Svakako sami pokušajte riješiti zadatke prije konzultiranja ovih rješenja.

Tomislav Petković, Zagreb, 21. 3. 2008.

1. a) $E_1 = \sum_n x_1^2[n] = 1+4+1 = 6,$

$E_2 = 1+4+1 = 6$

b) $P_1 = \frac{1}{N} \sum_{n=0}^{N-1} x_1^2[n] = \frac{1}{5}(1+2^2+1) = \frac{6}{5}$

$P_2 = \frac{1}{5}(1+4+1) = \frac{6}{5}$

c) $x_1 * x_2 = \{1, 2, -1, \underline{-4}, -1, 2, 1\}$

2. $x_1[n] * x_2[n] \circ\!\!-\!\!\bullet\ X_1(z) \cdot X_2(z)$, $X_1(z) = \frac{z}{(z-1)^2}$, $X_2(z) = \frac{z}{z+1}$

$$Y(z) = X_1(z)X_2(z) = \frac{z^2}{(z+1)(z-1)^2} = \frac{Az}{z-1} + \frac{Bz}{(z-1)^2} + \frac{Cz}{z+1}$$

$$C = \frac{z+1}{z} Y(z) \Big|_{z=-1} = \frac{-1}{(-1-1)^2} = -\frac{1}{4}$$

$$B = \frac{(z-1)^2}{z} Y(z) \Big|_{z=1} = \frac{1}{1+1} = \frac{1}{2}$$

A je najjednostavnije odrediti uvrštavanjem u jednadžbu

$$z = A(z-1)(z+1) + \frac{1}{2}(z+1) - \frac{1}{4}(z-1)^2$$

$$4z = 4A(z^2-1) + 2z + 2 - z^2 + 2z - 1 \Rightarrow A = 1/4$$

$$Y(z) = \frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2} - \frac{1}{4}\frac{z}{z+1}$$

$$y[n] = \frac{1}{4} 1^n + \frac{1}{2} \cdot n - \frac{1}{4}(-1)^n$$

3. $$x_1[n] * x_2[n] = \sum_{i=-\infty}^{+\infty} x_1[i] x_2[n-i] = \sum_{i=0}^{n} \left(\frac{1}{5}\right)^i (-1)^{n-i} = (-1)^n \sum_{i=0}^{n} \left(-\frac{1}{5}\right)^i =$$

$$= (-1)^n \frac{1-(1/5)^{n+1}}{1-(-1/5)} = \frac{5}{6}(-1)^n + \frac{5}{6}\left(\frac{1}{5}\right)^{n+1} = \frac{5}{6}(-1)^n + \frac{1}{6}\left(\frac{1}{5}\right)^n, \quad n \geq 0$$

4. $x[n] * x[-n] = \sum_{i=-\infty}^{+\infty} \left(\frac{1}{4}\right)^i \underset{i \geq 0}{\mu[i]} \left(\frac{1}{4}\right)^{-n+i} \underset{-n+i \geq 0 \Rightarrow i \geq n}{\mu[-n+i]}$

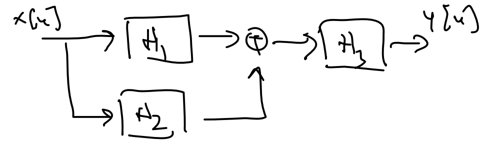
razlikujemo dva slučaja

$$n > 0: \quad \sum_{i=n}^{+\infty} \left(\frac{1}{4}\right)^i \left(\frac{1}{4}\right)^{-n+i} = \left(\frac{1}{4}\right)^{-n} \sum_{i=n}^{+\infty} \left(\frac{1}{4}\right)^{2i} = \left(\frac{1}{4}\right)^{-n} \frac{\left(\frac{1}{4}\right)^{2n}}{1-\left(\frac{1}{4}\right)^2} = \frac{16}{15}\left(\frac{1}{4}\right)^n$$

$$n < 0: \quad \sum_{i=0}^{+\infty} \left(\frac{1}{4}\right)^i \left(\frac{1}{4}\right)^{-n+i} = \left(\frac{1}{4}\right)^{-n} \sum_{i=0}^{+\infty} \left(\frac{1}{4}\right)^{2i} = \left(\frac{1}{4}\right)^{-n} \frac{1}{1-\left(\frac{1}{4}\right)^2} = \frac{16}{15}\left(\frac{1}{4}\right)^{-n}$$

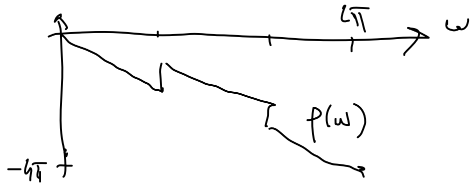
dobivamo

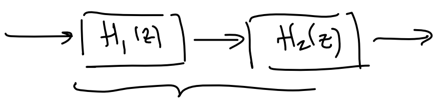
$$x[n] * x[-n] = \begin{cases} \frac{16}{15}\left(\frac{1}{4}\right)^n, & n \geq 0 \\ \frac{16}{15}\left(\frac{1}{4}\right)^{-n}, & n \leq 0 \end{cases}$$

5. 

$$H(z) = (H_1(z) + H_2(z))H_3(z) =$$
$$= 1 + 3z^{-1} + 4z^{-2} + 3z^{-3} + z^{-4}$$

$$H(e^{j\omega}) = 1 + 3e^{-j\omega} + 4e^{-2j\omega} + 3e^{-3j\omega} + e^{-4j\omega} =$$
$$= e^{-2j\omega}\left(e^{+2j\omega} + 3e^{+j\omega} + 4 + 3e^{-j\omega} + e^{-2j\omega}\right) =$$
$$= e^{-2j\omega}\left(\underbrace{4 + 6\cos(\omega) + 2\cos(2\omega)}_{\in \mathbb{R}}\right)$$



6. 

$$H(z) = H_1(z) \cdot H_2(z)$$
$$h[n] = h_1[n] * h_2[n]$$

	1	$\sqrt{6}$	-1
-2	-2	$-2\sqrt{6}$	2
5	5	$5\sqrt{6}$	-5
$2(\sqrt{6}-\sqrt{2})$	$2(\sqrt{6}-\sqrt{2})$	$2(6-4\sqrt{3})$	$-2(\sqrt{6}-\sqrt{2})$

zbrojenjem elemenata na dijagonalama dobivamo koeficijente od $h[n]$ i $H(z)$

$$H(z) = \frac{1}{18}\left(-2 + (2\sqrt{6}+5)z^{-1} + (2+7\sqrt{6}-2\sqrt{2})z^{-2} + (7-4\sqrt{3})z^{-3} - 2(\sqrt{6}-\sqrt{2})z^{-4}\right)$$

$$h[n] = \frac{1}{18}\left\{\underline{-2}, 2\sqrt{6}+5, 2+7\sqrt{6}-2\sqrt{2}, 7-4\sqrt{3}, -2(\sqrt{6}-\sqrt{2})\right\}$$

7. $6y[n]-5y[n-1]+y[n-2]=n^2u[n]$

za linearnost zahtijevamo homogenost i aditivnost, odnosno da je
$L[u_1[n]]=y_1[n]$ i $L[u_2[n]]=y_2[n]$ onda mora vrijediti

$$L[\alpha u_1[n]+\beta u_2[n]]=\alpha y_1[n]+\beta y_2[n]$$

neka je $6y[n]+5y[n-1]+y[n-2]=n^2(\underbrace{\alpha u_1[n]+\beta u_2[n]}_{u[n]})$

moramo pokazati da vrijedi (ili ne) $y[n]\overset{?}{=}\alpha y_1[n]+\beta y_2[n]$

$$\begin{cases} 6y_1[n]+5y_1[n-1]+y_1[n-2]=n^2u_1[n] \\ 6y_2[n]+5y_2[n-1]+y_2[n-2]=n^2u_2[n] \end{cases}$$

$$n^2(\alpha u_1[n]+\beta u_2[n])=\alpha n^2u_1[n]+\beta n^2u_2[n]=$$
$$=6(\alpha y_1[n]+\beta y_2[n])-5(\alpha y_1[n]+\beta y_2[n])+\alpha y_1[n]+\beta y_2[n]$$
$$\overset{?}{=}6y[n]-5y[n-1]+y[n-2]$$

sada je i $y[n]=\alpha y_1[n]+\beta y_2[n]$ te je sustav LINEARAN

za vremensku nepromjenjivost zahtijevamo da odziv i pobuda ostaju isti prilikom pomaka, odnosno da je $y[n]=L[u[n]]$ onda je i $y[n+m]=L[u[n+m]]$, $\forall m\in\mathbb{Z}$

$$\begin{cases} 6y[n]-5y[n-1]+y[n-2]=n^2u[n] \\ 6y[n+m]-5y[n+m-1]+y[n+m-2]=n^2u[n+m] \end{cases}$$

Zamijenom $n+m=n'$ u drugoj jednadžbi vidimo da su jednadžbe iste za $m=0$, odnosno sustav je vremenski nepromjenjiv

Odredimo i odziv za pobudu $u[n]=2\delta[n-1]$. Moramo riješiti jednadžbu:

$$6y[n]-5y[n-1]+y[n-2]=2\delta[n-1]$$

$$6q^2-5q+1=0 \Rightarrow q=\frac{5\pm\sqrt{25-24}}{12}=\frac{5\pm1}{12}\Rightarrow q_1=\frac{1}{2},\ q_2=\frac{1}{3}$$

$$y[n]=C_1\left(\frac{1}{2}\right)^n+C_2\left(\frac{1}{3}\right)^n$$

$$\begin{cases} n=0: & 6(C_1+C_2)=0 \\ n=1: & 6\left(\frac{1}{2}C_1+\frac{1}{3}C_2\right)-5(\overbrace{C_1+C_2}^{=0})=2 \end{cases}$$

$$C_1=-C_2 \wedge 3C_1+2C_2=2 \Rightarrow C_1=2,\ C_2=-2$$

$$y[n]=2\left(\frac{1}{2}\right)^n+(-2)\left(\frac{1}{3}\right)^n,\quad n\geq 0$$

8. Vidi predavanje „UVOD U OBRADU INFORMACIJA" na stranicama predmeta.

9. Vidi predavanje „REPREZENTACIJA SIGNALA I SUSTAVA" na stranicama predmeta.

10. Neka je D prostor ulaznih signala i neka je K prostor izlaznih signala. Svako preslikavanje $S: D \to K$ nazivamo sustav. Za $u \in D$ i $y \in K$ pišemo $y = S(u)$.

Neka su D i K vektorski prostori nad poljem $\mathbb{C}$. Sustav $S: D \to K$ je linearan ako za $u, v \in D$ i $\alpha \in \mathbb{C}$ vrijedi:

a) $S(\alpha u) = \alpha S(u)$ HOMOGENOST

b) $S(u+v) = S(u) + S(v)$ ADITIVNOST

1) $y(t) = 5u(t)$ je LINEARAN

2) $y(t) = 5u(t) + 2$ je NELINEARAN

3) $y(t) = 5u^2(t)$ je NELINEARAN

4) $y(t) = 5tu(t) + 10t^2 u(t-2)$ je LINEARAN

11) BIBO stabilnost: Sustav je BIBO stabilan ako za svaku ograničenu pobudu daje ograničeni odziv, odnosno

$$|u(t)| < B \Rightarrow |S(u(t))| < \alpha B, \quad \alpha \in \mathbb{R}$$

Nužan i dovoljan uvjet za LTI sustave jest postojanje L^1 norme impulsnog odziva, odnosno:

a) $\int_{-\infty}^{+\infty} |h(t)|\,dt < \infty$

b) $\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$

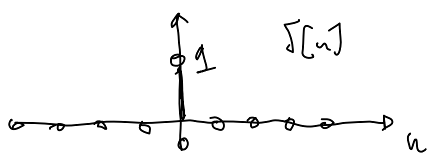
1) $h(t) = e^{-t}\mu(t) \Rightarrow \int_{-\infty}^{+\infty} e^{-t}\mu(t)\,dt = \int_0^{+\infty} e^{-t}\,dt = -e^{-t}\Big|_0^{+\infty} = 0 - (-1) = 1 < \infty$

2) $h[n] = 2^{-n}\mu[n] \Rightarrow \sum_{n=-\infty}^{+\infty} 2^{-n}\mu[n] = \sum_{n=0}^{+\infty} 2^{-n} = 1 \cdot \frac{1}{1-\frac{1}{2}} = 2 < \infty$

Oba sustava su STABILNA!

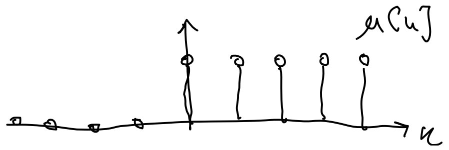
12) DISKRETNI JEDINIČNI IMPULS

$$\delta[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{inače} \end{cases}$$

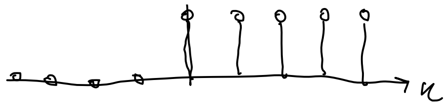


DISKRETNA JEDINIČNA STEPENICA

$$\mu[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{inače} \end{cases}$$



$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & \text{inače} \end{cases}$$



DISKRETNA HARMONIJSKA FUNKCIJA

$x[n] = e^{j(\Omega k + \theta)}$, Ω je frekvencija, θ je faza

DISKRETNA EKSPONENCIJALNA FUNKCIJA

$x[n] = (\rho e^{j\Omega})^n$

13) $X(\omega) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \text{DTFT}\left[x[n]\right] \quad \wedge \quad x[n] \in \mathbb{R}$

a) $\text{Re}\, X(\omega) = \text{Re} \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[n] \underbrace{\text{Re}\, e^{-j\omega n}}_{\cos(\omega n)} = \sum_{n=-\infty}^{+\infty} x[n] \cos(\omega n) =$

$= \sum_{n=-\infty}^{+\infty} x[n] \cos(-\omega n) = \text{Re}\, X(-\omega) \Rightarrow \text{Re}\, X(\omega)$ je parna funkcija

b) $\text{Im}\, X(\omega) = \text{Im} \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x[n] \underbrace{\text{Im}\, e^{-j\omega n}}_{\sin(\omega n)} = \sum_{n=-\infty}^{+\infty} x[n] \sin(\omega n) =$

$= -\sum_{n=-\infty}^{+\infty} x[n] \sin(-\omega n) = -\text{Im}\, X(-\omega) \Rightarrow \text{Im}\, X(\omega)$ je neparna funkcija

c) $|X(\omega)| = \sqrt{\text{Re}^2 X(\omega) + \text{Im}^2 X(\omega)}$ je zbog kvadrata očigledno parna funkcija

d) $\arg X(\omega) = \arctan \frac{\text{Im}\, X(\omega)}{\text{Re}\, X(\omega)} + \begin{cases} 0, & \text{Re}\, X(\omega) > 0 \\ \pi, & \text{inače} \end{cases}$

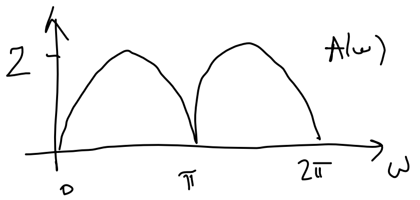
$\arg X(-\omega) = -\arctan \frac{\text{Im}\, X(\omega)}{\text{Re}\, X(\omega)} + \begin{cases} 0, & \text{Re}\, X(\omega) > 0 \\ -\pi, & \text{inače} \end{cases}$

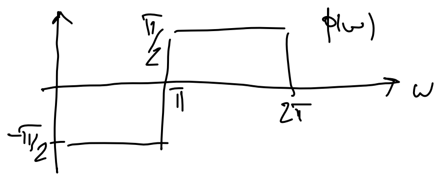
te je faza neparna funkcija

14) a) $x[n] = \begin{cases} 1, & n \in \{-1, 1\} \\ 0, & \text{inače} \end{cases} = \{-1, \underline{0}, 1\}$

$\text{DTFT}\left[x[n]\right] = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = -e^{+j\omega} + e^{-j\omega} = -2j \sin(\omega)$

$A(\omega) = |-2\sin(\omega)| = 2|\sin(\omega)|, \quad \phi(\omega) = \begin{cases} -\pi/2 & 0 < \omega < \pi \\ +\pi/2 & -\pi < \omega < 0 \end{cases}$

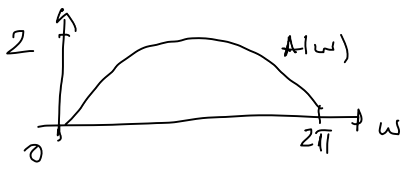


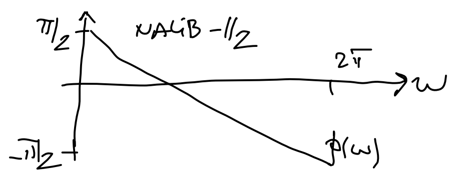


b) $x[n] = \begin{cases} 1, & n = 0 \\ -1, & n = 1 \\ 0, & \text{inače} \end{cases}$ $\quad \text{DTFT}\left[x[n]\right] = 1 - e^{-j\omega} = e^{-j\omega/2}\left(e^{+j\omega/2} - e^{-j\omega/2}\right) =$

$= e^{-j\omega/2} \cdot 2j \sin(\omega/2).$

$A(\omega) = 2|\sin(\omega/2)|, \quad \phi(\omega) = -\omega/2 + \pi/2$





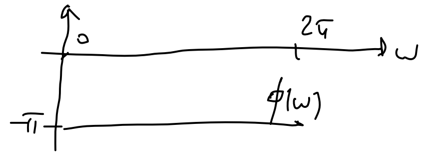
c) $x[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$

$\text{DTFT}\left[x[n]\right] = e^{+j\omega} - 2 + e^{-j\omega} = -2 + 2\cos(\omega)$

c) $x[n] = \delta[n+1] - 2\delta[n] + \delta[n-1]$

$$DTFT[x[n]] = e^{+j\omega} - 2 + e^{-j\omega} = -2 + 2\cos(\omega)$$

$$A(\omega) = 2 - 2\cos(\omega) \quad , \quad \phi(\omega) = \pi \quad (\text{mote i } -\pi)$$

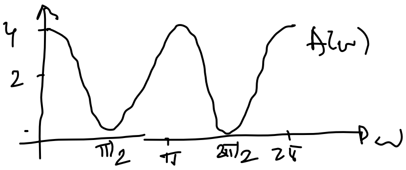


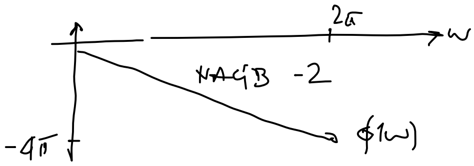
d) $x[n] = \delta[n] + 2\delta[n-2] + \delta[n-4] = \{ \underline{1}, 0, 2, 0, 1 \}$

$$DTFT[x[n]] = 1 + 2e^{-2j\omega} + e^{-4j\omega} = e^{-2j\omega}(e^{+2j\omega} + 2 + e^{-2j\omega}) =$$

$$= e^{-2j\omega}(2 + 2\cos(2\omega))$$

$$A(\omega) = 2 + 2\cos(2\omega) \quad , \quad \phi(\omega) = -2\omega$$





15. a) $h[n] = \{\underline{1}, 2, 3, 4\}$

$$H(\omega) = 1 + 2e^{-j\omega} + 3e^{-2j\omega} + 4e^{-3j\omega}$$

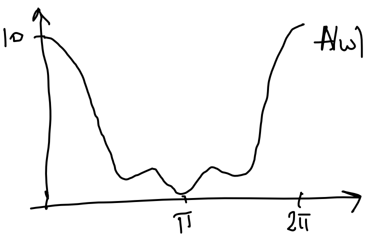
$$\mathrm{Re}\, H(\omega) = 1 + 2\cos(\omega) + 3\cos(2\omega) + 4\cos(3\omega)$$

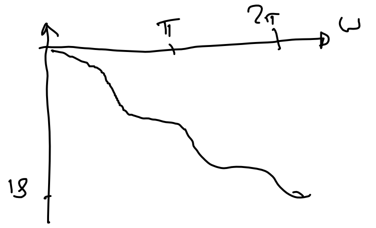
$$\mathrm{Im}\, H(\omega) = -2\sin(\omega) - 3\sin(2\omega) - 4\sin(3\omega)$$

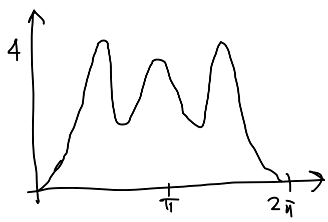
$$A(\omega) = 2\sqrt{2 + 11\cos^2(\omega) + 4\cos(\omega) + 8\cos(\omega)}, \quad \phi(\omega) = \mathrm{arctan}\frac{\mathrm{Im}\, H(\omega)}{\mathrm{Re}\, H(\omega)}$$

$$\tau(\omega) = -\frac{d}{d\omega}\phi(\omega) = \frac{\mathrm{Im}\, H(\omega)\,(\mathrm{Re}\, H(\omega))' - (\mathrm{Im}\, H(\omega))'\,\mathrm{Re}\, H(\omega)}{A^2(\omega)} =$$

$$= \frac{32 + 76\cos^2(\omega) + 44\cos(\omega) + 48\cos^3(\omega)}{8 + 44\cos^2(\omega) + 16\cos(\omega) + 32\cos^3(\omega)}$$







b) Kao i a) zadatak. Samo je potrebno primijeniti teorem o pomaku!

$$h_b[n] = \{1, 2, 3, \underline{4}\} = h_a[n+3] \Rightarrow H_b(\omega) = H_a(\omega)\cdot e^{3j\omega}$$

Vidimo da vrijedi:

$$|H_a(\omega)| = |H_b(\omega)|, \quad \phi_a(\omega) = \phi_b(\omega) - 3\omega, \quad \tau_a(\omega) = \tau_b(\omega) - 3$$

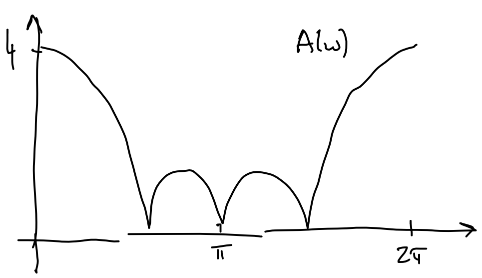
NAPOMENA: a) i b) dio 15. zadatka zahtijevaju više od 25 minuta ako se ručno rješavaju!

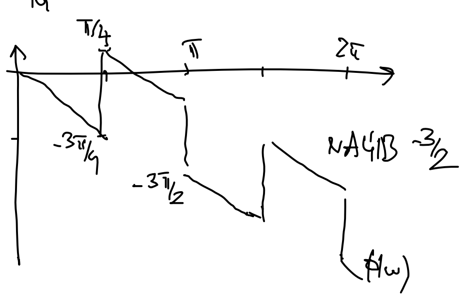
c) $h[n] = \{\underline{1}, 1, 1, 1\}$, $\mathrm{DTFT}[h[n]] = 1 + e^{-j\omega} + e^{-2j\omega} + e^{-3j\omega} =$

$$= e^{-3/2 j\omega}\left(e^{3/2 j\omega} + e^{1/2 j\omega} + e^{-1/2 j\omega} + e^{-3/2 j\omega}\right) =$$

$$= e^{-3/2 j\omega}\left(2\cos\left(\tfrac{1}{2}\omega\right) + 2\cos\left(\tfrac{3}{2}\omega\right)\right)$$

$$A(\omega) = \left|2\cos\left(\tfrac{\omega}{2}\right) + 2\cos\left(\tfrac{3\omega}{2}\right)\right|, \quad \phi(\omega) = -\tfrac{3}{2}\omega, \quad \tau(\omega) = -\frac{d\phi}{d\omega} = \tfrac{3}{2}$$

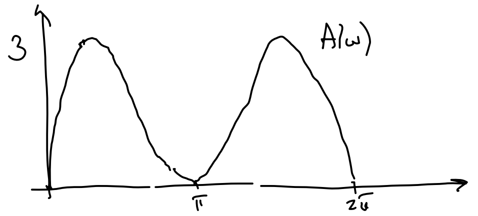


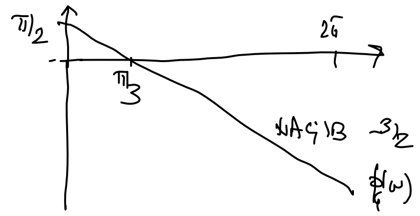


d) $h[n] = \{\underline{1}, 1, -1, -1\}$

$$DTFT[h[n]] = 1 + e^{-j\omega} - e^{-2j\omega} - e^{-3j\omega} = e^{-\frac{3}{2}j\omega + j\frac{\pi}{2}} \underbrace{\left(2\sin\left(\frac{\omega}{2}\right) + 2\sin\left(\frac{3\omega}{2}\right)\right)}_{R(\omega)}$$

$$A(\omega) = |R(\omega)| = \left|2\sin\left(\frac{\omega}{2}\right) + 2\sin\left(\frac{3\omega}{2}\right)\right|$$



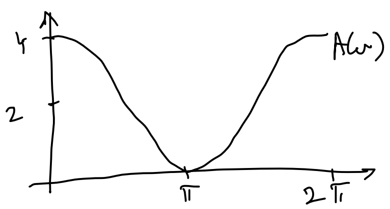


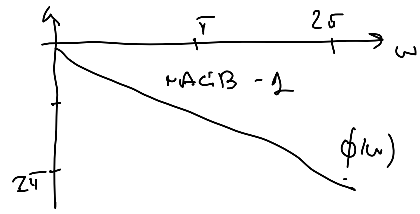
$$\tau = -\frac{d}{d\omega}\phi_g(\omega) = -\frac{d}{d\omega}\left(-\frac{3}{2}\omega + \frac{\pi}{2}\right) = \frac{3}{2}$$

e) $h[n] = \{\underline{1}, 2, 1\} \Rightarrow DTFT[h[n]] = 1 + 2e^{-j\omega} + e^{-2j\omega} =$

$$= e^{-j\omega}(2 + 2\cos(\omega))$$

$A(\omega) = 2 + 2\cos(\omega)$, $\phi(\omega) = -\omega$, $\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = 1$





f) $h[n] = \{\underline{1}, 2, -1\} \Rightarrow DTFT[h[n]] = 1 + 2e^{-j\omega} - e^{-2j\omega} =$

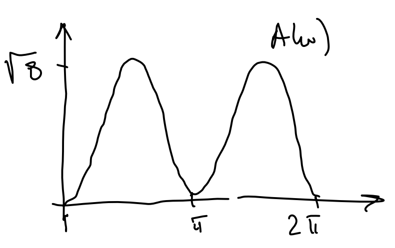
$$= e^{-j\omega}(2 + 2j\sin(\omega))$$

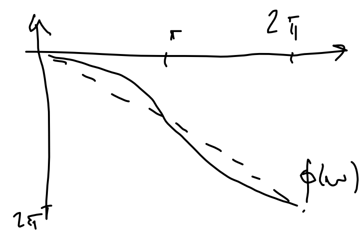
← ZBOG 'j' OVO NIJE GENERALIZIRANA AMPLITUDA

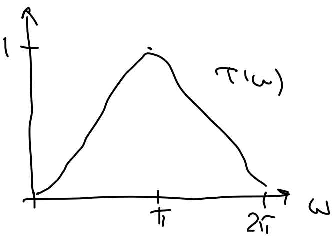
OVO TAKOĐER NIJE FAZA

$A(\omega) = \sqrt{4 + 4\sin^2(\omega)}$, $\phi(\omega) = -\omega + \mathrm{arctg}(\sin(\omega))$

$$\tau(\omega) = -\frac{d\phi(\omega)}{d\omega} = 1 - \frac{\cos(\omega)}{1 + \sin^2(\omega)}$$





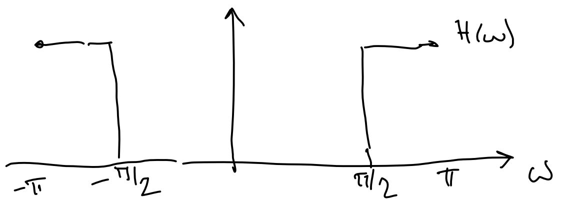


Kauzalni sustavi su sustavi kod kojih odziv dolazi nakon pobude. Za LTI sustave $\tau(\omega)$ je vrijeme potrebno za prolazak signala kroz sustav. Za kauzalne sustave je stoga uvijek $\tau(\omega) > 0$. Kako je $\tau(\omega) = -\frac{d\phi(\omega)}{d\omega}$ za kauzalne sustave je faza padajuća funkcija!

za kauzalne sustave je stoga uvijek $\tau(\omega) \geq 0$, kako je $\tau(\omega) = -\frac{d\varphi}{d\omega}$

za kauzalne sustave je faza padajuća funkcija!

16. a)



$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)\, e^{+j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} e^{j\omega n}\, d\omega + \frac{1}{2\pi}\int_{\pi/2}^{\pi} e^{j\omega n}\, d\omega =$$

$$= \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{\pi/2}^{\pi} = \frac{1}{\pi n}\frac{1}{2j}\left(e^{-jn\pi/2} - e^{jn\pi/2}\right) +$$

$$+ \frac{1}{\pi n}\frac{1}{2j}\left(-e^{-jn\pi} + e^{jn\pi}\right) = \frac{1}{n\pi}\left(\sin(n\pi) - \sin\left(\frac{n\pi}{2}\right)\right)$$

Za faze $e^{\pm j2\omega}$ primjenjujući teorem o pomaku te dobivamo odziv $\frac{1}{(n\pm 2)\pi}\left(\sin((n\pm 2)\pi) - \sin\left((n\pm 2)\frac{\pi}{2}\right)\right)$, odnosno

$$H(\omega)e^{+2j\omega} \circ\!\!-\!\!\circ \frac{1}{(n+2)\pi}\left(\sin(n\pi) - \sin\left((n+2)\frac{\pi}{2}\right)\right)$$

$$H(\omega)\, e^{-2j\omega} \circ\!\!-\!\!\circ \frac{1}{(n-2)\pi}\left(\sin(n\pi) - \sin\left((n-2)\frac{\pi}{2}\right)\right)$$

b) $$h[n] = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} 1\cdot e^{+j\omega n}\, d\omega = \frac{1}{2\pi}\frac{1}{jn} e^{j\omega n}\Big|_{-\pi/2}^{\pi/2} = \frac{1}{n\pi}\frac{1}{2j}\left(e^{jn\pi/2} - e^{-jn\pi/2}\right) =$$

$$= \frac{1}{n\pi}\sin\left(n\frac{\pi}{2}\right)$$

$$h[n\pm 2] = \frac{1}{(n\pm 2)\pi}\sin\left((n\pm 2)\frac{\pi}{2}\right) \quad \text{za faze } e^{\pm j2\omega}$$

c) $$h[n] = \frac{1}{n\pi}\left(\sin\left(n\frac{3\pi}{4}\right) - \sin\left(n\frac{\pi}{4}\right)\right)$$

$$h[n\pm 2] = \frac{1}{(n\pm 2)\pi}\left(\sin\left((n\pm 2)\frac{3\pi}{4}\right) - \sin\left((n\pm 2)\frac{\pi}{4}\right)\right)$$

Minimalno fazni sustavi, dekonvolucija i inverzni sustav

21. ožujak, 2008.
14:11

DEFINICIJE:

Diskretni LTI sustav je stabilan ako mu se svi polovi nalaze unutar jedinične kružnice.

Diskretni LTI sustav je minimalno-fazni ako mu se svi polovi i sve nule nalaze unutar jedinične kružnice.

Diskretni LTI sustav je maksimalno-fazni ako mu se svi polovi i sve nule nalaze izvan jedinične kružnice.

POLOVI su nul-točke nazivnika prijenosne funkcije.

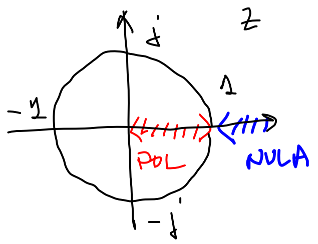
NULE su nul-točke brojnika prijenosne funkcije.

17 a) $H(z) = r \dfrac{1 - r^{-1} z^{-1}}{1 - r z^{-1}}, \quad 0 < r < 1$

$$1 - r^{-1} z^{-1} = 0 \Rightarrow z = \frac{1}{r} > 1 \text{ jer je } 0 < r < 1$$

$$1 - r z^{-1} = 0 \Rightarrow z = r < 1$$

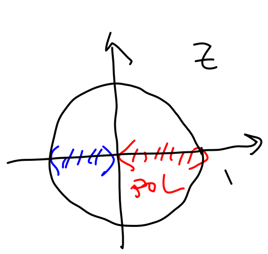
Kako je pol $z = r$ unutar jedinične kružnice sustav je stabilan, ali zbog nule $z = \frac{1}{r}$ nije minimalno fazni.



b) $H(z) = \dfrac{1-r}{1+r} \dfrac{1 + r z^{-1}}{1 - r z^{-1}}, \quad 0 < r < 1$

$1 + r z^{-1} = 0 \Rightarrow z = -r$ NULA

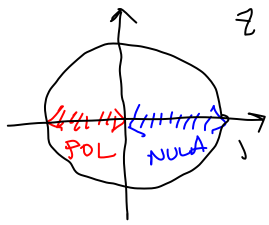
$1 - r z^{-1} = 0 \Rightarrow z = r$ POL



Kako je $0 < r < 1$ i nula $-r$ i pol r su unutar jedinične kružnice te je sustav stabilan i minimalno fazni.

Kako je $0<r<1$ i nula $-r$ i pol r su unutar jedinične kružnice te je sustav stabilan i minimalno fazni.

c) $H(z) = \dfrac{1-r}{1+r}\,\dfrac{1-r z^{-1}}{1+r z^{-1}}, \quad 0<r<1$



$1 - r z^{-1} = 0 \Rightarrow z = r$ NULA

$1 + r z^{-1} = 0 \Rightarrow z = -r$ POL

Kako su i pol i nula unutar jedinične kružnice sustav je stabilan i minimalno fazni.

13. $H_1(z) = \dfrac{z+b}{z+a}, \quad H_2(z) = \dfrac{bz+1}{z+a}, \quad |a|<1, \quad |b|<1$

$n_1 = -b$ $\quad$ $n_2 = -\dfrac{1}{b}$

$p_1 = -a$ $\quad$ $p_2 = -a$

Sustavi $H_1(z)$ i $H_2(z)$ imaju isti pol, no recipročnu nulu.

Oba sustava su stabilna jer je $|p_1|<1$ i $|p_2|<1$, no samo sustav $H_1(z)$ je minimalno fazni jer je $|n_1|<1$. Sustav $H_2(z)$ ne može biti minimalno fazni jer je $\forall\, b, |b|<1$ nula $n_2 = -\frac{1}{b}$ izvan jedinične kružnice.

$$H_1(e^{j\omega}) = \frac{e^{j\omega}+b}{e^{j\omega}+a}, \quad H_2(e^{j\omega}) = \frac{b e^{j\omega}+1}{e^{j\omega}+a}$$

$$A_1^2(\omega) = |H_1(e^{j\omega})|^2 = \left|\frac{b+\cos(\omega)+j\sin(\omega)}{a+\cos(\omega)+j\sin(\omega)}\right|^2 = \frac{b^2+2b\cos(\omega)+1}{a^2+2a\cos(\omega)+1}$$

$$A_2^2(\omega) = |H_2(e^{j\omega})|^2 = \left|e^{+j\omega}\frac{b+\cos(\omega)-j\sin(\omega)}{a+\cos(\omega)+j\sin(\omega)}\right|^2 = \frac{b^2+2b\cos(\omega)+1}{a^2+2a\cos(\omega)+1}$$

Vidimo da su amplitudne karakteristike jednake, $A_1(\omega) = A_2(\omega)$

$$\varphi_1(\omega) = \operatorname{arctg}\frac{\sin(\omega)}{b+\cos(\omega)} - \operatorname{arctg}\frac{\sin(\omega)}{a+\cos(\omega)}$$

$$\varphi_1(\omega) = \operatorname{arctg} \frac{\sin(\omega)}{b+\cos(\omega)} - \operatorname{arctg} \frac{\sin(\omega)}{a+\cos(\omega)}$$

$$\varphi_2(\omega) = -\omega - \operatorname{arctg} \frac{\sin(\omega)}{b+\cos(\omega)} - \operatorname{arctg} \frac{\sin(\omega)}{a+\cos(\omega)}$$

Vidimo da drugi sustav unosi veći fazni pomak, odnosno $H_2(z)$ nije minimalne faze.

19. $$u[n] = \frac{1}{h[0]}\left(y[n] - \sum_{i=1}^{n} u[n-i]\,h[i]\right)$$

Sumu računamo samo za $n \geq i$. Ako je $n < 0$ vrijednost sume je NULA!

a) $y[n] = \underbrace{\{\underline{1}, 3, 6, 9, 12, 9, 5\}}_{7 \text{ uzoraka}}$, $h[n] = \underbrace{\{\underline{1}, 1, 1\}}_{3 \text{ uzorka}}$

Ulaz $u(n)$ ima najmanje $7-3+1=5$ uzoraka različitih od nule!

$$u[0] = \frac{1}{h[0]}\left(y[0] - \cancel{\sum_{i=1}^{0} u[n-i]h[i]}\right) = \frac{y[0]}{h[0]} = \frac{1}{1} = 1$$

$\emptyset$

$$u[1] = \frac{1}{h[0]}\left(y[0] - \sum_{i=1}^{1} u[n-i]h[i]\right) = \frac{1}{1}(3 - 1\cdot 1) = 2$$

$$u[2] = \frac{1}{h[0]}\left(y[0] - \sum_{i=1}^{2} u[n-i]h[i]\right) = \frac{1}{1}(6-(2\cdot 1 - 1\cdot 1)) = 3$$

$$u[3] = \frac{1}{h[0]}\left(y[0] - \sum_{i=1}^{3} u[n-i]h[i]\right) = \frac{1}{1}(9-(3\cdot 1 - 2\cdot 1 - 1\cdot 0)) = 4$$

$$u[4] = \frac{1}{h[0]}\left(y[0] - \sum_{i=1}^{4} u[n-i]h[i]\right) = \frac{1}{1}(12-(4\cdot 1 - 3\cdot 1 - 2\cdot 0 - 1\cdot 0)) = 5$$

$$u[5] = \frac{1}{h[0]}\left(y[0] - \sum_{i=1}^{5} u[n-i]h[i]\right) = \frac{1}{1}(9-(5\cdot 1 - 4\cdot 1 - \dots)) = 0$$

Svi ostali uzorci [illegible] od nule! Izračunajmo sada dekonvoluciju dijeljenjem u Z domeni. Vrijedi

$$Y(z) = H(z)\cdot U(z) \Rightarrow U(z) = \frac{Y(z)}{H(z)}$$

$$Y(z) = 1 + 3z^{-1} + 6z^{-2} + 9z^{-3} + 12z^{-4} + 9z^{-5} + 5z^{-6}$$

$$H(z) = 1 + z^{-1} + z^{-2}$$

$$(1+3z^{-1}+6z^{-2}+9z^{-3}+12z^{-4}+9z^{-5}+5z^{-6}) : (1+z^{-1}+z^{-2}) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 5z^{-4}$$

$$
\begin{array}{l}
-\ \underline{1 + z^{-1} + z^{-2}} \\
\quad 2z^{-1} + 5z^{-2} \\
\quad -\ \underline{2z^{-1} + 2z^{-2} + 2z^{-3}} \\
\qquad 3z^{-2} + 7z^{-3} \\
\qquad -\ \underline{3z^{-2} + 3z^{-3} + 3z^{-4}} \\
\qquad\quad 4z^{-3} + 9z^{-4} \\
\qquad\quad -\ \underline{4z^{-3} + 4z^{-4} + 4z^{-5}} \\
\qquad\qquad 5z^{-4} + 5z^{-5} \\
\qquad\qquad -\ \underline{5z^{-4} + 5z^{-5} + 5z^{-6}} \\
\qquad\qquad\qquad \emptyset
\end{array}
$$

Za razliku od prethodnog postupka kada ostatak postane $\emptyset$ možemo prestati s dijeljenjem u Z-domeni!

b) $y[n]=\{\underline{5},9,12,9,6,3,1\}$, $h[n]=\{\underline{1},1,1\} \Rightarrow x[n]=\{\underline{5},4,3,2,1\}$

c) $y[n]=\{\underline{1},1,2,1,2,1,1\}$, $h[n]=\{\underline{1},-1,1\} \Rightarrow x[n]=\{\underline{1},2,3,2,1\}$

d) $y[n]=\{\underline{1},4,8,10,8,4,1\}$, $h[n]=\{\underline{1},2,1\} \Rightarrow x[n]=\{\underline{1},2,3,2,1\}$

e) $y[n]=\{\underline{1},0,2,0,2,1\}$, $h[n]=\{\underline{1},0,1\} \Rightarrow x[n]=\{\underline{1},0,1,0,1\}$

20) Prema definiciji minimalno-fazni LTI diskretni sustavi imaju sve polove i sve nule unutar jedinične kružnice. Kako inverzni sustav ima polove na mjestu nula polaznog sustava stabilnost inverznog sustava je osigurana ako su nule polaznog sustava bile unutar jedinične kružnice.

Dakle, diskretni LTI sustav je invertibilan ako i samo ako je minimalno-fazni.

FIR (finite impulse response) sustavi su diskretni LTI sustavi koji imaju konačan impulsni odziv, odnosno čija prijenosna funkcija $H(z)$ nema nazivnika. FIR sustavi dakle nemaju polova i uvijek su stabilni!

FIR sustav je invertibilan ako su sve njegove nule unutar jedinične kružnice, odnosno ako je sustav minimalno-fazni.

a) $h[n]=\{\underline{1},2\} \Rightarrow H(z)=1+2z^{-1} \Rightarrow n=-2$ nije minimalno-fazni

$$H^{-1}(z)=\frac{1}{1+2z^{-1}}$$ nije stabilan

Što se tiče kauzalnosti dobivenog inverza ako bi redefinirali Z transformaciju preko sume $\sum_{n=-\infty}^{1} x[n]z^{-n}$ dobivena funkcija $H^{-1}(z)$ može se interpretirati kao prijenosna funkcija stabilnog nekauzalnog sustava, koji naravno opet nije izvediv!

b) $h[n]=\{\underline{2},1\} \Rightarrow H(z)=2+z^{-1} \Rightarrow n=-\frac{1}{2}$

b) $h[n] = \{\underline{2}, 1\} \Rightarrow H(z) = 2 + z^{-1} \Rightarrow n = -\frac{1}{2}$

$H(z)$ je minimalno-fazni

$H^{-1}(z) = \dfrac{1}{2+z^{-1}} \Rightarrow p = -\frac{1}{2}$, $H^{-1}(z)$ je stabilan

c) $h[n] = \{\underline{1}, 2, 1\} \Rightarrow H(z) = 1 + 2z^{-1} + z^{-2} \Rightarrow n_{1,2} = -1$

$H(z)$ je na granici stabilnosti (BIBO nestabilan) i nije minimalno-fazni

$H^{-1}(z) = \dfrac{1}{1+2z^{-1}+z^{-2}} \Rightarrow p_{1,2} = -1$, $H^{-1}(z)$ je na granici stabilnosti

d) $h[n] = \{\underline{1}, 1, 1\} \Rightarrow H(z) = 1 + z^{-1} + z^{-2} \Rightarrow n_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$

$H(z)$ je minimalno fazni

$H^{-1}(z) = \dfrac{1}{1+z^{-1}+z^{-2}} \Rightarrow p_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} j$, $H^{-1}(z)$ je stabilan

e) $h[n] = \{\underline{2}, -7, 3\} \Rightarrow H(z) = 2 - 7z^{-1} + 3z^{-2} \Rightarrow n_1 = 3, n_2 = \frac{1}{2}$

$H(z)$ nije minimalno fazni

$H^{-1}(z) = \dfrac{1}{2-7z^{-1}+3z^{-2}} \Rightarrow p_1 = 3, p_2 = \frac{1}{2}$, $H^{-1}(z)$ nije stabilan

f) $h[n] = \{\underline{9}, 6, 2\} \Rightarrow H(z) = 9 + 6z^{-1} + 2z^{-2} \Rightarrow n_{1,2} = -\frac{1}{3} \pm \frac{1}{3} j$

$H(z)$ je minimalno fazni

$H^{-1}(z) = \dfrac{1}{9+6z^{-1}+2z^{-2}} \Rightarrow p_{1,2} = -\frac{1}{3} \pm \frac{1}{3} j$, $H^{-1}(z)$ je stabilan

g) $h[n] = \{\underline{1}, 6, 18\} \Rightarrow H(z) = 1 + 6z^{-1} + 18z^{-2} \Rightarrow n_{1,2} = -3 \pm 3j$

$H(z)$ nije minimalno fazni

$H^{-1}(z) = \dfrac{1}{1+6z^{-1}+18z^{-2}} \Rightarrow p_{1,2} = -3 \pm 3j$, $H^{-1}(z)$ nije stabilan

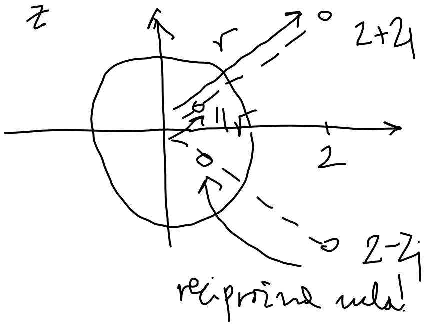
21) U prethodnom zadatku smo vidjeli da neminimalno fazni

21) U prethodnom zadatku smo vidjeli da neminimalno fazni FIR sustavi nemaju stabilan inverz. No za takve neminimalno fazne FIR sustave možemo pronaći pripadni minimalno fazni sustav tako da nule koje se nalaze izvan jedinične kružnice zamijenimo s recipročnima koje će se nalaziti unutar jedinične kružnice. Tako dobiveni sustav je minimalno fazni i ima jednaku amplitudnu karakteristiku kao i polazni sustav.

a) $h[n] = \{\underline{1}, -4, 8\} \Rightarrow H(z) = 1 - 4z^{-1} + 8z^{-2} \Rightarrow n_{1,2} = 2 \pm 2j$

Kako se obje nule nalaze izvan jedinične kružnice sustav nije minimalno fazni. Nule koje se nalaze izvan jedinične kružnice moramo zamijeniti recipročnima.

$$H(z) = 1 - 4z^{-1} + 8z^{-2} = (1 + (-2-2j)z^{-1})(1 + (-2+2j)z^{-1})$$



$2+2j = \sqrt{8} \cdot e^{+j\pi/4} \rightarrow \frac{1}{\sqrt{8}} e^{+j\pi/4} = \frac{1}{4} + \frac{1}{4}j$

$2-2j = \sqrt{8} \cdot e^{-j\pi/4} \rightarrow \frac{1}{\sqrt{8}} e^{-j\pi/4} = \frac{1}{4} - \frac{1}{4}j$

$$H_{mf}(z) = (z^{-1} + (-2-2j))(z^{-1} + (-2+2j)) = z^{-2} - 4z^{-1} + 8$$

$$H_{mf}^{-1}(z) = \frac{1}{8 - 4z^{-1} + z^{-2}}$$

Razmotrimo sada sustav $H(z) \cdot H_{mf}^{-1}(z)$. Kako su amplitudne karakteristike sustava $H(z)$ i $H_{mf}(z)$ iste možemo pisati

$$|H(e^{j\omega})| = |H_{mf}(e^{j\omega})| = A(\omega)$$

Sada je

$$H(e^{j\omega}) = A(\omega) e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = A(\omega)\, e^{j\phi(\omega)}$$

$$H_{mf}(e^{j\omega}) = A(\omega)\, e^{j\phi_{mf}(\omega)} \quad \left(\text{uz } |\phi_{mf}(\omega)| < |\phi(\omega)|\right)$$

te je

$$H(e^{j\omega}) \cdot H_{mf}^{-1}(e^{j\omega}) = A(\omega) e^{j\phi(\omega)} \cdot \frac{1}{A(\omega)} e^{-j\phi_{mf}(\omega)} =$$

$$= \frac{A(\omega)}{A(\omega)} \cdot e^{j(\phi(\omega) - \phi_{mf}(\omega))} = 1 \cdot e^{j(\phi(\omega) - \phi_{mf}(\omega))}$$

Vidimo, dakle, da kaskada $H(z) \cdot H_{mf}^{-1}(z)$ mora imati jediničnu amplitudnu karakteristiku, dok je fazna karakteristika oblika $\phi(\omega) - \phi_{mf}(\omega)$. Takvi sustavi se nazivaju svepropusnim sustavima, a koeficijenti uz potencije od z brojnika i nazivnika zadovoljavaju sljedeće svojstvo:

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_N z^{-N}}$$

$$a_0 = b_N, \quad a_1 = b_{N-1}, \quad a_2 = b_{N-2}, \quad \ldots, \quad a_N = b_0$$

U a) zadatku dobivena prijenosna funkcija je

$$H(z) \cdot H_{mf}^{-1}(z) = \frac{1 - 4z^{-1} + 8z^{-2}}{8 - 4z^{-1} + z^{-2}}$$ i ona zadovoljava navedeno

svojstvo (što je zgodno napomenuti radi provjere rezultata).

Odredimo i impulsni odziv dobivenog sustava:

$$\frac{1 - 4z^{-1} + 8z^{-2}}{8 - 4z^{-1} + z^{-2}} = \frac{z^2 - 4z + 8}{8z^2 - 4z + 1} = 8 + \frac{-63z^2 + 28z}{8z^2 - 4z + 1}$$

$$= 8 + \frac{Az}{z - \frac{1}{4} - \frac{1}{4}j} + \frac{Bz}{z - \frac{1}{4} + \frac{1}{4}j}$$

$$A = \frac{z^2 - 4z + 8}{8z^2 - 4z + 1} \cdot \frac{z - \frac{1}{4} - \frac{1}{4}j}{z} \Bigg|_{z = \frac{1}{4} + \frac{1}{4}j} = -\frac{63}{16} - \frac{49}{16}j$$

$$B = \frac{z^2 - 4z + 8}{8z^2 - 4z + 1} \cdot \frac{z - \frac{1}{4} + \frac{1}{4}j}{z} \Bigg|_{z = \frac{1}{4} - \frac{1}{4}j} = -\frac{63}{16} + \frac{49}{16}j$$

Prema tablici Z transformacija impulsni odziv je bila

$$h[n] = 8\delta[n] + \left(-\frac{63}{16} - \frac{49}{16}j\right)\left(\frac{1}{4} + \frac{1}{4}j\right)^n + \left(-\frac{63}{16} + \frac{49}{16}j\right)\left(\frac{1}{4} - \frac{1}{4}j\right)^n$$

b) $h[n] = \{\underline{1}, 4, 8\} \Rightarrow H(z) = (1 + (2+2j)z^{-1})(1 + (2-2j)z^{-1})$

Obje nule su izvan kružnice te sustav nije minimalno fazni.

$$H_{mf}(z) = (z^{-1} + (2+2j))(z^{-1} + (2-2j)) = z^{-2} + 4z^{-1} + 8$$

$$H_{mf}^{-1}(z) = \frac{1}{8 + 4z^{-2} + z^{-1}}$$

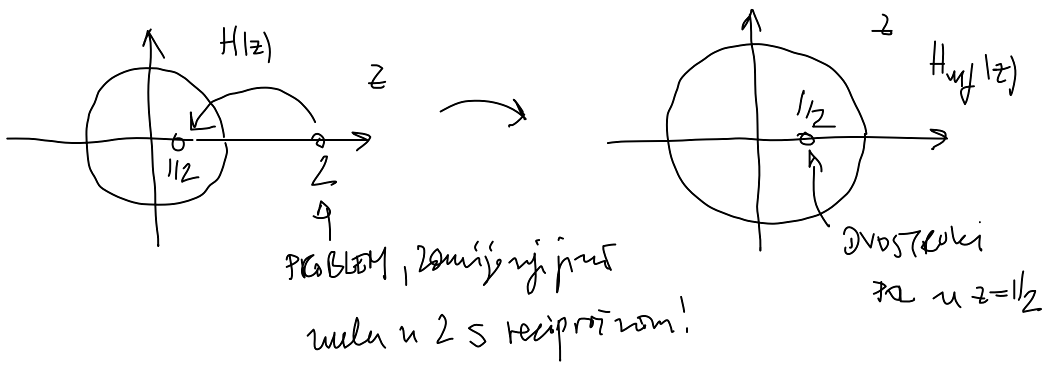
$$H(z) \cdot H_{mf}^{-1}(z) = \frac{1 + 4z^{-1} + 8z^{-2}}{8 + 4z^{-2} + z^{-1}} = \frac{z^2 + 4z + 8}{8z^2 + 4z + 1} = 8 + \frac{63z^2 + 28z}{8z^2 + 4z + 1} =$$

$$= 8 + \frac{\frac{49}{16} + \frac{63}{16}j}{z + \frac{1}{4} - \frac{1}{4}j} + \frac{\frac{49}{16} - \frac{63}{16}j}{z + \frac{1}{4} + \frac{1}{4}j}$$

$$h[n] = 8\delta[n] + \left(\frac{49}{16} + \frac{63}{16}j\right)\left(-\frac{1}{4} + \frac{1}{4}j\right)^n + \left(\frac{49}{16} - \frac{63}{16}j\right)\left(-\frac{1}{4} - \frac{1}{4}j\right)^n$$

c) $h[n] = \{\underline{2}, -5, 2\} \Rightarrow H(z) = 2 - 5z^{-1} + 2z^{-2} = \underbrace{(2 - z^{-1})}_{z=\frac{1}{2}}\underbrace{(1 - 2z^{-1})}_{z=2}$

Sustav nije minimalno fazni zbog nule u $z=2$.
Da bi ga učinili minimalno-faznim moramo nulu u $z=2$ zamijeniti recipročnom, odnosno umjesto $1-2z^{-1}$ stavljamo $z^{-1}-2$ član!



$$H_{mf}(z) = (2 - z^{-1})(z^{-1} - 2) = -4 + 4z^{-1} - z^{-1}$$

$$H_{mf}(z) = (2 - z^{-1})(z^{-1} - 2) = -4 + 4z^{-1} - z^{-1}$$

$$H^{-1}_{mf}(z) = \frac{1}{-4 + 4z^{-1} - z^{-1}}$$

$$H(z) \cdot H^{-1}_{mf}(z) = \frac{2 - 5z^{-1} + 2z^{-2}}{-4 + 4z^{-1} - z^{-2}} = \frac{(2 - z^{-1})(1 - 2z^{-1})}{(2 - z^{-1})(z^{-1} - 2)} = \frac{1 - 2z^{-1}}{-2 + z^{-1}}$$

Primijetite da je $H(z) \cdot H^{-1}_{mf}(z)$ svepropusni sustav (vidi komentar uz a) dio zadatka).

$$H(z) \cdot H^{-1}_{mf}(z) = \frac{1 - 2z^{-1}}{-2 + z^{-1}} = \frac{z - 2}{-2z + 1} = 2 + \frac{-3/2\, z}{z - 1/2}$$

$$h[n] = 2\delta[n] - 3/2\,(1/2)^n$$

NAPOMENA: Razmislite je li još jedno moguće rješenje ovog zadatka dano funkcijom $H_{mf}(z) = 4 - 4z^{-1} + z^{-2}$? U čemu je bitna razlika u odnosu na dobiveno rješenje? Koje rješenje je bolje i zašto?

d) $h[n] = \{\underline{2}, 5, 2\} \Rightarrow H(z) = 2 + 5z^{-1} + 2z^{-2} = \underbrace{(2 + z^{-1})}_{z = -\frac{1}{2}}\underbrace{(1 + z^{-1} 2)}_{z = -2}$

$$H_{mf}(z) = (2 + z^{-1})(z^{-1} + 2) = 4 + 4z^{-1} + 1$$

$$H^{-1}_{mf}(z) = \frac{1}{4 + 4z^{-1} + 1 \cdot z^{-2}}$$

$$H(z) \cdot H^{-1}_{mf}(z) = \frac{2 + 5z^{-1} + 2z^{-2}}{4 + 4z^{-1} + z^{-2}} = \frac{(2 + z^{-1})(1 + 2z^{-1})}{(2 + z^{-1})(z^{-1} + 2)} = \frac{1 + 2z^{-1}}{z^{-1} + 2} =$$

$$= \frac{z + 2}{1 + 2z} = \frac{1/2\, z + 1}{z + 1/2} = 2 + \frac{-3/2\, z}{z + 1/2}$$

$$h[n] = 2\delta[n] - \frac{3}{2}\left(-\frac{1}{2}\right)^n$$

e) $h[n] = \{ \underline{2}, -3, -2 \} \Rightarrow H(z) = 2 - 3z^{-1} - 2z^{-2} = \underbrace{(1-2z^{-1})}_{z=2} \underbrace{(2+z^{-1})}_{z=-\frac{1}{2}}$

$H_{mf}(z) = (z^{-1} - 2)(2 + z^{-1}) = -4 + z^{-2}$

$H_{mf}^{-1}(z) = \frac{1}{-4+z^{-2}}$

$H(z) H_{mf}^{-1}(z) = \frac{2-3z^{-1}-2z^{-2}}{-4+z^{-2}} = \frac{(1-2z^{-1})(2+z^{-1})}{(z^{-1}-2)(2+z^{-1})} = \frac{1-2z^{-1}}{z^{-1}-2} =$

$= \frac{z-2}{1-2z} = \frac{-\frac{1}{2}z+1}{z-1/2} = -2 + \frac{3/2 z}{z - 1/2}$

$h[n] = -2\delta[n] + 3/2 \left(\frac{1}{2}\right)^n$

f) $h[n] = \{ \underline{2}, 3, -2 \} \Rightarrow H(z) = 2 + 3z^{-1} - 2z^{-2} = \underbrace{(2-z^{-1})}_{z=\frac{1}{2}} \underbrace{(1+2z^{-1})}_{z=-2}$

$H_{mf}(z) = (2 - z^{-1})(z^{-1} + 2) = 4 - z^{-2}$

$H_{mf}^{-1}(z) = \frac{1}{4-z^{-2}}$

$H(z) \cdot H_{mf}^{-1}(z) = \frac{2+3z^{-1}-2z^{-2}}{4-z^{-2}} = \frac{(2-z^{-1})(1+2z^{-1})}{(2-z^{-1})(z^{-1}+2)} = \frac{1+2z^{-1}}{z^{-1}+2} =$

$= \frac{z+2}{1+2z} = \frac{1/2 z + 1}{z + 1/2} = 2 + \frac{-3/2 z}{z + 1/2}$

$h[n] = 2\delta[n] - 3/2 \left(-\frac{1}{2}\right)^n$

22. $W_N \overset{DEF}{=} e^{-2\pi j \frac{1}{N}}$

$$DFT_N\left[x[n]\right] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk} = \sum_{n=0}^{N-1} x[n]\, e^{-2\pi j \frac{nk}{N}} = X[k]$$

$$IDFT_N\left[X(k)\right] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, e^{+2\pi j \frac{nk}{N}} = x[n]$$

Matrica W_N je kvadratna $N \times N$ matrica koja u i-tom stupcu i j-tom retku ima element $W_N^{(i-1)(j-1)}$. Matrica W_N^{-1} pak u i-tom stupcu i j-tom retku ima element $\frac{1}{N} W_N^{-(i-1)(j-1)}$.

Sada još za $N=3$ moramo pokazati da vrijedi $W_N \cdot W_N^H = N \cdot I$. Pri tome oznaka H podrazumijeva Hermitski transponiranu matricu - matricu transponiramo te zatim svaki element konjugiramo.

$$W_3 = \begin{bmatrix} W_3^{0\cdot 0} & W_3^{0\cdot 1} & W_3^{0\cdot 2} \\ W_3^{0\cdot 1} & W_3^{1\cdot 1} & W_3^{1\cdot 2} \\ W_3^{0\cdot 2} & W_3^{1\cdot 2} & W_3^{2\cdot 2} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{1} & W_3^{2} \\ 1 & W_3^{2} & W_3^{4} \end{bmatrix}$$

$$W_3^H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix}$$

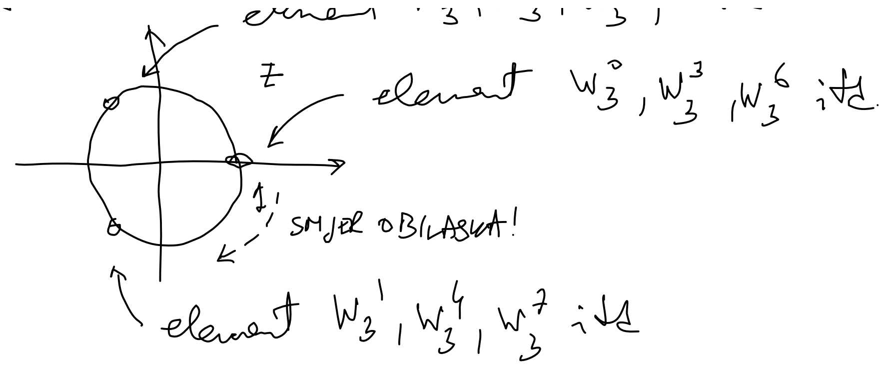
W_N je kompleksna eksponencijala pa je $(W_N^k)^* = W_N^{-k}$

$$W_3\, W_3^H = \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{1} & W_3^{2} \\ 1 & W_3^{2} & W_3^{4} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & W_3^{-1} & W_3^{-2} \\ 1 & W_3^{-2} & W_3^{-4} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = 3\, I$$

NAPOMENA: Kada računate kompleksnu eksponencijalu W_N zamislite jediničnu kružnicu u z-ravnini (jer je $|W_N| = 1$) na kojoj je smješteno N jednoliko udaljenih točaka, npr. za $N=3$ na kružnici postavljamo 3 točke, u $z=0$, $z = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$ i $z = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$

element $W_3^2, W_3^5, W_3^8, \dots$ itd.

z — element W_3^0, W_3^3, W_3^6 itd.

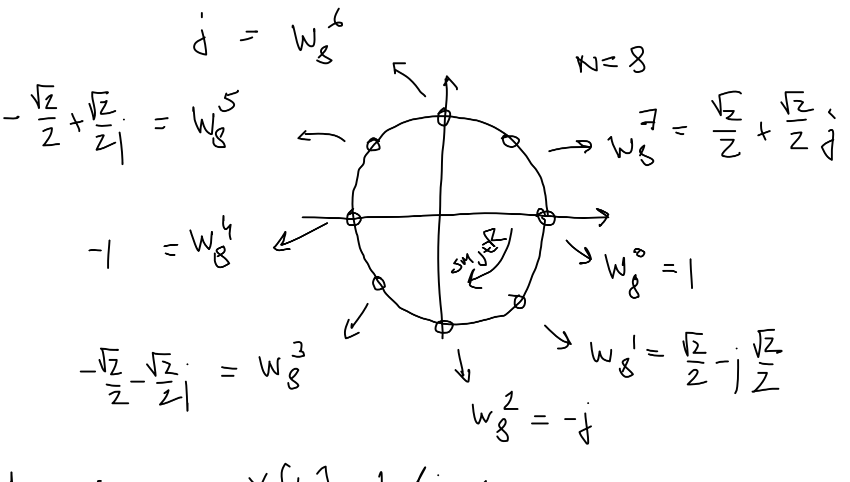


23) Kod računanja DFT_N transformacije moramo zapamtiti da i signal i spektar imaju točno N uzoraka! Stoga N uvijek mora biti zadan!

a) $x[n] = \{\underline{1}, 0, 2, 1, 1, 0, 1, 1\}$, $N=8$

$$X[k] = \sum_{n=0}^{7} x[n] W_8^{nk} = 1 \cdot W_8^{0 \cdot k} + 2W_8^{2 \cdot k} + 1 \cdot W_8^{3k} + 1 \cdot W_8^{4 \cdot k} + 1 \cdot W_8^{6 \cdot k} + 1 \cdot W_8^{7 \cdot k}$$

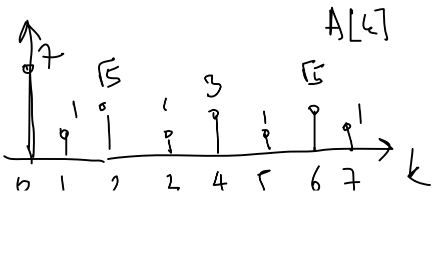
Eksponencijale W_8^k je najjednostavnije odrediti crtanjem jedinične kružnice u Z ravnini.

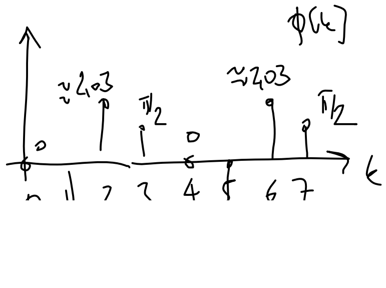


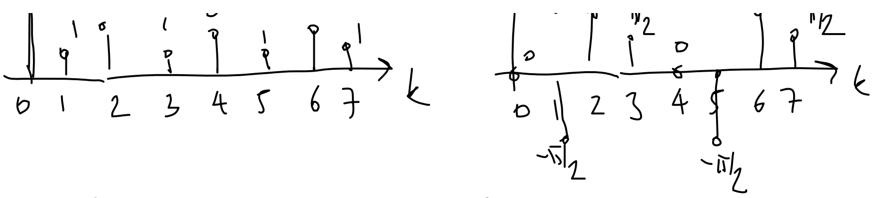
Uvrštavanjem u X[k] dobivamo

$$X[k] = \{\underline{7}, -j, -1+2j, j, 3, -j, -1-2j, j\}$$

Amplitudni i fazni spektar imaju samo 8 uzoraka i računamo ih kao $A[k] = |X[k]|$ i $\phi[k] = \arg X[k]$







$A[k] = \{ \underline{7}, 1, \sqrt{5}, 1, 3, 1, \sqrt{5}, 1 \}$

$\phi[k] = \{ \underline{0}, -\pi/2, 2{,}03, \pi/2, 0, -\pi/2, 2{,}03, \pi/2 \}$

b) $x[n] = \{ \underline{-1}, 1, 2, 0, 1 \}, \quad N=5$

$$DFT_5[x[n]] = \sum_{n=0}^{4} x[n] W_5^{nk} = -1 + W_5^{k} + 2W_5^{2k} + W_5^{4k}$$

Za $N=5$ vrijednosti W_5, iako se mogu izraziti analitički, ne poprimaju vrijednosti koje su zgodne za računanje, npr.

$$W_5 = e^{-2\pi j/5} = \cos\left(\frac{2\pi}{5}\right) - j\sin\left(\frac{2\pi}{5}\right) = \frac{-1+\sqrt{5}}{4} - j\,\frac{\sqrt{10+2\sqrt{5}}}{4}$$

Stoga se za vrijednosti N koje nisu 2, 3, 4, 6 i 8 uobičajeno DFT_N računa numerički! Za $N = 2,3,4,6,8$ kompleksne eksponencijale W_N se mogu prikazati kao kombinaciju sinusa i kosinusa standardnih kuteva. Za rješavanje zadatka je stoga potrebno znati vrijednosti sinusa i kosinusa za kuteve $\pi/4$, $\pi/3$, $\pi/2$ i njihovih višekratnika.

$X[k] = \{ \underline{-3}, -2-1{,}1756j, -2+1{,}9021j, -2-1{,}9021j, -2+1{,}1766j \}$

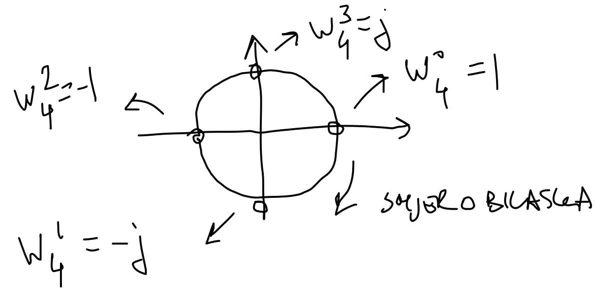
$A[k] = \{ \underline{3}, 2{,}3199, 2{,}7601, 2{,}7601, 2{,}3199 \}$

$\phi[k] = \{ \underline{0^\circ}, -149{,}6^\circ, 136{,}4^\circ, -136{,}4^\circ, 149{,}6^\circ \}$

c) $x[n] = \{\underline{0}, 2, 0, -2\}$, $N=4$

$$X[k] = DFT_4\,[x[n]] = \sum_{n=0}^{3} x[n] W_4^{nk} = 2W_4^{k} - 2W_4^{3k}$$

Za $N=4$ kompleksne eksponencijale poprimaju najjednostavnije vrijednosti za računanje, i to ± 1 i $\pm j$ kako je prikazano slikom:



$X[k] = \{\underline{0}, -4j, 0, 4j\}$

$A[k] = \{\underline{0}, 4, 0, 4\}$

$\phi[k] = \{\underline{0}, -\pi/2, 0, \pi/2\}$

d) $x[n] = \{\underline{1}, 0, 0, 0, -1\}$, $N=5$

$$X[k] = DFT_5[x[n]] = \sum_{n=0}^{4} x[n] W_5^{nk} = 1 - W_5^{4k}$$

$X[k] = \{\underline{0}, 0{,}6910-0{,}9511j, 1{,}8090-0{,}5878j, 1{,}8090+0{,}5878j, 0{,}6910+0{,}9511j\}$

$A[k] = \{\underline{0}, 1{,}1756, 1{,}9021, 1{,}9021, 1{,}1756\}$

$\phi[k] = \{\underline{0^\circ}, -54^\circ, -18^\circ, 18^\circ, 54^\circ\}$

e) $x[n] = \{\underline{1}, 0, 0, 0, 0, 0, -1\}$, $N=7$

$$X[k] = DFT_7\,[x[n]] = \sum_{n=0}^{6} x[n] W_7^{nk} = 1 - W_7^{6k}$$

$X[k] = \{\underline{0}, 0{,}3765-0{,}7818j, 1{,}2225-0{,}9749j, 1{,}9010-0{,}4339j,$
$1{,}9010+0{,}4339j, 1{,}2225+0{,}9749j, 0{,}3765+0{,}7818j\}$

$A[k] = \{\underline{0}, 0{,}8678, 1{,}5637, 1{,}9499, 1{,}9499, 1{,}5637, 0{,}8678\}$

$\phi[k] = \{\underline{0^\circ}, -64{,}3^\circ, -38{,}6^\circ, -12{,}9^\circ, 12{,}9^\circ, 38{,}6^\circ, 64{,}3^\circ\}$

NAPOMENA: Kao i za DTFT za realne signale amplitudni

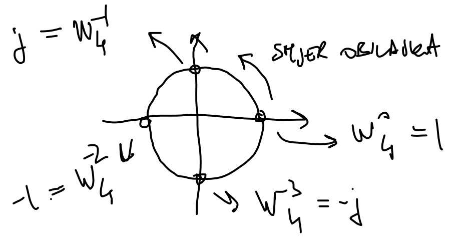
NAPOMENA: Kao i za DTFT za realne signale amplitudni spektar je parna funkcija dok je fazni spektar neparna funkcija. To svojstvo možete koristiti za provjeru rezultata samo pripazite na to da se simetrija drugačije ponaša za parni i neparni N!

24) $\mathrm{IDFT}_N[X(k)] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]\, W_N^{-nk} = x[n]$

a) $X[k] = \{2, 1, 0, 1\}$, $N = 4$

$$x[n] = \frac{1}{4}\sum_{k=0}^{3} X[k]\, W_4^{-nk} = \frac{1}{4}\left(2 + W_4^{-n} + W_4^{-3n}\right)$$

Kao i za DFT_N možemo W_4^{-k} računati pomoću jedinične kružnice samo što moramo promijeniti smjer obilaska.



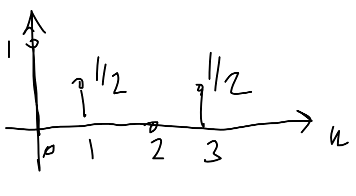
$x[0] = \frac{1}{4}(2+1+1) = 1$

$x[1] = \frac{1}{4}(2+j-j) = \frac{1}{2}$

$x[2] = \frac{1}{4}(2-1-1) = 0$

$x[3] = \frac{1}{4}(2-j+j) = \frac{1}{2}$

Dobili smo signal $x[n] = \{1, \frac{1}{2}, 0, \frac{1}{2}\}$



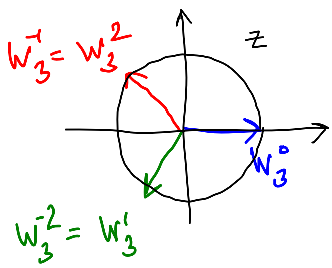
5) $X[k] = \{2, 0, 2, 0, 2, 0\}$, $N=6$

$$x[n] = \mathrm{IDFT}_6\left[X[k]\right] = \frac{1}{6}\sum_{k=1}^{5} X(k)\,W_6^{-nk} = \frac{1}{6}\left(2 + 2W_6^{-2n} + 2W_6^{-4n}\right)$$

Još jedno svojstvo kompleksne eksponencijale je korisno ovdje, odnosno $W_{Na}^{a} = W_N$ (jer je $W_{Na}^{a} = e^{-2\pi j\frac{a}{Na}} = e^{-2\pi j\frac{1}{N}} = W_N$).

Prema tome možemo pisati

$$x(n) = \frac{1}{6}\left(2 + 2W_3^{-n} + 2W_3^{-2n}\right)$$



Dodatno, primijetimo da je $\sum_{i=0}^{N-1} W_N^{i} = 0$, odnosno u našem slučaju $W_3^{0} + W_3^{-1} + W_3^{-2} = 0$. Samo za $n=0$ i $n=3$ dobivamo zbroj koji se ne poništava!

$$x[n] = \{1, 0, 0, 1, 0, 0\}$$

25)

$$X[k] = \mathrm{DFT}_4\left[x[n]\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ x[3] \end{bmatrix}$$

U zadatku se traži od nas da izračunamo DFT_4 transformaciju 4 kompleksna signala koji odgovaraju retcima matrice W_4.

Signali su:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$x_1[n] = \{1, 1, 1, 1\}$

$x_2[n] = \{1, -j, -1, j\}$

$x_3[n] = \{1, -1, 1, -1\}$

$x_4[n] = \{1, j, -1, -j\}$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}\begin{bmatrix} 1 \\ -j \\ -1 \\ j \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ j \\ -1 \\ -j \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 0 \end{bmatrix}$$

Kažemo da su retci matrice DFT_4 transformacije međusobno ortogonalni (vidi dokaz zadatak 34*)

26) Treba dokazati da vrijedi:

$$x[n] \in \mathbb{R} \wedge X[k] = \sum_{n=0}^{N-1} x[n] W_N^{nk} \Rightarrow X[k] = X^*[N-k]$$

Signal $x[n]$ je realan te je $x^*[n] = x[n]$. Izračunajmo najprije čemu je jednak $X^*[k]$:

$$X^*[k] = \left(\sum_{n=0}^{N-1} x[n] W_N^{nk} \right)^* = \sum_{n=0}^{N-1} x^*[n] W_N^{-nk} =$$

$$= \sum_{n=0}^{N-1} x[n] W_N^{-nk}$$

Kako je W_N periodično s periodom N možemo pisati

$$W_N^{-nk} = W_N^{-nk+nN} = W_N^{n(N-k)}$$

Sada je

$$X^*[k] = \sum_{n=0}^{N-1} x[n] W_N^{-nk} = \sum_{n=0}^{N-1} x[n] W_N^{n(N-k)} = X[N-k]$$

Konjugiranjem gornjeg izraza dobivamo

$$X[k] = X^*[N-k]$$

27) Za DFT_N transformaciju i signal i spektar imaju konačan broj uzoraka. Pitanje je onda što u kontekstu DFT_N transformacije znači paran niz! Uobičajena definicija parnog signala $f[n]: \mathbb{Z} \to \mathbb{R}$ jest da je $f[n]$ paran niz ako vrijedi $f[n] = f[-n]$. No za DFT_N uzorci s kojima mi računamo su definirani za $n \in \{0, 1, 2, \dots, N-1\}$. Želimo li odrediti uzorke

$f[n] = f[-n]$. No za DFT_N uzorci s kojima radimo su definirani za $n \in \{0,1,2,\dots,N-1\}$. Želimo li odrediti uzorke izvan tog intervala niz proširujemo na cijeli $\mathbb{Z}$ tako da bude periodičan.

Promatramo li dakle signal $x[n]$ definiran za $n \in \{0,1,\dots,N-1\}$ najprije ga periodički proširimo na cijeli $\mathbb{Z}$. Tada vrijedi $x[n] = x[n+iN]$, $i \in \mathbb{Z}$. No niz je ~~također~~ paran što znači da također vrijedi $x[n] = x[-n] = x[-n+iN]$. Za $i=1$ dobivamo $x[n] = x[N-n]$.

DFT_N transformacija signala $x[n]$ je

$$X[k] = DFT_N\left[x[n]\right] = \sum_{n=0}^{N-1} x[n]\, W_N^{nk}$$

Ako je signal realan i paran onda je i $x[n] = x[N-n]$. Grupirajmo sada sve takve parove iz sume:

$$x[n]\,W_N^{nk} + x[N-n]\,W_N^{(N-n)k} = x[n]\left(W_N^{nk} + \overset{1}{\cancel{W_N^{Nk}}}\cdot W_N^{-nk}\right) =$$

$$= x[n]\left(\cos\left(2\pi\frac{nk}{N}\right) + j\sin\left(2\pi\frac{nk}{N}\right) + \cos\left(2\pi\frac{nk}{N}\right) - j\sin\left(2\pi\frac{nk}{N}\right)\right) =$$

$$= 2x[n]\cos\left(2\pi\frac{nk}{N}\right)$$

Za N paran uvijek možemo spariti $x[n]$ i $x[N-n]$, no za neparan N jedan član sumacije ostaje nesparen! No u tom slučaju zbog neparnosti N imamo nesparen baš nulti član

$$x[0]\,W_N^{0k} \in \mathbb{R}$$

koji je realan. Time je pokazano da je DFT_N parnog realnog niza opet realna.

23) $x[n]$ je neparan realan niz, stoga vrijedi $x[n] = -x[N-n]$. Raspišimo sada DFT_N:

N je paran:

$$X[k] = \sum_{n=0}^{N-1} x[n]\,W_N^{nk} = \sum_{n=0}^{N/2-1} x[n]\,W_N^{nk} + \sum_{n=N/2}^{N-1} x[n]\,W_N^{nk} =$$

$$\sum^{N/2-1} \dots W^{nk} \quad \sum^{N/2-1} \dots W^{(N-n)k}$$

$$X[k] = \sum_{n=0} \ldots W_N = \sum_{n=0} x[n] W_N + \sum_{n=N/2} x[n] W_N$$

$$= \sum_{n=0}^{N/2-1} x[n] W_N^{nk} + \sum_{n=0}^{N/2-1} x[N-n] W_N^{(N-n)k} =$$

$$= \sum_{n=0}^{N/2-1} \left(x[n] W_N^{nk} + x[N-n] W_N^{(N-n)k} \right) =$$

$$= \sum_{n=0}^{N/2-1} x[n] \left(W_N^{nk} - W_N^{Nk} W_N^{-nk} \right) = \sum_{n=0}^{N/2-1} x[n] \cdot j2 \sin\left(2\pi \frac{nk}{N}\right)$$

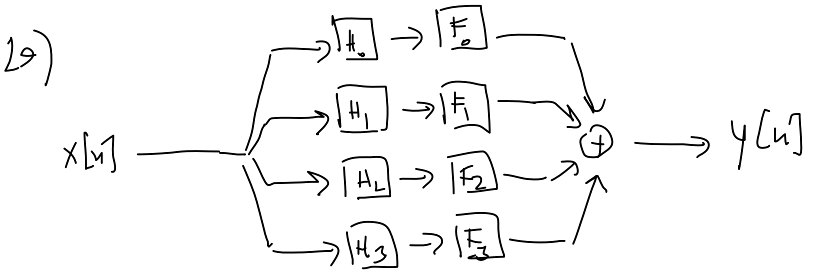
N je neparan:

$$X[k] = x[0] W_N^{0\cdot k} + \sum_{n=1}^{\frac{N-1}{2}} x[n] W_N^{nk} + \sum_{n=\frac{N+1}{2}}^{N-1} x[n] W_N^{nk}$$

$$= \phi + \sum_{n=1}^{N-1/2} x[n] \left(W_N^{nk} - W_N^{Nk} W_N^{-nk} \right) =$$

$$= \sum_{n=1}^{\frac{N-1}{2}} x[n]\, 2j \sin\left(2\pi \frac{nk}{N}\right)$$

29)



$$\begin{cases} F_k(z) = 1 + z^{-1} W_N^{-k} + z^{-2} W_N^{-2k} + \ldots + z^{-(N-1)} W_N^{-k(N-1)} \\ H_k(z) = 1 + z W_N^{k} + z^{2} W_N^{2k} + \ldots + z^{N-1} W_N^{k(N-1)} \end{cases}$$

$$H(z) = H_0(z) F_0(z) + H_1(z) F_1(z) + H_2(z) F_2(z) + H_3(z) F_3(z)$$

Izračunajmo prvo $H_0(z) \cdot F_0(z)$ za $N=4$:

$$H_0(z) \cdot F_0(z) = \left(1 + z^{-1} W_4^{-0} + z^{-2} W_4^{-2\cdot 0} + z^{-3} W_4^{-3\cdot 0}\right)\left(1 + z W_4^{0} + z^2 W_4^{0\cdot 2} + z^3 W_4^{0\cdot 3}\right) =$$

$$= \left(1 + z^{-1} + z^{-2} + z^{-3}\right)\left(1 + z + z^2 + z^3\right) = z^{-3} + 2z^{-2} + 3z^{-1} + 4 + 3z + 2z^2 + z^3$$

Na jednak način možemo računati preostale $H_k(z) \cdot F_k(z)$. No primijetimo da su koeficijenti uz potencije od z^{-m} u $F_k(z)$ i z^{N-m} u $H_k(z)$ jednaki jer je $W_N^{-mk} = W_N^{-mk+Nk} = W_N^{(N-m)k}$, odnosno impulsni odziv $H_k(z) F_k(z)$ je autokorelacija k-tog retka matrice impulsni odziv $H_k(z)F_k(z)$ je autokorelacija k-tog retka matrice DFT_N transformacije. Autokorelacija takvog kompleksnog signala je Hermitska funkcija što znači da je dovoljno izračunati pola impulsnog odziva dok se preostali dio odredi iz konjugirane simetrije niza.

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

KONJUGIRANA SIMETRIJA

$$H_0(z)*F_0(z) \circ\!\!-\!\!\bullet\; h_0[n]*h_0^*[4-n] = \{1, 2, 3, \underline{4}, 3, 2, 1\}$$

$$H_1(z)*F_1(z) \circ\!\!-\!\!\bullet\; h_1[n]*h_1^*[4-n] = \{j, -2, -3j, \underline{4}, 3j, -2, -j\}$$

$$H_2(z)*F_2(z) \circ\!\!-\!\!\bullet\; h_2[n]*h_2^*[4-n] = \{-1, 2, -3, \underline{4}, -3, 2, -1\}$$

$$H_3(z)*F_3(z) \circ\!\!-\!\!\bullet\; h_3[n]*h_3^*[4-n] = \{-j, -2, 3j, \underline{4}, -3j, -2, j\}$$

$$\Sigma \quad \{0, 0, 0, \underline{16}, 0, 0, 0\} = 16\delta[n]$$