Obrada informacija

2. međuispit – 28. travnja 2008.

1. Što je to FFT-algoritam? Kolika je asimptotska složenost korijen-2 FFT algoritma ako njime računamo diskretnu Fourierovu transformaciju u 2^n točaka, $n \in \mathbb{N}$? Nacrtajte graf toka signala za diskretnu Fourierovu transformaciju u dvije točke (DFT_2). Što je DFT-leptir?

2. Navedite definiciju linearne i cirkularne konvolucije. Zadana su dva signala

$$x_1[n] = \{\underline{1}, 2, 3, 4, 5\} \quad \text{i} \quad x_2[n] = \{\underline{1}, 0, 0, 0, -1\}.$$

Izračunajte $x_1[n] * x_2[n]$ i $x_1[n] \textcircled{5}\, x_2[n]$. Koji N moramo odabrati da vrijedi $x_1[n] * x_2[n] = x_1[n] \textcircled{N}\, x_2[n]$ za $0 \leq n < N$? Izračunajte $x_1[n] \textcircled{N}\, x_2[n]$ za odabrani N!

3. Projekcijskom metodom odredite impulsni odziv u $N = 5$ uzoraka FIR filtra tipa 1 ako je željena amplitudno-frekvencijska karakteristika filtra

$$A_d(\omega) = \begin{cases} 1, & -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ 0, & \text{inače} \end{cases}.$$

Fazna karakteristika je jednaka nuli!

4. Napišite matricu DCT-I_4 transformacije ($N = 4$). Objasnite vezu DCT-I_4 i DFT_6 transformacija. Korištenjem te veze parno proširite signal $x[n] = \{\underline{2}, 1, 0, 1\}$, izračunajte DFT_6 proširenog niza te usporedite dobivenu transformaciju s $\text{DCT-I}\big[x[n]\big]$.

Napomena: DCT-I transformacija je za $N > 1$ dana izrazom

$$X[k] = \sqrt{\frac{2 - \delta[k] - \delta[k - N + 1]}{N - 1}} \sum_{n=0}^{N-1} x[n] \frac{1}{\sqrt{1 + \delta[n] + \delta[n - N + 1]}} \cos \frac{nk\pi}{N - 1}.$$

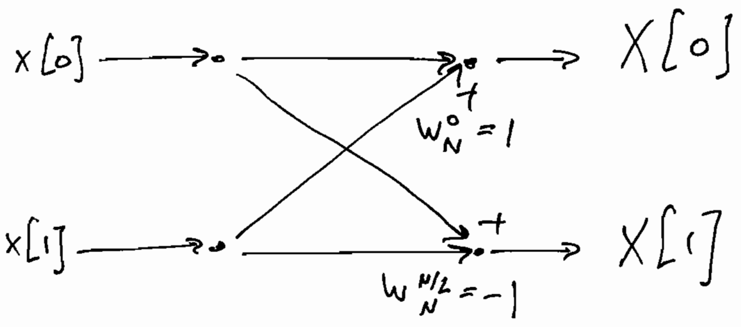
5. Nacrtajte strukturu DCT filtarskog sloga s decimacijom za $N = 4$. Za svaki filtarski element u slogu napišite prijenosnu funkciju u $\mathcal{Z}$ domeni te navedite vezu koeficijenata prijenosnih funkcija i matrice $\mathcal{C}_4$ (izračunajte elemente matrice $\mathcal{C}_4$) diskretne kosinusne transformacije u četiri točke.

Napomena: DCT ili DCT-II transformacija je dana izrazom

$$X[k] = \sqrt{\frac{2 - \delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cos \frac{(2n + 1)k\pi}{2N}.$$

2 međuispit, 2008. 4. 28. Davor Petrinović

① FFT algoritam, odnosno brza Fourierova transformacija je efikasni postupak za računanje DFT_N transformacije. Asimptotska složenost FFT algoritma je $O(N \log_2 N)$, što za $N=2^n$ postaje $O(2^n \log_2 2^n) = O(n \cdot 2^n)$.



DFT leptir odnosno DFT_2

②

$$x_1[n] * x_2[n] = \sum_{i=-\infty}^{+\infty} x_1[i]\, x_2[n-i]$$ LINEARNA KONVOLUCIJA

$$x_1[n] \, ⓝ \, x_2[n] = \sum_{i=0}^{N-1} x_1[i]\, x_2[\langle n-i\rangle_N]$$ CIRKULARNA KONVOLUCIJA DULJINE N

$x_1[n] = \{\underline{1}, 2, 3, 4, 5\}$, $x_2[n] = \{\underline{1}, 0, 0, 0, -1\}$

$x_1[n] * x_2[n] = \{\underline{1}, 2, 3, 4, 4, -2, -3, -4, -5\}$

$x_1[n] \, ⑤ \, x_2[n] = \{\underline{-1}, -1, -1, -1, 4\}$

Računamo li linearnu konvoluciju $x_1 * x_2$ dva konačna signala duljina l_1 i l_2 odabirom $N \geq l_1 + l_2 - 1$ postižemo jednakost linearne i cirkularne konvolucije za $0 \leq n < N$, odnosno vrijedi

$$x_1[n] * x_2[n] = x_1[n] \, ⓝ \, x_2[n]$$

Odaberimo npr. $N = 5+5-1 = 9$ i izračunajmo $x_1[n] \, ⑨ \, x_2[n]$

$x_1[i]$	1	2	3	4	5	0	0	0	0	$x_1 ⑨ x_2$
	1	0	0	0	0	-1	0	0	0	1
	0	1	0	0	0	0	-1	0	0	2
	0	0	1	0	0	0	0	-1	0	3
	0	0	0	1	0	0	0	0	-1	4
	-1	0	0	0	1	0	0	0	0	4
	0	-1	0	0	0	1	0	0	0	-2
	0	0	-1	0	0	0	1	0	0	-3
	0	0	0	-1	0	0	0	1	0	-4
	0	0	0	0	-1	0	0	0	1	-5

$$x_1 ⑨ x_2 = \{\underline{1}, 2, 3, 4, 4, -2, -3, -4, -5\} = x_1 * x_2$$

(3)

$$A_d(\omega) = \begin{cases} 1, & -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ 0, & \text{inače} \end{cases}$$

FIR filtar 4 reda projektiramo koristeći projekcijske metode. Filtar je TIP I, odnosno

$$A(\omega) = \sum_{m=0}^{2} a[m] \cos(\omega m)$$

$$a[0] = \frac{1}{\pi} \int_0^{\pi} A_d(\omega)\, d\omega = \frac{1}{\pi} \int_0^{2\pi/3} d\omega = \frac{1}{\pi} \cdot \frac{2\pi}{3} = \frac{2}{3}$$

$$a[m] = \frac{2}{\pi} \int_0^{\pi} A_d(\omega) \cos(\omega m) d\omega = \frac{2}{\pi} \int_0^{2\pi/3} \cos(\omega m)\, d\omega = \frac{2}{\pi m} \sin\left(\frac{2\pi}{3} m\right)$$

$$a[1] = \frac{2}{\pi} \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{\pi}$$

$$a[2] = \frac{2}{2\pi} \sin\left(\frac{2\pi}{3} \cdot 2\right) = -\frac{\sqrt{3}}{2\pi}$$

$$A(\omega) = \frac{2}{3} + \frac{\sqrt{3}}{\pi}\cos(\omega) - \frac{\sqrt{3}}{2\pi}\cos(2\omega)$$

Dobiveni FIR filtar nije kauzalan, no možemo ga učiniti kauzalnim ako njegov impulsni odziv zakasnimo za dva uzorka. Tada je

$$H(\omega) = e^{-2j\omega}\left(\frac{2}{3} + \frac{\sqrt{3}}{\pi}\cos(\omega) - \frac{\sqrt{3}}{2\pi}\cos(2\omega)\right)$$

$$= -\frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3}}{2\pi}e^{-j\omega} + \frac{2}{3}e^{-2j\omega} + \frac{\sqrt{3}}{2\pi}e^{-3j\omega} - \frac{\sqrt{3}}{4\pi}e^{-4j\omega}$$

$$h[n] = \left\{\underline{-\frac{\sqrt{3}}{4\pi}}, \frac{\sqrt{3}}{2\pi}, \frac{2}{3}, \frac{\sqrt{3}}{2\pi}, -\frac{\sqrt{3}}{4\pi}\right\}$$

$$\approx \{\underline{-0{,}1378}, 0{,}2757, 0{,}6667, 0{,}2757, -0{,}1378\}$$

(4) DCT-I

$$X[k] = \sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}} \sum_{n=0}^{N-1} \frac{x[n]}{\sqrt{1+\delta[n]+\delta[n-N+1]}} \cos\frac{nk\pi}{N-1}$$

Za $N=4$ je matrica transformacije C_I dimenzija 4×4 koja u i-tom retku i j-tom stupcu ima element

$$\sqrt{\frac{2-\delta[i-1]+\delta[i-4]}{3}} \cdot \frac{1}{\sqrt{1+\delta[j-1]+\delta[j-4]}} \cdot \cos\frac{(i-1)(j-1)\pi}{3}$$

$$C_I = \begin{bmatrix} \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}\cos\phi & \sqrt{\frac{1}{3}}\cos\phi & \frac{1}{\sqrt{3}}\cos\phi & \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}\cos\phi \\ \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}\cos\phi & \sqrt{\frac{2}{3}}\cos\frac{\pi}{3} & \sqrt{\frac{2}{3}}\cos\frac{2\pi}{3} & \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}\cos\frac{3\pi}{3} \\ \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}\cos\phi & \sqrt{\frac{2}{3}}\cos\frac{2\pi}{3} & \sqrt{\frac{2}{3}}\cos\frac{4\pi}{3} & \sqrt{\frac{2}{3}}\frac{1}{\sqrt{2}}\cos\frac{6\pi}{3} \\ \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}\cos\phi & \sqrt{\frac{1}{3}}\cos\frac{3\pi}{3} & \sqrt{\frac{1}{3}}\cos\frac{6\pi}{3} & \sqrt{\frac{1}{3}}\frac{1}{\sqrt{2}}\cos\frac{9\pi}{3} \end{bmatrix}$$

$$C_I = \begin{bmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{bmatrix} \approx \begin{bmatrix} 0{,}4082 & 0{,}5774 & 0{,}5774 & 0{,}4082 \\ 0{,}5774 & 0{,}4082 & -0{,}4082 & -0{,}5774 \\ 0{,}5774 & -0{,}4082 & -0{,}4082 & 0{,}5774 \\ 0{,}4082 & -0{,}5774 & 0{,}5774 & -0{,}4082 \end{bmatrix}$$

$$x[n] = \{\underline{2}, 1, 0, 1\}$$

$$DCT\text{-}I_4[x[n]] = \frac{1}{\sqrt{6}}\{\underline{3+\sqrt{2}}, 1+\sqrt{2}, -1+3\sqrt{2}, 1-\sqrt{2}\}$$

$$\approx \{\underline{1{,}8021}, 0{,}9856, 1{,}3238, -0{,}1691\}$$

Želimo li $DCT\text{-}I_4$ transformaciju računati preko DFT_6 transformacije proširenog niza potrebno je simetrično i parno proširiti zadani niz. Dobivamo niz

$$y[n] = \{0, 1, \underline{2}, 1, 0, 1\}$$

Kako razmatramo ortonormiranu $DCT\text{-}I_4$ transformaciju ne računamo $DFT_6[y[n]]$ nego dodatno članove niza $y[n]$ za $n=0$ i $n=N-1=3$ množimo s $\sqrt{2}$. Označimo sa $z[n]$ signal

$$z[n] = \{\underline{2\sqrt{2}}, 1, 0, \sqrt{2}, 0, 1\}.$$

Tada vrijedi

$$DCT\text{-}I_4[x[n]] = \frac{1}{2}\cdot DFT_6[z[n]]\cdot\sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}}$$

$$DFT_6[z[n]] = 2\sqrt{2} + \underbrace{W_6^k} + \sqrt{2}\,\underbrace{W_2^k}_{(-1)^k} + \underbrace{W_6^{5k}} = 2\sqrt{2} + \sqrt{2}(-1)^k + 2\cos\frac{k\pi}{3}$$

Moramo pokazati da jednakost vrijedi za $k=0,1,2,3$

$$k=0: \frac{1}{2}\cdot\left(2\sqrt{2}+\sqrt{2}+2\right)\cdot\sqrt{\frac{1}{3}} = \frac{3+\sqrt{2}}{\sqrt{6}} \approx 1{,}8021$$

$$k=1: \frac{1}{2}\left(2\sqrt{2}-\sqrt{2}+2\cdot\frac{1}{2}\right)\cdot\sqrt{\frac{2}{3}} = \frac{1+\sqrt{2}}{\sqrt{6}} \approx 0{,}9856$$

$$k=2: \frac{1}{2}\left(2\sqrt{2}+\sqrt{2}+2\cdot\left(-\frac{1}{2}\right)\right)\cdot\sqrt{\frac{2}{3}} = \frac{3\sqrt{2}-1}{\sqrt{6}} \approx 1{,}3238$$

$$k=3: \frac{1}{2}\left(2\sqrt{2}-\sqrt{2}-2\right)\sqrt{\frac{1}{3}} = \frac{1-\sqrt{2}}{\sqrt{6}} \approx -0{,}1691$$

(5) DCT-II

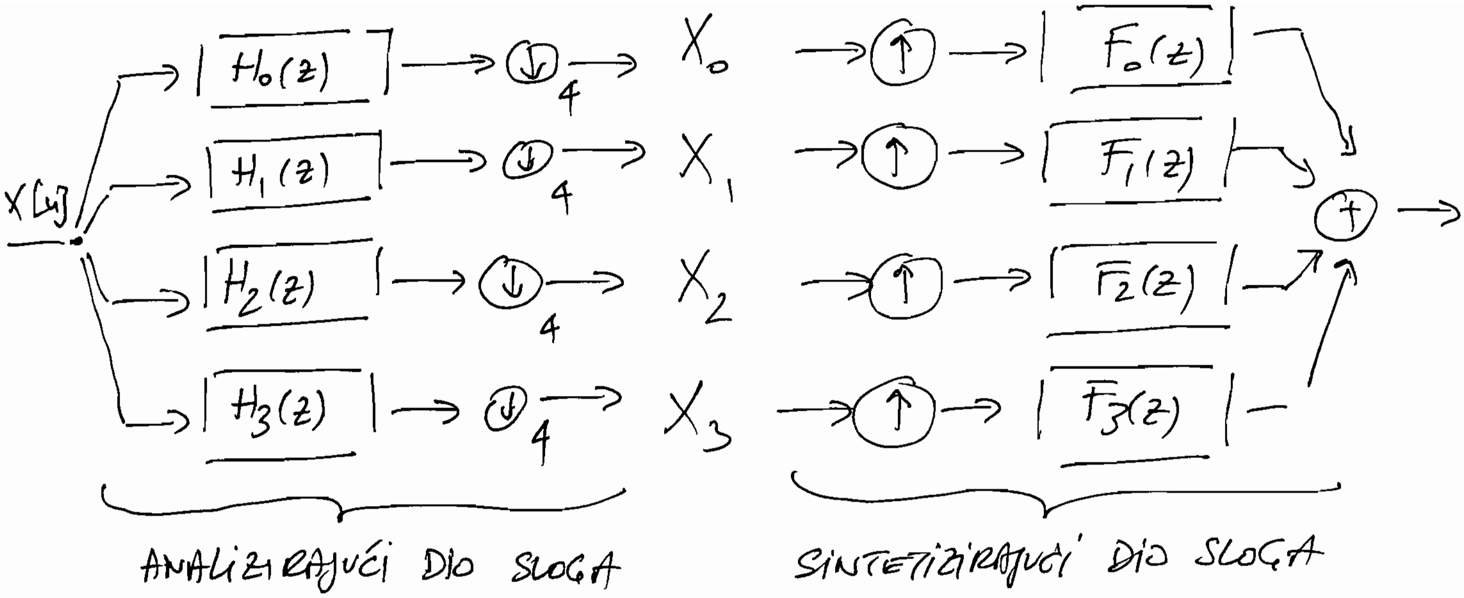
$$X[k] = \sqrt{\frac{2-\delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cos\frac{(2n+1)k\pi}{2N}$$

Za $N=4$ je matrica transformacije C_4 dimenzije 4×4 i u i-tom retku i j-tom stupcu ima element

$$\sqrt{\frac{2-\delta[i-1]}{4}}\cdot\cos\frac{(i-1)(2j-1)\pi}{8}$$

$$C_4 = \begin{bmatrix} \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 \\ \frac{1}{\sqrt{2}}\cos\frac{\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{5\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{7\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{2\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{6\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{10\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{14\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{9\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{15\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{21\pi}{8} \end{bmatrix} \approx$$

$$\approx \begin{bmatrix} 0{,}5000 & 0{,}5000 & 0{,}5000 & 0{,}5000 \\ 0{,}6533 & 0{,}2706 & -0{,}2706 & -0{,}6533 \\ 0{,}5000 & -0{,}5000 & -0{,}5000 & 0{,}5000 \\ 0{,}2706 & -0{,}6533 & 0{,}6533 & -0{,}2706 \end{bmatrix}$$



Koeficijenti $H_i(z)$ odgovaraju retcima matrice $\mathcal{C}_4$ dok koeficijenti $F_i(z)$ odgovaraju stupci matrice $\mathcal{C}_4^{-1} = \mathcal{C}_4^T$.

Dobivamo:

$$H_0(z) = \frac{1}{2} + \frac{1}{2} z + \frac{1}{2} z^2 + \frac{1}{2} z^3$$

$$H_1(z) = \frac{1}{\sqrt{2}} \cos\frac{\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{3\pi}{8} z + \frac{1}{\sqrt{2}} \cos\frac{5\pi}{8} z^2 + \frac{1}{\sqrt{2}} \cos\frac{7\pi}{8} z^3$$

$$H_2(z) = \frac{1}{\sqrt{2}} \cos\frac{2\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{6\pi}{8} z + \frac{1}{\sqrt{2}} \cos\frac{10\pi}{8} z^2 + \frac{1}{\sqrt{2}} \cos\frac{14\pi}{8} z^3$$

$$H_3(z) = \frac{1}{\sqrt{2}} \cos\frac{3\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{9\pi}{8} z + \frac{1}{\sqrt{2}} \cos\frac{15\pi}{8} z^2 + \frac{1}{\sqrt{2}} \cos\frac{21\pi}{8} z^3$$

$$F_0(z) = \frac{1}{2} + \frac{1}{2} z^{-1} + \frac{1}{2} z^{-2} + \frac{1}{2} z^{-3}$$

$$F_1(z) = \frac{1}{\sqrt{2}} \cos\frac{\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{3\pi}{8} z^{-1} + \frac{1}{\sqrt{2}} \cos\frac{5\pi}{8} z^{-2} + \frac{1}{\sqrt{2}} \cos\frac{7\pi}{8} z^{-3}$$

$$F_2(z) = \frac{1}{\sqrt{2}} \cos\frac{2\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{6\pi}{8} z^{-1} + \frac{1}{\sqrt{2}} \cos\frac{10\pi}{8} z^{-2} + \frac{1}{\sqrt{2}} \cos\frac{14\pi}{8} z^{-3}$$

$$F_3(z) = \frac{1}{\sqrt{2}} \cos\frac{3\pi}{8} + \frac{1}{\sqrt{2}} \cos\frac{9\pi}{8} z^{-1} + \frac{1}{\sqrt{2}} \cos\frac{15\pi}{8} z^{-2} + \frac{1}{\sqrt{2}} \cos\frac{21\pi}{8} z^{-3}$$