ELEMENTI :

1. DOMENA fja više varijabli :

2. NIVO KRIVULJE :
* preko onog c $(c = z)$, pa možda $c>0, c<0, c=0 \ldots$

3. LIMESI :
* prelazak u polarne koordinate

4. PARC. DERIVACIJA FJE :

$$\frac{\partial f}{\partial x}(x_0, y_0) = \lim_{x \to x_0} \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

a) TANGENCIJALNA RAVNINA :

EXPLICITNO:

$$z - z_0 = \frac{\partial f}{\partial x}(x_0, y_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0)$$

$z = f(x, y)$ $\quad T_0(x_0, y_0, z_0)$

$z_0 = f(x_0, y_0)$

IMPLICITNO:

3 var.

$$\frac{\partial F}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial F}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial F}{\partial z}(x_0, y_0, z_0)(z - z_0)$$

$F(x, y, z) = 0$ $\quad T(x_0, y_0, z_0)$

$f(x_0, y_0, z_0) = 0$

b) JDBA NORMALE na plohu :

$$\frac{x - x_0}{\frac{\partial f}{\partial x}(x_0, y_0)} = \frac{y - y_0}{\frac{\partial f}{\partial y}(x_0, y_0)} = \frac{z - z_0}{-1}$$

c) DIFERENCIJABILNOST FJA više varijabli :

Neka je $f(x_1, x_2, \ldots, x_m)$ fja n-varijabli

Označimo sa $\vec{h} = (h_1, h_2, \ldots, h_m)$

Neka je $\vec{x} = (x_1, \ldots, x_m)$ neka točka

Kažemo da je fja diferencijabilna u točki $\vec{x}$ ako postoji vektor $\vec{y}$ i fja $\sigma(\vec{h})$ tako da vrijedi:

$$f(\vec{x} + \vec{h}) - f(\vec{x}) - \vec{y} \cdot \vec{h} = \sigma(\vec{h})$$ te je

$$\lim_{\vec{h} \to 0} \frac{\sigma(\vec{h})}{\|\vec{h}\|} = 0$$

* za fje 2-var. vrijedi: $\nabla f(x, y) = \frac{\partial f}{\partial x}(x, y)\vec{i} + \frac{\partial f}{\partial y}(x, y)\vec{j}$

⑤ ODREDIT PRVI DIFERENCIJAL FJE:

a)
$$\boxed{(dz)_T = \frac{\partial f}{\partial x}(x_0,y_0)\,dx + \frac{\partial f}{\partial y}(x_0,y_0)\,dy}$$

b) DRUGI DIFERENCIJAL:

$$\boxed{(d^2z)_T = \frac{\partial^2 f}{\partial x^2}(x_0,y_0)\,dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(dx\cdot dy) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(dy)^2}$$

c) PRIMJENA DIFERENCIJALA ZA IZRAČ. PRIBLIŽNE VRIJEDNOSTI:

2 var: $f(x+\Delta x, y+\Delta y) \simeq f(x,y) + \frac{\partial f}{\partial x}(x,y)\,\Delta x + \frac{\partial f}{\partial y}(x,y)\,\Delta y$

⑥ DERIVACIJE SLOŽENE FJE:

2 slučaja:

1) $z = f(x,y)$, $x = \varphi(t)$, $y = \psi(t)$

$$\frac{dz}{dt}(t_0) = \frac{\partial z}{\partial x}(x_0,y_0)\frac{dx}{dt}(t_0) + \frac{\partial z}{\partial y}(x_0,y_0)\cdot\frac{dy}{dt}(t_0)$$

$x_0 = \varphi(t_0)$

$y_0 = \psi(t_0)$

2) $z = f(u,v)$, $u = \varphi(x,y,w)$, $v = \psi(x,y,w)$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\frac{\partial z}{\partial w} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial w} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial w}$$

⑦ DERIVACIJA VEKTORSKE FJE:

$\vec{f}(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

$\vec{f}'(t) = x'(t)\vec{i} + y'(t)\vec{j} + z'(t)\vec{k}$

JDBA TANGENTE:

$$t \ldots \frac{x-x_0}{x'(t_0)} = \frac{y-y_0}{y'(t_0)} = \frac{z-z_0}{z'(t_0)}$$

8) INTEGRALI OVISNI O PARAMETRU:

$$I(\alpha)=\int_{\varphi(\alpha)}^{\psi(\alpha)} f(x,\alpha)\,dx$$

$$\Rightarrow I'(\alpha)=\frac{dI}{d\alpha}=f(\psi(\alpha),\alpha)\cdot\psi'(\alpha)-f(\varphi(\alpha),\alpha)\cdot\varphi'(\alpha)+\int_{\varphi(\alpha)}^{\psi(\alpha)}\frac{\partial f}{\partial\alpha}(x,\alpha)\,dx$$

9) USMJERENE DERIVACIJE:

DOKAZ: Neka je zadana točka $T(x_0,y_0)$ i jedinični vekt. $\vec{h}=h_1\vec{i}+h_2\vec{j}$. Tada usmjerena derivacija fje f u toč. $T(x_0,y_0)$ u smjeru vektora $\vec{h}$ definiramo:

(x_0,y_0) $\vec{h}=h_1\vec{i}+h_2\vec{j}$

$$\begin{cases} x=x_0+th_1 \\ y=y_0+th_2 \end{cases}$$

→ param. oblik jdbe pravca u ravnini

$$\frac{\partial f}{\partial\vec{h}}=\lim_{t=0}\frac{f(x_0+th_1,y_0+th_2)-f(x_0,y_0)}{t}$$

FORMULA: $$\frac{\partial f}{\partial\vec{h}}(x,y)=\nabla f(x,y)\cdot\vec{h}$$

→ vekt. okomit na tang. ravninu

10) RADIJALNE FJE:

Za fju $F:\mathbb{R}^3\to\mathbb{R}$ ($\to 2,3,4,5\ldots$) kažemo da je radijalna ako postoji fja $f:\mathbb{R}\to\mathbb{R}$ t.d. $F(x,y,z)=f(\sqrt{x^2+y^2+z^2})$
$=f(\|x\vec{i}+y\vec{j}+z\vec{k}\|)$

$$\left.\begin{array}{l} r=\sqrt{x^2+y^2+z^2} \\ \vec{r}=x\vec{i}+y\vec{j}+z\vec{k} \end{array}\right\} \nabla r=\frac{\vec{r}}{r}$$ } (dokaz - parc.)

NAJVAŽNIJA FORMULA:

$$\nabla f(r)=f'(r)\cdot\nabla r$$

$$\boxed{\nabla f(r)=f'(r)\cdot\frac{\vec{r}}{r}}$$

VRIJEDI:

① $\nabla(f\cdot g)=(\nabla f)g+f(\nabla g)$

② $\nabla\left(\frac{f}{g}\right)=\frac{(\nabla f)g-f(\nabla g)}{g^2}$

→ KO DERIVACIJE... :)

(11) DERIVACIJA IMPLICITNO ZADANIH FJA:

pr. $F(x,y) = x^2+y^2-1$

$$f'(x_0) = -\frac{\frac{\partial F}{\partial x}(x_0,y_0)}{\frac{\partial F}{\partial y}(x_0,y_0) \neq 0}$$ TM. o implicit. FJI.

FORMULE:

$$\frac{\partial f}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$T(0,0)$
$z = f(x,y)$

$$\frac{\partial f}{\partial x} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

(12) TAYLOROVA FORMULA za fje 2 varijable:

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left[f'_x(x_0,y_0)(x-x_0) + f'_y(x_0,y_0)(y-y_0)\right] +$$
$$+ \frac{1}{2!}\left[f''_{xx}(x_0,y_0)(x-x_0)^2 + 2f''_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f''_{yy}(x_0,y_0)(y-y_0)^2\right]$$

n-ti član ... $+ \frac{1}{n!}\left[(x-x_0)\frac{d}{dx} + (y-y_0)\frac{d}{dy}\right]^n f(x_0,y_0$ ... $(n+1)$ ...

(13) KVADRATNA FORMA i SYLVESTEROV TM.:

$\hookrightarrow Q(h,k,l) = ah^2 + bk^2 + cl^2 + 2lhk + 2fkl + 2dhl$ (3 var)

$$A = \begin{array}{c|cc} & h & k \\ \hline h & a & b \\ k & b & c \end{array} \Bigg\} \; Q(h,k) = ah^2 + 2bhk + ck^2$$

$\Rightarrow$ 1. $D_1 = a > 0$, $D_2 = \begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0$ } POZITIVNO DEFINITNO

2. $D_1 < 0$, $D_2 > 0$ } NEGATIVNO DEFINITNA

3. $D_2 < 0$ } INDEFINITNO