\hat{k}$

& $d = |\vec{PQ}| = \sqrt{\left(-\frac{9}{7}\right)^2+\left(\frac{19}{14}\right)^2+\left(-\frac{18}{7}\right)^2}$

$= \sqrt{\left(\frac{-18}{14}\right)^2+\left(\frac{19}{14}\right)^2+\left(\frac{-36}{14}\right)^2}$

$= \frac{1}{14}\sqrt{18^2+19^2+36^2}$

$= \frac{1}{14}\sqrt{1981} \simeq 3.18$