2. MI ~~2007~~ / 2008

1. FFT (Fast Fourier Transform) je efikasni postupak za računanje DFT_N transformacije. Asimptotska složenost FFT algoritma je $O(N \log_2 N)$. Za $N = 2^k$ postaje $O(k\, 2^k \log_2 2^k) = O(k\, 2^k)$.

DFT leptir je graf toka za DFT_2

2. $x_1[n] * x_2[n] = \sum_{i=-\infty}^{\infty} x_1[i]\, x_2[n-i]$ $\quad x_1[n] \circledast_N x_2[n] = \sum_{i=0}^{N-1} x_1[i]\, x_2[(n-i)_N]$

$x_1[n] = \{1, 2, 3, 4\}$ $\quad x_2[n] = \{1, 0, 0, 0, -1\}$

$x_1[n] * x_2[n] = \{1, 2, 3, 4, 4, -2, -3, -4, -5\}$

$x_1[n] \circledast_5 x_2[n] = \{-1, -1, -1, -1, 4\}$

Jednakost linearne i cirkularne konvolucije dobivamo za $0 \le n < N$, gdje je N dan izrazom $N \ge L_1 + L_2 - 1$

$N = 5 + 5 - 1 = 9 \Rightarrow x_1[n] \circledast_9 x_2[n]$

1	2	3	4	5	0	0	0	0	$x_1 \circledast_9 x_2$
1	0	0	0	0	-1	0	0	0	1
0	1	0	0	0	0	-1	0	0	2
0	0	1	0	0	0	0	-1	0	3
0	0	0	1	0	0	0	0	-1	4
-1	0	0	0	1	0	0	0	0	4
0	-1	0	0	0	1	0	0	0	-2
0	0	-1	0	0	0	1	0	0	-3
0	0	0	-1	0	0	0	1	0	-4
0	0	0	0	-1	0	0	0	1	-5

$x_1 \circledast_9 x_2 = \{1, 2, 3, 4, 4, -2, -3, -4, -5\}$

③ $N=5$

$$A_d(\omega) = \begin{cases} 1, & -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ 0, & \text{inače} \end{cases}$$

$$A(\omega) = \sum_{m=0}^{2} a[m]\cos(\omega m)$$

$$a[0] = \frac{1}{\pi}\int_0^{\pi} A_d(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} d\omega = \frac{1}{\pi}\cdot\frac{2\pi}{3} = \frac{2}{3}$$

$$a[m] = \frac{2}{\pi}\int_0^{\pi} A_d(\omega)\cos(\omega m)\,d\omega = \frac{2}{\pi}\int_0^{\frac{2\pi}{3}} \cos(\omega m)\,d\omega = \frac{2}{\pi m}\sin\left(\frac{2\pi}{3}m\right)$$

$$a[1] = \frac{2\pi}{3}\sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{\pi}$$

$$a[2] = \frac{2}{2\pi}\sin\left(\frac{2\pi}{3}2\right) = -\frac{\sqrt{3}}{2\pi}$$

$$A(\omega) = \frac{2}{3} + \frac{\sqrt{3}}{2}\cos(\omega) - \frac{\sqrt{3}}{2\pi}\cos(2\omega)$$

Filter sa $A(\omega)$ nije kauzalan, ali možemo zakasniti njegov impulsni odziv za dva uzorka.

$$H(\omega) = e^{-2j\omega}\left[\frac{2}{3} + \frac{\sqrt{3}}{\pi}\cos\omega - \frac{\sqrt{3}}{2\pi}\cos(2\omega)\right]$$

$$= -\frac{\sqrt{3}}{4\pi} + \frac{\sqrt{3}}{2\pi}e^{-j\omega} + \frac{2\pi}{3}e^{-2j\omega} + \frac{\sqrt{3}}{2\pi}e^{-3j\omega} - \frac{\sqrt{3}}{4\pi}e^{-4j\omega}$$

$$h[n] = \left\{-\frac{\sqrt{3}}{4\pi}, \frac{\sqrt{3}}{2\pi}, \frac{2}{3}, \frac{\sqrt{3}}{2\pi}, -\frac{\sqrt{3}}{4\pi}\right\}$$

$$\approx \left\{\underline{-0.1378}, 0.2757, 0.6667, 0.2757, -0.1378\right\}$$

(1) DCT-I

$$X[k]=\sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}}\cdot\sum_{n=0}^{N-1}\frac{x[n]}{\sqrt{1+\delta[n]+\delta[n-N+1]}}\cos\frac{nk\pi}{N-1}$$

$$N=4 \Rightarrow \sqrt{\frac{2-\delta[i-1]-\delta[i-4]}{3}}\cdot\frac{1}{\sqrt{1+\delta[j-1]+\delta[j-4]}}\cdot\cos\frac{(i-1)(j-1)\pi}{3}$$

$$C_I=\begin{array}{c|cccc} i \backslash j & 1 & 2 & 3 & 4 \\ \hline 1 & \sqrt{\frac{2}{3}}\cdot\frac{1}{2}\cos 0 & \sqrt{\frac{1}{3}}\cos 0 & \sqrt{\frac{1}{3}}\cos 0 & \sqrt{\frac{2}{3}}\frac{1}{2}\cos 0 \\ 2 & \sqrt{\frac{2}{3}}\cdot\sqrt{\frac{1}{2}}\cos 0 & \sqrt{\frac{2}{3}}\cos\frac{\pi}{3} & \sqrt{\frac{2}{3}}\cos\frac{2\pi}{3} & \sqrt{\frac{2}{3}}\frac{1}{\sqrt2}\cos\pi \\ 3 & \sqrt{\frac{2}{3}}\cdot\frac{1}{\sqrt2}\cos 0 & \sqrt{\frac{2}{3}}\cos\frac{2\pi}{3} & \sqrt{\frac{2}{3}}\cos\frac{4\pi}{3} & \sqrt{\frac{2}{3}}\sqrt{\frac{1}{2}}\cos 2\pi \\ 4 & \sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{2}}\cos 0 & \sqrt{\frac{1}{3}}\cos\pi & \sqrt{\frac{1}{3}}\cos 2\pi & \sqrt{\frac{1}{3}}\sqrt{\frac{1}{2}}\cos 3\pi \end{array} = \begin{bmatrix} \frac{1}{\sqrt6} & \frac{1}{\sqrt3} & \frac{1}{\sqrt3} & \frac{1}{\sqrt6} \\ \frac{1}{\sqrt3} & \frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt3} \\ \frac{1}{\sqrt3} & -\frac{1}{\sqrt6} & -\frac{1}{\sqrt6} & \frac{1}{\sqrt3} \\ \frac{1}{\sqrt6} & -\frac{1}{\sqrt3} & \frac{1}{\sqrt3} & -\frac{1}{\sqrt6} \end{bmatrix}$$

$x[n]=\{\underline{2},1,0,1\}$

$DCT\text{-}I_4=\frac{1}{\sqrt6}\{3+\sqrt2,\ 1+\sqrt2,\ -1+3\sqrt2,\ 1-\sqrt2\}\approx\{1{,}8021,\ 0{,}9856,\ 1{,}3238,\ -0{,}1691\}$

$DCT\text{-}I_4$ možemo računati preko DFT_6 tako da simetrično i parno proširimo zadani niz. Novi niz je $y[n]=\{0,1,2,1,0,1\}$.

Kako koristimo ortonormirana $DCT\text{-}I_4$ transformaciju, članove niza $y[n]$ za $n=0$ i $n=3$ $(N-1)$ množimo sa $\sqrt2$. Dobivamo $z[n]=\{2\sqrt2,1,0,\sqrt2,0,1\}$

$$DCT\text{-}I_4\{x[n]\}=\frac{1}{2}DFT_6\{z[n]\}\cdot\sqrt{\frac{2-\delta[k]-\delta[k-N+1]}{N-1}}$$

$$DFT_6\{z[n]\}=2\sqrt2+W_6^k+\sqrt2\,W_2^k+W_6^{5k}=2\sqrt2+\sqrt2(-1)^k+2\cos\frac{k\pi}{3}$$

Pokazujemo da jednakost vrijedi za $k=0,1,2,3$

$$k=0:\ \frac{1}{2}(2\sqrt2+\sqrt2+2)\cdot\sqrt{\frac{1}{3}}=\frac{3+\sqrt2}{6}\approx 1{,}8021$$

$$k=1:\ \frac{1}{2}\left(2\sqrt2-\sqrt2+2\cdot\frac{1}{2}\right)\sqrt{\frac{2}{3}}=\frac{1+\sqrt2}{6}\approx 0{,}9856$$

$$k=2:\ \frac{1}{2}\left(2\sqrt2+\sqrt2+2\cdot\left(-\frac{1}{2}\right)\right)\cdot\sqrt{\frac{2}{3}}=\frac{3\sqrt2-1}{\sqrt6}\approx 1{,}3238$$

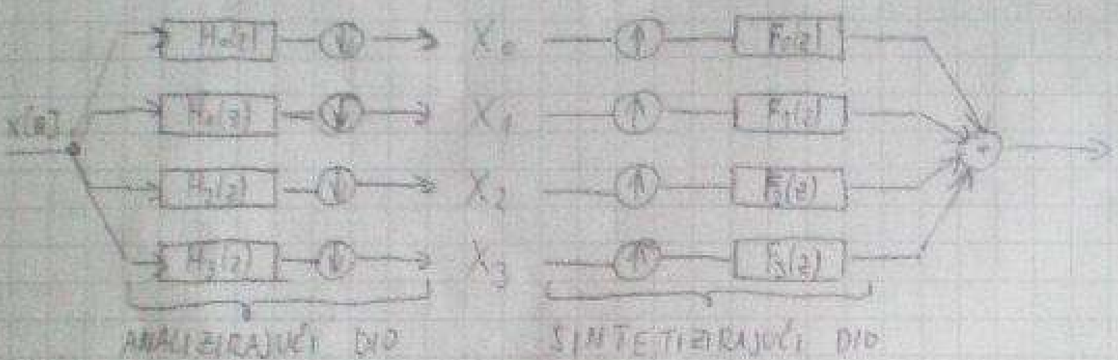
$$k=3:\ \frac{1}{2}(2\sqrt2-\sqrt2-2)\cdot\sqrt{\frac{1}{3}}=\frac{1-\sqrt2}{\sqrt6}\approx -0{,}1691$$

(5) DCT - II

$$X[k] = \sqrt{\frac{2-\delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cos \frac{(2n+1)k\pi}{2N}$$

$$C_4 \;\; u \;\; C[i,j] = \sqrt{\frac{2-\delta[i-1]}{4}} \cos \frac{(i-1)(2j-1)\pi}{8}$$

$$C_4 = \begin{bmatrix} \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 & \frac{1}{2}\cos 0 \\ \frac{1}{\sqrt{2}}\cos\frac{\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{5\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{7\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{2\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{6\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{10\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{14\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{9\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{15\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{21\pi}{8} \end{bmatrix} \approx \begin{bmatrix} 0,5 & 0,5 & 0,5 & 0,5 \\ 0,6533 & 0,2706 & -0,2706 & -0,6533 \\ 0,50 & -0,5 & -0,5 & 0,5 \\ 0,2706 & -0,6533 & 0,653 & -0,2706 \end{bmatrix}$$



Koeficijenti $H_i(z)$ odgovaraju retcima C_4

Koeficijenti $F_i(z)$ odgovaraju retcima $C_4^{-1} = C_4^T$

1) $H_0(z) = \frac{1}{2} + \frac{1}{2}z + \frac{1}{2}z^2 + \frac{1}{2}z^3$

$H_1(z) = \frac{1}{\sqrt{2}}\cos\frac{\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8}z + \frac{1}{\sqrt{2}}\cos\frac{5\pi}{8}z^2 + \frac{1}{\sqrt{2}}\cos\frac{7\pi}{8}z^3$

$H_2(z) = \frac{1}{\sqrt{2}}\cos\frac{2\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{6\pi}{8}z + \frac{1}{\sqrt{2}}\cos\frac{10\pi}{8}z^2 + \frac{1}{\sqrt{2}}\cos\frac{14\pi}{8}z^3$

$H_3(z) = \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{9\pi}{8}z + \frac{1}{\sqrt{2}}\cos\frac{15\pi}{8}z^2 + \frac{1}{\sqrt{2}}\cos\frac{21\pi}{8}z^3$

2) $F_0(z) = \frac{1}{2} + \frac{1}{2}z^{-1} + \frac{1}{2}z^{-2} + \frac{1}{2}z^{-3}$

$F_1(z) = \frac{1}{\sqrt{2}}\cos\frac{\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8}z^{-1} + \frac{1}{\sqrt{2}}\cos\frac{5\pi}{8}z^{-2} + \frac{1}{\sqrt{2}}\cos\frac{7\pi}{8}z^{-3}$

$F_2(z) = \frac{1}{\sqrt{2}}\cos\frac{2\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{6\pi}{8}z^{-1} + \frac{1}{\sqrt{2}}\cos\frac{10\pi}{8}z^{-2} + \frac{1}{\sqrt{2}}\frac{14\pi}{8}z^{-3}$

$F_3(z) = \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} + \frac{1}{\sqrt{2}}\cos\frac{9\pi}{8}z^{-1} + \frac{1}{\sqrt{2}}\cos\frac{15\pi}{8}z^{-2} + \frac{1}{\sqrt{2}}\frac{21\pi}{8}z^{-3}$

2 MI 2008/2009.

(1) $x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$

$x_1[n] \circledast x_2[n] = \sum_{k=0}^{N-1} x_1[k]\, x_2[(n-k)_N] = \sum_{k=0}^{N-1} x_1[k]\, x_2[(n-k)_N]$; $(n-k)_N = (n-k) \bmod N$

$x_1[n] = \{5, -5, 5, -5, 5\}$ $\qquad x_2[n] = \{2, 0, 2, 0, -2\}$

$$x_1 * x_2 = \begin{array}{c|ccccc} & 5 & -5 & 5 & -5 & 5 \\ \hline 2 & 10 & -10 & 10 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 10 & -10 & 10 & -10 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -10 & 10 & -10 & 10 & -10 \end{array}$$

$= \{10, -10, 20, -20, 10, 0, 0, 10, -10\}$

$$x_1 \circledast x_2 = \begin{array}{|ccccc|c} 5 & -5 & 5 & -5 & 5 & \\ \hline 2 & -2 & 0 & 2 & 0 & 10 \\ 0 & 2 & 2 & 0 & 2 & -10 \\ 2 & 0 & 2 & -2 & 0 & 30 \\ 0 & 2 & 0 & 2 & -2 & -30 \\ -2 & 0 & 2 & 0 & 2 & 10 \end{array}$$

$= \{10, -10, 30, -30, 10\}$

$N \geq 5+5-1 \Rightarrow N \geq 9$

$x_1 \circledast_{15} x_2 = \{10, -10, 20, -20, 10, 0, 0, 10, -10, 0, 0, 0, 0, 0, 0\}$

(2) $A = \begin{cases} \frac{3}{2\pi}|\omega| & , -\frac{2\pi}{3} < \omega < \frac{2\pi}{3} \\ 0 & , \text{inače} \end{cases}$

$A(\omega) = \sum_{m=0}^{M'} a[m]\cos(\omega m)$

$a[0] = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_d(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\pi} A_d(\omega)\,d\omega = \frac{1}{\pi}\int_0^{\pi} \frac{3}{2\pi}|\omega|\,d\omega = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} \frac{3}{2\pi}\omega\,d\omega$

$= \frac{1}{\pi}\frac{3}{2\pi}\frac{\omega^2}{2}\Big|_0^{\frac{2\pi}{3}} = \frac{3}{4\pi^2}\cdot\left(\frac{2\pi}{3}\right)^2 = \frac{1}{3}$

$a[m] = \frac{1}{\pi}\int_{-\pi}^{\pi} A_d(\omega)\cos(\omega m)\,d\omega = \frac{2}{\pi}\int_0^{\frac{2\pi}{3}} \frac{3}{2\pi}\omega\cos(\omega m)\,d\omega = \frac{3}{\pi^2}\int_0^{\frac{2\pi}{3}} \omega\cos(\omega m)\,d\omega$

$= \frac{3}{\pi^2}\left(\omega\frac{\sin\omega m}{m}\Big|_0^{\frac{2\pi}{3}} - \int_0^{\frac{2\pi}{3}}\frac{\sin\omega m}{m}\,d\omega\right) = \frac{3}{\pi^2}\left(\frac{2\pi}{3m}\sin\left(\frac{2m\pi}{3}\right) + \frac{1}{m^2}\cos(\omega m)\Big|_0^{\frac{2\pi}{3}}\right)$

$= \frac{2}{\pi m}\sin\left(\frac{2m\pi}{3}\right) + \frac{3}{\pi^2 m^2}\cos\left(\frac{2\pi m}{3}\right) - \frac{3}{\pi^2 m^2}$

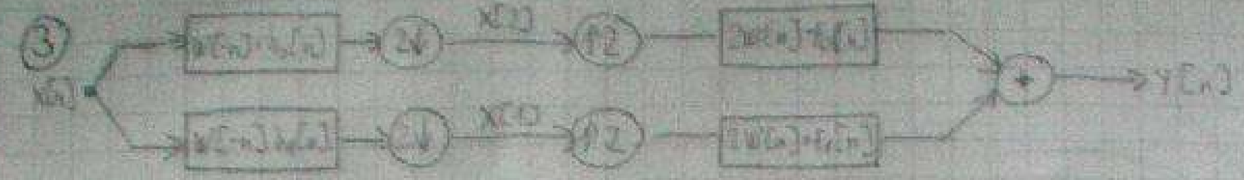
$a[1] = \frac{2\sqrt{3}\pi - 9}{2\pi^2} \approx 0{,}095384 \qquad a[2] = -\frac{4\sqrt{3}\pi + 9}{8\pi^2} \approx -0{,}38965$

$h[n] = \begin{cases} \frac{1}{2}a\left[\frac{N-1}{2} - n\right] & , 0 \leq n \leq \frac{N-1}{2} \\ a[0] & , n = \frac{N-1}{2} \\ \frac{1}{2}a\left[n - \frac{N-1}{2}\right] & , \frac{N-1}{2} < n < N \end{cases} \Rightarrow \frac{a[2]}{2}, \frac{a[1]}{2}, a[0], \frac{a[1]}{2}, \frac{a[2]}{2}$

$h[n] = \left\{-\frac{4\sqrt{3}\pi + 9}{16\pi^2}, \frac{2\sqrt{3}\pi - 9}{4\pi^2}, \frac{1}{3}, \frac{2\sqrt{3}\pi - 9}{4\pi^2}, -\frac{4\sqrt{3}\pi + 9}{16\pi^2}\right\}$

$= \{-0{,}1948, 0{,}0477, 0{,}3333, 0{,}0477, -0{,}1948\}$

3



$w[n] = \sin \frac{(2n+1)\pi}{4N} \Rightarrow w[n] = \sin \frac{2n+1}{8}\pi = \{\underline{\sin \frac{\pi}{8}}, \sin \frac{3\pi}{8}, \sin \frac{5\pi}{8}, \sin \frac{7\pi}{8}\}$

$= \{\underline{0{,}3827}, 0{,}9239, 0{,}9239, 0{,}3827\}$

$w[-n] = \{\sin \frac{7\pi}{8}, \sin \frac{5\pi}{8}, \sin \frac{3\pi}{8}, \sin \frac{\pi}{8}\}$

IMPULSNI ODZIVI

$w[-n] \cdot h_0[n] = \{\cos \frac{9\pi}{8} \sin \frac{7\pi}{8}, \cos \frac{3\pi}{8} \sin \frac{5\pi}{8}, \cos \frac{15\pi}{8} \sin \frac{3\pi}{8}, \underline{\cos \frac{21\pi}{8} \sin \frac{\pi}{8}}\}$

$= \{-0{,}3536, -0{,}8536, -0{,}3536, \underline{0{,}1464}\}$

$w[n] \cdot h_1[n] = \{\cos \frac{27\pi}{8} \sin \frac{\pi}{8}, \cos \frac{45\pi}{8} \sin \frac{3\pi}{8}, \cos \frac{45\pi}{8} \sin \frac{5\pi}{8}, \cos \frac{9\pi}{8} \sin \frac{7\pi}{8}\}$

$= \{-0{,}1464, -0.3536, 0.8536, \underline{-0{,}3536}\}$

$2w[n] \cdot f_0[n] = \{\underline{\cos \frac{15\pi}{8} \sin \frac{\pi}{8}}, \cos \frac{9\pi}{8} \sin \frac{3\pi}{8}, \cos \frac{3\pi}{8} \sin \frac{5\pi}{8}, \cos \frac{9\pi}{8} \sin \frac{7\pi}{8}\}$

$= \{\underline{0{,}1464}, -0{,}3536, -0{,}8536, -0.3536\}$

$2w[n] \cdot f_1[n] = \{\underline{\cos \frac{9\pi}{8} \sin \frac{\pi}{8}}, \cos \frac{15\pi}{8} \sin \frac{3\pi}{8}, \cos \frac{25\pi}{8} \sin \frac{5\pi}{8}, \cos \frac{21\pi}{8} \sin \frac{7\pi}{8}\}$

$= \{\underline{-0{,}3536}, 0{,}8536, -0{,}3536, -0{,}1464\}$

$H_0(z) = 0{,}1464 - 0{,}3536z - 0{,}8536z^2 - 0{,}3536z^3$

$H_1(z) = -0{,}3536 + 0{,}8536z - 0{,}3536z^2 - 0{,}1464z^3$

$F_0(z) = 0{,}1464 - 0{,}3536z^{-1} - 0{,}8536z^{-2} - 0{,}3536z^{-3}$

$F_1(z) = -0{,}3536 + 0{,}8536z^{-1} - 0{,}3536z^{-2} - 0{,}1464z^{-3}$

④ LINEARNOST – neka je L neki 2D prostorni diskretni sustav, neka su $I_1(x,y): \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ i $I_2(x,y): \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ dvije slike te neka su $a,b\in\mathbb{R}$ neke konstante. Sustav L je linearan ako za $\forall a,b\in\mathbb{R}$ i $\forall I_1, I_2$ vrijedi

$$L[aI_1(x,y)+bI_2(x,y)] = aL[I_1(x,y)]+bL[I_2(x,y)]$$

PROSTORNA NEPROMJENJIVOST – neka je L neki 2D prostorno diskretni sustav, neka je $I(x,y): \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ slika i neka je $J(x,y): \mathbb{Z}\times\mathbb{Z}\to\mathbb{R}$ odziv sustava na sliku I. Sustav $L[I(x,y)] = J(x,y)$ prostorno nepromjenjiv ako za svaku $I(x,y)\in\mathbb{R}^2$ i za $\forall (a,b)\in\mathbb{Z}^2$, $a,b\in\mathbb{Z}$

$$L[I(\cancel{x,y}+b)] = J(x+a, y+b)$$

$x+a,\ y+b$

$$J(x,y) = 10\,I(x-2, y+2)+2$$

<u>Linearnost</u>

$$L[I_1(x,y)] = 5I_1(x,y)+1 = J_1$$
$$L[I_2(x,y)] = 5I_2(x,y)+1 = J_2$$

$$\left.\begin{aligned} &L[aI_1(x,y)+bI_2(x,y)] = 5[aI_1(x,y)+bI_2(x,y)]+1\\ &L[aI_1(x,y)+bI_2(x,y)] = a(5I_1(x,y)+1)+b(5I_2(x,y)+1)\end{aligned}\right\}\ \text{NIJE LINEARAN}$$

<u>Prostorna nepromjenjivost</u>

$$\left.\begin{aligned} &L[I(x+a, y+b)] = 5I(x+a, y+b)+1\\ &J(x+a, y+b) = 5I(x+a, y+b)+1\end{aligned}\right\}\ \text{PROSTORNO NEPROMJENJIV}$$

⑤ FUNKCIJA UDALJENOSTI

Udaljenost na skupu X je preslikavanje $d: X \times X \to R$ koje za svaka tri elementa $x, y, z \in X$ zadovoljava sljedeća svojstva:

1) d je pozitivno definitna $[\, d(x,y) \geq 0 \wedge d(x,y)=0 \Leftrightarrow x=y \,]$

2) d je simetrično $[\, d(x,y) = d(y,x) \,]$

3) d zadovoljava nejednakost trokuta $[\, d(x,y) + d(y,z) \geq d(x,z) \,]$

$d_E(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}$ → Euklidska udaljenost

$d_4(A,B) = |x_A - x_B| + |y_A - y_B|$ → city block udaljenost

$d_8(A,B) = \max(|x_A - x_B|, |y_A - y_B|)$ → chessboard udaljenost

A(1,2)

B(5,7)

$d_E(A,B) = \sqrt{16+25} = \sqrt{41} \approx 6{,}4031$

$d_4(A,B) = |1-5| + |2-7| = 4+5 = 9$

$d_8(A,B) = \max(4,5) = 5$

2. MI 2009./2010.

① FFT je efikasni postupak za računanje DFT_N transformacije. Asimptotska složenost mu je $O(N\log_2 N)$, što je brže od direktnog računanja složenosti $O(N^2)$.

$$DFT_{2N}[x[n]] = \sum_{n=0}^{2N-1} x[n] W_{2N}^{nk} = \sum_{n=0}^{N-1} x[2n] W_{2N}^{2nk} + \sum_{n=0}^{N-1} x[2n+1] W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x[2n] W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x[2n+1] W_N^{nk} = DFT_N[g[n]] + W_{2N}^{k} DFT_N[h[n]]$$

② DCT - II $\Rightarrow X[k] = \sqrt{\frac{2-\delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cos\frac{(2n+1)k\pi}{2N}$

$$C_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt2}\cos\frac{\pi}{8} & \frac{1}{\sqrt2}\cos\frac{3\pi}{8} & \frac{1}{\sqrt2}\cos\frac{5\pi}{8} & \frac{1}{\sqrt2}\cos\frac{7\pi}{8} \\ \frac{1}{\sqrt2}\cos\frac{2\pi}{8} & \frac{1}{\sqrt2}\cos\frac{6\pi}{8} & \frac{1}{\sqrt2}\cos\frac{10\pi}{8} & \frac{1}{\sqrt2}\cos\frac{14\pi}{8} \\ \frac{1}{\sqrt2}\cos\frac{3\pi}{8} & \frac{1}{\sqrt2}\cos\frac{9\pi}{8} & \frac{1}{\sqrt2}\cos\frac{15\pi}{8} & \frac{1}{\sqrt2}\cos\frac{21\pi}{8} \end{bmatrix} \approx \begin{bmatrix} 0{,}5 & 0{,}5 & 0{,}5 & 0{,}5 \\ 0{,}6533 & 0{,}2706 & -0{,}2706 & -0{,}6533 \\ 0{,}5 & -0{,}5 & -0{,}5 & 0{,}5 \\ 0{,}2706 & -0{,}6533 & 0{,}6533 & -0{,}2706 \end{bmatrix}$$

DCT - II i DFT su povezani parnim proširenjem tako da se susjedni članovi DFT ponište.

$x[n] = \{\underline{0},1,0,1\} \Rightarrow y[n] = \{\underline{0},0,0,1,0,0,0,1,0,1,0,0,0,1,0,0\}$

$X[k] = C_4 x[n] = \{1, \frac{1}{\sqrt2}(\cos\frac{3\pi}{8}+\cos\frac{7\pi}{8}), 0, \frac{1}{\sqrt2}(\cos\frac{9\pi}{8}+\cos\frac{21\pi}{8})\} =$

$= \{1, -0.3827, 0, -0.9239\}$

$Y[k] = \sum_{n=0}^{15} y[n] W_{16}^{nk} = W_{16}^{3k} + W_{16}^{7k} + W_{16}^{9k} + W_{16}^{13k} = W_{16}^{3k} + W_{16}^{7k} + W_{16}^{-7k} + W_{16}^{-3k}$

$= (W_{16}^{3k} + W_{16}^{-3k}) + (W_{16}^{7k} + W_{16}^{-7k}) = 2\cos(2\pi\frac{3k}{16}) + 2\cos(2\pi\frac{7k}{16})$

$2\,DCT\text{-}II_4[x[n]] = \sqrt{\frac{2-\delta[k]}{4}}\, DFT_{16}[y[n]]$

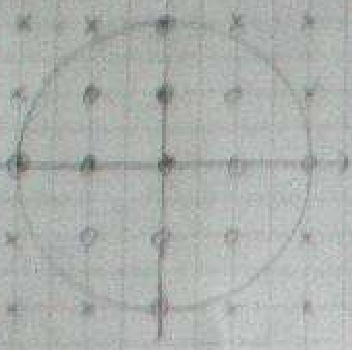
$k=0 \quad 2\cdot 1 = 2 \Leftrightarrow \sqrt{\frac{2-1}{4}}(2+2) = \frac{1}{2}\cdot 4 = 2$

$k=1 \quad \frac{2}{\sqrt2}(\cos\frac{3\pi}{8}+\cos\frac{7\pi}{8}) \Leftrightarrow \sqrt{\frac{2-0}{4}}(2\cos\frac{3\pi}{8}+2\cos\frac{7\pi}{8}) = \frac{2}{\sqrt2}(\cos\frac{3\pi}{8}+\cos\frac{7\pi}{8})$

$k=2 \quad 2\cdot 0 = 0 \Leftrightarrow \sqrt{\frac{2-0}{4}}(2\cos\frac{6\pi}{8}+2\cos\frac{14\pi}{8}) = \frac{1}{\sqrt2}\cdot 0 = 0$

$k=3 \quad \frac{2}{\sqrt2}[\cos(\frac{9\pi}{8})+\cos(\frac{21\pi}{8})] \Leftrightarrow \sqrt{\frac{2-0}{4}}(2\cos\frac{9\pi}{8}+2\cos\frac{21\pi}{8})$

③ Otipkavanje slike je pretvorba iz prostorno-kontinuirane slike u prostorno-diskretnu odabiranjem samo onih vrijednosti prostorno-kontinuirane slike koji se nalaze na nekoj prostornoj rešetci.



(0,2)

(-1,1), (0,1), (1,1)

(-2,0), (-1,0), (0,0), (1,0), (2,0)

(-1,-1), (0,-1), (1,-1)

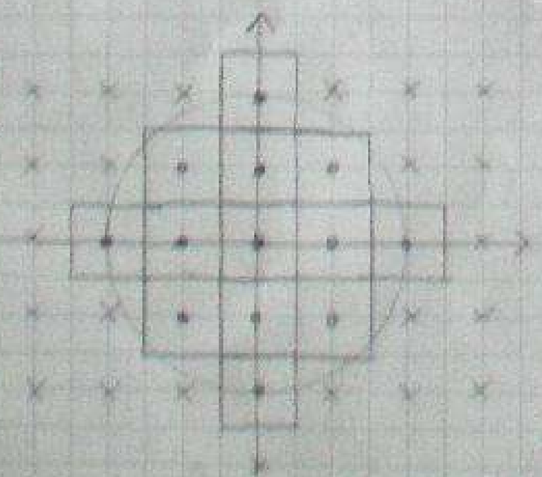
(0,-2)

dobiveni uzorci $\{(0,2), (0,1), (0,0), (0,-1), (0,-2), (1,1), (1,0), (1,-1), (2,0), (-2,0), (-1,1), (-1,0), (-1,-1)\}$

VORONOI SUSJEDSTVO – neka je G skup točaka u $\mathbb{R}^n$. Voronoi susjedstvo u G svakog elementa $g \in G$ je skup

$$N_G(g) = \{v \in \mathbb{R}^n \mid \forall h \in G, \|v-g\| \le \|v-h\|\}$$

Za $G = \mathbb{Z}^2$ Voronoi susjedstva svake točke su kvadrati koji su u skupu $\{(v_1, v_2) \in \mathbb{R}^2 \mid \max(|v_1-g_1|, |v_2-g_2|) < \frac{1}{2}\}$



Postoje točke unutar Voronoi susjedstva koje ne zadovoljavaju jednadžbu $x^2+y^2 \le 4$

4 **FUNKCIJA UDALJENOSTI** - Udaljenost na skupu X je preslikavanje $d : X \times X \to \mathbb{R}$ koje za svaka tri elementa $x,y,z \in X$ zadovoljava sljedeća svojstva:

1) d je pozitivno-definitna tj. $d(x,y) \geq 0 \wedge d(x,y) = 0 \Leftrightarrow x = y$

2) d je simetrična tj. $d(x,y) = d(y,x)$

3) d zadovoljava nejednakost trokuta tj. $d(x,y) + d(y,z) \geq d(x,z)$

$A = (x_A, y_A)$, $B = (x_B, y_B)$ $\quad | \quad A_T = (10,10)$, $B_T = (20,20)$

$$d_E(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2}\,\Big|_T = \sqrt{(10-20)^2 + (10-20)^2} = \sqrt{200} \approx 14,14$$

$$d_4(A,B) = |x_A - x_B| + |y_A - y_B|\,\Big|_T = |10-20| + |10-20| = 20$$

$$d_8(A,B) = \max\left(|x_A - x_B|, |y_A - y_B|\right)\Big|_T = \max(10,10) = 10$$

5 Neka je L neki 2D prostorno-diskretni sustav i neka su $I_1(x,y) : \mathbb{Z}^2 \to \mathbb{R}$ i $I_2(x,y) : \mathbb{Z}^2 \to \mathbb{R}$ dvije slike te neka su $a,b \in \mathbb{Q}$ konstante. Sustav L je linearan ako za $\forall a, \forall b, \forall I_1, \forall I_2$ vrijedi

$$L\left[a I_1(x,y) + b I_2(x,y)\right] = a L\left[I_1(x,y)\right] + b L\left[I_2(x,y)\right]$$

$$J_1(x,y) = \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I_1(\chi, \nu), \quad J_2(x,y) = \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I_2(\chi, \nu)$$

$$L\left[a I_1(x,y) + b I_2(x,y)\right] = \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y)\left[a I_1(\chi,\nu) + b I_2(\chi,\nu)\right]$$

$$= a \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I_1(\chi,\nu) + b \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I_2(\chi,\nu)$$

$$= a J_1(x,y) + b J_2(x,y) \Rightarrow \text{sustav je } \underline{\text{linearan}}$$

$$I(x,y) = \begin{cases} 1, & 0 \leq x < 3,\ 0 \leq y < 3 \\ 0, & \text{inače} \end{cases} \Rightarrow I_x(x) = I_y(y) = \{\ldots 0,0,0,1,1,1,0,0,0 \ldots\}$$

$$J(x,y) = \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I(\chi,\nu) = \sum_{\chi=0}^{x} \sum_{\nu=0}^{y} (\chi - x)(\nu - y) I_x(\chi) I_y(\nu)$$

$$= \left(\sum_{\chi=0}^{x} (\chi - x) I_x(\chi)\right)\left(\sum_{\nu=0}^{y} (\nu - y) I_y(\nu)\right) = J_x(x)\, J_y(y)$$

Određujemo odziv na $I_x = I_y$

$$I_1(x) = \{\ldots 0, 0, 0, 0, 1, 1, 1, 0, 0, 0, \ldots\}$$

$$\sum_{l=0}^{x} (x-n) I_1(l) = \{\ldots 0, 0, 0, 0, 0, -1, -3, -6, -9, \ldots\}$$

$$J_x(x) = \begin{cases} 3(1-x), & x \geq 2 \\ -1, & x = 1 \\ 0, & \text{inače} \end{cases}$$

$$J_y(y) = \begin{cases} 3(1-y), & y \geq 2 \\ -1, & y = 1 \\ 0, & \text{inače} \end{cases}$$

$$J(x,y) = J_x(x) \cdot J_y(y)$$

$$J(x,y) = \begin{cases} 9(1-x)(1-y), & x \geq 2 \wedge y \geq 2 \\ -3(1-x), & x \geq 2 \wedge y = 1 \\ -3(1-y), & y \geq 2 \wedge x = 1 \\ 0, & \text{inače} \end{cases}$$

$$J(x,y) = \begin{bmatrix} & \vdots & \vdots & & & & \\ & 0 & 0 & \iddots & & & \\ & 0 & 6 & 18 & & & \\ & 0 & 3 & 9 & 12 & & \\ & 0 & 1 & 3 & 6 & 9 & \ldots \\ \ldots & \underline{0} & 0 & 0 & 0 & 0 & \ldots \\ & \vdots & & & & & \end{bmatrix}$$

2. MI 2009./2010.

① FFT je efikasni postupak za računanje DFT_N transformacije. Asimptotska složenost mu je $O(N \log_2 N)$, što je brže od direktnog računanja složenosti $O(N^2)$.

$$DFT_{2N}[x[n]] = \sum_{n=0}^{2N-1} x[n] W_{2N}^{nk} = \sum_{n=0}^{N-1} x[2n] W_{2N}^{2nk} + \sum_{n=0}^{N-1} x[2n+1] W_{2N}^{(2n+1)k}$$

$$= \sum_{n=0}^{N-1} x[2n] W_N^{nk} + W_{2N}^{k} \sum_{n=0}^{N-1} x[2n+1] W_N^{nk} = DFT_N[y[n]] + W_{2N}^{k} DFT_N[z[n]]$$

② DCT-II $\Rightarrow X[k] = \sqrt{\frac{2-\delta[k]}{N}} \sum_{n=0}^{N-1} x[n] \cos \frac{(2n+1)k\pi}{2N}$

$$C_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\sqrt{2}}\cos\frac{\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{5\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{7\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{2\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{6\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{10\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{14\pi}{8} \\ \frac{1}{\sqrt{2}}\cos\frac{3\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{9\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{15\pi}{8} & \frac{1}{\sqrt{2}}\cos\frac{21\pi}{8} \end{bmatrix} \approx \begin{bmatrix} 0,5 & 0,5 & 0,5 & 0,5 \\ 0,6533 & 0,2706 & -0,2706 & -0,6533 \\ 0,5 & -0,5 & -0,5 & 0,5 \\ 0,2706 & -0,6533 & 0,6533 & -0,2706 \end{bmatrix}$$

DCT-II i DFT su povezani parnim proširenjem tako da se [illegible] DFT

$x[n] = \{0,1,0,1\} \Rightarrow y[n] = \{0,0,0,1,0,0,0,1,0,1,0,0,0,1,0,0\}$

$X[k] = C_4 x[n] = \{1, \frac{1}{\sqrt{2}}(\cos\frac{3\pi}{8} + \cos\frac{7\pi}{8}), 0, \frac{1}{\sqrt{2}}(\cos\frac{9\pi}{8} + \cos\frac{21\pi}{8})\} =$

$= \{1, -0.3827, 0, -0.9239\}$

$Y[k] = \sum_{n=0}^{15} y[n] W_{16}^{nk} = W_{16}^{3k} + W_{16}^{7k} + W_{16}^{9k} + W_{16}^{13k} = W_{16}^{3k} + W_{16}^{7k} + W_{16}^{-7k} + W_{16}^{-3k}$

$= (W_{16}^{3k} + W_{16}^{-3k}) + (W_{16}^{7k} + W_{16}^{-7k}) = 2\cos(2\pi \frac{3k}{16}) + 2\cos(2\pi \frac{7k}{16})$

$2\, DCT\text{-}II_4[x[n]] = \sqrt{\frac{2-\delta[k]}{4}} DFT_{16}[y[n]]$

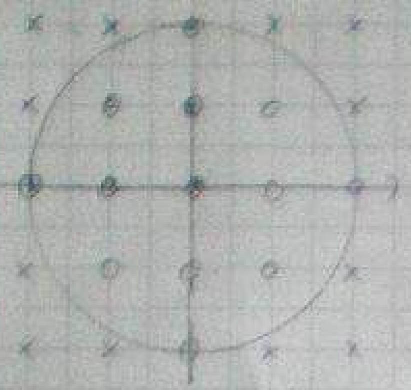
$k=0 \quad 2 \cdot 1 = 2 \Leftrightarrow \sqrt{\frac{2-1}{4}}(2+2) = \frac{1}{2} \cdot 4 = 2$

$k=1 \quad \frac{2}{\sqrt{2}}(\cos\frac{3\pi}{8} + \cos\frac{7\pi}{8}) \Leftrightarrow \sqrt{\frac{2}{4}}(2\cos\frac{3\pi}{8} + 2\cos\frac{7\pi}{8}) = \frac{2}{\sqrt{2}}(\cos\frac{3\pi}{8} + \cos\frac{7\pi}{8})$

$k=2 \quad 2 \cdot 0 = 0 \Leftrightarrow \sqrt{\frac{2}{4}}(2\cos\frac{6\pi}{8} + 2\cos\frac{14\pi}{8}) = \ldots \cdot 0 = 0$

$k=3 \quad \frac{2}{\sqrt{2}}[\cos\frac{9\pi}{8} + \cos\frac{21\pi}{8}] \Leftrightarrow \sqrt{\frac{2}{4}}(2\cos\frac{9\pi}{8} + 2\cos\frac{21\pi}{8})$

③ Otipkavanje slike je pretvorba iz prostorno-kontinuirane slike u prostorno-diskretnu odabiranjem samo onih vrijednosti prostorno-kontinuirane slike koji se nalaze na nekoj prostornoj rešetci.



(0,2)

(-1,1), (0,1), (1,1)

(-2,0), (-1,0), (0,0), (1,0), (2,0)

(-1,-1), (0,-1), (-1,1)

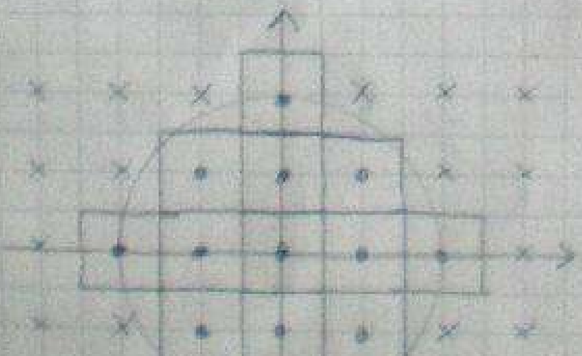
(0,-2)

dobiveni uzorci $\{(0,2), (0,1), (0,0), (0,-1), (0,-2), (1,1), (1,0), (1,-1), (2,0), (-2,0),$
$(-1,1), (-1,0), (-1,-1)\}$

VORONOI SUSJEDSTVO – neka je G skup točaka u $\mathbb{R}^n$. Voronoi susjedstvo svakog elementa $g \in G$ je skup

$$N_G(g) = \{v \in \mathbb{R}^n \mid \forall h \in G, \|v-g\| \leq \|v-h\|\}$$

Za $G = \mathbb{Z}^2$ Voronoi susjedstva svake točke su kvadrati koji su u skupu $\{(v_1, v_2) \in \mathbb{R}^2 \mid \max(|v_1 - g_1|, |v_2 - g_2|) < \frac{1}{2}\}$



Postoje točke unutar Voronoi susjedstva koje ne zadovoljavaju

④ FUNKCIJA UDALJENOSTI - Udaljenost na skupu X je preslikavanje $d: X \times X \to \mathbb{R}$ koje za svaka tri elementa $x, y, z \in X$ zadovoljava sljedeća svojstva

1) d je pozitivno-definitna tj. $d(x,y) \geq 0 \wedge d(x,y) = 0 \Leftrightarrow x = y$

2) d je simetrična tj. $d(x,y) = d(y,x)$

3) d zadovoljava nejednakost trokuta tj. $d(x,y) + d(y,z) \geq d(x,z)$

$A = (x_A, y_A)$, $B = (x_B, y_B)$ | $A_1 = (10,10)$, $B_1 = (20,20)$

$d_E(A,B) = \sqrt{(x_A - x_B)^2 + (y_A - y_B)^2} \Big|_1 = \sqrt{(10-20)^2 + (10-20)^2} = \sqrt{200} \approx 14{,}14$

$d_4(A,B) = |x_A - x_B| + |y_A - y_B| \Big|_1 = |10-20| + |10-20| = 20$

$d_8(A,B) = \max\{|x_A - x_B|, |y_A - y_B|\} \Big|_1 = \max\{10, 10\} = 10$

⑤ Neka je L neki 2D prostorno-diskretni sustav i neka su $I_1(x,y): \mathbb{Z}^2 \to \mathbb{R}$ i $I_2(x,y): \mathbb{Z}^2 \to \mathbb{R}$ dvije slike te neka su $a, b \in \mathbb{R}$ konst.

Sustav L je linearan ako za $\forall a, \forall b, \forall I_1, \forall I_2$ vrijedi

$$L\left[aI_1(x,y) + bI_2(x,y)\right] = aL\left[I_1(x,y)\right] + bL\left[I_2(x,y)\right]$$

$$J_1(x,y) = \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I_1(\chi, \psi) \quad ; \quad J_2(x,y) = \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I_2(\chi, \psi)$$

$$L\left[aI_1(x,y) + bI_2(x,y)\right] = \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y)\left[aI_1(\chi,\psi) + bI_2(\chi,\psi)\right]$$

$$= a \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I_1(\chi, \psi) + b \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I_2(\chi, \psi)$$

$$= aJ_1(x,y) + bJ_2(x,y) \Rightarrow \text{sustav je linearan}$$

$$I(x,y) = \begin{cases} 1, & 0 \leq x < 3,\ 0 \leq y < 3 \\ 0, & \text{inače} \end{cases} \Rightarrow I_x(x) = I_y(y) = \ldots 0, 0, 1, 1, 1, 0, 0, \ldots$$

$$J(x,y) = \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I(\chi, \psi) = \sum_{\chi=0}^{x} \sum_{\psi=0}^{y} (\chi - x)(\psi - y) I_x(\chi) I_y(\psi)$$

$$= \left(\sum_{\chi=0}^{x} (\chi - x) I_x(\chi)\right)\left(\sum_{\psi=0}^{y} (\psi - y) I_y(\psi)\right) = J_x(x)\, J_y(y)$$

Odredjemo odziv na $I_x = I_y$

$$I_x(\chi) = \{\ldots 0, 0, 0, 0, \underline{1}, 1, 1, 0, 0, 0, \ldots\}$$

$$\sum_{\chi=0}^{x} (\chi - x) I_x(\chi) = \{\ldots 0, 0, 0, 0, \underline{0}, -1, -3, -6, -9, \ldots\}$$

$$J_x(x) = \begin{cases} 3(1-x), & x \geq 2 \\ -1, & x = 1 \\ 0, & \text{inače} \end{cases}$$

$$J_y(y) = \begin{cases} 3(1-y), & y \geq 2 \\ -1, & y = 1 \\ 0, & \text{inače} \end{cases}$$

$$J(x,y) = J_x(x) \cdot J_y(y)$$

$$J(x,y) = \begin{cases} 9(1-x)(1-y), & x \geq 2 \wedge y \geq 2 \\ -3(1-x), & x \geq 2 \wedge y = 1 \\ -3(1-y), & y \geq 2 \wedge x = 1 \\ 0, & \text{inače} \end{cases}$$

$$J(x,y) = \begin{bmatrix} & \vdots & \vdots & & & & \\ & 0 & 9 & & \iddots & & \\ & 0 & 6 & 18 & & & \\ & 0 & 3 & 9 & 18 & & \\ & 0 & 1 & 3 & 6 & 9 & \ldots \\ \ldots & \underline{0} & 0 & 0 & 0 & 0 & \ldots \\ & \vdots & & & & & \end{bmatrix}$$