MEĐUISPIT IZ MATEMATIČKE ANALIZE 2

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RJEŠENJA

1. (a) (i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{4xy}{x^2+y^2} = \left[\begin{array}{c}\text{POLARNE KOORDINATE} \\ x=r\cos\varphi,\ y=r\sin\varphi \\ r\to 0\end{array}\right] = \lim\limits_{r\to 0} \dfrac{4\cancel{r^2}\cos\varphi\sin\varphi}{\cancel{r^2}} = \lim\limits_{r\to 0} 2\sin(2\varphi) =$

$= 2\cdot\sin(2\varphi)$

Po definiciji funkcije f, da bi bila neprekidna u $(0,0)$, mora biti $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

Za $\varphi = \frac{\pi}{4} \Rightarrow$ limes je $2\cdot\sin\left(2\cdot\frac{\pi}{4}\right) = 2\cdot\sin\left(\frac{\pi}{2}\right) = 2$.

$\Rightarrow$ funkcija nije neprekidna u $(0,0)$.

(ii) Po definiciji: $\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\frac{4\cdot h\cdot 0}{h^2+0^2} - 0}{h} =$

$$= \lim_{h\to 0} \frac{0}{h} = 0$$

$\Rightarrow$ postoji parcijalna derivacija $\dfrac{\partial f}{\partial x}(0,0)$.

(b) Tvrdnja je <u>netočna</u>. Protuprimjer je funkcija f iz (a) dijela zadatka. Na isti način kao u (ii) se pokaže da postoji $\dfrac{\partial f}{\partial y}(0,0)$, međutim pod (i) smo vidjeli da f nije neprekidna u $(0,0)$ pa posebno nije niti diferencijabilna.

(c) $$\frac{\partial G}{\partial x} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial x} = 2x\cdot\frac{\partial g}{\partial u} + \frac{1}{y}\cdot\frac{\partial g}{\partial v}$$

$$\frac{\partial G}{\partial y} = \frac{\partial g}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial g}{\partial v}\cdot\frac{\partial v}{\partial y} = 3y\cdot\frac{\partial g}{\partial u} - \frac{x}{y^2}\cdot\frac{\partial g}{\partial v}$$

2. (a) $\dfrac{\partial f}{\partial \vec{h}}(P_0) \overset{\text{def.}}{=} \lim\limits_{t\to 0} \dfrac{f(P_0 + t\cdot\vec{h}_0) - f(P_0)}{t}$, gdje je $\vec{h}_0 = \dfrac{\vec{h}}{\|\vec{h}\|}$.

<u>TVRDNJA:</u> $\dfrac{\partial f}{\partial \vec{h}}(P_0) = \nabla f(P_0)\cdot\vec{h}_0$

<u>DOKAZ:</u> Označimo $P_0 = (x_0, y_0)$, $\vec{h}_0 = (h_1, h_2)$.

Po definiciji usmjerene derivacije, želimo pronaći brzinu promjene vrijednosti funkcije f kada se gibamo po pravcu određenom točkom P_0 i vektorom smjera $\vec{h}$, čiji parametarski oblik je $\begin{cases} x = x_0 + s\cdot h_1 \\ y = y_0 + s\cdot h_2 \end{cases}$, $s\in\mathbb{R}$. P_0 se postiže za $s=0$.

$z = f(x,y) = f(x_0 + s\cdot h_1, y_0 + s\cdot h_2)\ \Big/ \dfrac{\partial}{\partial s}$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s} = f_x\cdot h_1 + f_y\cdot h_2$$

$$\Rightarrow \frac{\partial f}{\partial \vec{h}}(P_0) = \frac{\partial z}{\partial s}(0) = f_x(x_0,y_0)\cdot h_1 + f_y(x_0,y_0)\cdot h_2 = \nabla f(P_0)\cdot\vec{h}_0 \quad \square$$

(b) TVRDNJA: za $\forall \vec{h} \in V^2$ vrijedi: $\frac{\partial f}{\partial \vec{h}}(P_0) \in \left[-\|\nabla f(P_0)\|, \|\nabla f(P_0)\|\right]$.

DOKAZ: Po tvrdnji (a) imamo: $\frac{\partial f}{\partial \vec{h}}(P_0) = \nabla f(P_0) \cdot \vec{h_0} = \|\nabla f(P_0)\| \cdot \underbrace{\|\vec{h_0}\|}_{=1} \cdot \underbrace{\cos(\measuredangle \nabla f(P_0), \vec{h_0})}_{\in[-1,1]}$

$\Rightarrow \frac{\partial f}{\partial \vec{h}}(P_0) \in \left[-\|\nabla f(P_0)\|, \|\nabla f(P_0)\|\right]$. $\square$

(c) Po (b) dijelu, usmjerena derivacija funkcije f u smjeru najbržeg rasta iznosi $\|\nabla f(P_0)\|$.

$\nabla f(x,y) = [2xy \quad x^2+3y^2] \Rightarrow \nabla f(1,2) = [4 \quad 13]$

$\|\nabla f(1,2)\| = \sqrt{4^2+13^2} = \sqrt{16+169} = \underline{\underline{\sqrt{185}}}$

(d) $\nabla g(P_0) \cdot \vec{\jmath} = \frac{\partial g}{\partial y}(P_0)$ je minimalna vrijednost usmjerene derivacije u P_0.

Po (b) dijelu znamo da se minimalna vrijednost postiže za $\vec{h_0} = -\frac{\nabla f(P_0)}{\|\nabla f(P_0)\|}$, a maksimalna za $\frac{\nabla f(P_0)}{\|\nabla f(P_0)\|}$. Dakle, $\vec{\jmath} = \frac{-\nabla f(P_0)}{\|\nabla f(P_0)\|} \Leftrightarrow -\vec{\jmath} = \frac{\nabla f(P_0)}{\|\nabla f(P_0)\|}$

Zaključujemo, maksimalna vrijednost usmjerene derivacije iz točke P_0 je

$\nabla g(P_0) \cdot (-\vec{\jmath}) = -\frac{\partial g}{\partial y}(P_0)$. //

3.
(a) Tražimo tangencijalnu ravninu na plohu $z(x,y)$ u točki $A(0,0,0)$.

Jednadžba tangencijalne ravnine u $A(0,0,0)$ glasi:

$$z - 0 = \frac{\partial z}{\partial x}(0,0) \cdot (x-0) + \frac{\partial z}{\partial y}(0,0) \cdot (y-0)$$

Definiramo: $F(x,y,z) = 2x + 3y + \sin(4x+5y) + z^2 + \sin z$

Jer je $F(x,y,z(x,y)) = 0$, vrijede formule: $\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$ i $\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$.

$\Rightarrow \frac{\partial z}{\partial x} = \frac{-(2+4\cos(4x+5y))}{2z+\cos z}$ & $\frac{\partial z}{\partial y} = \frac{-(3+5\cos(4x+5y))}{2z+\cos z}$

$\Rightarrow \frac{\partial z}{\partial x}(0,0) = -6$, $\frac{\partial z}{\partial y} = -8$

$\Rightarrow$ tangencijalna ravnina je $z = -6x - 8y$. //

(b) $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right) = -\frac{\partial}{\partial x}\left(\frac{2+4\cos(4x+5y)}{2z+\cos z}\right) = \frac{-16\sin(4x+5y)\cdot(2z+\cos z) - (2+4\cos(4x+5y))(2-\sin z)\frac{\partial z}{\partial x}}{(2z+\cos z)^2} =$

pa je $\frac{\partial^2 f}{\partial x^2}(0,0,0) = -72$.

Slično se izračuna $\frac{\partial^2 f}{\partial x \partial y}(0,0,0) = -96$ i $\frac{\partial^2 f}{\partial y^2}(0,0,0) = -128$ pa je

$$d^2 f = -72(dx)^2 - 192\,dx\,dy - 128(dy)^2.$$

4.

(a) Neka je $Q(h,k) = ah^2 + 2bhk + ck^2$.

TVRDNJA: Ako je $a>0$ i $ac-b^2>0$, onda je kvadratna forma $Q(h,k)$ pozitivno definitna.

DOKAZ: Treba vidjeti da je $Q(h,k)>0$, za $(h,k)\neq(0,0)$.

- Ako je $k=0$, onda $Q(h,k) = ah^2 > 0$ jer $a>0$ i $h\neq 0$ u ovom slučaju.
- Neka je sada $k\neq 0$.

$$Q(h,k) = \underbrace{k^2}_{>0}\cdot\left(a\cdot\left(\frac{h}{k}\right)^2 + 2b\frac{h}{k} + c\right)$$

Uz supstituciju $t := \frac{h}{k}$ imamo: $f(t) = at^2 + 2bt + c$

Diskriminanta ove kvadratne funkcije je $4b^2-4ac = 4(b^2-ac)<0$ jer je po pretp. $ac-b^2>0$. Jer je $a>0$ radi se o kvadratnoj funkciji čija slika je podskup $(0,+\infty)$. □

(b) Potražimo prvo stacionarne točke:

$$\nabla f(x,y) = \begin{bmatrix} 2xy+2y^2+\frac{1}{2}y & x^2+4xy+\frac{1}{2}x\end{bmatrix} = \begin{bmatrix}0 & 0\end{bmatrix}$$

$$\Longleftrightarrow$$

$$\begin{cases} y\cdot\left(2x+2y+\frac{1}{2}\right)=0 \\ x\cdot\left(x+4y+\frac{1}{2}\right)=0\end{cases}$$

1° $x=0$ & $y=0$

$\boxed{T_1(0,0)}$

2° $x=0$ & $y\neq 0$

$y\cdot\left(2y+\frac{1}{2}\right)=0 \quad /:y$

$y=-\frac{1}{4}$

$\boxed{T_2\left(0,-\frac{1}{4}\right)}$

3° $x\neq 0$ & $y=0$

$x\cdot\left(x+\frac{1}{2}\right)=0 \quad /:x$

$x=-\frac{1}{2}$

$\boxed{T_3\left(-\frac{1}{2},0\right)}$

4° $x\neq 0$ & $y\neq 0$

$$\begin{cases} 2x+2y+\frac{1}{2}=0 \\ x+4y+\frac{1}{2}=0 \quad /\cdot(-2)\end{cases}$$

$2y+\frac{1}{2}-8y-1=0$

$-6y=\frac{1}{2} \quad /:(-6)$

$y=-\frac{1}{12}$

$x-\frac{1}{3}+\frac{1}{2}=0$

$x=-\frac{1}{6}$

$\boxed{T_4\left(-\frac{1}{6},-\frac{1}{12}\right)}$

Izračunajmo druge derivacije:

$$\frac{\partial^2 f}{\partial x^2} = 2y, \quad \frac{\partial^2 f}{\partial x\partial y} = 2x+4y+\frac{1}{2},$$

$$\frac{\partial^2 f}{\partial y^2} = 4x$$

$$\Rightarrow D^2 f = \begin{bmatrix} 2y & 2x+4y+\frac{1}{2} \\ 2x+4y+\frac{1}{2} & 4x\end{bmatrix}$$

$\det(D^2f(T_1)) = \begin{vmatrix} 0 & 1/2 \\ 1/2 & 0\end{vmatrix} = -1/4 < 0$ sedlasta točka

$\det(D^2f(T_2)) = \begin{vmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 0\end{vmatrix} = -1/4 < 0$ sedlasta točka

$\det(D^2f(T_3)) = \begin{vmatrix} 0 & -1/2 \\ -1/2 & -2\end{vmatrix} = -1/4 < 0$ sedlasta točka

$\det(D^2f(T_4)) = \begin{vmatrix} \underset{<0}{-\frac{1}{6}} & -\frac{1}{6} \\ -\frac{1}{6} & -\frac{2}{3}\end{vmatrix} = \frac{1}{9}-\frac{1}{36} = \frac{3}{36} = \frac{1}{12} > 0 \Rightarrow$ točka maksimuma

5.

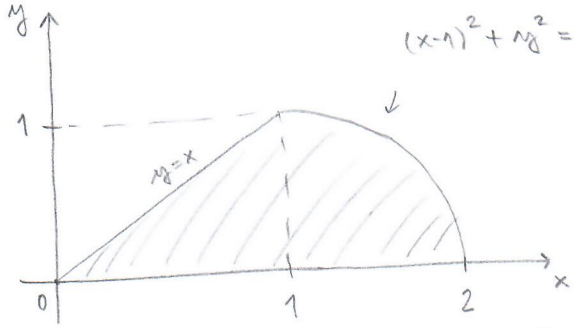
(a) $$J \overset{\text{def.}}{=} \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\[2ex] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

(b) $x(r,\varphi) = r \cdot \cos\varphi$

$y(r,\varphi) = r \cdot \sin\varphi$

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \varphi} \\[2ex] \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\cdot\sin\varphi \\[2ex] \sin\varphi & r\cdot\cos\varphi \end{vmatrix} = r\cdot\big(\underbrace{\cos^2\varphi + \sin^2\varphi}_{=1}\big) = r$$

(c)



$(x-1)^2 + y^2 = 1 \Rightarrow x^2+y^2 = 2x$ u polarnim koordinatama:

$r^{\cancel{2}} = 2\cancel{r}\cos\varphi$

$\boxed{r = 2\cos\varphi}$

$\varphi \in [0, \frac{\pi}{4}]$

$$\Rightarrow \int_0^1 dy \int_y^{1+\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx = \int_0^{\pi/4} d\varphi \int_0^{2\cos\varphi} \sqrt{r^2}\cdot \overset{\substack{\text{Jacobijan}\\ \shortparallel}}{r}\, dr = \int_0^{\pi/4} \frac{1}{3} r^3 \Big|_0^{2\cos\varphi} d\varphi =$$

$$= \frac{8}{3}\int_0^{\pi/4} \cos^3\varphi\, d\varphi = \frac{8}{3}\int_0^{\pi/4} (1-\sin^2\varphi)\cos\varphi\, d\varphi = \begin{bmatrix} t = \sin\varphi \\ dt = \cos\varphi\, d\varphi \end{bmatrix} =$$

$0 \mapsto 0$

$\pi/4 \mapsto \sqrt{2}/2$

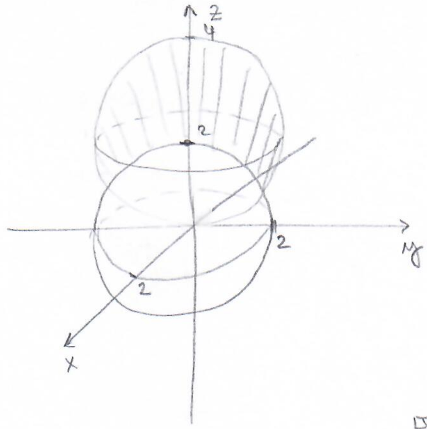
$$= \frac{8}{3}\int_0^{\sqrt{2}/2} 1 - t^2\, dt = \frac{8}{3}\cdot\left(t - \frac{t^3}{3}\right)\Big|_0^{\frac{\sqrt{2}}{2}} = \frac{10\sqrt{2}}{9}$$

(6.)

$$x^2+y^2+z^2 \leq 4z$$

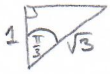
$$\Longleftrightarrow \quad x^2+y^2+(z-2)^2 \leq 4$$

$$x^2+y^2+z^2 \geq 4$$



$$x^2+y^2+(z-2)^2=4 \ \cap \ x^2+y^2+z^2=4$$

$$= \{ (x,y,z) \in \mathbb{R}^3 \mid z=1,\ x^2+y^2=3 \}$$



Koristimo sferne koordinate: $\theta \in [0, \frac{\pi}{3}]$ jer je $tg(\frac{\pi}{3})=\sqrt{3}$.

$\varphi \in [0, 2\pi]$

$r \in [2, 4\cos\theta]$ što dobijemo uvrštavanjem sfernih koordinata u jednadžbu $x^2+y^2+(z-2)^2=4$.

$J = r^2 \sin\theta$

$$\Rightarrow \quad V = \iiint_V dV = \int_0^{2\pi} d\varphi \int_0^{\pi/3} \sin\theta \, d\theta \int_2^{4\cos\theta} r^2 \, dr =$$

$$= 2\pi \int_0^{\pi/3} \sin\theta \left(\frac{64}{3}\cos^3\theta - \frac{8}{3} \right) d\theta =$$

$$= \frac{2\pi}{3} \left(64 \frac{\cos^4\theta}{4} \Big|_{\frac{\pi}{3}}^{0} + 8\cos\theta \Big|_0^{\pi/3} \right) =$$

$$= \frac{2\pi}{3} (16 - 1 + 4 - 8) = \frac{22\pi}{3}$$