1 (a)

$$\mathcal{D}_f = \mathbb{R}^3 \quad \left(x^2 + 4y^2 + 9z^2 \geq 0 \quad \forall (x,y,z) \in \mathbb{R}^3 \right)$$

$$\mathrm{Im}\, f = \langle -\infty, 5] \qquad f(x,y,z) = 5 - \underbrace{\sqrt{x^2+4y^2+9z^2}}_{\geq 0} \leq 5 \quad \forall (x,y,z) \in \mathbb{R}^3$$

i $\forall C \in \langle -\infty, 5]$ postoji $(x,y,z) = (5-C, 0, 0)$ t.d.

$$f(5-C,0,0) = 5 - \sqrt{\underbrace{(5-C)^2}_{\geq 0}} = 5 - (5-C) = C$$

Nivo-plohe za $C \in \langle -\infty, 5]$

$$5 - \sqrt{x^2+4y^2+9z^2} = C$$

$$\underbrace{5-C}_{\geq 0} = \sqrt{x^2+4y^2+9z^2} \iff \underbrace{(5-C)^2}_{\text{nova konstanta}} = x^2+4y^2+9y^2$$

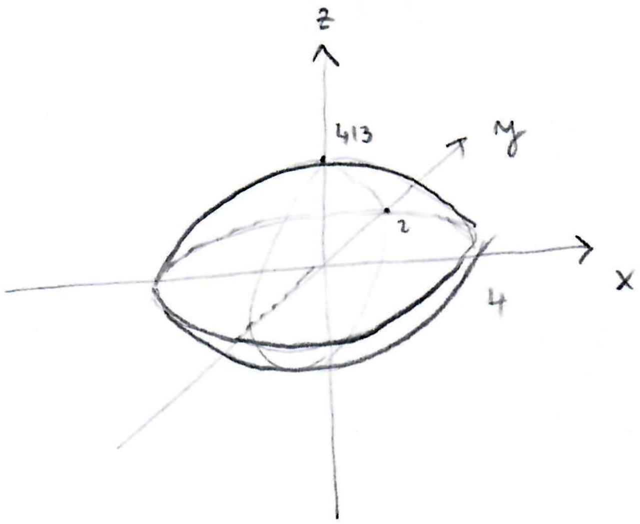
$\Rightarrow$ to su elipsoidi

① (b) nivo-ploha kroz $T(\sqrt{3}, 1, 1)$ je

$$5 - \sqrt{x^2 + 4y^2 + 9z^2} = f(\sqrt{3}, 1, 1) = 5 - \sqrt{16} = 1$$

$$4 = \sqrt{x^2 + 4y^2 + 9z^2}$$

$$\boxed{16 = \underbrace{x^2 + 4y^2 + 9z^2}_{g(x,y,z)}} \quad \text{elipsoid}$$



tangencijalna ravnina ..

$$\nabla g = (2x, 8y, 18z)$$

$$\nabla g(\sqrt{3}, 1, 1) = (2\sqrt{3}, 8, 18) = \vec{n}$$

Tangencijalna ravnina glasi ..

$$(x - \sqrt{3})\, 2\sqrt{3} + (y-1) \cdot 8 + (z-1) \cdot 18 = 0$$

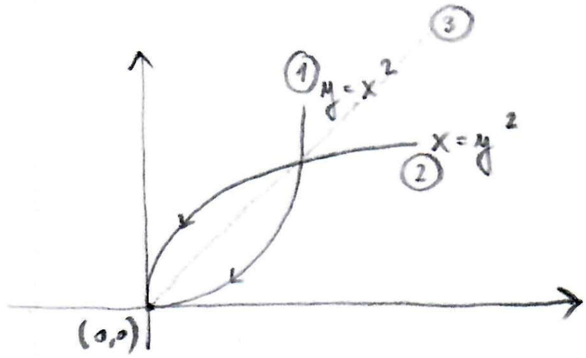
$$2\sqrt{3}x + 8y + 18z = 32$$

② (a)

$$\lim_{\vec{x}\to\vec{a}} f(\vec{x}) = L \iff (\forall \varepsilon>0)(\exists \delta>0)\Big[(0<|\vec{x}-\vec{a}|<\delta) \Rightarrow$$

$$\Rightarrow |f(\vec{x})-L|<\varepsilon\Big]$$

② (b) Navedena tvrdnja općenito NIJE TOČNA, jer nemamo informacija što se događa po ostalih beskonačno mnogo približavanja ishodištu. (npr. $y=x$)



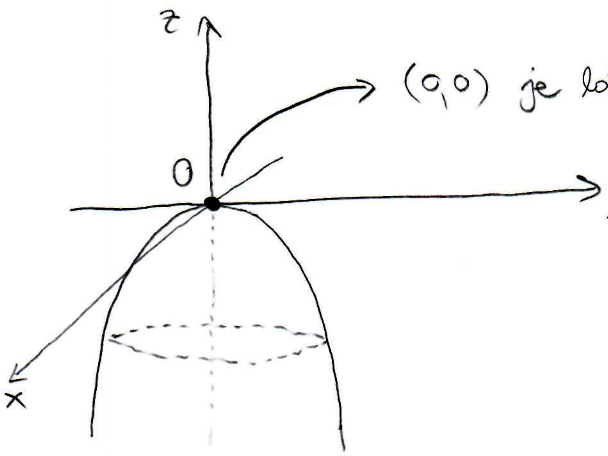
② (c) $$z(r\cos\varphi, r\sin\varphi) = \frac{r^2}{\sqrt{r^4\cos^4\varphi + r^4\sin^4\varphi}} \quad r\neq 0$$

$$= \frac{1}{\sqrt{\cos^4\varphi+\sin^4\varphi}} \quad r\neq 0$$

$\Rightarrow$ Limes ovisi o kutu φ, pa zaključujemo da NE POSTOJI, odnosno funkcija z ima prekid u (0,0)

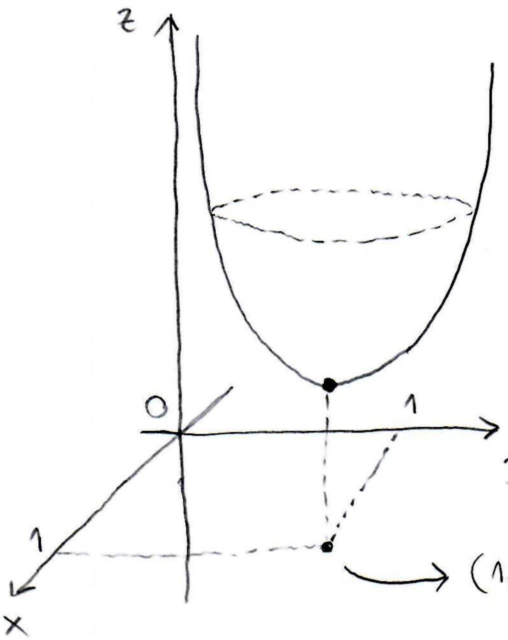
③ (a) Na primjer,

$$f(x,y) = -x^2 - y^2$$



(0,0) je lokalni (i globalni) maksimum

$$g(x,y) = (x-1)^2 + (y-1)^2$$



(1,1) je lokalni (i globalni) minimum

③ (b) $f(x) = \frac{1}{2}\ln(x^2) + \frac{1}{2}\ln(y^2) - x^2 - y^2 + xy$

$$f'_x = \frac{1}{2}\cdot\frac{1}{x^2}\cdot 2x - 2x + y = 0$$

$$\frac{1}{x} - 2x + y = 0 \quad \Big| \cdot x$$

$$\boxed{1 - 2x^2 + yx = 0}$$

$$f'_y = \frac{1}{2}\,\frac{1}{y^2}\cdot 2y - 2y + x = 0$$

$$\frac{1}{y} - 2y + x = 0 \quad \Big| \cdot y$$

$$\boxed{1 - 2y^2 + xy = 0}$$

$\ominus \Rightarrow$

$$-2x^2 + 2y^2 = 0$$

$$x^2 = y^2$$

$x = y$

$1 - 2x^2 + x^2 = 0$

$x^2 = 1$

$$\boxed{\begin{array}{l} T_1\,(1,1) \\ T_2\,(-1,-1) \end{array}}$$

$x = -y$

$1 - 2x^2 - x^2 = 0$

$-3x^2 = -1$

$x^2 = \frac{1}{3}$

$$\boxed{\begin{array}{l} T_3\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right) \\ T_4\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \end{array}}$$

$$f''_{xx} = -\frac{1}{x^2} - 2$$

$$f''_{yy} = -\frac{1}{y^2} - 2$$

$$f''_{xy} = 1$$

$$H_f(T_{1,2}) = \begin{vmatrix} -3 & 1 \\ 1 & -3 \end{vmatrix} = 9 - 1 = 8 > 0$$

$f''_{xx}(T_{1,2}) = -3 < 0$ } LOKALNI MAKSIMUMI

$$H_f(T_{3,4}) = \begin{vmatrix} -5 & 1 \\ 1 & -5 \end{vmatrix} = 25 - 1 = 24 > 0$$

$; f''_{xx}(T_{3,4}) = -5 < 0$ } LOKALNI MAKSIMUMI

4 (a)

$$\underbrace{(x, y, z)}_{\text{pravokutne koord}} = \underbrace{(r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)}_{\text{sferne koord}}$$

$$r \in [0, +\infty\rangle$$
$$\varphi \in [0, 2\pi\rangle$$
$$\theta \in [0, \pi]$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\varphi \qquad \frac{\partial x}{\partial \varphi} = -r\sin\theta\sin\varphi \qquad \frac{\partial x}{\partial \theta} = r\cos\theta\cos\varphi$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\varphi \qquad \frac{\partial y}{\partial \varphi} = r\sin\theta\cos\varphi \qquad \frac{\partial y}{\partial \theta} = r\cos\theta\sin\varphi$$

$$\frac{\partial z}{\partial r} = \cos\theta \qquad \frac{\partial z}{\partial \varphi} = 0 \qquad \frac{\partial z}{\partial \theta} = -r\sin\theta$$

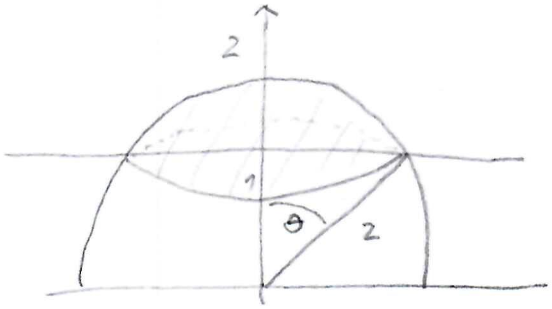
Jacobijeva matrica prijelaza iz pravokutnih u sferne koordinate je:

$$J(r, \varphi, \theta) = \begin{bmatrix} \sin\theta\cos\varphi & -r\sin\theta\sin\varphi & r\cos\theta\cos\varphi \\ \sin\theta\sin\varphi & r\sin\theta\cos\varphi & r\cos\theta\sin\varphi \\ \cos\theta & 0 & -r\sin\theta \end{bmatrix}$$

Jacobijan tada glasi $|J| = r^2\sin\theta$

④ (b) $(x,y,z) = (r\sin\theta\cos\varphi,\ r\sin\theta\sin\varphi,\ r\cos\theta)$

$$z=1 \Rightarrow r\cos\theta = 1 \Rightarrow r = \frac{1}{\cos\theta}$$



$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\int_0^{2\pi} d\varphi \int_0^{\pi/3} \sin\theta\, d\theta \int_{\frac{1}{\cos\theta}}^{2} \sqrt{r^2}\, r^2\, dr$$

$$= 2\pi \int_0^{\pi/3} \sin\theta \left.\frac{r^4}{4}\right|_{\frac{1}{\cos\theta}}^{2} d\theta = 2\pi \left[\int_0^{\pi/3} \left(4\sin\theta - \frac{\sin\theta}{4\cos^4\theta}\right) d\theta \right]$$

$\cos\theta = t$ supst

$$= 2\pi \left[-4\cos\theta \Big|_0^{\pi/3} + \frac{1}{4} \frac{(\cos\theta)^{-3}}{-3} \Big|_0^{\pi/3} \right]$$

$$= 2\pi \left(-4\left(\frac{1}{2} - 1\right) - \frac{1}{12}(8-1) \right)$$

$$= 2\pi \cdot \frac{17}{12} = \frac{17\pi}{6}$$

(5) (a) T_1: istinita

DOKAZ: nužan uvjet konvergencije, obrat po kontrapoziciji (TM 5.1.2)

T2: lažna

PROTUPRIMJER: $\sum_{n=1}^{\infty} \frac{1}{n}$

T3: istinita

DOKAZ: nužan uvjet konvergencije (TM 5.1.2.)

(i) $\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{n}{2n+1} = \frac{1}{2} < 1 \Rightarrow$ prema Cauchyjevom kriteriju zadani red konvergira

(5) (b)

(ii) $$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{2n}{2n+1}\right)^n$$

$$= \lim_{n\to\infty} \left[\left(1 + \frac{-1}{2n+1}\right)^{2n+1}\right]^{\frac{n}{2n+1}}$$

$$= \left[e^{-1}\right]^{\lim_{n\to\infty} \frac{n}{2n+1}} = e^{-\frac{1}{2}} \neq 0$$

$\Rightarrow$ red divergira

Pazi: Cauchyev kriterij ne daje odluku jer

$$\lim_{n\to\infty} \sqrt[n]{a_n} = 1$$

⑥ <u>1 način:</u>

$$y' = \frac{2xy}{x^2} - \frac{y^2}{x^2}$$

$$y' = 2\left(\frac{y}{x}\right) - \left(\frac{y}{x}\right)^2$$

$\Rightarrow$ <u>supst.</u> $z = \frac{y}{x} \Rightarrow y = zx$

$y' = z'x + z$

$$\Rightarrow xz' + z = 2z - z^2$$

$$xz' - z + z^2 = 0$$

$$xz' = z - z^2$$

$z = 0$ i $z = 1$ su rješenja

$$\frac{z'}{z - z^2} = \frac{1}{x} \quad \Big| \int \to \text{parcijalni razlomci} \quad \frac{1}{z - z^2} = \frac{1}{z} + \frac{1}{1-z}$$

$$\int \frac{1}{z - z^2}\, dz = \ln|x| + \ln|c|$$

$$\ln|z| - \ln|1 - z| = \ln|cx|$$

$$\ln\left|\frac{z}{1-z}\right| = \ln|cx|$$

$$\frac{z}{1-z} = Cx$$

$$\left(1 - \frac{y}{x}\right)Cx = \frac{y}{x}$$

$$Cx - Cy = \frac{y}{x} \Rightarrow \boxed{y = \frac{Cx^2}{1 + Cx}}$$

Cauchyjev uvjet $y(2) = 1 \Rightarrow 1 = \frac{4C}{1 + 2C} \Rightarrow C = \frac{1}{2} \Rightarrow \boxed{y(x) = \frac{x^2}{2 + x}}$

⑥ 2. način

$$x^2 y' = 2xy - y^2$$

Bernoullijeva $y' + f(x) y = g(x) y^{\alpha}$, $\alpha \in \mathbb{R}$, $\alpha \neq 0, \alpha \neq 1$

$$\Rightarrow \quad y' = \frac{2xy}{x^2} - \frac{y^2}{x^2}$$

$$y' = 2\frac{y}{x} - \frac{y^2}{x^2}$$

$$y' - \frac{2}{x} y = -\frac{1}{x^2} y^2 \quad \Rightarrow \quad \boxed{\alpha = 2}, \quad f(x) = -\frac{2}{x}, \quad g(x) = -\frac{1}{x^2}$$

$\Rightarrow$ supst. $\quad z = y^{1-2} = \frac{1}{y} \quad \Rightarrow \quad z' = -\frac{1}{y^2} y'$

$/ : y^2$

$$-\frac{1}{y^2} y' + \frac{2}{x} \frac{1}{y} = \frac{1}{x^2}$$

$$\underbrace{z' + \frac{2}{x}}_{f(x)} z = \underbrace{\frac{1}{x^2}}_{g(x)}$$

$\Rightarrow$ Opće rješenje Cauchyevog problema uz uvjet $y(2) = 1$

$$z(x) = e^{-\int_2^x \frac{2}{s} ds} \left(1 + \int_2^x \frac{1}{t^2} e^{\int_2^t \frac{2}{s} ds} dt \right)$$

$$= e^{-2 \ln s \Big|_2^x} \left(1 + \int_2^x \frac{1}{t^2} e^{2 \ln |s| \Big|_2^t} dt \right) =$$

$$= e^{-2 \ln x + 2 \ln 2} \left(1 + \int_2^x \frac{1}{t^2} e^{2(\ln |t| - \ln 2)} dt \right)$$

$$= 4x^{-2}\left(1+\int_2^x \frac{1}{t^2}\cdot t^2\cdot\frac{1}{4}\,dt\right)$$

$$= \frac{4}{x^2}\left(1+\frac{1}{4}\,t\Big|_2^x\right)$$

$$= \frac{4}{x^2}\left(1+\frac{1}{4}x-\frac{1}{2}\right)$$

$$= \frac{4+x-2}{x^2} = \frac{2+x}{x^2}$$

$$\Rightarrow \quad z(x) = \frac{2+x}{x^2}$$

$$\Rightarrow \quad \boxed{y(x) = \frac{x^2}{2+x}}$$

⑦ $$\frac{d}{dy}\left(\frac{\sin^2 x}{y^2}\right) = \frac{d}{dx}\left(\frac{\alpha\,(x - \sin x\cos x)}{y^3} + \cos y\right)$$

$$\sin^2 x \cdot (-2)\, y^{-3} = \frac{\alpha}{y^3} - \frac{\alpha}{y^3}\underbrace{\left(\cos^2 x - \sin^2 x\right)}_{\cos(2x)} \quad \Big|\cdot y^3$$

$$-2\sin^2 x = \alpha - \alpha\cos 2x$$

$$\alpha = \frac{-2\sin^2 x}{1-\cos(2x)} = \frac{-2\cancel{\sin^2 x}}{2\cancel{\sin^2 x}} = -1$$

$$\boxed{\alpha = -1}$$

Tražimo funkciju $u(x,y)$ t.d. $\frac{\partial u}{\partial x} = \frac{\sin^2 x}{y^2}$

$$\frac{\partial u}{\partial y} = \frac{-x + \sin x \cos x}{y^3} + \cos y$$

$\Big| \int dy$

$$u(x,y) = (-x + \sin x \cos x) \cdot \left(-\frac{1}{2y^2}\right) + \sin y + C(x)$$

$$u(x,y) = \left(-x + \frac{1}{2}\sin(2x)\right) \cdot \left(\frac{-1}{2y^2}\right) + \sin y + C(x)$$

$$u(x,y) = \frac{-\frac{1}{2}\sin(2x) + x}{2y^2} + \sin y + C(x)$$

$$\Rightarrow \frac{\sin^2 x}{y^2} = \frac{\partial u}{\partial x} = \frac{1}{2y^2}\left(-\frac{1}{2} \cdot \cos(2x) \cdot 2 + 1\right) + C'(x)$$

$$\sin^2 x = -\frac{1}{2}\cos(2x) + \frac{1}{2} + C'(x) \quad \Big| \cdot 2$$

$$2\sin^2 x + \cos^2 x - \sin^2 x = 1 + 2C'(x)$$

$$0 = C'(x)$$

$$\Leftrightarrow C(x) = C \in \mathbb{R}$$

$$\Rightarrow u(x,y) = \frac{x - \frac{1}{2}\sin(2x)}{2y^2} + \sin y = C \in \mathbb{R}$$

8 (a)

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x + x + 1 & e^{-x} + x - 2 \\ e^x + 1 & -e^{-x} + 1 \end{vmatrix}$$

$$= \left(e^x + x + 1\right)\left(-e^{-x} + 1\right) - \left(e^{-x} + x - 2\right)\left(e^x + 1\right)$$

Za $x = 0 \Rightarrow W(y_1, y_2) = 2$

Prema Teoremu 7.2.2, postoji x za koji je $W \neq 0$

$\Rightarrow y_1, y_2$ su linearno nezavisne

Opće rješenje je oblika $y(x) = C_1 y_1 + C_2 y_2$

- Uvjet da rješenje prolazi točkom $T(0,1)$ znači $y(0) = 1$
- Uvjet da je kut tangente u $T(0,1)$ s osi ordinate $\frac{\pi}{6}$ znači da je kut tangente s osi x jednak $\frac{\pi}{3}$, odnosno $y'(0) = \sqrt{3}$

- Uvrštavanjem $x=0$ i $y=1$ u funkciju:

$$C_1(1 + 0 + 1) + C_2(1 + 0 - 2) = 1$$

$$\Rightarrow \boxed{2C_1 - C_2 = 1}$$

Deriviranjem y $\Rightarrow$ $y'(x) = C_1(e^x+1) + C_2(-e^{-x}+1)$

Uvrstimo $x=0$ i $y'(0)=\sqrt{3}$

$$\Rightarrow \quad C_1(1+1) + C_2(-1+1) = \sqrt{3}$$

$$\boxed{2C_1 = \sqrt{3}}$$

$\Rightarrow$ Rješavanjem sustava dobivamo $C_1 = \frac{\sqrt{3}}{2}$ i $C_2 = \sqrt{3} - 1$

8 (b)

1° HOMOGENA JDBA

$$y'' + 9y = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$\Rightarrow y_H = C_1 \cos(3x) + C_2 \sin(3x)$$

2° VARIJACIJA KONSTANTI

$$y = C_1 \cos(3x) + C_2 \sin(3x)$$

$$y' = -3C_1 \sin(3x) + 3C_2 \cos(3x) + \overbrace{\left(C_1' \cos(3x) + C_2' \sin(3x)\right)}^{=0}$$

$$y'' = -9C_1 \cos(3x) - 9C_2 \sin(3x) + \underbrace{\left(-3C_1' \sin(3x) + 3C_2' \cos(3x)\right)}_{= \frac{3}{\cos(3x)}}$$

$$\begin{cases} C_1' \cos(3x) + C_2' \sin(3x) = 0 & \Big| \cdot \sin(3x) \\ -3C_1' \sin(3x) + 3C_2' \cos(3x) = \dfrac{3}{\cos(3x)} & \Big| \cdot \dfrac{\cos(3x)}{3} \end{cases}$$

$$\left.\begin{cases} C_1' \cos(3x)\sin(3x) + C_2' \sin^2(3x) = 0 \\ -C_1' \sin(3x)\cdot\cos(3x) + C_2' \cos^2(3x) = 1 \end{cases}\right\} +$$

$$\Rightarrow C_2'(x) = 1$$

$$\boxed{C_2(x) = x + D_2}$$

$$C_1'(x)\cos(3x) = -\sin(3x)$$

$$C_1'(x) = -\frac{\sin(3x)}{\cos(3x)}$$

$$\boxed{C_1(x) = \frac{1}{3}\ln|\cos 3x| + D_1}$$

$$\Rightarrow \boxed{y = \left(\frac{1}{3}\ln|\cos 3x| + D_1\right)\cos(3x) + \left(x + D_2\right)\sin(3x)}$$