1. a)

Ako takav a postoji onda je jedinstven i jednak limesu $\lim_{(x,y)\to(0,0)} f(x,y)$

Računamo:

$$\lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = \lim_{r\to 0} \frac{r^3(\cos^3\varphi + \sin^3\varphi)}{r^2} = \lim_{r\to 0} r(\cos^3\varphi + \sin^3\varphi)$$

Budući da je $\cos^3\varphi + \sin^3\varphi$ ograničena, po teoremu o sendviču dobivamo da je traženi limes jednak 0.

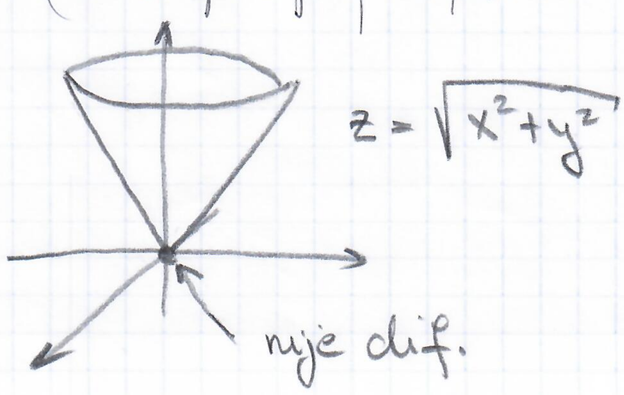
b) Definicija iz skripte.

c) T1: Istinita. Dokaz u skripti.

T2: Lažna:

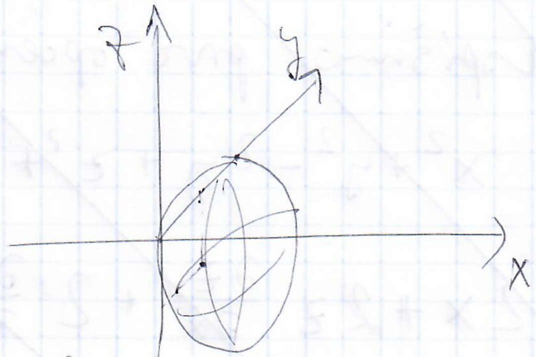
Protuprimjer: $f(x,y) = \sqrt{x^2+y^2}$

neprekidna u $(0,0)$ no nije diferencijabilna u $(0,0)$ (ne postoji parcijalna derivacija $\frac{\partial f}{\partial x}(0,0)$)



2. a) Zapišimo zadanu jednadžbu u obliku $x^2+(y-1)^2+(z+1)^2=2$.

$\Rightarrow$ Ploha je sfera radijusa $\sqrt{2}$ s centrom u $(0,1,-1)$.



b) Ploha je implicitno zadana s

$F(x,y,z)=0$, gdje je $F(x,y,z)=x^2+(y-1)^2+(z+1)^2-2$.

$\nabla F(x,y,z)=(2x,2y-2,2z+2)$, $\nabla F(0,2,0)=(0,2,2)$

Tangencijalna ravnina na sferu u točki $(0,2,0)$ zadana je s $(x-0,y-2,z-0)\cdot\nabla F(0,2,0)=0$, odnosno $2y+2z-4=0$

c)

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{2y-2}{2z+2}$$

$$\frac{\partial z}{\partial y}(0,2) = -\frac{2}{0+2} = -1$$

$$\frac{\partial^2 z}{\partial y^2} = -\frac{2(2z+2)-(2y-2)\cdot 2\frac{\partial z}{\partial y}}{(2z+2)^2}$$

$$\frac{\partial^2 z}{\partial y^2}(0,2) = -\frac{2\cdot 2-2\cdot 2\cdot(-1)}{(0+2)^2} = -2$$

d) $z = z(x,y)$ nije jedinstveno definirana u točkama plohe u kojima je $\frac{\partial F}{\partial z} = 0$

$$\Rightarrow \quad 2z+2=0$$

$$z=-1$$

3. a) Definicija iz skripte.

b) Dokaz iz skripte.

c) Nužan uvjet za lokalni ekstrem je

$\frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial y}(x,y)$ pa potražimo prvo takve točke.

$$\frac{\partial f}{\partial x}(x,y) = 3x^2 + 3y^2 - 15 = 0 \qquad (1)$$

$$\frac{\partial f}{\partial y}(x,y) = 6xy - 12 = 0 \qquad (2)$$

Zbrajanjem (1) i (2) dobivamo

$3(x+y)^2 - 27 = 0$, odnosno $x+y = \pm 3$.

Uvrštavanjem $x = -y \pm 3$ u (2) i rješavanjem kvadratne jednadžbe dobivamo četiri kandidata:

$\boxed{\begin{matrix} x=1 \\ y=2 \end{matrix}}$ $\boxed{\begin{matrix} x=2 \\ y=1 \end{matrix}}$ $\boxed{\begin{matrix} x=-1 \\ y=-2 \end{matrix}}$ $\boxed{\begin{matrix} x=-2 \\ y=-1 \end{matrix}}$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad \frac{\partial^2 f}{\partial x \partial y} = 6y \qquad \frac{\partial^2 f}{\partial y^2} = 6x$$

$$\Rightarrow H_f(x,y) = \begin{bmatrix} 6x & 6y \\ 6y & 6x \end{bmatrix}$$

$$H_f(1,2) = \begin{bmatrix} 6 & 12 \\ 12 & 6 \end{bmatrix} \quad \left.\begin{array}{l} 6 > 0 \\ 6 \cdot 6 - 12^2 < 0 \end{array}\right\} \text{ indefinitna forma}$$

$$H_f(2,1) = \begin{bmatrix} 12 & 6 \\ 6 & 12 \end{bmatrix} \quad \left.\begin{array}{l} 12 > 0 \\ 12 \cdot 12 - 6^2 > 0 \end{array}\right\} \text{ pozitivno definitna forma}$$

$$H_f(-1,-2) = \begin{bmatrix} -6 & -12 \\ -12 & -6 \end{bmatrix} \quad \left.\begin{array}{l} 6 < 0 \\ (-6)(-6) - 12^2 < 0 \end{array}\right\} \text{ indefinitna forma}$$

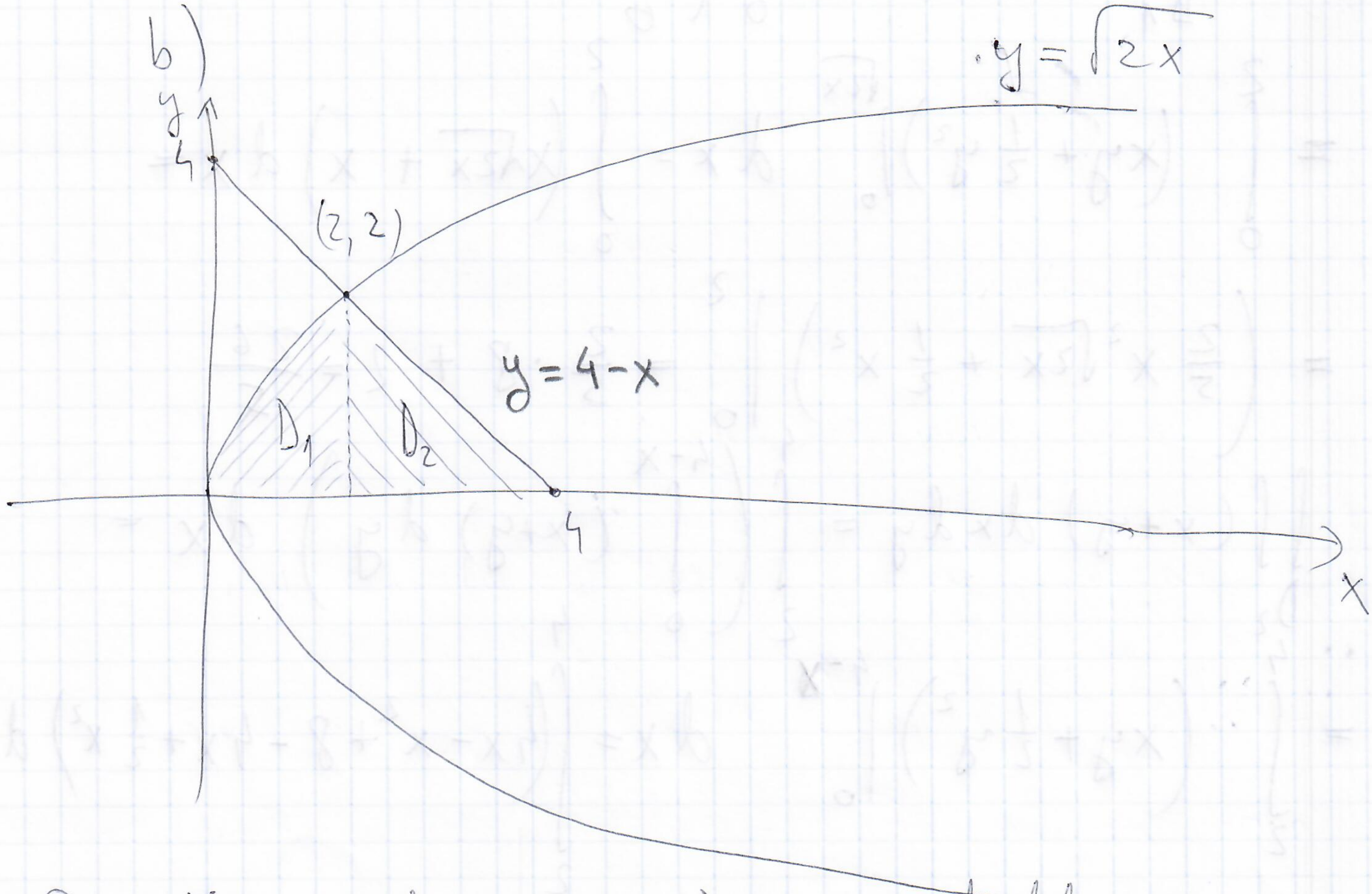
$$H_f(-2,-1) = \begin{bmatrix} -12 & -6 \\ -6 & -12 \end{bmatrix} \quad \left.\begin{array}{l} -12 < 0 \\ (-12)(-12) - (-6)^2 > 0 \end{array}\right\} \text{ negativno definitna forma}$$

$\Rightarrow$ $(2,1)$ je lokalni minimum, a

$(-2,-1)$ je lokalni maksimum

4. a) Definicija iz skripte.

b)



Osjenčano područje je D, no podijelili smo ga na dva dijela zbog lakšeg računanja.

$$\iint_D (x+y)\,dx\,dy = \iint_{D_1} (x+y)\,dx\,dy + \iint_{D_2} (x+y)\,dx\,dy$$

Primjenom Fubinijevog teorema računamo integrale na D_1 i D_2.

1. način:

$$\iint_{D_1} (x+y)\,dx\,dy = \int_0^2 \left(\int_0^{\sqrt{2x}} (x+y)\,dy \right) dx =$$

$$= \int_0^2 \left(xy + \frac{1}{2}y^2 \right)\Big|_0^{\sqrt{2x}} dx = \int_0^2 \left(x\sqrt{2x} + x \right) dx =$$

$$= \left(\frac{2}{5}x^2\sqrt{2x} + \frac{1}{2}x^2 \right)\Big|_0^2 = \frac{2}{5}\cdot 8 + 2 = \frac{26}{5}$$

$$\iint_{D_2} (x+y)\,dx\,dy = \int_2^4 \left(\int_0^{4-x} (x+y)\,dy \right) dx =$$

$$= \int_2^4 \left(xy + \frac{1}{2}y^2 \right)\Big|_0^{4-x} dx = \int_2^4 \left(4x - x^2 + 8 - 4x + \frac{1}{2}x^2 \right) dx =$$

$$= \left(-\frac{1}{6}x^3 + 8x \right)\Big|_2^4 = -\frac{64}{6} + 32 + \frac{8}{6} - 16 =$$

$$= -\frac{56}{6} + 16$$

$$\iint_D (x+y)\,dx\,dy = \frac{16}{5} - \frac{\cancel{56}^{28}}{\cancel{6}_3} + 18 = \frac{178}{15}$$

2. način:

$$\int_0^2 dy \int_{\frac{y^2}{2}}^{4-y} (x+y)\,dx = \int_0^2 \left(\frac{x^2}{2} + yx \right)\Big|_{\frac{y^2}{2}}^{4-y} dy$$

$$= \int_0^2 \left(8 - \cancel{4y} + \frac{y^2}{2} + \cancel{4y} - y^2 - \frac{y^4}{8} - \frac{y^3}{2} \right) dy$$

$$= \left(8y - \frac{1}{6}y^3 - \frac{1}{40}y^5 - \frac{1}{8}y^4 \right)\Big|_0^2 = \frac{178}{15}$$

5. a) $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \Rightarrow f^{(n)}(0) = 1$

$$\Rightarrow \sum_{n=0}^{\infty} x^n \frac{f^{(n)}(0)}{n!} = \sum_{n=0} \frac{x^n}{n!}$$ je Taylorov

razvoj funkcije f oko 0.

b) Promatramo ostatak $f(x) - \sum_{k=1}^{n} x^k \frac{f^{(k)}(0)}{n!} = R_n^L(x)$

u Lagrangeovom obliku $R_n^L(x) = \frac{f^{(n+1)}(c_x)}{n!} x^{n+1}$,

gdje je c_x između 0 i x

$$|R_n^L(x)| = \left| \frac{e^{c_x}}{n!} x^{n+1} \right| \leq \frac{e^{|x|}}{n!} |x|^{n+1} \xrightarrow{n \to \infty} 0, \forall x \in \mathbb{R}$$

Budući da ostatak konvergira u 0 za svaki $x \in \mathbb{R}$, područje konvergencije reda je cijeli $\mathbb{R}$.

c) Promatrajmo funkciju $g(x) = x e^x = x f(x)$.

Iz razvoja funkcije f u Taylorov red dobivamo razvoj funkcije g, $g(x) = \sum_{n=0}^{\infty} x^{n+1} \frac{1}{n!}$.

Deriviranjem član po član dobivamo

$$g'(x) = \sum_{n=0} x^n \frac{n+1}{n!},$$ no znamo da je

$$g'(x) = e^x + x e^x$$

$$\Rightarrow \sum_{n=0} \frac{1}{2^n} \frac{n+1}{n!} = g'\left(\frac{1}{2}\right) = e^{\frac{1}{2}} + \frac{1}{2} e^{\frac{1}{2}} = \frac{3}{2}\sqrt{e}$$

6\. a) $P(x,y) = 2x^2 + y^2$

$Q(x,y) = -2xy$

$P(tx,ty) = 2(tx)^2 + (ty)^2 =$

$= t^2(2x^2+y^2) = t^2 P(x,y)$

$Q(tx,ty) = -2txty = t^2(-2xy) = t^2 Q(x,y)$

$\Rightarrow$ Jednadžba je homogena stupnja homogenosti 2.

b)

$2x^2+y^2-2xyy'=0$, odnosno

$y' - \frac{1}{2x} y = xy^{-1}$

$\Rightarrow$ Jednadžba je Bernoullijeva

c) Množenjem s $2y$ i uvođenjem supstitucije $z=y^2$ dobivamo

~~$\frac{y}{y}$~~ $z' - \frac{1}{x} z = 2x$

$z(x) = e^{\int \frac{1}{x}dx}\left(c + \int 2x e^{-\int \frac{1}{x}dx}\right) = e^{\ln x}\left(c + \int 2x e^{-\ln x}\, dx\right) =$

$= x\left(c + \int 2\, dx\right) = x(c+2x) = cx + 2x^2$

$y^2 = cx + 2x^2$ je implicitno zadano rješenje

2. način: $y' - \frac{1}{2}\frac{y}{x} = \frac{x}{y}$ supst. $u = \frac{y}{x}$

$y' = u'x + u$

$$u'x + u - \frac{1}{2}u = \frac{1}{u}$$

$$\frac{du}{dx} \cdot x = \frac{2-u^2}{2u}$$

$$\int \frac{2u}{u^2-2}\,du = -\int \frac{dx}{x} \quad / \int$$

$$\ln|u^2-2| = -\ln|x| + \ln C$$

$$\ln\left|\frac{y^2}{x^2} - 2\right| = \ln\left|\frac{C}{x}\right| \quad / e$$

$$\frac{y^2 - 2x^2}{x^2} = \frac{C}{x} \quad / \cdot x^2$$

$$y^2 = Cx + 2x^2$$

7.

$$\frac{\partial P}{\partial y} = 2y\cos(xy^2) - y^2\sin(xy^2)\cdot 2xy$$

$$\frac{\partial Q}{\partial x} = 2y\cos(xy^2) - 2xy\sin(xy^2)\cdot y^2$$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\Rightarrow$ jednadžba je egzaktna

$$U(x,y) = \int_0^x P(s,y)\,ds + \int_0^y Q(0,t)\,dt + C =$$

$$= \int_0^x \left[2s + y^2\cos(sy^2)\right] ds + \int_0^y 3t^2\,dt + C =$$

$$= s^2 + \sin(sy^2)\Big|_0^x + t^3\Big|_0^y + C =$$

$$= x^2 + \sin(xy^2) + y^3 + C$$

Rješenja $y = y(x)$ implicitno su zadana jednadžbama $x^2 + \sin(xy^2) + y^3 = c$.

8. a) Neka $r_1 \neq r_2$. Promatramo Wronskijan funkcija $e^{r_1 x}$ i $e^{r_2 x}$.

$$\frac{d}{dx} e^{r_1 x} = r_1 e^{r_1 x}, \quad \frac{d}{dx} e^{r_2 x} = r_2 e^{r_2 x}$$

$$W\left(e^{r_1 x}, e^{r_2 x}\right) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix} = (r_2 - r_1) e^{(r_1 + r_2)x} \neq 0, \forall x \in \mathbb{R}$$

$\Rightarrow$ $e^{r_1 x}$ i $e^{r_2 x}$ su linearno nezavisne

b) Rješavamo metodom desne strane.

Prvo tražimo partikularna rješenja za

$$(p_1) \qquad y'' - 5y' + 6y = 3x$$

i

$$(p_2) \qquad y'' - 5y' + 6y = e^{2x}$$

(p_1) Potražimo rješenje u obliku $y_p(x) = Ax + B$

$$y_p'(x) = A, \quad y_p''(x) = 0$$

$$3x = y_p'' - 5y_p' + 6y_p = -5A + 6Ax + 6B$$

$$\Rightarrow 6A = 3, \quad 6B - 5A = 0$$

$$\Rightarrow A = \frac{1}{2}, \quad B = \frac{5}{12}$$

(p_2) Potražimo rješenje u obliku

$$y_p(x) = Cxe^{2x}$$

$$y_p'(x) = Ce^{2x} + 2Cxe^{2x}$$

$$y_p''(x) = 4Ce^{2x} + 4Cxe^{2x}$$

$$e^{2x} = y_p'' - 5y_p' + 6y_p = -Ce^{2x} \Rightarrow C = -1.$$

Linearno nezavisna rješenja homogene jednadžbe su ~~oblika~~ e^{2x} i e^{3x} pa konačno rješenje tražimo u obliku

$$\underline{y}(x) = Ae^{2x} + Be^{3x} - xe^{2x} + \frac{1}{2}x + \frac{5}{12}$$

$$y'(x) = 2Ae^{2x} + 3Be^{3x} - e^{2x} - 2xe^{2x} + \frac{1}{2}$$

$$0 = y(0) = A + B + \frac{5}{12}$$

$$1 = y'(0) = 2A + 3B - 1 + \frac{1}{2}$$

$$\left.\begin{array}{l} A + B = -\frac{5}{12} \\ 2A + 3B = \frac{3}{2} \end{array}\right| \Rightarrow \begin{array}{l} B = \frac{7}{3} \\ A = -\frac{11}{4} \end{array}$$

$$y(x) = -\frac{11}{4}e^{2x} + \frac{7}{3}e^{3x} - xe^{2x} + \frac{1}{2}x + \frac{5}{12}$$

b) 2. način (metoda varijacije konstanti

Linearno nezavisna rješenja homogene jednadžbe su e^{2x} i e^{3x} pa potražimo konačno rješenje u obliku:

$$y(x) = A(x)e^{2x} + B(x)e^{3x}.$$

Dobivamo sljedeći sustav uvjeta:

$$A'(x)e^{2x} + B'(x)e^{3x} = 0 \qquad (\mathrm{I})$$

$$A'(x)2e^{2x} + B'(x)3e^{3x} = 3x + e^{2x} \qquad (\mathrm{II})$$

$$(-2)\cdot(\mathrm{I}) + 1\cdot(\mathrm{II}) \Rightarrow B'(x)e^{3x} = 3x + e^{2x}$$

$$\Rightarrow B'(x) = 3xe^{-3x} + e^{-x}$$

$$\Rightarrow B(x) = -xe^{-3x} - e^{-x} - \frac{1}{3}e^{-3x} + b$$

$$\Rightarrow A'(x) = -B(x)e^{x} = -3xe^{-2x} - 1$$

$$A(x) = \frac{3}{2}xe^{-2x} - x + \frac{3}{4}e^{-2x} + a$$

$$y(x) = \frac{3}{2}x - xe^{2x} + \frac{3}{4} + ae^{2x} - x - e^{2x} - \frac{1}{3} + be^{3x}$$

$$y'(x) = \frac{3}{2} - e^{2x} - 2xe^{2x} + 2ae^{2x} - 1 - 2e^{2x} + 3be^{3x}$$

$$0 = y(0) = \frac{3}{4} + a - 1 - \frac{1}{3} + b$$

$$1 = y'(0) = \frac{3}{2} - 1 + 2a - 1 - 2 + 3b$$

$$a + b = \frac{7}{12}$$

$$2a + 3b = \frac{7}{2} = \frac{42}{12}$$

$$= b = \frac{28}{12} = \frac{7}{3}$$

$$a = -\frac{21}{12} = -\frac{7}{4}$$

$$y(x) = \frac{1}{2}x - xe^{2x} - \frac{11}{4}e^{2x} + \frac{7}{3}e^{3x} + \frac{5}{12}$$