MATEMATIČKA ANALIZA 2

Međuispit (26.4.2021.)

– RJEŠENJA ZADATAKA –

1. (a) Kažemo da je funkcija $f : \mathbb{R}^2 \to \mathbb{R}$ NEPREKIDNA u točki (x_0, y_0) ako postoji limes $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ i vrijedi

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0).$$

(b) Promotrimo ponašanje sljedećih dviju restrikcija od f:

$$\lim_{x\to 0} f(x,x^3) = \lim_{x\to 0} \frac{x^4}{4x^4+3x^4} = \lim_{x\to 0} \frac{x^4}{7x^4} = \frac{1}{7},$$

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{x\cdot 0}{4x^4+3\cdot 0} = \lim_{x\to 0} 0 = 0.$$

2. način: polarne koord.

$$\lim_{r\to 0} \frac{r^2\cos\varphi\sin\varphi}{4r^4\cos^4\varphi + 3r^{\frac{4}{3}}\sin^{\frac{4}{3}}\varphi}$$

$$= \lim_{r\to 0} \frac{r^{\frac{2}{3}}\cos\varphi\sin\varphi}{4r^{\frac{8}{3}}\cos^4\varphi + 3\sin^{\frac{4}{3}}\varphi}$$

$\Rightarrow$ limes ovisi o φ

Budući da se dobivene vrijednosti razlikuju, ne postoji limes od f u $(0,0)$ pa f u toj točki nije neprekidna.

(c) TOČNO

Ako je f diferencijabilna u (x_0, y_0), onda postoje $\frac{\partial f}{\partial x}(x_0,y_0), \frac{\partial f}{\partial y}(x_0,y_0) \in \mathbb{R}$ te vrijedi

$$f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y + \sigma(\Delta x, \Delta y), \quad (*)$$

gdje

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\sigma(\Delta x,\Delta y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0. \qquad (**)$$

No, zbog $(**)$ mora vrijediti i $\lim_{(\Delta x,\Delta y)\to(0,0)} \sigma(\Delta x,\Delta y) = 0$ pa puštanjem $(*)$ na limes kada $(\Delta x,\Delta y)\to(0,0)$ dobivamo

$$\lim_{(\Delta x,\Delta y)\to(0,0)} f(x_0+\Delta x, y_0+\Delta y) = f(x_0,y_0)$$

$$\Leftrightarrow \lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0),$$

tj. f je neprekidna u (x_0,y_0).

(d) Prema (c) podzadatku, funkcija iz (b) nije diferencijabilna. Naime, ta funkcija nije diferencijabilna u $(0,0)$ jer u toj točki nije neprekidna.

2. (a) Definiramo funkciju

$$F(x,y,z) = x^2 + y^2 + z^2 - 2x + 4y - 6z + 5.$$

Budući da je

$$\frac{\partial F}{\partial z}(x,y,z) = 2z - 6$$

i posebno,

$$\frac{\partial F}{\partial z}(1,-2,0) = -6 \neq 0,$$

prema teoremu o implicitnoj funkciji, z se može izraziti kao funkcija varijabli x i y na nekoj okolini točke $(1,-2)$.

(b) Vektor normale tangencijalne ravnine na zadanu plohu u točki (x,y,z) je jednak

$$\nabla F(x,y,z) = (2x-2,\ 2y+4,\ 2z-6)$$

i prema uvjetu zadatka taj vektor mora biti kolinearan s vektorom normale ravnine $x - 2y + 2z = 3$, tj. mora postojati $\lambda \in \mathbb{R}$ takav da

$$(2x-2,\ 2y+4,\ 2z-6) = \lambda(1,-2,2)$$

$$\Rightarrow \quad x = \tfrac{1}{2}(\lambda+2), \quad y = \tfrac{1}{2}(-2\lambda-4) = -\lambda-2, \quad z = \tfrac{1}{2}(2\lambda+6) = \lambda+3$$

Budući da tražene točke moraju ležati na plohi

$$\frac{1}{4}(\lambda+2)^2 + (\lambda+2)^2 + (\lambda+3)^2 - (\lambda+2) - 4(\lambda+2) - 6(\lambda+3) + 5 = 0$$

$$5(\lambda^2+4\lambda+4) + 4(\lambda^2+6\lambda+9) - 20(\lambda+2) - 24(\lambda+3) + 20 = 0$$

$$9\lambda^2 - 36 = 0$$

$$\lambda^2 = 4$$

$\lambda_1 = 2$ $\qquad\qquad$ $\lambda_2 = -2$

$\boxed{(x,y,z) = (2,-4,5)}$ $\qquad\qquad$ $\boxed{(x,y,z) = (0,0,1)}$

3. (a) USMJERENA DERIVACIJA funkcije f u točki $T_0 \in \mathbb{R}^2$ u smjeru vektora $\vec{v} \in V^2$ je broj

$$\frac{\partial f}{\partial \vec{v}}(T_0) = \lim_{h \to 0} \frac{f(T_0 + h\vec{v}_0) - f(T_0)}{h},$$

gdje je $\vec{v}_0 = \frac{1}{\|\vec{v}\|}\vec{v}$ jedinični vektor u smjeru vektora $\vec{v}$.

Definirajmo funkciju

$$g(h) := f(T_0 + h\vec{v}_0).$$

Tada iz definicije usmjerene derivacije slijedi

$$\frac{\partial f}{\partial \vec{v}}(T_0) = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0).$$

S druge strane, uz $T_0 = (x_0, y_0)$ i $\vec{v}_0 = v_{0x}\vec{i} + v_{0y}\vec{j}$ te stavljanjem

$$x(h) := x_0 + h v_{0x}, \quad y(h) := y_0 + h v_{0y},$$

vidimo da je

$$g(h) = f(x(h), y(h))$$

pa $g'(0)$ možemo računati i koristeći lančano pravilo:

$$g'(0) = \frac{\partial f}{\partial x}(x(0), y(0)) \cdot \frac{dx}{dh}(0) + \frac{\partial f}{\partial y}(x(0), y(0)) \cdot \frac{dy}{dh}(0)$$

$$= \frac{\partial f}{\partial x}(T_0)\, v_{0x} + \frac{\partial f}{\partial y}(T_0)\, v_{0y}$$

$$\Rightarrow \frac{\partial f}{\partial \vec{v}}(T_0) = \nabla f(T_0) \cdot \vec{v}_0$$

(b) Uočimo da treba izračunati usmjerenu derivaciju funkcije f u točki T_1 u smjeru vektora $\overrightarrow{T_1T_2}$. Imamo:

$$\frac{\partial f}{\partial x}(x,y) = 6xy - 2y + 5 \quad \Rightarrow \quad \frac{\partial f}{\partial x}(T_1) = 15,$$

$$\frac{\partial f}{\partial y}(x,y) = 3x^2 - 2x - 3 \quad \Rightarrow \quad \frac{\partial f}{\partial y}(T_1) = 5,$$

$$\vec{v} = \overrightarrow{T_1T_2} = 3\vec{i} - 4\vec{j} \quad \Rightarrow \quad \vec{v}_0 = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}$$

$$\Rightarrow \frac{\partial f}{\partial \vec{v}}(T_1) = \nabla f(T_1) \cdot \vec{v}_0 = 15 \cdot \frac{3}{5} + 5 \cdot \left(-\frac{4}{5}\right) = 5$$

(c) Treba odrediti smjer u kojem f najbrže pada iz točke T_1. To je u smjeru

$$-\nabla f(T_1) = -15\vec{\imath} - 5\vec{\jmath}$$

te usmjerena derivacija u tom smjeru iznosi

$$-\|\nabla f(T_1)\| = -\sqrt{15^2+5^2} = -5\sqrt{10}.$$

(d) Tražimo smjer u kojem je usmjerena derivacija od f u T_1 jednaka nuli. Iz

$$\frac{\partial f}{\partial \vec{v}}(T_1) = \nabla f(T_1) \cdot \vec{v}_0 = 0$$

vidimo da je $\vec{v} \perp \nabla f(T_1)$ pa je traženi smjer

$$\vec{v} = 5\vec{\imath} - 15\vec{\jmath}.$$

4\. (a) Lagrangeova funkcija:

$$L(x,y,\lambda) = e^{x^2+xy+y^2} + \lambda(x^2+y^2-2)$$

Tražimo njene stacionarne točke:

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,\lambda) = e^{x^2+xy+y^2}\cdot(2x+y) + 2\lambda x = 0 \\ \frac{\partial L}{\partial y}(x,y,\lambda) = e^{x^2+xy+y^2}\cdot(x+2y) + 2\lambda y = 0 \\ \frac{\partial L}{\partial \lambda}(x,y,\lambda) = x^2+y^2-2 = 0 \end{cases}$$

Zbrajanjem prvih dviju jednadžbi slijedi

$$e^{x^2+xy+y^2}\cdot 3(x+y) + 2\lambda(x+y) = 0$$

$$\Rightarrow (x+y)\left(3e^{x^2+xy+y^2} + 2\lambda\right) = 0,$$

odakle dobivamo dva slučaja:

1° $x+y=0 \Rightarrow x=-y$

Uvrštavanjem u treću jednadžbu slijedi

$$y^2+y^2-2=0$$

$$y^2 = 1$$

$y_1 = 1$, $x_1 = -1$

$e^{2-1}\cdot(-2+1) - 2\lambda = 0$

$\Rightarrow \lambda_1 = -\frac{1}{2}e$

$y_2 = -1$, $x_2 = 1$

$e^{2-1}\cdot(2-1) + 2\lambda = 0$

$\Rightarrow \lambda_2 = -\frac{1}{2}e$

$\Rightarrow$ stacionarne točke su $T_1\left(-1,1,-\frac{1}{2}e\right)$ i $T_2\left(1,-1,-\frac{1}{2}e\right)$

2° $3e^{x^2+xy+y^2} + 2\lambda = 0 \Rightarrow 2\lambda = -3e^{x^2+xy+y^2}$

Uvrštavanjem u npr. prvu jednadžbu dobivamo

$$e^{x^2+xy+y^2}\cdot(2x+y) - 3e^{x^2+xy+y^2}x = 0 \quad \Big/ : e^{x^2+xy+y^2} > 0$$

$$\Rightarrow -x+y=0 \Rightarrow x=y$$

Sada iz treće jednadžbe slijedi

$$y^2 + y^2 - 2 = 0$$
$$y^2 = 1$$

$y_3 = 1$

$x_3 = 1$

$e^{2+1} \cdot (2+1) + 2\lambda = 0$

$\lambda_3 = -\frac{3}{2} e^3$

$y_4 = -1$

$x_4 = -1$

$e^{2+1} \cdot (-2-1) - 2\lambda = 0$

$\lambda_4 = -\frac{3}{2} e^3$

$\Rightarrow$ stacionarne točke su $T_3\left(1, 1, -\frac{3}{2}e^3\right)$ i $T_4\left(-1, -1, -\frac{3}{2}e^3\right)$

Ispitujemo karakter dobivenih točaka:

$$\frac{\partial^2 L}{\partial x^2}(x,y,\lambda) = e^{x^2+xy+y^2}(2x+y)^2 + e^{x^2+xy+y^2} \cdot 2 + 2\lambda$$

$$\frac{\partial^2 L}{\partial y^2}(x,y,\lambda) = e^{x^2+xy+y^2}(x+2y)^2 + e^{x^2+xy+y^2} \cdot 2 + 2\lambda$$

$$\frac{\partial^2 L}{\partial x \partial y}(x,y,\lambda) = e^{x^2+xy+y^2}(2x+y)(x+2y) + e^{x^2+xy+y^2}$$

$$\Rightarrow d^2L(T) = \frac{\partial^2 L}{\partial x^2}(T)(dx)^2 + 2\frac{\partial^2 L}{\partial x \partial y}(T)\, dx\, dy + \frac{\partial^2 L}{\partial y^2}(T)(dy)^2$$

Diferencijal uvjeta: $x^2 + y^2 = 2 \quad /d \quad \Rightarrow \quad 2x\,dx + 2y\,dy = 0 \Rightarrow x\,dx + y\,dy = 0$

U stacionarnim točkama imamo:

$$d^2L(T_{1,2}) = \left(e^{2-1} \cdot 1^2 + e^{2-1} \cdot 2 - e\right)(dx)^2 + 2\left(e^{2-1} \cdot (-1) + e^{2-1}\right) dx\,dy$$
$$+ \left(e^{2-1} \cdot 1^2 + e^{2-1} \cdot 2 - e\right)(dy)^2$$
$$= 2e(dx)^2 + 2e(dy)^2 > 0 \text{ za } (dx, dy) \neq (0,0)$$

$\Rightarrow T_{1,2}$ uvjetni lokalni minimumi

$f(T_1) = f(T_2) = e$

$$d^2L(T_{3,4}) = \left(e^{2+1} \cdot 3^2 + e^{2+1} \cdot 2 - 3e^3\right)(dx)^2 + 2\left(e^{2+1} \cdot 3^2 + e^{2+1}\right) dx\,dy$$
$$+ \left(e^{2+1} \cdot 3^2 + e^{2+1} \cdot 2 - 3e^3\right)(dy)^2$$
$$= 8e^3(dx)^2 + 20e^3\,dx\,dy + 8e^3(dy)^2 = \left[\begin{array}{l}\text{iz diferencijala uvjeta} \\ dx + dy = 0 \Rightarrow dy = -dx\end{array}\right]$$
$$= 16e^3(dx)^2 - 20e^3(dx)^2$$
$$= -4e^3(dx)^2 < 0 \text{ za } (dx, dy) \neq (0,0)$$

$\Rightarrow T_{3,4}$ uvjetni lokalni maksimumi

$f(T_3) = f(T_4) = e^3$

(b) Budući da je funkcija f neprekidna, a skup $\{(x,y)\in\mathbb{R}^2 : x^2+y^2\leq 2\}$ omeđen i zatvoren, ona na tom skupu mora postizati minimum i maksimum.

Kandidati za točke ekstrema su tzv. kritične točke:

1) točke ekstrema od f na rubu skupa, tj. na kružnici $x^2+y^2=2$

Iz (a) podzadatka znamo da su to točke $T_{1,2,3,4}$.

2) stacionarne točke od f unutar skupa (ako postoje)

Računamo:

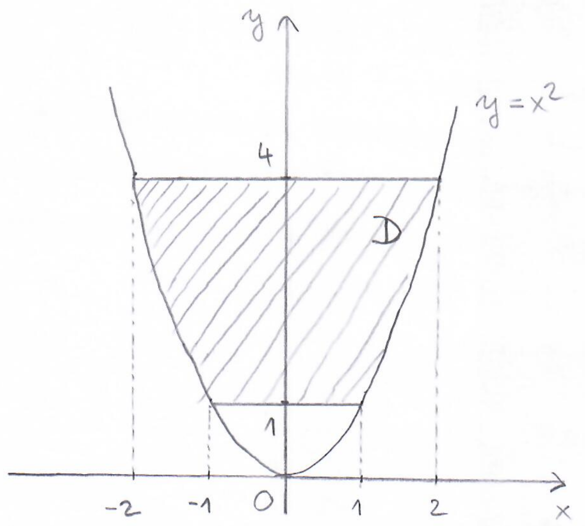
$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = e^{x^2+xy+y^2}(2x+y)=0 & \Big|: e^{x^2+xy+y^2}>0 \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = e^{x^2+xy+y^2}(x+2y)=0 & \Big|: e^{x^2+xy+y^2}>0 \end{cases}$$

$$\Rightarrow \begin{cases} 2x+y=0 \\ x+2y=0 \end{cases} \Rightarrow (x,y)=(0,0)$$

$T_5(0,0)$ je jedina stacionarna točka od f i nalazi se unutar skupa

Budući da je $f(T_5)=e^0=1$, $f(T_{1,2})=e$ i $f(T_{3,4})=e^3$, slijedi da f u $(0,0)$ postiže globalni minimum, dok u $(1,1)$ i $(-1,-1)$ postiže globalni maksimum na zadanom skupu.

5.



(a)

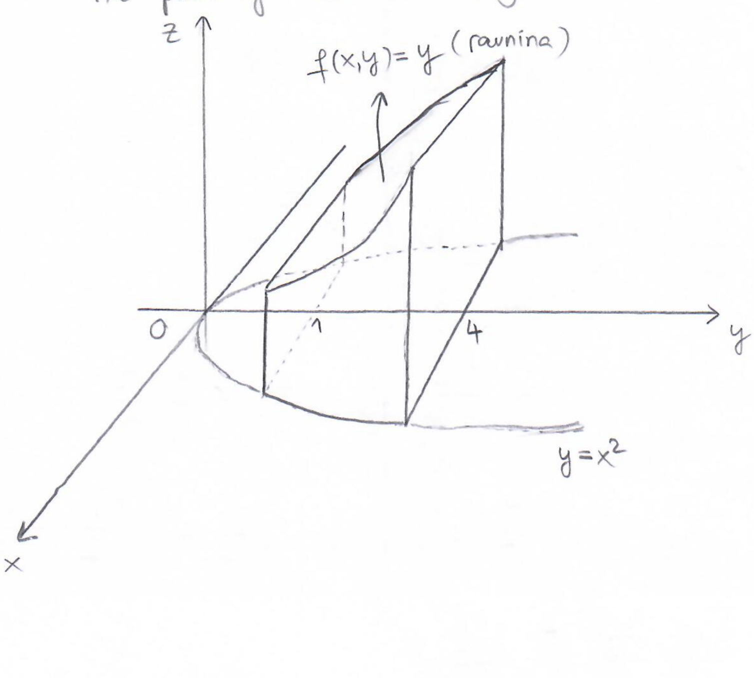
$$I = \iint\limits_D y \, dx\, dy = \int_1^4 \int_{-\sqrt{y}}^{\sqrt{y}} y \, dx\, dy$$

$$= \int_{-2}^{-1}\int_{x^2}^{4} y \, dy\,dx + \int_{-1}^{1}\int_{1}^{4} y \, dy\,dx + \int_{1}^{2}\int_{x^2}^{4} y \, dy\,dx$$

$$I = \int_1^4 \int_{-\sqrt{y}}^{\sqrt{y}} y \, dx\, dy = \int_1^4 y \cdot \left(x \Big|_{-\sqrt{y}}^{\sqrt{y}} \right) dy = \int_1^4 2y\sqrt{y}\, dy$$

$$= 2\int_1^4 y^{\frac{3}{2}}\, dy = 2 \cdot \frac{1}{\frac{5}{2}}\, y^{\frac{5}{2}} \Big|_1^4 = \frac{4}{5}\cdot\left(2^5 - 1\right) = \frac{124}{5}$$

(b) Gornjim je integralom izražen volumen ispod grafa funkcije $f(x,y) = y$ na području D u xOy ravnini:



(6.) (a) $$\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J|\,du\,dv\,dw,$$

gdje V' praslika područja V po preslikavanju

$$(u,v,w) \longmapsto (x(u,v,w), y(u,v,w), z(u,v,w)),$$

a J Jacobijan tog preslikavanja

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

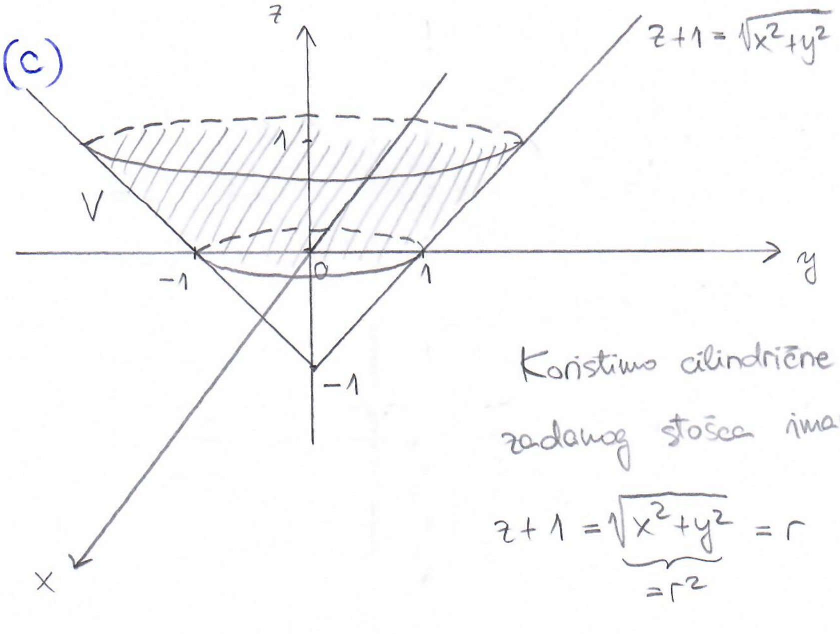
(b) Cilindrične koordinate:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = z, \qquad r \geq 0,\ \varphi \in [0, 2\pi],\ z \in \mathbb{R}.$$

Jacobijan je prema (a) podzadatku jednak

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1\cdot r\cdot \underbrace{(\cos^2\varphi + \sin^2\varphi)}_{=1} = r$$

(c)



Koristimo cilindrične koordinate. U točkama zadanog stošca imamo

$$z + 1 = \sqrt{\underbrace{x^2+y^2}_{=r^2}} = r \implies z = r - 1$$

Računamo:

$$\iiint\limits_V y^2\,dV = \left[\begin{array}{lll} \text{Cilindrične koordinate:} & & \\ x = r\cos\varphi & \varphi \in [0, 2\pi] & |J| = |r| = r \\ y = r\sin\varphi & z \in [0, 1] & \text{jer } r \geq 0 \\ z = z & r \in [0, z+1] & \end{array}\right]$$

$$= \int_0^{2\pi}\int_0^1\int_0^{z+1} r^2\sin^2\varphi \cdot r\,dr\,dz\,d\varphi = \int_0^{2\pi}\int_0^1 \sin^2\varphi\left(\frac{1}{4}r^4\Big|_0^{z+1}\right)dz\,d\varphi$$

$$= \frac{1}{4}\int_0^{2\pi}\int_0^1 \sin^2\varphi\,(z+1)^4\,dz\,d\varphi = \frac{1}{4}\int_0^{2\pi}\sin^2\varphi\left(\frac{1}{5}(z+1)^5\Big|_0^1\right)d\varphi$$

$$= \frac{1}{4}\cdot\frac{1}{5}\left(2^5 - 1^5\right)\int_0^{2\pi}\sin^2\varphi\,d\varphi = \frac{31}{20}\int_0^{2\pi}\frac{1}{2}\left(1 - \cos 2\varphi\right)d\varphi$$

$$= \frac{31}{20}\left(\frac{1}{2}\left(\varphi - \frac{1}{2}\sin 2\varphi\right)\Big|_0^{2\pi}\right) = \frac{31}{20}\cdot\frac{1}{2}\cdot\left(2\pi - 0 \underbrace{- \frac{1}{2}\sin 4\pi + \frac{1}{2}\sin 0}_{=0}\right)$$

$$= \frac{31}{20}\cdot\frac{1}{2}\cdot 2\pi = \frac{31\pi}{20}$$

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2. način:

$$\int_0^{2\pi}\sin^2\varphi\,d\varphi\int_0^1 r^3\,dr\int_0^1 dz + \int_0^{2\pi}\sin^2\varphi\,d\varphi\int_1^2 r^3\,dr\int_{r-1}^1 dz$$

$$= \pi\cdot\frac{1}{4}\cdot 1 + \pi\int_1^2 r^3(2-r)\,dr$$

$$= \frac{\pi}{4} + \pi\left(\frac{r^4}{2} - \frac{r^5}{5}\right)\Big|_1^2 = \frac{\pi}{4} + \frac{13\pi}{10} = \frac{31\pi}{20}$$