ZAVRŠNI ISPIT IZ MATEMATIČKE ANALIZE 2

28.06.2021.

RJEŠENJA ZADATAKA

1. a) Teorem: Leibnizov kriterij konvergencije

Ako za red $\sum_{n=1}^{\infty} (-1)^n a_n$ vrijedi:

(i) $a_n \geq 0 \quad \forall n \in \mathbb{N}$,

(ii) niz (a_n) je padajući,

(iii) $\lim_{n\to\infty} a_n = 0$

tada je taj red konvergentan.

Dokaz:

Za $n \in \mathbb{N}$ promatramo $(2n)$-tu parcijalnu sumu zadanog reda

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \ldots + (a_{2n-3} - a_{2n-2}) + (a_{2n-1} - a_{2n})$$

$$= S_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\overset{(ii)}{\geq 0}} \geq S_{2n-2}$$

Dakle, niz (S_{2n}) je rastući. Nadalje,

$$S_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{\overset{(ii)}{\geq 0}} - \ldots - \underbrace{(a_{2n-2} - a_{2n-1})}_{\overset{(ii)}{\geq 0}} - \underbrace{a_{2n}}_{\overset{(i)}{\geq 0}} \leq a_1,$$

tj. niz (S_{2n}) je omeđen odozgo pa je i on konvergentan,

tj. postoji $\lim_{n\to\infty} S_{2n} = S$.

Za niz (S_{2n+1}) neparnih parcijalnih suma imamo

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \underbrace{\lim_{n\to\infty} a_{2n+1}}_{\overset{(iii)}{=0}} = S.$$

Dakle, niz (S_n) je konvergentan pa red $\sum_{n=1}^{\infty} (-1)^n a_n$ konvergira po definiciji. $\square$

1. b) $\displaystyle\sum_{n=3}^{\infty} \frac{(-1)^n}{\ln(\ln n)}$

1°) $a_n = \dfrac{1}{\ln(\ln n)} > 0 \quad \forall n \in \mathbb{N},\ n \geq 3$

2°) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{1}{\ln(\ln n)} = \dfrac{1}{\infty} = 0$

3°) $n+1 > n$

$\ln(n+1) > \ln(n)$

$\ln(\ln(n+1)) > \ln(\ln n)$

$\dfrac{1}{\ln(\ln(n+1))} < \dfrac{1}{\ln(\ln n)}$

$a_{n+1} < a_n \quad \forall n \in \mathbb{N},\ n \geq 3$, niz padajući!

Red KONVERGIRA prema Leibnizovom kriteriju.

1. c) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n} \rightarrow$ DIVERGIRA

$\displaystyle\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)} \rightarrow ?$

Prvi red je minoranta traženog reda jer

$n > \ln n > \ln(\ln n) \quad \ldots \quad n \geq 3$

$\dfrac{1}{n} < \dfrac{1}{\ln(\ln n)} \quad \ldots \quad n \geq 3$

Dakle, zadani red DIVERGIRA jer ima divergentnu minorantu (po poredbenom kriteriju)

2.a) $\sum_{n=1}^{\infty} \frac{n}{n+1} \frac{x^n}{3^{n+1}}$

Prema D'Alembertovom kriteriju red konvergira za :

$$\left| \frac{\frac{n+1}{n+2} \cdot \frac{x^n \cdot x}{3^{n+1} \cdot 3}}{\frac{n}{n+1} \cdot \frac{x^n}{3^{n+1}}} \right| = \frac{(n+1)^2 \cdot |x|}{n(n+2) \cdot 3} \xrightarrow{n \to \infty} \frac{|x|}{3} < 1$$

$\Rightarrow \quad |x| < 3 \quad , \quad x \in \langle -3, 3 \rangle$

Ispitujmo konvergenciju na rubovima :

- $x = -3$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \cdot \frac{(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} (-1)^n \cdot \frac{n}{n+1}$$

NUK : $\lim_{n \to \infty} a_n \neq 0 \quad \Rightarrow$ Red divergira .

- $x = 3$

$$\sum_{n=1}^{\infty} \frac{n}{n+1} \cdot \frac{3^n}{3^{n+1}} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{n}{n+1}$$

NUK : $\lim_{n \to \infty} a_n \neq 0 \quad \Rightarrow$ Red divergira .

Područje konvergencije ovog reda je $I = \langle -3, 3 \rangle$.

2.b)

$$\left(\sum_{n=1}^{\infty} \frac{n}{n+1} \frac{x^n}{3^{n+1}}\right)' = \sum_{n=1}^{\infty} \frac{n^2}{(n+1)\, 3^{n+1}} x^{n-1}$$

$$a_n = \frac{(n+1)^2}{(n+2)\cdot 3^{n+2}}$$

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+2)\cdot 3^{n+2}}}{\frac{(n+2)^2}{(n+3)\cdot 3^{n+3}}}$$

$$= \lim_{n\to\infty} \frac{3\,(n+3)(n+1)^2}{(n+2)^3} = 3$$

ili: $R=3$ jer se deriviranjem / integriranjem ne mijenja polumjer konv.

$$\boxed{R=3}$$

2.c)

$$\int\left(\sum_{n=1}^{\infty} \frac{n}{n+1} \frac{x^n}{3^{n+1}}\right) dx = \sum_{n=1}^{\infty} \frac{n}{(n+1)^2\, 3^{n+1}} x^{n+1}$$

$$a_n = \frac{n-1}{n^2\cdot 3^n}$$

Ispitajmo područje konvergencije dobivenog reda:

$$\left|\frac{\frac{n+1}{(n+2)^2\, 3^{n+2}}\cdot x^{n+2}}{\frac{n}{(n+1)^2\cdot 3^{n+1}}\cdot x^{n+1}}\right| = \frac{(x+1)^3\cdot |x|}{n(n+2)^2\cdot 3} \xrightarrow{n\to\infty} \frac{|x|}{3} < 1$$

$\Rightarrow |x| < 3$, $x \in \langle -3, 3 \rangle$

Ispitajmo konvergenciju na rubovima :

- $x=-3$

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^2 \cdot \cancel{3^{n+1}}} \cdot (-3)^{n+1} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{(n+1)^2}$$

Alterniran red → Leibniz :

i) $\frac{n}{(n+1)^2} \geq 0 \quad \forall n \in \mathbb{N}$ ✓

ii) $\frac{n}{(n+1)^2} > \frac{n+1}{(n+2)^2} \quad \forall n \in \mathbb{N}$ ✓

iii) $\lim_{n\to\infty} \frac{n}{(n+1)^2} = 0$ ✓

Prema Leibnitzovom kriteriju red konvergira.

- $x=3$

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)^2 \cdot 3^{n+1}} \cdot 3^{n+1} = \sum_{n=1}^{\infty} \frac{n}{(n+1)^2} \sim \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergira}$$

$\Longrightarrow$ red divergira po poredbenom kriteriju (limes varijanta)

Područje konvergencije zadanog reda potencija je $I = [-3, 3\rangle$.

Integriranjem reda potencije područje konvergencije može se promijeniti, tj. polumjer ostaje isti ali se može promijeniti ponašanje u rubovima.

3. a) $\frac{x+y^2}{x^2}\,dx - \frac{2y}{x}\,dy = 0$

Provjerimo egzaktnost:

$$\left.\begin{aligned}\frac{\partial P}{\partial y}(x,y) &= \frac{\partial}{\partial y}\left(\frac{x+y^2}{x^2}\right) = \frac{2y}{x^2}\\ \frac{\partial Q}{\partial x}(x,y) &= \frac{\partial}{\partial x}\left(-\frac{2y}{x}\right) = \frac{2y}{x^2}\end{aligned}\right\} \frac{\partial P}{\partial y}(x,y) = \frac{\partial Q}{\partial x}(x,y) \checkmark$$

Riješimo egzaktnu jednadžbu:

$$\frac{\partial u}{\partial x}(x,y) = \frac{x+y^2}{x^2} \Big/ \int \qquad u(x,y) = \int \frac{x+y^2}{x^2}\,dx = \int\left(\frac{1}{x} + \left(\frac{y}{x}\right)^2\right)dx$$

$$u(x,y) = \ln x - \frac{y^2}{x} + \varphi(y) \Big/ \frac{\partial}{\partial y}$$

$$\frac{\partial u}{\partial y}(x,y) = -\frac{2y}{x}$$

$$=$$

$$\frac{\partial u}{\partial y}(x,y) = -\frac{2y}{x} + \varphi'(y)$$

$$-\frac{2y}{x} = -\frac{2y}{x} + \varphi'(y)$$

$$\varphi'(y) = 0$$

$$\varphi(y) = C$$

$$u(x,y) = \ln x - \frac{y^2}{x} + C$$

Opće rješenje:

$$\boxed{\ln x - \frac{y^2}{x} = C, \quad C \in \mathbb{R}}$$

|2|:

$$u(x,y) = \int_1^x \frac{x+y^2}{x^2}\,dx - \int_0^y \frac{2y}{1}\,dy$$

$$= \left(\ln x - \frac{y^2}{x}\right)\Big|_1^x - y^2\Big|_0^y$$

$$= \ln x - \frac{y^2}{x} + \cancel{y^2} - \cancel{y^2} + 0$$

$$= \ln x - \frac{y^2}{x}$$

opće rj: $\boxed{u(x,y) = C}$

2. način: $\frac{1}{x} + \frac{y^2}{x^2} - \frac{2y}{x}\cdot y' = 0 \Big/ \cdot \frac{-x}{2y}$

$y' - \frac{1}{2x}y = \frac{1}{2y}$ Bernoulli: $\alpha = -1$, supst. $z = y^2$

$$\Rightarrow \boxed{y^2 = Cx + x\ln x}$$

3.b) $xy^2y' = y^3 + 2x \quad \Big| : xy^2, \quad y \neq 0 \qquad y=0$ nije rješenje

$$y' = \frac{1}{x}y + \frac{2}{y^2}$$

$$\boxed{y' - \frac{1}{x}y = \frac{2}{y^2}} \Big/ \cdot 3y^2 \qquad \text{Bernoulli} \quad \alpha = -2$$

supst. $\quad z = y^{1-\alpha} = y^3$

$$3y^2 \cdot y' - \frac{3}{x}y^3 = 6 \qquad z' = 3y^2 \cdot y'$$

$$\boxed{z' - \frac{3}{x}z = 6} \quad \text{LDJ 1. reda}$$

hom:

$$z' = \frac{3}{x}z$$

$$\frac{dz}{z} = \frac{3}{x}dx$$

$$\ln|z| = 3\ln|x| + \ln|c|$$

$$\boxed{z = cx^3, \quad c \in \mathbb{R}}$$

nehom: $\quad z = c(x) \cdot x^3$

$$z' - \frac{3}{x}z = 6$$

$$c'(x) \cdot x^3 + \cancel{c(x) \cdot 3x^2} - \cancel{\frac{3}{x} \cdot c(x) \cdot x^3} = 6$$

$$c'(x) = \frac{6}{x^3} \Big/ \int$$

$$c(x) = 6 \cdot \frac{x^{-2}}{-2} + C = -\frac{3}{x^2} + C$$

$$\boxed{z = \left(C - \frac{3}{x^2}\right) \cdot x^3 = Cx^3 - 3x}$$

Opće rješenje:

$$\boxed{y = \sqrt[3]{z} = \sqrt[3]{Cx^3 - 3x}} \Rightarrow \boxed{\frac{y^3}{x^3} + \frac{3}{x^2} = C}$$

2. način: $(y^3 + 2x)\,dx - xy^2\,dy = 0$

nije egzaktna, ali postoji Eulerov multipl. $\mu(x) = \frac{1}{x^4}$

$$\Rightarrow \left(\frac{y^3}{x^4} + \frac{2}{x^3}\right)dx - \frac{y^2}{x^3}dy = 0 \Rightarrow u(x,y) = -\frac{1}{x^2} - \frac{y^3}{3x^3} = C \Big/ \cdot -3$$

4.a) $\boxed{\text{Teorem } 6.7.2.}$

4. b) $\begin{cases} y' = \sqrt[3]{(y-1)^2} \\ y(3) = y_0 \end{cases}$

$$y' = \sqrt[3]{(y-1)^2}$$

$$\frac{dy}{dx} = \sqrt[3]{(y-1)^2} \quad \Big| : \sqrt[3]{(y-1)^2} \quad , \; y \neq 1 \Rightarrow \boxed{y=1} \text{ je rješenje}$$

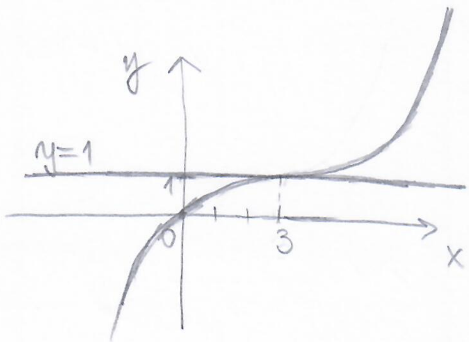
$$\frac{dy}{\sqrt[3]{(y-1)^2}} = dx$$

$$3 \cdot \sqrt[3]{(y-1)} = x + C$$

$$\sqrt[3]{y-1} = \frac{x}{3} + \frac{C}{3}$$

$$\boxed{y = \frac{(x+C)^3}{27} + 1}$$

OPĆE RJ.



(i) (CP) s $y(3) = 1$ ima 2 rješenja

$\boxed{y=1}$

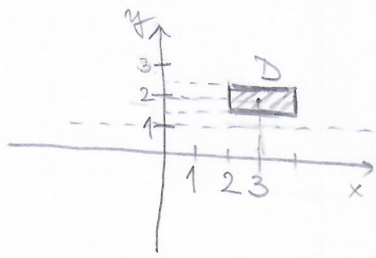
$1 = \frac{(3+C)^3}{27} + 1 \Rightarrow C = -3$

$$\boxed{y = \frac{(x-3)^3}{27} + 1}$$

(ii) $f(x,y) = \sqrt[3]{(y-1)^2}$ nepr. na $\mathbb{R}$

$\frac{\partial f}{\partial y} = \frac{2}{3}(y-1)^{-\frac{1}{3}}$ nije omeđena u $y=1$

$y(3) = 2 \Rightarrow D$ je oko $T(3,2)$



$$D = \{(x,y) \in \mathbb{R} : |x-3| < 1, |y-2| < \tfrac{1}{2}\}$$

5.a) DEF. 7.2.4.

5.b) Iskrena. Dokaz teorema 7.2.2.

5.c)
$$\begin{cases} y''' + 3y'' + 3y' + y = 0 \\ y(0) = 3, \ y'(0) = 1, \ y''(0) = 4 \end{cases}$$

Pripadna karakteristična jednadžba je:

$$r^3 + 3r^2 + 3r + 1 = 0$$
$$(r+1)^3 = 0$$
$$r = -1 \quad \text{kratnosti } 3$$

$$\Rightarrow y_H = C_1 e^{-x} + C_2 x e^{-x} + C_3 x^2 e^{-x}$$

$$y(0) = C_1 \Rightarrow \boxed{C_1 = 3}$$

$$y' = -3e^{-x} + C_2 e^{-x} - C_2 x e^{-x} + 2C_3 x e^{-x} - C_3 x^2 e^{-x}$$

$$y'(0) = -3 + C_2 \Rightarrow -3 + C_2 = 1 \Rightarrow \boxed{C_2 = 4}$$

$$y'' = 3e^{-x} - 4e^{-x} - 4e^{-x} + 4xe^{-x} + 2C_3 e^{-x} - 2C_3 x e^{-x}$$
$$- 2C_3 x e^{-x} + C_3 x^2 e^{-x}$$

$$y''(0) = 3 - 4 - 4 + 2C_3 = 4 \Rightarrow C_3 = \frac{9}{2}$$

$$\Rightarrow \boxed{y = e^{-x}\left(3 + 4x + \frac{9}{2}x^2\right)}$$

6. a) $y_1 = \sin^2 x$

$y_2 = \sin 2x$

Računamo Wronskijan

$$W = \begin{vmatrix} \sin^2 x & \sin 2x \\ 2\sin x \cos x & 2\cos 2x \end{vmatrix} \xRightarrow{\text{npr. } x=\frac{\pi}{2}} \begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} = -2 \neq 0$$

Funkcije y_1 i y_2 su linearno nezavisne.

b) $y'' + 4y + 2 = \sin 2x \Rightarrow y'' + 4y = -2 + \sin 2x$

1°) Homogena jednadžba:

$$y'' + 4y = 0$$

Karakteristična jednadžba:

$$r^2 + 4 = 0$$

$$r_{1,2} = \pm 2i$$

$$y_H = C_1 \sin 2x + C_2 \cos 2x \quad , C_{1,2} \in \mathbb{R}$$

2°) Partikularno rješenje (oblik desne strane)

(1) $y_{P_1} = A \Rightarrow 4A = -2 \, , \; A = -\frac{1}{2}$

$$y_{P_1} = -\frac{1}{2}$$

(2) $y_{P_2} = (B\sin 2x + C\cos 2x) \cdot x$

$$y_{p_2}' = B\sin 2x + C\cos 2x + (2B\cos 2x - 2C\sin 2x)\cdot x$$

$$y_{p_2}'' = 4B\cos 2x - 4C\sin 2x + (-4B\sin 2x - 4C\cos 2x)\cdot x$$

$$4B\cos 2x - 4C\sin 2x = \sin 2x$$

$$B = 0\ ,\quad C = -\frac{1}{4}$$

$$\Rightarrow y_{p_2} = -\frac{1}{4}x\cos 2x$$

Opće rješenje :

$$y = y_H + y_{p_1} + y_{p_2}$$

$$\boxed{y = C_1\sin 2x + C_2\cos 2x - \frac{1}{2} - \frac{1}{4}x\cos 2x\ ,\ C_{1,2}\in\mathbb{R}}$$

2. način : MVK

$$+\begin{cases} C_1'(x)\sin 2x + C_2'(x)\cos 2x = 0 & /\cdot 2\sin 2x \\ C_1'(x)\,2\cos 2x - C_2'(x)\,2\sin 2x = -2 + \sin 2x & /\cdot\cos 2x \end{cases}$$

$$C_1'(x)\cdot 2 = -2\cos 2x + \sin 2x\cos 2x\ ,\quad C_2'(x) = -\frac{C_1'(x)\sin 2x}{\cos 2x}$$

$$C_1'(x) = -\cos 2x + \frac{1}{4}\sin 4x \quad /\int dx \qquad C_2'(x) = \sin 2x - \frac{1}{2}\underbrace{\sin^2 2x}_{\frac{1}{2}-\frac{1}{2}\cos 4x}$$

$$C_1(x) = -\frac{1}{2}\sin 2x - \frac{1}{16}\cos 4x + D_1$$

$$C_2(x) = -\frac{1}{2}\cos 2x - \frac{1}{4}x + \frac{1}{16}\sin 4x + D_2$$

$$\Rightarrow y = \left(-\frac{1}{2}\sin 2x - \frac{1}{16}\cos 4x + D_1\right)\sin 2x + \left(-\frac{1}{2}\cos 2x - \frac{1}{4}x + \frac{1}{16}\sin 4x + D_2\right)\cos 2x$$

$$\boxed{y = D_1\sin 2x + D_2\cos 2x - \frac{1}{2} - \frac{1}{4}x\cos 2x}$$