EXERCISES FOR CHAPTER 3: Infinite Series

SET 3.1

1. Express $\dfrac{4}{k(k+2)}$ in partial fractions and hence find the sum $\sum_{k=1}^{n} \dfrac{4}{k(k+2)}$.

Determine if $\lim_{n\to\infty} \sum_{k=1}^{n} \dfrac{4}{k(k+2)}$ exists.

Solution

$\dfrac{4}{k(k+2)} = \dfrac{A}{k} + \dfrac{B}{k+2}$. By the cover-up method $A = 2$ and $B = -2$. Hence

$$\sum_{k=1}^{n} \frac{4}{k(k+2)} = \left(\frac{2}{1} - \frac{2}{3}\right) + \left(\frac{2}{2} - \frac{2}{4}\right) + \cdots + \left(\frac{2}{n-1} - \frac{2}{n+1}\right) + \left(\frac{2}{n} - \frac{2}{n+2}\right)$$

$$= 2 + 1 - \frac{2}{n+1} - \frac{2}{n+2}$$

$$= 3 - \frac{2n+3}{(n+1)(n+2)}$$

$$= \frac{n(3n+5)}{(n+1)(n+2)}$$

Thus, $\sum_{k=1}^{\infty} \dfrac{4}{k(k+2)} = \lim_{n\to\infty} \dfrac{n(3n+5)}{(n+1)(n+2)} = 3$.

2. Show that $\dfrac{5}{1\times 2} + \dfrac{5}{2\times 3} + \dfrac{5}{3\times 4} + \cdots$ exists and find its value.

Solution

$\dfrac{5}{k(k+1)} = \dfrac{A}{k} + \dfrac{B}{k+1}$. By the cover-up method $A = 5$ and $B = -5$. Hence

$$\sum_{k=1}^{n} \frac{5}{k(k+1)} = \left(\frac{5}{1} - \frac{5}{2}\right) + \left(\frac{5}{2} - \frac{5}{3}\right) + \cdots + \left(\frac{5}{n-1} - \frac{5}{n}\right) + \left(\frac{5}{n} - \frac{5}{n+1}\right)$$

$$= 5 - \frac{5}{n+1}$$

$$= \frac{5n}{n+1}$$

Thus, $\sum_{k=1}^{\infty} \dfrac{5}{k(k+1)} = \lim_{n\to\infty} \dfrac{5n}{n+1} = 5$.

3. Find the value of the infinite sum $\dfrac{1}{1\times 4} + \dfrac{1}{2\times 5} + \dfrac{1}{3\times 6} + \cdots + \dfrac{1}{n(n+3)} + \cdots$.

Solution

$\dfrac{1}{n(n+3)} = \dfrac{A}{n} + \dfrac{B}{n+3}$. By the cover-up method $A = \dfrac{1}{3}$ and $B = -\dfrac{1}{3}$. Hence

$$\sum_{n=1}^{N}\frac{1}{n(n+3)}=\frac{1}{3}\left(\frac{1}{1}-\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{2}-\frac{1}{5}\right)+\frac{1}{3}\left(\frac{1}{3}-\frac{1}{6}\right)+\frac{1}{3}\left(\frac{1}{4}-\frac{1}{7}\right)\cdots+$$
$$+\frac{1}{3}\left(\frac{1}{N-2}-\frac{1}{N+1}\right)+\frac{1}{3}\left(\frac{1}{N-1}-\frac{1}{N+2}\right)+\frac{1}{3}\left(\frac{1}{N}-\frac{1}{N+3}\right)$$
$$=\frac{1}{3}\left(1+\frac{1}{2}+\frac{1}{3}-\frac{1}{N+1}-\frac{1}{N+2}-\frac{1}{N+3}\right)$$
$$=\frac{N(11N^2+48N+49)}{18(N+1)(N+2)(N+3)}$$

Thus $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)}=\lim_{N\to\infty}\frac{N(11N^2+48N+49)}{18(N+1)(N+2)(N+3)}=\frac{11}{18}$.

4. Evaluate $\displaystyle\frac{1}{1\times 3\times 5}+\frac{1}{2\times 4\times 6}+\cdots+\frac{1}{n(n+2)(n+4)}+\cdots$.

Solution

$\displaystyle\frac{1}{n(n+2)(n+4)}=\frac{A}{n}+\frac{B}{n+2}+\frac{C}{n+4}$. By the cover-up method $A=\frac{1}{8}$, $B=-\frac{1}{4}$ and $C=\frac{1}{8}$. We then write

$$\frac{1}{n(n+2)(n+4)}=\frac{1}{8n}-\frac{2}{8(n+2)}+\frac{1}{8(n+4)}$$
$$=\frac{1}{8n}-\frac{1}{8(n+2)}+\frac{1}{8(n+4)}-\frac{1}{8(n+2)}$$
$$=\frac{1}{8}\left(\frac{1}{n}-\frac{1}{(n+2)}\right)+\frac{1}{8}\left(\frac{1}{(n+4)}-\frac{1}{(n+2)}\right)$$

so that

$$\sum_{n=1}^{N}\frac{1}{n(n+2)(n+4)}=\frac{1}{8}\sum_{1}^{N}\left(\frac{1}{n}-\frac{1}{(n+2)}\right)-\frac{1}{8}\sum_{1}^{N}\left(\frac{1}{(n+2)}-\frac{1}{(n+4)}\right)$$
$$=\frac{1}{8}\left((1-\frac{1}{3})+(\frac{1}{2}-\frac{1}{4})+(\frac{1}{3}-\frac{1}{5})+\cdots+(\frac{1}{N-1}-\frac{1}{N+1})+(\frac{1}{N}-\frac{1}{N+2})\right)+$$
$$-\frac{1}{8}\left((\frac{1}{3}-\frac{1}{5})+(\frac{1}{4}-\frac{1}{6})+(\frac{1}{5}-\frac{1}{7})+\cdots+(\frac{1}{N+1}-\frac{1}{N+3})+(\frac{1}{N+2}-\frac{1}{N+4})\right)$$
$$=\frac{1}{8}\left(1+\frac{1}{2}-\frac{1}{N+1}-\frac{1}{N+2}\right)+$$
$$-\frac{1}{8}\left(\frac{1}{3}+\frac{1}{4}-\frac{1}{N+3}-\frac{1}{N+4}\right)$$
$$=\frac{N(N+5)(11N^2+55N+62)}{96(N+1)(N+2)(N+3)(N+4)}$$

Thus, $\sum_{n=1}^{\infty}\frac{1}{n(n+2)(n+4)}=\lim_{N\to\infty}\frac{N(N+5)(11N^2+55N+62)}{96(N+1)(N+2)(N+3)(N+4)}=\frac{11}{96}$.

5. Find $\frac{2}{1\times 3\times 5}+\frac{3}{3\times 5\times 7}+\cdots+\frac{n+1}{(2n-1)(2n+1)(2n+3)}+\cdots$.

Solution

$\frac{n+1}{(2n+1)(2n-1)(2n+3)}=\frac{A}{2n+1}+\frac{B}{2n-1}+\frac{C}{2n+3}$. By the cover-up method, $A=-\frac{1}{8}$, $B=\frac{3}{16}$ and $C=-\frac{1}{16}$. Hence we may write:

$$\frac{n+1}{(2n+1)(2n-1)(2n+3)}=-\frac{2}{16}\frac{1}{2n+1}+\frac{3}{16}\frac{1}{2n-1}-\frac{1}{16}\frac{1}{2n+3}$$
$$=\frac{2}{16}\left(\frac{1}{2n-1}-\frac{1}{2n+1}\right)+\frac{1}{16}\left(\frac{1}{2n-1}-\frac{1}{2n+3}\right)$$

Then,

$$\sum_{n=1}^{N}\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{2}{16}\left((1-\frac{1}{3})+(\frac{1}{3}-\frac{1}{5})+(\frac{1}{5}-\frac{1}{7})+\cdots+(\frac{1}{2N-1}-\frac{1}{2N+1})\right)$$
$$+\frac{1}{16}\left((1-\frac{1}{5})+(\frac{1}{3}-\frac{1}{7})+(\frac{1}{5}-\frac{1}{9})+\cdots+(\frac{1}{2N-3}-\frac{1}{2N+1})+(\frac{1}{2N-1}-\frac{1}{2N+3})\right)$$

$$\sum_{n=1}^{N}\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\frac{2}{16}\left(1-\frac{1}{2N+1}\right)+\frac{1}{16}\left(1+\frac{1}{3}-\frac{1}{2N+1}-\frac{1}{2N+3})\right)$$
$$=\frac{N(5N+7)}{6(2N+3)(2N+1)}$$

Hence, $\sum_{n=1}^{\infty}\frac{n+1}{(2n-1)(2n+1)(2n+3)}=\lim_{N\to\infty}\frac{N(5N+7)}{6(2N+3)(2N+1)}=\frac{5}{24}$.

6. Express $\frac{k+3}{k(k^2-1)}$ in partial fractions and hence find the sum $\sum_{k=2}^{n}\frac{k+3}{k(k^2-1)}$.

Determine if $\lim_{n\to\infty}\sum_{k=2}^{n}\frac{k+3}{k(k^2-1)}$ exists.

Solution

$\frac{k+3}{k(k^2-1)}=\frac{k+3}{k(k-1)(k+1)}=\frac{A}{k}+\frac{B}{k-1}+\frac{C}{k+1}$. By the cover-up method, $A=-3$, $B=2$ and $C=1$. Hence we may write:

$$\frac{k+3}{k(k^2-1)}=-\frac{3}{k}+\frac{2}{k-1}+\frac{1}{k+1}=2\left(\frac{1}{k-1}-\frac{1}{k}\right)-\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

so that

$$\sum_{2}^{N}\frac{k+3}{k(k^2-1)} = 2\left((1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+\cdots+(\frac{1}{N-1}-\frac{1}{N})\right)$$
$$-\left((\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\cdots+(\frac{1}{N}-\frac{1}{N+1})\right)$$
$$= 2\left(1-\frac{1}{N}\right)-\left(\frac{1}{2}-\frac{1}{N+1}\right)$$
$$= \frac{2(N-1)}{N}-\frac{N-1}{2(N+1)}$$
$$= \frac{(N-1)(3N+4)}{2N(N+1)}$$

and hence $\sum_{2}^{\infty}\frac{k+3}{k(k^2-1)} = \lim_{N\to\infty}\frac{(N-1)(3N+4)}{2N(N+1)} = \frac{3}{2}$.

7. Investigate, by using partial fractions, whether the integrals (a) $\int_5^{\infty}\frac{1}{x^2-5x+6}dx$ and (b) $\int_5^{\infty}\frac{x+1}{x^2-5x+6}dx$ exist.

Solution

(a) $\frac{1}{x^2-5x+6} = \frac{1}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2}$. By the cover up method, $A=1$ and $B=-1$, so that $I = \lim_{\Lambda\to\infty}\int_5^{\Lambda}\frac{1}{x^2-5x+6}dx = \lim_{\Lambda\to\infty}\int_5^{\Lambda}(\frac{1}{x-3}-\frac{1}{x-2})dx$. Then,

$$I = \lim_{\Lambda\to\infty}(\ln|x-3|-\ln|x-2|)\Big|_5^{\Lambda}$$
$$= \lim_{\Lambda\to\infty}\ln\left|\frac{x-3}{x-2}\right|_5^{\Lambda}$$
$$= \lim_{\Lambda\to\infty}\ln\frac{\Lambda-3}{\Lambda-2}-\ln\frac{2}{3}$$
$$= 0+\ln\frac{3}{2}$$

The integral converges. (Notice that $\lim_{\Lambda\to\infty}\ln\frac{\Lambda-3}{\Lambda-2} = \ln\lim_{\Lambda\to\infty}\frac{\Lambda-3}{\Lambda-2} = \ln 1 = 0$.)

(b) $\frac{x+1}{x^2-5x+6} = \frac{x+1}{(x-3)(x-2)} = \frac{A}{x-3}+\frac{B}{x-2}$. By the cover up method, $A=4$ and $B=-3$, so that $I = \lim_{\Lambda\to\infty}\int_5^{\Lambda}\frac{x+1}{x^2-5x+6}dx = \lim_{\Lambda\to\infty}\int_5^{\Lambda}(\frac{4}{x-3}-\frac{3}{x-2})dx$. Then,

$$I = \lim_{\Lambda\to\infty} (4\ln|x-3| - 3\ln|x-2|)\Big|_5^\Lambda$$
$$= \lim_{\Lambda\to\infty} (4\ln|\Lambda-3| - 3\ln|\Lambda-2|) - (4\ln 2 - 3\ln 3)$$
$$= \lim_{\Lambda\to\infty} (4\ln|\Lambda-3| - 3\ln|\Lambda-2|) - \ln\frac{16}{27}$$

To evaluate the limit, $\lim_{\Lambda\to\infty}(4\ln|\Lambda-3| - 3\ln|\Lambda-2|)$ write

$$\lim_{\Lambda\to\infty}(4\ln|\Lambda-3| - 3\ln|\Lambda-2|) = \lim_{\Lambda\to\infty}(4\ln|\Lambda-3| - 4\ln|\Lambda-2|) + \lim_{\Lambda\to\infty}\ln|\Lambda-2|$$
$$= 4\lim_{\Lambda\to\infty}\ln\left|\frac{\Lambda-3}{\Lambda-2}\right| + \lim_{\Lambda\to\infty}\ln|\Lambda-2|$$
$$= 0 + \lim_{\Lambda\to\infty}\ln|\Lambda-2|$$
$$= \infty$$

The integral diverges. (Again, $\lim_{\Lambda\to\infty} 4\ln\frac{\Lambda-3}{\Lambda-2} = 4\ln\lim_{\Lambda\to\infty}\frac{\Lambda-3}{\Lambda-2} = 4\ln 1 = 0$.)

SET 3.2

Apply the divergence test to the series below to determine those that diverge. If the test is inconclusive you should say so.

1. (a) $\sum_1^\infty \frac{n+1}{n}$ (b) $\sum_1^\infty \frac{n^2}{n^2+1}$ (c) $\sum_1^\infty \frac{n}{2n+5}$

Solution

(a) $\lim_{n\to\infty}\frac{n+1}{n} = 1$, so series diverges.

(b) $\lim_{n\to\infty}\frac{n^2+1}{n^2} = 1$, so series diverges.

(c)) $\lim_{n\to\infty}\frac{n}{2n+5} = \frac{1}{2}$, so series diverges.

2. (a) $\sum_1^\infty \left(1+(-1)^n\right)$ (b) $\sum_1^\infty (-e)$ (c) $\sum_2^\infty \left(\ln(n+2) - \ln n\right)$

Solution

(a) $\lim_{n\to\infty}\left(1+(-1)^n\right)$ does not exist so series diverges.

(b) $\lim_{n\to\infty} u_n = -e \neq 0$, so series diverges.

(c) $\lim_{n\to\infty}\left(\ln(n+2) - \ln n\right) = \lim_{n\to\infty}\ln\left(\frac{n+2}{n}\right) = \ln\lim_{n\to\infty}\left(\frac{n+2}{n}\right) = \ln 1 = 0$, inconclusive.

3. (a) $\sum_1^\infty e^{-n}$ (b) $\sum_1^\infty e^n$ (c) $\sum_1^\infty ne^{-n}$

Solution

(a) $\lim_{n\to\infty} e^{-n} = 0$, so test is inconclusive. (Series is geometric and converges.)

(b) $\lim_{n\to\infty} e^n = \infty$, so series diverges.

(c) $\lim_{n\to\infty} ne^{-n} = 0$, so test in inconclusive.

4. (a) $\sum_1^\infty (\sqrt{n^2+n} - n)$ (b) $\sum_1^\infty \left(\frac{1}{\sqrt{n+1}-\sqrt{n}}\right)$ (c) $\sum_1^\infty \left(\frac{1}{\sqrt{2}}\right)^n$

Solution

(a) $\lim_{n\to\infty}\left(\sqrt{n^2+n} - n\right) = \frac{1}{2}$ by the example in the text so series diverges.

(b)

$$\lim_{n\to\infty}\frac{1}{\sqrt{n+1}-\sqrt{n}} = \lim_{n\to\infty}\frac{1}{\sqrt{n+1}-\sqrt{n}}\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+1}+\sqrt{n}}{n+1-n}$$

$$= \lim_{n\to\infty}\sqrt{n+1}+\sqrt{n}$$

$$= \infty$$

so series diverges.

(c) $\lim_{n\to\infty}\left(\frac{1}{\sqrt{2}}\right)^n = 0$, so test is inconclusive. (Series is geometric and converges.)

5. (a) $\sum_1^\infty (1 - \frac{(-1)^n}{n})$ (b) $\sum_1^\infty \sqrt[n]{n}$ (c) $\sum_1^\infty \frac{e^n - 1}{\pi^n}$

(a) $\lim_{n\to\infty}\left(1 - \frac{(-1)^n}{n}\right) = 1$, so series diverges.

(b) $\lim_{n\to\infty}\sqrt[n]{n} = 1$, so series diverges.

(c) $\lim_{n\to\infty}\frac{e^n - 1}{\pi^n} = 0$, so test is inconclusive. (Series is geometric and converges.)

6. (a) $\sum_1^\infty \left(\frac{n-2}{n}\right)^n$ (b) $\sum_1^\infty \left(\frac{n+3}{n}\right)^n$ (c) $\sum_1^\infty \left(\frac{n}{n+3}\right)^n$

Solution

(a) $\lim_{n\to\infty}\left(\frac{n-2}{n}\right)^n = e^{-2}$, so series diverges.

(b) $\lim_{n\to\infty}\left(\frac{n+3}{n}\right)^n = e^3$, so series diverges.

(c) $\lim_{n\to\infty}\left(\frac{n}{n+3}\right)^n = e^{-3}$, so series diverges.

7. (a) $\sum_2^\infty \frac{1}{\ln n}$ (b) $\sum_1^\infty \ln\left(\frac{n+1}{n}\right)$ (c) $\sum_1^\infty \ln\left(\frac{n}{n+1}\right)$

Solution

(a) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$, so test is inconclusive.

(b) $\lim_{n\to\infty} \ln(\frac{n+1}{n}) = 0$, so test is inconclusive.

(c) $\lim_{n\to\infty} \ln(\frac{n}{n+1}) = 0$, so test is inconclusive.

8. (a) $\sum_{1}^{\infty} \cos\left(\frac{1}{n}\right)$ (b) $\sum_{1}^{\infty} n\sin\left(\frac{1}{n}\right)$ (c) $\sum_{1}^{\infty} \frac{n}{n+2}$

Solution

(a) $\lim_{n\to\infty} \cos\frac{1}{n} = 1$, so series diverges.

(b) $\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$, so series diverges.

(c) $\lim_{n\to\infty} \frac{n}{n+2} = 1$, so series diverges.

9. (a) $\sum_{n=1}^{\infty} \frac{n!}{5^n}$ (b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$ (c) $\sum_{1}^{\infty} \frac{2n!+7}{n^{10}+n!}$

Solution

(a) $\lim_{n\to\infty} \frac{n!}{5^n} = \infty$, so series diverges.

(b) $\lim_{n\to\infty} \frac{n^n}{n!} = \infty$, so series diverges.

(c) $\lim_{n\to\infty} \frac{2n!+7}{n^{10}+n!} = 2$, so series diverges.

10. $\sum_{1}^{\infty} \left(\frac{2n+7}{3n-2}\right)^n$

Solution

$$\lim_{n\to\infty} \left(\frac{2n+7}{3n-2}\right)^n = \lim_{n\to\infty} \frac{(2n)^n (1+\frac{7}{2n})^n}{(3n)^n (1-\frac{2}{3n})^n} = \left(\frac{2}{3}\right)^n \frac{e^{7/2}}{e^{-2/3}} = 0$$, test is inconclusive.

SET 3.3

Apply, if possible, the integral test on the following series. If the test cannot be applied, state why.

1. (a) $\sum_{1}^{\infty}\frac{n}{n+1}$ (b) $\sum_{1}^{\infty}\frac{n}{n^2+1}$ (c) $\sum_{1}^{\infty}\frac{n^4}{4n^5+2}$

Solution

(a) Terms are increasing because $\frac{d}{dn}\left(\frac{n}{n+1}\right)=\frac{(n+1)-n}{(n+1)^2}=\frac{1}{(n+1)^2}>0$, test cannot be applied.

(b) Terms are decreasing because $\frac{d}{dn}\left(\frac{n}{n^2+1}\right)=\frac{(n^2+1)-n\times 2n}{(n^2+1)^2}=\frac{1-n^2}{(n^2+1)^2}<0$ and $\int_1^{\infty}\frac{n}{n^2+1}dn=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{n}{n^2+1}dn=\frac{1}{2}\lim_{\Lambda\to\infty}\ln(n^2+1)\Big|_1^{\Lambda}=\infty$ so series diverges.

(c) $\frac{d}{dn}\frac{n^4}{4n^5+2}=\frac{4n^3(4n^5+2)-n^4 20n^4}{(4n^5+2)^2}=\frac{8n^3-4n^8}{(4n^5+2)^2}=\frac{4n^3(2-n^5)}{(4n^5+2)^2}<0$ for $n>1$ so terms are decreasing. $\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{n^4}{4n^5+2}=\frac{1}{20}\lim_{\Lambda\to\infty}\ln(4\Lambda^5+2)-\frac{1}{20}\ln 6=\infty$, so series diverges.

2. (a) $\sum_{2}^{\infty}\frac{1}{n\ln n}$ (b) $\sum_{1}^{\infty}\frac{\ln n}{n^2}$ (c) $\sum_{1}^{\infty}\frac{\ln n}{\sqrt{n}}$

Solution

(a) Terms are clearly decreasing. The integral is $\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n\ln n}dn$. Make the substitution $u=\ln n\Rightarrow du=\frac{dn}{n}$ so that

$$\lim_{\Lambda\to\infty}\int_2^{\Lambda}\frac{1}{n\ln n}dn=\lim_{\Lambda\to\infty}\int_{\ln 2}^{\ln\Lambda}\frac{1}{u}du=\ln u\Big|_{\ln 2}^{\ln\Lambda}=\infty$$ so series diverges.

(b) $\frac{d}{dn}\frac{\ln n}{n^2}=\frac{\frac{1}{n}n^2-2n\ln n}{n^4}=\frac{1-2\ln n}{n^3}<0$ so terms are decreasing. The integral is

$$\lim_{\Lambda\to\infty}\int_2^{\Lambda}\frac{\ln n}{n^2}dn=\lim_{\Lambda\to\infty}\left(-\frac{\ln n}{n}\right)\Bigg|_2^{\Lambda}+\lim_{\Lambda\to\infty}\int_2^{\Lambda}\frac{1}{n^2}dn=\lim_{\Lambda\to\infty}\left(-\frac{\ln n}{n}\right)\Bigg|_2^{\Lambda}-\lim_{\Lambda\to\infty}\left(-\frac{1}{n}\right)\Bigg|_2^{\Lambda}=\frac{\ln 2}{2}+\frac{1}{2}$$

and so series converges.

(c) $\frac{d}{dn}\frac{\ln n}{\sqrt{n}}=\frac{\frac{\sqrt{n}}{n}-\ln n\frac{1}{2\sqrt{n}}}{n}=\frac{2-\ln n}{2n^{3/2}}<0$ for $n>e^2$. The integral is $\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{\ln n}{\sqrt{n}}dn$.

Using integration by parts,

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{\ln n}{\sqrt{n}}\,dn = 2\lim_{\Lambda\to\infty}\sqrt{n}\ln n\Big|_1^{\Lambda} - 2\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{\sqrt{n}}{n}\,dn = 2\lim_{\Lambda\to\infty}\sqrt{n}\ln n\Big|_1^{\Lambda} - 4\lim_{\Lambda\to\infty}\sqrt{n}\Big|_1^{\Lambda}$$

$$= \lim_{\Lambda\to\infty} 2\sqrt{n}(-2+\ln n)\Big|_1^{\Lambda} = \infty$$

and the series diverges.

3. (a) $\sum_1^{\infty}\frac{e^n}{(e^n+1)^2}$ (b) $\sum_1^{\infty} ne^{-n}$ (c) $\sum_1^{\infty}\frac{1}{2^n}$

Solution

(a) $\frac{d}{dn}\frac{e^n}{(e^n+1)^2} = \frac{e^n(e^n+1)^2 - e^n 2(e^n+1)e^n}{(e^n+1)^4} = \frac{e^n(1-e^n)}{(e^n+1)^3} < 0$ so terms are decreasing.

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{e^n}{(e^n+1)^2}\,dn = \lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{e^n}{(e^n+1)^2}\,dn$$

$$= -\lim_{\Lambda\to\infty}\frac{1}{e^n+1}\Big|_1^{\Lambda}$$

$$= \frac{1}{e+1}$$

so series converges.

(b) $\frac{d}{dn}ne^{-n} = \frac{e^n - ne^n}{e^{2n}} = \frac{e^n(1-n)}{e^{2n}} < 0$ for $n>1$ so terms are decreasing.

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda} ne^{-n}dn = \lim_{\Lambda\to\infty} -ne^{-n}\Big|_1^{\Lambda} + \lim_{\Lambda\to\infty}\int_1^{\Lambda} e^{-n}dn = \lim_{\Lambda\to\infty} -ne^{-n}\Big|_1^{\Lambda} - \lim_{\Lambda\to\infty} e^{-n}\Big|_1^{\Lambda}$$

$$= 1 + e^{-1}$$

so series converges.

(c) The terms are obviously decreasing. The integral gives

$\lim_{\Lambda\to\infty}\int_1^{\Lambda} 2^{-n}\,dn = \lim_{\Lambda\to\infty}\frac{2^{-n}}{-\ln 2}\Big|_1^{\Lambda} = \frac{1}{2\ln 2}$ and so series converges.

4. (a) $\sum_1^{\infty}\frac{1}{2n+1}$ (b) $\sum_1^{\infty}\frac{1}{n+5}$ (c) $\sum_1^{\infty} ne^{-n^2}$

Solution

(a) The terms are obviously decreasing. The integral gives

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{2n+1}dn = \frac{1}{2}\lim_{\Lambda\to\infty}\ln(2n+1)\Big|_1^{\Lambda} = \infty$ and so series diverges.

(b) The terms are obviously decreasing. The integral gives

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n+5}dn = \lim_{\Lambda\to\infty}\ln(n+5)\Big|_1^{\Lambda} = \infty$ and so series diverges.

(c) $\frac{d}{dn}ne^{-n^2} = \frac{e^{n^2} - ne^{n^2}(2n)}{e^{2n^2}} = \frac{1-2n^2}{e^{n^2}} < 0$ and so terms are decreasing.

$\lim_{\Lambda\to\infty}\int_1^{\Lambda} ne^{-n^2}\,dn = -\frac{1}{2}\lim_{\Lambda\to\infty} e^{-n^2}\Big|_1^{\Lambda} = \frac{1}{2}$ so series converges.

5. (a) $\sum_2^{\infty}\frac{\ln n}{n}$ (b) $\sum_2^{\infty}\frac{n}{\ln n}$ (c) $\sum_1^{\infty}\frac{e^{-n}}{e^{-n}+1}$

Solution

(a) $\frac{d}{dn}\frac{\ln n}{n} = \frac{1-\ln n}{n^2} < 0$ for $n > 2$ so terms are decreasing. Making the substitution $u = \ln n \Rightarrow du = \frac{dn}{n}$ the integral is $\lim_{\Lambda\to\infty}\int_2^{\Lambda}\frac{\ln n}{n}dn = \lim_{\Lambda\to\infty}\int_{\ln 2}^{\ln\Lambda} u\,du = \lim_{\Lambda\to\infty}\frac{(\ln n)^2}{2})\Big|_2^{\Lambda} = \infty$ and the series diverges.

(b) $\frac{d}{dn}\frac{n}{\ln n} = \frac{\ln n - 1}{(\ln n)^2} > 0$ for $n > 2$ so test cannot be applied.

(c) The general term can be re-written as $\frac{e^n}{e^n}\left(\frac{e^{-n}}{e^{-n}+1}\right) = \frac{1}{1+e^n}$ and is obviously decreasing. The integral is

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{e^{-n}}{e^{-n}+1}dn = -\lim_{\Lambda\to\infty}\ln(e^{-n}+1)\Big|_1^{\Lambda} = -\lim_{\Lambda\to\infty}\ln(e^{-\Lambda}+1) + \ln(e^{-1}+1) = \ln(e^{-1}+1)$ and so series converges.

6. (a) $\sum_1^{\infty}\frac{1}{e^n+1}$ (b) $\sum_1^{\infty}\frac{1}{(n+1)(n+2)}$ (c) $\sum_1^{\infty}\frac{1}{n^2+1}$

Solution

(a) Terms are clearly decreasing.

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{e^n+1}dn = \lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{e^{-n}}{e^{-n}+1}dn = -\lim_{\Lambda\to\infty}\ln(e^{-n}+1)\Big|_1^{\Lambda} = -\lim_{\Lambda\to\infty}\ln(e^{-\Lambda}+1) + \ln(e^{-1}+1)$

$= \ln(e^{-1}+1)$

and so series converges.

(b) Terms are clearly decreasing.

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{(n+1)(n+2)}dn = \lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^2+2n+2}dn = \lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{(n+1)^2+1}dn$

$= \lim_{\Lambda\to\infty}\arctan(n+1)\Big|_1^{\Lambda} = \frac{\pi}{2} - \arctan 2$

and so series converges.

(c) Terms are clearly decreasing.

$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^2+1}dn = \lim_{\Lambda\to\infty}\arctan(n)\Big|_1^{\Lambda} = \frac{\pi}{2} - \arctan 1 = \frac{\pi}{4}$

and so series converges.

7. (a) $\sum_{1}^{\infty}\frac{1}{n^2+4n+5}$ (b) $\sum_{1}^{\infty}\sin\frac{\pi n}{2}$ (c) $\sum_{1}^{\infty}\frac{n^2}{n^4+1}$

Solution

(a) Terms are clearly decreasing.

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^2+4n+5}dn=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{(n+2)^2+1}dn=\lim_{\Lambda\to\infty}\arctan(n+2)\big|_1^{\Lambda}=\frac{\pi}{2}-\arctan 3$$

and so series converges.

(b) Test cannot be applied. The terms are not positive and are not decreasing.

(c) $\frac{d}{dn}\frac{n^2}{n^4+1}=\frac{2n(n^4+1)-n^2 4n^3}{(n^4+1)^2}=\frac{2n(1-n^4)}{(n^4+1)^2}<0$ so terms are decreasing. The integral is $\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{n^2}{n^4+1}dn<\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{n^2}{n^4}dn=\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^2}dn=-\lim_{\Lambda\to\infty}\frac{1}{n}\bigg|_1^{\Lambda}=1$ and so the series converges.

8. (a) $\sum_{1}^{\infty}\frac{1}{1+\sqrt{n}}$ (b) $\sum_{1}^{\infty}\frac{n}{2^n}$ (c) $\sum_{1}^{\infty}\frac{1}{n^3+1}$

Solution

(a) Terms are clearly decreasing. The integral is $\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{1+\sqrt{n}}dn$. Make the substitution $u=1+\sqrt{n}\Rightarrow du=\frac{1}{2\sqrt{n}}dn$. Then

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{1+\sqrt{n}}dn=2\lim_{\Lambda\to\infty}\int_2^{\Lambda}\frac{u-1}{u}du=2\lim_{\Lambda\to\infty}\left((\Lambda-2)-\ln\frac{\Lambda}{2}\right)=\infty$$, and so series diverges.

(b)

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}n2^{-n}\,dn=\lim_{\Lambda\to\infty}\frac{n2^{-n}}{-\ln 2}\bigg|_1^{\Lambda}+\frac{1}{\ln 2}\lim_{\Lambda\to\infty}\int_1^{\Lambda}2^{-n}\,dn=\lim_{\Lambda\to\infty}\frac{n2^{-n}}{-\ln 2}\bigg|_1^{\Lambda}+\frac{1}{\ln 2}\lim_{\Lambda\to\infty}\frac{2^{-n}}{-\ln 2}\bigg|_1^{\Lambda}$$

$$=-\frac{1}{\ln 2}\lim_{\Lambda\to\infty}\frac{\Lambda}{2^{\Lambda}}-\frac{1}{(\ln 2)^2}\lim_{\Lambda\to\infty}\frac{\Lambda}{2^{\Lambda}}+\frac{1+\ln 2}{2(\ln 2)^2}$$

Both limits are clearly zero and so series converges.

(c) $\frac{d}{dn}\frac{1}{n^3+1}=-\frac{3n^2}{(n^3+1)^2}<0$ so the terms are decreasing. The integral is

$$\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^3+1}dn<\lim_{\Lambda\to\infty}\int_1^{\Lambda}\frac{1}{n^3}dn=-\frac{1}{2}\lim_{\Lambda\to\infty}\frac{1}{n^2}\bigg|_1^{\Lambda}=\frac{1}{2}$$ and so the series converges.

9. For what values of p does the following series converge: $\sum_{2}^{\infty}\frac{1}{n(\ln n)^p}$?

Solution

Terms are obviously decreasing. $\int_2^\infty \frac{dn}{n(\ln n)^p} = \lim_{\Lambda\to\infty}\int_2^\Lambda \frac{dn}{n(\ln n)^p}$. Call $u = \ln n \Rightarrow du = \frac{dn}{n}$ and so the integral is $\int_2^\infty \frac{dn}{n(\ln n)^p} = \lim_{\Lambda\to\infty}\int_{\ln 2}^{\ln\Lambda} \frac{du}{u^p} = \lim_{\Lambda\to\infty} \left.\frac{u^{1-p}}{1-p}\right|_{\ln 2}^{\ln\Lambda} = \lim_{\Lambda\to\infty} \frac{(\ln\Lambda)^{1-p}}{1-p} - \frac{\ln 2^{1-p}}{1-p}$.
The limit is zero if $p > 1$ and infinite if $p < 1$. If $p = 1$ the integral must be redone to give

$$\int_2^\infty \frac{dn}{n(\ln n)} = \lim_{\Lambda\to\infty}\int_{\ln 2}^{\ln\Lambda} \frac{du}{u} = \lim_{\Lambda\to\infty} \ln u\Big|_{\ln 2}^{\ln\Lambda} = \lim_{\Lambda\to\infty}(\ln\ln\Lambda) - \ln\ln 2 \to \infty .$$

Hence the series converges for $p > 1$ and diverges for $p \le 1$.

SET 3.4

Apply the ratio test to these series. If the ratio test is inconclusive apply a different test.

1. (a) $\sum_1^\infty \frac{5}{3^n}$ (b) $\sum_1^\infty \frac{5^n}{3^n}$ (c) $\sum_1^\infty \frac{e^n}{\pi^n}$

Solution

(a) $\rho = \lim_{n\to\infty} \frac{\frac{5}{3^{n+1}}}{\frac{5}{3^n}} = \frac{1}{3} < 1$ so series converges.

(b) $\rho = \lim_{n\to\infty} \frac{\frac{5^{n+1}}{3^{n+1}}}{\frac{5^n}{3^n}} = \frac{5}{3} > 1$ so series diverges.

(c) $\rho = \lim_{n\to\infty} \frac{\frac{e^{n+1}}{\pi^{n+1}}}{\frac{e^n}{\pi^n}} = \frac{e}{\pi} < 1$ so series converges.

2. (a) $\sum_1^\infty \left(\frac{n+1}{n}\right)^n$ (b) $\sum_1^\infty \frac{7^n}{4^n+1}$ (c) $\sum_1^\infty e^{-n} n!$

Solution

(a) $\rho = \lim_{n\to\infty} \frac{\left(\frac{n+2}{n+1}\right)^{n+1}}{\left(\frac{n+1}{n}\right)^n} = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \frac{e}{e} = 1$ so test is inconclusive. (Series diverges by the divergence test.)

(b) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{7^{n+1}}{4^{n+1}+1}}{\dfrac{7^n}{4^n+1}} = \lim_{n\to\infty} \dfrac{7^{n+1}}{7^n}\dfrac{4^n+1}{4^{n+1}+1} = 7\lim_{n\to\infty} \dfrac{4^n(1+\dfrac{1}{4^n})}{4^{n+1}(1+\dfrac{1}{4^{n+1}})} = \dfrac{7}{4} > 1$, series diverges.

(c) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)!}{e^{n+1}}}{\dfrac{n!}{e^n}} = \lim_{n\to\infty} \dfrac{(n+1)}{e} = \infty$, series diverges.

3. (a) $\sum_1^\infty \dfrac{2^n}{n!}$ (b) $\sum_1^\infty \dfrac{1+n^2}{n^2}$ (c) $\sum_1^\infty \dfrac{2^n}{3^n+1}$

Solution

(a) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}}{(n+1)!}}{\dfrac{2^n}{n!}} = \lim_{n\to\infty} \dfrac{2}{n+1} = 0$, series converges.

(b)

$$\rho = \lim_{n\to\infty} \frac{\dfrac{1+(n+1)^2}{(n+1)^2}}{\dfrac{1+n^2}{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \left(\frac{1+(n+1)^2}{1+n^2}\right) = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 \frac{(n+1)^2}{n^2} \left(\frac{1+\dfrac{1}{(n+1)^2}}{1+\dfrac{1}{n^2}}\right) = 1$$

so test is inconclusive. (Series diverges by the divergence test.)

(c) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}}{3^{n+1}+1}}{\dfrac{2^n}{3^n+1}} = \lim_{n\to\infty} \dfrac{2^{n+1}}{2^n}\dfrac{3^n+1}{3^{n+1}+1} == 2\lim_{n\to\infty} \dfrac{3^n(1+\dfrac{1}{3^n})}{3^{n+1}(1+\dfrac{1}{3^{n+1}})} = \dfrac{2}{3} < 1$, series converges.

4. (a) $\sum_2^\infty \dfrac{\ln n}{n}$ (b) $\sum_2^\infty \dfrac{\ln n}{\sqrt{n}}$ (c) $\sum_1^\infty ne^{-n}$

Solution

(a) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{\ln(n+1)}{(n+1)}}{\dfrac{\ln n}{n}} = \lim_{n\to\infty} \dfrac{\ln(n+1)}{\ln n}\dfrac{n}{n+1} = 1$ by using L' Hôpital's rule on each fraction. Hence test is inconclusive. (Series diverges by the integral test.)

(b) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{\ln(n+1)}{\sqrt{n+1}}}{\dfrac{\ln n}{\sqrt{n}}} = \lim_{n\to\infty} \dfrac{\ln(n+1)}{\ln n}\sqrt{\dfrac{n}{n+1}} = 1\times\sqrt{1} = 1$ by using L' Hôpital's rule on each fraction. Hence test is inconclusive. (Series diverges by the integral test.)

(c) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{n+1}{e^{n+1}}}{\dfrac{n}{e^n}} = \lim_{n\to\infty} \dfrac{e^n}{e^{n+1}}\dfrac{n+1}{n} = \dfrac{1}{e} < 1$, series converges.

5. (a) $\sum_1^\infty \frac{1+n^4}{n!}$ (b) $\sum_1^\infty \frac{(2n)!}{3^n}$ (c) $\sum_2^\infty e^{-n}\ln n$

Solution

(a) $\rho = \lim_{n\to\infty} \frac{\frac{1+(n+1)^4}{(n+1)!}}{\frac{1+n^4}{n!}} = \lim_{n\to\infty} \frac{1}{n+1}\frac{1+(n+1)^4}{1+n^4} = 0\times 1 = 0 < 1$, series converges.

(b) $\rho = \lim_{n\to\infty} \frac{\frac{(2n+2)!}{3^{n+1}}}{\frac{(2n)!}{3^n}} = \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{3} \to \infty$, series diverges.

(c) $\rho = \lim_{n\to\infty} \frac{\frac{\ln(n+1)}{e^{n+1}}}{\frac{\ln n}{e^n}} = \lim_{n\to\infty} \frac{1}{e}\frac{\ln(n+1)}{\ln n} = \frac{1}{e} < 1$ (used L' Hôpital's rule), so series converges.

6. (a) $\sum_1^\infty \frac{5n}{3^n}$ (b) $\sum_1^\infty \frac{n}{n^2+1}$ (c) $\sum_1^\infty \left(\frac{2n+1}{5n-3}\right)^n$

Solution

(a) $\rho = \lim_{n\to\infty} \frac{\frac{5(n+1)}{3^{n+1}}}{\frac{5n}{3^n}} = \lim_{n\to\infty} \frac{3^n}{3^{n+1}}\frac{n+1}{n} = \frac{1}{3} < 1$, series converges.

(b) $\rho = \lim_{n\to\infty} \frac{\frac{n+1}{(n+1)^2+1}}{\frac{n}{n^2+1}} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\frac{n+1}{n} = \lim_{n\to\infty} \frac{n^2+1}{(n+1)^2+1}\lim_{n\to\infty}\frac{n+1}{n} = 1$ by using L' Hôpital's rule, so test is inconclusive. (Series diverges by the integral test.)

(c)

$$\rho = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+2}\right)^{n+1}}{\left(\frac{2n+1}{5n-3}\right)^n} = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{5n+2}\right)^{n+1}}{\left(\frac{2n+1}{5n-3}\right)^{n+1}}\frac{2n+1}{5n-3}$$

$$= \lim_{n\to\infty} \frac{(2n)^{n+1}(1+\frac{3}{2n})^{n+1}}{(5n)^{n+1}(1+\frac{2}{5n})^{n+1}}\frac{(5n)^{n+1}(1-\frac{3}{5n})^{n+1}}{(2n)^{n+1}(1+\frac{1}{2n})^{n+1}}\frac{2n+1}{5n-3}$$

$$= \frac{e^{3/2}}{e^{2/5}}\frac{e^{-3/5}}{e^{1/2}}\frac{2}{5}$$

$$= \frac{2}{5} < 1$$

hence series converges.

7. (a) $\sum_1^\infty \frac{2^{n+1}n^3}{3^n}$ (b) $\sum_1^\infty \left(\frac{5}{9n+1}\right)^n$ (c) $\sum_1^\infty \left(\frac{5n+3}{2n-1}\right)^n$

Solution

(a) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+2}(n+1)^3}{3^{n+1}}}{\dfrac{2^{n+1}n^3}{3^n}} = \dfrac{2}{3}\lim_{n\to\infty}\left(\dfrac{n+1}{n}\right)^3 = \dfrac{2}{3}\left(\lim_{n\to\infty}\dfrac{n+1}{n}\right)^3 = \dfrac{2}{3} < 1$ so series converges.

(b)

$$\rho = \lim_{n\to\infty} \frac{\left(\dfrac{5}{9n+10}\right)^{n+1}}{\left(\dfrac{5}{9n+1}\right)^{n}} = 5\lim_{n\to\infty}\left(\frac{9n+1}{9n+10}\right)^n \frac{1}{9n+10} = 5\lim_{n\to\infty}\frac{(1+\dfrac{1}{9n})^n}{(1+\dfrac{10}{9n})^n}\frac{1}{9n+10}$$

$$= 5\frac{e^{1/9}}{e^{10/9}}\lim_{n\to\infty}\frac{1}{9n+10} = 0 < 1$$

so series converges.

(c) Working as in 6 (c) the limit is $\rho = \dfrac{5}{2} > 1$ so series diverges.

8. (a) $\displaystyle\sum_1^\infty \frac{2^n+1}{3^n}$ (b) $\displaystyle\sum_1^\infty \frac{3^n}{2^n+1}$ (c) $\displaystyle\sum_1^\infty \frac{n^3}{n!}$

Solution

(a) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{2^{n+1}+1}{3^{n+1}}}{\dfrac{2^n+1}{3^n}} = \dfrac{1}{3}\lim_{n\to\infty}\dfrac{2^{n+1}+1}{2^n+1} = \dfrac{1}{3}\lim_{n\to\infty}\dfrac{2^n(1+\dfrac{1}{2^n})}{2^{n+1}(1+\dfrac{1}{2^{n+1}})} = \dfrac{2}{3} < 1$, series converges.

(b) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{3^{n+1}}{2^{n+1}+1}}{\dfrac{3^n}{2^n+1}} = 3\lim_{n\to\infty}\dfrac{2^n+1}{2^{n+1}+1} = 3\lim_{n\to\infty}\dfrac{2^n(1+\dfrac{1}{2^n})}{2^{n+1}(1+\dfrac{1}{2^{n+1}})} = \dfrac{3}{2} > 1$, series diverges.

(c) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^3}{(n+1)!}}{\dfrac{n^3}{n!}} = \lim_{n\to\infty}\dfrac{1}{n+1}\left(\dfrac{n+1}{n}\right)^3 = 0\times 1 = 0 < 1$, series converges.

9. (a) $\displaystyle\sum_1^\infty \frac{n^n}{n!}$ (b) $\displaystyle\sum_1^\infty \frac{n!}{(2n)^n}$ (c) $\displaystyle\sum_1^\infty \frac{n^n}{(n!)^2}$

Solution

(a) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{(n+1)^{n+1}}{(n+1)!}}{\dfrac{n^n}{n!}} = \lim_{n\to\infty}\dfrac{n!}{(n+1)!}\left(\dfrac{n+1}{n}\right)^{n+1} n = \lim_{n\to\infty}\dfrac{n}{n+1}\left(1+\dfrac{1}{n}\right)^{n+1} = 1\times e = e > 1$,

series diverges.

(b) $$\rho = \lim_{n\to\infty} \frac{\dfrac{(n+1)!}{(2n+2)^{n+1}}}{\dfrac{n!}{(2n)^n}} = \lim_{n\to\infty} \frac{(n+1)!}{n!}\left(\frac{2n}{2n+2}\right)^{n+1}\frac{1}{2n} = \lim_{n\to\infty}\frac{n+1}{2n}\left(1-\frac{1}{n+1}\right)^{n+1}$$

$$= \frac{1}{2}\times e^{-1} < 1$$

and the series converges.

(c)

$$\rho = \lim_{n\to\infty} \frac{\dfrac{(n+1)^{n+1}}{((n+1)!)^2}}{\dfrac{n^n}{(n!)^2}} = \lim_{n\to\infty}\left(\frac{n!}{(n+1)!}\right)^2\left(\frac{n+1}{n}\right)^{n+1} n = \lim_{n\to\infty}\frac{n}{(n+1)^2}\left(1+\frac{1}{n}\right)^{n+1} = 0\times e = 0 < 1,$$

series converges.

10. (a) $\sum_1^\infty \frac{5^n+1}{3^n}$ (b) $\sum_1^\infty \frac{3^n+1}{n5^n}$ (c) $\sum_1^\infty \frac{n}{3^n}$

(a) $$\rho = \lim_{n\to\infty} \frac{\dfrac{5^{n+1}+1}{3^{n+1}}}{\dfrac{5^n+1}{3^n}} = \frac{1}{3}\lim_{n\to\infty}\frac{5^{n+1}+1}{5^n+1} = \frac{1}{3}\lim_{n\to\infty}\frac{5^{n+1}(1+\frac{1}{5^{n+1}})}{5^n(1+\frac{1}{5^n})} = \frac{1}{3}\times 5 = \frac{5}{3} > 1$$

and so the series diverges.

(b)

$$\rho = \lim_{n\to\infty} \frac{\dfrac{3^{n+1}+1}{(n+1)5^{n+1}}}{\dfrac{3^n+1}{n5^n}} = \frac{1}{5}\lim_{n\to\infty}\frac{n}{n+1}\frac{3^{n+1}+1}{3^n+1} = \frac{1}{5}\lim_{n\to\infty}\frac{n}{n+1}\frac{3^{n+1}(1+\frac{1}{3^{n+1}})}{3^n(1+\frac{1}{3^n})} = \frac{1}{5}\times 1\times 3 = \frac{3}{5} < 1,$$

series converges.

(c) $\rho = \lim_{n\to\infty} \frac{\frac{n+1}{3^{n+1}}}{\frac{n}{3^n}} = \frac{1}{3}\lim_{n\to\infty}\frac{n+1}{n} = \frac{1}{3}\times 1 = \frac{1}{3} < 1$ and the series converges.

SET 3.5

Determine the convergence or divergence of the series by using one of the comparison tests.

1. (a) $\sum_1^\infty \frac{1}{\sqrt{n(n+1)}}$ (b) $\sum_1^\infty \frac{10}{3^n n!}$ (c) $\sum_2^\infty \frac{2}{\ln n}$

Solution

(a) Series behaves like the divergent $\sum_1^\infty \frac{1}{n}$. Using the limit comparison test we have that

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n(n+1)}}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n}{\sqrt{n(n+1)}}=1$$ and so the series diverges.

(b) Since $\frac{10}{3^n n!}<\frac{10}{3^n}$ and the series $\sum_1^\infty \frac{10}{3^n}$ converges being geometric, the other series converges as well.

(c) Since $\frac{2}{\ln n}>\frac{1}{\ln n}>\frac{1}{n}$ and the series $\sum_2^\infty \frac{1}{n}$ diverges, the other series diverges as well.

2. (a) $\sum_1^\infty \frac{1}{3^n+1}$ (b) $\sum_1^\infty \frac{\sqrt{n}}{n^2+1}$ (c) $\sum_2^\infty \left(\frac{\ln n}{n}\right)^2$

Solution

(a) $3^n+1>3^n \Rightarrow \frac{1}{3^n+1}<\frac{1}{3^n}$. Hence series converges because $\sum_1^\infty \frac{1}{3^n}$ does.

(b) $\frac{\sqrt{n}}{n^2+1}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}$. Hence series converges because $\sum_1^\infty \frac{1}{n^{3/2}}$ does.

(c) For large enough n, $\ln n<n^{1/8}\Rightarrow (\ln n)^2<n^{1/4}\Rightarrow \left(\frac{\ln n}{n}\right)^2<\frac{n^{1/4}}{n^2}=\frac{1}{n^{3/4}}$. Hence series converges.

3. (a) $\sum_2^\infty \frac{5}{n^4-1}$ (b) $\sum_1^\infty \frac{2^n-1}{3^n-1}$ (c) $\sum_1^\infty \frac{2}{3n+1}$

Solution

(a) Use the limit comparison test with the convergent series $\sum_1^\infty \frac{5}{n^4}$.

$$\lim_{n\to\infty}\frac{\frac{5}{n^4-1}}{\frac{5}{n^4}}=\lim_{n\to\infty}\frac{n^4}{n^4-1}=1$$ so the series converges.

(b) Use the direct comparison test with the convergent series $\sum_1^\infty \frac{2^n}{2.5^n}$.

$\frac{2^n-1}{3^n-1}<\frac{2^n}{3^n-1}<\frac{2^n}{2.5^n}$ (because $3^n-1>2.5^n$ for $n>1$) and so the series converges.

(c) $\frac{2}{3n+1}>\frac{2}{3n+n}=\frac{1}{2n}$ and so series diverges because $\sum_1^\infty \frac{1}{2n}$ does.

4. (a) $\sum_2^\infty \frac{\ln n}{n^{3/2}}$ (b) $\sum_1^\infty \frac{7}{n^2+n+1}$ (c) $\sum_1^\infty \frac{2n+3}{n3^n}$

Solution

(a) $\frac{\ln n}{n^{3/2}}<\frac{n^{1/4}}{n^{3/2}}=\frac{1}{n^{5/4}}$ and so series converges since $\sum_2^\infty \frac{1}{n^{5/4}}$ does.

(b) $\frac{7}{n^2+n+1}<\frac{7}{n^2}$ and so series converges since $\sum_1^\infty \frac{7}{n^2}$ does.

(c) Compare with the convergent $\sum_1^\infty \frac{1}{3^n}$ using the limit comparison test:

$$\lim_{n\to\infty}\frac{\frac{2n+3}{n3^n}}{\frac{1}{3^n}}=\lim_{n\to\infty}\frac{2n+2}{n}=2$$ and so series converges.

5. (a) $\sum_2^\infty \frac{\ln n}{n^2}$ (b) $\sum_1^\infty \sin^2\frac{1}{n}$ (c) $\sum_2^\infty \frac{n+5}{2n^2-1}$

Solution

(a) $\frac{\ln n}{n^2}<\frac{n^{1/4}}{n^2}=\frac{1}{n^{7/4}}$ and so the series converges because $\sum_1^\infty \frac{1}{n^{7/4}}$ does.

(b) Compare with the convergent series $\sum_1^\infty \frac{1}{n^2}$:

$$\lim_{n\to\infty}\frac{\sin^2\frac{1}{n}}{\frac{1}{n^2}}=\lim_{n\to\infty}\left(\frac{\sin\frac{1}{n}}{\frac{1}{n}}\right)^2=1$$

and so the series converges.

(c) The series behaves as the divergent $\sum_2^\infty \frac{1}{n}$ and since $\lim_{n\to\infty}\frac{\frac{n+5}{2n^2-1}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{n^2+5n}{2n^2-1}=\frac{1}{2}$ the series diverges.

6. (a) $\sum_1^\infty \frac{1}{2^n-1}$ (b) $\sum_1^\infty \frac{1}{\sqrt{n(n+1)(n+2)}}$ (c) $\sum_2^\infty \frac{n}{\sqrt{n^3-1}}$

Solution

(a) Use the limit comparison test with the convergent series $\sum_1^\infty \frac{1}{2^n}$:

$$\lim_{n\to\infty}\frac{\frac{1}{2^n-1}}{\frac{1}{2^n}}=1$$ and so the series converges.

(b) Use the limit comparison test with the convergent series $\sum_1^\infty \frac{1}{n^{3/2}}$:

$$\lim_{n\to\infty}\frac{\frac{1}{\sqrt{n(n+1)(n+2)}}}{\frac{1}{n^{3/2}}}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n(n+1)(n+2)}}=1$$ and so the series converges.

(c) Use the limit comparison test with the divergent series $\sum_1^\infty \frac{1}{n^{1/2}}$:

$$\lim_{n\to\infty}\frac{\frac{n}{\sqrt{n^3-1}}}{\frac{1}{n^{1/2}}}=\lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3-1}}=1$$ and so the series diverges.

7. (a) $\sum_1^\infty \frac{\sqrt{n+1}}{\sqrt{n^2+1}}$ (b) $\sum_2^\infty \frac{\ln n}{n^2+1}$ (c) $\sum_2^\infty \frac{1}{\ln\ln n}$

Solution

(a) Use the limit comparison test with the divergent series $\sum_1^\infty \frac{1}{n^{1/2}}$:

$$\lim_{n\to\infty}\frac{\frac{\sqrt{n+1}}{\sqrt{n^2+1}}}{\frac{1}{n^{1/2}}}=\lim_{n\to\infty}\frac{\sqrt{n^2+n}}{\sqrt{n^2+1}}=1$$ and so the series diverges.

(b) $\frac{\ln n}{n^2+1}<\frac{n^{1/4}}{n^2}=\frac{1}{n^{7/4}}$ and so series converges since $\sum_1^\infty \frac{1}{n^{7/4}}$ does.

(c) $\ln(\ln n)<(\ln n)^{1/2}<\left(n^{1/2}\right)^{1/2}=n^{1/4}$. Hence $\frac{1}{\ln(\ln n)}>\frac{1}{n^{1/4}}$ and so the series diverges.

8. (a) $\sum_2^\infty \sin\frac{\pi}{n}$ (b) $\sum_1^\infty \frac{2^{1/n}}{n^{1/2}}$ (c) $\sum_1^\infty \frac{1}{n!}$

Solution

(a) Compare with the divergent $\sum_2^\infty \frac{1}{n}$: $\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{1}{n}}=\pi$ and so the series diverges.

(b) Compare with the divergent series $\sum_1^\infty \frac{1}{n^{1/2}}$:

$$\lim_{n\to\infty}\frac{\frac{2^{1/n}}{n^{1/2}}}{\frac{1}{n^{1/2}}}=\lim_{n\to\infty}2^{1/n}=1$$ and so the series diverges.

(c) $\frac{1}{n!}<\frac{1}{n^2}$ for $n>3$ and so the series converges.

9. (a) $\sum_1^\infty \frac{1+\cos n}{4^n+n^4}$ (b) $\sum_1^\infty \frac{4^n}{n^4}$ (c) $\sum_1^\infty \frac{n^4}{4^n}$

Solution

(a) We have that $\frac{1+\cos n}{4^n+n^4}<\frac{2}{4^n+n^4}$. Compare the series $\sum_1^\infty \frac{1}{4^n+n^4}$ with $\sum_1^\infty \frac{1}{4^n}$:

$$\lim_{n\to\infty}\frac{\frac{2}{4^n+n^4}}{\frac{1}{4^n}}=\lim_{n\to\infty}\frac{2\times 4^n}{4^n+n^4}=\lim_{n\to\infty}\frac{2\times 4^n}{4^n(1+\frac{n^4}{4^n})}=2$$. Hence $\sum_1^\infty \frac{1+\cos n}{4^n+n^4}$ converges.

(b) Because $n^4 < 2^n$ for $n > 16$ (proof by induction), $\frac{4^n}{n^4} > \frac{4^n}{2^n}$. Since $\sum_1^\infty \frac{4^n}{2^n}$ diverges so does $\sum_1^\infty \frac{4^n}{n^4}$.

(c) $\frac{n^4}{4^n} < \frac{2^n}{4^n}$ for $n > 16$. Since $\sum_1^\infty \frac{2^n}{4^n}$ converges so does $\sum_1^\infty \frac{n^4}{4^n}$.

10. (a) $\sum_1^\infty \frac{3^n}{n!}$ (b) $\sum_1^\infty \frac{1}{2^n n!}$ (c) $\sum_1^\infty \frac{n!}{5^n}$

Solution

(a) $3^n < \left(\frac{n}{2}\right)!$ for $n \geq 44$. Hence $\frac{3^n}{n!} < \frac{\left(\frac{n}{2}\right)!}{n!} < \frac{1}{n^2}$ and so series converges.

(b) $\lim_{n\to\infty} \frac{\frac{1}{2^n n!}}{\frac{1}{n!}} = 0$. Since $\sum_1^\infty \frac{1}{n!}$ converges so does $\sum_1^\infty \frac{1}{2^n n!}$.

(c) $5^n < \left(\frac{n}{2}\right)!$ for $n \geq 130$. Hence $\frac{n!}{5^n} > \frac{n!}{\left(\frac{n}{2}\right)!} > n^2$. Hence series diverges.

11. (a) $\sum_1^\infty \frac{1}{n^n}$ (b) $\sum_1^\infty \frac{\sin^2 n}{n!}$ (c) $\sum_1^\infty \frac{3}{n+3}$

Solution

(a) Since $n^n > n^2$ we have that $\frac{1}{n^n} < \frac{1}{n^2}$. But $\sum_1^\infty \frac{1}{n^2}$ converges and thus so does $\sum_1^\infty \frac{1}{n^n}$.

(b) Since $\frac{\sin^2 n}{n!} < \frac{1}{n!}$, series converges.

(c) Using the limit comparison test with the diverging series $\sum_1^\infty \frac{1}{n}$: we have

$\lim_{n\to\infty} \frac{\frac{3}{n+3}}{\frac{1}{n}} = 3$ and so series diverges.

SET 3.6

Apply any test to determine the convergence or divergence of the following infinite series.

1. (a) $\sum_1^\infty ne^{-n^2}$ (b) $\sum_1^\infty \frac{3^k}{4^k+1}$ (c) $\sum_1^\infty \frac{k!}{5^k}$

Solution

(a) $\frac{d}{dn} ne^{-n^2} = \frac{e^{n^2} - ne^{n^2}(2n)}{e^{2n^2}} = \frac{1-2n^2}{e^{n^2}} < 0$ and so terms are decreasing.

$\lim\limits_{\Lambda\to\infty}\int_1^{\Lambda} ne^{-n^2}\,dn = -\frac{1}{2}\lim\limits_{\Lambda\to\infty} e^{-n^2}\Big|_1^{\Lambda} = \frac{1}{2}$ so series converges.

(b) By the ratio test, $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{3^{k+1}}{4^{k+1}+1}}{\dfrac{3^k}{4^k+1}} = \lim\limits_{k\to\infty}\dfrac{3^{k+1}}{3^k}\lim\limits_{k\to\infty}\dfrac{4^k+1}{4^{k+1}+1} = \dfrac{3}{4} < 1$ so series converges.

(c) By the ratio test, $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{(k+1)!}{5^{k+1}}}{\dfrac{k!}{5^k}} = \lim\limits_{k\to\infty}\dfrac{5^k}{5^{k+1}}\lim\limits_{k\to\infty}(k+1) = \infty$ so series diverges.

2. (a) $\sum_1^\infty \frac{1}{e^k}$ (b) $\sum_1^\infty \frac{k^3}{k^4+k+1}$ (c) $\sum_1^\infty \frac{3^k+2^k}{5^k}$

Solution

(a) By the ratio test, $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{1}{e^{k+1}}}{\dfrac{1}{e^k}} = \dfrac{1}{e} < 1$ so series converges.

(b) Series behaves like the divergent $\sum_1^\infty \frac{1}{k}$: $\lim\limits_{k\to\infty}\dfrac{\dfrac{k^3}{k^4+k+1}}{\dfrac{1}{k}} = \lim\limits_{k\to\infty}\dfrac{k^4}{k^4+k+1} = 1$ so series diverges.

(c) By the ratio test, $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{3^{k+1}+2^{k+1}}{5^{k+1}}}{\dfrac{3^k+2^k}{5^k}} = \lim\limits_{k\to\infty}\dfrac{5^k}{5^{k+1}}\lim\limits_{k\to\infty}\dfrac{3^{k+1}+2^{k+1}}{3^k+2^k} = \dfrac{3}{5} < 1$ so series converges.

3. (a) $\sum_1^\infty \frac{\arctan k}{k}$ (b) $\sum_1^\infty \frac{1}{2^k k!}$ (c) $\sum_1^\infty \frac{3^k}{k^3}$

Solution

(a) Compare with $\sum_1^\infty \frac{1}{k}$: $\lim\limits_{k\to\infty}\dfrac{\dfrac{\arctan k}{k}}{\dfrac{1}{k}} = \lim\limits_{k\to\infty}\arctan k = \dfrac{\pi}{2}$ and so series diverges.

(b) By the ratio test $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{1}{2^{k+1}(k+1)!}}{\dfrac{1}{2^k k!}} = \dfrac{1}{2}\times\lim\limits_{k\to\infty}\dfrac{1}{k+1} = 0 < 1$ and the series converges.

(c) By the ratio test $\rho = \lim\limits_{k\to\infty}\dfrac{\dfrac{3^{k+1}}{(k+1)^3}}{\dfrac{3^k}{k^3}} = 3\times\lim\limits_{k\to\infty}\left(\dfrac{k}{k+1}\right)^3 = 3 > 1$ and the series diverges.

4. (a) $\sum_1^\infty \frac{2^k}{3^k\sqrt{k}}$ (b) $\sum_1^\infty \frac{1+|\cos k|}{k}$ (c) $\sum_1^\infty \frac{k^4}{4^k}$

Solution

(a) By the ratio test $\rho = \lim_{k\to\infty} \frac{\frac{2^{k+1}}{3^{k+1}\sqrt{k+1}}}{\frac{2^k}{3^k\sqrt{k}}} = \frac{2}{3} < 1$ and the series converges.

(b) $\frac{1+|\cos k|}{k} > \frac{1}{k}$ and so the series diverges by the direct comparison test.

(c) By the ratio test $\rho = \lim_{k\to\infty} \frac{\frac{(k+1)^4}{4^{k+1}}}{\frac{k^4}{4^k}} = \frac{1}{4} < 1$ and the series converges.

5. (a) $\sum_1^\infty \frac{1}{k^{1+\frac{1}{k}}}$ (b) $\sum_1^\infty \frac{1}{2^k+1}$ (c) $\sum_1^\infty \left(\frac{k-2}{k}\right)^k$

Solution

(a) Compare with $\sum_1^\infty \frac{1}{k}$: $\lim_{k\to\infty} \frac{\frac{1}{k^{1+1/k}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k^{1+1/k}} = \lim_{k\to\infty} \frac{1}{k^{\frac{1}{k}}} = 1$ and so series diverges by the limit comparison test.

(b) $\frac{1}{2^k+1} < \frac{1}{2^k}$ and so series converges since $\sum_1^\infty \frac{1}{2^k}$ does.

(c) $\lim_{k\to\infty} \left(\frac{k-2}{k}\right)^k = e^{-2} \neq 0$ so series diverges by the divergence test.

6. (a) $\sum_2^\infty \frac{1}{\sqrt{k^2-1}}$ (b) $\sum_2^\infty \frac{1}{\ln k}$ (c) $\sum_2^\infty \frac{1}{\ln(\ln k)}$

Solution

(a) Series behaves like $\sum_2^\infty \frac{1}{k}$: $\lim_{k\to\infty} \frac{\frac{1}{\sqrt{k^2-1}}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{\sqrt{k^2-1}} = \lim_{k\to\infty} \frac{k}{k\sqrt{1-\frac{1}{k^2}}} = 1$ and so series diverges.

(b) $\ln k < k^{1/2} \Rightarrow \frac{1}{\ln k} > \frac{1}{k^{1/2}}$ and so series diverges.

(c) $\ln(\ln k) < \ln k < k \Rightarrow \frac{1}{\ln(\ln k)} > \frac{1}{\ln k} > \frac{1}{k}$ and so series diverges since $\sum_2^\infty \frac{1}{k}$ does.

7. (a) $\sum_1^\infty \frac{k}{7^k}$ (b) $\sum_1^\infty \frac{\sqrt{k}+1}{k+1}$ (c) $\sum_2^\infty \frac{(2k)!}{(k!)^2}$

Solution

(a) $\rho = \lim_{k\to\infty}\dfrac{\dfrac{k+1}{7^{k+1}}}{\dfrac{k}{7^k}} = \lim_{k\to\infty}\dfrac{7^k}{7^{k+1}}\lim_{k\to\infty}\dfrac{k+1}{k!} = \dfrac{1}{7}\times 1 = \dfrac{1}{7} < 1$ and the series converges.

(b) Compare with $\sum_1^\infty \dfrac{1}{\sqrt{k}}$: $\lim_{k\to\infty}\dfrac{\dfrac{\sqrt{k}+1}{k+1}}{\dfrac{1}{\sqrt{k}}} = \lim_{k\to\infty}\dfrac{k+\sqrt{k}}{k+1} = 1$ and so series diverges since $\sum_1^\infty \dfrac{1}{\sqrt{k}}$ does.

(c) By the ratio test series diverges because:

$$\rho = \lim_{k\to\infty}\frac{\dfrac{(2k+2)!}{((k+1)!)^2}}{\dfrac{(2k)!}{(k!)^2}} = \lim_{k\to\infty}\frac{(2k+2)!}{(2k)!}\lim_{k\to\infty}\frac{(k!)^2}{((k+1)!)^2} = \lim_{k\to\infty}\frac{(2k+2)(2k+1)}{(k+1)^2} = 4\times 1 > 1$$

8. (a) $\sum_1^\infty \dfrac{k!}{e^k}$ (b) $\sum_1^\infty \dfrac{k}{(k+1)(k+2)(k+3)}$ (c) $\sum_1^\infty \dfrac{1}{\sqrt{k}+k}$

Solution

(a) By the ratio test the series diverges because:

$$\rho = \lim_{k\to\infty}\frac{\dfrac{(k+1)!}{e^{k+1}}}{\dfrac{k!}{e^k}} = \lim_{k\to\infty}\frac{e^k}{e^{k+1}}\lim_{k\to\infty}\frac{(k+1)!}{k!} = \frac{1}{e}\times\infty = \infty$$

(b) $\dfrac{k}{(k+1)(k+2)(k+3)} < \dfrac{k}{k^3} = \dfrac{1}{k^2}$ and so the series converges by the direct comparison test.

(c) Compare with $\sum_1^\infty \dfrac{1}{k}$: $\lim_{k\to\infty}\dfrac{\dfrac{1}{k+\sqrt{k}}}{\dfrac{1}{k}} = \lim_{k\to\infty}\dfrac{k}{k+\sqrt{k}} = 1$ and so series diverges since $\sum_1^\infty \dfrac{1}{k}$ does.

9. (a) $\sum_1^\infty \dfrac{k^{100}}{k!}$ (b) $\sum_1^\infty \dfrac{1}{\ln k + k}$ (c) $\sum_1^\infty \dfrac{4^k}{5^k+2^k}$

Solution

(a) By the ratio test the series converges because:

$$\rho = \lim_{k\to\infty}\frac{\dfrac{(k+1)^{100}}{(k+1)!}}{\dfrac{k^{100}}{k!}} = \lim_{k\to\infty}\left(\frac{k+1}{k}\right)^{100}\lim_{k\to\infty}\frac{k!}{(k+1)!} = 1\times 0 = 0$$

(b) Comparing with the series $\sum_1^{\infty}\frac{1}{k}$: $\rho=\lim_{k\to\infty}\frac{\frac{1}{\ln k+k}}{\frac{1}{k}}=\lim_{n\to\infty}\frac{k}{\ln k+k}=1$ and so the series diverges.

(c) By the ratio test the series converges because:

$$\rho=\lim_{k\to\infty}\frac{\frac{4^{k+1}}{5^{k+1}+2^{k+1}}}{\frac{4^k}{5^k+2^k}}=\lim_{k\to\infty}\frac{4^{k+1}}{4^k}\lim_{k\to\infty}\frac{5^k+2^k}{5^{k+1}+2^{k+1}}=4\times\frac{1}{5}=\frac{4}{5}<1$$

10. (a) $\sum_1^{\infty}\frac{1}{1+2+\cdots+k}$ (b) $\sum_1^{\infty}\frac{1}{1^2+2^2+\cdots+k^2}$ (c) $\sum_1^{\infty}\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{k}}$

Solution

(a) Since $1+2+\cdots+k=\frac{k(k+1)}{2}>\frac{k^2}{2}$, we have that $\frac{1}{1+2+\cdots+k}<\frac{2}{k^2}$. But $\sum_1^{\infty}\frac{2}{k^2}$ converges so by the direct comparison test so does $\sum_1^{\infty}\frac{1}{1+2+\cdots+k}$.

(b) Since $1^2+2^2+\cdots+k^2>k^2$, we have that $\frac{1}{1^2+2^2+\cdots+k^2}<\frac{1}{k^2}$ and so series converges since $\sum_1^{\infty}\frac{1}{k^2}$ converges.

(c) From Chapter 2 we know that $1+\frac{1}{2}+\cdots+\frac{1}{k}<1+\ln k<1+k$. Hence $\frac{1}{1+\frac{1}{2}+\cdots+\frac{1}{k}}>\frac{1}{k+1}$. But $\sum_1^{\infty}\frac{1}{1+k}$ diverges (using the integral test for example) and thus so does our series.

11. (a) $\sum_1^{\infty}k\left(\frac{1}{2}\right)^k$ (b) $\sum_1^{\infty}\frac{k!2^k}{3^k}$ (c) $\sum_1^{\infty}\frac{(k+2)!}{k!2^k}$

Solution

(a) Using the ratio test series converges because

$$\rho=\lim_{n\to\infty}\frac{(k+1)\left(\frac{1}{2}\right)^{k+1}}{k\left(\frac{1}{2}\right)^k}=\frac{1}{2}$$

(b) $\rho=\lim_{n\to\infty}\frac{\frac{(k+1)!2^{k+1}}{3^{k+1}}}{\frac{k!2^k}{3^k}}=\frac{2}{3}\lim_{n\to\infty}\frac{(k+1)!}{k!}=\frac{2}{3}\lim_{n\to\infty}(k+1)=\infty$ and so series diverges.

(c) $\rho = \lim_{n\to\infty} \dfrac{\dfrac{(k+3)!}{2^{k+1}(k+1)!}}{\dfrac{(k+2)!}{2^k k!}} = \dfrac{1}{2}\lim_{n\to\infty}\dfrac{k!(k+3)!}{(k+1)!(k+2)!} = \dfrac{1}{2}\lim_{n\to\infty}\dfrac{(k+3)}{(k+1)} = \dfrac{1}{2} < 1$ and so series converges.

12. (a) $\displaystyle\sum_1^\infty \frac{\sqrt{k+1}-\sqrt{k-1}}{\sqrt{k}}$ (b) $\displaystyle\sum_1^\infty \frac{\sqrt{k^2+1}-\sqrt{k^2-1}}{k}$ (c) $\displaystyle\sum_1^\infty \frac{(k^2+1)^4}{(k^3+2)^3}$

Solution

(a) Using the limit comparison test and comparing with the divergent series $\displaystyle\sum_1^\infty \frac{1}{k}$ we have that:

$$\begin{aligned}
\lim_{k\to\infty}\frac{\dfrac{\sqrt{k+1}-\sqrt{k-1}}{\sqrt{k}}}{\dfrac{1}{k}} &= \lim_{k\to\infty}\sqrt{k}(\sqrt{k+1}-\sqrt{k-1}) \\
&= \lim_{k\to\infty}\frac{\sqrt{k}(\sqrt{k+1}-\sqrt{k-1})(\sqrt{k+1}+\sqrt{k-1})}{\sqrt{k+1}+\sqrt{k-1}} \\
&= \lim_{k\to\infty}\frac{2\sqrt{k}}{\sqrt{k+1}+\sqrt{k-1}} = \lim_{k\to\infty}\frac{2\sqrt{k}}{\sqrt{k}(\sqrt{1+1/\sqrt{k}}+\sqrt{1-1/\sqrt{k}})} \\
&= 1
\end{aligned}$$

and so the series diverges.

(b) Compare with the convergent series $\displaystyle\sum_1^\infty \frac{1}{k^2}$.

$$\begin{aligned}
\lim_{k\to\infty}\frac{\dfrac{\sqrt{k^2+1}-\sqrt{k^2-1}}{k}}{\dfrac{1}{k^2}} &= \lim_{k\to\infty} k(\sqrt{k^2+1}-\sqrt{k^2-1}) \\
&= \lim_{k\to\infty}\frac{k(\sqrt{k^2+1}-\sqrt{k^2-1})(\sqrt{k^2+1}+\sqrt{k^2-1})}{\sqrt{k^2+1}+\sqrt{k^2-1}} \\
&= \lim_{k\to\infty}\frac{2k}{\sqrt{k^2+1}+\sqrt{k^2-1}} = \lim_{k\to\infty}\frac{2k}{k(\sqrt{1+1/k^2}+\sqrt{1-1/k^2})} \\
&= 1
\end{aligned}$$

and so series converges.

(c) Compare with the divergent series $\displaystyle\sum_1^\infty \frac{1}{k}$.

$$\lim_{k\to\infty}\frac{\dfrac{(k^2+1)^4}{(k^3+2)^3}}{\dfrac{1}{k}}=\lim_{k\to\infty}\frac{k(k^2+1)^4}{(k^3+2)^3}$$

$$=\lim_{k\to\infty}\frac{k^9(1+1/k^2)^4}{k^9(1+2/k^3)^3}$$

$$=1$$

and so the series diverges.

13. (a) $\sum_1^\infty \frac{(5k)!}{10^k(3k)!(2k)!}$ (b) $\sum_1^\infty \frac{9^k(3k)!k!}{(4k)!}$ (c) $\sum_1^\infty \frac{4^k(k!)^4}{(4k)!}$

Solution

(a) By the ratio test

$$\rho=\lim_{k\to\infty}\frac{\dfrac{(5k+5)!}{10^{k+1}(3k+3)!(2k+2)!}}{\dfrac{(5k)!}{10^k(3k)!(2k)!}}$$

$$=\frac{1}{10}\lim_{k\to\infty}\frac{(5k+5)(5k+4)(5k+3)(5k+2)(5k+1)}{(3k+3)(3k+2)(3k+1)(2k+2)(2k+1)}$$

$$=\frac{1}{10}\frac{5^5}{3^3 2^2}>1$$

and so series diverges.

(b) By the ratio test,

$$\rho=\lim_{k\to\infty}\frac{\dfrac{9^{k+1}(3k+3)!(k+1)!}{(4k+4)!}}{\dfrac{9^k(3k)!k!}{(4k)!}}$$

$$=9\lim_{k\to\infty}\frac{(3k+3)(3k+2)(3k+1)(k+1)}{(4k+4)(4k+3)(4k+2)(4k+1)}$$

$$=9\times\frac{3^3}{4^4}<1$$

so series converges.

(c) By the ratio test,

$$\rho = \lim_{k\to\infty} \frac{\dfrac{4^{k+1}((k+1)!)^4}{(4k+4)!}}{\dfrac{4^k(k!)^4}{(4k)!}}$$
$$= 4\lim_{k\to\infty}\frac{(k+1)^4}{(4k+4)(4k+3)(4k+2)(4k+1)}$$
$$= \frac{1}{4^3} < 1$$

so series converges.

14. (a) $\sum_1^\infty \sin\left(\frac{\pi}{2n^2}\right)$ (b) $\sum_1^\infty\left(1-\cos\frac{\pi}{n}\right)$ (c) $\sum_1^\infty \cos^2\frac{(n-1)\pi}{n}$

Solution

(a) For large n the general term behaves as $\frac{\pi}{2n^2}$ so we compare with the convergent series $\sum_1^\infty\frac{1}{n^2}$. By the limit comparison test and using L' Hôpital's rule,

$$\lim_{n\to\infty}\frac{\sin\frac{\pi}{2n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{\frac{-2\pi}{2n^3}\cos\frac{\pi}{2n^2}}{\frac{-2}{n^3}} = \frac{\pi}{2},$$

hence the series converges.

(b) We will be able to see when we study MacLaurin series that the general term will behave as $\frac{\pi}{2n^2}$ so we compare with the convergent series $\sum_1^\infty\frac{1}{n^2}$. By the limit comparison test and using L' Hôpital's rule,

$$\lim_{n\to\infty}\frac{1-\cos\frac{\pi}{n}}{\frac{1}{n^2}} = \lim_{n\to\infty}\frac{-\frac{\pi}{n^2}\sin\frac{\pi}{n}}{\frac{-2}{n^3}} = \frac{\pi}{2}\lim_{n\to\infty}\frac{\sin\frac{\pi}{n}}{\frac{1}{n}} = \frac{\pi}{2}\lim_{n\to\infty}\frac{\frac{-\pi}{n^2}\cos\frac{\pi}{n}}{\frac{-1}{n^2}} = \frac{\pi^2}{2},$$

hence series converges.

(c) $\lim_{n\to\infty}\cos^2\frac{(n-1)\pi}{n} = \cos^2\pi = 1 \neq 0$ and so series diverges by the divergence test.

15. The series of positive terms $\sum_1^\infty u_n$ converges. Deduce that the series (a) $\sum_1^\infty u_n^2$ and (b) $\sum_1^\infty\left(u_n - \frac{1}{n^\alpha}\right)^2$ where $\alpha > \frac{1}{2}$ also converge. (c) Hence or otherwise prove that the series $\sum_1^\infty \frac{u_n}{n^\alpha}$ $(\alpha > \frac{1}{2})$ converges.

Solution

(a) Since $\sum_1^\infty u_n$ converges $\lim_{n\to\infty} u_n = 0$ and so for sufficiently large *n*, $u_n < 1$. Then $u_n^2 < u_n$ and so $\sum_1^\infty u_n^2$ converges by the direct comparison test.

(b) Since $(u_n - \frac{1}{n^\alpha})^2 = u_n^2 + \frac{1}{n^{2\alpha}} - \frac{2u_n}{n^\alpha} < u_n^2 + \frac{1}{n^{2\alpha}}$ (since $\frac{2u_n}{n^\alpha} > 0$) and $\sum_1^\infty u_n^2$ converges and $\sum_1^\infty \frac{1}{n^{2\alpha}}$ also converges being a *p*-series with $p = 2\alpha > 1$ we have that $\sum_1^\infty \left(u_n - \frac{1}{n^\alpha} \right)^2$ converges as well by the comparison test.

(c) $\sum_1^\infty \left(u_n - \frac{1}{n^\alpha} \right)^2 = \sum_1^\infty (u_n^2 + \frac{1}{n^{2\alpha}} - 2\frac{u_n}{n^\alpha})$. All the series converge and thus so must $\sum_1^\infty \frac{u_n}{n^\alpha}$ provided $\alpha > \frac{1}{2}$.

16. For what value(s) of *p* does the series $\sum_2^\infty \frac{1}{n^p \ln n}$ converge?

Solution

For $p > 1$, $\frac{1}{n^p \ln n} < \frac{1}{n^p}$ and the series converges by comparison with $\sum_2^\infty \frac{1}{n^p}$. For $p = 1$ the series is $\sum_2^\infty \frac{1}{n \ln n}$ that diverges as shown earlier (Set 3.3 Ex. 2(a)) by the integral test. For $p < 1$, $n^p < n$ and so $\frac{1}{n^p \ln n} > \frac{1}{n \ln n}$ that diverges. Hence series converges for $p > 1$ and diverges for $p \le 1$.

17. Examine the convergence or divergence of the two series (a) $\sum_2^\infty \frac{1}{(\ln n)^p}$ and (b) $\sum_2^\infty \frac{1}{(\ln n)^n}$.

Solution

(a) The series clearly diverges for $p \le 0$ so we may assume that *p* is positive. In the text we stated that for *n* sufficiently large, $\ln n < n^\alpha$. This can be proved by examining the function $f(x) = \ln x - x^\alpha$ and seeing that it has a maximum at $x = \left(\frac{1}{\alpha} \right)^{1/\alpha}$. Then, $\frac{1}{\ln n} > \frac{1}{n^\alpha}$ and so $\frac{1}{(\ln n)^p} > \frac{1}{n^{\alpha p}}$. Choose α such that $\alpha p < 1$. Then series diverges by the direct comparison test since $\sum_2^\infty \frac{1}{n^{\alpha p}}$ diverges when $\alpha p < 1$.

(b) Since $\ln n > 2$ for $n > e^2 \approx 7.4$, then $\frac{1}{\ln n} < \frac{1}{2}$ and so $\frac{1}{(\ln n)^n} < \frac{1}{2^n}$. Hence series converges by the direct comparison test since $\sum_2^\infty \frac{1}{2^n}$ converges.

18. Using Euler's result $\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6}$ find the sum $\sum_1^\infty \frac{1}{(2n-1)^2}$.

Solution

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6} = \sum_{n\ even}^\infty \frac{1}{n^2} + \sum_{n\ odd}^\infty \frac{1}{n^2} = \sum_1^\infty \frac{1}{(2n)^2} + \sum_1^\infty \frac{1}{(2n-1)^2} = \frac{1}{4}\sum_1^\infty \frac{1}{n^2} + \sum_1^\infty \frac{1}{(2n-1)^2}$$

$$\frac{\pi^2}{6} = \frac{1}{4}\frac{\pi^2}{6} + \sum_1^\infty \frac{1}{(2n-1)^2}$$

$$\sum_1^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{6} - \frac{1}{4}\frac{\pi^2}{6}$$

$$= \frac{\pi^2}{8}$$

19. Euler has shown that the series $\frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \cdots$ i.e. the sum of the reciprocals of the prime numbers diverges. Use this fact to explain why the number of prime numbers is infinite.

Solution

If the number of primes were finite the infinite series would converge. It diverges, however, and so the number of primes is infinite.