MATEMATIČKA ANALIZA 2

Ljetni ispitni rok (9.7.2020.)

– RJEŠENJA ZADATAKA –

1. (a) $f(x,y)=\sin(2x+3y)$

$$\vec{h}=\sqrt{3}\,\vec{i}-\vec{j} \Rightarrow \vec{h}_0=\frac{1}{\|\vec{h}\|}\vec{h}=\frac{1}{\sqrt{3+1}}(\sqrt{3}\,\vec{i}-\vec{j})=\frac{\sqrt{3}}{2}\vec{i}-\frac{1}{2}\vec{j}$$

$$\frac{\partial f}{\partial x}(x,y)=\cos(2x+3y)\cdot 2 \Rightarrow \frac{\partial f}{\partial x}(-6,4)=2\cos(-12+12)=2$$

$$\frac{\partial f}{\partial y}(x,y)=\cos(2x+3y)\cdot 3 \Rightarrow \frac{\partial f}{\partial y}(-6,4)=3\cos(-12+12)=3$$

$$\Rightarrow \frac{\partial f}{\partial \vec{h}}(-6,4)=\nabla f(-6,4)\cdot\vec{h}_0=(2\vec{i}+3\vec{j})\cdot\left(\frac{\sqrt{3}}{2}\vec{i}-\frac{1}{2}\vec{j}\right)=\sqrt{3}-\frac{3}{2}$$

Funkcija f najbrže pada iz točke $(-6,4)$ u smjeru jediničnog vektora

$$-\frac{1}{\|\nabla f(-6,4)\|}\nabla f(-6,4)=-\frac{1}{\sqrt{4+9}}(2\vec{i}+3\vec{j})=-\frac{2}{\sqrt{13}}\vec{i}-\frac{3}{\sqrt{13}}\vec{j}$$

te je minimalna vrijednost usmjerene derivacije jednaka $-\|\nabla f(-6,4)\|=-\sqrt{13}$.

(b) Teorem.

Neka je $U\subseteq\mathbb{R}^2$ otvoren i konveksan skup te $f:U\to\mathbb{R}$ diferencijabilna funkcija. Tada za svake dvije točke $\vec{a},\vec{b}\in U$ postoji točka $\vec{c}$ na njihovoj spojnici takva da

$$f(\vec{b})-f(\vec{a})=\nabla f(\vec{c})\cdot(\vec{b}-\vec{a}).$$

Dokaz.

Definiramo funkciju

$$g:\mathbb{R}\to\mathbb{R},\quad g(t)=f(\vec{a}+t(\vec{b}-\vec{a})).$$

Funkcija g je diferencijabilna i vrijedi $g(0)=f(\vec{a})$, $g(1)=f(\vec{b})$.

Prema lančanom pravilu

$$g'(t) = \nabla f(\vec{a} + t(\vec{b} - \vec{a})) \cdot \frac{d}{dt}(\vec{a} + t(\vec{b} - \vec{a}))$$

$$= \nabla f(\vec{a} + t(\vec{b} - \vec{a})) \cdot (\vec{b} - \vec{a}).$$

Primjenom Lagrangeovog teorema o srednjoj vrijednosti na funkciju jedne varijable g slijedi da postoji $s \in \langle 0, 1 \rangle$ takav da

$$g(1) - g(0) = g'(s)(1-0),$$

odnosno,

$$f(\vec{b}) - f(\vec{a}) = \nabla f(\vec{a} + s(\vec{b} - \vec{a})) \cdot (\vec{b} - \vec{a})$$

pa tvrdnja teorema slijedi stavljanjem $\vec{c} = \vec{a} + s(\vec{b} - \vec{a})$.

Q.E.D.

(c) Neka su $\vec{a}, \vec{b} \in U$ proizvoljni. Prema Lagrangeovom teoremu srednje vrijednosti slijedi da postoji $\vec{c} \in U$ na spojnici tih točaka takav da

$$f(\vec{b}) - f(\vec{a}) = \underbrace{\nabla f(\vec{c})}_{= \vec{0}} \cdot (\vec{b} - \vec{a}) = 0$$

$$\Rightarrow f(\vec{b}) = f(\vec{a}),$$

odakle zbog proizvoljnosti $\vec{a}$ i $\vec{b}$ slijedi da je f konstantna na U.

2. (a) $f(x,y) = e^{x^2} + \ln\frac{1}{xy} = e^{x^2} - \ln(xy)$ $\qquad T(1,1)$

$$\frac{\partial f}{\partial x}(x,y) = 2xe^{x^2} - \frac{1}{xy}\cdot y = 2xe^{x^2} - \frac{1}{x} \quad \Rightarrow \frac{\partial f}{\partial x}(1,1) = 2e-1$$

$$\frac{\partial f}{\partial y}(x,y) = 0 - \frac{1}{xy}\cdot x = -\frac{1}{y} \quad \Rightarrow \frac{\partial f}{\partial y}(1,1) = -1$$

$$\Rightarrow df(T) = (2e-1)dx - dy$$

$$f(1.02, 0.9) = f(1+0.02, 1-0.1)$$

$$\approx f(1,1) + \frac{\partial f}{\partial x}(1,1)\cdot 0.02 + \frac{\partial f}{\partial y}(1,1)\cdot(-0.1) = e + (2e-1)\cdot 0.02 + 0.1$$

$$= 1.04e + 0.08$$

(b) $$\frac{\partial^2 f}{\partial x^2}(x,y) = 2e^{x^2} + 2x\cdot 2xe^{x^2} + \frac{1}{x^2} = (2+4x^2)e^{x^2} + \frac{1}{x^2}$$

$$\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{1}{y^2}$$

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y) = 0$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}(1,1) = 6e+1, \quad \frac{\partial^2 f}{\partial y^2}(1,1) = 1, \quad \frac{\partial^2 f}{\partial x \partial y}(1,1) = \frac{\partial^2 f}{\partial y \partial x}(1,1) = 0$$

$$T_2(x,y) = f(1,1) + \left(\frac{\partial f}{\partial x}(1,1)(x-1) + \frac{\partial f}{\partial y}(1,1)(y-1)\right)$$
$$+ \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(1,1)(x-1)^2 + 2\cdot\frac{\partial^2 f}{\partial x \partial y}(1,1)(x-1)(y-1) + \frac{\partial^2 f}{\partial y^2}(1,1)(y-1)^2\right)$$
$$= e + (2e-1)(x-1) - (y-1) + \frac{1}{2}(6e+1)(x-1)^2 + \frac{1}{2}(y-1)^2$$

$$f(1.02, 0.9) \approx T_2(1.02, 0.9)$$
$$= e + (2e-1)\cdot 0.02 + 0.1 + \frac{1}{2}(6e+1)\cdot(0.02)^2 + \frac{1}{2}\cdot(-0.1)^2$$
$$= 1.052e + 0.0852$$

3. $f(x,y,z) = xy + y^3 - z^2$ $\qquad y - z = 1, \quad y - x = 5$

Lagrangeova funkcija:

$$L(x,y,z,\lambda,\mu) = xy + y^3 - z^2 + \lambda(y-z-1) + \mu(y-x-5)$$

$$\begin{cases} L'_x = y - \mu = 0 & \Rightarrow \mu = y \\ L'_y = x + 3y^2 + \lambda + \mu = 0 & \\ L'_z = -2z - \lambda = 0 & \Rightarrow \lambda = -2z = -2y + 2 \\ L'_\lambda = y - z - 1 = 0 & \Rightarrow z = y - 1 \\ L'_\mu = y - x - 5 = 0 & \Rightarrow x = y - 5 \end{cases}$$

Uvrstimo sve u drugu jednadžbu:

$$\cancel{y} - 5 + 3y^2 \cancel{-2y} + 2 + \cancel{y} = 0$$

$$\Rightarrow 3y^2 - 3 = 0 \Rightarrow y^2 = 1$$

$\to y_1 = 1,\ x_1 = -4,\ z_1 = 0,\ \lambda_1 = 0,\ \mu_1 = 1$

$\to y_2 = -1,\ x_2 = -6,\ z_2 = -2,\ \lambda_2 = 4,\ \mu_2 = -1$

Drugi diferencijal Lagrangeove funkcije:

$$L''_{xx} = 0,\ L''_{yy} = 6y,\ L''_{zz} = -2,\ L''_{xy} = L''_{yx} = 1,\ L''_{yz} = L''_{zy} = 0,\ L''_{zx} = L''_{xz} = 0$$

$$\Rightarrow d^2L(x,y,z) = 6y(dy)^2 - 2(dz)^2 + 2dx\,dy$$

Diferencijali uvjeta:

$$\left.\begin{aligned} y - z &= 1 \quad /d \quad \Rightarrow dy = dz \\ y - x &= 5 \quad /d \quad \Rightarrow dy = dx \end{aligned}\right\} \Rightarrow dx = dy = dz \Rightarrow d^2L(x,y,z) = 6y(dy)^2$$

Imamo

$$d^2L(-4,1,0) = 6(dy)^2 > 0 \quad \text{za} \quad (dx,dy,dz) \neq (0,0,0)$$

$\Rightarrow (-4,1,0)$ lokalni uvjetni minimum

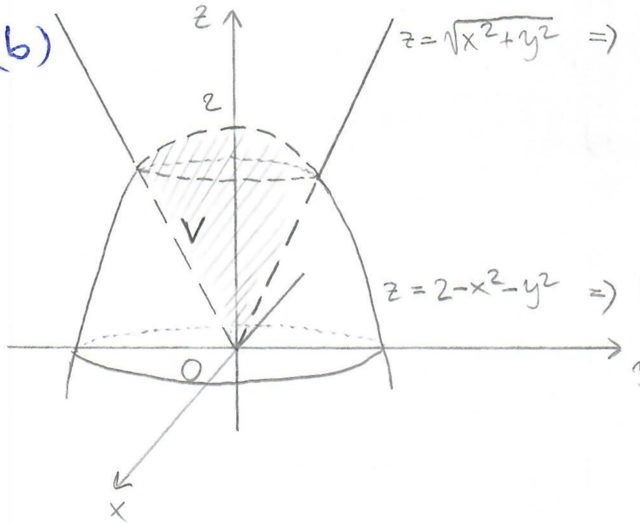
$$d^2L(-6,-1,-2) = -6(dy)^2 < 0 \quad \text{za} \quad (dx,dy,dz) \neq (0,0,0)$$

$\Rightarrow (-6,-1,-2)$ lokalni uvjetni maksimum

4. (a) $\left.\begin{aligned} x &= r\cos\varphi \\ y &= r\sin\varphi \\ z &= z \end{aligned}\right\} \Rightarrow J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\varphi & -r\sin\varphi & 0 \\ \sin\varphi & r\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix}$

$$= \begin{vmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r\cos^2\varphi + r\sin^2\varphi = r$$

(b)



$z=\sqrt{x^2+y^2} \Rightarrow z=r$

$z=2-x^2-y^2 \Rightarrow z=2-r^2$

Presjek zadanih ploha:

$$\begin{cases} z=\sqrt{x^2+y^2} \\ z=2-(x^2+y^2) \end{cases} \Rightarrow z=2-z^2$$

$$z^2+z-2=0$$

$$(z-1)(z+2)=0$$

$$\boxed{z_1=1} \qquad z_2=-2$$

$$\Rightarrow r\in[0,1]$$

Koristimo cilindrične koordinate:

$$V=\int_0^{2\pi}\int_0^1\int_r^{2-r^2} r\,dz\,dr\,d\varphi = \int_0^{2\pi}\int_0^1 r\left(z\Big|_r^{2-r^2}\right)dr\,d\varphi$$

$$=\int_0^{2\pi}\int_0^1 (-r^3-r^2+2r)\,dr\,d\varphi = \int_0^{2\pi}\left(-\frac{1}{4}r^4-\frac{1}{3}r^3+r^2\right)\Big|_0^1 d\varphi$$

$$=\int_0^{2\pi}\frac{5}{12}\,d\varphi = \frac{5}{12}\left(\varphi\Big|_0^{2\pi}\right) = \frac{5\pi}{6}$$

2. način

Znamo da se zadane plohe sijeku u ravnini $z = 1$ te u točkama presjeka vrijedi i $x^2+y^2=1$.

Zato je projekcija presjeka B na Oxy ravninu jedinični krug sa središtem u ishodištu. Traženi volumen je jednak razlici volumena ispod grafova funkcija $z = 2-x^2-y^2$ i $z=\sqrt{x^2+y^2}$ na području B:

$$V = \iint_B (2-(x^2+y^2))\,dx\,dy - \iint_B \sqrt{x^2+y^2}\,dx\,dy$$

$$= \begin{bmatrix} \text{područje } B \text{ parametriziramo u} \\ \text{polarnim koordinatama} \end{bmatrix}$$

$$= \int_0^{2\pi}\int_0^1 (2-r^2)r\,dr\,d\varphi - \int_0^{2\pi}\int_0^1 r\cdot r\,dr\,d\varphi$$

$$= \int_0^{2\pi}\int_0^1 (-r^3-r^2+2r)\,dr\,d\varphi = \begin{bmatrix} \text{isti račun kao u} \\ \text{prvom rješenju} \end{bmatrix} = \frac{5\pi}{6}.$$

5. (a) Teorem.

Ako za red $\sum_{n=1}^{\infty} (-1)^n a_n$ vrijedi:

(i) $a_n \geq 0 \quad \forall n \in \mathbb{N}$,

(ii) niz (a_n) je padajući,

(iii) $\lim_{n\to\infty} a_n = 0$,

tada je taj red konvergentan.

Dokaz.

Za $n \in \mathbb{N}$ promatramo $(2n)$-tu parcijalnu sumu zadanog reda:

$$S_{2n} = (a_1 - a_2) + (a_3 - a_4) + \dots + (a_{2n-3} - a_{2n-2}) + (a_{2n-1} - a_{2n})$$

$$= S_{2n-2} + \underbrace{(a_{2n-1} - a_{2n})}_{\overset{(ii)}{\geq} 0} \geq S_{2n-2}.$$

Dakle, niz (S_{2n}) je rastući. Nadalje,

$$S_{2n} = a_1 - \underbrace{(a_2 - a_3)}_{\overset{(ii)}{\geq} 0} - \dots - \underbrace{(a_{2n-2} - a_{2n-1})}_{\overset{(ii)}{\geq} 0} - \underbrace{a_{2n}}_{\overset{(i)}{\geq} 0} \leq a_1,$$

tj. niz (S_{2n}) je omeđen odozgo pa je on i konvergentan, tj. postoji

$$\lim_{n\to\infty} S_{2n} = S.$$

Za niz (S_{2n+1}) neparnih parcijalnih sume imamo

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (S_{2n} + a_{2n+1}) = \lim_{n\to\infty} S_{2n} + \underbrace{\lim_{n\to\infty} a_{2n+1}}_{\overset{(iii)}{=} 0} = S.$$

Dakle, niz (S_n) je konvergentan pa red $\sum_{n=1}^{\infty} (-1)^n a_n$ konvergira po definiciji.

Q.E.D.

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(x+2)^n}{2^n(n+2)}$

Prema d'Alembertovom kriteriju zadani red (apsolutno) konvergira za

$$\lim_{n\to\infty} \left| \frac{\frac{(x+2)^{n+1}}{2^{n+1}(n+3)}}{\frac{(x+2)^n}{2^n(n+2)}} \right| = \lim_{n\to\infty} \frac{1}{2}\cdot \overbrace{\frac{n+2}{n+3}}^{\to 1} |x+2| = \frac{1}{2}|x+2| < 1$$

$$\Leftrightarrow |x+2| < 2$$

$$\Leftrightarrow x \in \langle -4, 0\rangle$$

Ispitujemo konvergenciju u rubovima:

- $x = -4$

$$\sum_{n=0}^{\infty} \frac{(-4+2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$$

Za $a_n := \frac{1}{n+2}$ imamo

(i) $a_n \geq 0 \quad \forall n \in \mathbb{N}_0$,

(ii) $a_{n+1} = \frac{1}{n+3} < \frac{1}{n+2} = a_n \quad \forall n \in \mathbb{N}_0 \Rightarrow (a_n)$ je padajući niz,

(iii) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+2} = 0$,

pa prema Leibnitzovom kriteriju slijedi da red $\sum_{n=0}^{\infty} \frac{(-1)^n}{n+2}$ konvergira.

- $x = 0$

$$\sum_{n=0}^{\infty} \frac{(0+2)^n}{2^n(n+2)} = \sum_{n=0}^{\infty} \frac{1}{n+2}$$

Zbog $\displaystyle\lim_{n\to\infty} \frac{\frac{1}{n+2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+2} = 1 \in \langle 0, \infty\rangle$

i činjenice da harmonijski red divergira, slijedi i da red $\sum_{n=0}^{\infty} \frac{1}{n+2}$ divergira.

Područje konvergencije zadanog reda potencija je $[-4, 0\rangle$.

6. $xy = a \quad / \frac{d}{dx}$

$y + xy' = 0$ ⇝ diferencijalna jednadžba zadane familije

Diferencijalna jednadžba ortogonalne familije glasi

$$y + x \cdot \left(-\frac{1}{y'}\right) = 0$$

$$y\,dy = x\,dx \quad / \int$$

$$\frac{1}{2}y^2 + C = \frac{1}{2}x^2 \qquad C \in \mathbb{R}$$

$$\Rightarrow x^2 - y^2 = C \qquad C \in \mathbb{R}$$

(ortogonalna familija jednakostraničnih hiperbola u prvom i trećem kvadrantu je ponovno familija jednakostraničnih hiperbola)

7. (a) Ako je $\mu = \mu(y)$ Eulerov multiplikator diferencijalne jednadžbe

$$P(x,y)dx + Q(x,y)dy = 0,$$

onda uvjet egzaktnosti povlači

$$\frac{\partial(\mu P)}{\partial y} = \frac{\partial(\mu Q)}{\partial x}$$

$$\Rightarrow \mu'(y)P(x,y) + \mu(y)\frac{\partial P}{\partial y}(x,y) = \mu(y)\frac{\partial Q}{\partial x}(x,y)$$

$$\Rightarrow \mu' \cdot P = \mu \cdot (Q'_x - P'_y)$$

$$\Rightarrow \frac{1}{\mu} \cdot \frac{d\mu}{dy} = \frac{1}{P}(Q'_x - P'_y)$$

$$\Rightarrow \frac{1}{\mu}d\mu = \frac{1}{P}(Q'_x - P'_y)dy \quad / \int$$

$$\Rightarrow \ln|\mu(y)| = \int \frac{1}{P}(Q'_x - P'_y)\,dy$$

(b) Imamo

$$P(x,y)=\frac{1}{x+y} \Rightarrow P'_y(x,y)=-\frac{1}{(x+y)^2},$$

$$Q(x,y)=\frac{2\ln(x+y)}{y}+\frac{1}{x+y} \Rightarrow Q'_x(x,y)=\frac{2}{y(x+y)}-\frac{1}{(x+y)^2},$$

pa diferencijalna jednadžba za Eulerov multiplikator oblika $\mu=\mu(y)$ glasi

$$\frac{1}{\mu}\cdot\frac{d\mu}{dy}=\frac{1}{\frac{1}{x+y}}\left(\frac{2}{y(x+y)}-\cancel{\frac{1}{(x+y)^2}}+\cancel{\frac{1}{(x+y)^2}}\right)=\frac{2}{y}$$

$$\Rightarrow \quad \frac{1}{\mu}d\mu=\frac{2}{y}dy \quad /\int$$

$$\Rightarrow \quad \ln|\mu|=2\ln|y|+\ln C \qquad C>0$$

$$\ln|\mu|=\ln C|y|^2 \qquad C>0$$

$$\mu=Cy^2 \qquad C\in\mathbb{R}$$

Dakle, traženi Eulerov multiplikator je $\mu=y^2$ (određujemo ga do na množenje konstantom) pa imamo egzaktnu jednadžbu:

$$\frac{y^2}{x+y}dx+\left(2y\ln(x+y)+\frac{y^2}{x+y}\right)dy=0$$

Odredimo njen prvi integral (rješenje):

$$\begin{cases} \frac{\partial u}{\partial x}(x,y)=\frac{y^2}{x+y} & /\int dx \Rightarrow u(x,y)=y^2\ln(x+y)+\varphi(y) \quad /\frac{\partial}{\partial y} \\ \frac{\partial u}{\partial y}(x,y)=2y\ln(x+y)+\frac{y^2}{x+y} & \Rightarrow \frac{\partial u}{\partial y}(x,y)=2y\ln(x+y)+\frac{y^2}{x+y}+\varphi'(y) \end{cases}$$

$$\cancel{2y\ln(x+y)}+\cancel{\frac{y^2}{x+y}}=\cancel{2y\ln(x+y)}+\cancel{\frac{y^2}{x+y}}+\varphi'(y)$$

$$\Rightarrow \varphi'(y)=0 \Rightarrow \varphi(y)=C, \quad C\in\mathbb{R}$$

Rješenje zadane jednadžbe je

$$y^2\ln(x+y)=C, \quad C\in\mathbb{R}$$

8. (a) Imamo

$$L(y_p + y_h) = Ly_p + \underbrace{Ly_h}_{=0} = Ly_p = \left[\begin{array}{l} y_p \text{ partikularno} \\ \text{rješenje} \end{array}\right] = f,$$

tj. $y_p + y_h$ je rješenje jednadžbe $Ly = f$.

(b) $$\begin{cases} y'' - 2y' + y = -\frac{1}{x^2}e^x \\ y(1) = 0 \\ y'(1) = 0 \end{cases}$$

1° Homogena jednadžba

$$y'' - 2y' + y = 0$$

Karakteristična jednadžba:

$$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r_{1,2} = 1$$

Opće rješenje homogene jednadžbe:

$$y_h = C_1 e^x + C_2 x e^x, \qquad C_{1,2} \in \mathbb{R}$$

2° Varijacija konstanti

$$y(x) = C_1(x) e^x + C_2(x) x e^x$$

$$y' = C_1 e^x + C_2(e^x + xe^x) + \underbrace{C_1' e^x + C_2' x e^x}_{=0}$$

$$y'' = C_1 e^x + C_2(2e^x + xe^x) + \underbrace{C_1' e^x + C_2'(e^x + xe^x)}_{= -\frac{1}{x^2}e^x}$$

$$\Rightarrow \begin{cases} C_1' e^x + C_2' x e^x = 0 & \Big| \cdot (-1) \\ C_1' e^x + C_2'(e^x + xe^x) = -\frac{1}{x^2}e^x & + \end{cases} \Rightarrow C_2' e^x = -\frac{1}{x^2}e^x$$

$$\Rightarrow C_2' = -\frac{1}{x^2} \quad / \int dx$$

$$\Rightarrow C_2 = \frac{1}{x} + D_1, \quad D_1 \in \mathbb{R}$$

$$C_1' e^x = -C_2' x e^x = \frac{1}{x} e^x$$

$$\Rightarrow C_1' = \frac{1}{x} \quad / \int dx$$

$$\Rightarrow C_1 = \ln|x| + D_2, \quad D_2 \in \mathbb{R}$$

$$\Rightarrow y = (\ln|x| + D_2)\, e^x + \left(\frac{1}{x} + D_1\right) x e^x$$

$$= e^x \ln|x| + e^x + D_1 x e^x + D_2 e^x \qquad D_{1,2} \in \mathbb{R}$$

Iz početnih uvjeta:

$$0 = y(1) = 0 + e + D_1 e + D_2 e \quad \Rightarrow \quad D_1 + D_2 = -1$$

$$y' = e^x \ln|x| + e^x \cdot \frac{1}{x} + e^x + D_1 (e^x + x e^x) + D_2 e^x$$

$$0 = y'(1) = 0 + e + e + D_1 \cdot 2e + D_2 \cdot e \quad \Rightarrow \quad 2D_1 + D_2 = -2$$

$$\Rightarrow \begin{cases} D_1 + D_2 = -1 & / \cdot (-1) \\ 2D_1 + D_2 = -2 & + \end{cases} \quad \Rightarrow \quad D_1 = -1, \quad D_2 = 0$$

$$\Rightarrow y = e^x \ln|x| + e^x - x e^x$$