ISPIT IZ MATEMATIKE 2
12.09.2016.

1. **(5 bodova)**
a) (2b) Neka su $\vec{a}$ i $\vec{b} \in V^3$. Napišite što je duljina, nosač i orijentacija vektorskog umnoška $\vec{a} \times \vec{b}$.
b) (3b) Dokažite sljedeće tvrdnje:
(T1) $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ za sve vektore $\vec{u}, \vec{v} \in V^3$.
(T2) Neka su vektori $\vec{i}, \vec{j}, \vec{k}$ vektori kanonske baze prostora V^3. Vektor $\vec{i} \times \vec{j}$ i vektor $\vec{i} \times (\vec{i} \times (\vec{i} \times \vec{j}))$ su kolinearni vektori.

2. **(5 bodova)** Jednadžbama

$$p_1 \cdots \quad \frac{x-1}{2} = \frac{y+2}{1} = \frac{z-3}{4},$$

$$p_2 \cdots \quad \frac{x+1}{2} = \frac{y-2}{1} = \frac{z+3}{4}$$

zadana su dva paralelna pravca. Odredite jednadžbu ravnine π s obzirom na koju su p_1 i p_2 zrcalno simetrični.

3. **(5 bodova)**
a) (2b) Dokažite da limes $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+2y^6}$ ne postoji.
b) (3b) Ispitajte istinitost sljedeće tvrdnje:
Ako funkcija dviju varijabla ima obje parcijalne derivacije u nekoj točki T_0, tada je ta funkcija nužno diferencijabilna u T_0.
Ako je tvrdnja točna ju dokažite, a ako je netočna opovrgnite ju protuprimjerom. Obrazložite svoje tvrdnje.

4. **(4 boda)** Zadana je funkcija dviju varijabla $f(x,y) = e^{xy}\sin(x+y)$.
a) (2b) Odredite jedinični vektor u smjeru najbržeg rasta funkcije iz točke $T(0, \frac{\pi}{2})$.
b) (2b) Ako uvedemo parametarsku zamjenu varijabli $x = 2t+1$, $y = 4t^3$, koristeći pravilo za složeno deriviranje funkcije više varijabla izračunajte $\frac{\partial f}{\partial t}|_{t=1}$.

5. **(5 bodova)** Odredite uvjetne lokalne ekstreme funkcije

$$f(x,y) = x^2 + y^2 - 2x - 4y$$

uz uvjet $xy - y - 2x + 1 = 0$.

6. **(5 bodova)**
a) (1b) Navedite koja je tvrdnja jedina istinita od navedenih:
(T1) Redovi $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$ oba konvergiraju.
(T2) Redovi $\sum \frac{1}{n}$, $\sum \frac{1}{n^2}$ oba divergiraju.
(T3) Red $\sum \frac{1}{n}$ divergira, a red $\sum \frac{1}{n^2}$ konvergira.
(T4) Red $\sum \frac{1}{n}$ konvergira, a red $\sum \frac{1}{n^2}$ divergira.

b) (4b) Koristeći odgovor pod a) i kriterij o uspoređivanju redova s pozitivnim članovima dokažite sljedeće tvrdnje:
(T5) Red $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ konvergira kada je $p > 2$.
(T6) Red $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ divergira kada je $p \in \langle 0, 1 \rangle$.

7. (6 bodova)
a) (4b) Razvijte u Taylorov red oko točke $c = 2$ funkciju

$$f(x) = \frac{x-2}{5-x}$$

te odredite područje konvergencije toga reda.
b) (2b) Izračunajte sumu reda

$$\sum_{n=0}^{\infty}(n+1)\left(\frac{2}{3}\right)^n.$$

8. (6 bodova)
a) (2b) Neka je zadana diferencijalna jednadžba u obliku

$$P(x,y)\,dx + Q(x,y)dy = 0$$

koja nije egzaktna. Izvedite formulu za Eulerov multiplikator μ u slučaju da je μ funkcija ovisna samo o varijabli y.
b) (4b) Riješite diferencijalnu jednadžbu

$$(y^4\cos x + y)dx + (2y^3\sin x - x)dy = 0.$$

9. (**4 boda**)
Nađite ono partikularno rješenje diferencijalne jednadžbe

$$\frac{dy}{dx} = 2x(x^2 + y)$$

koje prolazi točkom $T(0,3)$.

10. (**5 bodova**)
a) (3b) Odredite opće rješenje diferencijalne jednadžbe

$$y'' + y = \frac{\sin x}{\cos^2 x}.$$

b) (2b) Zadana je nehomogena linearna diferencijalna jednadžba s konstantnim koeficijentima.

$$y'' + a_1y' + a_0y = f(x).$$

Neka su y_1 i y_2 dva linearno nezavisna rješenja pripadne homogene jednadžbe. Detaljno obrazložite zašto sustav dobiven u metodi varijacije konstanti

$$\begin{cases} C_1'y_1 + C_2'y_2 = 0 \\ C_1'y_1' + C_2'y_2' = f(x). \end{cases}$$

ima za rješenje točno jedan par funkcija C_1' i C_2'.

Napomena: Vrijeme pisanja je **150 minuta**.

1, a) $\vec{a}\times\vec{b}$ je vektor koji zadovoljava sljedeća svojstva:

1) $\vec{a}\times\vec{b}\perp\vec{a}$ & $\vec{a}\times\vec{b}\perp\vec{b}$ — nosač

2) $|\vec{a}\times\vec{b}| = |\vec{a}|\cdot|\vec{b}|\cdot\sin\angle(\vec{a},\vec{b})$ — duljina

3.) $(\vec{a},\vec{b},\vec{a}\times\vec{b})$ je desna baza — orijentacija

b) T1 $\vec{u}\times\vec{v}\perp\vec{u}$ (po def.) $\Rightarrow$ $\vec{u}\cdot(\vec{u}\times\vec{v}) = |\vec{u}|\cdot|\vec{u}\times\vec{v}|\cdot\cos 90^\circ$

$= 0$

T2 $\vec{i}\times\vec{j}=\vec{k}$

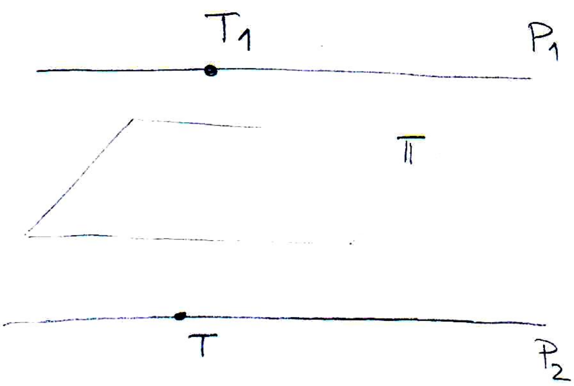
$\vec{i}\times(\vec{i}\times(\vec{i}\times\vec{j})) = \vec{i}\times(\vec{i}\times\vec{k}) = \vec{i}\times(-\vec{j}) = -\vec{k}$

$\Rightarrow$ $\vec{i}\times\vec{j}$ & $\vec{i}\times(\vec{i}\times(\vec{i}\times\vec{j}))$ su kolinearni.

2

$p_1 \equiv \dfrac{x-1}{2} = \dfrac{y+2}{1} = \dfrac{z-3}{4}$

$p_2 \equiv \dfrac{x+1}{2} = \dfrac{y-2}{1} = \dfrac{z+3}{4}$



$\vec{s} = (2,1,4)$

$T_1 = (1,-2,3) \in p_1$

$T_2 = (-1,2,-3) \in p_2$

Neka je T točka na p_2 t.d. $\overrightarrow{T_1T}\perp\vec{s}$.

$T = (2t-1, t+2, 4t-3)$

$\overrightarrow{T_1T} = (2t-2, t+4, 4t-6)$

$\overrightarrow{T_1T}\cdot\vec{s} = 0 \Rightarrow 4t-4+t+4+16t-24=0$

$21t-24=0$

$t = \frac{24}{21} = \frac{8}{7}$

$\overrightarrow{T_1T} = \left(\frac{2}{7}, \frac{36}{7}, \frac{-10}{7}\right)$

$\overrightarrow{T_1T}\perp\Pi \Rightarrow$ Vektor normale te ravnine je $(1,18,-5)$.

$\Pi \ldots\ x+18y-5z=D$ $\quad$ $P=$ polovište $\overline{T_1T}$ $\quad$ $P=\left(\frac{8}{7}, \frac{4}{7}, \frac{16}{7}\right)$

$T = \left(\frac{9}{7}, \frac{22}{7}, \frac{11}{7}\right)$

$\Pi \ldots \quad x+18y-5z=0$

drugo rješenje:

$$\vec{n}_1 = \vec{s} \times \overrightarrow{T_1T_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 4 \\ 2 & -4 & 6 \end{vmatrix} = 2\,(11\vec{i} - 2\vec{j} - 5\vec{k})$$

$$\vec{n}_1 \in \Pi \quad \Rightarrow \quad \vec{n} = \vec{n}_1 \times \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 11 & -2 & -5 \\ 2 & 1 & 4 \end{vmatrix} = -3\,(\vec{i} + 18\vec{j} - 5\vec{k})$$

$S = \text{polovište } \overline{T_1T_2} = (0,0,0)$

$\Rightarrow \Pi \equiv 1\,(x-0) + 18\,(y-0) - 5\,(z-0) = 0$

$\Pi \ldots \quad x+18y-5z=0$

3. a) $\lim\limits_{(y^3,y)\to(0,0)} \dfrac{3xy^3}{x^2+2y^6} = \lim\limits_{y\to 0} \dfrac{3y^3y^3}{y^6+2y^6} = 1$

$\lim\limits_{(x,0)\to(0,0)} \dfrac{3xy^3}{x^2+2y^6} = 0$

$\Longrightarrow$ limes ne postoji

b) Netočno! Primjer: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$

$\dfrac{\partial f}{\partial x}(0,0) = \lim\limits_{t\to 0} \dfrac{f(t,0)-f(0,0)}{t} = \dfrac{0}{t} = 0$

$\dfrac{\partial f}{\partial y}(0,0) = \lim\limits_{t\to 0} \dfrac{f(0,t)-f(0,0)}{t} = \dfrac{0}{t} = 0$

$\Rightarrow$ postoje parc. der. u $(0,0)$.

f nije neprekidna u 0. $\lim\limits_{(x,x)\to(0,0)} \dfrac{x^2}{x^2+x^2} = \dfrac{1}{2}$ $\neq$

$\lim\limits_{(x,0)\to(0,0)} \dfrac{x\cdot 0}{x^2+0^2} = 0$

$\Longrightarrow$ ne postoji $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ pa f nije nepr. u 0.

(4.) $f(x,y) = e^{xy}\sin(x+y)$

a) $\nabla f(x,y) = \big(y e^{xy}\sin(x+y) + e^{xy}\cdot\cos(x+y),$

$x e^{xy}\sin(x+y) + e^{xy}\cdot\cos(x+y) \big)$

$$h_0 = \frac{\nabla f(0,\frac{\pi}{2})}{\|\nabla f(0,\frac{\pi}{2})\|} = \frac{\left(\frac{\pi}{2}\sin\frac{\pi}{2} + \cos(\frac{\pi}{2}),\ \cos\frac{\pi}{2}\right)}{\| -\|\,-\|} =$$

$$= \frac{\left(\frac{\pi}{2}, 0\right)}{\frac{\pi}{2}} = (1,0)$$

$$\frac{\partial f}{\partial h}\left(0,\frac{\pi}{2}\right) = \|\nabla f(0,\tfrac{\pi}{2})\| = \frac{\pi}{2}$$

b) $x = 2t+1$ $\quad x = 3$ $\quad \frac{\partial x}{\partial t} = 2$

$y = 4t^3$ $\quad y = 4$ $\quad \frac{\partial y}{\partial t} = 12t^2$

(1)

$$\left.\frac{\partial f}{\partial t}\right|_{t=1} = \left.\frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}\right|_{t=1} = \left(4e^{12}\sin 7 + e^{12}\cos 7\right)\cdot 2$$

$$+\left(3e^{12}\sin 7 + e^{12}\cos 7\right)\cdot 12 = 44\, e^{12}\sin 7 + 14\cdot e^{12}\cos 7$$

5. $f(x,y) = x^2+y^2-2x-4y$

UVJET: $xy-y-2x+1=0$

$\varphi(x,y) = xy-y-2x+1$

$L(x,y,\lambda) = x^2+y^2-2x-4y+\lambda(xy-y-2x+1)$

$\nabla L(x,y,\lambda) = (2x-2+\lambda y-2\lambda,\ 2y-4+\lambda x-\lambda,\ xy-y-2x+1)$

$2x-2+\lambda y-2\lambda=0 \Rightarrow x = \frac{2\lambda+2-\lambda y}{2}$

$2y-4+\lambda x-\lambda=0 \Rightarrow 2y-4+\frac{2\lambda^2+2\lambda-\lambda^2 y}{2}-\lambda=0$

$xy-y-2x+1=0$

$2y-4+\lambda^2+\cancel{\lambda}-\frac{\lambda^2 y}{2}-\cancel{\lambda}=0$

$y\left(2-\frac{\lambda^2}{2}\right)=4-\lambda^2$

$y\cdot\frac{4-\lambda^2}{2}=4-\lambda^2 \Rightarrow y=2$ ili $\lambda=\pm2$

Za $y=2 \Rightarrow x=1$, ali $\varphi(1,2)=\cancel{2}-\cancel{2}-2+1=-1\neq0$

$\Rightarrow \lambda=\pm2$

Za $\lambda=2$ $\quad x=\frac{4+2-2y}{2}=3-y$

$y(3-y)-y-6+2y+1=0$

$3y-y^2-6-y+2y+1=0$

$-y^2+4y-5=0$

$y^2-4y+5=0 \quad y\notin\mathbb{R}$ jer $\Delta=16-20=-4<0$

Za $\boxed{\lambda=-2}$ $\quad x=\frac{-4+2+2y}{2}=y-1$

$y(y-1)-y-2(y-1)+1=0$

$y^2-y-y-2y+3=0 \Rightarrow y^2-4y+3=0 \qquad y_1=1,\ y_2=3$

$x_1=0,\ x_2=2$

$T_1 = (0,1)$

$T_2 = (2,3)$

$L''_{xx} = 2$

$L''_{xy} = \lambda$ $\qquad d^2L = 2(dx)^2 + 2\lambda\, dx\,dy + 2(dy)^2$

$L''_{yy} = 2$ $\qquad = 2(dx)^2 - 4\,dx\,dy + 2(dy)^2$

$= 2\,(dx-dy)^2 \geq 0$

Za $dx = dy$ je $d^2L = 0$

$xy - y - 2x + 1 = 0$

$(y-2)\,dx + (x-1)\,dy = 0$

$$dy = \frac{2-y}{x-1}\,dx$$

$$d^2L = 2(dx)^2 - 4\,dx \cdot \frac{2-y}{x-1}\,dx + 2 \cdot \frac{(2-y)^2}{(x-1)^2}(dx)^2$$

Za $(0,1)$ je $d^2L = 2(dx)^2 + 4(dx)^2 + 2(dx)^2 > 0$ za

$(dx, dy) \neq (0,0)$

Za $(2,3)$ je $d^2L = 2(dx)^2 + 4(dx)^2 + 2(dx)^2 > 0$ za

$(dx, dy) \neq (0,0)$

T_1 i T_2 su strogi lokalni minimumi

6. a) T3

b) T5 $n^p > n^2$ za $p > 2$

$$\frac{1}{n^p} < \frac{1}{n^2} \quad \text{za } p > 2$$

Iz a) usporednim kriterijem dobijemo $\sum_{n\geq 1} \frac{1}{n^p}$ konv.

T6 $n^p < n$ za $p \in \langle 0, 1 \rangle$

$$\frac{1}{n^p} > \frac{1}{n}$$

Kako $\sum_{n\geq 1} \frac{1}{n}$ divergira $\Longrightarrow \sum_{n\geq} \frac{1}{n^p}$ divergira

7. a) $f(x) = \dfrac{x-2}{5-x} = \dfrac{x-2}{5-(x-2)-2} = \dfrac{x-2}{3-(x-2)} = \dfrac{x-2}{3\left(1-\frac{x-2}{3}\right)}$

$$\frac{1}{1-\frac{x-2}{3}} = \sum_{n=0}^{+\infty} \left(\frac{x-2}{3}\right)^n \quad \text{za } \left|\frac{x-2}{3}\right| < 1$$

$$\frac{x-2}{\left(1-\frac{x-2}{3}\right)} \cdot \frac{1}{3} = \sum_{n=0}^{+\infty} \left(\frac{x-2}{3}\right)^{n+1}$$

$$f(x) = \sum_{n=0}^{+\infty} \left(\frac{x-2}{3}\right)^{n+1}$$

Konvergira za $\left|\frac{x-2}{3}\right| < 1$, divergira za $\left|\frac{x-2}{3}\right| > 1$.

Za $x=5$ i $x=-1$ red divergira.

Područje konvergencije je $\langle -1, 5 \rangle$.

b) $f(x) = \sum_{n=0}^{+\infty} \left(\frac{x-2}{3}\right)^{n+1}$ $/ ^{\prime}$

$$f'(x) = \sum_{n=0}^{+\infty} (n+1)\left(\frac{x-2}{3}\right)^{n} \cdot \frac{1}{3}$$

$$\left(\frac{x-2}{5-x}\right)' = \frac{1}{3} \sum_{n=0}^{+\infty} (n+1)\left(\frac{x-2}{3}\right)^{n}$$

$$\frac{(5-x)+(x-2)}{(5-x)^2} \cdot 3 = \sum_{n=0}^{+\infty} (n+1)\left(\frac{x-2}{3}\right)^{n} \qquad x=4$$

$$\sum_{n=0}^{+\infty} (n+1)\left(\frac{2}{3}\right)^{n} = 3 \cdot \frac{1+2}{1^2} = 9$$

8. b) $(y^4\cos x + y)\,dx + (2y^3\sin x - x)\,dy = 0$

$$P(x,y) = y^4\cos x + y, \quad Q(x,y) = 2y^3\sin x - x$$

$$\frac{\partial P}{\partial y}(x,y) = 4y^3\cos x + 1, \quad \frac{\partial Q}{\partial x}(x,y) = 2y^3\cos x - 1$$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x} \Rightarrow \text{nije egzaktna}$$

$$\frac{1}{P}(P'_y - Q'_x) = \frac{4y^3\cos x + 1 - 2y^3\cos x + 1}{y^4\cos x + y} = \frac{2y^3\cos x + 2}{y^4\cos x + y} =$$

$$= \frac{2}{y} \cdot \frac{y^3\cos x + 1}{y^3\cos x + 1} = \frac{2}{y} \quad \text{funkcija varijable } y$$

$$\ln\mu(y) = -\int \frac{1}{P}(P'_y - Q'_x)\,dy = -\int \frac{2}{y}\,dy = \ln\frac{1}{y^2}$$

$$\mu(y) = \frac{1}{y^2}$$

Množenjem s μ dobijemo $\left(y^2\cos x + \frac{1}{y}\right)dx + \left(2y\sin x - \frac{x}{y^2}\right)dy = 0$

$$u(x,y) = \int_{x_0}^{x}\left(y^2\cos x + \frac{1}{y}\right)dx + \int_{y_0}^{y}\left(2y\sin x_0 - \frac{x_0}{y^2}\right)dy \qquad y_0 = 1,\ x_0 = 0$$

$$= \int_0^x \left(y^2\cos x + \frac{1}{y}\right)dx = \left(y^2\sin x + \frac{x}{y}\right)\Big|_0^x = y^2\sin x + \frac{x}{y}$$

$$u(x,y) = y^2\sin x + \frac{x}{y} = C$$

a) $\mu(x,y)P(x,y)\,dx+\mu(x,y)Q(x,y)\,dy=0$

Z warunku egzaktności: $\frac{\partial(\mu P)}{\partial y}=\frac{\partial(\mu Q)}{\partial x}$

$\Rightarrow \mu_y' P+\mu P_y'=\mu_x' Q+\mu Q_x'$

$\mu(x,y)=\mu(y)$ $\quad\Rightarrow \mu' P+\mu P_y'=\mu Q_x'$

$\Rightarrow \mu' P=\mu(Q_x'-P_y')$

$\Rightarrow \frac{\mu'}{\mu}=\frac{Q_x'-P_y'}{P} \Big/ \int dy$

$\ln\mu(y)=-\int\frac{1}{P}(P_y'-Q_x')\,dy$

9.

$$\frac{dy}{dx} = 2x(x^2+y)$$

$$y(0)=3$$

$$y' - 2xy = 2x^3$$

homogene $\quad y' - 2xy = 0$

$$\frac{dy}{dx} = 2xy \Rightarrow \frac{dy}{y} = 2x\,dx \quad / \int$$

$$\ln y = x^2 + \ln C$$

$$\ln \frac{y}{C} = x^2$$

$$\frac{y}{C} = e^{x^2} \Rightarrow y_h = C e^{x^2}$$

$$y = C(x) e^{x^2}$$

$$C'(x) e^{x^2} + \cancel{C(x) e^{x^2} \cdot 2x} = 2x^3 + \cancel{2x \cdot C(x) e^{x^2}}$$

$$C'(x) = 2x^3 e^{-x^2} \Rightarrow C(x) = \int 2x^3 \cdot e^{-x^2} dx = \begin{bmatrix} -x^2 = t \\ \frac{dt}{dx} = -2x \end{bmatrix}$$

$$= \int t e^t dt = \begin{bmatrix} u = t & du = dt \\ dv = e^t dt & v = e^t \end{bmatrix}$$

$$= te^t - \int e^t dt = -e^{-x^2} - x^2 e^{-x^2} + C$$

$$y(x) = -1 - x^2 + Ce^{x^2} \qquad y(0)=3 \Rightarrow 3 = -1 + C \Rightarrow \boxed{C=4}$$

$$\Rightarrow y(x) = -1 - x^2 + 4e^{x^2}$$

(10.) a) $y'' + y = \frac{\sin x}{\cos^2 x}$

homogena: $y'' + y = 0$

$r^2 + 1 = 0 \Rightarrow r_{1,2} = \pm i \qquad y_h = C_1 \cos x + C_2 \sin x$

Opće rješenje $y = C_1(x)\cos x + C_2(x)\sin x$

$C_1'(x)y_1(x) + C_2'(x)y_2(x) = 0 \qquad \Rightarrow C_1'(x)\cos x + C_2'(x)\sin x = 0 \ /\cdot \sin x$

$C_1'(x)y_1'(x) + C_2'(x)y_2'(x) = \frac{\sin x}{\cos^2 x} \qquad -C_1'(x)\sin x + C_2'(x)\cos x = \frac{\sin x}{\cos^2 x} \ /\cdot \cos x$

$\Rightarrow C_2'(x) = \operatorname{tg} x \qquad \Rightarrow C_2(x) = -\ln|\cos x| + C_2$

$C_1'(x) = \frac{-1}{\cos x} \cdot \frac{\sin^2 x}{\cos^2 x} = -\operatorname{tg}^2 x = -\frac{\sin^2 x}{\cos^2 x} = -\frac{1-\cos^2 x}{\cos^2 x} = -\frac{1}{\cos^2 x} + 1$

$C_1(x) = x - \operatorname{tg} x + C_1$

$y(x) = (x - \operatorname{tg} x + C_1)\cos x + (-\ln|\cos x| + C_2)\sin x$

b)

$$C_1' y_1 + C_2' y_2 = 0$$
$$C_1' y_1' + C_2' y_2' = f(x)$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} C_1' \\ C_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(x) \end{bmatrix} \qquad (*)$$

Kako su y_1 i y_2 linearno nezavisni $\Rightarrow$ Wronskijana

$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0$, a onda je rj. sustava (*) jedinstveno.