1

(a) Kada bi F bila neprekidna moralo bi vrijediti:

$$F(0,0) = \lim_{(x,y)\to(0,0)} F(x,y) = \lim_{(x,y)\to(0,0)} f(x,y)$$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = \lim_{r\to 0} \frac{\sin(r^2)}{r^2} = 1$$

$\Rightarrow$ Jedini kandidat za F je:

$$F(x,y) = \begin{cases} \dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 1, & (x,y) = (0,0) \end{cases}$$

i taj F uistinu je neprekidan.

(b) Budući da je neprekidnost nužan uvjet za diferencijabilnost, za (b) je dovoljno provjeriti je li F iz (a) dijela diferencijabilna.

Računamo parcijalne derivacije

$$\frac{\partial F}{\partial x}(0,0) = \lim_{x\to 0} \frac{F(x,0)-F(0,0)}{x} = \lim_{x\to 0} \frac{\frac{\sin x^2}{x^2} - 1}{x} =$$

$$= \lim_{x\to 0} \frac{\sin x^2 - x^2}{x^3} \overset{(L'H)}{=} \lim_{x\to 0} \frac{2x\cos x^2 - 2x}{3x^2} =$$

$$= \lim_{x\to 0} \frac{2\cos x^2 - 2}{3x} \overset{(L'H)}{=} \lim_{x\to 0} \frac{-4x\sin x^2}{3} = 0$$

Slično $\dfrac{\partial F}{\partial y}(0,0) = 0$

Sada znamo da je za diferencijabilnost u $(0,0)$ dovoljno provjeriti

$$\lim_{(x,y)\to(0,0)} \frac{|F(x,y) - F(0,0)|}{\sqrt{x^2+y^2}} = 0$$

Računamo:

$$\lim_{(x,y)\to(0,0)} \frac{|F(x,y) - F(0,0)|}{\sqrt{x^2+y^2}} \underset{r^2 = x^2+y^2}{=} \lim_{r\to 0} \frac{\left|\frac{\sin r^2}{r^2} - 1\right|}{r} =$$

$$= \lim_{r\to 0} \frac{\sin r^2 - r^2}{r^3} =$$

$$\overset{(L'H)}{=} \lim_{r\to 0} \frac{2r\cos r^2 - 2r}{3r^2}$$

$$= \lim_{r\to 0} \frac{2\cos r^2 - 2}{3r}$$

$$\overset{(L'H)}{=} \lim \frac{-4r\sin r^2}{3} = 0$$

$\Rightarrow$ F je diferencijabilna u $(0,0)$

$\Leftarrow$ F je diferencijabilna na $\mathbb{R}^2 \setminus \{(0,0)\}$ kao umnožak / kompozicija diferencijabilnih f-ja

$\Rightarrow$ F je diferencijabilna

(2)

(a) $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{-e^{z-x}}{e^{z-x}+e^{z-y}} = \frac{e^{-x}}{e^{-x}+e^{-y}}$$

$$\frac{\partial z}{\partial x}(0,0) = \frac{1}{2}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{-e^{-x}\left(e^{-x}+e^{-y}\right) - e^{-x}\left(-e^{-x}\right)}{\left(e^{-x}+e^{-y}\right)^2}$$

$$\frac{\partial^2 z}{\partial x^2}(0,0) = -\frac{1}{4}$$

[Napomena: Funkcija se može eksplicitno izraziti kao $z = -\ln\left(e^{-x}+e^{-y}\right)$]

② (b)

$$F'_x(0,0)(x-x_0) + F'_y(0,0)(y-y_0) + F'_z(0,0)(z-z_0) = 0$$

$$-1\,(x-0) - 1\,(y-0) + 2\,(z-(-\ln 2)) = 0 \qquad [z_0 = -\ln 2]$$

$$\boxed{-x - y + 2z + 2\ln 2 = 0}$$

(c) Jdn. tangencijalne ravnine u proizvoljnoj točki (x_0, y_0, z_0)

$$z_x\big|_{(x_0,y_0)}(x-x_0) + z_y\big|_{(x_0,y_0)}(y-y_0) = z - z_0$$

$$\pi \ldots \quad e^{z_0-x_0}x + e^{z_0-y_0}y - z = x_0 e^{z_0-x_0} + y_0 e^{z_0-y_0} - z_0$$

Budući da vrijedi

$$\vec{n}_\pi \cdot (\vec{i} + \vec{j} + \vec{k}) = (e^{z_0-x_0}\vec{i} + e^{z_0-y_0}\vec{j} - \vec{k})(\vec{i} + \vec{j} + \vec{k})$$

$$= e^{z_0-x_0} + e^{z_0-y_0} - 1$$

$$= [(x_0, y_0, z_0) \text{ leži na plohi}]$$

$$= 0$$

$\Rightarrow$ $\vec{n}_\pi \perp \vec{i} + \vec{j} + \vec{k}$, tj. tangencijalna ravnina je okomita

na ravninu $x + y + z = 0$.

(3)

(a) Funkcija $f : D \subset \mathbb{R}^3 \to \mathbb{R}$ na omeđenom i zatvorenom skupu D poprima <u>globalni</u> minimum i <u>globalni</u> maksimum. Točke u kojima se ti ekstremi mogu postići su <u>kritične točke u D</u> i <u>rub od D</u>. (Tm 3.5.3)

(b) globalni ekstremi se poprimaju na rubu → na sferi $x^2+y^2+z^2=52$

f je linearna fja : $\vec{n} = (4,0,6)$

pravac kroz ishodište sa $\vec{c} = \vec{n}$: $\frac{x}{4} = \frac{y}{0} = \frac{z}{6} = t$

$\Rightarrow \quad x = 4t \quad , y = 0 \quad , z = 6t$

$x^2+y^2+z^2 = 52$

$16t^2 + 0 + 36t^2 = 52$

$t^2 = 1$

$t = \pm 1$

$\Rightarrow \quad T_1(4,0,6)$ globalni maksimum $f_{max} = 52$

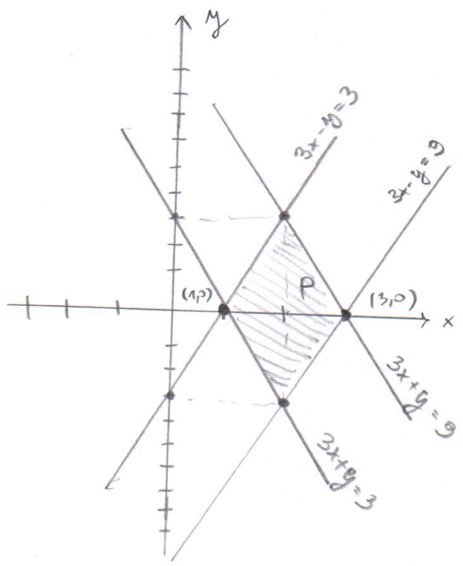
$T_2(-4,0,-6)$ globalni minimum $f_{min} = -52$

4

(a) Za $x = x(u,v)$ i $y = y(u,v)$ Jacobijan definiramo kao

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

(b)



Vrhovi od P leže na pravcima $3x+y=3$, $3x+y=9$

$3x-y=3$, $3x-y=9$

pa uvodimo zamjenu varijabli $u = 3x+y$, $v = 3x-y$

$$\Rightarrow \quad x = \frac{1}{6}(u+v), \quad y = \frac{1}{2}(u-v)$$

$$\Rightarrow \text{ Jacobijan } \quad J = \begin{vmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{6}$$

$$\Rightarrow \iint_P \frac{\ln(3x+y)}{9x^2-y^2}\,dx\,dy = \int_3^9\int_3^9 \left|-\frac{1}{6}\right| \frac{\ln(u)}{uv}\,du\,dv = \frac{1}{6}\left(\int_3^9 \frac{1}{v}\,dv\right)\left(\int_3^9 \frac{\ln(u)}{u}\,du\right)$$

$$= \frac{1}{6}(\ln(v))\Big|_3^9 \left(\int_{\ln 3}^{\ln 9} t\,dt\right) = \frac{1}{6}\cdot \ln 3 \cdot \frac{1}{2}\left(\ln^2 9 - \ln^2 3\right)$$

$$= \frac{1}{12}\ln 3 \cdot 3\ln^2 3 = \frac{1}{4}\ln^3 3$$

(5)

(a) Promatramo f-ju $f:[1,\infty\rangle \to [0,\infty\rangle$, $f(x)=\frac{1}{x^p}$

koja je neprekidna i padajuća na svojoj domeni za $p>0$.

Računamo:

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x^p}\,dx = \frac{1}{1-p}\,x^{1-p}\Big|_1^\infty = \begin{cases} \infty, & p<1 \\ \frac{1}{p-1}, & p>1 \end{cases}$$

Posebno, za $p=1$ imamo:

$$\int_1^\infty \frac{1}{x}\,dx = \ln|x|\Big|_1^\infty = \infty,$$

pa prema integralnom kriteriju slijedi da red konvergira za

$p>1$, a divergira za $0<p\leq 1$.

U slučaju $p\leq 0$ imamo:

$$\lim_{n\to\infty} n^{-p} = \begin{cases} 1, & p=0 \\ \infty, & p<0 \end{cases}$$

pa zbog narušenog nužnog uvjeta konvergencije zadani red

u ovom slučaju divergira.

Zaključak:

$$\sum_{n=1}^\infty \frac{1}{n^p} \begin{cases} \text{konvergira za } p>1 \\ \text{divergira za } p\leq 1 \end{cases}$$

5 (b)

(i) $$\sum_{n=1}^{\infty}\left(\sqrt{n^3+2n+1}-n\sqrt{n}\right)\cdot\frac{\sqrt{n^3+2n+1}+n\sqrt{n}}{\sqrt{n^3+2n+1}+n\sqrt{n}}=$$

$$=\sum_{n=1}^{\infty}\frac{n^3+2n+1-n^3}{\sqrt{n^3+2n+1}+n\sqrt{n}}\sim\sum_{n=1}^{\infty}\frac{\cancel{2}\cancel{n}}{\cancel{2}\cancel{n}\sqrt{n}}\sim\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}$$

$\Rightarrow p=\frac{1}{2}<1 \Rightarrow$ red divergira!

(ii) $\sin\frac{1}{\sqrt{n}}\sim\frac{1}{\sqrt{n}}$ kada $n\to\infty$ (jer $\frac{1}{\sqrt{n}}\to 0$)

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\sin^2\left(\frac{1}{\sqrt{n}}\right)\sim\sum_{n=1}^{\infty}\frac{\left(\frac{1}{\sqrt{n}}\right)^2}{\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n}}=\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$$

$\Rightarrow p=\frac{3}{2}>1 \Rightarrow$ red konvergira

6 (a)

$P(x,y) = Ax + By$
$Q(x,y) = Cx + Dy$

$$\left.\begin{aligned} P(tx,ty) &= Atx + Bty = t(Ax+By) = tP(x,y) \\ Q(tx,ty) &= Ctx + Dty = t(Cx+Dy) = tQ(x,y) \end{aligned}\right\} \alpha = 1$$

$\Rightarrow$ $P(x,y)dx + Q(x,y)dx = 0$ je diferencijabilna jdn. homogenog stupnja i stupanj joj je $\alpha = 1$

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(b) za $C \neq 0$ i $D = 0$

$$\Rightarrow (Ax + By)\,dx + Cx\,dy = 0 \quad /:dx$$

$$Ax + By + Cx\,y' = 0 \quad /:Cx$$

$$y' + \left(\frac{B}{Cx}\right)y = -\frac{A}{C}$$

(6)

(c)

1 način:

$$(Ax+By)\,dx + (Cx+Dy)\,dy = 0 \quad \Big| :dx \quad \Big|_{D=0}$$

$$\Rightarrow \quad Ax+By + Cxy' = 0 \quad \Big| :x$$

$$A + B\frac{y}{x} + Cy' = 0$$

Supstitucija $z(x) = \frac{y}{x}$ nam daje:

$$A + Bz + Cz'x + Cz = 0$$

$$Cxz' = -Bz - Cz - A$$

pa separacijom varijabli dobivamo:

$$\frac{dz}{(B+C)z + A} = -\frac{dx}{Cx} \quad \Big| \int$$

$$\frac{1}{(B+C)} \ln\left((B+C)z + A\right) = \ln x^{-\frac{1}{c}} + C_0$$

$$(B+C)z + A = C_1 x^{-\frac{(B+C)}{c}} \Rightarrow (B+C)\frac{y}{x} + A = C_1\frac{x^{-\frac{B}{c}}}{x} \Big| \cdot \frac{x}{B+C}$$

$$\Rightarrow \quad y(x) = C_1 x^{-\frac{B}{c}} - \frac{A}{B+C}x \quad , \; C_1 \in \mathbb{R}$$

⑥ (c)

2. način:

iz (b) dijela zadatka imamo $y' + \left(\frac{B}{Cx}\right) y = -\frac{A}{C}$

što je linearna diferencijalna jdn prvog reda za $f(x) = \frac{B}{Cx}$, $g(x) = -\frac{A}{C}$

Rješenje takve dif. jdn glasi:

$$y(x) = e^{-\int f(x)\,dx}\left(M + \int g(x)\, e^{\int f(x)\,dx}\,dx\right), \quad M \in \mathbb{R}$$

$$\Rightarrow \quad \int f(x)\,dx = \int \frac{B}{Cx}\,dx = \frac{B}{C}\ln|x|$$

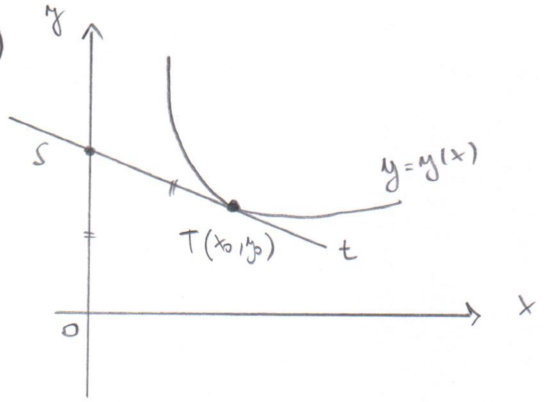
$$\Rightarrow \quad y(x) = e^{-\frac{B}{C}\ln|x|}\left(M + \int -\frac{A}{C}\, e^{\frac{B}{C}\ln|x|}\,dx\right)$$

$$= x^{-\frac{B}{C}}\left(M - \frac{A}{C}\int x^{\frac{B}{C}}\,dx\right)$$

$$= x^{-\frac{B}{C}}\left(M - \frac{A}{C}\,\frac{x^{\frac{B}{C}+1}}{\frac{B}{C}+1}\right)$$

$$= M x^{-\frac{B}{C}} - \frac{A}{B+C}\,x, \quad M \in \mathbb{R}$$

7



$t \dots \quad y - y_0 = y'(x_0)\,(x - x_0)$

$S(0, y_S) \Rightarrow y_S = y_0 - x_0 \cdot y'(x_0)$

$|OS| = |ST| \Rightarrow y_0 - x_0 y'(x_0) = \sqrt{x_0^2 + (x_0 y'(x_0))}$

$$\forall x_0, y_0 \Rightarrow \quad y - x \cdot y' = \sqrt{x^2 + x^2 y'^2} \quad \Big/^2$$

$$y^2 - 2xyy' + x^2 y'^2 = x^2 + x^2 y'^2$$

$$2xyy' = y^2 - x^2 \quad \Big/ : xy$$

$$2y' = \frac{y}{x} - \frac{x}{y} \quad , \quad z = \frac{y}{x} \quad , \quad y' = z'x + z$$

(može i Bernoulli, $\alpha = -1$)

$$2z'x + 2z = z - \frac{1}{z}$$

$$2z'x = -\frac{z^2 + 1}{z}$$

$$\int \frac{2z\,dz}{z^2+1} = -\int \frac{dx}{x}$$

$$\ln|z^2+1| = -\ln|x| + C$$

$$\frac{y^2}{x^2} + 1 = \frac{C}{x}$$

$$\boxed{y^2 + x^2 = Cx}$$ familija kružnica

8 (a) $y_1 = e^{2x}$

$y_2 = \operatorname{sh} 2x = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2x} & \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \\ 2e^{2x} & e^{2x} + e^{-2x} \end{vmatrix} = e^{2x}\left(e^{2x} + e^{-2x}\right) - e^{2x}\left(e^{2x} - e^{-2x}\right)$$

$$= e^{2x}\left[\cancel{e^{2x}} + e^{-2x} - \cancel{e^{2x}} + e^{-2x}\right] = e^{2x}\left[2e^{-2x}\right] = 2 \neq 0$$

$W \neq 0 \quad \forall x \Rightarrow y_1$ i y_2 su linearno nezavisne

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(b) $y'' - 4y = \operatorname{sh} 2x$

$y'' - 4y = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$

1 način: homogena jj.

$r^2 - 4 = 0$

$r_{1,2} = \pm 2$

$y_h = C_1 e^{2x} + C_2 e^{-2x}$

metoda VK

$$\begin{cases} C_1'(x)\, e^{2x} + C_2'(x)\, e^{-2x} = 0 \\ 2C_1'(x)\, e^{2x} + C_2'(x)\left(-2e^{-2x}\right) = \frac{1}{2}\left(e^{2x} - e^{-2x}\right) \end{cases}$$

Pomnožimo prvu jdn sa 2 i zbrojimo:

$$4C_1'(x)\, e^{2x} = \frac{1}{2}\left(e^{2x} - e^{-2x}\right)$$

$$C_1'(x) = \frac{1}{8}\left(1 - e^{-4x}\right) \quad \Big| \int$$

$$\boxed{C_1(x) = \frac{1}{8}x + \frac{1}{32}e^{-4x} + D_1}$$

$$C_2'(x)\, e^{-2x} = -C_1'(x)\, e^{2x}$$

$$C_2'(x) = -\frac{1}{8}\left(e^{4x} - 1\right) \quad \Big| \int$$

$$\boxed{C_2(x) = -\frac{1}{32}e^{4x} + \frac{1}{8}x + D_2}$$

$$y(x) = \left(\frac{1}{8}x + \frac{1}{32}e^{-4x} + D_1\right)e^{2x} + \left(-\frac{1}{32}e^{4x} + \frac{1}{8}x + D_2\right)e^{-2x}$$

$$= E_1 e^{2x} + E_2 e^{-2x} + \frac{1}{8}x e^{2x} + \frac{1}{8}x e^{-2x}, \quad E_1, E_2 \in \mathbb{R}$$

2. način: metoda desne strane

$$y_p = Axe^{2x} + Bxe^{-2x} \quad (\text{jer su } r_{1,2} = \pm 2 \text{ rj. karakt. jedn.})$$

$$y_p' = Ae^{2x} + 2Axe^{2x} + Be^{-2x} - 2Bxe^{-2x}$$

$$= (A + 2Ax)e^{2x} + (B - 2Bx)e^{-2x}$$

$$y_p'' = 2Ae^{2x} + (2A + 4Ax)e^{2x} - 2Be^{-2x} + (-2B + 4Bx)e^{-2x}$$

$$\Rightarrow e^{2x}(4A + \cancel{4Ax}) + e^{-2x}(-4B + \cancel{4Bx}) - \cancel{4Axe^{2x}} - \cancel{4Bxe^{-2x}} = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}$$

$$\left.\begin{aligned} 4A = \frac{1}{2} &\Rightarrow A = \frac{1}{8} \\ -4B = -\frac{1}{2} &\Rightarrow B = \frac{1}{8} \end{aligned}\right\} \quad y_p = \frac{1}{8}xe^{2x} + \frac{1}{8}xe^{-2x}$$

$$\Rightarrow y = y_h + y_p = C_1 e^{2x} + C_2 e^{-2x} + \frac{1}{8}xe^{2x} + \frac{1}{8}xe^{-2x}, \quad C_1, C_2 \in \mathbb{R}$$