MATEMATIČKA ANALIZA 2

Jesenski ispitni rok (9.9.2019.)

RJEŠENJA ZADATAKA

1. (a) $$\frac{\partial f}{\partial x}(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$$

$$\frac{\partial f}{\partial y}(x_0,y_0)=\lim_{h\to 0}\frac{f(x_0,y_0+h)-f(x_0,y_0)}{h}$$

(b) $f(x,y)=x^2y^3$

$$\frac{\partial f}{\partial x}(1,1)=\lim_{h\to 0}\frac{f(1+h,1)-f(1,1)}{h}=\lim_{h\to 0}\frac{(1+h)^2\cdot 1^3-1^2\cdot 1^3}{h}$$

$$=\lim_{h\to 0}\frac{h^2+2h+\cancel{1}-\cancel{1}}{h}=\lim_{h\to 0}(h+2)=2$$

$$\frac{\partial f}{\partial y}(1,1)=\lim_{h\to 0}\frac{f(1,1+h)-f(1,1)}{h}=\lim_{h\to 0}\frac{1^2\cdot(1+h)^3-1^2\cdot 1^3}{h}$$

$$=\lim_{h\to 0}\frac{h^3+3h^2+3h+\cancel{1}-\cancel{1}}{h}=\lim_{h\to 0}(h^2+3h+3)=3$$

(c) $$f(x+h,y+k)\approx f(x,y)+h\frac{\partial f}{\partial x}(x,y)+k\frac{\partial f}{\partial y}(x,y)$$

$$(0.99)^2(1.01)^3=f(0.99,1.01)=f(1-0.01,1+0.01)$$

$$\approx f(1,1)-0.01\cdot\frac{\partial f}{\partial x}(1,1)+0.01\cdot\frac{\partial f}{\partial y}(1,1)$$

$$=1-0.01\cdot 2+0.01\cdot 3=1+0.01=1.01$$

(2.) (a) Vektor smjera tangente na krivulju u točki $T_0(x(t_0), y(t_0), z(t_0))$:

$$\vec{s} = (x'(t_0), y'(t_0), z'(t_0)) = \left(-\frac{1}{t_0^2}, 1, 3t_0^2\right).$$

Prema uvjetu zadatka ovaj vektor mora biti okomit na vektor normale zadane ravnine, $\vec{n} = 2\vec{i} + \frac{1}{2}\vec{j}$:

$$\vec{n} \| \vec{s} \Rightarrow \vec{n} \cdot \vec{s} = 0$$

$$\Rightarrow -\frac{2}{t_0^2} + \frac{1}{2} = 0$$

$$\Rightarrow t_0^2 = 4$$

$$\Rightarrow t_0 = 2 \qquad (t_0 \in [1,3])$$

Dakle, tražena točka je $T_0\left(\frac{1}{2}, 2, 8\right)$.

(b)
$$\frac{dw}{dt}(\sqrt{\pi}) = \frac{\partial w}{\partial x}(x(\sqrt{\pi}), y(\sqrt{\pi}), z(\sqrt{\pi}))\frac{dx}{dt}(\sqrt{\pi})$$
$$+ \frac{\partial w}{\partial y}(x(\sqrt{\pi}), y(\sqrt{\pi}), z(\sqrt{\pi}))\frac{dy}{dt}(\sqrt{\pi})$$
$$+ \frac{\partial w}{\partial z}(x(\sqrt{\pi}), y(\sqrt{\pi}), z(\sqrt{\pi}))\frac{dz}{dt}(\sqrt{\pi})$$
$$= \Big(-5\sin(x(\sqrt{\pi})y(\sqrt{\pi}))\,y(\sqrt{\pi}) - \cos(x(\sqrt{\pi})z(\sqrt{\pi}))z(\sqrt{\pi})\Big)\cdot\left(-\frac{1}{(\sqrt{\pi})^2}\right)$$
$$+ \Big(-5\sin(x(\sqrt{\pi})y(\sqrt{\pi}))\cdot x(\sqrt{\pi}) - 0\Big)\cdot 1$$
$$+ \Big(0 - \cos(x(\sqrt{\pi})z(\sqrt{\pi}))x(\sqrt{\pi})\Big)\cdot 3(\sqrt{\pi})^2$$
$$= \left(-5\sin 1\cdot\sqrt{\pi} - \cos(\pi)\cdot(\sqrt{\pi})^3\right)\cdot\left(-\frac{1}{\pi}\right)$$
$$-5\sin 1\cdot\frac{1}{\sqrt{\pi}} - \cos(\pi)\cdot\frac{1}{\sqrt{\pi}}\cdot 3\pi$$
$$= \cancel{\frac{5}{\sqrt{\pi}}\sin 1} - \sqrt{\pi} - \cancel{\frac{5}{\sqrt{\pi}}\sin 1} + 3\sqrt{\pi} = 2\sqrt{\pi}$$

(3.) (a) Neka je $T_0(x_0, y_0)$ točka lokalnog maksimuma diferencijabilne funkcije $f: \mathbb{R}^2 \to \mathbb{R}$.

Tada funkcije

$$g_1, g_2 : \mathbb{R} \to \mathbb{R}, \quad g_1(x) = f(x, y_0), \quad g_2(y) = f(x_0, y),$$

imaju lokalne maksimume u točkama x_0, y_0 redom pa mora vrijediti:

$$0 = g_1'(x_0) = \frac{\partial f}{\partial x}(x_0, y_0),$$

$$0 = g_2'(y_0) = \frac{\partial f}{\partial y}(x_0, y_0),$$

tj. T_0 je stacionarna točka od f.

(b) Funkciju f možemo razviti u Taylorov polinom oko točke T_0:

$$f(x,y) = f(T_0) + \Big[\underbrace{f_x'(T_0)}_{=0}(x - x_0) + \underbrace{f_y'(T_0)}_{=0}(y - y_0)\Big]$$

$$+ \frac{1}{2!}\Big[f_{xx}''(T_c)(x-x_0)^2 + 2f_{xy}''(T_c)(x-x_0)(y-y_0)$$

$$+ f_{yy}''(T_c)(y-y_0)^2\Big]$$

$$= f(T_0) + \frac{1}{2!} d^2 f(T_c),$$

gdje je T_c neka točka na spojnici točaka (x,y) i T_0.

Za točke (x,y) u dovoljno maloj okolini točke T_0 vrijedit će $d^2 f(x,y) < 0$ pa posebno i $d^2 f(T_c) < 0$ (zbog $d^2 f(T_0) < 0$ i neprekidnosti drugog diferencijala).

Zato za sve takve točke imamo

$$f(x,y) = f(T_0) + \frac{1}{2!}\underbrace{d^2 f(T_c)}_{<0} < f(T_0),$$

odakle po definiciji slijedi da je T_0 lokalni maksimum od f.

(c) $x^2+y^2+z^2-2x+4y-6z-11=0 \quad |\partial_x$

$\Rightarrow 2x+2zz_x-2-6z_x=0$

$\Rightarrow (z-3)z_x=1-x$

$\Rightarrow z_x=\dfrac{1-x}{z-3}=0 \Rightarrow \boxed{x=1}$

$x^2+y^2+z^2-2x+4y-6z-11=0 \quad |\partial_y$

$\Rightarrow 2y+2zz_y+4-6z_y=0$

$\Rightarrow (z-3)z_y=-2-y$

$\Rightarrow z_y=\dfrac{-2-y}{z-3}=0 \Rightarrow \boxed{y=-2}$

U dobivenoj stacionarnoj točki računamo vrijednost funkcije z:

$1^2+(-2)^2+z^2-2-8-6z-11=0$

$z^2-6z-16=0$

$(z+2)(z-8)=0$

$\Rightarrow z_1=-2,\ z_2=8$ (dvije funkcije $z=z(x,y)$ određene zadanom implicitnom jednadžbom koje imaju stacionarnu točku $(1,-2)$)

U dobivenoj stacionarnoj točki računamo druge parcijalne derivacije:

(zanimaju nas 2. parcijalne derivacije u stacionarnim točkama)

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{1-x}{z-3}\right)=\frac{-1\cdot(z-3)-(1-x)\cdot \overset{0}{\overset{\|}{z_x}}}{(z-3)^2}$$

$$=-\frac{1}{z-3}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{-2-y}{z-3}\right)=\frac{-1\cdot(z-3)-(-2-y)\cdot \overset{0}{\overset{\|}{z_y}}}{(z-3)^2}=-\frac{1}{z-3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{-2-y}{z-3}\right) = -\frac{-2-y}{(z-3)^2} \cdot \overset{0}{\overset{\shortparallel}{z_x}} = 0$$

Zato za Hesseovu matricu od z imamo

$$H_z(x,y) = \begin{bmatrix} -\frac{1}{z-3} & 0 \\ 0 & -\frac{1}{z-3} \end{bmatrix}$$

$$\Rightarrow H_{z_1}(1,-2) = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & \frac{1}{5} \end{bmatrix} \quad \left.\begin{aligned} \Delta_1 &= \frac{1}{5} > 0 \\ \Delta_2 &= \frac{1}{25} > 0 \end{aligned}\right\} \Rightarrow H_{z_1}(1,-2) > 0$$

$\Rightarrow (1,-2)$ je lokalni minimum funkcije z_1

$$\Rightarrow H_{z_2}(1,-2) = \begin{bmatrix} -\frac{1}{5} & 0 \\ 0 & -\frac{1}{5} \end{bmatrix} \quad \left.\begin{aligned} \Delta_1 &= -\frac{1}{5} < 0 \\ \Delta_2 &= \frac{1}{25} > 0 \end{aligned}\right\} \Rightarrow H_{z_2}(1,-2) < 0$$

$\Rightarrow (1,-2)$ je lokalni maksimum funkcije z_2

2. način

Preuredimo zadanu implicitnu jednadžbu:

$$(x^2-2x) + (y^2+4y) + (z^2-6z) = 11$$

$$(x-1)^2 + (y+2)^2 + (z-3)^2 = 25$$

Dobili smo jednadžbu sfere sa središtem $(1,-2,3)$ radijusa 5.

Odavde slijedi da se maksimalna vrijednost z iznosi $z_{max} = 3+5 = 8$

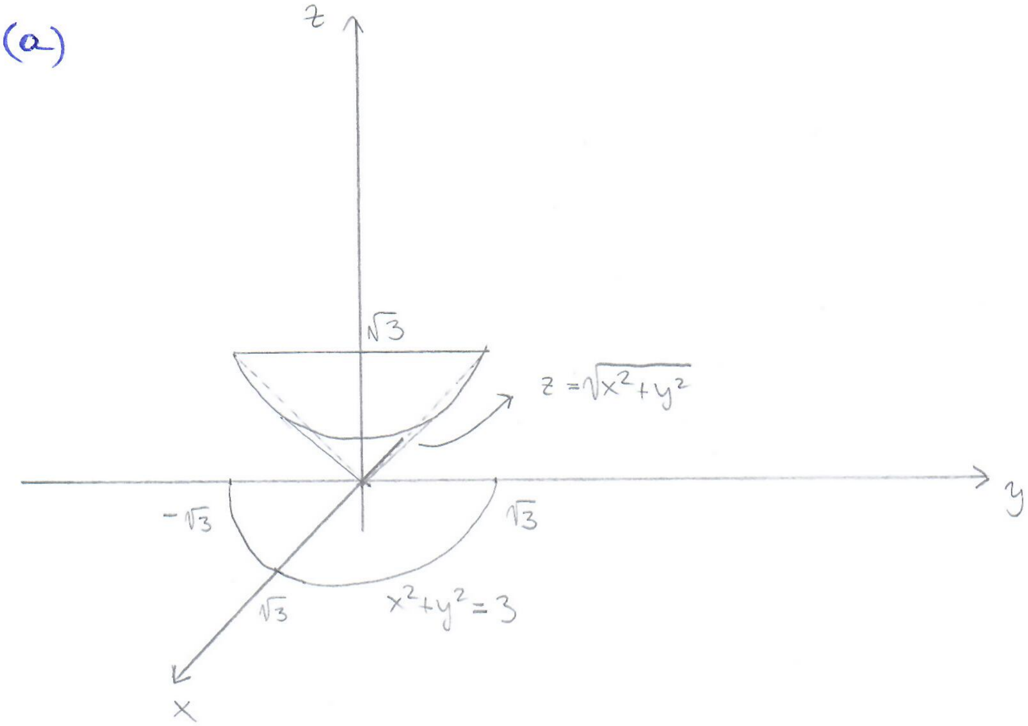
(i to je maksimum funkcije $z = 3+\sqrt{25-(x-1)^2-(y+2)^2}$).

Slično, minimalna vrijednost od z iznosi $z_{min} = 3-5 = -2$

(i to je minimum funkcije $z = 3-\sqrt{25-(x-1)^2-(y+2)^2}$).

④ $I = \int_0^{\sqrt{3}} \int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{3}} \sqrt{x^2+y^2}\, dz\, dy\, dx$

(a)



(b) Projekcija područja integracije na Oxy ravninu je polukružnica sa središtem u ishodištu radijusa $\sqrt{3}$ koju parametriziramo u polarnim koordinatama:

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r\in[0,\sqrt{3}], \quad \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\Rightarrow z \in \left[\sqrt{x^2+y^2}, \sqrt{3}\right] = \left[r, \sqrt{3}\right]$$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \int_r^{\sqrt{3}} r\cdot r\, dz\, dr\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{\sqrt{3}} \int_r^{\sqrt{3}} r^2\, dz\, dr\, d\varphi$$

(c) $x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$J = r^2\sin\theta$

$\varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\theta \in \left[0, \frac{\pi}{4}\right]$

$r \in \left[0, \frac{\sqrt{3}}{\cos\theta}\right]$ u ravnini $z=\sqrt{3}$

$$\Rightarrow I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{4}}\int_{0}^{\frac{\sqrt{3}}{\cos\theta}} \underbrace{\sqrt{r^2\sin^2\theta}}_{=r\sin\theta} \cdot r^2\sin\theta \, dr\, d\theta\, d\varphi$$

$\left(\sin\theta \geq 0 \text{ za } \theta \in \left[0, \frac{\pi}{4}\right]\right)$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{4}}\int_{0}^{\frac{\sqrt{3}}{\cos\theta}} r^3\sin^2\theta \, dr\, d\theta\, d\varphi$$

(d) Prema (b) dijelu

$$I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\sqrt{3}}\int_{r}^{\sqrt{3}} r^2 \, dz\, dr\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\sqrt{3}} r^2 z\Big|_{r}^{\sqrt{3}} dr\, d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{0}^{\sqrt{3}} \left(\sqrt{3}r^2 - r^3\right) dr\, d\varphi = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\sqrt{3}}{3}r^3 - \frac{1}{4}r^4\right)\Big|_{0}^{\sqrt{3}} d\varphi$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(3 - \frac{9}{4}\right) d\varphi = \frac{3}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi = \frac{3\pi}{4}$$

5. (a) Teorem. Neka je $(a_n)_{n\in\mathbb{N}}$ niz realnih brojeva takav da

1) $a_n \geq 0 \quad \forall n \in \mathbb{N}$

2) $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

3) $\lim\limits_{n\to\infty} a_n = 0$

Tada red $\sum\limits_{n=1}^{\infty} (-1)^n a_n$ konvergira.

Dokaz.

Promotrimo $(2n)$-tu parcijalnu sumu tog reda:

$$S_{2n} = -a_1 + a_2 - a_3 + a_4 - \dots - a_{2n-3} + a_{2n-2} - a_{2n-1} + a_{2n}$$

$$= S_{2n-2} + \underbrace{(-a_{2n-1} + a_{2n})}_{\leq 0} \leq S_{2n-2}.$$

Dakle, niz $(S_{2n})_{n\in\mathbb{N}}$ je padajuć. S druge strane,

$$S_{2n} = -a_1 + \underbrace{(a_2 - a_3)}_{\geq 0} + \dots + \underbrace{(a_{2n-2} - a_{2n-1})}_{\geq 0} + \underbrace{a_{2n}}_{\geq 0} \geq -a_1$$

za sve $n \in \mathbb{N}$. Dakle, niz $(S_{2n})_{n\in\mathbb{N}}$ je odozdo ograničen pa je konvergentan, tj. postoji $S := \lim\limits_{n\to\infty} S_{2n}$.

Sada za neparne parcijalne sume slijedi

$$\lim_{n\to\infty} S_{2n+1} = \lim_{n\to\infty} (\underbrace{S_{2n}}_{\to S} + \underbrace{a_{2n+1}}_{\to 0}) = S$$

pa slijedi da zadani red konvergira.

Q.E.D.

(b) Prema d'Alambertovom kriteriju red konvergira za

$$\left| \frac{\frac{(-1)^{n+1}}{(n+1)^2+4}(x+1)^{2n+3}}{\frac{(-1)^n}{n^2+4}(x+1)^{2n+1}} \right| = \frac{n^2+4}{(n+1)^2+4}|x+1|^2 \xrightarrow{n\to\infty} |x+1|^2 < 1$$

$$\Rightarrow |x+1|^2 < 1 \Rightarrow |x+1| < 1 \Rightarrow x \in \langle -2, 0 \rangle$$

Ispitujemo konvergenciju u rubovima:

- $x=-2$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}(-1)^{2n+1} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}$$

Za niz $a_n := \frac{1}{n^2+4}$ vrijedi:

1) $a_n \geq 0 \quad \forall n \in \mathbb{N}$

2) $n < n+1 \Rightarrow n^2+4 < (n+1)^2+4 \Rightarrow \underbrace{\frac{1}{n^2+4}}_{a_n} > \underbrace{\frac{1}{(n+1)^2+4}}_{a_{n+1}} \quad \forall n \in \mathbb{N}$

3) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \frac{1}{n^2+4} = 0$

pa prema Leibnizovom kriteriju dobiveni red konvergira.

- $x=0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4} \cdot 1^{2n+1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+4}$$

Konvergenciju ovog reda smo već utvrdili.

Dakle, područje konvergencije ovog reda je $I=[-2,0]$.

6. $y' + \frac{2}{x} y = x^4 y^4 \rightsquigarrow$ Bernoullijeva jednadžba, $n=4$

Supstitucija: $z = y^{1-4} = y^{-3}$

$$\Rightarrow z' = -3 y^{-4} y'$$

Jednadžba postaje:

$$y^{-4} y' + \frac{2}{x} y^{-3} = x^4$$

$$\Rightarrow -\frac{1}{3} z' + \frac{2}{x} z = x^4$$ (linearna ODJ 1. reda)

1° Homogena jednadžba

$$-\frac{1}{3} z' + \frac{2}{x} z = 0$$

$$\frac{dz}{z} = \frac{6dx}{x} \quad / \int$$

Slučaj $z=0 \Rightarrow \frac{1}{y^3} = 0$ nije moguć.

$$\ln|z| = 6 \ln|x| + \ln C \qquad C>0$$

$$|z| = C|x|^6 \qquad C>0$$

$$z = C x^6 \qquad C \neq 0$$

2° Varijacija konstanti

$$z(x) = C(x) x^6$$

$$z' = C' x^6 + 6 C x^5$$

$$\Rightarrow -\frac{1}{3} z' + \frac{2}{x} z = -\frac{1}{3} C' x^6 - 2\cancel{C x^5} + 2\cancel{C x^5} = -\frac{1}{3} C' x^6 = x^4$$

$$\Rightarrow C' = -3 \cdot \frac{1}{x^2} \quad / \int dx$$

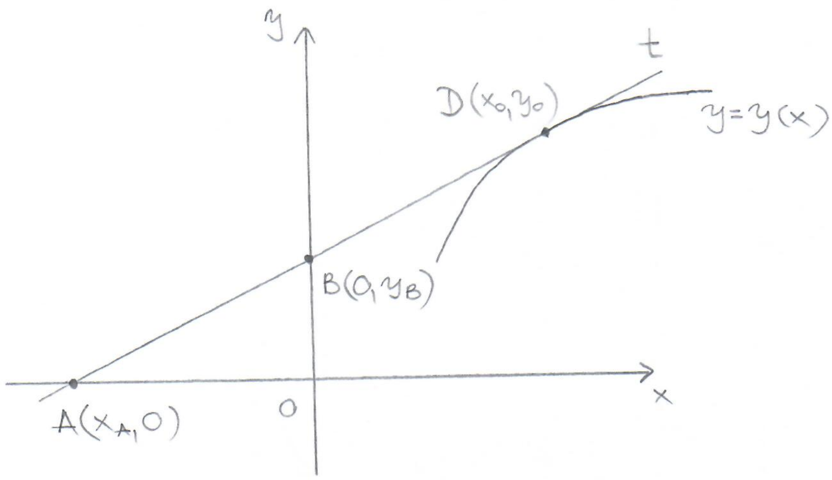
$$\Rightarrow C = \frac{3}{x} + D \qquad D \in \mathbb{R}$$

$$\Rightarrow z = 3x^5 + D x^6 \qquad D \in \mathbb{R}$$

Opće rješenje:

$$y^{-3} = 3x^5 + Dx^6 \Rightarrow \boxed{y = \frac{1}{\sqrt[3]{3x^5 + Dx^6}}, \quad D \in \mathbb{R}}$$

7.



Tangenta na krivulju u točki (x_0, y_0):

$$t \ldots \quad y - y_0 = y'(x_0)(x - x_0)$$

U točki presjeka tangente s osi apscisa imamo

$$y = 0 \implies x_A = x_0 - \frac{1}{y'(x_0)} y_0$$

U točki presjeka tangente s osi ordinata imamo

$$x = 0 \implies y_B = y_0 - y'(x_0) x_0$$

Iz uvjeta da je B polovište dužine $\overline{AD}$:

$$\left.\begin{aligned} 0 = \tfrac{1}{2}(x_A + x_D) &\implies 2x_0 - \frac{1}{y'(x_0)} y_0 = 0 \\ y_B = \tfrac{1}{2}(y_A + y_D) &\implies y_0 - y'(x_0) y_0 = \tfrac{1}{2} y_0 \end{aligned}\right\} \implies y_0 = 2y'(x_0) x_0$$

Budući da je (x_0, y_0) proizvoljna točka krivulje:

$$y = 2y'x$$

$$\frac{1}{y} dy = \frac{1}{2x} dx \quad / \int$$

• $y = 0$ nije rješenje (početni uvjet)

$$\ln|y| = \frac{1}{2}\ln|x| + \ln C \qquad C > 0$$

$$y = C\sqrt{x} \qquad C \neq 0$$

Iz početnog uvjeta: $y(3) = 1 \implies C = \frac{1}{\sqrt{3}}$

$$\implies y = \sqrt{\frac{x}{3}} \implies \boxed{y^2 = \frac{1}{3} x}$$

8. (a) $W(y_1, y_2)(x) := \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$

Za funkcije $y_1 = e^{rx}$, $y_2 = xe^{rx}$ računamo

$$W(y_1, y_2)(x) = \begin{vmatrix} e^{rx} & xe^{rx} \\ re^{rx} & e^{rx} + rxe^{rx} \end{vmatrix} \begin{matrix} | \cdot (-r) \\ + \end{matrix}$$

$$= \begin{vmatrix} e^{rx} & xe^{rx} \\ 0 & e^{rx} \end{vmatrix} = e^{2rx} \neq 0 \quad \forall x \in \mathbb{R}, \forall r \in \mathbb{R}$$

pa vidimo da su te funkcije linearno nezavisne za svaki $r \in \mathbb{R}$.

2. način

Linearnu nezavisnost možemo ispitati i po definiciji. Naime, neka su $\alpha_1, \alpha_2 \in \mathbb{R}$ proizvoljni skalari takvi da

$$\alpha_1 y_1 + \alpha_2 y_2 = 0$$

$$\Rightarrow \alpha_1 e^{rx} + \alpha_2 x e^{rx} = 0 \qquad | : e^{rx} > 0$$

$$\Rightarrow \alpha_1 + \alpha_2 x = 0$$

Budući da ova jednakost mora vrijediti za sve $x \in \mathbb{R}$, ona posebno vrijedi za

$$x = 0 \Rightarrow \alpha_1 = 0$$

$$x = 1 \Rightarrow \alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_2 = 0$$

Dakle, $\alpha_1 = \alpha_2 = 0$ pa po definiciji slijedi da su funkcije y_1 i y_2 linearno nezavisne.

(b) Provjeravamo direktnim računom:

$$y = 5e^{r_0x} + 2xe^{r_0x}$$

$$y' = 5r_0e^{r_0x} + 2e^{r_0x} + 2r_0xe^{r_0x}$$

$$y'' = 5r_0^2e^{r_0x} + 2r_0e^{r_0x} + 2r_0e^{r_0x} + 2r_0^2xe^{r_0x}$$

$$= 5r_0^2e^{r_0x} + 4r_0e^{r_0x} + 2r_0^2xe^{r_0x}$$

$$\Rightarrow y'' + a_1y' + a_0y =$$

$$= \underbrace{(r_0^2 + a_1r_0 + a_0)}_{=0}5e^{r_0x} + \underbrace{(r_0^2 + a_1r_0 + a_0)}_{=0}2e^{r_0x} + (2r_0 + a_1)2e^{r_0x}$$

jer je po pretpostavci r_0 nultočka karakteristične jednadžbe

Nadalje, budući da je r_0 dvostruka realna nultočka funkcije $f(r) = r^2 + a_1r + a_0$, r_0 mora biti apscisa tjemena te parabole, tj. njena stacionarna točka pa je i

$$f'(r_0) = 2r_0 + a_1 = 0.$$

Dakle, $y'' + a_1y' + a_0y = 0$ pa je zadani y rješenje zadane diferencijalne jednadžbe.

(c) $y'' - 4y' + 4y = \frac{e^{2x}}{x^2}$

1° Homogena jednadžba

$$y'' - 4y' + 4y = 0$$

Karakteristična jednadžba: $r^2 - 4r + 4 = 0 \Rightarrow (r-2)^2 = 0 \Rightarrow r_{1,2} = 2$

$$\Rightarrow y_h = C_1 e^{2x} + C_2 x e^{2x}, \quad C_{1,2} \in \mathbb{R}$$

2° Partikularno rješenje (varijacija konstanti)

$$y(x) = C_1(x) e^{2x} + C_2(x) x e^{2x}$$

$$y' = C_1 \cdot 2e^{2x} + C_2 (e^{2x} + 2x e^{2x}) + \underbrace{C_1' e^{2x} + C_2' x e^{2x}}_{=0}$$

$$y'' = C_1 \cdot 4e^{2x} + C_2 (2e^{2x} + 2e^{2x} + 4x e^{2x}) + \underbrace{C_1' \cdot 2e^{2x} + C_2' (2x+1) e^{2x}}_{= \frac{e^{2x}}{x^2}}$$

$$\begin{cases} C_1' e^{2x} + C_2' x e^{2x} = 0 & | \cdot (-2) \\ C_1' 2e^{2x} + C_2' (2x+1) e^{2x} = \frac{e^{2x}}{x^2} & + \end{cases}$$

$$\Rightarrow C_2' e^{2x} = \frac{e^{2x}}{x^2} \quad | : e^{2x} \neq 0$$

$$\Rightarrow C_2' = \frac{1}{x^2} \quad \Big| \int dx$$

$$\Rightarrow \boxed{C_2 = -\frac{1}{x} + D_2, \; D_2 \in \mathbb{R}}$$

$$\Rightarrow C_1' = -C_2' \cdot x = -\frac{1}{x} \Rightarrow \boxed{C_1 = -\ln|x| + D_1, \; D_1 \in \mathbb{R}}$$

Opće rješenje:

$$\boxed{y = -e^{2x} \ln|x| - e^{2x} + D_1 e^{2x} + D_2 x e^{2x}, \quad D_{1,2} \in \mathbb{R}}$$