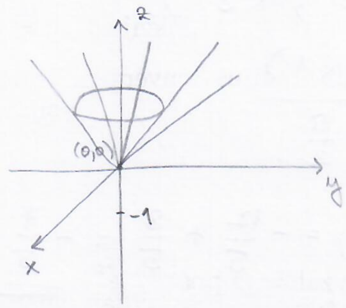
LJETNI ISPITNI ROK IZ MATEMATIČKE ANALIZE 2

15. srpnja 2019.

RJEŠENJA

1.

(a) Skica grafa: pogledamo presjek grafa funkcije $z = \sqrt{2x^2+y^2}$ sa xz i yz ravninama.



Funkcija nije neprekinuta u $(0,0)$ jer $\lim\limits_{(x,y)\to(0,0)} g(x,y) \neq -1$. To vidimo ako se u $(0,0)$ približimo po pravcu (x,x), dobivamo $\lim\limits_{x\to 0} \sqrt{2x^2+x^2} = \lim\limits_{x\to 0} \sqrt{3}\cdot|x| = 0$. $(\neq -1)$.

(b) Funkcija $f: \mathcal{D}(f) \to \mathbb{R}$ je diferencijabilna u $(x_0,y_0) \in \mathcal{D}(f)$ ako postoji vektor u $\mathbb{R}^2$ $\nabla f(x_0,y_0)$ sa koji vrijedi:

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(x_0+h_1, y_0+h_2) - f(x_0,y_0) - \nabla f(x_0,y_0)\cdot(h_1,h_2)}{\sqrt{h_1^2+h_2^2}} = 0.$$

(c) $f(x,y) = 2x+3y$

T: $f(x,y)$ je diferencijabilna u točki $(2,1)$.

DOKAZ: Znamo da je $\nabla f(x_0,y_0)$ dan sa $\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) \\ \frac{\partial f}{\partial y}(x_0,y_0) \end{bmatrix}$ pa računamo:

$f(2,1) = 7$

$\frac{\partial f}{\partial x} = 2$

$\frac{\partial f}{\partial y} = 3$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(2+h_1, 1+h_2) - f(2,1) - \frac{\partial f}{\partial x}(2,1)\cdot h_1 - \frac{\partial f}{\partial y}(2,1)\cdot h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{2(2+h_1)+3(1+h_2) - 7 - 2h_1 - 3h_2}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{0}{\sqrt{h_1^2+h_2^2}} = \underline{\underline{0}}$$

$\square$

2.

$f(x,y) = x^2 \cdot e^y$

(a) Tražimo usmjerenu derivaciju funkcije f iz točke $P(2,0)$ u smjeru vektora $\vec{PQ}$, gdje je $Q(\frac{1}{2},2)$.

Usmjerenu derivaciju računamo po formuli: $\frac{\partial f}{\partial \vec{v}}(P) = \nabla f(P) \cdot \vec{v_0}$ gdje je $\vec{v_0} = \frac{\vec{v}}{\|\vec{v}\|}$,

$\vec{v} = \vec{PQ} = (\frac{1}{2},2) - (2,0)$

$\vec{v} = (-\frac{3}{2}, 2)$

$\|\vec{v}\| = \sqrt{\frac{9}{4}+4} = \frac{5}{2}$

$\Rightarrow \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2xe^y, x^2e^y)$

$\Rightarrow \frac{\partial f}{\partial \vec{v}}(x,y) = (2xe^y, x^2e^y) \cdot \frac{(-\frac{3}{2}, 2)}{\frac{5}{2}}$

$\Rightarrow \frac{\partial f}{\partial \vec{v}}(2,0) = (4,4) \cdot (-\frac{3}{5}, \frac{4}{5}) = -\frac{12}{5} + \frac{16}{5} = \underline{\underline{\frac{4}{5}}}$

(b) Jer je $\frac{\partial f}{\partial \vec{v}}(\vec{x}) = \nabla f(\vec{x}) \cdot \frac{v}{\|v\|} = \|\nabla f(\vec{x})\| \cdot \underbrace{\frac{\|v\|}{\|v\|}}_{=1} \cdot \underbrace{\cos(\measuredangle \nabla f(\vec{x}), v)}_{\in[-1,1]} \in [-\|\nabla f(\vec{x})\|, \|\nabla f(\vec{x})\|]$,

vidimo da je vrijednost usmjerene derivacije u smjeru najbržeg rasta jednaka

$\|\nabla f(2,0)\| = \|(4,4)\| = \sqrt{4^2+4^2} = \underline{\underline{4\sqrt{2}}}$.

(c)

$f(x_0,y_0) = x_0^2 \cdot e^{y_0}$

$\frac{\partial f}{\partial x}(x_0,y_0) = 2x_0e^{y_0}$

$\frac{\partial f}{\partial y}(x_0,y_0) = x_0^2 \cdot e^{y_0}$

$\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) = 2x_0e^{y_0}$

$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) = 2e^{y_0}$

$\frac{\partial^2 f}{\partial y^2}(x_0,y_0) = x_0^2e^{y_0}$

$\Rightarrow T_2(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) +$

$+ \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x-x_0)(y-y_0) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2\right) =$

$= x_0^2 \cdot e^{y_0} + 2x_0e^{y_0}(x-x_0) + x_0^2 \cdot e^{y_0}(y-y_0) +$

$\frac{1}{2}\left(2e^{y_0}(x-x_0)^2 + 4x_0e^{y_0}(x-x_0)(y-y_0) + x_0^2e^{y_0}(y-y_0)^2\right)$

Približnu vrijednost izraza $(2.1)^2 \cdot e^{-0.1}$ pomoću $T_2(x,y)$ računamo kao $T_2(2.1,-0.1) = 4 + \cancel{4\cdot 0.1} + \cancel{4\cdot(-0.1)} +$

$(x_0,y_0) = (2,0)$

$\frac{1}{\cancel{2}}\cdot(\cancel{2}\cdot 0.1^2 + \cancel{2}\cdot 4 \cdot 0.1 \cdot (-0.1) + {}^2\cancel{4}\cdot 0.1^2) =$

$= \underline{3.99}$

3.

(a) Hesseova matrica funkcije $f:\mathbb{R}^2\to\mathbb{R}$ u točki $T_0(x_0,y_0)$: $H_f(T_0)=\begin{bmatrix} \frac{\partial^2 f}{\partial x^2}(T_0) & \frac{\partial^2 f}{\partial y\partial x}(T_0) \\ \frac{\partial^2 f}{\partial x\partial y}(T_0) & \frac{\partial^2 f}{\partial y^2}(T_0)\end{bmatrix}$.

(b) Dovoljni uvjeti da bi stacionarna točka $T_0(x_0,y_0)$ bila lokalni minimum :

$$\frac{\partial^2 f}{\partial x^2}(T_0)>0 \quad i \quad \det H_f(T_0)>0.$$

(c) $f(x,y)=x^2+y^2+\frac{2}{xy}$

$\nabla f(x,y)=0$

$$\begin{cases} \frac{\partial f}{\partial x}=2x-\frac{2}{yx^2}=0 \\ \frac{\partial f}{\partial y}=2y-\frac{2}{xy^2}=0\end{cases} \Rightarrow \begin{cases} x^3y=1 \\ xy^3=1 \Rightarrow x=\frac{1}{y^3}\end{cases} \Rightarrow \frac{1}{y^8}=1 \Rightarrow \underline{y=\pm 1}$$

$\Rightarrow T_1(1,1)$ i $T_2(-1,-1)$ su stacionarne točke.

$$H_f=\begin{bmatrix} 2+\frac{4}{yx^3} & \frac{2}{x^2y^2} \\ \frac{2}{x^2y^2} & 2+\frac{4}{xy^3}\end{bmatrix}, \quad H_f(1,1)=\begin{bmatrix} 6>0 & 2 \\ 2 & 6\end{bmatrix}, \quad H_f(-1,-1)=\begin{bmatrix} 6 & 2 \\ 2 & 6\end{bmatrix}$$

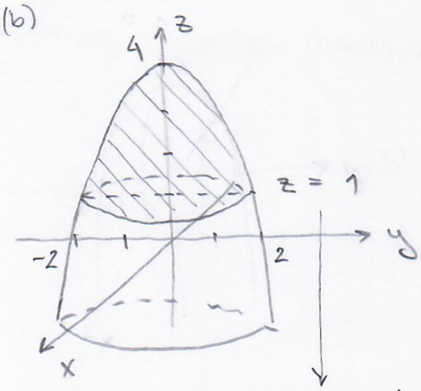
$\det H_f(1,1)=\det H_f(-1,-1)=32>0$

$\Rightarrow T_1$ i T_2 su točke lokalnog minimuma.

4.

(a) $\left.\begin{array}{l} x=r\cos\varphi \\ y=r\sin\varphi \\ z=z\end{array}\right\}$ $J=\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z}\end{vmatrix}=\begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1\end{vmatrix}=1\cdot(r\cos^2\varphi+r\sin^2\varphi)=\underline{\underline{r}}$

(b)



$4-x^2-y^2=1$

$x^2+y^2=3$

$(r=\sqrt{3})$

Koristimo cilindrične koordinate :

$$V=\iiint\limits_V r\,dr\,d\varphi\,dz=\int_0^{2\pi}d\varphi\int_0^{\sqrt{3}}r\,dr\int_1^{4-r^2}dz$$

$$=2\pi\int_0^{\sqrt{3}}r(4-r^2-1)\,dr$$

$$=2\pi\int_0^{\sqrt{3}}(3r-r^3)\,dr$$

$$=2\pi\left(3\frac{r^2}{2}-\frac{1}{4}r^4\right)\Big|_0^{\sqrt{3}}=2\pi\left(\frac{9}{2}-\frac{9}{4}\right)=\underline{\underline{\frac{9\pi}{2}}}$$

5.

(a) Polumjer konvergencije reda potencija $\sum\limits_{n=0}^{\infty} a_n(x-x_0)^n$ je $r = \dfrac{1}{\limsup\limits_{n\to\infty}\sqrt[n]{|a_n|}}$.

(b) $$f(x) = \frac{3}{(1-x)(2+x)} = \frac{1}{1-x} + \frac{1}{2+x} = \sum_{n=0}^{\infty} x^n + \frac{1}{2}\cdot\frac{1}{1-\left(\frac{-x}{2}\right)} =$$

$$= \sum_{n=0}^{\infty} x^n + \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n}\cdot x^n = \sum_{n=0}^{\infty}\left(1+\frac{(-1)^n}{2^{n+1}}\right)x^n.$$

$$r = \frac{1}{\limsup\limits_{n\to\infty}\sqrt[n]{\left|1+\underbrace{\frac{(-1)^n}{2^{n+1}}}_{\to 0}\right|}} = 1$$

(c) $$\sum_{n=0}^{\infty}\left(1+\frac{(-1)^n}{2^{n+1}}\right)\cdot\frac{1}{2^n} = f\left(\tfrac{1}{2}\right) = \frac{3}{\left(1-\frac{1}{2}\right)\cdot\left(2+\frac{1}{2}\right)} = \frac{3}{\frac{1}{2}\cdot\frac{5}{2}} = \frac{12}{5}.$$

6.

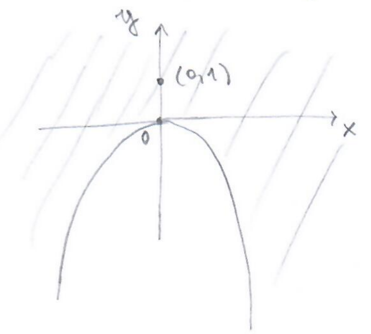
(a) Da bi Cauchyjeva zadaća $\begin{cases} y' = f(x,y) \\ y(x_0) = y_0 \end{cases}$ lokalno imala egzistenciju i jedinstvenost rješenja treba postojati pravokutnik oko točke (x_0,y_0) na kojem je $f(x,y)$ neprekinuta i $\frac{\partial f}{\partial y}$ omeđena.

(b) $(x_0,y_0) = (0,1)$

$$\begin{cases} y' = x + \sqrt{y+4x^2} = f(x,y) \\ y(0) = 1 \end{cases}$$

$\mathcal{D}_f = \{(x,y) : y + 4x^2 \geq 0\}$

$y \geq -4x^2$



$$\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y+4x^2}}$$

Za $D = \langle -\frac{1}{2}, \frac{1}{2}\rangle \times \langle \frac{1}{2}, \frac{3}{2}\rangle$:

- f je neprekinuta jer je $D \subseteq \mathcal{D}_f$ $\left(y + 4x^2 \geq \frac{1}{2} > 0\right)$, a f je po svojoj definiciji neprekinuta na domeni
- $\left|\frac{\partial f}{\partial y}(x,y)\right| = \dfrac{1}{2\underbrace{\sqrt{y+4x^2}}_{\geq \frac{1}{2}}} \leq \dfrac{1}{2\sqrt{1/2}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$, $\forall (x,y) \in D$

Zadovoljeni su uvjeti Picardovog teorema $\Rightarrow$ postoji jedinstveno rješenje $y(x)$ na okolini $x_0 = 0$.

(c) $n = 3$, $y(1) = ?$, $h = \frac{1-0}{3} = \frac{1}{3}$, $y_{n+1} = y_n + h\cdot f(x_n,y_n)$

n	x_n	y_n	$f(x_n,y_n) = x_n + \sqrt{y_n + 4x_n^2}$
0	0	1	$0 + \sqrt{1+0} = 1$
1	$1/3$	$4/3$	$1/3 + \sqrt{4/3 + 4\cdot 1/9} = 1/3 + 4/3 = 5/3$
2	$2/3$	$17/9$	$2/3 + \sqrt{17/9 + 4\cdot 4/9} = 2/3 + \frac{\sqrt{33}}{3}$
3	1	$\frac{19+\sqrt{33}}{9}$	

$$\Rightarrow y(1) \approx \frac{19+\sqrt{33}}{9}$$

⑦

(a) $y' = \frac{y}{x} + e^{-\frac{y}{x}}$

Uvedimo supstituciju: $z = \frac{y}{x}$

$z \cdot x = y \quad /'$

$y' = z' \cdot x + z$

$z' \cdot x + \cancel{z} = \cancel{z} + e^{-z}$ ← separacija varijabli

$\int e^{z} dz = \int \frac{1}{x} dx$

$e^{z} = \ln|x| + C$

$\Rightarrow e^{y/x} = \ln|x| + C \quad , \ C \in \mathbb{R}$.

(b) $(ye^{x} + 2x)dx + e^{x}dy = 0$

uvjet egzaktnosti:

$\frac{\partial P}{\partial y} = e^{x} = \frac{\partial Q}{\partial x}$ ✓

$\Rightarrow u(x,y) = \int_0^x (ye^{x} + 2x)dx + \int_0^y e^{0} dy$

$= (ye^{x} + x^{2})\Big|_0^x + y\Big|_0^y$

$= ye^{x} + x^{2} - \cancel{y} + \cancel{y}$

$\Rightarrow$ Rj: $ye^{x} + x^{2} = C, \quad C \in \mathbb{R}$

⑧

(a) $y_1(x)$ i $y_2(x)$ su rješenja jednadžbe

$$y^{(4)} - 9y''' - y'' + 9y' = 0$$

gornje jednadžbe za $\forall \alpha, \beta \in \mathbb{R}$.

(T:) $\alpha y_1(x) + \beta y_2(x)$ su rj. gornje jednadžbe za $\forall \alpha,\beta \in \mathbb{R}$.

DOKAZ: To slijedi iz linearnosti derivacije: $(\alpha y_1 + \beta y_2)' = \alpha y_1' + \beta y_2'$.

Uzastopnim primjenjivanjem derivacije dobivamo:

$\alpha y_1^{(4)} + \beta y_2^{(4)} - 9\alpha y_1''' - 9\beta y_2''' - \alpha y_1'' - \beta y_2'' + 9\alpha y_1' + 9\beta y_2' = 0$

$\alpha \cdot \underbrace{(y_1^{(4)} - 9y_1''' - y_1'' + 9y_1')}_{=0} + \beta \cdot \underbrace{(y_2^{(4)} - 9y_2''' - y_2'' + 9y_2')}_{=0} = 0$

jer i kako su y_1 i y_2 rj. početne jednadžbe.

(b) Pogledajmo karakterističnu jednadžbu:

$r^4 - 9r^3 - r^2 + 9r = 0$

$r(r^3 - 9r^2 - r + 9) = 0$

$r_1 = 0 \qquad r(r^2 - 1) - 9(r^2 - 1) = 0$

$(r^2 - 1)(r - 9) = 0$

$r_2 = 1, r_3 = -1, r_4 = 9$

$\Rightarrow$ 4 linearno nezavisna rj. su $\{1, e^{x}, e^{-x}, e^{9x}\}$

Jer je to baza za prostor rješenja imamo: $y = C_1 + C_2e^{x} + C_3e^{-x} + C_4e^{9x}, \ C_i \in \mathbb{R}$.

↑ opće rješenje

(c) Linearnu nezavisnost ispitujemo Wronskijanom:

$$W(1, e^x, e^{-x}, e^{9x}) = \begin{vmatrix} 1 & e^x & e^{-x} & e^{9x} \\ 0 & e^x & -e^{-x} & 9e^{9x} \\ 0 & e^x & e^{-x} & 81e^{9x} \\ 0 & e^x & -e^{-x} & 729e^{9x} \end{vmatrix} = \begin{vmatrix} e^x & -e^{-x} & 9e^{9x} \\ e^x & e^{-x} & 81e^{9x} \\ e^x & -e^{-x} & 729e^{9x} \end{vmatrix} \begin{matrix} / \cdot (-1) \\ + \\ + \end{matrix} =$$

$$= \begin{vmatrix} e^x & -e^{-x} & 9e^{9x} \\ 0 & 2e^{-x} & 72e^{9x} \\ 0 & 0 & 720e^{9x} \end{vmatrix} = e^x \cdot 2 \cdot 720 \cdot e^{8x} = 1440\, e^{9x} \neq 0$$

$\Rightarrow$ funkcije su linearno nezavisne.