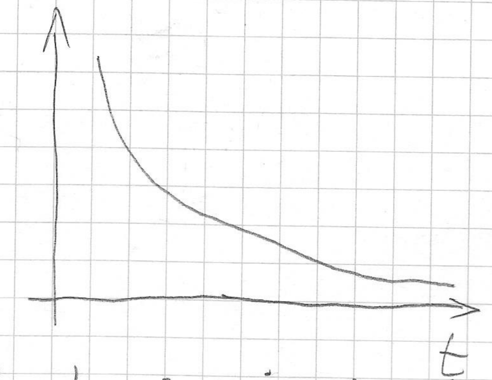
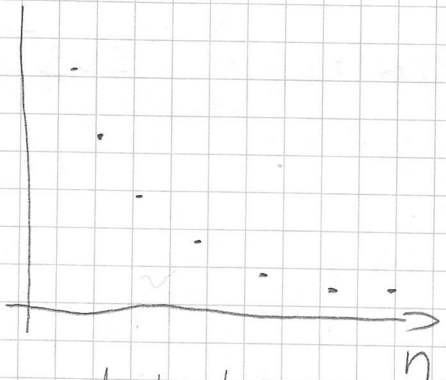
Signali i sustavi

prof. Ana Sović

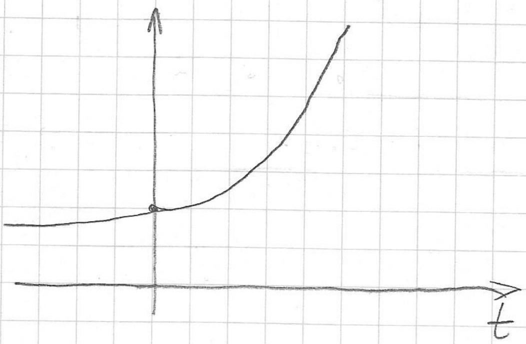


kontinuirani signal

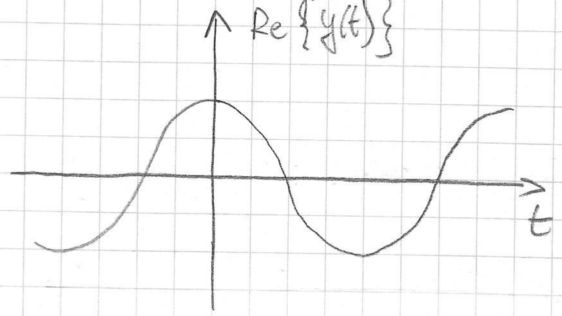


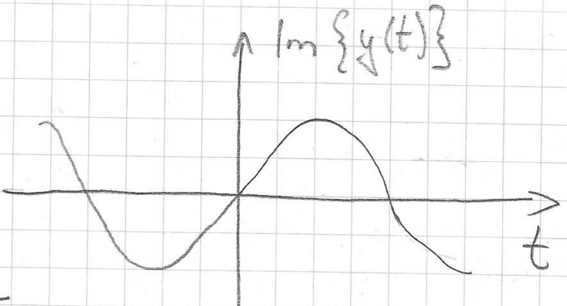
diskretni signal

$$y(t) = e^{t}$$



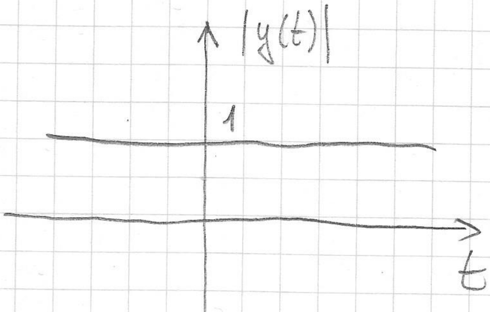
$$y(t) = e^{jt} = \cos t + j\sin t$$

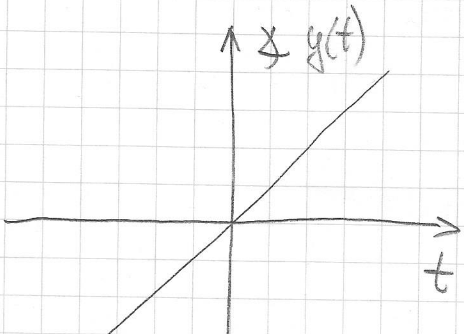


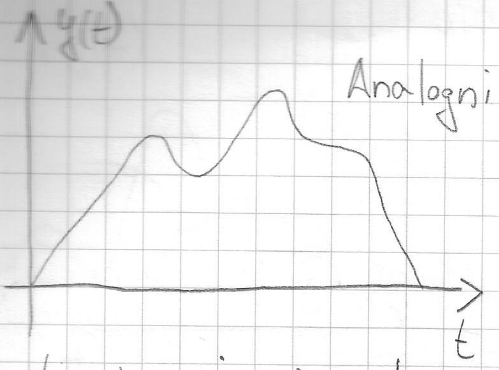


$$|y(t)| = \sqrt{Re^2 + Im^2} = \sqrt{\cos^2 t + \sin^2 t} = 1$$

$$\measuredangle y(t) = \operatorname{arctg} \frac{\sin t}{\cos t} = t$$

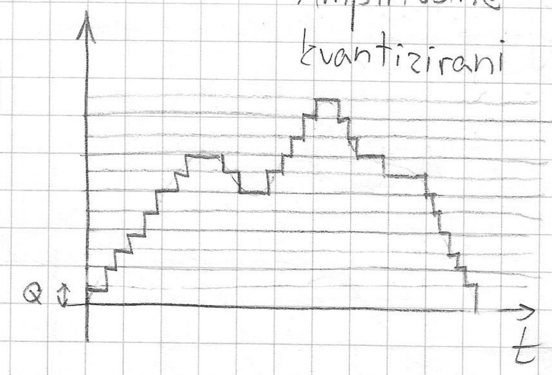






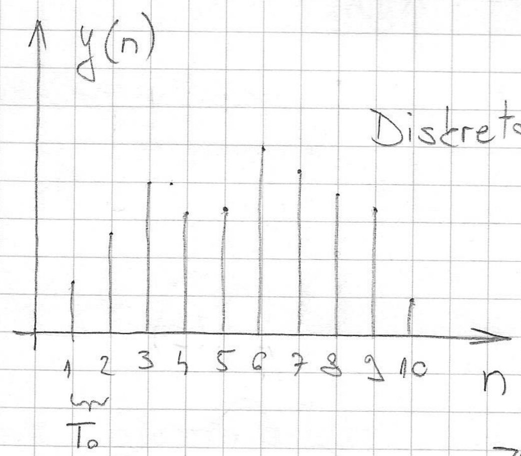
Analogni signal

Amplitudne kvantizirani signal



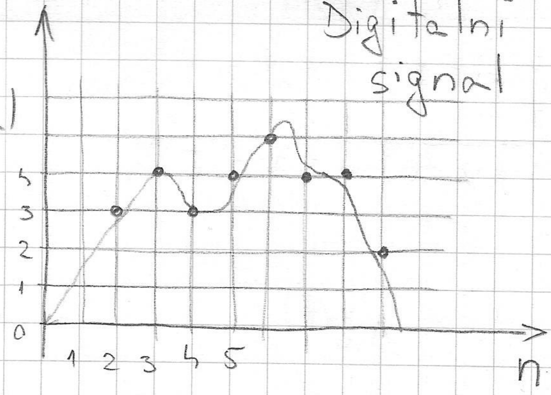
OTIPKAVANJE
OČITAVANJE
UZORKOVANJE

Kontinuirani signal $t \in \mathbb{R}$



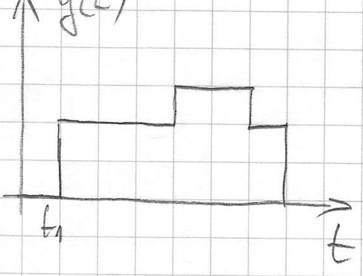
Distretan signal

Digitalni signal



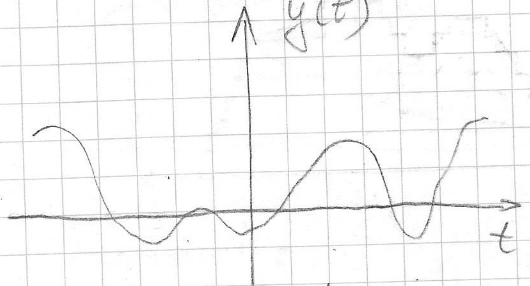
$n \in \mathbb{Z}$

$$y(n) = y(nT_0)$$

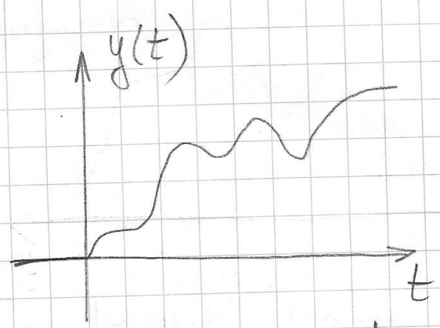


$y(t_1^-) \neq y(t_1^+)$ -> Diskontinuitet

Podjela signala prema trajanju

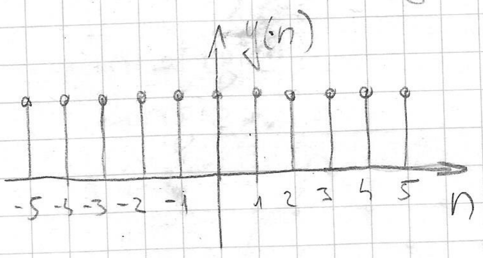


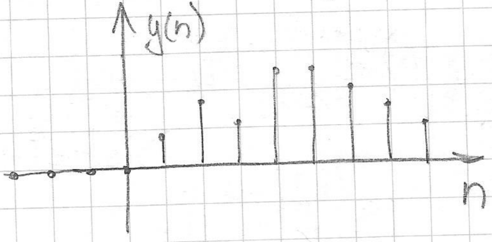
Svevremenski signal



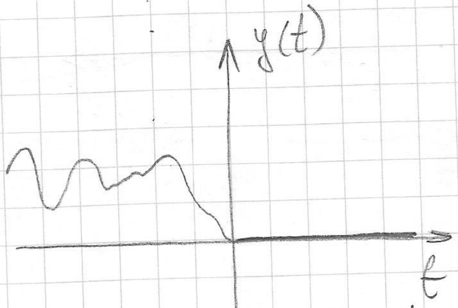
$y(t)=0,\ t<0$

Kauzalni signal



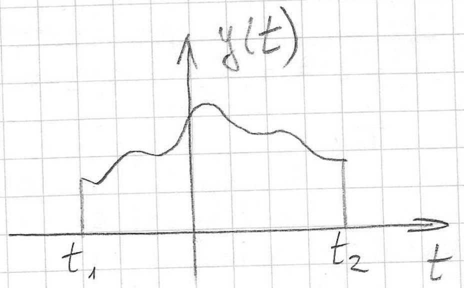


$y(n)=0,\ n<0$



$y(t)=0,\ t>0$

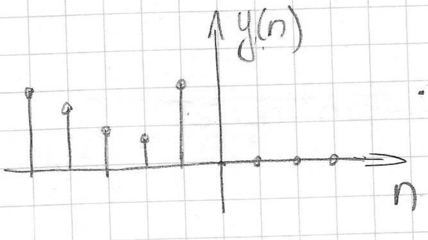
Antikauzalni signal



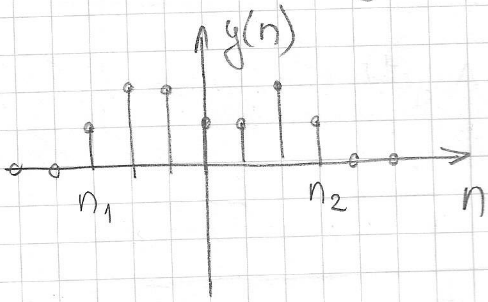
$t_1<0$

$t_2>0$

Nekauzalni signal

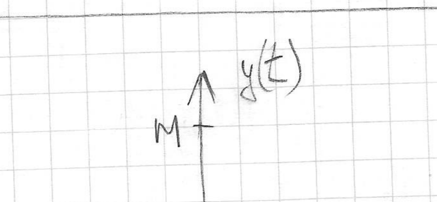


$y(n)=0,\ n>0$



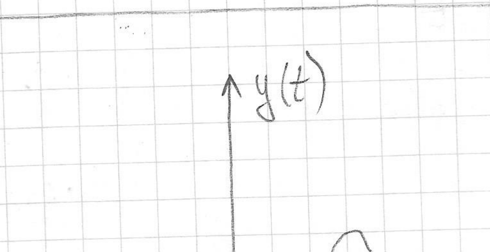
$n_1<0$

$n_2>0$

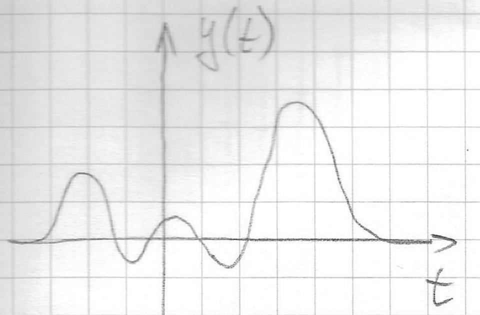


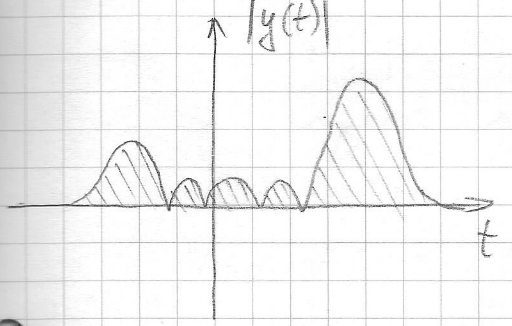
Omeđen

$|y(t)|<M<\infty$



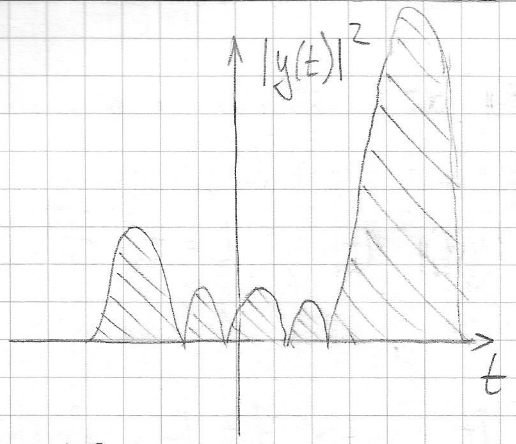
Konačnog trajanja





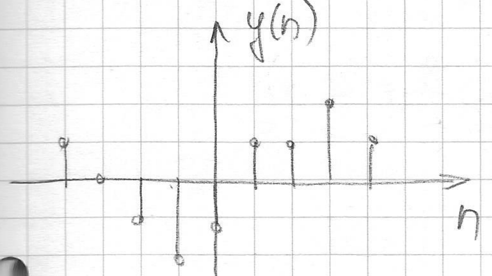
$$\int_{-\infty}^{+\infty} |y(t)|\, dt < \infty$$

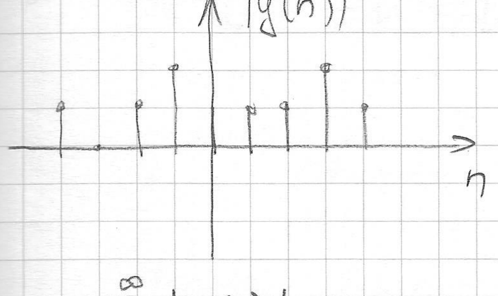
Apsolutno integrabilan



$$\int_{-\infty}^{+\infty} |y(t)|^2\, dt < \infty$$

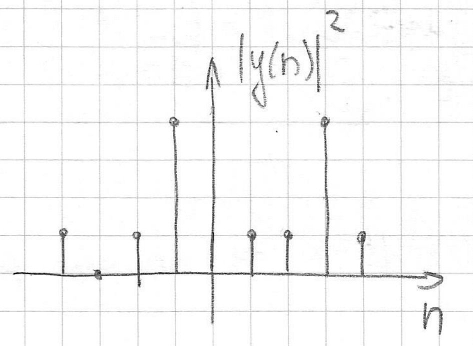
Kvadratno integrabilan





$$\sum_{n=-\infty}^{\infty} |y(n)| < \infty$$

Apsolutno zbrojiv



$$\sum_{n=-\infty}^{\infty} |y(n)|^2 < \infty$$

Kvadratno zbrojiv

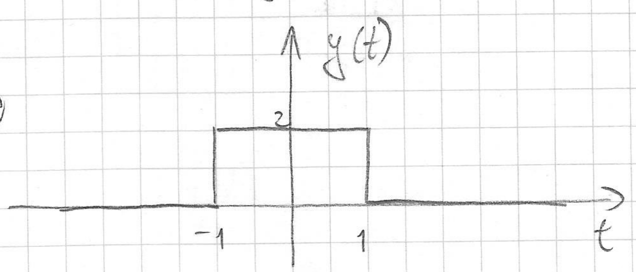
$$E_{[t_1,t_2]} = \int_{t_1}^{t_2} |y(t)|^2 \, dt$$

$$P_{[t_1,t_2]} = \frac{1}{L} \int_{t_1}^{t_2} |y(t)|^2 \, dt \quad , \quad L = t_2 - t_1$$

$$E = \int_{-\infty}^{\infty} |y(t)|^2 \, dt \quad \rightarrow \text{ totalna energija}$$

$$P = \lim_{L \to \infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |y(t)|^2 \, dt \quad \rightarrow \text{ totalna srednja snaga}$$

Z

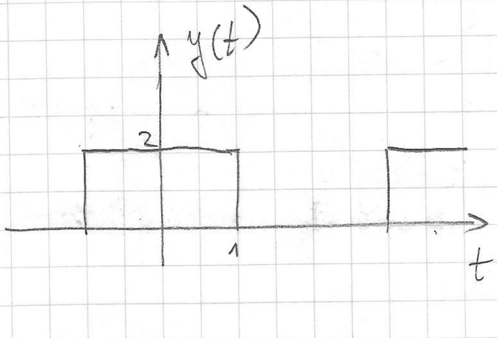


$$E = \int_{-1}^{1} |y(t)|^2 \, dt = \int_{-1}^{1} |2|^2 \, dt = 4 \int_{-1}^{1} dt = 4 \, t \Big|_{-1}^{1} = 4(1-(-1)) = 8$$

$$P = \lim_{L \to \infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} |y(t)|^2 \, dt = \lim_{L \to \infty} \frac{1}{L} \int_{-1}^{1} |2|^2 \, dt = \lim_{L \to \infty} \frac{8}{L} = 0$$

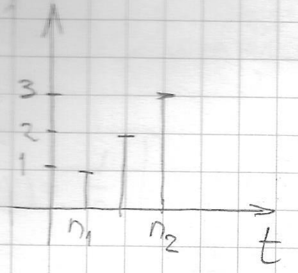
$$P = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} |y(t)|^2 \, dt \quad \rightarrow \text{ snaga periodičnog signala perioda } T_0$$

Z



$$E = \infty$$

$$P = \frac{1}{4} \int_{-1}^{1} 2^2 \, dt = \frac{8}{4} = 2$$



$$E = \sum_{n_1}^{n_2} |y(n)|^2 = 1^2 + 2^2 + 3^2 = 14$$

$$\forall n \in \mathbb{Z}, \quad E_\infty = \sum_{n=-\infty}^{\infty} |y(n)|^2$$

Srednja snaga periodičnog niza perioda N:

$$\forall n \in \mathbb{Z}, \quad P_{\tilde{y}} = \frac{1}{N} \sum_{n=b}^{b+N-1} |\tilde{y}(n)|^2 = \frac{1}{N} \sum_{n=\langle N \rangle} |\tilde{y}(n)|^2$$

(2) $y(n) = \begin{cases} \left(\frac{1}{2}\right)^n, & n \geq 0 \\ 3^n, & n < 0 \end{cases}$ $\quad E_\infty = ?$ $\quad \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \quad \left[\,|q| < 1\,\right]$

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n} = \frac{1}{1 - \left(\frac{1}{2}\right)^2} = \frac{4}{3}$$

$$\sum_{n=-\infty}^{-1} 3^{2n} = \left| k = -n \right| = \sum_{k=1}^{\infty} 3^{-2k} = \sum_{k=1}^{\infty} \left(3^{-2}\right)^k = \sum_{k=0}^{\infty} \left[3^{-2}\right]^k - \underbrace{\left[3^{-2}\right]^0}_{1} =$$

$$= \frac{1}{1 - 3^{-2}} - 1 = \frac{9}{8} - 1 = \frac{1}{8}$$

$$E_\infty = \frac{4}{3} + \frac{1}{8} = \frac{35}{24}$$

Uzlazna i silazna diferencija vremenski diskretnih signala

uzlazna diferencija: $\Delta u(n) = u(n+1) - u(n) \quad \forall n \in \mathbb{Z}$

silazna diferencija $\nabla u(n) = u(n) - u(n-1) \quad \forall n \in \mathbb{Z}$

Akumulacija vremenski diskretnih signala

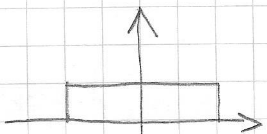
$$y(n)=\sum_{m=-\infty}^{n} u(m)$$

$$y(n)=\underbrace{\sum_{m=-\infty}^{n_0-1} u(m)}_{y(n_0-1)} + \sum_{m=n_0}^{n} u(m) = y(n_0-1) + \sum_{m=n_0}^{n} u(m)$$

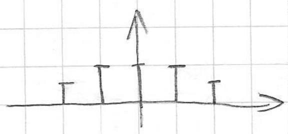
Konvolucija:

$$\forall t \in R,\quad z(t)=(x*y)(t)=\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$u(t)=u(-t)$

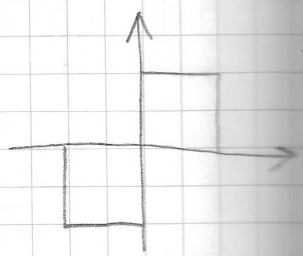


$v(n)=v(-n)$



Parne funkcije

$u(t)=-u(-t)$



$v(n)=-v(-n)$

Neparne f-je

$$+\left/\begin{array}{l} u(t)=u_p(t)+u_n(t) \\ u(-t)=u_p(-t)+u_n(-t) \end{array}\right.$$

$$=u_p(t)-u_n(t)$$

$$u(t)+u(-t)=2u_p(t) \qquad u(t)-u(-t)=2u_n(t)$$

$$u_p(t)=\frac{u(t)+u(-t)}{2} \qquad u_n(t)=\frac{u(t)-u(-t)}{2}$$

② $u(n) = n \cdot \sin\left(\frac{2\pi}{3}n\right) + n^3$

$$u_p(n) = \frac{1}{2}\left[n\sin\left(\frac{2\pi}{3}n\right) + n^3 - n\sin\left(-\frac{2\pi}{3}n\right) - n^3\right] =$$

$$= \frac{1}{2}\left[2n\sin\left(\frac{2\pi}{3}n\right)\right] = n\sin\left(\frac{2\pi}{3}n\right)$$

$$u_n(n) = \frac{1}{2}\left[n\sin\left(\frac{2\pi}{3}n\right) + n^3 + n\sin\left(-\frac{2\pi}{3}n\right) + n^3\right] =$$

$$= \frac{1}{2}\left[2n^3\right] = n^3$$

Konjugirano - kompleksni brojevi

$2 + 3j \Rightarrow 2 - 3j$

$3e^{j5} = 3\cos 5 + j3\sin 5 \Rightarrow 3\cos 5 - j3\sin 5 = 3e^{-j5}$

$u(t) = u^*(-t)$ $\qquad$ $u(t) = -u^*(-t)$

$v(n) = v^*(-n)$ $\qquad$ $u(n) = -u^*(-n)$

Konjugirano simetrični $\qquad$ Konjugirano nesimetrični

$$u_{KS}(t) = \frac{1}{2}\left[u(t) + u^*(-t)\right]$$

$$u_{KN}(t) = \frac{1}{2}\left[u(t) - u^*(-t)\right]$$

(2) $f(t) = e^{j\left(t+\frac{\pi}{3}\right)}$

$$f_{KS}(t) = \frac{1}{2}\left[e^{j\left(t+\frac{\pi}{3}\right)} + e^{-j\left(-t+\frac{\pi}{3}\right)}\right] =$$

$$= \frac{1}{2}\left[e^{j\left(t+\frac{\pi}{3}\right)} + e^{j\left(t-\frac{\pi}{3}\right)}\right] = e^{jt}\,\frac{e^{j\frac{\pi}{3}} + e^{-j\frac{\pi}{3}}}{2} =$$

$$= e^{jt}\cdot\frac{1}{2}\left[\cos\left(\frac{\pi}{3}\right) + j\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{3}\right) - j\sin\left(\frac{\pi}{3}\right)\right] =$$

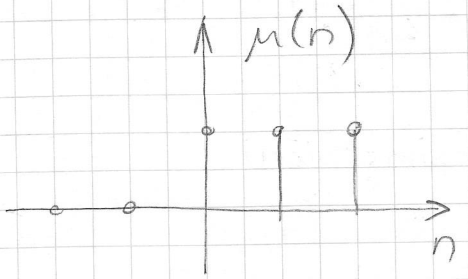
$$= e^{jt}\cos\left(\frac{\pi}{3}\right) \qquad f_{KA}(t) = e^{jt}\sin\left(\frac{\pi}{3}\right)$$

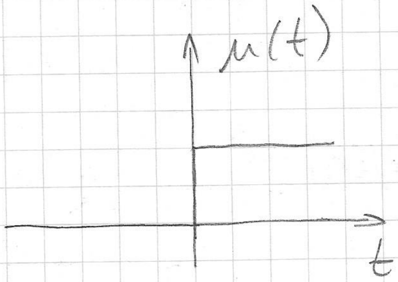
Konjugirana simetričnost kompleksnih brojeva

$$f_{KS}(-t) = e^{-jt}\cos\frac{\pi}{3} = \cos(t)\cos\left(\frac{\pi}{3}\right) - j\sin(t)\cos\left(\frac{\pi}{3}\right)$$

$$f_{KS}(t) = \qquad = \cos(t)\cos\left(\frac{\pi}{3}\right) + j\sin(t)\cos\left(\frac{\pi}{3}\right)$$

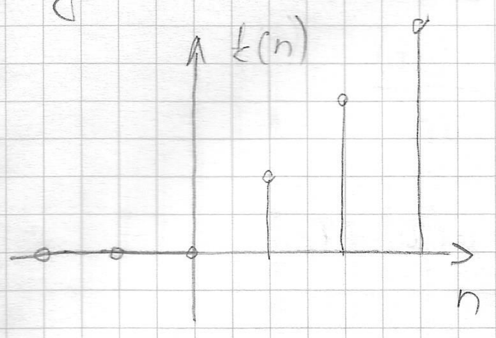
Step funkcija

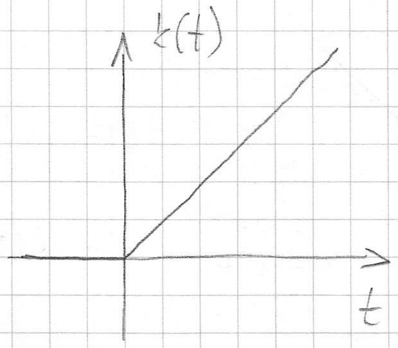




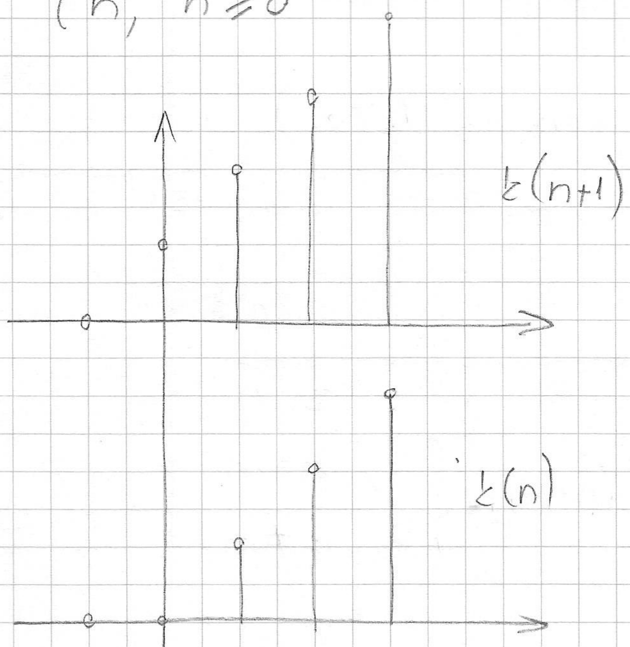
$$\mu(n) = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases} \qquad \mu(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

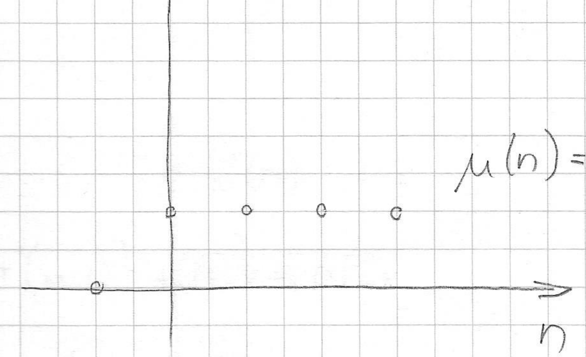
Jedinična kosina



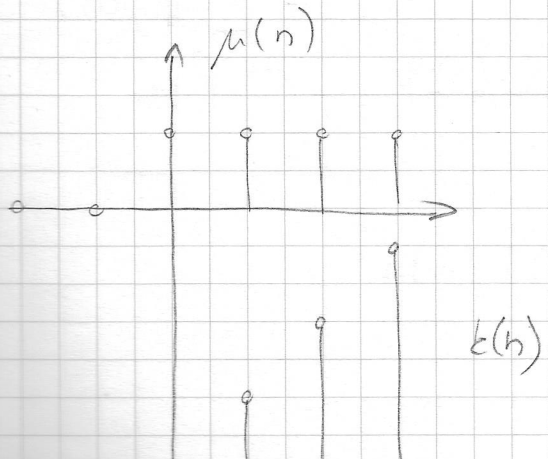


$$k(n) = \begin{cases} 0, & n < 0 \\ n, & n \geq 0 \end{cases} \qquad k(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$$





$\mu(n) = k(n+1) - k(n)$ - diferencija

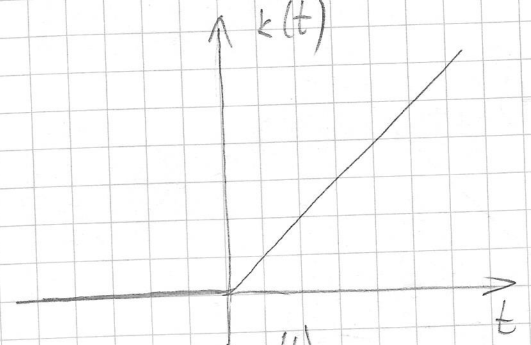


$n = 1 \quad k(n) = 1$

$n = 2 \quad k(n) = 2$

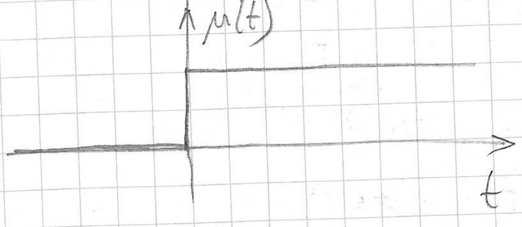
$$\boxed{k(n) = \sum_{m=-\infty}^{n-1} \mu(m)}$$

Akumulacija

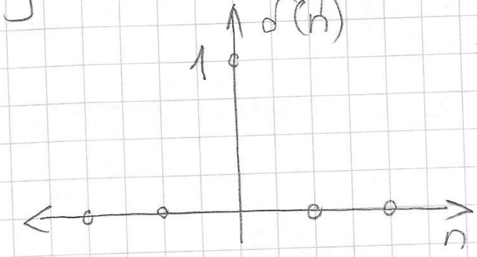


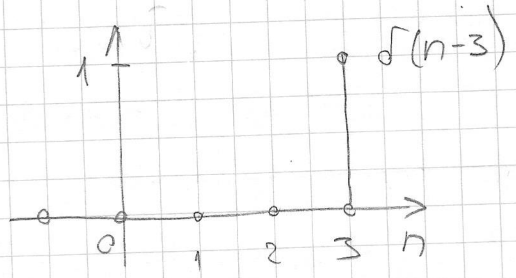
$$\mu(t) = \frac{d\,k(t)}{dt}$$

$$k(t) = \int_{-\infty}^{t} \mu(\tau)\, d\tau$$



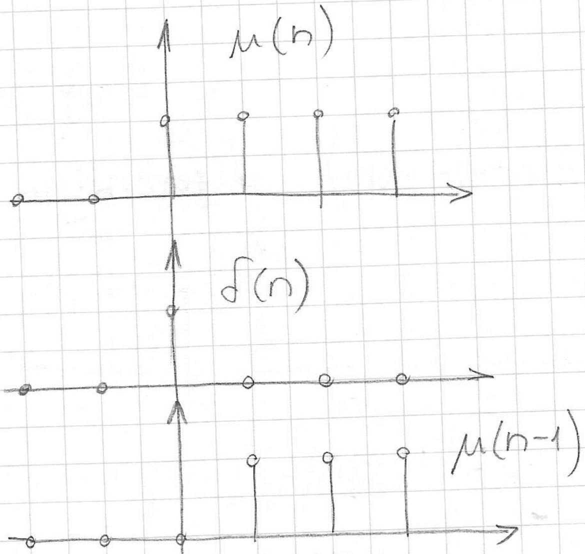
Jedinični impuls (kroneckerova delta)





$$\delta(n) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

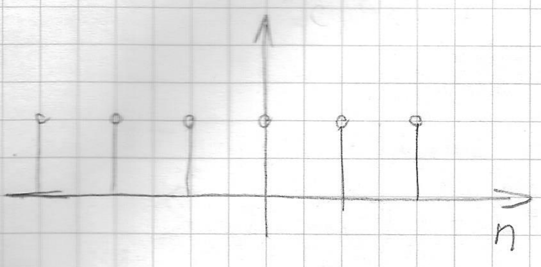
$$\delta(n-m) = \begin{cases} 1, & n = m \\ 0, & n \neq m \end{cases}$$



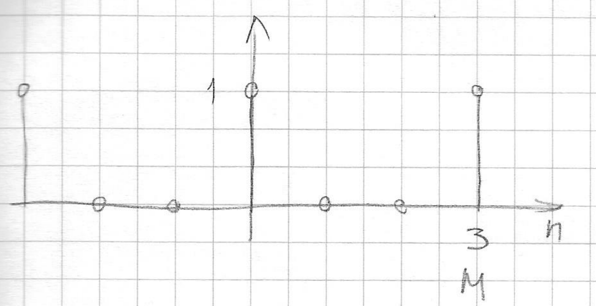
$$\mu(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \ldots$$

$$= \sum_{m=-\infty}^{n} \delta(m)$$

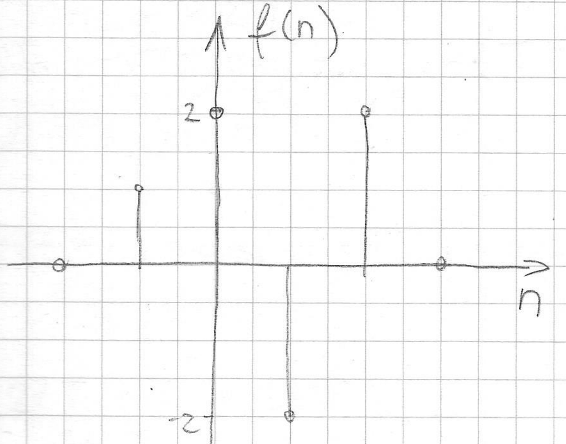
$$\delta(n) = \mu(n) - \mu(n-1)$$

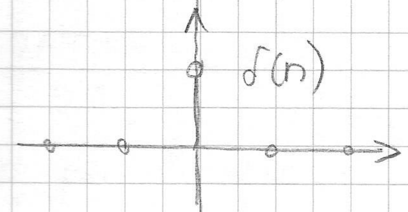


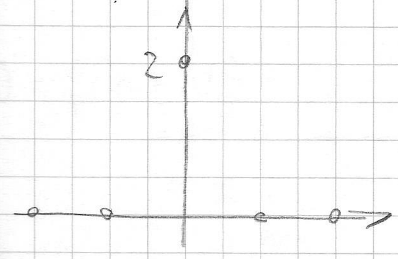
$$\mathrm{comb}(n) = 1 = $$
$$= \sum_{m=-\infty}^{\infty} \delta(n-m)$$



$$\mathrm{comb}_M(n) = \sum_{m=-\infty}^{\infty} \delta(n-mM)$$







$$f(n)\cdot\delta(n) = f(0)\cdot\delta(n)$$

$$f(n)\,\delta(n-m) = f(m)\,\delta(n-m)$$

Auditorne

$\sin(\omega t)$ $\quad 2k\pi,\ k\in\mathbb{Z}$

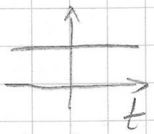
$\cos(\omega t)$ $\quad 2k\pi$

$tg(\omega t)$ $\quad k\pi$

$ctg(\omega t)$ $\quad k\pi$

Periodičan signal:

$$f(t)=f(t+T),\quad T\in\mathbb{R}$$



A $\quad T$

① a)

$$f(t)=\cos^2(t)=\frac{1+\cos(2t)}{2}=\frac{1}{2}+\frac{1}{2}\cos(2t)$$

$$\cos(2t)=\cos(2(t+T))=\cos(2t+2T)=\cos(2t)\underbrace{\cos(2T)}_{=1}-\underbrace{\sin(2t)\sin(2T)}_{=0}$$

$\cos(2T)=1$

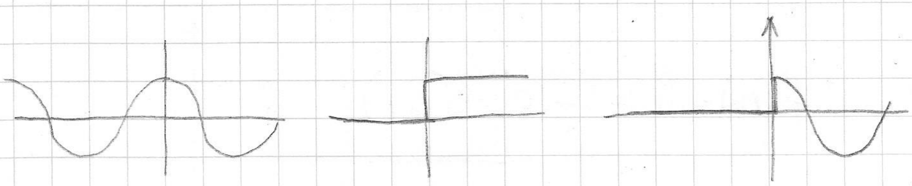
$\sin(2T)=0$

$2T=2k\pi$

$T=k\pi$

$$\boxed{T_0=\pi}$$

b) $f(t)=\cos(2\pi t)\cdot\mu(t)$



Nije periodičan

c) $e^{j\pi t} = \cos(\pi t) + j\sin(\pi t)$

$$e^{j\pi(t+T)} = e^{j\pi t}\underbrace{e^{j\pi T}}_{=1}$$

$$e^{j\pi T} = 1$$

$$\cos(\pi T) + \underbrace{j\sin(\pi T)}_{=0} = 1$$

$$\cos(\pi T) = 1$$

$$\pi T = 2k\pi$$

$$T = 2k$$

$$\boxed{T_0 = 2}$$

d) $f(t) = \cos(t^2)$

$$f(t+T) = \cos((t+T)^2) = \cos(t^2 + \underbrace{2tT + T^2}_{=2k\pi})$$

$$2tT + T^2 = 2k\pi$$

$$\underset{T \in \mathbb{R}}{\downarrow}$$

$$\frac{2tT + T^2}{2\pi} = k$$

Nije periodičan

$$f(n) = f(n+N)$$

$$\underset{N \in \mathbb{Z}}{\downarrow}$$

2 a) $\cos(\pi n + \frac{\pi}{4}) = \cos(\pi(n+N) + \frac{\pi}{4}) = \cos(\pi n + \pi N + \frac{\pi}{4})$

$$\pi N = 2k\pi$$

$$N = 2k$$

$$N_0 = 2$$

$$\cos\left(n+\frac{\pi}{4}\right)=\cos\left(n+N+\frac{\pi}{4}\right)$$

$$N = 2k\pi$$

Nije periodičan jer N mora biti cijeli broj

b) $$\cos\left(\frac{\pi n^2}{8}\right)=\cos\left(\frac{\pi(n+N)^2}{8}\right)=\cos\left(\frac{\pi n^2}{8}+\frac{\pi n N}{4}+\frac{\pi N^2}{8}\right)$$

$$\frac{\cancel{\pi} n N}{4}+\frac{\cancel{\pi} N^2}{8}=2k\cancel{\pi}$$

$$\frac{N}{8}(2n+N)=2k$$

$$N=8$$

$$\cos(2t)+\sin(3t)$$

$$2T=2k\pi \qquad 3T=2k\pi$$

$$T=k\pi \qquad T=\frac{2k\pi}{3}$$

$$T_0=\pi=\frac{3\pi}{3} \qquad T_0=\frac{2\pi}{3}$$

$$T_0=\frac{6\pi}{3}=2\pi$$

Najmanji zajednički višekratnik

$$E=\int_{-\infty}^{\infty}|x(t)|^2\,dt$$

③ b) $x(t)=t\mu(t)$

$$E=\int_{-\infty}^{\infty}|t\mu(t)|^2\,dt=\int_0^{\infty}t^2\,dt=\frac{t^3}{3}\Big|_0^{\infty}=\infty$$

$$E=\sum_{n=-\infty}^{\infty}|x(n)|^2$$

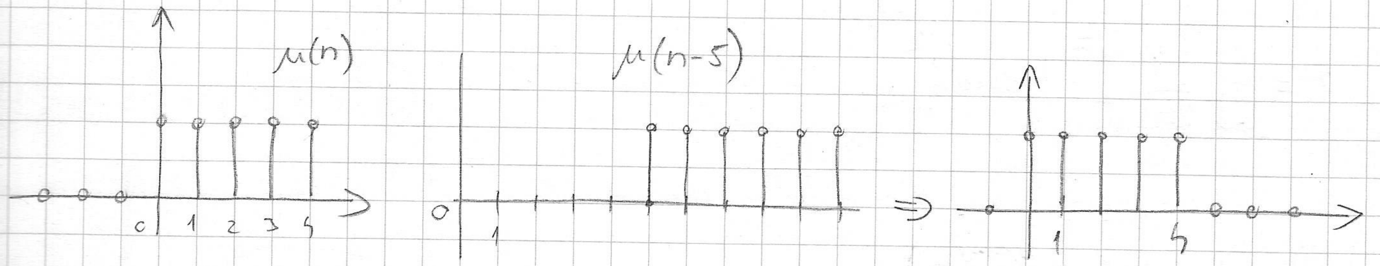
(1) a) $x(n)=(-0{,}5)^n\mu(n)$

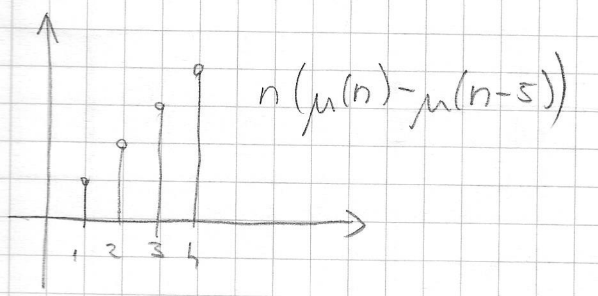
$$E=\sum_{0}^{\infty}\left|(-0{,}5)^n\right|^2=\sum_{n=0}^{\infty}(-0{,}5)^{2n}=\sum_{n=0}^{\infty}\left(\frac{1}{4}\right)^n$$

$$\sum_{n=0}^{\infty}q^n=\frac{1}{1-q}\qquad\sum_{n=0}^{N}q^n=\frac{1-q^{N+1}}{1-q}$$

$$E=\frac{1}{1-\frac{1}{4}}=\frac{4}{3}$$

b) $x(n)=n(\mu(n)-\mu(n-5))$





$$E=\sum_{n=0}^{4}|x(n)|^2=1^2+2^2+3^2+4^2=30$$

$$P=\lim_{L\to\infty}\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}|x(t)|^2\,dt$$

⑤ a) $x(t) = e^{j2\pi t} \sin\left(t + \frac{\pi}{3}\right)$

$$P = \lim_{L\to\infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \left| e^{j2\pi t} \sin\left(t+\frac{\pi}{3}\right)\right|^2 dt$$

$$\left|e^{j2\pi t}\right|^2 \cdot \left|\sin\left(t+\frac{\pi}{3}\right)\right|^2$$

$$\left|e^{j2\pi t}\right| = \left|\cos(2\pi t) + j\sin(2\pi t)\right| = \sqrt{\cos^2(2\pi t) + \sin^2(2\pi t)} = 1$$

$$P = \lim_{L\to\infty} \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} \sin^2\left(t+\frac{\pi}{3}\right) dt = \lim_{L\to\infty} \frac{1}{L}\left[\frac{L}{2} + \frac{1}{4}\sin L\right] =$$

$$= \lim_{L\to\infty}\left[\frac{1}{2} + \frac{\sin L}{4L}\right] = \frac{1}{2} + 0 = \frac{1}{2}$$

$$P = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=-L}^{L} |x(n)|^2$$

⑥ a) $x(n) = \mu(n)$

$$P = \lim_{L\to\infty} \frac{1}{2L+1} \sum_{n=0}^{L} 1 = \lim_{L\to\infty} \frac{L+1}{2L+1} = \frac{1}{2}$$

Nastavak predavanja

⑦ $f(n) = 2\cos\left(\frac{2\pi}{3} n\right)$

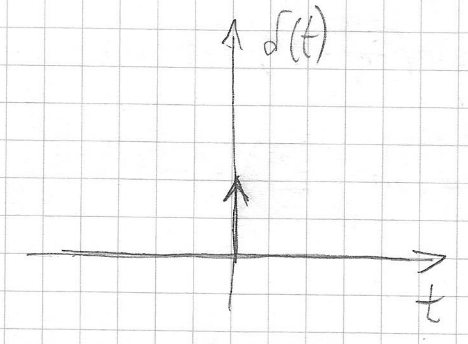
$$f(n)\cdot\delta(\underbrace{n-3}_{n=3}) = 2\cos\left(\frac{2\pi}{3}\cdot 3\right)\cdot\delta(n-3) = 2\cdot\delta(n-3)$$

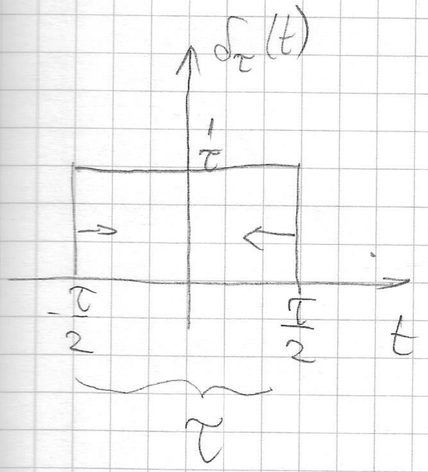
$$\sum_{n=-\infty}^{\infty} f(n)\delta(n-3) = 2$$

Diracova delta funkcija

$$\delta(t) = 0 \quad , t \neq 0$$

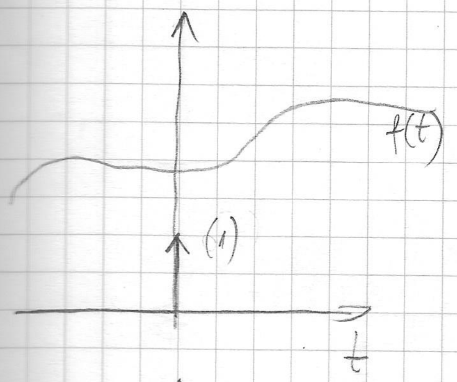
$$\int_{-\infty}^{\infty} \delta(t)\,dt = 1$$

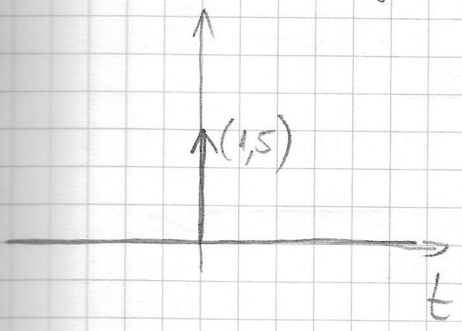




$$\delta(t) = \lim_{\tau \to 0} \delta_\tau(t)$$

$$f(t) \cdot \delta(t) = f(0) \cdot \delta(t)$$





$$f(t)\,\delta(\underbrace{t-t_0}_{t=t_0}) = f(t_0)\,\delta(t-t_0)$$

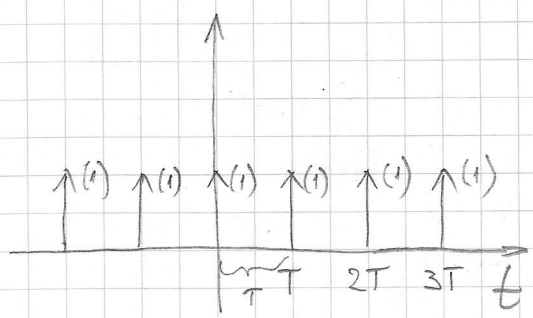
$$\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t) = f(0)$$

② $f(t)=\sin\left(\frac{3\pi}{4}t\right)$

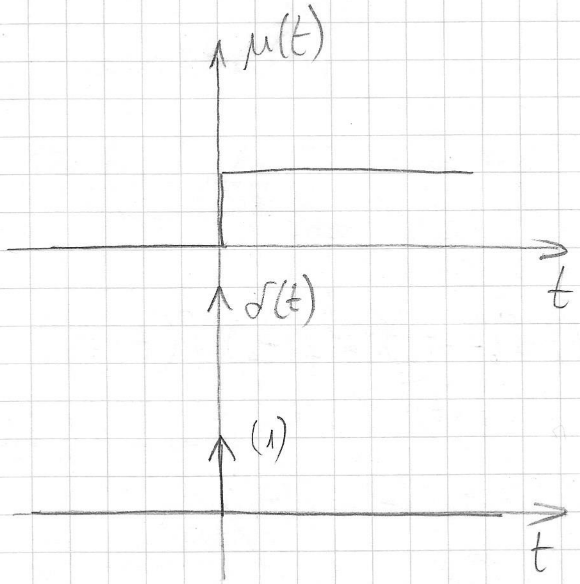
$$f(t)\delta(t-2)=\sin\left(\frac{3\pi}{4}\cdot 2\right)\delta(t-2)=\sin\left(\frac{3\pi}{2}\right)\delta(t-2)=-\delta(t-2)$$

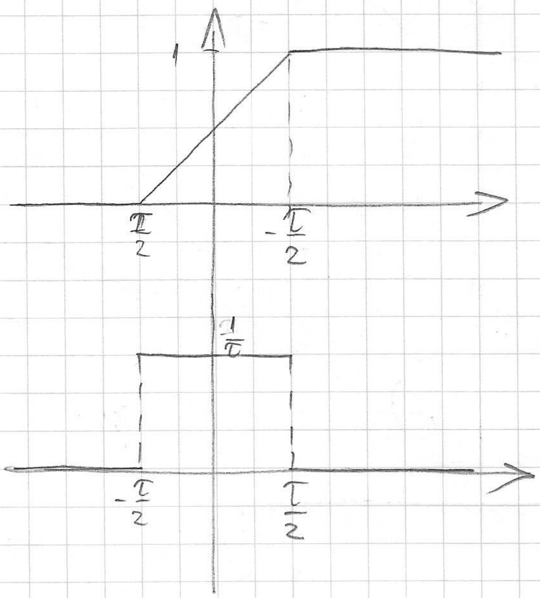
$$\int_{-\infty}^{\infty} f(t)\delta(t-2)\,dt=\sin\left(\frac{3\pi}{2}\right)=-1$$



$$\mathrm{comb}_T(t)=\sum_{m=-\infty}^{\infty}\delta(t-mT)$$

$$f(t)\sum_{m=-\infty}^{\infty}\delta(t-mT)=\sum_{m=-\infty}^{\infty}f(mT)\delta(t-mT)$$



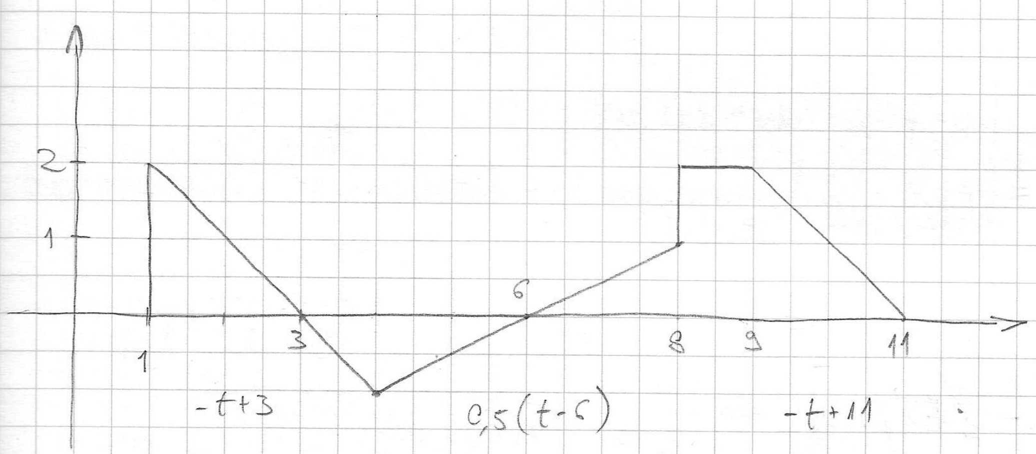


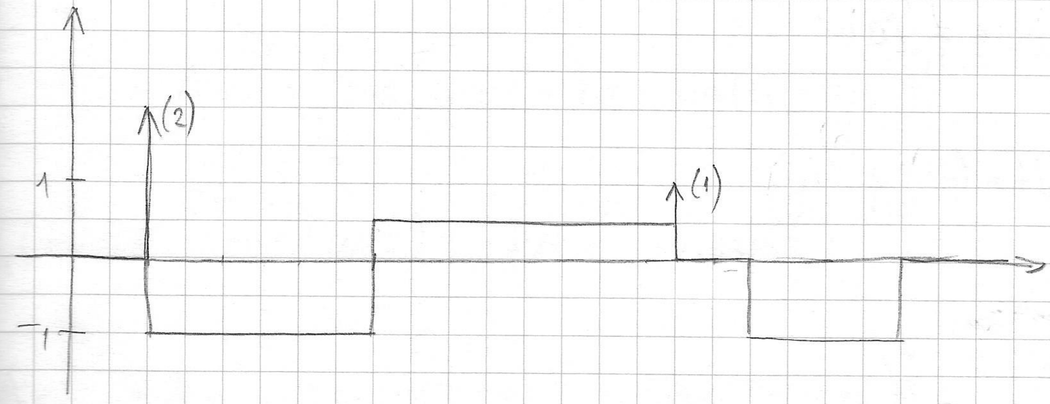
$$\delta(t)=\frac{d}{dt}\mu(t)$$

$$\mu(t)=\int_{-\infty}^{t}\delta(\lambda)\,d\lambda$$

$$\int_{-\infty}^{\infty} f(t)\mu'(t)\,dt = \begin{vmatrix} u=f(t) & dv=\mu'(t)dt \\ du=f'(t)dt & v=\mu(t) \end{vmatrix} = f(t)\mu(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \mu(t) f'(t)\,dt =$$

$$= f(\infty) - \int_0^{\infty} f'(t)\,dt = f(\infty) - f(t)\Big|_0^{\infty} = f(\infty) - f(\infty) + f(0) = f(0)$$





$$g(t) = (-t+3)\left[\mu(t-1) - \mu(t-4)\right] +$$
$$+ 0{,}5(t-6)\left[\mu(t-4) - \mu(t-8)\right] +$$
$$+ 2\left[\mu(t-8) - \mu(t-9)\right] +$$
$$+ (-t+11)\left[\mu(t-9) - \mu(t-11)\right]$$

$$g'(t) = -1\left[\mu(t-1) - \mu(t-4)\right] + (-t+3)\left[\delta(t-1) - \delta(t-4)\right] +$$
$$+ 0{,}5\left[\mu(t-4) - \mu(t+8)\right] + 0{,}5(t-6)\left[\delta(t-4) - \delta(t-8)\right] +$$
$$+ 2\left[\delta(t-8) - \delta(t-9)\right] - 1\cdot\left[\mu(t-9) - \mu(t-11)\right) +$$
$$+ (-t+11)\left[\delta(t-9) - \delta(t-11)\right] = \rightarrow \text{druga strana}$$

$$= (-t+3)\,\delta(t-1) - (-t+3)\,\delta(t-4)$$

$t=1 \qquad t=4$

$-1+3 \qquad -4+3$

$$2\,\delta(t-1) - \cancel{(-\delta(t-4))} - \cancel{1\delta(t-4)} - \underbrace{1\delta(t-8) + 2\delta(t-8)}_{1\cdot\delta(t-8)} - 2\cancel{\delta(t-9)} +$$

$$+\ 2\cancel{\delta(t-9)}$$

$$\int_{-\infty}^{\infty} f(t)\,\delta'(t)\,dt = \left|\begin{matrix} u = f(t) & dv = \delta'(t)dt \\ du = f'(t)dt & v = \delta(t) \end{matrix}\right| = f(t)\,\delta(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(t)\,f'(t)\,dt =$$

$$= -f'(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta''(t)\,dt = f''(0)$$

$$\int_{-\infty}^{\infty} f(t)\,\delta^{(n)}(t)\,dt = (-1)^n f^{(n)}(0)$$

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)\,g(t-\tau)\,d\tau$$

$$f(t) * \delta(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(\underset{\tau = t}{t-\tau})\,d\tau = f(t)$$

$$f(t) * \delta(t-t_1) = \int_{-\infty}^{\infty} f(\tau)\,\delta(\underset{\tau = t-t_1}{t-t_1-\tau})\,d\tau = f(t-t_1)$$

$$\frac{d}{dt}\left(2t\cdot\mu(3t-5)\right)=2\mu(3t-5)+2t\left[\delta(3t-5)\cdot 3\right]$$

Kontinuirani sinusoidni signal

$$f(t)=A\cos(\omega_0 t+\theta)$$

Kontinuirani kompleksni eksponencijalni signal

$$f(t)=C\cdot e^{s_0 t}=$$

$$=\underbrace{A\cdot e^{j\theta}}\cdot\underbrace{e^{(\sigma_0+j\omega_0)t}}=Ae^{\sigma_0 t}\cdot e^{j(\omega_0 t+\theta)}=$$

$$=Ae^{\sigma_0 t}\cdot\left[\cos(\omega_0 t+\theta)+j\sin(\omega_0 t+\theta)\right]$$

$$f_1(t)=Ae^{\sigma_0 t}\cos(\omega_0 t)+jAe^{\sigma_0 t}\sin(\omega_0 t)$$

$$Ae^{s_0^* t}=Ae^{\sigma_0 t}\left[\cos(\omega_0 t+\theta)-j\sin(\omega_0 t+\theta)\right]$$

$$Ae^{s_0 t}+Ae^{s_0^* t}=2Ae^{\sigma_0 t}\cos(\omega_0 t+\theta)$$

$$s_0=\sigma_0+j\omega_0$$

$$f_a(t) = C e^{(\sigma_0 + j\omega_0)t}$$

$$f(n) = C e^{(\sigma_0 + j\omega_0)nT} = C e^{(\sigma_0 T + j\omega_0 T)n} =$$

$$= C\Big[\underbrace{e^{s_0 T} \cdot e^{j\omega_0 T}}_{=q}\Big]^n = C \cdot q^n$$

$$\omega_0 T = \Omega_0$$

$$q = \underbrace{e^{s_0 T}}_{|q|} \cdot e^{j\Omega_0} = |q|\, e^{j\Omega_0}$$

$$f(n) = C \cdot q^n = A e^{j\theta} q^n = A e^{j\theta} |q|^n e^{j\Omega_0 n} = A \cdot |q|^n \cdot e^{j(\Omega_0 n + \theta)} =$$

$$= A \cdot |q|^n \left[\cos(\Omega_0 n + \theta) + j \sin(\Omega_0 n + \theta)\right]$$

$|q| < 1 \Rightarrow$ Prigušenje

$|q| > 1 \Rightarrow$ raspirivanje

Periodičnost sinusnog niza

$$f(n) = f(n+N)$$

$$u(n) = \cos(\Omega_0 n + \theta) = \cos(\Omega_0(n+N) + \theta) =$$

$$= \cos(\Omega_0 n + \theta + \Omega_0 N) =$$

$$= \cos(\Omega_0 n + \theta)\underbrace{\cos(\Omega_0 N)}_{1} - \underbrace{\sin(\Omega_0 n + \theta)\sin(\Omega_0 N)}_{0}$$

$$\cos \Omega_0 N = 0$$

$$\Omega_0 N = 2k\pi \Rightarrow N = \frac{2k\pi}{\Omega_0} = \frac{2k\pi}{\ldots} = \frac{k}{\ldots}$$

$$\cos\left(\frac{2\pi}{9}n\right) = \cos\left(\frac{2\pi}{9}n + \frac{2\pi}{9}N\right)$$

$$\frac{2\pi}{9}N = 2k\pi$$

$$N = 9k$$

$$N_0 = 9$$

$$\cos\left(\frac{5}{9}n\right)$$

$$\frac{5}{9}N = 2k\pi$$

$$N = \frac{2k\pi \cdot 9}{5}, \quad N \notin \mathbb{Z}$$

$$\cos(\Omega_0 n) = \cos\left((\Omega_0 + 2k\pi)n\right) = \cos(\Omega_0 n + 2k\pi n)$$

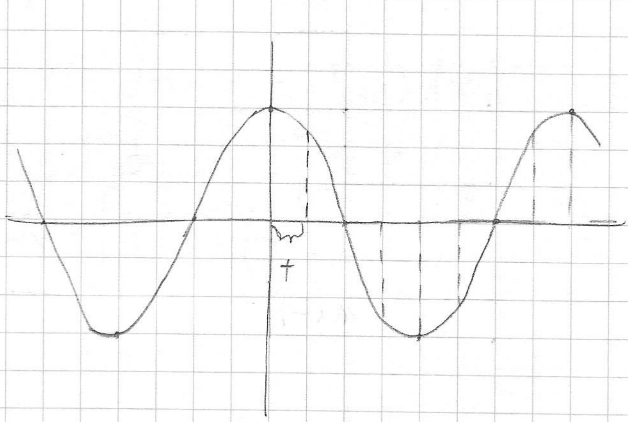
$$-\pi \leq \Omega_0 \leq \pi$$

$$-\pi \leq 2\pi F_0 \leq \pi$$

$$-\frac{1}{2} \leq F_0 \leq \frac{1}{2}$$

$$x_a(t)$$

$$x(n) = x_a(t)\Big|_{t=nT} = x_a(nT)$$



$$x_a(t) = \cos(2\pi f t + \theta)$$

$$x(n) = \cos(2\pi f nT + \theta) =$$

$$= \cos\left(\frac{2\pi f}{f_s} \cdot n + \theta\right) =$$

$$= \cos\left(\frac{2\pi\omega}{\omega_s} n + \theta\right) =$$

$$= \cos(\Omega n + \theta)$$

$$f_s = \frac{1}{T}$$

$$\omega = 2\pi f$$

$$\omega_s = 2\pi f_s$$

$$\frac{2\pi w}{w_s} \le \pi$$

$$2\pi w \le \pi w_s$$

$$\boxed{2w \le w_s}$$

$$2\cdot 2\pi f \le f_s \cdot 2\cdot \pi$$

$$\boxed{2f \le f_s}$$

Auditorne III. tjedan

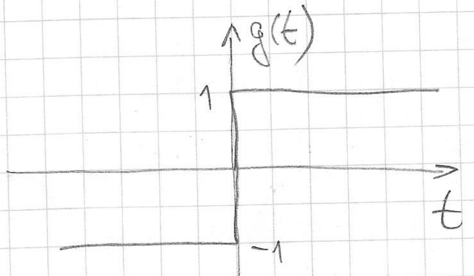
5 a) $$\int_0^\infty \delta(\underset{t=2}{t-2})\, t^2\, dt = \int_0^\infty 4\,\delta(t-2)\, dt = 4$$

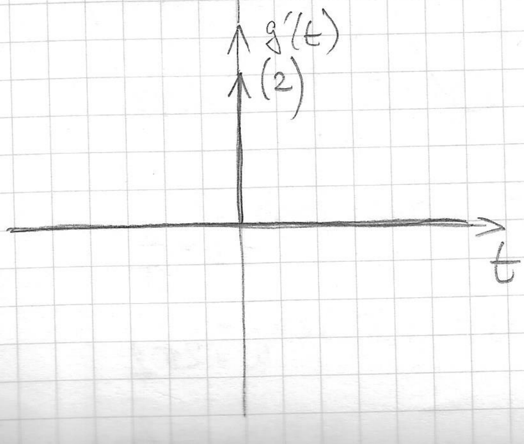
b) $$\int_{-\infty}^{\infty} \mu(t-1)\,\delta(\underset{t=0}{t})\cos t\, dt = \int_{-\infty}^{\infty} \underbrace{\overbrace{\mu(0-1)}^{0}\cdot \cos 0}_{0}\, dt = 0$$

6 $$g(t) = \begin{cases} 1, & t \geq 0 \\ -1, & t < 0 \end{cases}$$

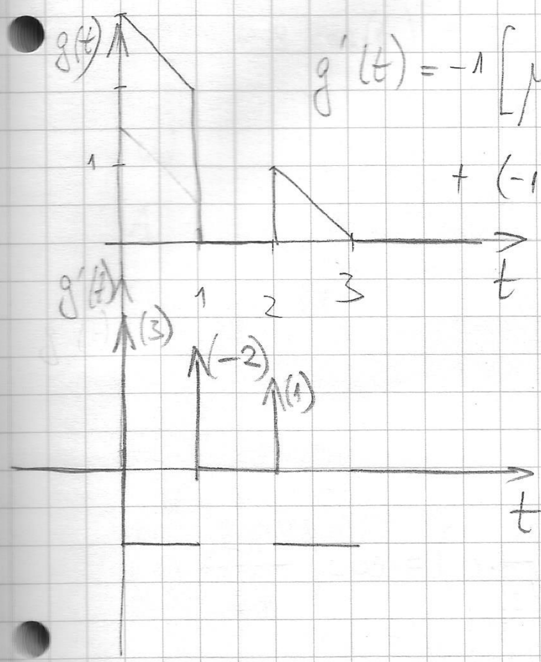
$$g(t) = -1 + 2\mu(t)$$

$$g'(t) = 2\delta(t)$$





⑦ b) $g(t) = (3-t)\big(\mu(t)-\mu(t-1)\big) + (3-t)\big(\mu(t-2)-\mu(t-3)\big)$



$$g'(t) = -1\left[\mu(t)-\mu(t-1)\right] + (3-t)\left[\delta(t)-\delta(t-1)\right] +$$

$$+ (-1)\left[\mu(t-2)-\mu(t-3)\right] + (3-t)\left[\delta(t-2)-\delta(t-3)\right] =$$

$$= \mu(t-1) - \mu(t) + (3-t)\delta(t) - (3-t)\delta(t-1) +$$

$$+ \mu(t-3) - \mu(t-2) + (3-t)\delta(t-2) - (3-t)\delta(t-3) =$$

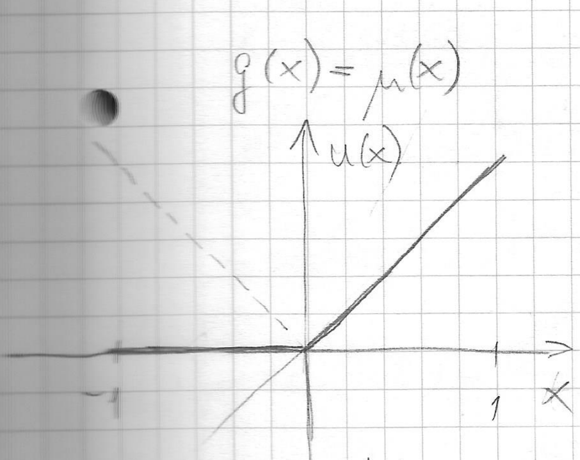
$$= \mu(t-1) - \mu(t) + 3\delta(t) - 2\delta(t-1) +$$

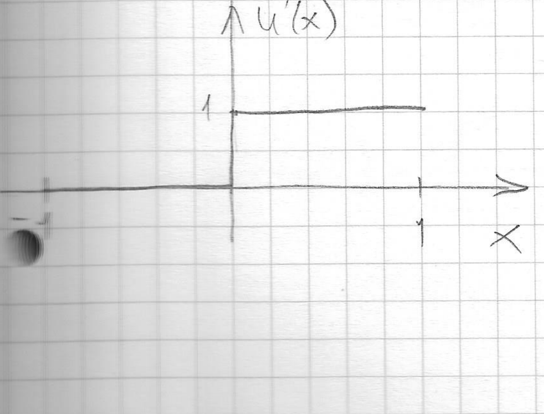
$$+ \mu(t-3) - \mu(t-2) + \delta(t-2)$$

⑧ $$\int_I u(x)\varphi'(x)\,dx = -\int_I g(x)\varphi(x)\,dx$$

$$u(x) = \frac{1}{2}\left(|x|+x\right)$$

$$g(x) = \mu(x)$$





$$\int_{-1}^{1} u(x)\varphi'(x)\,dx = \int_0^1 x\,\varphi'(x)\,dx = \left| \begin{matrix} u=x & dv=\varphi'(x)dx \\ du=dx & v=\varphi(x) \end{matrix} \right|$$

$$= x\varphi(x)\Big|_0^1 - \int_0^1 \varphi(x)\,dx = 0 - \int_0^1 \varphi(x)\,dx$$

$$-\int_{-1}^{1} \mu(x)\varphi(x)\,dx = -\int_0^1 \varphi(x)\,dx \rightarrow =$$

(9) a) $\frac{56\,000 \text{ bit/s}}{16 \text{ bit}} = 3{,}5 \text{ kHz}$

b) $\frac{100 \cdot 10^6 \text{ bit/s}}{16 \text{ bit}} = 6{,}25 \text{ MHz}$

(10) $f_s = 10 \text{ kHz}$

$$x(n) = \cos\left(\frac{\pi}{8} n\right)$$

$$x(t) = \cos(\omega_0 \underset{\substack{\downarrow \\ nT}}{t}) \quad \overset{x(nT)}{\downarrow}\cos(\omega_0 T \cdot n) = \cos\left(\omega_0 \frac{1}{f_s} n\right) = \cos\left(\frac{2\pi f_0}{f_s} n\right)$$

$$x(n) = \cos\left(\frac{2\pi f_0}{10^4} n\right)$$

$$\frac{\cancel{\pi}}{8} = \frac{2\cancel{\pi} f_0}{10^4}$$

$$\frac{10^4}{16} = f_0 = 625 \text{ Hz}$$

$$\cos\left[\left(\frac{\pi}{8} + 2\pi k\right) n\right] = \cos\left(\frac{2\pi f_0}{10^4} n\right)$$

$$\frac{\pi}{8} + 2\pi k = \frac{2\pi f_0}{10^4}$$

$$f_0 = \frac{10^4}{2}\left(\frac{1}{8} + 2k\right)$$

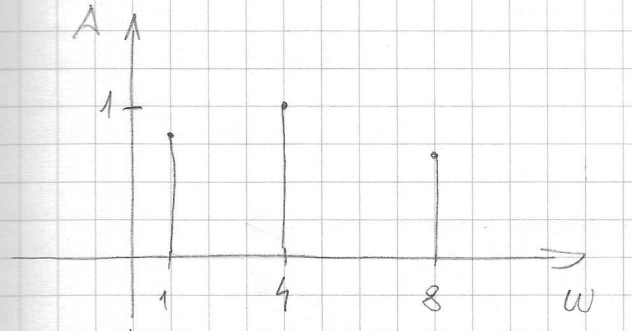
$k = 0 \Rightarrow f_0 = 625 \text{ Hz}$

$k = 1 \Rightarrow f_0 = 10^4\left(\frac{1}{16} + 1\right) = 10\,625 \text{ Hz}$

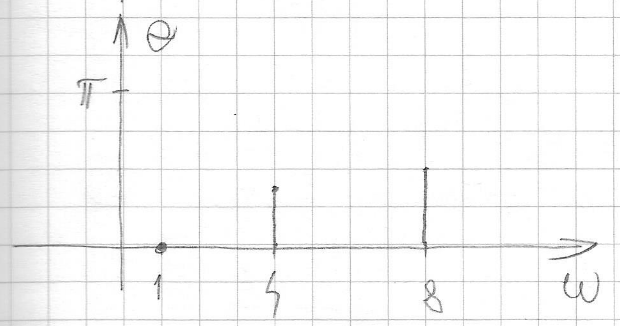
Nastavak predavanja

$$f(t) = 0{,}8\cos(t) + \cos\left(4t + \frac{\pi}{3}\right) + 0{,}7\cos\left(8t + \frac{\pi}{2}\right)$$

ω	1	4	8
A	0,8	1	0,7
θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$



Amplitudno-frekvencijska karakteristika



Fazno-frekvencijska karakteristika

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$
$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t) \quad \Big| \; + \; (-)$$

$$2\cos(\omega t) = e^{j\omega t} + e^{-j\omega t}$$

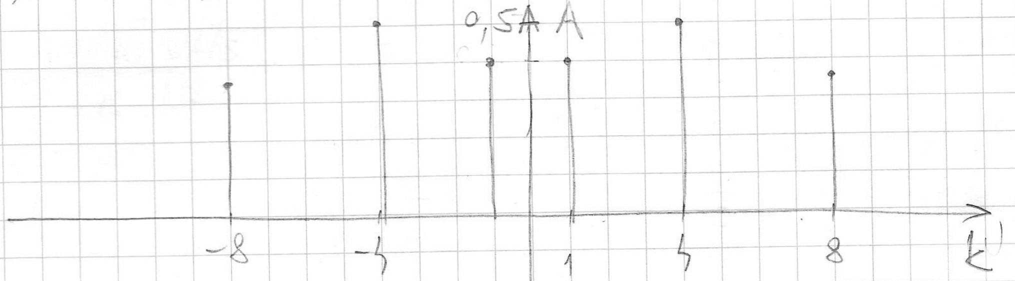
$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

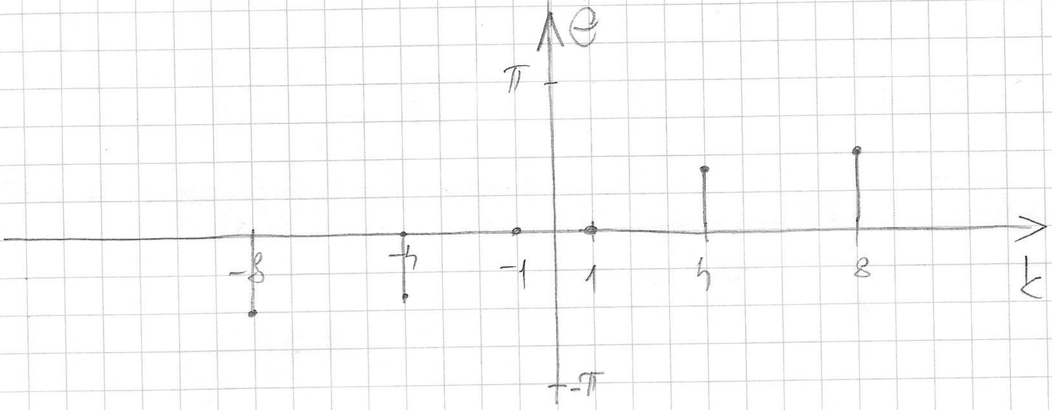
$$2j\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{1}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j} = \frac{1}{2} e^{-j\frac{\pi}{2}}\left[e^{j\omega t} - e^{-j\omega t}\right] \qquad \left(\frac{1}{2j}\cdot\frac{j}{j} = -\frac{1}{2}j = \frac{1}{2}e^{-j\frac{\pi}{2}}\right)$$

$$0{,}8\cdot\frac{1}{2}\left[e^{jt}+e^{-jt}\right]+1\cdot\frac{1}{2}\left[e^{j(4t+\frac{\pi}{3})}+e^{-j(4t+\frac{\pi}{3})}\right]+0{,}7\cdot\frac{1}{2}\left[e^{j(8t+\frac{\pi}{2})}+e^{-j(8t+\frac{\pi}{2})}\right]=$$

$$=0{,}4\,e^{jt}+0{,}4\,e^{j(-t)}+0{,}5\,e^{j\frac{\pi}{3}}e^{j4t}+0{,}5\,e^{-j\frac{\pi}{3}}e^{j(-4t)}+0{,}35\,e^{j\frac{\pi}{2}}e^{j8t}+0{,}35\,e^{j(-\frac{\pi}{2})}e^{j(-8t)}$$





$$f(t)=\sum_{k=-\infty}^{\infty}|c_k|\,e^{j\theta_k}e^{jt\omega_0 k}$$ – Vremenski kontinuiran Fourierov red

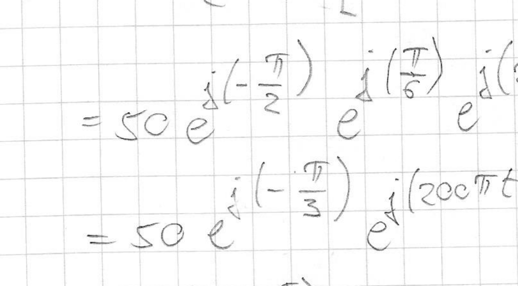
$$f(t)=100\sin\left(200\pi t+\frac{\pi}{6}\right)=$$

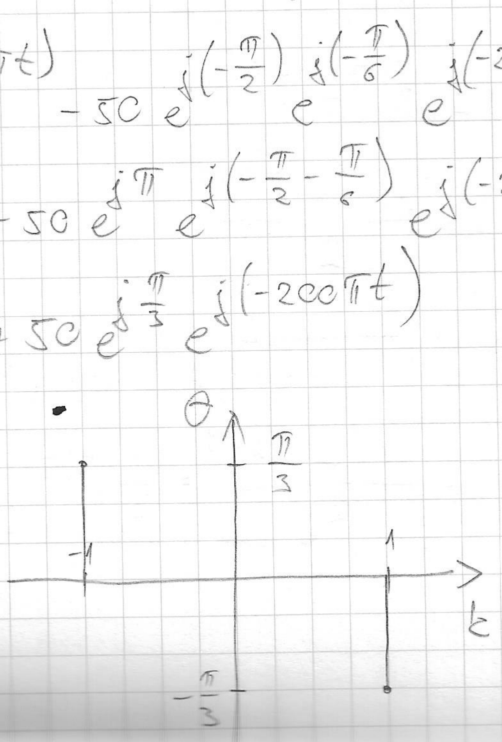
$$=100\cdot\frac{1}{2}e^{-j\frac{\pi}{2}}\left[e^{j(200\pi t+\frac{\pi}{6})}-e^{-j(200\pi t+\frac{\pi}{6})}\right]=$$

$$=50\,e^{j(-\frac{\pi}{2})}e^{j(\frac{\pi}{6})}e^{j(200\pi t)}-50\,e^{j(-\frac{\pi}{2})}e^{j(-\frac{\pi}{6})}e^{j(-200\pi t)}=$$

$$=50\,e^{j(-\frac{\pi}{3})}e^{j(200\pi t)}+50\,e^{j\pi}e^{j(-\frac{\pi}{2}-\frac{\pi}{6})}e^{j(-200\pi t)}=$$

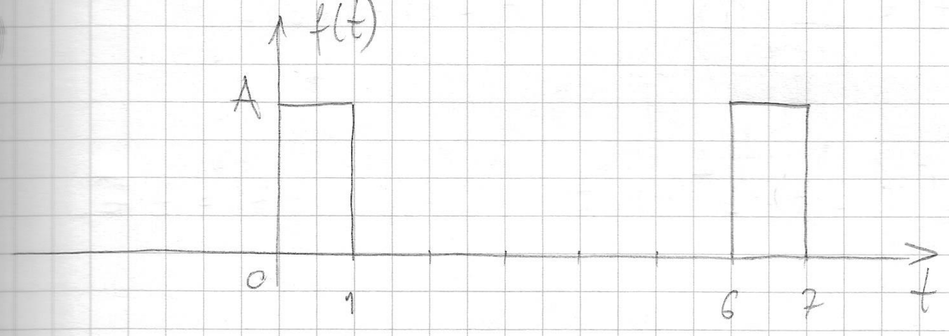
$$=50\,e^{j(-\frac{\pi}{3})}e^{j(200\pi t)}+50\,e^{j\frac{\pi}{3}}e^{j(-200\pi t)}$$





$$f(t) = \sum_{k=-\infty}^{\infty} F_k \, e^{jt\omega_0 k}$$

$$F_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jt\omega_0 k}\, dt$$



$$F_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-jk\omega_0 t}\, dt =$$

$$= \frac{1}{6}\int_0^1 A\, e^{-j\frac{2\pi}{6}kt}\, dt = \frac{A}{6}\int_0^1 e^{-j\frac{2\pi}{6}kt}\, dt =$$

$$= \frac{A}{6}\, \frac{e^{-j\frac{2\pi}{6}kt}}{-j\frac{2\pi}{6}k}\Bigg|_0^1 = \frac{A}{6}\, \frac{1}{-j\frac{2\pi}{6}k}\left[e^{-j\frac{2\pi}{6}k\cdot 1} - e^{-j\frac{2\pi}{6}k\cdot 0}\right] =$$

$$= \frac{A}{6}\, \frac{1}{-j\frac{2\pi}{6}k}\, e^{-j\frac{2\pi}{6}k\cdot\frac{1}{2}}\left[\underbrace{e^{-j\frac{2\pi}{6}k\cdot\frac{1}{2}} - e^{j\frac{2\pi}{6}k\cdot\frac{1}{2}}}_{-2j\sin\left(\frac{2\pi}{6}k\frac{1}{2}\right)}\right] =$$

$$= \frac{A}{6}\, \frac{\sin\left(\frac{2\pi}{6}k\frac{1}{2}\right)}{\frac{2\pi}{6}k\frac{1}{2}}\, e^{-j\frac{2\pi}{6}k\cdot\frac{1}{2}} =$$

$$= \frac{A}{6}\cdot \operatorname{sinc}\left(\frac{2\pi}{6}k\frac{1}{2}\right) e^{-j\frac{2\pi}{6}k\cdot\frac{1}{2}}$$

Parseval za CTFS

$$\int_{T_0} |f(t)|^2 dt < \infty$$ (signal konačne snage)

$$P(f) = \frac{1}{T_0} \int_{T_0} |f(t)|^2 dt = \sum_{k=-\infty}^{\infty} |F_k|^2 \qquad F_k = \text{CTFS}\{f(t)\}$$

$$P = \frac{1}{T_0} \int_{T_0} |f(t)|^2 dt = \frac{1}{T_0} \int_{T_0} f(t) \cdot f^*(t)\, dt \qquad |z|^2 = z \cdot z^*$$

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} \quad \Big/ ^*$$

$$f^*(t) = \sum_{k=-\infty}^{\infty} F_k^* e^{-jk\omega_0 t}$$

$$P = \frac{1}{T_0} \int_{T_0} \left[f(t) \sum_{k=-\infty}^{\infty} F_k^* e^{-jk\omega_0 t} \right] dt = \sum_{k=-\infty}^{\infty} F_k^* \underbrace{\frac{1}{T_0} \int_{T_0} f(t) e^{-jk\omega_0 t} dt}_{= F_k} =$$

$$= \sum_{k=-\infty}^{\infty} F_k^* F_k = \sum_{k=-\infty}^{\infty} |F_k|^2$$

Auditorne N tjedan

(1)

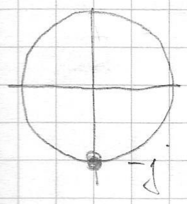
$$X_k = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{4}\int_{-0,5}^{3,5} x(t)\, e^{-jk\omega_0 t}\, dt =$$

$$= \frac{1}{4}\Bigg[\int_{-\frac{1}{2}}^{\frac{1}{2}} \delta(t)\, e^{-jk\frac{\pi}{2}t}\, dt + \int_{\frac{1}{2}}^{\frac{3}{2}} \delta(t-1)\, e^{-jk\frac{\pi}{2}t}\, dt + \int_{\frac{3}{2}}^{\frac{5}{2}} \delta(t-2)\, e^{-jk\frac{\pi}{2}t}\, dt +$$

$$- \int_{\frac{5}{2}}^{\frac{7}{2}} \delta(t-3)\, e^{-jk\frac{\pi}{2}t}\, dt\Bigg] = \frac{1}{4}\Big[1 + e^{-jk\frac{\pi}{2}} + e^{-jk\pi} - e^{-jk\frac{3\pi}{2}}\Big] =$$

$$= \frac{1}{4}\Big[1 + (-j)^k + (-1)^k - j^k\Big]$$

$$e^{-jk\frac{\pi}{2}} = \left(e^{-j\frac{\pi}{2}}\right)^k = (-j)^k \qquad x(t) = \sum_{k=-\infty}^{\infty} X_k\, e^{jk\omega_0 t}$$



(2)

$T_0 = 1$

$\omega_0 = 2\pi$

samo ako $k \neq 0$

$$X_k = \frac{1}{1}\int_{-0,5}^{0,5} x(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{1}\int_{-0,25}^{0,25} e^{-jk2\pi t}\, dt = \frac{e^{-j2k\pi t}}{-j2k\pi}\Bigg|_{-0,25}^{0,25} =$$

$$= \frac{1}{-j2k\pi}\Big[e^{-j2k\pi\cdot\frac{1}{4}} - e^{-j2k\pi\cdot(-\frac{1}{4})}\Big] = \frac{-e^{-j\frac{k\pi}{2}} + e^{j\frac{k\pi}{2}}}{+j2k\pi} = \frac{\sin\left(\frac{k\pi}{2}\right)}{k\pi} =$$

$$= \frac{1}{2}\cdot\frac{\sin\left(\frac{k\pi}{2}\right)}{\frac{k\pi}{2}}$$

$$X_0 = \int_{-0,25}^{0,25} 1\cdot e^{0}\, dt = t\Bigg|_{-0,25}^{0,25} = \frac{1}{2}$$

$$P = \frac{1}{T_0} \int_{T_0} x(t)\, x^*(t)\, dt = \int_{-0,25}^{0,25} |x(t)|^2\, dt = \int_{-0,25}^{0,25} 1\, dt = \frac{1}{2}$$

$$P = \sum_{k=-\infty}^{\infty} X_k X_k^* = \sum_{k=-\infty}^{\infty} |X_k|^2 = X_0^2 + \sum_{k=-\infty}^{-1} |X_k|^2 + \sum_{k=1}^{\infty} |X_k|^2 =$$

$$= \frac{1}{4} + 2 \sum_{k=1}^{\infty} |X_k|^2 = \frac{1}{4} + 2 \left[\left(\frac{1}{\pi}\right)^2 + 0 + \left(\frac{1}{3\pi}\right)^2 - 0 + \left(\frac{1}{5\pi}\right)^2 \dots \right] =$$

$$= \frac{1}{4} + 2 \frac{1}{\pi^2} \sum_{k=0}^{\infty} \left(\frac{1}{2k+1}\right)^2 = \frac{1}{4} + 2 \cdot \frac{1}{\pi^2} \cdot \frac{\pi^2}{8} = \frac{1}{2}$$

(3) $x(t) = 10\cos(50\pi t) + 5\sin(100\pi t) + \sin\left(150\pi t + \frac{2\pi}{3}\right) + \cos\left(200\pi t + \frac{\pi}{4}\right)$

$\omega_1 = 50\pi$ $\qquad \omega_2 = 100\pi$ $\qquad \omega_3 = 150\pi$ $\qquad \omega_4 = 200\pi$

$T_1 = \frac{2\pi}{50\pi} = \frac{1}{25}$ $\qquad T_2 = \frac{1}{50}$ $\qquad T_3 = \frac{1}{75}$ $\qquad T_4 = \frac{1}{100}$

$$T_0 = \frac{1}{25} \Rightarrow \omega_0 = 50\pi$$

$$\frac{1}{T_0} \int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

Ako je zadatak zadan samo preko sin i cos prebaci u eksponencijalu

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2} \qquad \sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

$$10\cos(50\pi t) = 10\cos(1 \cdot \omega_0 \cdot t) = 5\left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right]$$

$$5\sin(100\pi t) = 5\sin(2\omega_0 t) = \frac{5}{2j}\left[e^{j2\omega_0 t} - e^{-j2\omega_0 t}\right]$$

$$\sin\left(150\pi t + \frac{2\pi}{3}\right) = 1\sin\left(3\omega_0 t + \frac{2\pi}{3}\right) = \frac{1}{2j}\left[e^{j\left(3\omega_0 t + \frac{2\pi}{3}\right)} - e^{-j\left(3\omega_0 t + \frac{2\pi}{3}\right)}\right] =$$

$$= \frac{1}{2j}\left[e^{j\frac{2\pi}{3}} e^{j3\omega_0 t} - e^{-j\frac{2\pi}{3}} e^{-j3\omega_0 t}\right] = \frac{1}{2j}\left[\left(\frac{\sqrt{3}}{2}j - \frac{1}{2}\right) e^{j3\omega_0 t} - \left(-\frac{\sqrt{3}}{2}j - \frac{1}{2}\right) e^{-j3\omega_0 t}\right]$$

$$\cos\left(200\pi t+\frac{\pi}{4}\right)=\cos\left(4\omega_0 t+\frac{\pi}{4}\right)=\frac{1}{2}\left[e^{j\left(4\omega_0 t+\frac{\pi}{4}\right)}+e^{-j\left(4\omega_0 t+\frac{\pi}{4}\right)}\right]=$$

$$=\frac{1}{2}\left[e^{j\frac{\pi}{4}}e^{j4\omega_0 t}+e^{-j\frac{\pi}{4}}e^{-j4\omega_0 t}\right]$$

$$x(t)=5\left(e^{j\omega_0 t}+e^{-j\omega_0 t}\right)+\frac{5}{2j}\left(e^{j2\omega_0 t}-e^{-j2\omega_0 t}\right)+$$

$$+\frac{1}{2j}\left(e^{j\frac{2\pi}{3}}e^{j3\omega_0 t}-e^{-j\frac{2\pi}{3}}e^{-j3\omega_0 t}\right)+\frac{1}{2}\left(e^{j\frac{\pi}{4}}e^{j4\omega_0 t}+e^{-j\frac{\pi}{4}}e^{-j4\omega_0 t}\right)$$

CTFS

$$x(t)=\sum_{k=-\infty}^{\infty}X_k e^{j\omega_0 k t}$$

$X_1=5$

$X_{-1}=5$

$X_2=\frac{5}{2j}$

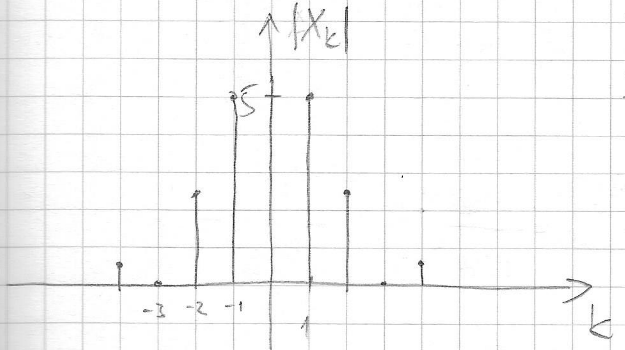
$X_{-2}=\frac{5}{2j}$

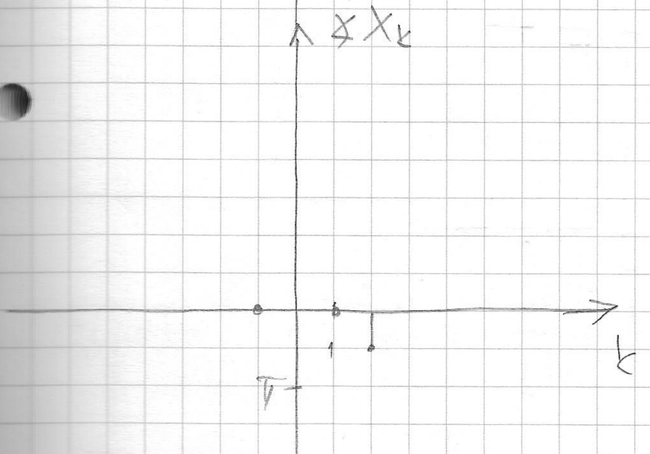
$X_3=\frac{e^{j\frac{2\pi}{3}}}{2j}$

$X_{-3}=\frac{e^{-j\frac{2\pi}{3}}}{2j}$

$X_4=\frac{e^{j\frac{\pi}{4}}}{2}$

$X_{-4}=\frac{e^{-j\frac{\pi}{4}}}{2}$





$T_s=0{,}02$

$\omega_s=100\pi$

Doći će do aliasinga jer

$\omega_s<2\omega_1, 2\omega_2, 2\omega_3$ i $2\omega_4$

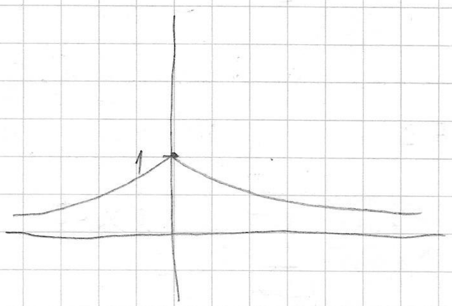
⑤ a) $x(t) = e^{-t}\mu(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-t} e^{-j\omega t}\, dt = \int_{0}^{\infty} e^{-t(1+j\omega)}\, dt =$$

$$= \left.\frac{e^{-t(1+j\omega)}}{-(1+j\omega)}\right|_{0}^{\infty} = \frac{+1}{1+j\omega} = \frac{1-j\omega}{1+\omega^2}$$

$$X(j\omega) = \frac{1-j\omega}{1+\omega^2}$$

$$|X(j\omega)| = \frac{|1-j\omega|}{|1+\omega^2|} = \frac{\sqrt{1+(-\omega)^2}}{1+\omega^2} = \frac{1}{\sqrt{1+\omega^2}}$$



$$\measuredangle X(j\omega) = \operatorname{arctg} \frac{\frac{-\omega}{\cancel{1+\omega^2}}}{\frac{1}{\cancel{1+\omega^2}}} = -\operatorname{arctg} \omega$$

$$\operatorname{Re}\{X(j\omega)\} = \frac{1}{1+\omega^2} \qquad \operatorname{Im}\{X(j\omega)\} = \frac{-\omega}{1+\omega^2}$$

$$F(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \quad - \quad \text{CTFT}$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega \quad - \quad \text{ICTFT}$$

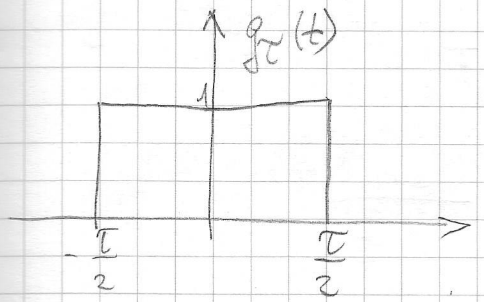
$$E = \int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} f(t) f^*(t)\, dt = \int_{-\infty}^{\infty} f(t)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{j\omega t}\, d\omega\right]^* dt =$$

$$= \int_{-\infty}^{\infty} f(t)\cdot\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(j\omega)\, e^{-j\omega t}\, d\omega\right] dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(j\omega)\left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt\right] d\omega =$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} F^*(j\omega)\, F(j\omega)\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega$$

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |f(t)|^2\, dt$$

Z

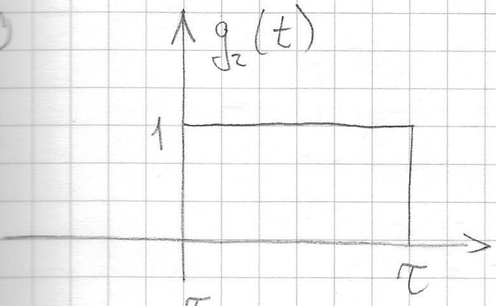


$$G(j\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt =$$

$$= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 1\cdot e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{-j\omega}\left[e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right] =$$

$$= \frac{1}{j\omega}\left[e^{j\omega\frac{\tau}{2}} - e^{-j\omega\frac{\tau}{2}}\right] = \frac{2\cancel{j}\sin(\omega\frac{\tau}{2})}{\cancel{j}\omega} = \frac{\tau\sin(\omega\frac{\tau}{2})}{\frac{\omega\tau}{2}}$$

Z



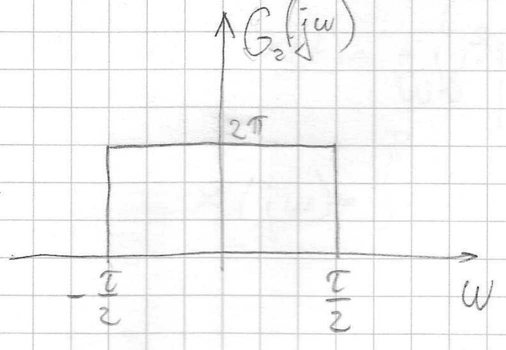
$$G_2(j\omega) = \int_0^{\tau} 1\cdot e^{-j\omega t}\, dt = \frac{e^{-j\omega t}}{-j\omega}\Bigg|_0^{\tau} = \frac{1}{-j\omega}\left[e^{-j\omega\tau} - e^{-j\omega 0}\right] =$$

$$= \frac{1}{-j\omega}\, e^{-j\omega\frac{\tau}{2}}\left[e^{-j\omega\frac{\tau}{2}} - e^{j\omega\frac{\tau}{2}}\right] = \frac{1}{-j\omega}\, e^{-j\omega\frac{\tau}{2}}\cdot\left[-2j\sin(\omega\tfrac{\tau}{2})\right] =$$

$$= \frac{2}{\omega}\sin(\omega\tfrac{\tau}{2})\, e^{-j\omega\frac{\tau}{2}} = \tau\,\frac{\sin(\omega\frac{\tau}{2})}{\frac{\omega\tau}{2}}\, e^{-j\omega\frac{\tau}{2}}$$

$$G_2(j\omega=0) = \int_0^{\tau} 1\cdot e^{0}\, dt = \tau$$

$$g_2(t) = g_1(t - t_0) \Rightarrow G_2(j\omega) = G_1(j\omega)\cdot e^{-j\omega t_0}$$



$$g_2(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} G_2(j\omega)\,e^{j\omega t}\,d\omega =$$

$$=\frac{1}{2\pi}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} 2\pi\, e^{j\omega t}\,d\omega = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} e^{j\omega t}\,d\omega =$$

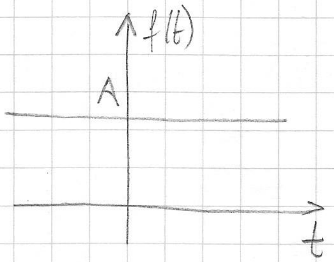
$$=\frac{e^{j\omega t}}{jt}\Bigg|_{-\frac{\tau}{2}}^{\frac{\tau}{2}} = \frac{1}{jt}\left[e^{j\frac{\tau}{2}t}-e^{-j\frac{\tau}{2}t}\right]=$$

$$=\frac{2\cancel{j}\sin(t\frac{\tau}{2})}{\cancel{j}t} = \frac{\tau\sin(t\frac{\tau}{2})}{t\frac{\tau}{2}}$$

VRIJEME	FREKVENCIJA
$g(t)$	$G(j\omega)$
$G(jt)$	$2\pi\, g(-\omega)$

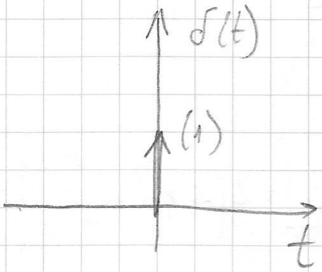
$g(t)=1 \qquad -\frac{\tau}{2}\le \tau \le \frac{\tau}{2} \qquad G(j\omega)=\tau\,\frac{\sin(\omega\frac{\tau}{2})}{\omega\frac{\tau}{2}}$

$g(-\omega)=2\pi \qquad -\frac{\tau}{2}\le -\omega \le \frac{\tau}{2} \qquad G(jt)=\tau\,\frac{\sin(t\frac{\tau}{2})}{t\frac{\tau}{2}}$



$$F(j\omega)=\int_{-\infty}^{\infty} A\, e^{-j\omega t}\,dt = A\,\frac{e^{-j\omega t}}{-j\omega}\Bigg|_{-\infty}^{\infty} =$$

$$= A\cdot\frac{1}{-j\omega}\left[e^{-j\omega\infty} - e^{-j\omega(-\infty)}\right]$$



$$F(j\omega)=\int_{-\infty}^{\infty} \underbrace{\delta(t)}_{t=0}\, e^{-j\omega t}\,dt = 1$$

$f(t)=1$

$F(j\omega)=2\pi\,\delta(\omega)$

$$f(t) = \sum_{-\infty}^{\infty} F_k e^{jk\omega_0 t}$$ CTFS

$$f(n) = \sum_{k=-\infty}^{\infty} F_k e^{jk\Omega_0 n}$$

$$e^{j(k+N)\Omega_0 n} = e^{j(k+N)\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} \underbrace{e^{jN\frac{2\pi}{N}n}}_{\substack{\cos(2\pi n) + j\sin(2\pi n) \\ 1 \quad + \quad j0}} = e^{jk\frac{2\pi}{N}n}$$

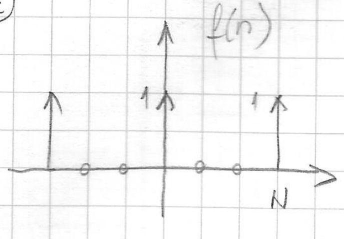
$$f(n) = \sum_{k=0}^{N-1} F_k e^{jk\frac{2\pi}{N}n}$$ DTFS

$$F_k = \frac{1}{N} \sum_{n=0}^{N-1} f(n) e^{-jk\frac{2\pi}{N}n}$$ IDTFS

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |f(n)|^2 = \sum_{k=0}^{N-1} |F_k|^2$$

$$\sum_{n=0}^{N-1} q^n = \frac{1-q^n}{1-q}$$

(2)



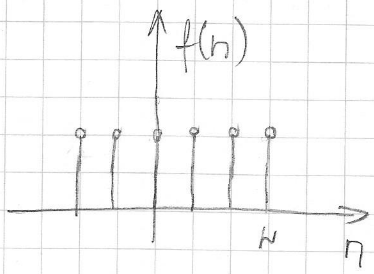
$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\, e^{-jk\frac{2\pi}{N}n} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} \delta(n)\, e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\cdot 1 = \frac{1}{N}$$

$n_0 = 0$

$$P = \frac{1}{N}\sum_{n=0}^{N-1} |f(n)|^2 = \sum_{k=0}^{N-1} |F_k|^2 = \frac{1}{N}$$

(2)



$$F_k = \frac{1}{N}\sum_{n=0}^{N-1} f(n)\, e^{-jk\frac{2\pi}{N}n} =$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} 1\cdot e^{-jk\frac{2\pi}{N}n} =$$

$$= \frac{1}{N}\,\frac{1-e^{-jk\frac{2\pi}{N}N}}{1-e^{-jk\frac{2\pi}{N}}} = 0$$

$$F_0 = \frac{1}{N}\sum_{n=0}^{N-1} 1\cdot 1 = \frac{1}{N}\cdot N = 1$$

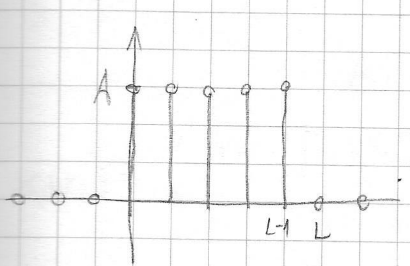
(DTFT) Vremenski diskretna Fourierova transformacija

$$F(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} f(n)\, e^{-j\Omega n} \qquad \text{DTFT}$$

$$f(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\Omega})\, e^{j\Omega n}\, d\Omega \qquad \text{IDTFT}$$

$$E = \sum_{n=-\infty}^{\infty} |f(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |F(e^{j\Omega})|^2\, d\Omega$$

2

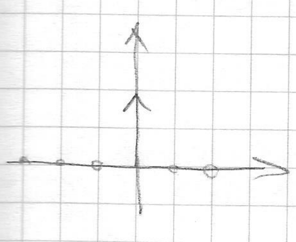


$$F(e^{j\Omega}) = \sum_{n=0}^{L-1} f(n)\, e^{-j\Omega n} =$$

$$= A \sum_{n=0}^{L-1} e^{-j\Omega n} = A\, \frac{1 - e^{-j\Omega L}}{1 - e^{-j\Omega}} =$$

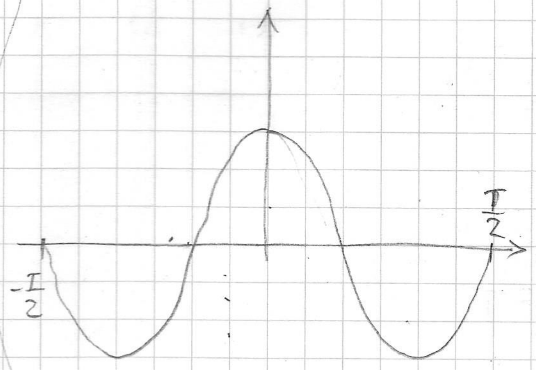
$$= A\, \frac{e^{-j\Omega 0} - e^{-j\Omega L}}{e^{-j\Omega 0} - e^{-j\Omega}} = A\, \frac{\sin\left(\frac{\Omega L}{2}\right)}{\sin\left(\frac{\Omega}{2}\right)}\, e^{-j\frac{\Omega}{2}(L-1)}$$

2



$$F(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} \delta(n)\, e^{-j\Omega n} = 1$$

$f(t) = \cos(\omega_0 t)$



$$F(j\omega) = \int_{t=-\infty}^{\infty} f(t) e^{-j\omega t} dt =$$

$$= \int_{-\frac{T}{2}}^{\frac{T}{2}} \cos(\omega_0 t) e^{-j\omega t} dt = \frac{1}{2} \int_{-\frac{T}{2}}^{\frac{T}{2}} (e^{j\omega_0 t} - e^{-j\omega_0 t}) e^{-j\omega t}$$

$\xrightarrow{CTFT}$ $K_1 \delta(\omega - \omega_0) + K_2 \delta(\omega + \omega_0)$

$$f(t) = \delta(t) \xleftrightarrow{CTFT} F(j\omega) = 1$$

$$f(t) = 1 \xleftrightarrow{CTFT} F(j\omega) = 2\pi\,\delta(\omega)$$

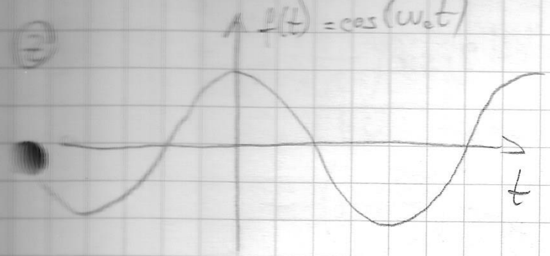
$$f(t) = \delta(t - t_0) \xleftrightarrow{CTFT} F(j\omega) = 1\, e^{j\omega t_0}$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 2\pi\,\delta(\omega - k\omega_0) e^{j\omega t} d\omega = e^{jk\omega_0 t}$$

$$f(t) = e^{jk\omega_0 t} \xleftrightarrow{CTFT} F(j\omega) = 2\pi\,\delta(\omega - k\omega_0)$$

$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{jk\omega_0 t} \xrightarrow{CTFT} \boxed{f(t) = \sum_{k=-\infty}^{\infty} F_k \cdot 2\pi\,\delta(\omega - k\omega_0)}$$

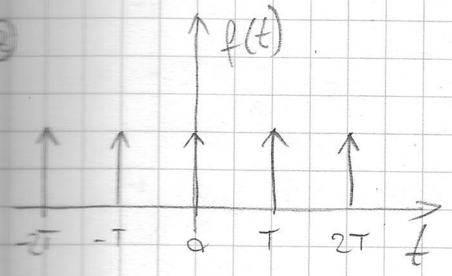
CTFT periodičkog signala

③ 

$$f(t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$F(j\omega) = 2\pi \cdot \frac{1}{2} \cdot \delta(\omega - \omega_0) + 2\pi \cdot \frac{1}{2} \cdot \delta(\omega + \omega_0)$$

$$f(t) = \sin(\omega_0 t) \Rightarrow F(j\omega) = \frac{1}{2j} 2\pi \delta(\omega - \omega_0) - \frac{1}{2j} 2\pi \delta(\omega + \omega_0)$$

② 

$$F_k = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T}$$

$$F(j\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{T} 2\pi \delta(\omega - k\omega_0)$$

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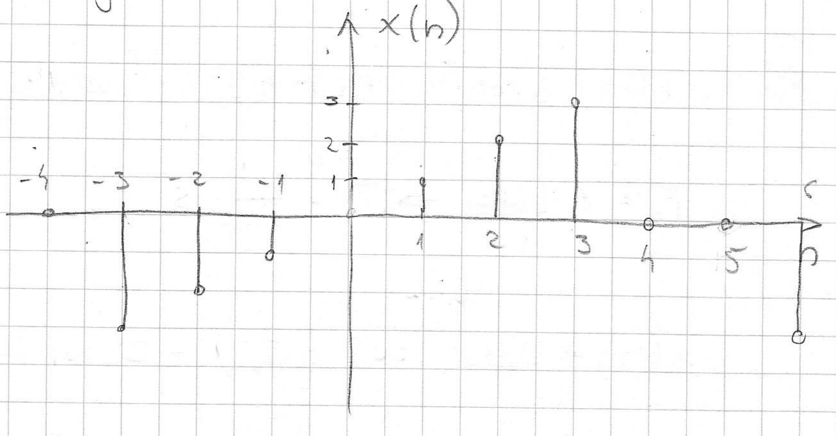
② $$x(n) = \begin{cases} n & |n| \leq 3 \\ 0 & n \in \{4, 5\} \end{cases}$$



$N = 9$

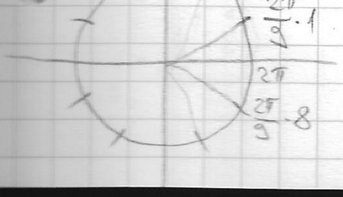
$$X_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-2\pi j kn/N} =$$

$$= \frac{1}{9} \sum_{n=0}^{8} x(n) e^{-j\frac{2\pi}{9}kn} = \frac{1}{9} \Big[0 + e^{-j\frac{2\pi}{9}k} + 2e^{-j\frac{2\pi}{9}k2} + 3e^{-j\frac{2\pi}{9}k3} + 0 + 0 +$$

$$-3e^{-j\frac{2\pi}{9}k6} - 2e^{-j\frac{2\pi}{9}k7} - e^{-j\frac{2\pi}{9}k8} \Big] =$$

$$= \frac{1}{9} \Big[\cos\left(\frac{2\pi}{9}k\right) - j\sin\left(\frac{2\pi}{9}k\right) + 2\cos\left(\frac{2\pi}{9}2k\right) - 2j\sin\left(\frac{2\pi}{9}2k\right) + 3\cos\left(\frac{2\pi}{9}3k\right) - 3j\sin\left(\frac{2\pi}{9}3k\right)$$

$$-3\cos\left(\frac{2\pi}{9}6k\right) + 3j\sin\left(\frac{2\pi}{9}6k\right) - 2\cos\left(\frac{2\pi}{9}7k\right) + 2\sin\left(\frac{2\pi}{9}7k\right) - \cos\left(\frac{2\pi}{9}8k\right) + j\sin\left(\frac{2\pi}{9}8k\right)$$



$$\cos\frac{2\pi}{9}1 = \cos\frac{2\pi}{9}8$$

$$\sin\frac{2\pi}{9}1 = -\sin\frac{2\pi}{9}8$$

$$= \frac{1}{9}\Big[\cos\left(\frac{2\pi}{9}k1\right) - j\sin\left(\frac{2\pi}{9}k1\right) + 2\cos\left(\frac{2\pi}{9}k\cdot 2\right) - 2j\sin\left(\frac{2\pi}{9}k\cdot 2\right) +$$

$$+ 3\cos\left(\frac{2\pi}{9}k\cdot 3\right) - 3j\sin\left(\frac{2\pi}{9}k3\right) - \cos\left(\frac{2\pi}{9}k\cdot 1\right) - j\sin\left(\frac{2\pi}{9}k\cdot 1\right) -$$

$$- 2\cos\left(\frac{2\pi}{9}k\cdot 2\right) - 2j\sin\left(\frac{2\pi}{9}k\cdot 2\right) - 3\cos\left(\frac{2\pi}{9}k\cdot 3\right) - j\sin\left(\frac{2\pi}{9}k\cdot 3\right)\Big] =$$

$$= \frac{1}{9}\Big[-2j\sin\left(\frac{2\pi}{9}k\cdot 1\right) - 4j\sin\left(\frac{2\pi}{9}k\cdot 2\right) - 6j\sin\left(\frac{2\pi}{9}k\cdot 3\right)\Big]$$

$$P = \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 = \sum_{k=0}^{N-1}|\tilde{F}_k|^2$$

$$= \frac{1}{9}\left[0^2 + 1^2 + 2^2 + 3^2 + 0^2 + 0^2 + (-3)^2 + (-2)^2 + (-1)^2\right] = \frac{28}{9}$$

3

$$x(t) = 2\cos(200\pi t) + 3\cos(500\pi t)$$

$$F_s = 1\ kHz \Rightarrow T_0 = \frac{1}{1000}\ s$$

$$x(n) = 2\cos(200\pi n\cdot T_0) + 3\cos(500\pi nT_0) =$$

$$= 2\cos\left(\frac{1}{5}\pi n\right) + 3\cos\left(\frac{1}{2}\pi n\right) = 2\cdot\frac{1}{2}\left(e^{j\frac{1}{5}\pi n} + e^{-j\frac{1}{5}\pi n}\right) +$$

$$+ 3\cdot\frac{1}{2}\left(e^{j\frac{1}{2}\pi n} + e^{-j\frac{1}{2}\pi n}\right) =$$

$\frac{1}{5}\pi N = 2k\pi$ $\qquad$ $\frac{1}{2}\pi N = 2k\pi$

$N = 10k$ $\qquad$ $N = 4k$

$N_0 = 10$ $\qquad$ $N_0 = 4$

$N = 20$

$$= e^{j\frac{2\pi}{20}n\cdot 2} + e^{j\frac{2\pi}{20}n(-2)} + \frac{3}{2}e^{j\frac{2\pi}{20}n5} + \frac{3}{2}e^{j\frac{2\pi}{20}n(-5)} =$$

$$= e^{j\frac{2\pi}{20}n\cdot 2} + e^{j\frac{2\pi}{20}n\cdot 18} + \frac{3}{2}e^{j\frac{2\pi}{20}n\cdot 5} + \frac{3}{2}e^{j\frac{2\pi}{2}n\cdot 15}$$

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1}x(n)\,e^{-j\frac{2\pi}{N}kn}$$

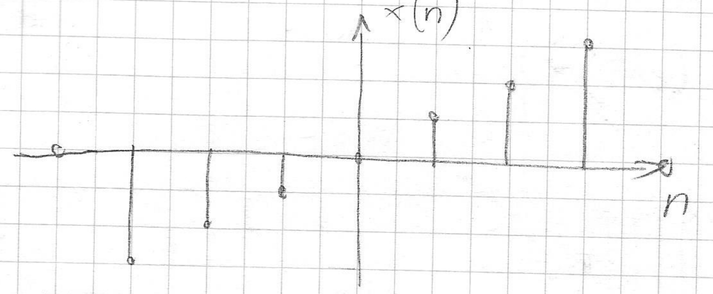
$F_2 = 1$ $\qquad$ $F_5 = \frac{3}{2}$

$$x(n) = \sum_{n=0}^{N-1}X_k\,e^{j\frac{2\pi}{N}kn}$$

$F_{18} = 1$ $\qquad$ $F_{15} = \frac{3}{2}$

1

$$x(n) = \begin{cases} n & |n| \leq 3 \\ 0 & \text{inače} \end{cases}$$



$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\Omega n} = -3e^{j\Omega 3} - 2e^{j\Omega 2} - e^{j\Omega} + 0 + e^{-j\Omega} + 2e^{-j\Omega 2} + 3e^{-j\Omega 3} =$$

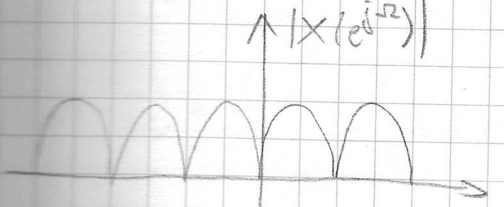
$$= \cancel{-3\cos(3\Omega)} - 3j\sin(3\Omega) \cancel{- 2\cos(2\Omega)} - 2j\sin(2\Omega) \cancel{- \cos(\Omega)} - j\sin(\Omega)$$

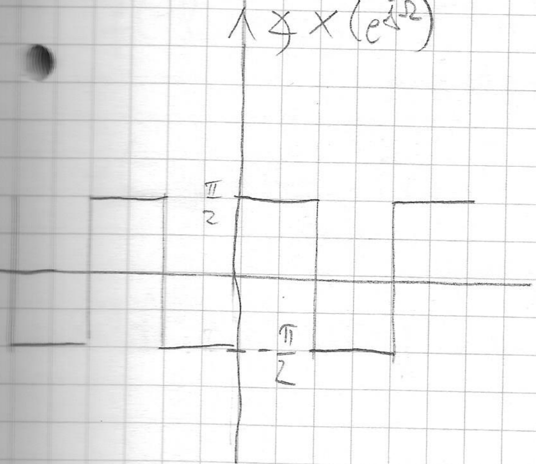
$$\cancel{+ \cos(\Omega)} - j\sin(\Omega) \cancel{+ 2\cos(2\Omega)} - 2j\sin(2\Omega) \cancel{+ 3\cos(3\Omega)} - 3j\sin(3\Omega)$$

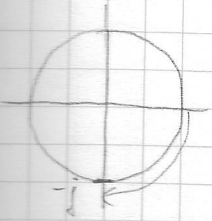
$$= -6j\sin(3\Omega) - 4j\sin(2\Omega) - 2j\sin(\Omega)$$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = (-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2 = 28$$

$$X(e^{j\Omega}) = 2j\sin\Omega$$

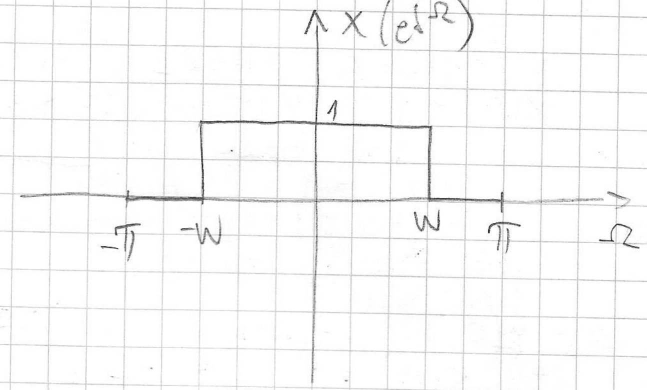






$$-j = e^{j(-\frac{\pi}{2})}$$

⑤ $$X(e^{j\Omega})=\begin{cases}1 & |\Omega|<W\\ 0 & W\le|\Omega|\le\pi\end{cases}$$



$$x(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\Omega})\,e^{j\Omega n}\,d\Omega=$$

$$=\frac{1}{2\pi}\int_{-W}^{W}e^{j\Omega n}\,d\Omega=\frac{1}{2\pi}\left.\frac{e^{j\Omega n}}{jn}\right|_{-W}^{W}=\frac{1}{2\pi}\frac{1}{jn}\left[e^{jWn}-e^{-jWn}\right]=$$

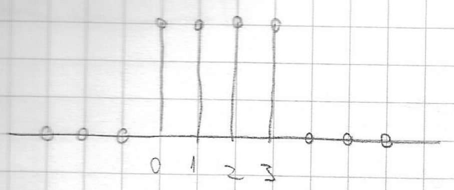
$$=\frac{1}{\pi n}\sin(Wn)=\frac{\frac{W}{\pi}\sin(Wn)}{Wn}$$

$$E=\frac{1}{2\pi}\int_{-W}^{W}|1|^2\,d\Omega=\frac{1}{2\pi}\Omega\Big|_{-W}^{W}=\frac{W}{\pi}$$

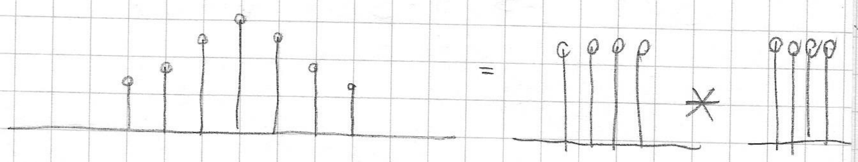
Nastavak predavanja

Postupak izvoda konvolucije:

1) Napiši izraz za konvoluciju
2) Fourierova transformacija konvolucije
3) Zamjena redoslijeda
4) Izlučivanje
5) Supstitucija
6) Izlučivanje



Spektar je:



$(f * g)(t)$	$F(j\omega)G(j\omega)$
$(f \circledast g)(t)$	$T_0 F(j\omega)G(j\omega)$
$(f * g)(n)$	$F(e^{j\Omega})G(e^{j\Omega})$
$(f \circledast g)(n)$	$N_0 (F j\omega) G(j\omega)$

Izvod svojstva dualnosti

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{+j\omega t}\, d\omega \qquad \tau = -t$$

$$f(-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(j\omega)\, e^{-j\omega \tau}\, d\omega \quad \Big| \ \omega \Rightarrow t$$

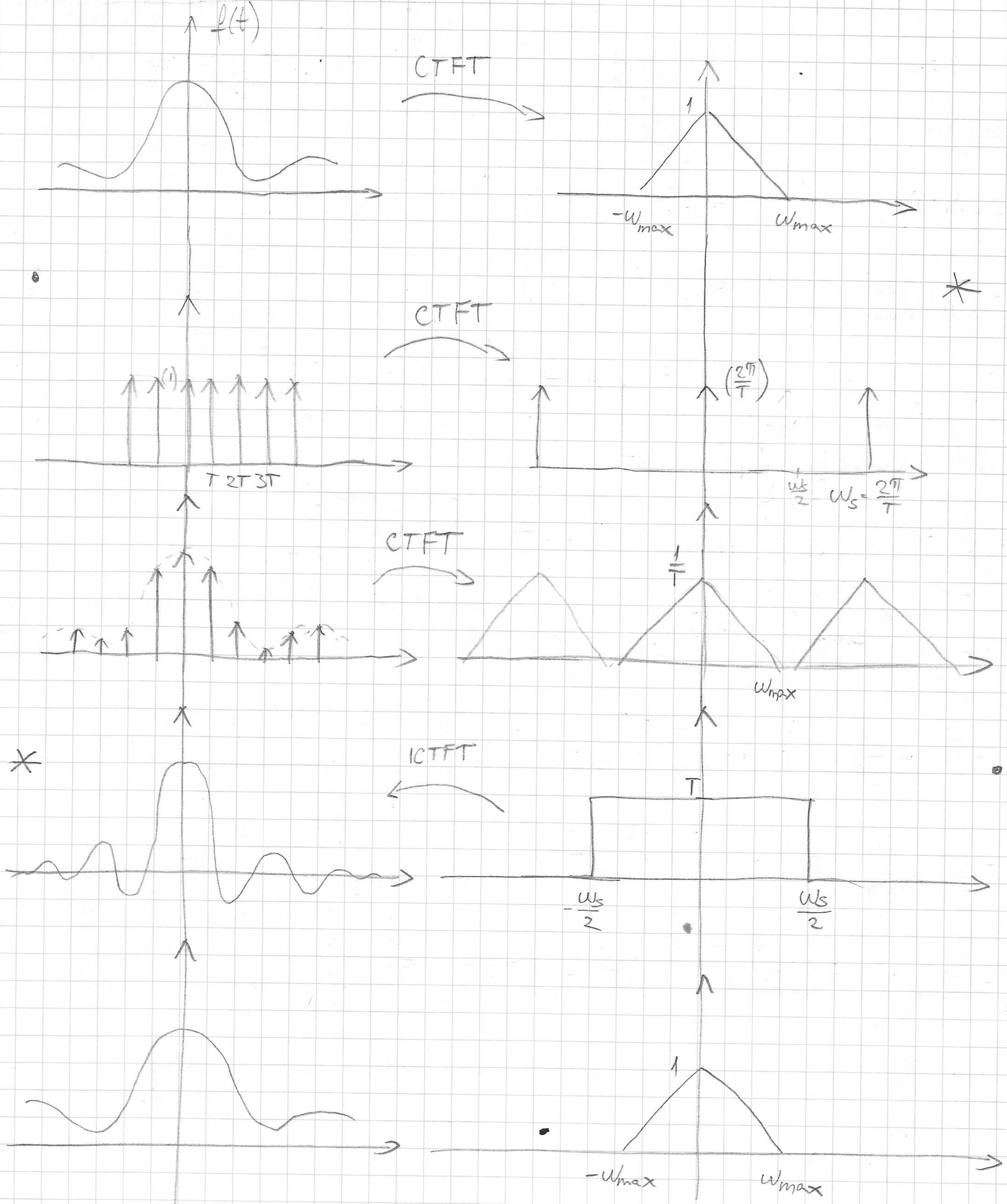
$$f(-\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(jt)\, e^{-jt\tau}\, dt \quad \Big| \ \tau \Rightarrow \omega$$

$$f(-\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(jt)\, e^{-j\omega t}\, dt$$

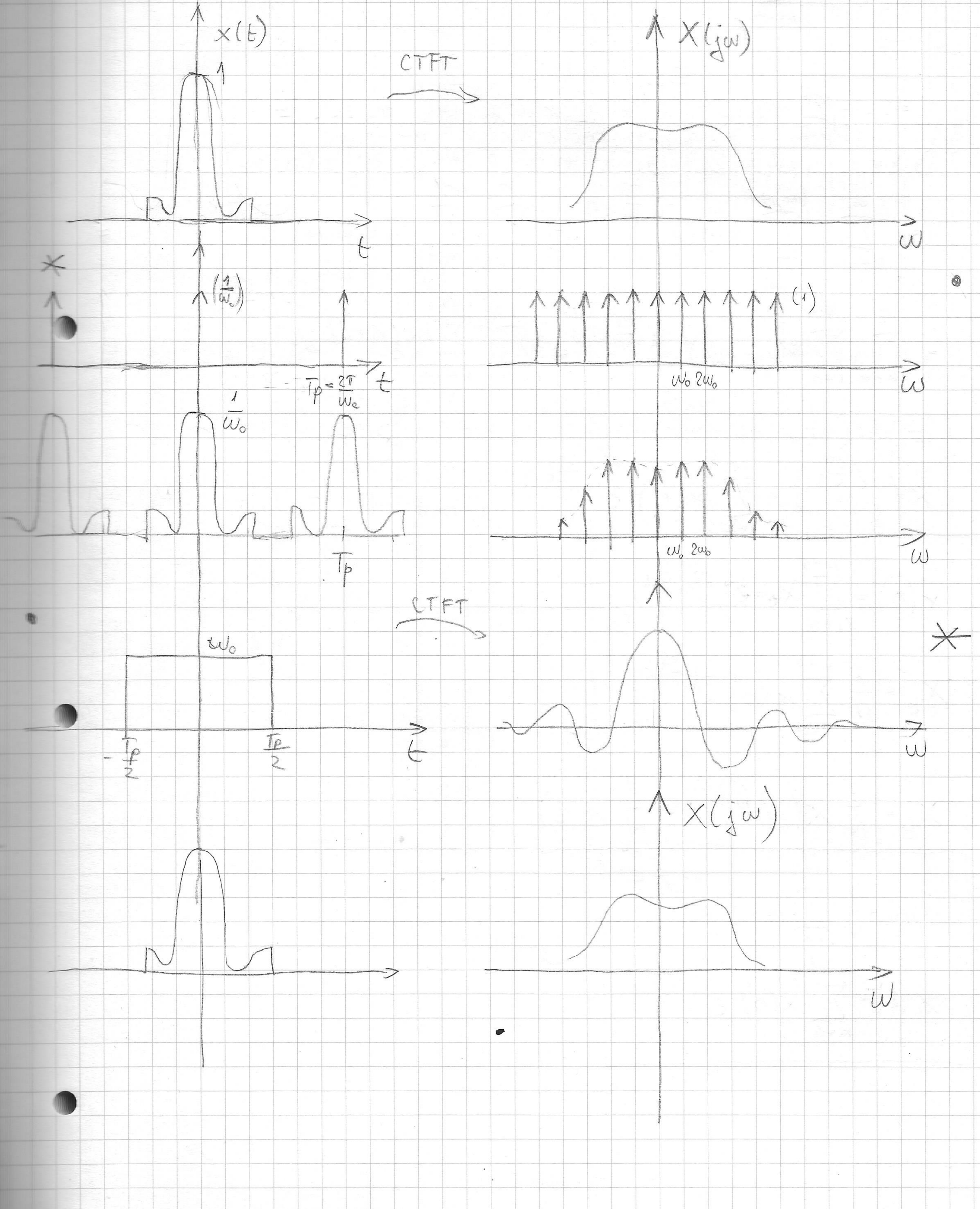
$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(jt)\, e^{-j\omega t}\, dt = \mathrm{DTFT}\{F(jt)\}$$

⑦ $3\delta(t-2) \rightarrow 3e^{-2j\omega}$

$3e^{-j2t} \rightarrow 2\pi \cdot 3\,\delta(-\omega-2)$



$$\boxed{\Omega = \omega T}$$



DFT

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\omega n}$$

periodičnost

$$X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=-\infty}^{+\infty} x(n) e^{-j\frac{2\pi}{N}kn} =$$

$$= \dots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \dots =$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=mN}^{(m+1)N-1} x(n) e^{-j\frac{2\pi}{N}kn} = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+mN) e^{-j\frac{2\pi}{N}k(n+mN)} =$$

$$= \sum_{m=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n+mN) e^{-j\frac{2\pi}{N}kn} = \sum_{n=0}^{N-1} \underbrace{\sum_{m=-\infty}^{\infty} x(n+mN)}_{\tilde{x}(n)} e^{-j\frac{2\pi}{N}kn}$$

$$X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} \tilde{x}(n) e^{-j\frac{2\pi}{N}kn}$$

$$\left.\begin{aligned} x(n) &= \sum_{n=0}^{N-1} X_k e^{j\frac{2\pi}{N}kn} \\ X_k &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \end{aligned}\right\} \text{DTFS}$$

$$X_k = \frac{1}{N} X\left(e^{j\frac{2\pi}{N}k}\right) = \frac{1}{N} X(k)$$

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{+j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

DFT

(3) $x(n) = \{\underline{2}, 1, 3, 1\}$ DFT = ?

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$N = 4$

$$X(k) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4}kn} = 2\cdot e^{0} + 1\cdot e^{-j\frac{2\pi}{4}k1} + 3e^{-j\frac{2\pi}{4}k2} + 1e^{-j\frac{2\pi}{4}k3} =$$

$$X(0) = 2 + 1 + 3 + 1 = 7$$

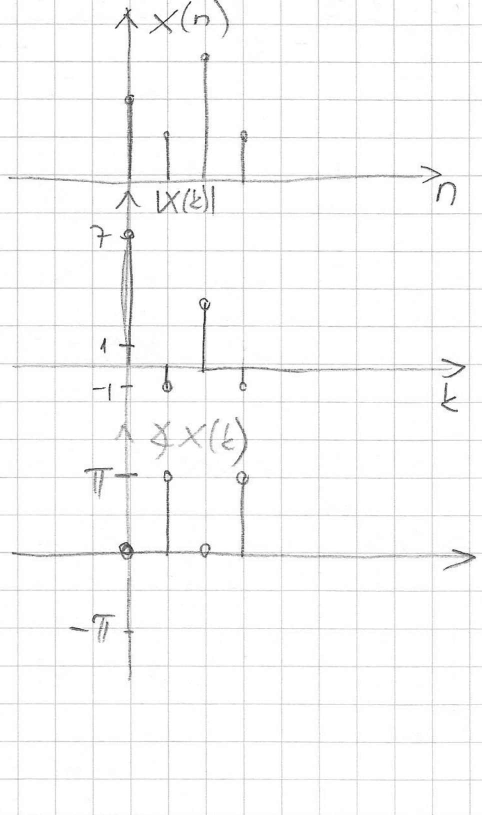
$$X(1) = 2 + 1\cdot e^{-j\frac{2\pi}{4}} + 3e^{-j\frac{2\pi}{4}\cdot 2} + 1e^{-j\frac{2\pi}{4}\cdot 3} = 2 + 0 - \cancel{j} - 3 + 0 + 0 + \cancel{j} = -1$$

$$2 + \cos\left(\frac{2\pi}{4}\right) - j\sin\left(\frac{2\pi}{4}\right) + 3\cos\left(\frac{4\pi}{4}\right) - 3j\sin\left(\frac{4\pi}{4}\right) + \cos\left(\frac{6\pi}{4}\right) - j\sin\left(\frac{6\pi}{4}\right)$$

$X(2) = 3$

$X(3) = -1$

$X(k) = \{\underline{7}, -1, 3, -1\}$



$$X(k) = \{\underline{7}, -1, 3, -1\}$$

$N = 4 \quad , \quad n = 0, 1, 2, 3$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}nk}$$

$$= \frac{1}{4}\left[7 e^{j\frac{2\pi}{4}n\cdot 0} - 1 e^{j\frac{2\pi}{4}n\cdot 1} + 3 e^{j\frac{2\pi}{4}n\cdot 2} - e^{j\frac{2\pi}{4}\cdot n\cdot 3}\right]$$

$$x(0) = \frac{1}{4}\left[7 - 1 + 3 - 1\right] = 2$$

$$x(1) = \frac{1}{4}\left[7 e^{j\frac{2\pi}{4}\cdot 1\cdot 0} - 1 e^{j\frac{2\pi}{4}\cdot 1} + 3 e^{j\frac{2\pi}{4}\cdot 2} - e^{j\frac{2\pi}{4}\cdot 3}\right] = \frac{4}{4} = 1$$

$$x(2) = \frac{1}{4}\left[7 - e^{j\frac{2\pi}{4}2} + 3 e^{j\frac{2\pi}{4}\cdot 2\cdot 2} - e^{j\frac{2\pi}{4}\cdot 2\cdot 3}\right] = 3$$

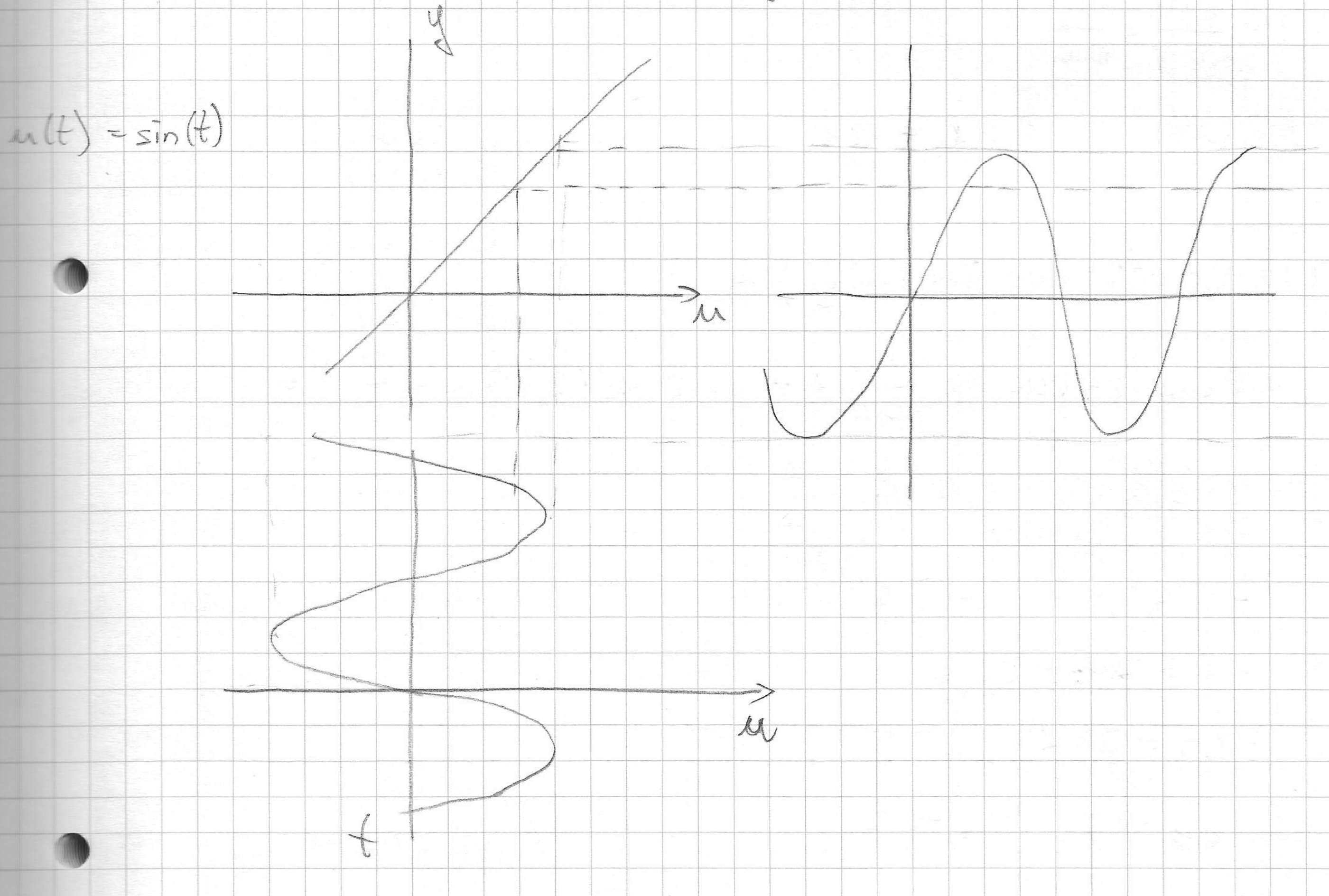
$$x(3) = \frac{1}{4}\left[7 - e^{j\frac{2\pi}{4}\cdot 3} + 3 e^{j\frac{2\pi}{4}\cdot 3\cdot 2} - e^{j\frac{2\pi}{4}\cdot 3\cdot 3}\right] = 1$$

$$x(n) = \{\underline{2}, 1, 3, 1\}$$

Sustavi

$$u(t) = \sin\left(2t + \frac{\pi}{3}\right) \longrightarrow \boxed{y(t) = 2\,u(t)} \longrightarrow y(t) = 2\sin\left(2t + \frac{\pi}{3}\right)$$

Kontinuirani sustavi $\rightarrow$ $u(t)$, $y(t)$



Diskretni sustavi $\rightarrow$ $u(n)$, $y(n)$

Auditorne - VIII. tjedan

$$x(t) = \sin(8000\pi t) + 2\cos\left(24000\pi t + \frac{\pi}{3}\right) + \sin(16000\pi t)$$

$f_s = 10\ kHz$

$\omega_s > 2\omega_{max}$

$$T_1 = \frac{2\pi}{8000\pi} = \frac{1}{4000}$$

$f_1 = 4000$ $\qquad f_2 = 12000$ $\qquad f_3 = 8000$

$$x_2(nT) = 2\cos\left(24000\pi nT + \frac{\pi}{3}\right) =$$

$$= 2\cos\left(24000\pi n \frac{1}{10000} + \frac{\pi}{3}\right) = 2\cos\left(2{,}4\pi n + \frac{\pi}{3}\right) =$$

$$= 2\cos\left((2{,}4\pi + 2k\pi)n + \frac{\pi}{3}\right)$$

$$-\pi \le 2{,}4\pi + 2k\pi \le \pi$$

$$-1 \le 2{,}4 + 2k \le 1$$

$$-3{,}4 \le 2k \le -1{,}4$$

$$k \ge -1{,}7 \quad k \le -0{,}7$$

$$\boxed{k = -1}$$

$$x_2(n) = 2\cos\left[(2{,}4\pi - 2\pi)n + \frac{\pi}{3}\right) = 2\cos\left(0{,}4\pi n + \frac{\pi}{3}\right)$$

$$x_2(t) = 2\cos\left(0{,}4\pi \cdot t \cdot 10000 + \frac{\pi}{3}\right) = 2\cos\left(4000\pi t + \frac{\pi}{3}\right)$$

$$x_3(nT) = \sin(1{,}6\pi n) = \sin\left[(1{,}6\pi + 2k\pi)n\right] = \sin(-0{,}4\pi n) = -\sin(0{,}4\pi n)$$

$$x_3(t) = -\sin(4000\pi t)$$

⑥ a) $x(n) = \cos\left(\frac{\pi}{2}n\right)$, $n = 0,1,2,3$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = 1\cdot e^{-j\frac{2\pi}{4}k\cdot 0} + 0\cdot e^{-j\frac{2\pi}{4}\cdot k\cdot 1} - 1\, e^{-j\frac{2\pi}{4}k\cdot 2} + 0\cdot e^{-j}$$

$x(0) = 1$

$$= 1 - 1\, e^{-j\pi k} = 1 - (-1)^k$$

$x(1) = 0$

$\cos(\pi k) - j\sin(\pi k)$

$1,-1,1,-1$ $\quad 0$

$x(2) = -1$

$x(3) = 0$

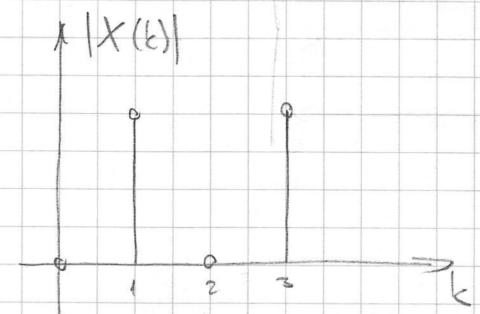
$X(0) = 0$

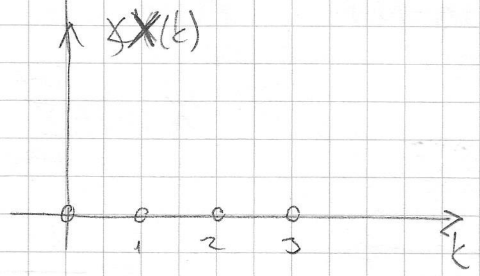
$X(1) = 2$

$X(2) = 0$

$X(3) = 2$

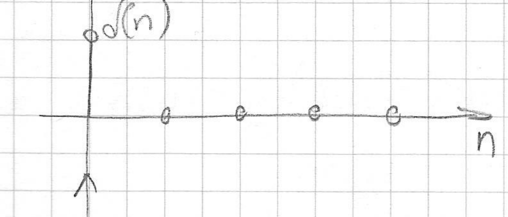
$X(k) = \{\underline{0}, 2, 0, 2\}$

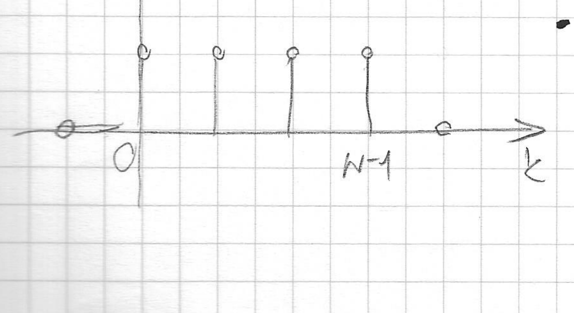




② a) $x(n) = \delta(n)$

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} = \sum_{0}^{N} \delta(n)\, e^{-j\frac{2\pi}{N}kn} = 1$$



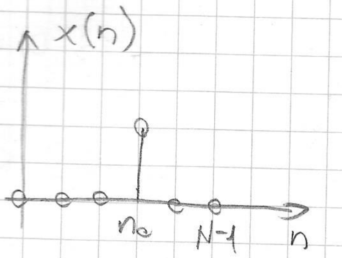


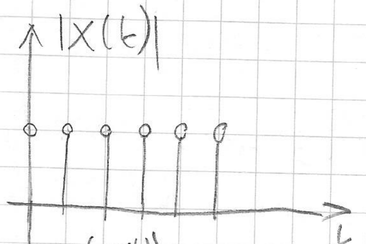
$x(n) = \delta(n-n_0)$

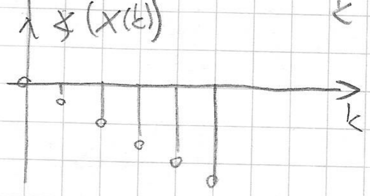
$$X(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\, e^{-j\frac{2\pi}{N}kn} = e^{-j\frac{2\pi}{N}kn_0}$$

, $n_0 \geq 0$, $n_0 \leq N-1$

(jer suma ide od 0 do $N-1$)

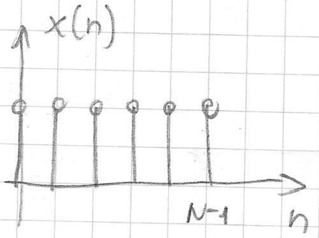






$x(n) = \mu(n) - \mu(n-N)$

$$\sum_{k=0}^{N-1} q^k = \frac{1-q^k}{1-q}$$

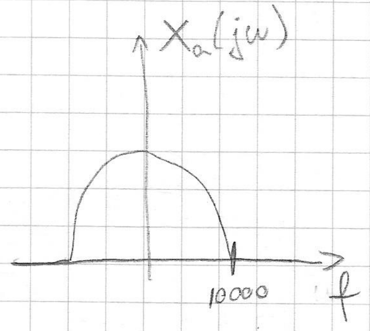


$$X(k) = \sum_{n=0}^{N-1} 1 \cdot e^{-j\frac{2\pi}{N}kn} =$$

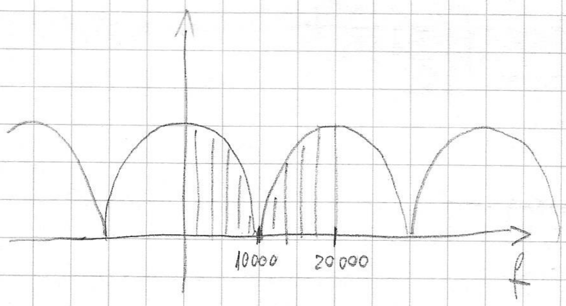
$$= \frac{1 - e^{-j\frac{2\pi}{N}kN}}{1 - e^{-j\frac{2\pi}{N}k}} = 0$$

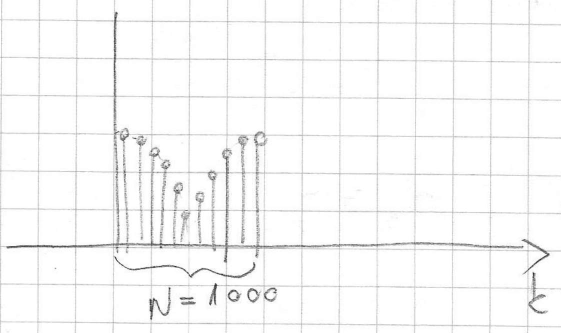
$$X(0) = \sum_{n=0}^{N-1} 1 \cdot e^{-j \cdot 0} = N$$

7



$f_s = 20\,000\ Hz$





$$\frac{20\,000}{1000} = \underline{20}$$

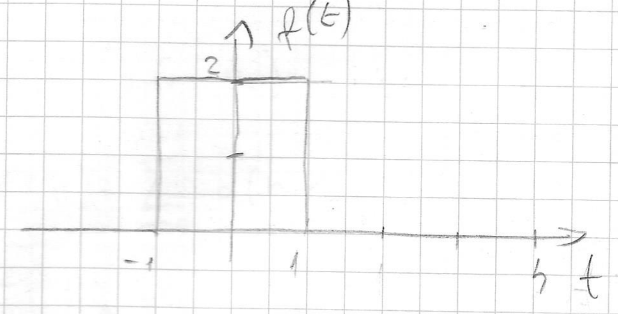
MI. 2014

1) c) $f(n) = \{16, 4, 0, 4, 16\}$

$f_2(n) = \{-8, -4, 0, 4, 8\}$

d) $f_3(n) = \frac{1}{T_s}\left(f(n) - f(n-1)\right) = \{-6, -6, 2, 6, \ldots\}$

2



$$F_k = \frac{1}{T_0}\int_{T_0} f(t)\, e^{-j\omega_0 k t}\, dt = \frac{1}{4}\int_{-1}^{1} e^{-j\frac{2\pi}{4}kt}\, dt = \frac{1}{2}\, \frac{e^{-j\frac{2\pi}{4}kt}}{-j\frac{2\pi}{4}k}\Bigg|_{-1}^{1} =$$

$$= \frac{1}{-j\frac{4\pi}{4}k}\left[e^{-j\frac{2\pi}{4}k} - e^{-j\frac{2\pi}{4}k(-1)}\right] = \frac{1}{-j\pi k}\cdot(-2j)\,\sin\frac{2\pi}{4}k =$$

$$= \frac{\sin\left(\frac{2\pi}{4}k\right)}{\frac{2\pi}{4}}$$

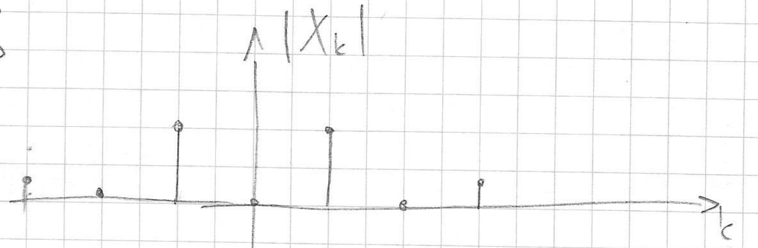
$k = \{-3, -2, -1, 0, 1, 2, 3\}$

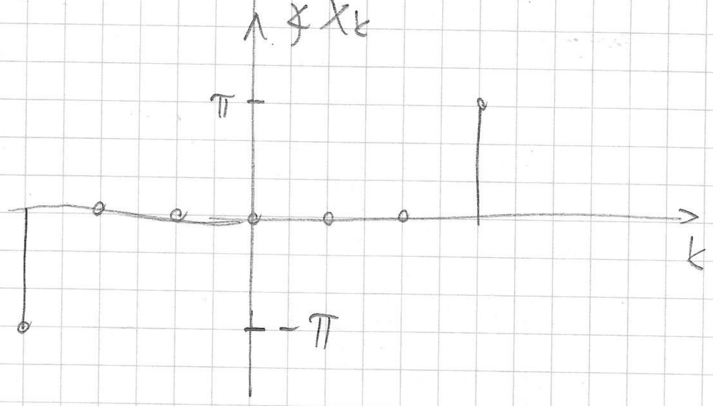
$X_{-3} = \frac{2}{4\pi}$ $\qquad X_1 = \frac{2}{\pi}$

$X_{-2} = 0$ $\qquad X_2 = 0$

$X_{-1} = \frac{2}{\pi}$ $\qquad X_3 = -\frac{2}{4\pi}$

$X_0 = 0$





$$f(t) = \sum_{k=-\infty}^{\infty} F_k e^{j\omega_0 k t} = \sum_{k=-\infty}^{-1} F_k e^{j\omega_0 k t} + F_0 + \sum_{k=1}^{\infty} F_k e^{j\omega_0 k t} =$$

$$= \sum_{k=1}^{\infty} F_{-k} e^{-j\omega_0 k t} + F_0 + \sum_{k=1}^{\infty} F_k e^{j\omega_0 k t} = \sum_{k=1}^{\infty} |F_k| e^{-j\sphericalangle F_k} e^{-j\omega_0 k t} + F_0 +$$

$$+ \sum_{k=1}^{\infty} |F_k| e^{j\sphericalangle F_k} e^{j\omega_0 k t} =$$

za realne signale vrijedi:

$$F_{-k} = |F_k| e^{j\sphericalangle F_{-k}} = |F_k| e^{-j\sphericalangle F_k}$$

↳ okreni

$$= \sum_{k=1}^{\infty} |F_k| \left[e^{-j(\omega_0 k t + \measuredangle F_k)} + e^{j(\omega_0 k t + \measuredangle F_k)} \right] + F_0 =$$

$$= F_0 + \sum_{k=1}^{\infty} |F_k| \cdot 2\cos(\omega_0 k t + \measuredangle F_k)$$

Nastavak predavanja

$$\underset{\{1,2,3\}}{u(n)} \longrightarrow \boxed{w(n) = 2u(n)+1} \longrightarrow \underset{\{3,5,7\}}{w(n)} \longrightarrow \boxed{y(n) = \frac{w(n)-1}{2}} \longrightarrow \underset{\{1,2,3\}}{y(n)}$$

$\rightarrow$ Kaskadni spoj

Inverzne f-je:

$w = 2u + 1$

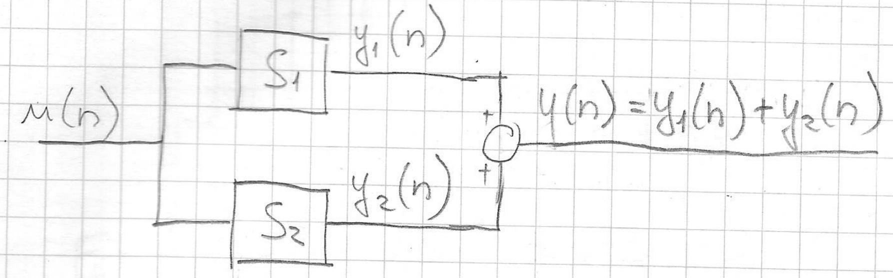
$u = 2w + 1$

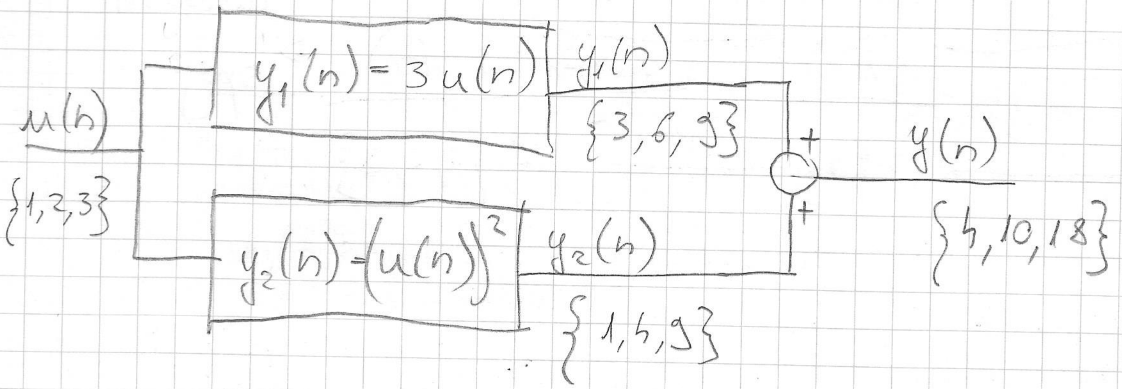
$2w = u - 1$

$w = \frac{u-1}{2}$ $\rightarrow$ inverzna f-ja od $w = 2u + 1$

$$\underset{\{1,2,3\}}{u(n)} \longrightarrow \boxed{w(n) = \frac{u(n)-1}{2}} \longrightarrow \underset{\{0,\frac{1}{2},1\}}{w(n)} \longrightarrow \boxed{y(n) = 2w(n)+1} \longrightarrow \underset{\{1,2,3\}}{y(n)}$$

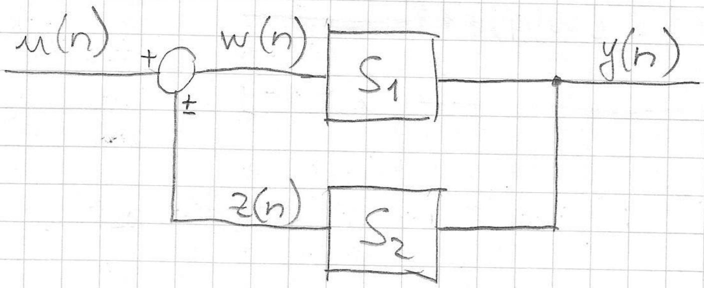
Paralelni spoj

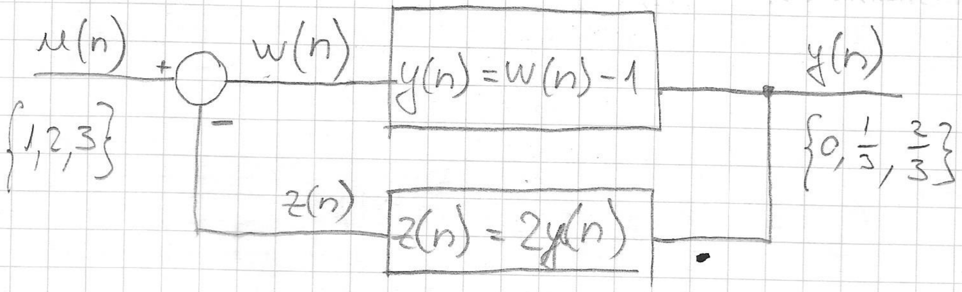




$$y(n) = y_1(n) + y_2(n) = 3u(n) + (u(n))^2$$

Povratna veza





$$y(n) = w(n) - 1 =$$

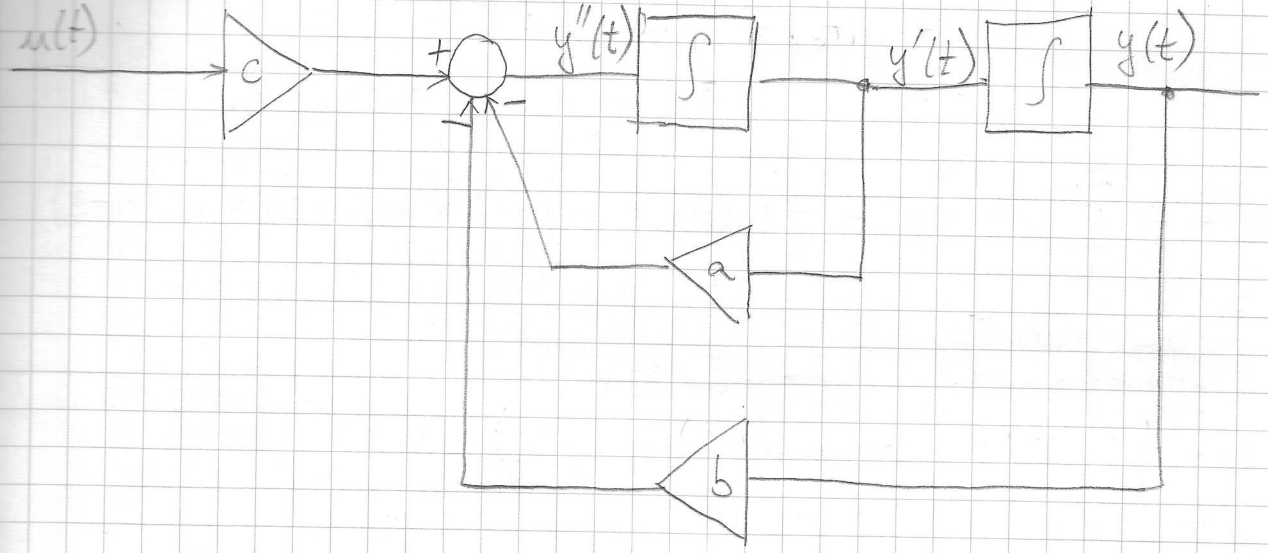
$$= u(n) - z(n) - 1 =$$

$$= u(n) - 2y(n) - 1$$

$$3y(n) = u(n) - 1$$

$$y(n) = \frac{u(n) - 1}{3}$$

$$y''(t) + a y'(t) + b y(t) = c\, u(t) \rightarrow y''(t) = c\, u(t) - a y'(t) - b y(t)$$



Svojstva sustava

Besmemorijski sustavi:

npr. $y(t) = \frac{1}{2} u(t)$

$$\forall t \in \mathbb{R} \qquad y(t) = S(u)(t) = S(u(t))$$

$y(t) = \frac{1}{2} u(t-1)$ -> memorijski sustav (ovisi o prošlosti)

$y(n) = u(n^2)$ -> memorijski sustav (ovisi o budućnosti)

$y(5) = u(25)$

$$y(t)=\int_{-\infty}^{t}u(\tau)\,d\tau=\int_{-\infty}^{t_0}u(\tau)\,d\tau+\int_{t_0}^{t}u(\tau)\,d\tau=$$

$$=y(t_0)+\int_{t_0}^{t}u(\tau)\,d\tau$$

$$y(n)=\frac{1}{4}\sum_{n=0}^{3}u(n-m)=\frac{1}{4}\big(u(n)+u(n-1)+u(n-2)+u(n-3)\big)$$

ima memoriju

$$y(0)=\frac{1}{4}\big(u(0)+u(-1)+u(-2)+u(-3)\big)$$

Kauzalnost sustava

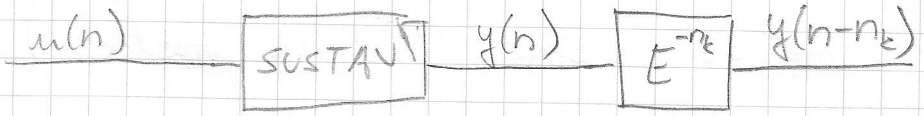
izlaz ovisi o sadašnjosti i prošlosti

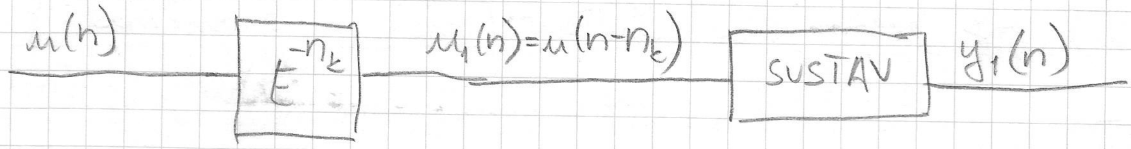
Kauzalan sustav je sustav koji poznaje prošlost i sadašnjost

Nekauzalan sustav je sustav koji poznaje prošlost, sadašnjost i budućnost

izlaz ovisi o sadašnjosti, prošlosti i budućnosti.

Vremenski stalni sustavi





Ako su $y(n-n_k)=y_1(n)$ sustav je vremenski stalan

inače je vremenski promjenjiv.

$$y(n) = \begin{cases} u\left(\frac{n}{3}\right), & n = 0, 3, 6 \dots \\ 0, & \text{inače} \end{cases}$$

$$u(n) = \{1, 2, 3\} \longrightarrow y(n) = \{1, 0, 0, 2, 0, 0, 3\}$$

$$u(n) \longrightarrow \boxed{\text{SUSTAV}} \longrightarrow y(n) = \begin{Bmatrix} u\left(\frac{n}{3}\right), & n = 0,3,6 \\ 0, & \text{inače} \end{Bmatrix} \Rightarrow y(n - n_k) = \begin{Bmatrix} u\left(\frac{n-n_k}{3}\right), & n = 0,3,6 \\ 0, & \text{inače} \end{Bmatrix}$$

$$u(n) \xrightarrow{\text{KAŠNJENJE}} u_1(n) = u(n - n_k) \xrightarrow{\text{SUSTAV}} y_1(n) = \begin{Bmatrix} u_1\left(\frac{n}{3}\right), & 0,3,6 \\ 0, & \text{inače} \end{Bmatrix} =$$

$$= \begin{Bmatrix} u\left(\frac{n}{3} - n_k\right), & n = 0,3,6 \\ 0 & \text{inače} \end{Bmatrix}$$

izrazi nisu jednaki

$\Rightarrow$ vremenski promjenjiv sustav

$$y(t) = 5t\,u(t)$$

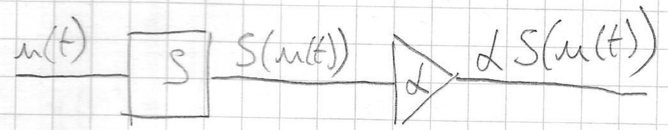
$$u(t) \xrightarrow{S} y(t) = 5t\,u(t) \xrightarrow{\text{KAŠNJ.}} y(t - t_k) = 5(t - t_k)\,u(t - t_k)$$

$$u(t) \xrightarrow{\text{KAŠNJ.}} u_1(t) = u(t - t_k) \xrightarrow{S} y_1(t) = 5t\,u_1(t) = 5t\,u(t - t_k)$$

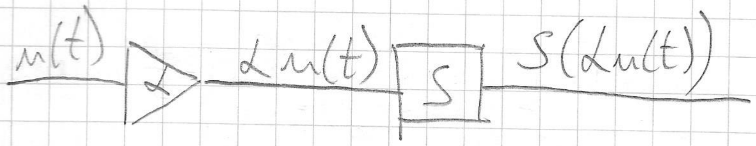
Vremenski promjenjiv.

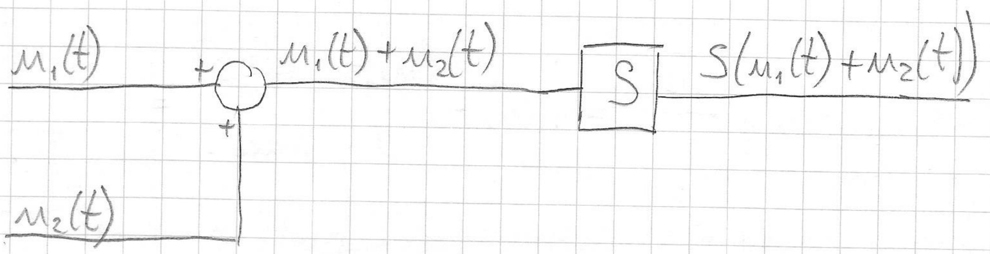
Linearnost sustava

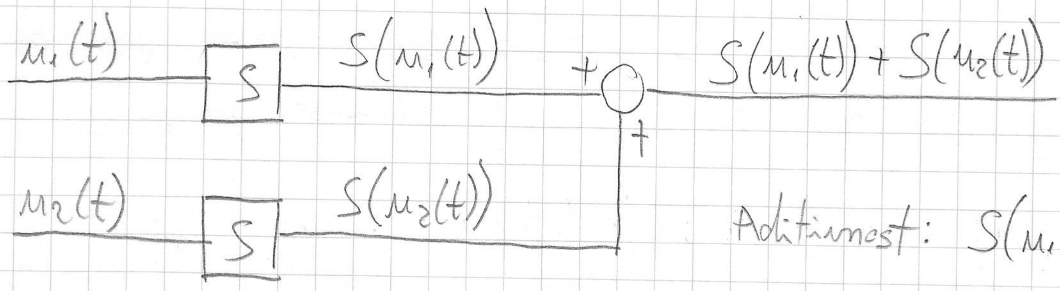
Sustav je linearan ako zadovoljava uvjete homogenosti i aditivnosti.



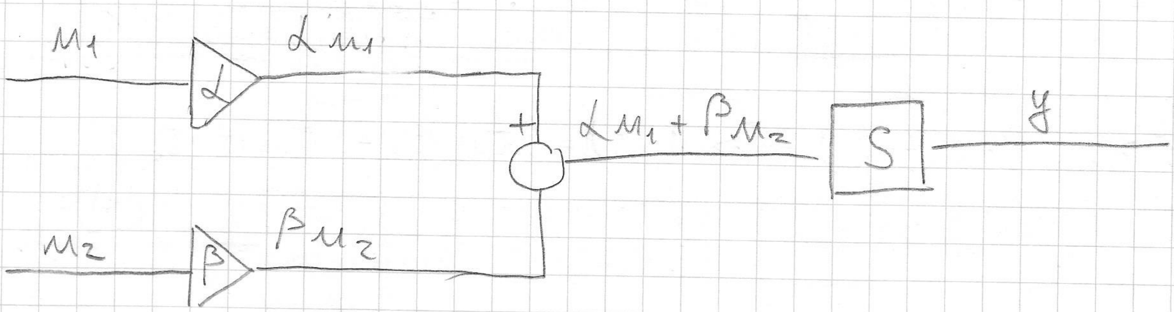
Homogenost: $S(\alpha u(t)) = \alpha S(u(t))$

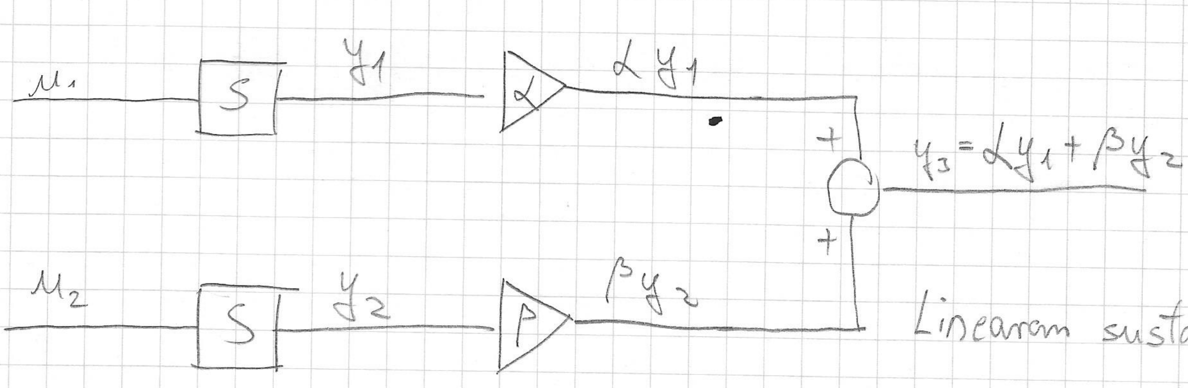






Aditivnost: $S(u_1(t) + u_2(t)) = S(u_1(t)) + S(u_2(t))$





Linearan sustav: $y = y_3$

Pr. $y(t) = 5t\,u(t)$

$$u_1(t) \longrightarrow \alpha u_1(t)$$
$$u_2(t) \longrightarrow \beta u_2(t)$$
$$\longrightarrow u(t) = \alpha u_1(t) + \beta u_2(t) \longrightarrow y(t) = 5t\left(\alpha u_1(t) + \beta u_2(t)\right)$$

$$u_1(t) \longrightarrow y_1(t) = 5t\,u_1(t) \longrightarrow \alpha y_1(t)$$
$$u_2(t) \longrightarrow y_2(t) = 5t\,u_2(t) \longrightarrow \beta y_2(t)$$
$$\longrightarrow y_3(t) = \alpha 5t\,u_1(t) + \beta 5t\,u_2(t)$$

$y(t) = y_3(t) \Rightarrow$ Sustav je linearan

Pr. $y(t) = \sin(u(t))$

$$u(t) = \alpha u_1(t) + \beta u_2(t) \longrightarrow y(t) = \sin\left(\alpha u_1(t) + \beta u_2(t)\right)$$

$$u(t) = \sin(u_1(t)) + \sin(u_2(t)) \longrightarrow \alpha \sin(u_1(t)) + \beta \sin(u_2(t)) = y_3(t)$$

$y(t) \neq y_3(t)$

Nije linearan

Linearnost integratora

$$y(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$$

$$\left.\begin{aligned} u_1 &\Rightarrow y_1(t) = \int_{-\infty}^{t} u_1(\tau)\,d\tau \\ u_2 &\Rightarrow y_2(t) = \int_{-\infty}^{t} u_2(\tau)\,d\tau \end{aligned}\right\} \Rightarrow y_3(t) = \alpha \int_{-\infty}^{t} u_1(\tau)\,d\tau + \beta \int_{-\infty}^{t} u_2(\tau)\,d\tau$$

$$\left.\begin{aligned} &\alpha u_1(t) \\ &\beta u_2(t) \end{aligned}\right\} \Rightarrow u(t) = \alpha u_1(t) + \beta u_2(t) \Rightarrow y(t) = \int_{-\infty}^{t} \left(\alpha u_1(\tau) + \beta u_2(\tau)\right) d\tau =$$

$$= \int_{-\infty}^{t} \alpha u_1(\tau)\,d\tau + \int_{-\infty}^{t} \beta u_2(\tau)\,d\tau$$

Linearan je

$$y(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = y(t_0) + \int_{t_0}^{t} u(\tau)\,d\tau$$

$$\left.\begin{aligned} u_1(t) &\rightarrow \alpha u_1(t) \\ u_2(t) &\rightarrow \beta u_2(t) \end{aligned}\right\} \Rightarrow u(t) = \alpha u_1(t) + \beta u_2(t) \Rightarrow y(t) = y(t_0) + \int_{t_0}^{t} \left(\alpha u_1(\tau) + \beta u_2(\tau)\right)$$

$$\left.\begin{aligned} u_1(t) &\Rightarrow y_1(t) = y(t_0) + \int_{t_0}^{t} u_1(\tau)\,d\tau \\ u_2(t) &\Rightarrow y_2(t) = y(t_0) + \int_{t_0}^{t} u_2(\tau)\,d\tau \end{aligned}\right\} \alpha y(t_0) + \alpha \int_{t_0}^{t} u_1(\tau)\,d\tau + \beta y(t_0) + \beta \int_{t_0}^{t} u_2(\tau)\,d\tau$$

Nije linearan

DZ: $y(n) = a u(n) + b$ → provjera linearnosti

Stabilnost sustava

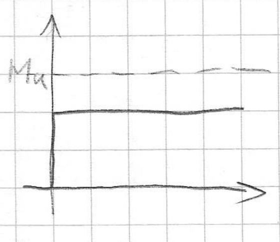
- BIBO stabilnost (Bounded Input Bounded Output)

Sustav je BIBO stabilan ako je za svaki omeđeni ulaz njegov izlaz također omeđen.

$$|u(t)| \leq M_u < \infty \longrightarrow |y(t)| \leq M_y < \infty$$

$$|u(n)| \leq M_u < \infty \longrightarrow |y(n)| \leq M_y < \infty$$

P. $y(n) = 7u(n) + 6$



$$y_n = 7M_u + 6 = M_y < \infty$$

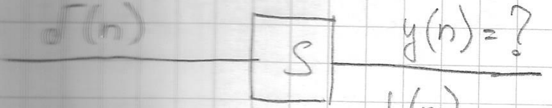
Sustav je BIBO stabilan

$$y(n) = \frac{1}{L+1} \sum_{n=0}^{L} u(n-m)$$

$$= \frac{1}{L+1} \underbrace{\sum_{n=0}^{L} M_u}_{(L+1)M_u} = M_u < \infty$$

$$y(t) = \int_{-\infty}^{t} u(\tau)\, d\tau \Rightarrow \int_{-\infty}^{t} 1\, d\tau = \tau \Big|_{-\infty}^{t} = t + \infty = \infty \Rightarrow \text{Nije BIBO stabilan}$$

Impulsni odziv vremenski diskretnog sustava



$h(n)$

impulsni odziv

početni uvjeti = 0

$$y(n) - 0{,}75\,y(n-1) = u(n)$$

$$u(n) = \delta(n)$$

$$y(n) = 0{,}75\,y(n-1) + u(n)$$

$$y(n) = 0{,}75\,y(n-1) + \delta(n)$$

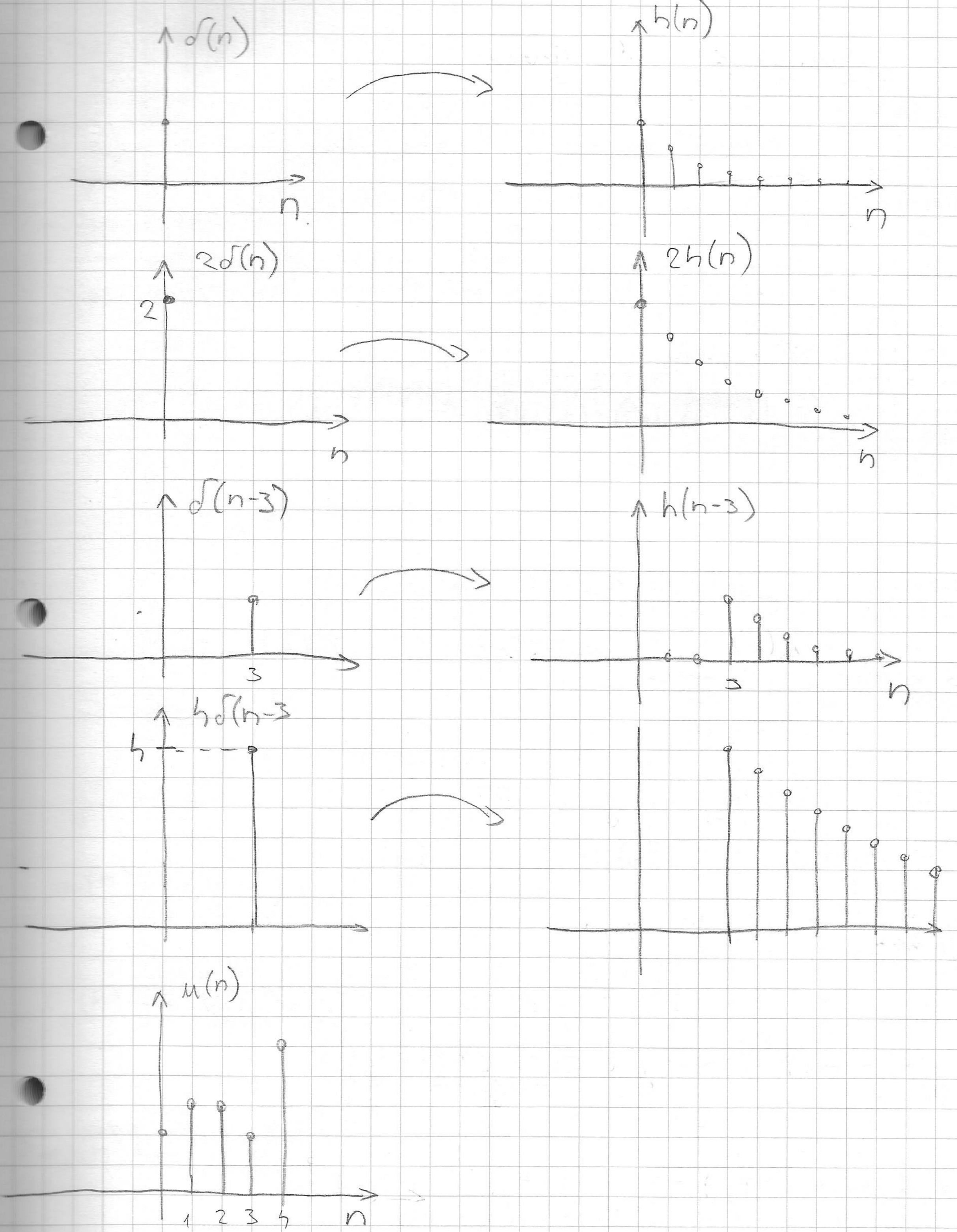
$n=0$ $\quad h(0) = 0{,}75\,y(-1) + \delta(0) = 1$

$n=1$ $\quad h(1) = 0{,}75\,h(0) + \delta(1) = 0{,}75$

$n=2$ $\quad h(2) = 0{,}75\,h(1) + \delta(2) = (0{,}75)^2$

$n=3$ $\quad h(3) = 0{,}75\,h(2) + \delta(3) = (0{,}75)^3$

$$h(n) = \begin{cases} (0{,}75)^n, & n \geq 0 \\ 0, & n < 0 \end{cases} = (0{,}75)^n \mu(n)$$



$$u(n) = u(0)\delta(n) + u(1)\delta(n-1) + u(2)\delta(n-2) + u(3)\delta(n-3) + u(4)\delta(n-4)$$

$$y(n) = u(0)h(n) + u(1)h(n-1) + u(2)h(n-2) + \dots$$

$$y(n) = \sum_{m=-\infty}^{\infty} u(m)h(n-m)$$

Auditorne - X. tjedan

1) $y(n) = 2^{u(n)}$ Nema memorije (izlaz ovisi samo o sadašnjosti)

2) $y(t) = u(t^2)$ Ima memoriju [npr. $y(2) = u(4)$] (izlaz ovisi o prošlosti ili budućnosti)

1. – kauzalan (ovisi o sadašnji. ili prošlosti) 2. – nije kauzalan (ovisi o budućnosti)

Vremenska stalnost:

$$y(n) = 2^{u(n)}$$

$$u(n) \rightarrow y(n) = 2^{u(n)} \longrightarrow 2^{u(n-n_k)}$$

Vremenski stalan s.

$$u(n) \rightarrow u_1(n) = u(n-n_k) \rightarrow y_1(n) = 2^{u(n-n_k)}$$

$$y(t) = u(t^2)$$

$$u(t) \rightarrow y(t) = u(t^2) \rightarrow y(t-t_k) = u\left((t-t_k)^2\right)$$

Vremenski promjenjiv s.

$$u(t) \rightarrow u_1(t) = u(t-t_k) \rightarrow y_1(t) = u\left(t^2 - t_k\right)$$

Linearnost sustava

$$y(t) = u(t^2)$$

$$\left.\begin{aligned} u_1(t) &\rightarrow y_1(t) = u_1(t^2) \rightarrow \alpha y_1(t) = \alpha u_1(t^2) \\ u_2(t) &\rightarrow y_2(t) = u_2(t^2) \rightarrow \beta y_2(t) = \beta u_2(t^2) \end{aligned}\right\} + \rightarrow y_3(t) = \alpha u_1(t^2) + \beta u_2(t^2)$$

$$\left.\begin{aligned} u_1(t) &\rightarrow \alpha u_1(t) \\ u_2(t) &\rightarrow \beta u_2(t) \end{aligned}\right\} + \; u(t) = \alpha u_1(t) + \beta u_2(t) \rightarrow y(t) = \alpha u_1(t^2) + \beta u_2(t^2)$$

Sustav je linearan

$$y(n) = 2^{u(n)}$$

$$\left.\begin{aligned} u_1(n) &\rightarrow y_1(n) = 2^{u_1(n)} \rightarrow \alpha 2^{u_1(n)} \\ u_2(n) &\rightarrow y_2(n) = 2^{u_2(n)} \rightarrow \beta 2^{u_2(n)} \end{aligned}\right\} + \rightarrow \alpha 2^{u_1(n)} + \beta 2^{u_2(n)}$$

$$\left.\begin{aligned} u_1(n) &\rightarrow \alpha u_1(n) \\ u_2(n) &\rightarrow \beta u_2(n) \end{aligned}\right\} u(n) = \alpha u_1(n) + \beta u_2(n) \rightarrow 2^{\alpha u_1(n) + \beta u_2(n)}$$

Sustav nije linearan

Stabilnost sustava (BIBO stabilnost)

$y(t) = u(t^2)$ -> BIBO stabilan

$y(n) = 2^{u(n)}$ -> BIBO stabilan

Inverzni sustav

$y(n) = 2^{u(n)}$

$u = 2^y$

$\log_2 u = \log_2 2^y = y \log_2 2$

$y(n) = \log_2 u(n)$

ulaz u 1. sustav — izlaz iz 1. sustava (ulaz u 2. sustav)

a) $\alpha u_1(n) + \beta u_2(n) \rightarrow \alpha w_1(n) + \beta w_2(n) \rightarrow \alpha y_1(n) + \beta y_2(n)$

$u(n-n_k) \rightarrow w(n-n_k) \rightarrow y(n-n_k)$

b) $w(n) = 2^{u(n)} \rightarrow y(n) = \log_2 w(n)$

$u(n) \rightarrow w(n) = 2^{u(n)} \rightarrow y(n) = \log_2 2^{u(n)} = u(n)$

7 a) $u_1(n) = (-1)^n$ $\quad y_1(n) = 1$

$u_2(n) = (-1)^{n+1}$ $\quad y_2(n) = 1$

$= -1(-1)^n$

$= -u_1(n)$

Vremenski stalan

b) $u_3(n) = (-1)^n$ $\quad y_3 = 1$

$u_4(n) = (-1)^{n+1}$ $\quad y_4 = -1$

Vremenski promjenjiv

8 $u_1(t) = \mu(t)$ $\quad y_1(t) = (1 - e^{-2t})\mu(t)$

$u_2(t) = 4\mu(t) - 4\mu(t-1)$ $\quad y_2(t) = ?$

$4\mu(t)$ $\quad 4(1 - e^{-2t})\mu(t)$

$\mu(t-1)$ $\quad (1 - e^{-2(t-1)})\mu(t-1)$

$-4\mu(t-1)$ $\quad -4(1 - e^{-2(t-1)})\mu(t-1)$

$u_2(t) = 4\mu(t) - 4\mu(t-1)$ $\quad y_2(t) = 4(1 - e^{-2t})\mu(t) - 4(1 - e^{-2(t-1)})\mu(t-1)$

$$y(n) = \underbrace{\sum_{m=-\infty}^{-1} u(m)h(n-m)}_{\substack{0 \\ \text{za kauzalne signale}}} + \sum_{m=0}^{n} u(m)h(n-m) + \sum_{m=n+1}^{+\infty} u(m)h(n-m)$$

$$\underbrace{\begin{matrix} h(n-(n+1)) = h(-1) \\ h(n-(n+2)) = h(-2) \\ h(n-(n+3)) = h(-3) \\ \vdots \end{matrix}}_{\substack{0 \\ \text{za kauzalne signale}}}$$

$$y(n) = \sum_{m=0}^{n} u(m)h(n-m) \quad \text{- za kauzalne signale}$$

2 LTI - linearni vremenski nepromjenjivi sustav

$$y(n) - 0{,}75\, y(n-1) = u(n)$$

$$u(n) = 0{,}5^n \mu(n)$$

$$h(n) = 0{,}75^n \mu(n)$$

$$y(n) = \sum_{m=0}^{n} u(m)h(n-m) = \sum_{m=0}^{n} 0{,}5^m \, 0{,}75^{n-m} = 0{,}75^n \sum_{m=0}^{n} \left(\frac{0{,}5}{0{,}75}\right)^m =$$

$$= 0{,}75^n \frac{1-\left(\frac{0{,}5}{0{,}75}\right)^{n+1}}{1-\left(\frac{0{,}5}{0{,}75}\right)} = 0{,}75^n \frac{\frac{0{,}75^{n+1} - 0{,}5^{n+1}}{0{,}75^{n+1}}}{\frac{0{,}75-0{,}5}{0{,}75}} =$$

$$= \cancel{0{,}75^n} \frac{\cancel{0{,}75} \cdot [0{,}75^{n+1} - 0{,}5^{n+1}]}{\cancel{0{,}75^{n+1}} [0{,}25]} = 4 \left[0{,}75^{n+1} - 0{,}5^{n+1}\right] \mu(n)$$

A.
$$y(n) = \frac{1}{M+1} \sum_{m=0}^{M} u(n-m)$$

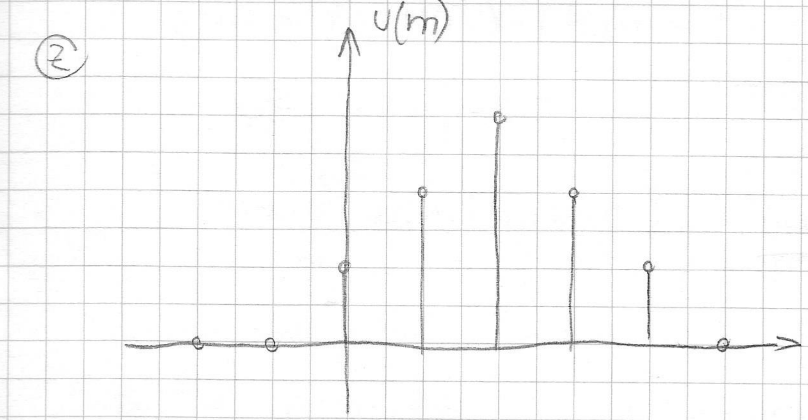
$\rightarrow M = 3$

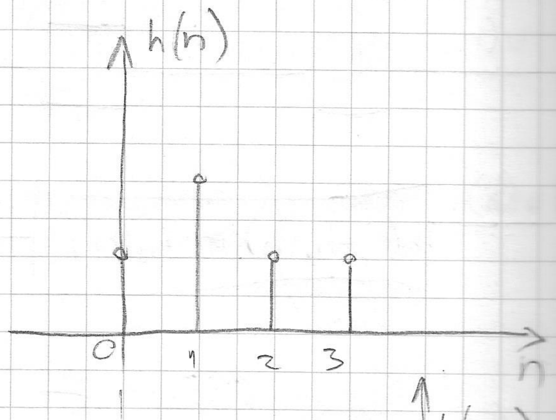
$$y(n) = \frac{1}{4} \sum_{m=0}^{3} u(n-m) = \sum_{m=0}^{3} \frac{1}{4} u(n-m) \Rightarrow h(m) = \frac{1}{4}$$

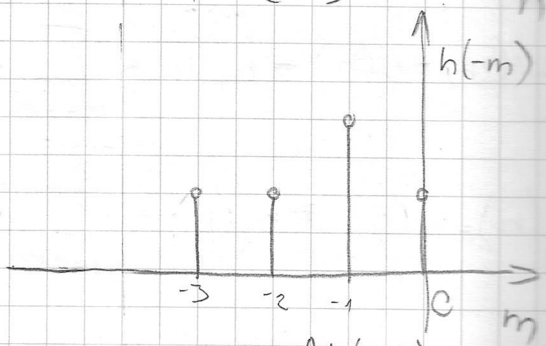
$$h(n) = \frac{1}{4}\delta(n) + \frac{1}{4}\delta(n-1) + \frac{1}{4}\delta(n-2) + \frac{1}{4}\delta(n-3)$$

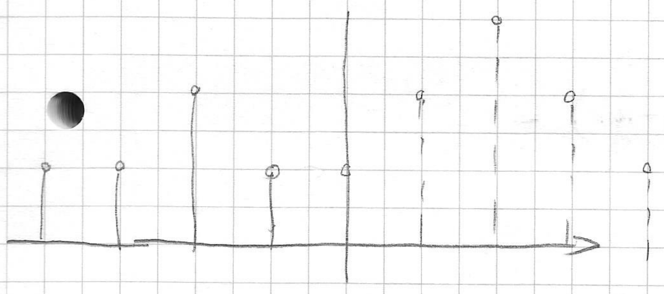
$$y(n) = \frac{1}{4}u(n) + \frac{1}{4}u(n-1) + \frac{1}{4}u(n-2) + \frac{1}{4}u(n-3)$$

(2)



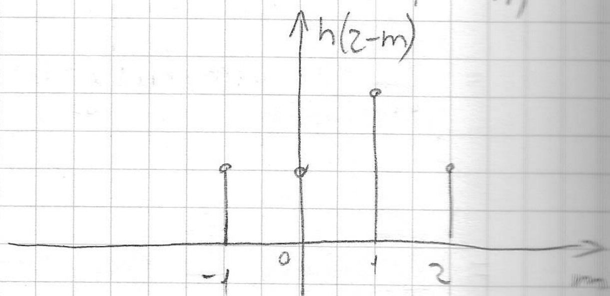


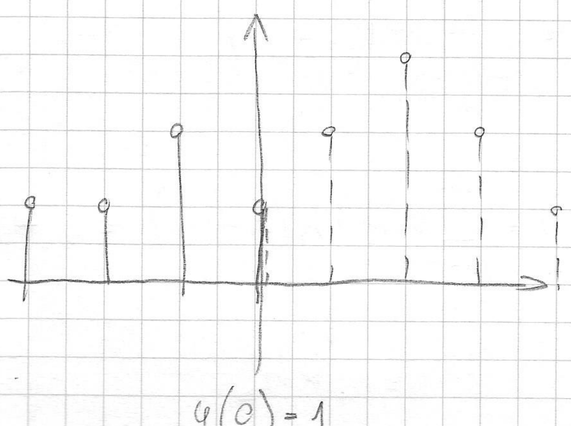




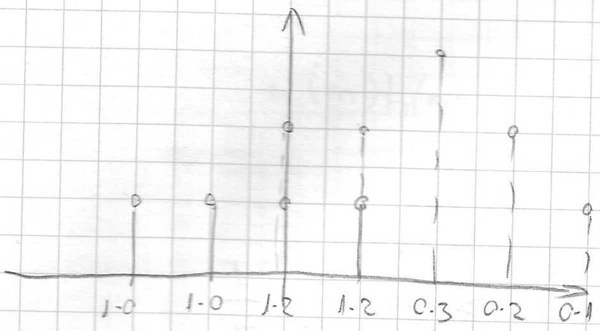
$h(-1-m)$

$y(-1) = 0$





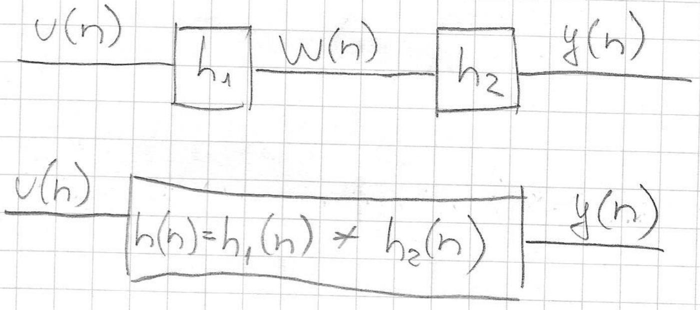
$y(0) = 1$



$y(1) = 4$

Svojstva konvolucije

$$y(n) = [\overbrace{v(n) * h_1(n)}^{w(n)}] * h_2(n) = v(n) * [h_1(n) * h_2(n)]$$



Asocijativnost

$$w(n) = v(n) * h_1(n) = \sum_{m=-\infty}^{\infty} v(m) h_1(n-m)$$

$$y(n) = w(n) * h_2(n) = \sum_{j=-\infty}^{\infty} w(j) h_2(n-j) =$$

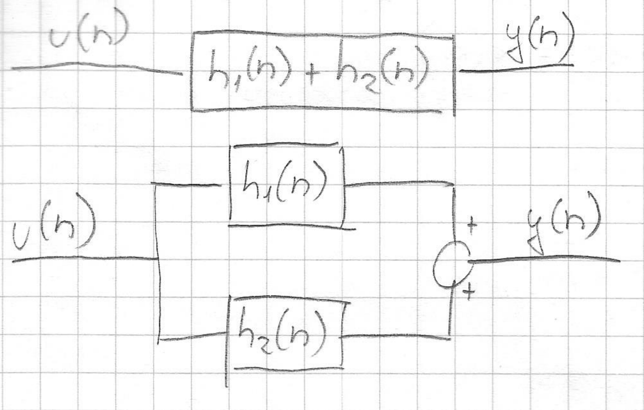
$$= \sum_{j=-\infty}^{\infty} \left[\sum_{m=-\infty}^{\infty} v(m) h_1(j-m) \right] h_2(n-j) =$$

$$= \sum_{m=-\infty}^{\infty} v(m) \sum_{j=-\infty}^{\infty} h_1(j-m) h_2(n-j) = \left| \begin{matrix} k = j-m \\ j = k+m \end{matrix} \right| =$$

$$= \sum_{m=-\infty}^{\infty} v(m) \sum_{k=-\infty}^{+\infty} h_1(k) h_2(n-m-k) = \sum_{m=-\infty}^{\infty} v(m) h(n-m) = v * h$$

$h = h_1 * h_2$

Distributivnost



$$y(n) = u(n) * (h_1(n) + h_2(n)) = u(n) * h_1(n) + u(n) * h_2(n)$$

$$y(n) = \sum_{m=-\infty}^{\infty} u(m) \cdot \left(h_1(n-m) + h_2(n-m)\right) =$$

$$= \sum_{m=-\infty}^{\infty} u(m) h_1(n-m) + \sum_{m=-\infty}^{\infty} u(m) h_2(n-m) = u * h_1 + u * h_2$$

Svojstvo pomaka

$$y(n) = x_1(n) * x_2(n)$$

$$y_1(n) = x_1(n-p) * x_2(n-q) =$$

$$= \sum_{m=-\infty}^{\infty} x_1(m-p)\, x_2(n-m-q) = \left| \begin{matrix} j = m-p \\ m = j+p \end{matrix} \right| =$$

$$= \sum_{j=-\infty}^{\infty} x_1(j)\, x_2(n-p-q-j) = y(n-p-q)$$

Konvolucija sa impulsom

$$x(n) * \delta(n) = \sum_{m=-\infty}^{\infty} x(m)\,\underbrace{\delta(n-m)}_{m=n} = x(n)$$

$$x(n) * \delta(n-p) = x(n-p)$$

(2) $x_1(n) = \{\underline{1}, 2, 3\}$

$x_2(n) = \{\underline{0}, 2, 1\} = 2\delta(n-1) + \delta(n-2)$

$x_1(n) * x_2(n) = ?$

$$x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)\,x_2(n-m) = 2\sum_{m=-\infty}^{\infty} x_1(m)\,\delta(n-m-1) + \sum_{m=-\infty}^{\infty} x_2(m)\,\delta(n-m-2)$$

$$= 2x_1(n-1) + x_2(n-2) =$$

$$= 2 \cdot \{\underline{0}, 1, 2, 3\} + \{\underline{0}, 0, 1, 2, 3\} =$$

$$= \{\underline{0}, 2, 4, 6\} + \{\underline{0}, 0, 1, 2, 3\} = \{\underline{0}, 2, 5, 8, 3\}$$

$$y(n) = x_1(n) * x_2(n) = \sum_{m=0}^{n} x_1(m)\, x_2(n-m)$$

$$0 \leq m \leq L_1 - 1$$

L_1, L_2 – broj uzoraka signala

$$0 \leq n-m \leq L_2 - 1 \quad / + m$$

$$m \leq n \leq L_2 - 1 + m$$

$$0 \leq m \leq L_2 - 1 + m \leq L_2 - 1 + L_1 - 1$$

$$0 \leq n \leq L_1 + L_2 - 2$$

$$\Downarrow$$

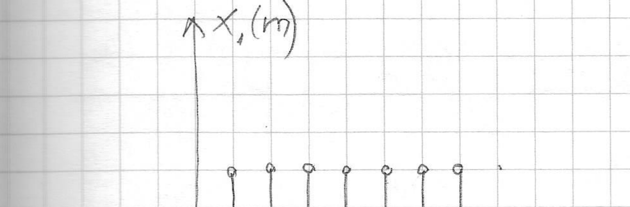
$\boxed{L_1 + L_2 - 1}$ → broj uzoraka nakon konvolucije

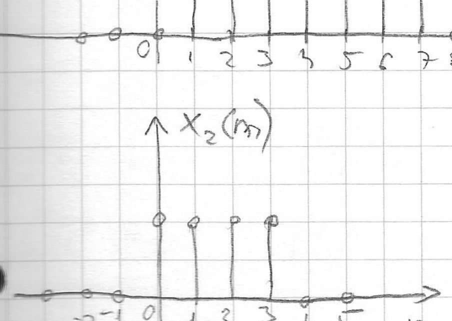
(2)

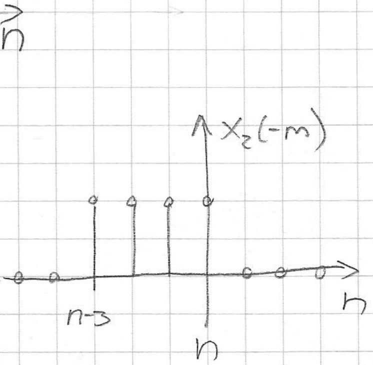
$$x_1(n) = \begin{cases} 1, & 1 \leq n \leq 7 \\ 0, & \text{inače} \end{cases}$$

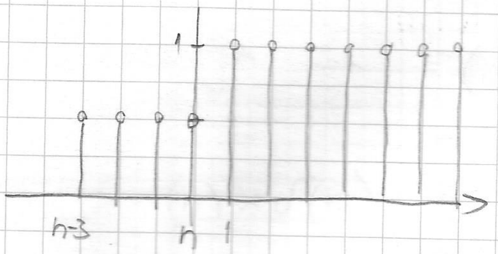
$$x_2(n) = \begin{cases} \frac{1}{2}, & 0 \leq n \leq 3 \\ 0, & \text{inače} \end{cases}$$

$$y(n) = x_1(n) * x_2(n) = \sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)$$



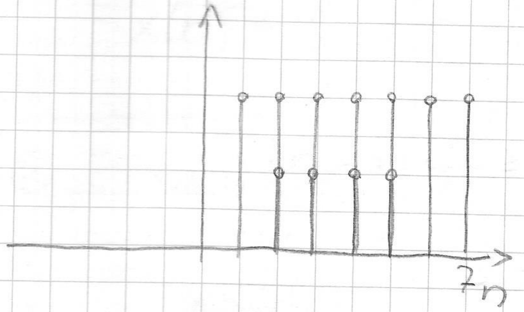






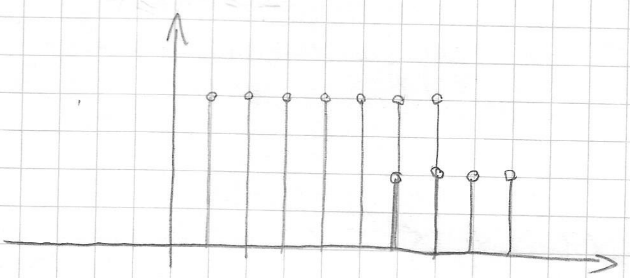
$1 \leq n < 4 \rightarrow$ djelomično preklamanje

$$y(n) = \sum_{m=1}^{n} 1 \cdot \frac{1}{2} = \frac{1}{2} n$$



$4 \leq n \leq 7 \rightarrow$ potpuno preklapanje

$$y(n) = \sum_{m=n-3}^{n} 1 \cdot \frac{1}{2} = \frac{4}{2} = 2$$



$8 \leq n \leq 10 \rightarrow$ djelomično preklapanje

$$y(n) = \sum_{m=n-3}^{7} 1 \cdot \frac{1}{2} = \frac{1}{2}\left[7-(n-3)+1\right] =$$

$$= \frac{1}{2}\left[7 - n + 4\right] = \frac{11}{2} - \frac{1}{2} n$$

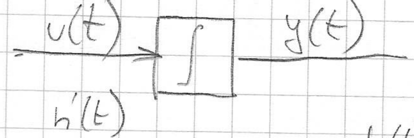
$$y(n) = \begin{cases} 0, & n \leq 0 \\ \frac{n}{2}, & 1 \leq n < 4 \\ 2, & 4 \leq n \leq 7 \\ \frac{11-n}{2}, & 8 \leq n \leq 10 \\ 0, & n > 10 \end{cases}$$

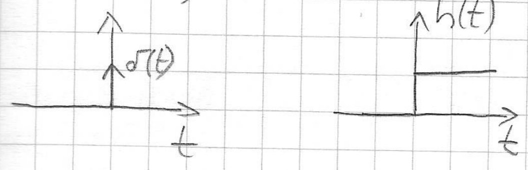
Impulsni odziv vremenski kontinuiranog signala

$$y(t) = \int_{-\infty}^{t} u(\tau)\,d\tau = y(0^-) + \int_{0^-}^{t} u(\tau)\,d\tau$$

Impulsni odziv

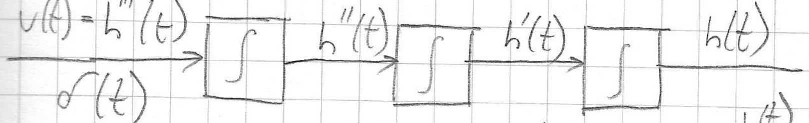
$$h(t) = 0 + \int_{0^-}^{t} \delta(\tau)\,d\tau = 1 \cdot \mu(t)$$

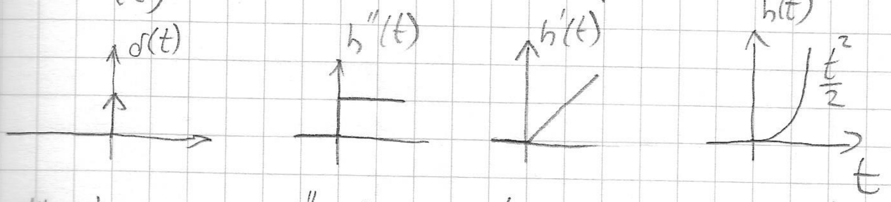




$h'(0^-) = 0 \qquad h(0^-) = 0$

$h'(0^+) = 0 \qquad h(0^+) = 1$





$h'''(0^-) = 0 \qquad h''(0^-) = 0 \qquad h'(0^-) = 0 \qquad h(0^-) = 0$

$h'''(0^+) = 0 \qquad h''(0^+) = 1 \qquad h'(0^+) = 0 \qquad h(0^+) = 0$

$$y'(t) + 2y(t) = u(t)$$

$$u(t) = \delta(t)$$

$$h'(t) + 2h(t) = \delta(t)$$

$$h'(t) - \delta(t) - 2h(t)$$

1) $t < 0 \Rightarrow h(t) = 0$

2) $t > 0 \Rightarrow h'(t) = -2h(t)$

$$h(t) = c e^{-2t}$$

3) $t = 0 \Rightarrow h'(t) = -2h(t) + \delta(t) \Big/ \int_{0^-}^{0^+}$

$$\int_{0^-}^{0^+} h'(t)\,dt = -2\int_{0^-}^{0^+} h(t)\,dt + \underbrace{\int_{0^-}^{0^+} \delta(t)\,dt}_{1}$$

$$h(0^+) - h(0^-) = -2 \cdot 0 + 1$$

$$h(0^+) - h(0^-) = 1$$

$$h(0^+) = 1$$

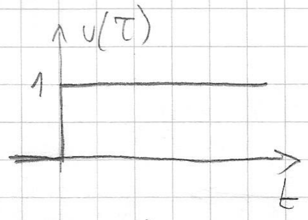
$$h(0^+) = c = 1$$

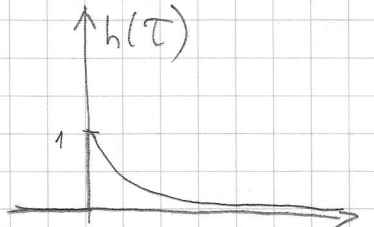
$$h(t) = e^{-2t}\mu(t)$$

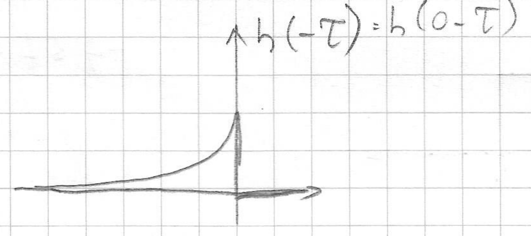
$$\boxed{y(t) = \int_{-\infty}^{+\infty} u(\tau)\, h(t-\tau)\, d\tau}$$

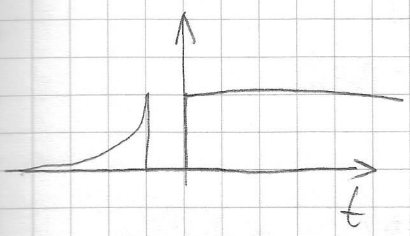
(6) $v(t) = \mu(t)$

$h(t) = e^{-3t}\mu(t)$

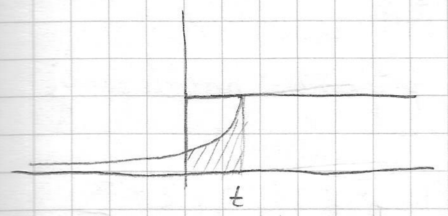








$t < 0 \Rightarrow y(t) = 0$

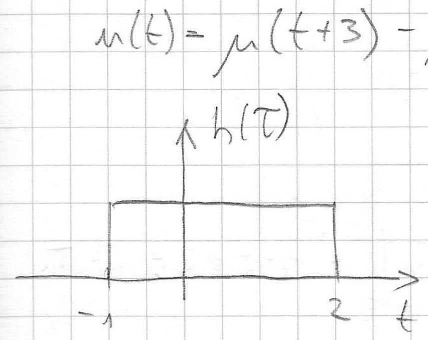


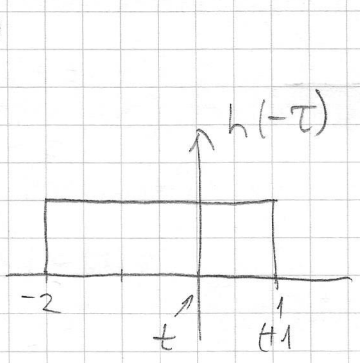
$$t \geq 0 \Rightarrow y(t) = \int_0^t 1\, e^{-3(t-\tau)}\, d\tau =$$

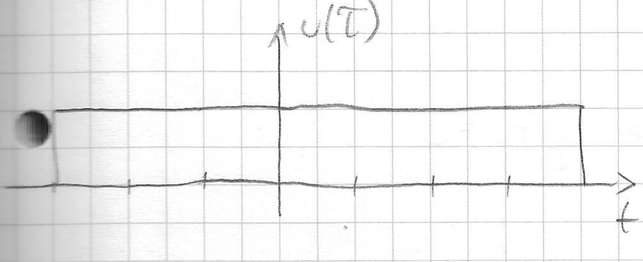
$$= e^{-3t} \left.\frac{e^{3\tau}}{3}\right|_0^t = \frac{e^{-3t}}{3}\left(e^{3t} - 1\right) = \frac{1}{3} - \frac{e^{-3t}}{3}$$

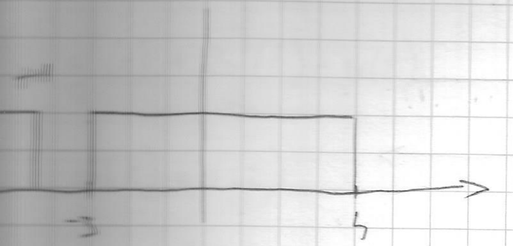
(7) $h(t) = \mu(t+1) - \mu(t-2)$

$u(t) = \mu(t+3) - \mu(t-4)$





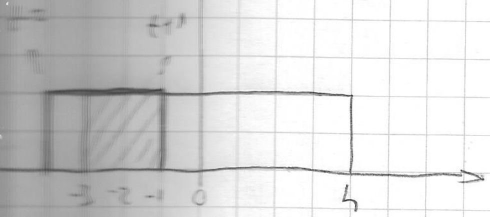




① $t+1=-3$

$t=-4$

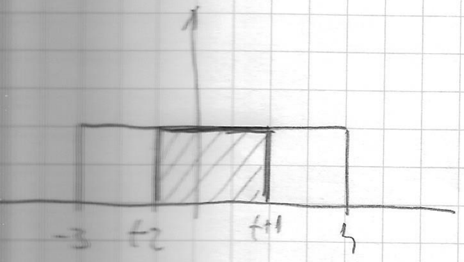
$t<-4 \Rightarrow y(t)=0$



$-4 \le t < -1$ $\qquad$ $t-2=-3$

Djelomično $\qquad$ $t=-1$

$$y(t)=\int_{-3}^{t+1} 1\cdot 1\, d\tau = \tau \Big|_{-3}^{t+1} = t+1+3 = t+4$$



$t-2=-3$ $\qquad$ $t+1=4$

$t=-1$ $\qquad$ $t=3$

$-1 \le t \le 3 \Rightarrow$ potpuno preklapanje

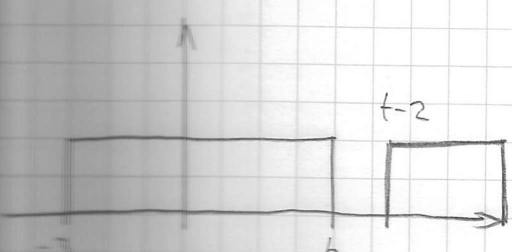
$$y(t)=\int_{t-2}^{t+1} 1\cdot 1\, d\tau = \tau \Big|_{t-2}^{t+1} = t+1-t+2 = 3$$

$t+1=4$ $\qquad$ $t-2=4$

$t=3$ $\qquad$ $t=6$

$3 < t \le 6$

$$y(t)=\int_{t-2}^{4} 1\cdot 1\, d\tau = \tau \Big|_{t-2}^{4} = 4-t+2 = 6-t$$



$t-2>4$

$t>6 \Rightarrow y(t)=0$

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \dots + a_n y(n-N) =$$

$$= b_0 u(n) + b_1 u(n-1) + b_2 u(n-2) + \dots + b_n u(n-N)$$

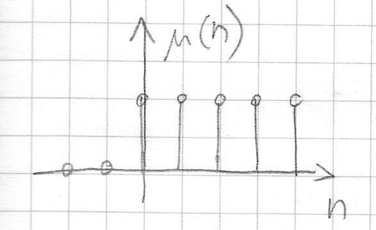
$$y(n) + \sum_{j=1}^{N} a_j y(n-j) = \sum_{j=0}^{N} b_j u(n-j)$$

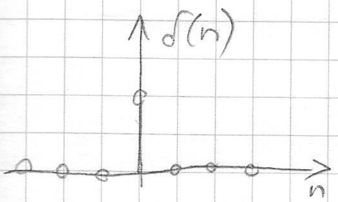
$$y(n+N) + a_1 y(n+N-1) + \dots + a_n y(n) =$$
$$= b_0 u(n+N) + \dots + b_n u(n)$$

Auditorne - XI. tjedan

① $\mu(n) \longrightarrow y(n) = (n+1)\mu(n)$

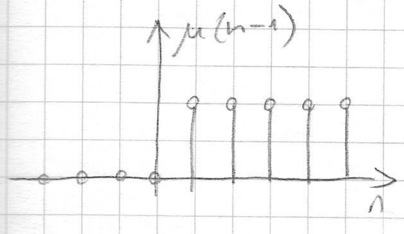
$\delta(n) \longrightarrow h(n) = ?$





$$\delta(n) = \mu(n) - \mu(n-1)$$

$$h(n) = (n+1)\mu(n) - [(n-1)+1]\mu(n-1) =$$
$$= (n+1)\mu(n) - n\mu(n-1)$$

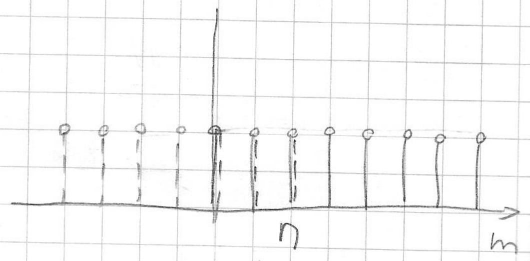


$$h(5) = 6\mu(5) - 5\mu(5) = 1$$

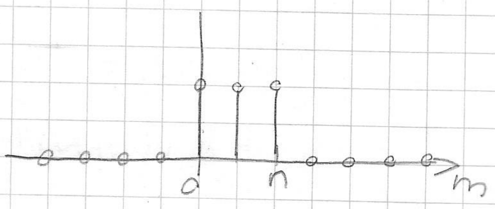
$$y(n) = \sum_{m=-\infty}^{+\infty} v(m)h(n-m) = \sum_{m=-\infty}^{+\infty} v(n-m)h(m)$$

③ $v(n) = \alpha^n \mu(n)$

$h(n) = \beta^n \mu(n)$



$$y(n) = \sum_{m=-\infty}^{+\infty} \alpha^m \mu(m) \beta^{n-m} \mu(n-m) =$$



$$= \sum_{m=0}^{n} \alpha^m \beta^{n-m} = \beta^n \sum_{m=0}^{n} \left(\frac{\alpha}{\beta}\right)^m =$$

$$= \beta^n \frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\frac{\alpha}{\beta}} = \cancel{\beta^n} \frac{\frac{\beta^{n+1}-\alpha^{n+1}}{\cancel{\beta^{n+1}}}}{\frac{\beta-\alpha}{\cancel{\beta}}} = \frac{\beta^{n+1}-\alpha^{n+1}}{\beta-\alpha}$$

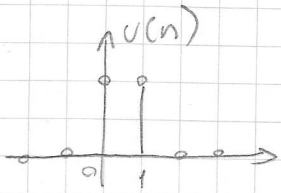
$$\sum_{n=0}^{N-1} q^n = \frac{1-q^N}{1-q}$$

$\alpha = \beta$

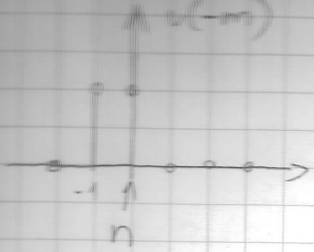
$$y(n) = \beta^n \sum_{m=0}^{n} \left(\frac{\beta}{\beta}\right)^m = (n+1)\beta^n = (n+1)\alpha^n$$

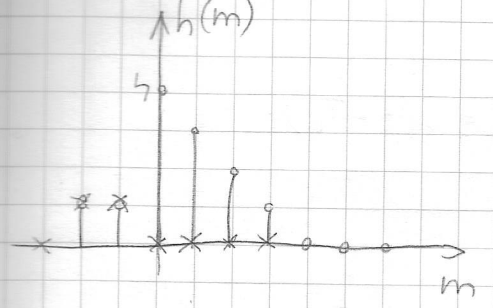
④ $h(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$

$v(n) = \delta(n) + \delta(n-1)$



$$y(n) = \sum_{m=-\infty}^{+\infty} h(m)v(n-m) = \sum_{m=-\infty}^{+\infty} v(m)h(n-m)$$

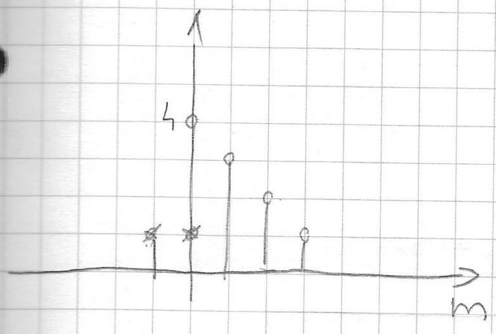




$n < 0$

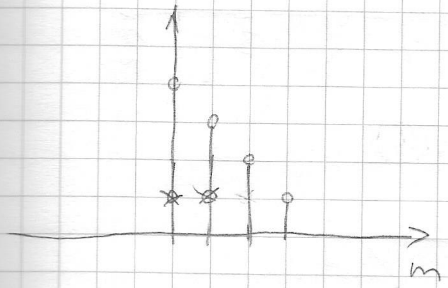
$y(n) = 0$

$n = -1$



$n = 0$

$y(n) = 4$



$n = 1$

$y(n) = 7$

$n = 2 \Rightarrow y(n) = 5$

$n = 3 \Rightarrow y(n) = 3$

$n = 4 \Rightarrow y(n) = 1$

$n \geq 5 \Rightarrow y(n) = 0$

6 $$y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{+\infty} u(t-\tau)\,h(\tau)\,d\tau$$

$$u(t) * \delta(t) = \int_{-\infty}^{+\infty} u(\tau)\,\delta(t-\tau)\,d\tau = u(t)$$

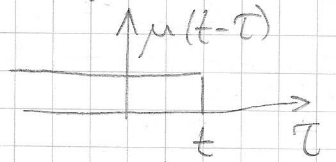
$t-\tau=0$

$\tau=t$

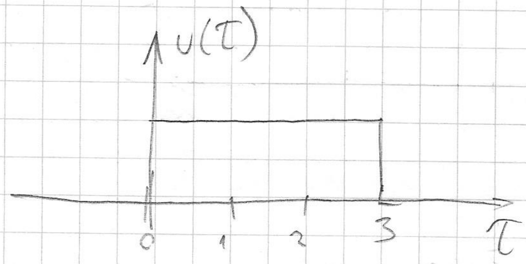
$$u(t) * \delta(t-T_0) = \int_{-\infty}^{+\infty} u(\tau)\,\delta(t-T_0-\tau)\,d\tau = u(t-T_0)$$

$\tau = t - T_0$

$$u(t) * \mu(t) = \int_{-\infty}^{+\infty} u(\tau)\,\mu(t-\tau)\,d\tau = \int_{-\infty}^{t} u(\tau)\,d\tau =$$

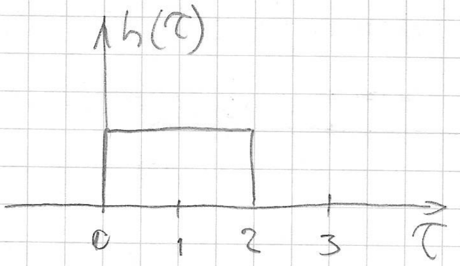


$$u(t) = \begin{cases} 1, & 0<t\leq 3 \\ 0, & \text{inače} \end{cases}$$



$$h(t) = \begin{cases} 1, & 0<t\leq 2 \\ 0, & \text{inače} \end{cases}$$

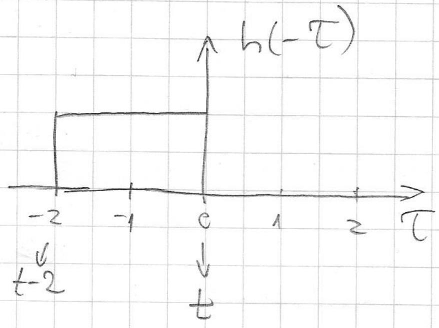
$$y(t) = \int_{-\infty}^{\infty} u(\tau)\,h(t-\tau)\,d\tau$$

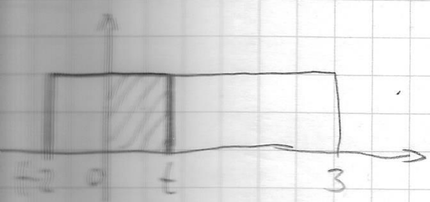


$t<0 \Rightarrow y(t) = 0$

$t=0 \qquad t-2=0$

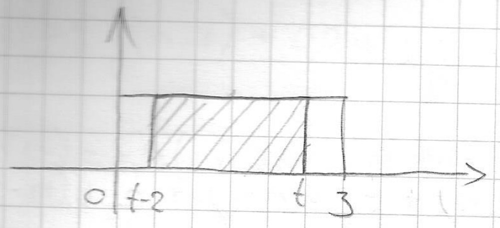
$0 \leq t < 2$





$$y(t) = \int_{0}^{t} 1\cdot 1\,d\tau = t$$

3)



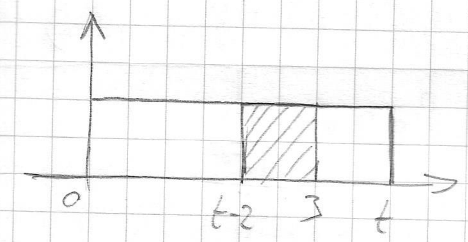
$t-2>0 \qquad t<3$

$t>2$

$2 \leq t < 3$

$$y(t)=\int_{t-2}^{t} d\tau = t-t+2=2$$

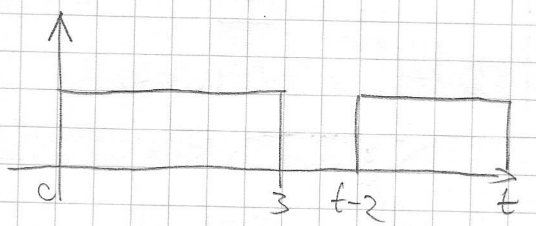
4)



$3 \leq t < 5$

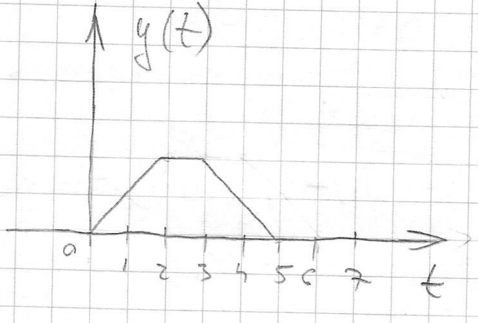
$$y(t)=\int_{t-2}^{3} d\tau = 3-t+2 = -t+5$$

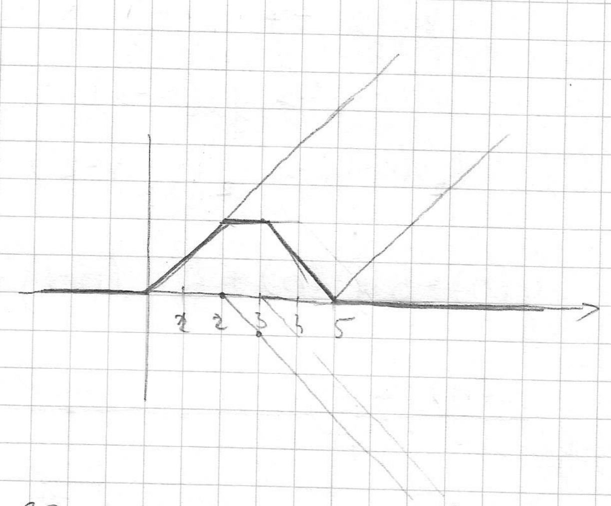
5)



$t \geq 5$

$y(t)=0$





Drugi način rješavanja:

$$v(t)=\mu(t)-\mu(t-3)$$

$$h(t)=\mu(t)-\mu(t-2)$$

$$y(t)=\int_{-\infty}^{+\infty} v(\tau)h(t-\tau)\,d\tau=\int_{-\infty}^{+\infty}[\mu(\tau)-\mu(\tau-3)][\mu(t-\tau)-\mu(t-\tau-2)]\,d\tau=$$

$$=\int_{-\infty}^{+\infty}\mu(\tau)\mu(t-\tau)\,d\tau-\int_{-\infty}^{+\infty}\mu(\tau)\mu(t-\tau-2)\,d\tau-\int_{-\infty}^{+\infty}\mu(\tau-3)\mu(t-\tau)\,d\tau+\int_{-\infty}^{+\infty}\mu(\tau-3)\mu(t-\tau-2)$$

$$=\underset{t\geq 0}{\mu(t)}\int_{0}^{t}d\tau-\underset{t-2\geq 0}{\mu(t-2)}\int_{0}^{t-2}d\tau-\underset{t\geq 3}{\mu(t-3)}\int_{3}^{t}d\tau+\underset{t-2\geq 3}{\mu(t-5)}\int_{3}^{t-2}d\tau=t\mu(t)-(t-2)\mu(t-2)-\mu(t-3)(t-3)+$$

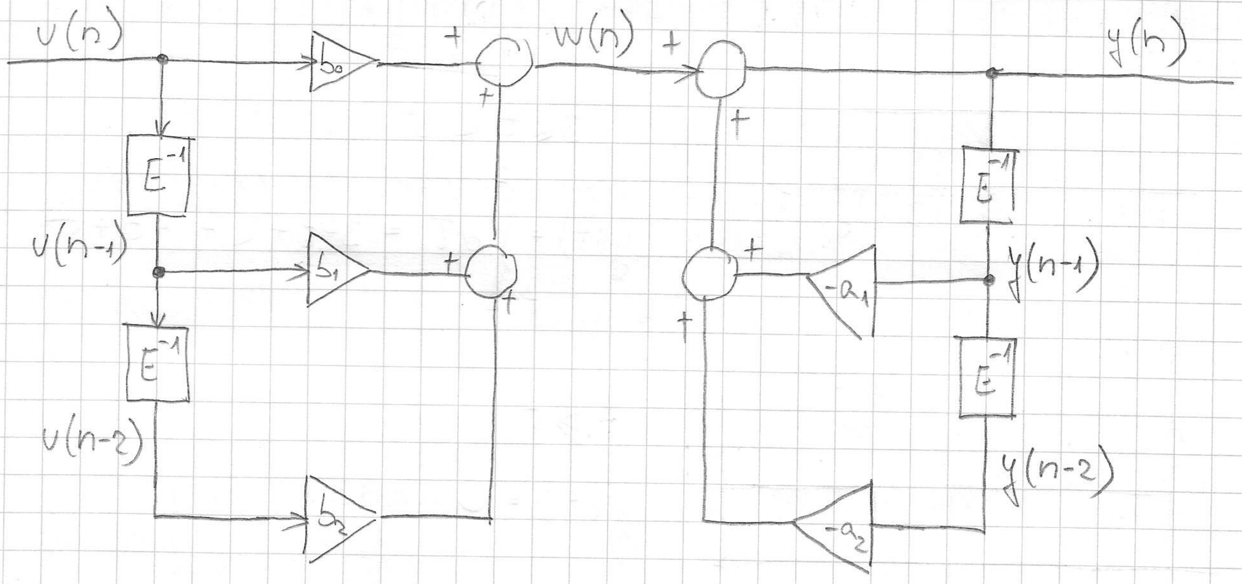
$$+\mu(t-5)(t-5)$$

Nastavak predavanja

Crtanje: direktna realizacija 1

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 u(n) + b_1 u(n-1) + b_2 u(n-2)$$

$$b_0 u(n) + b_1 u(n-1) + b_2 u(n-2) = w(n) \quad , \quad y(n) = w(n) - a_1 y(n-1) - a_2 y(n-2)$$



Drugi način: direktna realizacija 2

$$y_a(n) + a_1 y_a(n-1) + a_2 y(n-2) = u(n) \quad \rightarrow \text{pomoćna formula}$$

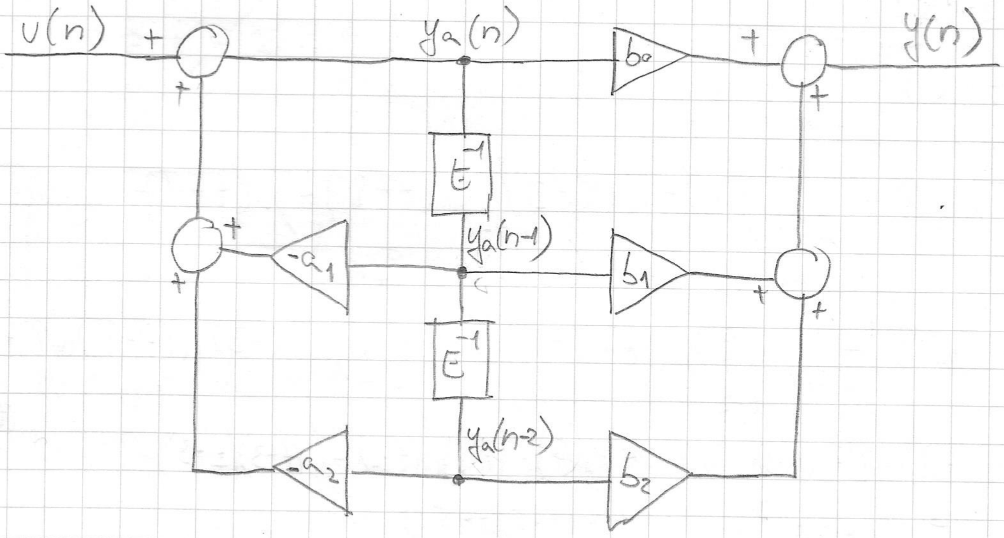
$$y_a(n) = u(n) - a_1 y_a(n-1) - a_2 y_a(n-2)$$

$$u(n) \rightarrow y_a(n)$$

$$u(n-1) \rightarrow y_a(n-1)$$

$$b_2 u(n-2) \rightarrow b_2 y_a(n-2)$$

$$y(n) = b_0 y_a(n) + b_1 y_a(n-1) + b_2 y_a(n-2)$$



$$y(n) - 0{,}9\,y(n-1) + 0{,}2\,y(n-2) = v(n) + v(n-1) + v(n-2)$$

$$v(n) = -9(-0{,}9)^n, \quad n \geq 0$$

$$y(-1) = 6 \qquad y(-2) = -3$$

Homogena jedn. $\quad y(n) - 0{,}9\,y(n-1) + 0{,}2\,y(n-2) = 0$

$$y_h(n) = C q^n$$

$$y_h(n-1) = C q^{n-1}$$

$$y_h(n-2) = C q^{n-2}$$

$$C q^n - 0{,}9\, C q^{n-1} + 0{,}2 \cdot C \cdot q^{n-2} = 0$$

$$C q^{n-2}\left[q^2 - 0{,}9q + 0{,}2\right] = 0$$

$$q^2 - 0{,}9q + 0{,}2 = 0$$

Koja dva broja kada ih pomnožimo daju 0,2, a kada ih zbrojimo $-0{,}9$

$$(q-0{,}5)(q-0{,}4) = 0$$

$$q = \frac{0{,}9 \pm \sqrt{0{,}81-0{,}8}}{2} = \frac{0{,}9 \pm 0{,}1}{2}$$

$$q_1 = 0{,}5 \qquad q_2 = 0{,}4$$

$$y_h(n) = C_1\, 0{,}5^n + C_2\, 0{,}4^n$$

Za slučaj da smo dobili iste q-eve:

npr.: $\quad q^2 - 4q + 4 = 0$

$$q_1 = +2 \quad q_2 = +2$$

$$y_h(n) = C_1\, 2^n + C_2\, n\, 2^n$$

Za kompleksne brojeve:

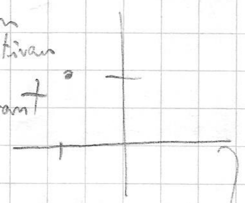
$$q^2 + q + 1 = 0$$

$$q_{12} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}j}{2}$$

$$y_h(n) = C_1\left(\frac{-1+\sqrt{3}j}{2}\right)^n + C_2\left(\frac{-1-\sqrt{3}j}{2}\right)^n$$

$$q_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$$

realni dio negativan
imaginarni pozitivan
II. kvadrant



$$|q_1| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$$

$$\sphericalangle q_1 = \operatorname{arctg} \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \operatorname{arctg}\frac{\sqrt{3}}{-1} = \frac{2\pi}{3}$$

$$q_1 = 1e^{j\frac{2\pi}{3}}$$

$$q_2 = -\frac{1}{2} - \frac{\sqrt{3}}{2}j$$

$$\left.\begin{aligned} |q_2| &= 1 \\ \sphericalangle q_2 &= -\frac{2\pi}{3} \end{aligned}\right\} \quad q_2 = 1e^{-j\frac{2\pi}{3}}$$

$$y_h(n) = \frac{C}{2}e^{+j\theta}\left(1e^{j\frac{2\pi}{3}}\right)^n + \frac{C}{2}e^{-j\theta}\left(1e^{-j\frac{2\pi}{3}}\right)^n$$

$$y_h(n) = \frac{C}{2}e^{j\theta}|1|^n e^{j\frac{2\pi}{3}n} + \frac{C}{2}e^{-j\theta}|1|^n e^{-j\frac{2\pi}{3}n} =$$

$$= \frac{C}{2}|1|^n \underbrace{\left[e^{j\left(\frac{2\pi}{3}n+\theta\right)} + e^{-j\left(\frac{2\pi}{3}n+\theta\right)}\right]}_{2\cos\left(\frac{2\pi}{3}n+\theta\right)} =$$

$$= \frac{C}{2}|1|^n \cdot 2\cos\left(\frac{2\pi}{3}n+\theta\right) = C|1|^n\cos\left(\frac{2\pi}{3}n+\theta\right)$$

$$y_h(n) = C\cdot|q|^n\cdot\cos(\beta n + \theta)$$

$$y(n)-0{,}9\,y(n-1)+0{,}2\,y(n-2)=u(n)+u(n-1)+u(n-2)$$

$$u(n)=-9(-0{,}9)^n\,,\quad n\geq 0$$

Partikularno rješenje: $y_p(n)=K\cdot(-0{,}9)^n$

$$K\cdot(-0{,}9)^n-0{,}9\cdot K(-0{,}9)^{n-1}+0{,}2\cdot K\cdot(-0{,}9)^{n-2}=-9(-0{,}9)^n-9(-0{,}9)^{n-1}-9(-0{,}9)^{n-2}$$

$$K\cdot\cancel{(-0{,}9)^{n-2}}\left[(-0{,}9)^2-0{,}9(-0{,}9)+0{,}2\right]=-9\cancel{(-0{,}9)^{n-2}}\left[(-0{,}9)^2+(-0{,}9)+1\right]$$

$$K=-4{,}5$$

$$y_p(n)=-4{,}5(-0{,}9)^n\,,\quad n\geq 0$$

Za pobudu: $u(n)=2n^3 \rightarrow y_p(n)=K_0+K_1 n+K_2 n^2+K_3 n^3$

$u(n)=2\cdot 0{,}5^n \rightarrow y_p(n)=K_1\, 0{,}5^n\cdot n$

ako je 0,5 vlastita frekvencija sustava

Totalno rješenje:

$$y_T(n)=y_h(n)+y_p(n)=$$

$$=C_1\,0{,}5^n+C_2\,0{,}4^n-4{,}5(-0{,}9)^n\,,\quad n\geq 0$$

$y(-1)=6$

$y(-2)=-3$

$$y(n)=u(n)+u(n-1)+u(n-2)+0{,}9y(n-1)-0{,}2y(n-2)$$

$$y(0)=-9+0+0+0{,}9\cdot 6-0{,}2\cdot(-3)=-3$$

pobuda počinje tek u 0

$$y(1)=-9(-0{,}9)-9+0+0{,}9(-3)-0{,}2\cdot 6=-4{,}8$$

$$y_T(0) = C_1 + C_2 - 4{,}5 = y(0) = -3$$

$$y_T(1) = C_1 \cdot 0{,}5 + C_2 \cdot 0{,}4 - 4{,}5(-0{,}9) = \overset{y(1)}{-4{,}8}$$

$C_1 = -94{,}5$

$C_2 = 96$

$$y_T(n) = \underbrace{-94{,}5 \cdot 0{,}5^n + 96 \cdot 0{,}4^n}_{\text{Prirodni odziv}} \underbrace{-4{,}5(-0{,}9)^n}_{\text{Prisilni odziv}}, \quad n \geq 0$$

Odziv linearnog sustava kao zbroj odziva nepobuđenog i odziva mirnog sustava.

$$y(n) - 0{,}9\,y(n-1) + 0{,}2\,y(n-2) = u(n) + u(n-1) + u(n-2)$$

$$y_h(n) = C_1 \cdot 0{,}5^n + C_2 \cdot 0{,}4^n = y_0(n)$$

nema pobude

$$y(n) - 0{,}9\,y(n-1) + 0{,}2\,y(n-2) = 0$$

$y(-1) = 6$

$y(-2) = -3$

$$y_h(-1) = C_1 \cdot 0{,}5^{-1} + C_2 \cdot 0{,}4^{-1} = 6$$

$$y_h(-2) = C_1 \cdot 0{,}5^{-2} + C_2 \cdot 0{,}4^{-2} = -3$$

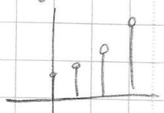
$C_1 = 18$

$C_2 = -12$

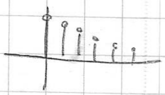
$$y_0(n) = 18 \cdot 0{,}5^n - 12 \cdot 0{,}4^n$$

Nestabilan sustav $|q| > 1$

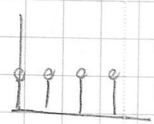
$y_0(n) = 2 \cdot 3^n$



$y_0(n) = 18 \cdot 0{,}5^n$



$y_0(n) = 5 \cdot 1^n$



Stabilan sustav $|q| < 1$

Granično stabilan sustav $|q| = 1$

Odziv mirnog sustava:

Početni uvjeti su 0

$y(-1)=0$

$y(-2)=0$

$$y(0) = u(0) + u(-1) + u(-2) + 0{,}9y(-1) - 0{,}2y(-2) =$$

$$= -9 + 0 + 0 + 0 - 0 = -9$$

$$y(1) = u(1) + u(0) + u(-1) + 0{,}9y(0) - 0{,}2y(-1) =$$

$$= -9\cdot(-0{,}9) - 9 + 0 + 0{,}9\cdot(-9) - 0{,}2\cdot 0 = -9$$

$$\left.\begin{aligned} y_m(n) &= C_1\, 0{,}5^n + C_2\cdot 0{,}4^n - 4{,}5\cdot(-0{,}9)^n \\ y_m(0) &= C_1 + C_2 - 4{,}5 = -9 \\ y_m(1) &= 0{,}5\cdot C_1 + 0{,}4\, C_2 + 4{,}5\cdot 0{,}9 = -9 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -112.5 \\ C_2 &= 108 \end{aligned}$$

$$y_m(n) = -112.5\cdot 0{,}5^n + 108\cdot 0{,}4^n - 4{,}5(-0{,}9)^n, \quad n \geq 0$$

$$y_T(n) = y_0(n) + y_m(n) = 18\cdot 0{,}5^n - 12\cdot 0{,}4^n - 112.5\cdot 0{,}5^n + 108\cdot 0{,}4^n - 4{,}5(-0{,}9)^n \quad n\geq 0$$

$$y_T(n) = -94{,}5\cdot 0{,}5^n + 96\cdot 0{,}4^n - 4{,}5(-0{,}9)^n, \quad n \geq 0$$

Određivanje impulsnog odziva

1. način:

$$h(n) - 0{,}9\,h(n-1) + 0{,}2\,h(n-2) = \delta(n) + \delta(n-1) + \delta(n-2)$$

$$h(n) - 0{,}9\,h(n-1) + 0{,}2\,h(n-2) = 0, \qquad n > 2$$

$$h(n) = C_1 \cdot 0{,}5^n + C_2 \cdot 0{,}4^n$$

$h(-1) = 0$

$h(-2) = 0$

$$h(0) = \overset{1}{\delta(0)} + \overset{0}{\delta(-1)} + \overset{0}{\delta(-2)} + 0{,}9\,\overset{0}{h(-1)} - 0{,}2\,\overset{0}{h(-2)} = \delta(0) = 1$$

$$h(1) = \overset{0}{\delta(1)} + \overset{1}{\delta(0)} + \overset{0}{\delta(-1)} + 0{,}9\,h(0) - 0{,}2\,h(-1) = 1 + 0{,}9 \cdot 1 = 1{,}9$$

$$h(2) = \overset{0}{\delta(2)} + \overset{0}{\delta(1)} + \overset{1}{\delta(0)} + 0{,}9\,h(1) - 0{,}2\,h(0) = 1 + 0{,}9 \cdot 1{,}9 - 0{,}2 \cdot 1 = 2{,}51$$

$$\left.\begin{aligned} h(1) &= C_1 \cdot 0{,}5^1 + C_2 \cdot 0{,}4^1 = 1{,}9 \\ h(2) &= C_1 \cdot 0{,}5^2 + C_2 \cdot 0{,}4^2 = 2{,}51 \end{aligned}\right\} \qquad \begin{aligned} C_1 &= 35 \\ C_2 &= -39 \end{aligned}$$

$$h(n) = \left(35 \cdot 0{,}5^n - 39 \cdot 0{,}4^n\right) \mu(n-1) + \delta(n), \qquad n \geq 0$$

zato jer rezultat vrijedi za $n > 0$

dodajemo kao rezultat za $n = 0$

2. način :

$$h_a(n) - 0,9\, h_a(n-1) + 0,2\, h_a(n-2) = \delta(n)$$

$$h(n) = b_0 h_a(n) + b_1 h_a(n-1) + b_2 h_a(n-2)$$

$n > 0$ $\quad h_a(n) - 0,9\, h_a(n-1) + 0,2\, h_a(n-2) = 0$

$$h_a(n) = C_1 \cdot 0,5^n + C_2 \cdot 0,4^n$$

$$h_a(0) = \delta(0) + 0,9\, h_a(-1) - 0,2\, h_a(-2) = 1$$

$$h_a(1) = \delta(1) + 0,9\, h_a(0) - 0,2\, h_a(-1) = 0,9$$

$$\left.\begin{aligned} C_1 + C_2 &= 1 \\ C_1 \cdot 0,5 + C_2 \cdot 0,4 &= 0,9 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 5 \\ C_2 &= -4 \end{aligned}$$

$$h_a(n) = 5 \cdot 0,5^n - 4 \cdot 0,4^n$$

$$h(n) = h_a(n) + h_a(n-1) + h_a(n-2)$$

$$h(n) = 5 \cdot 0,5^n - 4 \cdot 0,4^n + 5 \cdot 0,5^{n-1} - 4 \cdot 0,4^{n-1} + 5 \cdot 0,5^{n-2} - 4 \cdot 0,5^{n-1} =$$

$$= 5 \cdot 0,5^{n-1} \left(0,5^2 + 0,5 + 1\right) - 4 \cdot 0,4^{n-2} \left(0,4^2 + 0,4 + 1\right) =$$

$$= 35 \cdot 0,5^n - 39 \cdot 0,4^n \quad , \quad n \geq 2$$

$$h(0) = h_a(0) + h_a(-1) + h_a(-2) = 1$$

$$h(1) = h_a(1) + h_a(0) + h_a(-1) = 0,9 + 1 = 1,9$$

$$h(n) = \left(35 \cdot 0,5^n - 39 \cdot 0,4^n\right) \mu(n-1) + \delta(n)$$

Linearni vremenski stalni sustavi

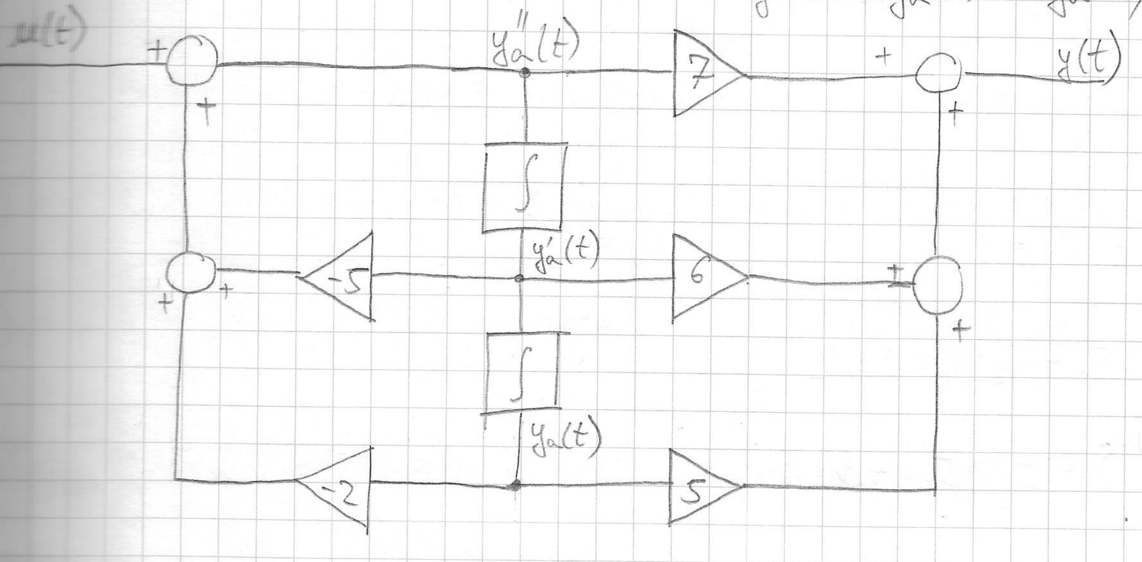
$$\frac{d^N y}{dt^N} + a_1 \frac{d^{N-1} y}{dt^{N-1}} + \dots + a_N y(t) = b_0 \frac{d^N u}{dt^N} + b_1 \frac{d^{N-1} u}{dt^{N-1}} + \dots + b_N u(t)$$

Blokovski dijagram (direktna realizacija 2:)

$$y''(t) + 5y'(t) + 2y(t) = 7u''(t) + 6u'(t) + 5u(t)$$

$$y_a''(t) + 5y_a'(t) + 2y_a(t) = u(t) \rightarrow y_a''(t) = u(t) - 5y_a'(t) - 2y_a(t)$$

$$y(t) = 7y_a''(t) + 6y_a'(t) + 5y_a(t)$$



$$y''(t) + 2y'(t) + 5y(t) = u(t)$$

$$y''(t) + 2y'(t) + 5y(t) = 0$$

$$y_h(t) = Ce^{st}$$

$$y_h'(t) = Cse^{st}$$

$$y_h''(t) = Cs^2e^{st}$$

$$Cs^2e^{st} + 2\cdot Cse^{st} + 5Ce^{st} = 0$$

$$Ce^{st}\left[s^2 + 2s + 5\right] = 0$$

$$s^2 + 2s + 5 = 0$$

$$s_{12} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j$$

$$y_h(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t} =$$

$$= \frac{C}{2} e^{j\Theta} e^{(-1+2j)t} + \frac{C}{2} e^{-j\Theta} e^{(-1-2j)t} =$$

$$= \frac{C}{2} e^{-t}\left[e^{j(2t+\Theta)} + e^{-j(2t+\Theta)}\right] =$$

$$= Ce^{-t}\cos(2t+\Theta)$$

Određivanje početnih uvjeta:

$$y''(t) + a_1 y'(t) + a_2 y(t) = b_0 u''(t) + b_1 u'(t) + b_2 u(t) \quad \Big/ \int_{0^-}^{t}$$

$$\int_{0^-}^{t} y''(\tau)d\tau + a_1\int_{0^-}^{t} y'(\tau)d\tau + a_2\int_{0^-}^{t} y(\tau)d\tau = b_0\int_{0^-}^{t} u''(\tau)d\tau + b_1\int_{0^-}^{t} u'(\tau)d\tau + b_2\int_{0^-}^{t} u(\tau)d\tau$$

$$y'(t) - y'(0^-) + a_1 y(t) - a_1 y(0^-) + a_2\int_{0^-}^{t} y(\tau)d\tau = b_0 u'(t) \underbrace{- \cancel{b_0 u'(0^-)} + b_1 u(t) - \cancel{b_1 u(0^-)}}_{\text{signal je kauzalan}} + b_2\int_{0^-}^{t} u(\tau)d\tau$$

$$\int_{0^-}^{t} y'(\tau)d\tau - \int_{0^-}^{t} y'(0^-)d\tau + a_1\int_{0^-}^{t} y(\tau)d\tau - a_1\int_{0^-}^{t} y(0^-)d\tau + a_2\int_{0^-}^{t}\int_{0^-}^{\tau} y(\lambda)d\lambda d\tau =$$

$$= b_0\int_{0^-}^{t} u'(\tau)d\tau + b_1\int_{0^-}^{t} u(\tau)d\tau + b_2\int_{0^-}^{t}\int_{0^-}^{\tau} u(\lambda)d\lambda d\tau$$

$$y(t) - y(0^-) - \cancel{y'(0^-)(t-0^-)} + \cancel{a_1\int_{0^-}^{t} y(\tau)d\tau} - \cancel{a_1 y(0^-)(t-0^-)} + \cancel{a_2\int_{0^-}^{t}\int_{0^-}^{\tau} y(\lambda)d\lambda d\tau} =$$

$$= b_0 u(t) - b_0 u(0^-) + \cancel{b_1\int_{0^-}^{t} u(\tau)d\tau} + \cancel{b_2\int_{0^-}^{t}\int_{0^-}^{\tau} u(\lambda)d\lambda d\tau}$$

$$\boxed{y(0^+) - y(0^-) = b_0 u(0^+)}$$

$$y'(t) - y'(0^-) + a_1 y(t) - a_1 y(0^-) + a_2\int_{0^-}^{t} y(\tau)d\tau = b_0 u'(t) + b_1 u(t) + b_2\int_{0^-}^{t} u(\tau)d\tau$$

$t = 0^+$

$$y'(0^+) - y'(0^-) + a_1 y(0^+) - a_1 y(0^-) + \underbrace{a_2\int_{0^-}^{0^+} y(\tau)d\tau}_{0} = b_0 u'(0^+) + b_1 u(0^+) + \underbrace{b_2\int_{0^-}^{0^+} u(\tau)d\tau}_{0}$$

$$y'(0^+) - y'(0^-) + a_1\left[y(0^+) - y(0^-)\right] = b_0 u'(0^+) + b_1 u(0^+)$$

Auditorne XII. tjedan

① $y(n+1) + 2y(n) = u(n) \qquad y(0) = 2$

$y(n+1) = u(n) - 2y(n)$

$n=0 \quad u_1(0) \longrightarrow y_1(1) = u_1(0) - 2y_1(0)$

$\quad\quad u_2(0) \longrightarrow y_2(1) = u_2(0) - 2y_2(0)$

poč. uvjet $\qquad$ poč. uvjet

$\alpha u_1(0) - 2\alpha y_1(0) + \beta u_2(0) - 2\beta y_2(0) =$

$= \alpha u_1(0) + \beta u_2(0) - 4(\alpha+\beta)$

$\alpha u_1(0) + \beta u_2(0) \longrightarrow y(1) = \alpha u_1(0) + \beta u_2(0) - 2y(0) = \alpha u_1(0) + \beta u_2(0) - 4$

③ $y(n+3) - y(n) = 0 \qquad y(0) = 0$

$\qquad\qquad y(1) = 0$

$\qquad\qquad y(2) = 1$

$y_h(n) = c\,q^n$

$c\,q^{n+3} - c\,q^n = 0$

$c\,q^n\left[q^3 - 1\right] = 0 \longrightarrow q^3 - 1 = 0$

$(q-1)(q^2+q+1) = 0$

$q_1 = 1 \qquad q_{2,3} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}\,j}{2}$

$q = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$

$|q| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$

$\measuredangle q = \operatorname{arctg} \frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \frac{2\pi}{3}$

$y_h(n) = C_1 \cdot 1^n + C_2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}j\right)^n + C_3\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}j\right)^n =$

$= C_1 \cdot 1^n + C_2\left(1 \cdot e^{j\frac{2\pi}{3}}\right)^n + C_3\left(1\,e^{-j\frac{2\pi}{3}}\right)^n = \qquad C_2 = \frac{c}{2}e^{j\theta} \qquad C_3 = \frac{c}{2}e^{-j\theta}$

$= C_1 \cdot 1^n + \frac{c}{2}e^{+j\theta}e^{j\frac{2\pi}{3}n} + \frac{c}{2}e^{-j\theta}e^{-j\frac{2\pi}{3}n} =$

$= C_1 \cdot 1^n + \frac{c}{2}1^n\left[e^{j\frac{2\pi}{3}n+\theta} + e^{-j\left(\frac{2\pi}{3}n+\theta\right)}\right] =$

$= C_1 \cdot 1^n + c \cdot 1^n \cos\left(\frac{2\pi}{3}n + \theta\right)$

$y_h(0) = C_1 + C\cos(\theta) = 0$

$y_h(1) = C_1 + C\cos\left(\frac{2\pi}{3} + \theta\right) = 0$

$y_h(2) = C_1 + C\cos\left(\frac{4\pi}{3} + \theta\right) = 1$

$y_h(n) = \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi}{3}n + \frac{2\pi}{3}\right)$

7) a) $y(n) - 6y(n-1) + 8y(n-2) = 4u(n)$

$y(n) - 6y(n-1) + 8y(n-2) = 0$

$1 - 6q^{-1} + 8q^{-2} = 0$

$q^2 - 6q + 8 = 0$

$(q-4)(q-2) = 0$

$q_1 = 4 \quad q_2 = 2$

$y_h(n) = C_1 \cdot 4^n + C_2 \cdot 2^n$

$u(n) = 2 - 3n \quad , \quad n \geq 0$

$y_p(n) = K_1 + K_2 n$

$K_1 + K_2 n - 6K_1 - 6K_2(n-1) + 8K_1 + 8K_2(n-2) = \underset{\overset{\downarrow}{4(2-3n)}}{8 - 12n}$

$K_1 + K_2 n - 6K_1 - 6K_2 n + 6K_2 + 8K_1 + 8K_2 n - 16K_2 = 8 - 12n$

$K_1 - 6K_1 + 6K_2 + 8K_1 - 16K_2 = 8$

$K_2 - 6K_2 + 8K_2 = -12 \rightarrow K_2 = -4$

$3K_1 - 10K_2 = 8$

$3K_1 = -32$

$K_1 = -\frac{32}{3}$

$\boxed{y_p(n) = -\frac{32}{3} - 4n}$

Oblik partikularnog rješenja

$u(n)$	$y_p(n)$
r^n	Kr^n
$A\cos\omega_0 n$	$K_1\cos\omega_0 n + K_2\sin\omega_0 n$
$A\sin\omega_0 n$	— " —
A	K
r^n, za $r = q$	$K \cdot n \cdot r^n$

$y_{prisilni}(n) = -\frac{32}{3} - 4n$

$y_T(n) = C_1 \cdot 4^n + C_2 \cdot 2^n - \frac{32}{3} - 4n$

$y(n) - 6y(n-1) + 8y(n-2) = 8 - 12n \rightarrow y(n) = 8 - 12n + 6y(n-1) - 8y(n-2)$

$y(-1) = 2$

$y(-2) = 1$

$y(0) = 8 + 6 \cdot y(-1) - 8y(-2) = 8 + 6 \cdot 2 - 8 \cdot 1 = 12$

$y(1) = 8 - 12 \cdot 1 + 6y(0) - 8y(-1) = -4 + 6 \cdot 12 - 8 \cdot 2 = 52$

$$\left.\begin{aligned} y_T(0) &= C_1 + C_2 - \frac{32}{3} = 12 \\ y_T(1) &= C_1 \cdot 4 + C_2 \cdot 2 - \frac{32}{3} - 4 = 52 \end{aligned}\right\} \quad \begin{aligned} C_1 &= \frac{32}{3} \\ C_2 &= 12 \end{aligned}$$

$$y_T(n) = \underbrace{\frac{32}{3} 4^n + 12 \cdot 2^n}_{\text{Prirodni odziv}} \underbrace{- \frac{32}{3} - 4n}_{\text{Prisilni odziv}}, \quad n \geq 0$$

b) $y(n) - 6y(n-1) + 8y(n-2) = 0$

$y_h(n) = C_1 \cdot 4^n + C_2 \cdot 2^n$

$y(-1) = 2$

$y(-2) = 1$

$$\left.\begin{aligned} y_h(-1) &= C_1 \cdot 4^{-1} + C_2 \cdot 2^{-1} = 2 \\ y_h(-2) &= C_1 \cdot 4^{-2} + C_2 \cdot 2^{-2} = 1 \end{aligned}\right\} \quad \begin{aligned} C_1 &= 0 \\ C_2 &= 4 \end{aligned}$$

$y_0(n) = 4 \cdot 2^n \rightarrow$ odziv nepobuđenog sustava

Odziv mirnog sustava:

$$y(n) - 6y(n-1) + 8y(n-2) = 8 - 12n$$

$$y(-1) = y(-2) = 0$$

$$y(0) = 8 + 6y(-1) - 8y(-2) = 8$$

$$y(1) = 8 - 12\cdot 1 + 6y(0) - 8y(-1) = -4 + 6\cdot 8 = 44$$

$$y_m(n) = C_1 4^n + C_2 2^n - \frac{32}{3} - 4n$$

$$\left.\begin{aligned} y_m(0) &= C_1 + C_2 - \frac{32}{3} = 8 \\ y_m(1) &= 4C_1 + 2C_2 - \frac{32}{3} - 4 = 44 \end{aligned}\right\} \begin{aligned} C_1 &= \frac{32}{3} \\ C_2 &= 8 \end{aligned}$$

$$y_m(n) = \frac{32}{3} 4^n + 8\, 2^n - \frac{32}{3} - 4n \quad , \quad n \geq 0$$

$$y_t(n) = \frac{32}{3} 4^n + 8\cdot 2^n - \frac{32}{3} - 4n + 4\cdot 2^n =$$

$$= \frac{32}{3} 4^n + 12\cdot 2^n - \frac{32}{3} - 4n \quad , \quad n \geq 0$$

3 a) $y(n) - \frac{1}{4} y(n-1) = u(n)$

$u(n) = \delta(n)$

$y(-1) = 0$

$h(n) = ?$

$n > 0 \qquad h(n) - \frac{1}{4} h(n-1) = 0$

$$q - \frac{1}{4} = 0$$

$$q = \frac{1}{4}$$

$$h(n) = c \cdot \left(\frac{1}{4}\right)^n$$

$$y(n) = u(n) + \frac{1}{5} y(n-1)$$

$$y(0) = u(0) + \frac{1}{5} y(-1) = 1$$

$$h(0) = C\left(\frac{1}{5}\right)^0 = 1 \Rightarrow C = 1$$

$$h(n) = 1 \cdot \left(\frac{1}{5}\right)^n \quad , \quad n \geq 0$$

) Mirni odziv : $u(n) = \left(\frac{1}{5}\right)^n \mu(n)$

$$y_m(n) = C\left(\frac{1}{5}\right)^n + y_p(n)$$

$$y_p(n) = K\left(\frac{1}{5}\right)^n \cdot n$$

(množimo sa n jer frekvencija pobude jednaka frekvenciji u partikularnom rješenju

$$K\left(\frac{1}{5}\right)^n \cdot n - \frac{1}{5} \cdot K\left(\frac{1}{5}\right)^{n-1} (n-1) = \left(\frac{1}{5}\right)^n$$

$$Kn - K(n-1) = 1$$

$$K = 1$$

$$y_p(n) = \left(\frac{1}{5}\right)^n \cdot n$$

$$y_m(n) = C\left(\frac{1}{5}\right)^n + \left(\frac{1}{5}\right)^n \cdot n$$

$$y(-1) = 0$$

$$y(0) = u(0) + \frac{1}{5} y(-1) = 1$$

$$y_m(0) = C = y(0) = 1$$

$$y_m(n) = \left(\frac{1}{5}\right)^n + \left(\frac{1}{5}\right)^n \cdot n \quad , \quad n \geq 0$$

Nastavak predavanja

$$y''(t) + \overset{a_1}{2}y'(t) + \overset{a_2}{5}y(t) = u(t) \qquad y(0^-) = -1 \qquad y'(0^-) = -1$$

$$u(t) = 5\mu(t) \qquad\qquad b_0 u''(t) + b_1 u'(t) + b_2 u(t)$$

$$b_0 = 0 \qquad a_1 = 2$$

$$b_1 = 0 \qquad a_2 = 5$$

$$b_2 = 1$$

$$y(0^+) - y(0^-) = b_0 u(0^+)$$

$$y(0^+) + 1 = 0 \cdot 5$$

$$y(0^+) = -1$$

$$y'(0^+) + 1 + 2\left[-1 + 1\right] = 0 \cdot 0 + 0 \cdot 5$$

$$y'(0^+) = -1$$

Određivanje početnih uvjeta za sustav 1. reda

$$y'(t) + a_1 y(t) = b_0 u'(t) + b_1 u(t) \quad \Big/ \int_{0^-}^{t}$$

$$\int_{0^-}^{t} y'(\tau)d\tau + a_1 \int_{0^-}^{t} y(\tau)d\tau = b_0 \int_{0^-}^{t} u'(\tau)d\tau + b_1 \int_{0^-}^{t} u(\tau)d\tau$$

$$y(t) - y(0^-) + a_1 \int_{0^-}^{t} y(\tau)d\tau = b_0 u(t) - \cancel{b_0 u(0^-)} + b_1 \int_{0^-}^{t} u(\tau)d\tau \quad \Big/ t = 0^+$$

$$y(0^+) - y(0^-) + \cancel{a_1 \int_{0^-}^{0^+} y(\tau)d\tau} = b_0 u(0^+) + \cancel{b_1 \int_{0^-}^{0^+} u(\tau)d\tau}$$

$$y(0^+) - y(0^-) = b_0 u(0^+)$$

Partikularno rješenje

$y''(t) + 2y'(t) + 5y(t) = u(t) \qquad u(t) = 5\mu(t)$

$y_p(t) = K$

$y_p'(t) = 0 \quad y_p''(t) = 0$

$0 + 2\cdot 0 + 5K = 5$

$K = 1$

$y_p(t) = 1\cdot\mu(t) = 1 \quad , t \geq 0$

Totalni odziv sustava

$y_T(t) = y_h(t) + y_h(t)$

$y_T(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t} + 1 \quad , t \geq 0$

$y(0^-) = -1 \qquad y(0^+) = -1$

$y'(0^-) = -1 \qquad y'(0^+) = -1$

$y_T(0^+) = C_1 + C_2 + 1 = -1$

$y_T'(t) = (-1+2j)C_1 e^{t} + (-1-2j)C_2 e^{(-1-2j)t}$

$y_T'(0^+) = (-1+2j)C_1 + (-1-2j)C_2 = -1$

$C_1 = -1 + j\cdot 0{,}75$

$C_2 = -1 - j\cdot 0{,}75$

$y_T(t) = (-1+j\cdot 0{,}75)e^{(-1+2j)t} + (-1-j\cdot 0{,}75)e^{(-1-2j)t} + 1 \quad , t \geq 0$

$y(t) = -2e^{-t}\cos(2t) - 1{,}5e^{-t}\sin(2t) + 1$

$$y''(t) + 2y'(t) + 5y(t) = u(t) \qquad y(0^-) = -1$$
$$y'(0^-) = -1$$

Nepobuđeni odziv:

$$y''(t) + 2y'(t) + 5y(t) = 0$$

$$y_h(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$$

$$y_h'(t) = C_1(-1+2j)e^{(-1+2j)t} + C_2(-1-2j)e^{(-1-2j)t}$$

$$\left.\begin{aligned} y_h(0^-) &= C_1 + C_2 = -1 \\ y_h'(0^-) &= C_1(-1+2j) + C_2(-1-2j) = -1 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -0{,}5 + j\cdot 0{,}5 \\ C_2 &= -0{,}5 - j\cdot 0{,}5 \end{aligned}$$

$$y_0(t) = (-0{,}5 + j0{,}5)e^{(-1+2j)t} + (-0{,}5 - j0{,}5)e^{(-1-2j)t}$$

$$\boxed{y_0(t) = -e^{-t}\cos(2t) - e^{-t}\sin(2t)}$$

$s = \alpha + j\beta$

$\alpha < 0 \longrightarrow \lim_{t\to\infty} y_0(t) = 0 \rightarrow$ STABILNI SUSTAV

$\alpha = 0 \longrightarrow$ MARGINALNO STABILAN

$\alpha > 0 \longrightarrow \lim_{t\to\infty} y_0(t) = \infty \rightarrow$ NESTABILAN SUSTAV

$s_1 = -1 + 2j \qquad s_2 = -1 - 2j$

$Re\{s_1\} = -1 < 0 \qquad Re\{s_2\} = -1 < 0$ Sustav je stabilan

Mirni sustav

$$y''(t) + 2y'(t) + 5y(t) = u(t)$$

$$y(0^-) = 0 \qquad u(t) = 5\mu(t)$$

$$y'(0^-) = 0$$

$$y_m(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t} + 1$$

$$y(0^+) - y(0^-) = b_0 u(0^+) \Rightarrow y(0^+) = 0$$

$$y'(0^+) - y'(0^-) + a_1\left[y(0^+) - y(0^-)\right] = b_0 u'(0^+) + b_1 u(0^+) \rightarrow y'(0^+) = 0$$

$$y_m'(t) = C_1(-1+2j)e^{(-1+2j)t} + C_2(-1-2j)e^{(-1-2j)t}$$

$$\left.\begin{aligned} y_m(0^+) &= C_1 + C_2 + 1 = 0 \\ y_m'(0^+) &= C_1(-1+2j) + C_2(-1-2j) = 0 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -0{,}5 + j0{,}25 \\ C_2 &= -0{,}5 - j0{,}25 \end{aligned}$$

$$y_m(t) = (-0{,}5 + j0{,}25)e^{(-1+2j)t} + (-0{,}5 - j0{,}25)e^{(-1-2j)t} + 1 \quad , t \geq 0$$

$$y_m(t) = -e^{-t}\cos(2t) - 0{,}5e^{-t}\sin(2t) + 1 \quad , t \geq 0$$

$$y_T(t) = y_0(t) + y_m(t)$$

$$y_T(t) = -e^{-t}\cos(2t) - e^{-t}\sin(2t) - e^{-t}\cos(2t) - 0{,}5e^{-t}\sin(2t) =$$

$$= -2e^{-t}\cos(2t) - 1{,}5e^{-t}\sin(2t) \quad , t \geq 0$$

Izračun impulsnog odziva

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + \dots + a_N y(t) = b_0 u^{(N)}(t) + \dots + b_N u(t)$$

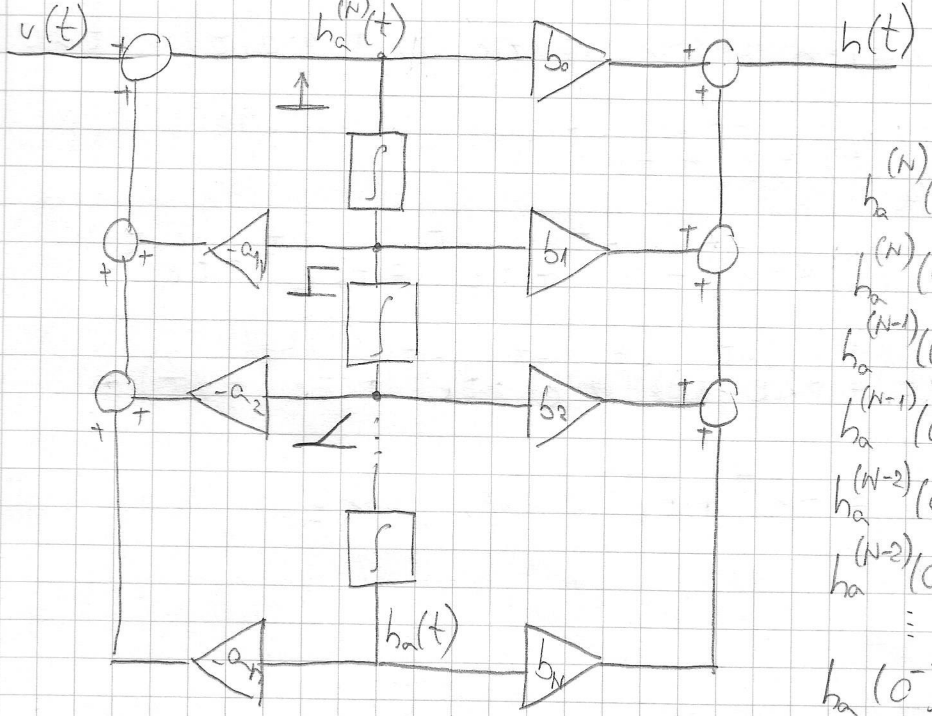
$t \geq 0^+$ $\quad y^{(n)}(t) + a_1 y^{(N-1)}(t) + \dots + a_N y(t) = 0$

$$h(t) = \sum_{i=1}^{N} c_i e^{s_i t}$$

$h(t) = \delta(t)$

$h(0^-) = 0, \quad h'(0^-) = 0, \dots, h^{(N-1)}(0^-) = 0$

$t = 0$ $\quad h_a^{(N)}(t) + \dots + a_N h_a(t) = u(t) \Rightarrow h_a^{N}(t) = u(t) - a_1 h_a^{(N-1)}(t) - \dots - a_N h_a(t)$



$h_a^{(N)}(0^-) = 0$

$h_a^{(N)}(0^+) = 0$

$h_a^{(N-1)}(0^-) = 0$

$h_a^{(N-1)}(0^+) = 1$ $\leftarrow$ jedino tu je 1

$h_a^{(N-2)}(0^-) = 0$

$h_a^{(N-2)}(0^+) = 0$

$\vdots$

$h_a(0^-) = 0$

$h_a(0^+) = 0$

$t \geq 0$ $\quad h(t) = \sum_{i=1}^{N} c_i e^{s_i t} + b_0 \delta(t)$

$h_a^{(N)}(0^-) = 0$

$h_a^{(N)}(0^+) = 0$

(7) $y''(t) + 2y'(t) + y(t) = u''(t) + u'(t) + u(t)$ $\qquad u(t) = \delta(t)$

$y''(t) + 2y'(t) + y(t) = 0$

$s^2 + 2s + 1 = 0$

$s = \frac{-2 \pm \sqrt{4-4}}{2} = -1$

$s_1 = -1 \quad s_2 = -1$

$h_a(t) = C_1 e^{-t} + C_2 t e^{-t}$ $\qquad h_a'(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t}$

$h_a(0^+) = 0$

$h_a'(0^+) = 1$

$\left.\begin{array}{l} h_a(0^+) = C_1 = 0 \\ h_a'(0^+) = -C_1 + C_2 = 1 \end{array}\right\} \begin{array}{l} C_1 = 0 \\ C_2 = 1 \end{array}$

$h(t) = h_a''(t) + h_a'(t) + h_a(t) + 1 \cdot \delta(t) \qquad t \geq 0$

$h_a'(t) = e^{-t} - t e^{-t}$

$h_a''(t) = -e^{-t} - e^{-t} + t e^{-t} = -2e^{-t} + t e^{-t}$

$h(t) = -2e^{-t} + t e^{-t} + e^{-t} - t e^{-t} + t e^{-t} + \delta(t) =$

$= -e^{-t} + t e^{-t} + \delta(t), \; t \geq 0$

Odziv sustava na pobudu eksponencijalom

$$u(t) = e^{st}$$

$$y(t) = u(t) * h(t) = e^{st} * h(t) = \int_{-\infty}^{\infty} h(\tau)\, u(t-\tau)\, d\tau =$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \underbrace{e^{s(t-\tau)}}_{e^{st} e^{-s\tau}}\, d\tau = e^{st} \underbrace{\int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau}_{H(s)}$$

$$y(t) = H(s)\, e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau$$

$$u(t) = U e^{st}$$

$$y(t) = Y e^{st}$$

$$y^{(N)}(t) + a_1 y^{(N-1)}(t) + \dots + a_N y(t) = b_0 u^{(N)}(t) + b_1 u^{(N-1)}(t) + \dots + b_N u(t)$$

$$y_p'(t) = Y s e^{st} \qquad u'(t) = U s e^{st}$$

$$y_p''(t) = Y s^2 e^{st} \qquad u''(t) = U s^2 e^{st}$$

$$Y s^N e^{st} + a_1 Y s^{N-1} e^{st} + \dots + a_N Y e^{st} = b_0 U s^N e^{st} + b_1 U s^{N-1} e^{st} + \dots + b_N U e^{st}$$

$$Y \cancel{e^{st}} \left[s^N + a_1 s^{N-1} + \dots + a_N \right] = U \cancel{e^{st}} \left[b_0 s^N + b_1 s^{N-1} + \dots + b_N \right]$$

$$Y = \underbrace{\frac{b_0 s^N + b_1 s^{N-1} + \dots + b_N}{s^N + a_1 s^{N-1} + \dots + a_N}}_{H(s)} \cdot U$$

$$H(s) = \frac{y_p(t)}{u(t)}\Bigg|_{u=e^{st}}$$

② $y'''(t) + 6y''(t) + 11y'(t) + 6y(t) = 3u''(t) + 12u'(t)$

$$s^3 Y + 6s^2 Y + 11sY + 6Y = 3s^2 U + 12sU$$

$$Y\left[s^3 + 6s^2 + 11s + 6\right] = U\left[3s^2 + 12s\right]$$

$y(t) = Ue^{st}$

$y_p(t) = Ye^{st}$

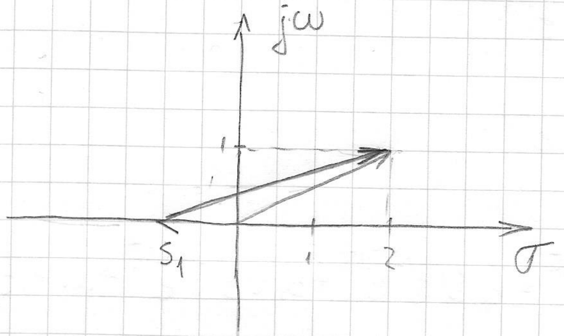
$$Y = \frac{3s^2 + 12s}{s^3 + 6s^2 + 11s + 6} \cdot U$$

$$H(s) = \frac{3s^2 + 12s}{s^3 + 6s^2 + 11s + 6} = \frac{3s\,[s+4]}{(s+1)(s+2)(s+3)}$$

$s = -1 \quad s = -2 \quad s = -3$ Polovi sustava

$s = 0 \quad s = -4$ Nule sustava

$s = \sigma + j\omega$



$u(t) = e^{(2+j)t}$

$s_1 = -1$

$2 + j - (-1) = 3 + j = \sqrt{10}\, e^{j\,\mathrm{arctg}\frac{1}{3}}$

$$H(s) = 3 \cdot \frac{|s|e^{j\measuredangle s}\,|s+4|e^{j\measuredangle(s+4)}}{|s+1|e^{j\measuredangle(s+1)}\,|s+2|e^{j\measuredangle(s+2)}\,|s+3|e^{j\measuredangle(s+3)}}$$

$$|H(s)| = 3\,\frac{|s|\,|s+4|}{|s+1|\,|s+2|\,|s+3|}$$

$$\measuredangle H(s) = \measuredangle s + \measuredangle(s+4) - \measuredangle(s+1) - \measuredangle(s+2) - \measuredangle(s+3)$$

② $y''(t) + 2y'(t) + 2y(t) = u(t)$

$u(t) = U e^{st}$

$y_p(t) = Y e^{st}$

$$H(s) = \frac{1}{s^2 + 2s + 2} = \frac{Y}{U}$$

$Y = H(s) \cdot U$

$$s_{12} = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$$

$$H(s) = \frac{s^N + a_1 s^{N-1} + \dots + a_N}{b_0 s^N + \dots + b_N}$$

$$H(j\omega) = \frac{(j\omega)^N + \dots + a_N}{b_0 (j\omega)^N + \dots + b_N}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} \, d\tau$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(t) e^{-j\omega t} \, dt$$

Auditorne vježbe

(15) $y''(t) + 5y'(t) + 6y(t) = u(t)$ $u(t) = 12t + 16$, $t \geq 0$

$y(0^-) = 3$, $y'(0^-) = -8$

$y''(t) + 5y'(t) + 6y(t) = 0$

$y_h(t) = Ce^{st}$

$y_h'(t) = Cse^{st}$

$y_h''(t) = Cs^2e^{st}$

Na kraju se dobije

$s^2 + 5s + 6 = 0$

$s = \frac{-5 \pm \sqrt{25-24}}{2} = \frac{-5 \pm 1}{2}$

$s_1 = -2$ $s_2 = -3$

Sustav je stabilan jer $Re\{s\} < 0$

$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$y_p(t) = K_0 + K_1 t$

$y_p'(t) = K_1$

$y_p''(t) = 0$

$0 + 5 \cdot K_1 + 6(K_0 + K_1 \cdot t) = 12t + 16$

$5K_1 + 6K_0 + 6K_1 t = 12t + 16$

$5K_1 + 6K_0 = 16$ $6K_1 t = 12t$

$10 + 6K_0 = 16$ $K_1 = 2$

$6K_0 = 6$

$K_0 = 1$

$y_p(t) = 1 + 2t$, $t \geq 0$

$y_T(t) = C_1 e^{-2t} + C_2 e^{-3t} + 1 + 2t$

$y_T'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + 2.$

$y(0^-)=3$ $\quad a_1=5 \quad a_2=6$

$y'(0^-)=-8$ $\quad b_0=0 \quad b_1=0 \quad b_2=1$

$a_0 \Delta y = b_0 \Delta u$

$y(0^+)-y(0^-)=0[u(0^+)-u(0^-)]$

$y(0^+)-3=0 \Rightarrow \boxed{y(0^+)=3}$

$a_0 \Delta y' + a_1 \Delta y = b_0 \Delta u' + b_1 \Delta u$

$y'(0^+)-y'(0^-)+5[y(0^+)-y(0^-)]=0$

$y'(0^+)+8+5[3-3]=0$

$\boxed{y'(0^+)=-8}$

$$\left.\begin{aligned} y_T(0^+) &= C_1 + C_2 + 1 = 3 \\ y_T'(0^+) &= -2C_1 - 3C_2 + 2 = -8 \end{aligned}\right\} \begin{aligned} C_2 &= 6 \\ C_1 &= -4 \end{aligned}$$

$$y_T(t) = \underbrace{-4e^{-2t} + 6e^{-3t}}_{\text{Prirodni/prijelazni odziv}} + \underbrace{1+2t}_{\text{Prisilni odziv}}, \quad t \geq 0$$

$y_0(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$y_0'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t}$

$$\left.\begin{aligned} y_0(0^-) &= C_1 + C_2 = 3 \\ y_0'(0^-) &= -2C_1 - 3C_2 = -8 \end{aligned}\right\} \begin{aligned} C_1 &= 1 \\ C_2 &= 2 \end{aligned}$$

$y_0(t) = e^{-2t} + 2e^{-3t}$

$$y_m(t) = C_1 e^{-2t} + C_2 e^{-3t} + 1 + 2t$$

$$y_m'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + 2$$

$$y_m(0^-) = 0 \qquad y_m'(0^-) = 0$$

$$y(0^+) - y(0^-) = b_0 \left[u(0^+) - u(0^-)\right]$$

$$y(0^+) - 0 = 0 \cdot [\ldots] \rightarrow \underline{y(0^+) = 0}$$

$$y'(0^+) - y'(0^-) + 5\left[y(0^+) - y(0^-)\right] = 0 + 0$$

$$y'(0^+) - 0 + 5 \cdot 0 = 0 \Rightarrow \underline{y'(0^+) = 0}$$

$$\left.\begin{aligned} y_m(0^+) &= C_1 + C_2 + 1 = 0 \\ y_m'(0^+) &= -2C_1 - 3C_2 + 2 = 0 \end{aligned}\right\} \quad \begin{aligned} C_1 &= -5 \\ C_2 &= 4 \end{aligned}$$

$$y_m(t) = -5e^{-2t} + 4e^{-3t} + 1 + 2t \quad , \quad t \geq 0$$

$$y_T(t) = e^{-2t} + 2e^{-3t} - 5e^{-2t} + 4e^{-3t} + 1 + 2t =$$

$$= -4e^{-2t} + 6e^{-3t} + 1 + 2t \quad , \quad t \geq 0$$

Impulsni odziv

$$y''(t) + 5y'(t) + 6y(t) = u(t) \qquad u(t) = \delta(t)$$

$$y(0^-) = 0 \quad , \quad y'(0^-) = 0$$

$$y(0^+) = 0 \quad , \quad y'(0^+) = 1$$

$$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t} = h_h(t)$$

$$h_a''(t) + 5h_a'(t) + 6h_a(t) = \delta(t)$$

$$h_h'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t}$$

$$h_a(0^+) = C_1 + C_2 = 0$$
$$h_a'(0^+) = -2C_1 - 3C_2 = 1$$
$C_1 = 1$

$C_2 = -1$

$$h_a(t) = 1 \cdot e^{-2t} - 1 e^{-3t}$$

$$h(t) = h_a(t) = e^{-2t} - e^{-3t}$$

da je ulaz bio $2u(t)$
$h(t)$ bi bio $2h_a(t)$

za $2u(t) + 5u'(t) \Rightarrow h(t) = 2h_a(t) + 5h_a'(t)$

ako broj derivacija ulaza jednak
broju derivacija izlaza na $h(t)$
treba još dodati $b_0 \delta(t)$

b_0 – broj uz najveću derivaciju
ulaza

ZI 2014.

(3) $$y''(t) + 4y'(t) + 5y(t) = 2u'(t) + u(t)$$

$$u(t) = \begin{cases} -t, & t < 0 \\ -t+1, & t > 0 \end{cases}$$

$$s^2 + 4s + 5 = 0$$
$$s = \frac{-4 \pm \sqrt{16-20}}{2} = \frac{-4 \pm 2j}{2} = -2 \pm j$$
$$s_1 = -2 + j \qquad s_2 = -2 - j$$
$$y_h(t) = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t}$$

$t < 0$ $\quad u(t) = -t$

$$y_p(t) = K_0 + K_1 t$$
$$y_p'(t) = K_1$$
$$y_p''(t) = 0$$

$$0 + 4K_1 + 5K_0 + 5K_1 t = -2 - t$$
$$4K_1 + 5K_0 = -2 \qquad 5K_1 = -1$$
$$5K_0 = -2 - 4K_1 \qquad K_1 = -\frac{1}{5}$$
$$K_0 = -\frac{6}{25}$$

$$\boxed{y_p(t) = -\frac{6}{25} - \frac{1}{5} t}$$

$t>0 \qquad y_p(t) = K_2 + K_3 t$

$$0 + 4K_3 + 5K_2 + 5K_3 t = 2(-1) + (-t+1)$$

$$4K_3 + 5K_2 = -1 \qquad 5K_3 = -1$$

$$K_2 = -\frac{1}{25} \qquad K_3 = -\frac{1}{5}$$

$$y_p(t) = -\frac{1}{25} - \frac{1}{5}t$$

$$y_T(t) = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t} - \frac{1}{25} - \frac{1}{5}t$$

$y_p(0^-) = -\frac{6}{25}$ [dobiveno iz partikularnog rješenja za $t<0$]

$$y_p'(0^-) = -\frac{1}{5}$$

$$y(0^+) - y(0^-) = 0 \qquad y'(0^+) - y'(0^-) + 4[y(0^+) - y(0^-)] = 2[u(0^+) - u(0^-)]$$

$$y(0^+) = -\frac{6}{25} \qquad y'(0^+) + \frac{1}{5} + 4 \cdot 0 = 2[1-0]$$

$$y(0^+) = -\frac{6}{25} \qquad y'(0^+) = 2 - \frac{1}{5} = \frac{9}{5}$$

$$y'(0^+) = \frac{9}{5}$$

$$y_T'(t) = (-2+j)C_1 e^{(-2+j)t} + (-2-j)C_2 e^{(-2-j)t} - \frac{1}{5}$$

$$\left.\begin{aligned} y_T(0^+) &= C_1 + C_2 - \frac{1}{25} = -\frac{6}{25} \\ y_T'(0^+) &= (-2+j)C_1 + (-2-j)C_2 - \frac{1}{5} = \frac{9}{5} \end{aligned}\right\} \quad \begin{aligned} C_1 &= -\frac{1}{10} + \frac{8}{10}j \\ C_2 &= -\frac{1}{10} - \frac{8}{10}j \end{aligned}$$

$$y(t) = \begin{cases} -\frac{6}{25} - \frac{1}{5}t, & t<0 \\ \left(-\frac{1}{10} + \frac{8}{10}j\right)e^{(-2+j)t} + \left(-\frac{1}{10} - \frac{8}{10}j\right)e^{(-2-j)t} - \frac{1}{25} - \frac{1}{5}t, & t>0 \end{cases}$$

Nastawak predawanja

$$y''(t) + 2y'(t) + 2y(t) = u(t)$$

$$H(s) = \frac{1}{s^2 + 2s + 2}$$

$$H(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 2} = \frac{1}{(2-\omega^2) + 2\omega j}$$

$$|H(j\omega)| = \frac{|1|}{|(2-\omega^2) + 2\omega j|} = \frac{1}{\sqrt{(2-\omega^2)^2 + (2\omega)^2}} = \frac{1}{\sqrt{4 - 4\omega^2 + \omega^4 + 4\omega^2}} = \frac{1}{\sqrt{4 - \omega^4}}$$

$$\measuredangle H(j\omega) = -\operatorname{arctg} \frac{2\omega}{2-\omega^2}$$

$$u(t) = Ue^{st} \qquad u(t) = Ue^{j\omega t} \qquad u(t) = Ue^{-j\omega t}$$

$$y_p(t) = H(s)Ue^{st} \qquad y_p(t) = H(j\omega)Ue^{j\omega t} \qquad y_p(t) = H(-j\omega)Ue^{-j\omega t}$$

$$y_p(0) = H(0)U$$

$$u(t) = U\cos(\omega t) = U\frac{e^{j\omega t} + e^{-j\omega t}}{2} = \frac{U}{2}e^{j\omega t} + \frac{U}{2}e^{-j\omega t}$$

$$y_p(t) = \frac{U}{2}e^{j\omega t}H(j\omega) + \frac{U}{2}e^{-j\omega t}H(-j\omega) = \frac{U}{2}H(j\omega)e^{j\omega t} + \frac{U}{2}\left[e^{j\omega t}H(j\omega)\right]^*$$

$$= 2\operatorname{Re}\left\{\frac{U}{2}e^{j\omega t}H(j\omega)\right\} = \operatorname{Re}\left(Ue^{j\omega t}H(j\omega)\right) =$$

$$= \operatorname{Re}\left[U \cdot |H(j\omega)|\, e^{j\measuredangle H(j\omega)}\, e^{j\omega t}\right] =$$

$$= \operatorname{Re}\left(U|H(j\omega)|\, e^{j(\omega t + \measuredangle H(j\omega))}\right) =$$

$$= U|H(j\omega)| \cdot \cos\left(\omega t + \measuredangle H(j\omega)\right)$$

$v(t) = 3\cos(\underbrace{2}_{\omega=2}t)$

$$|H(j2)| = \frac{1}{\sqrt{4+2^4}} = 0,2236$$

$$\measuredangle H(j2) = -\operatorname{arctg}\frac{2\cdot 2}{2-2^2} = -\operatorname{arctg}\frac{4}{-2} = -2.034$$

$$y_p(t) = 3\cdot 0,2236\cos(2t-2,034)$$

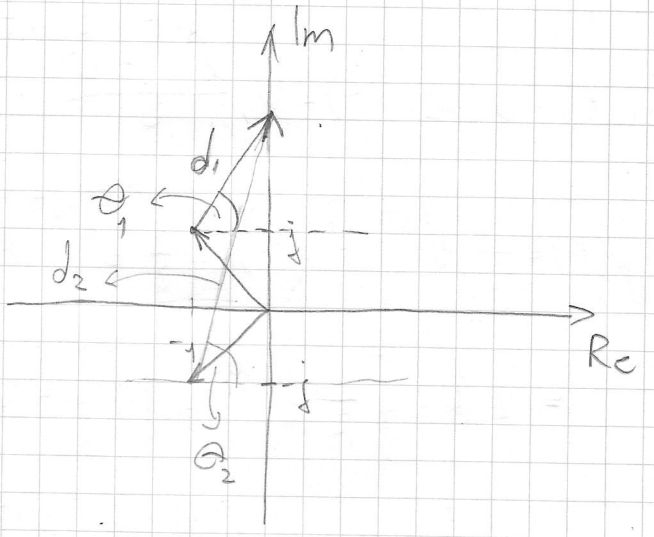
$$H(s) = \frac{1}{s^2+2s+2}$$

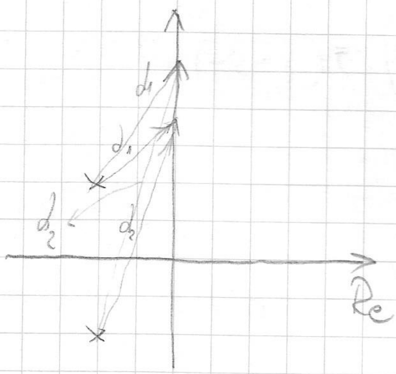
$$H(j\omega) = \frac{1}{(j\omega)^2+2j\omega+2} = \frac{1}{(j\omega-(-1+j))(j\omega-(-1-j))} =$$

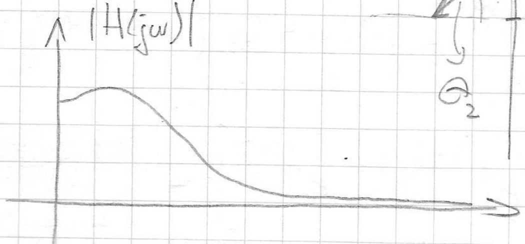
$$= \frac{1}{|j\omega-(-1+j)|\cdot|j\omega-(-1-j)|\,e^{j\measuredangle(-1+j)}\,e^{j\measuredangle(-1-j)}} =$$

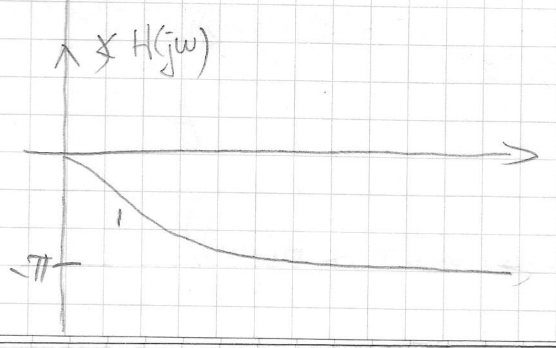
$$= \frac{1}{d_1\cdot d_2\, e^{j\theta_1}\, e^{j\theta_2}}$$

$$|H(j\omega)| = \frac{1}{d_1 d_2} \qquad \measuredangle H(j\omega) = -(\theta_1+\theta_2)$$









$\theta_1 \to \frac{\pi}{2}$

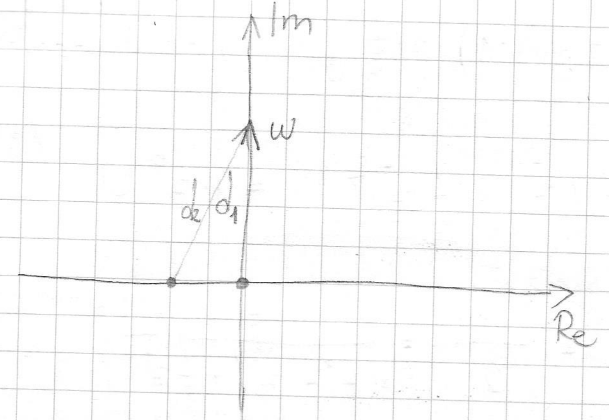
$\theta_2 \to \frac{\pi}{2}$

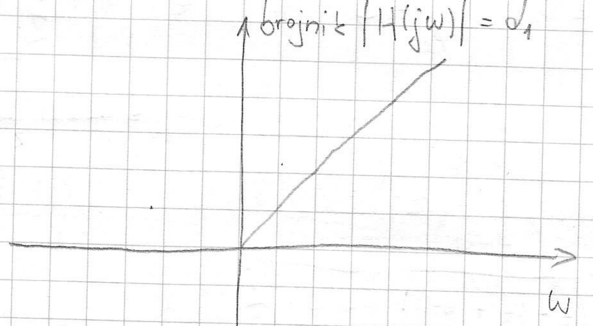
$\measuredangle H(j\omega) \to -\pi$

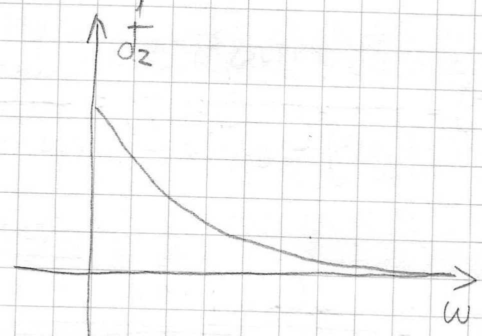
$$H(s) = \frac{s}{s+1}$$

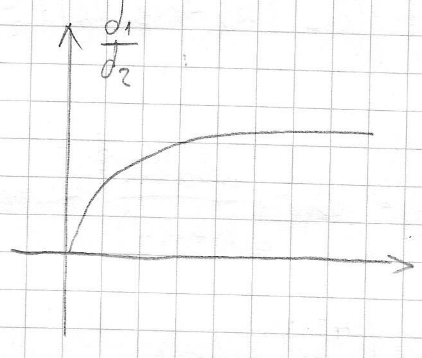
$$H(j\omega) = \frac{j\omega}{j\omega + 1}$$

$$|H(j\omega)| = \frac{|\omega|}{\sqrt{1+\omega^2}}$$





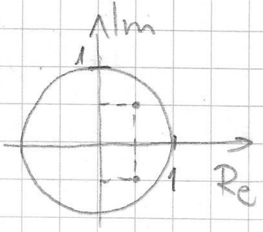




$$y(n) - y(n-1) + \frac{1}{2} y(n-2) = u(n)$$

$$\left(1 \cdot z^{0} - 1 z^{-1} + \frac{1}{2} z^{-2}\right) y = 1 \cdot z^{0} U$$

$$H(z) = \frac{Y}{U} = \frac{1 \cdot z^{0}}{1 \cdot z^{0} - z^{-1} + \frac{1}{2} z^{-2}} \Big/ \cdot z^{2} = \frac{z^{2}}{z^{2} - z + \frac{1}{2}}$$



$$z_{o_1} = 0 \qquad z_{o_2} = 0 \qquad z^{2} - z + \frac{1}{2} = 0$$

$$z_{1,2} = \frac{1 \pm \sqrt{1-2}}{2} = \frac{1 \pm j}{2} \Rightarrow \text{sustav je stabilan}$$

$$H(z) = \frac{(z-0)^{2}}{\left(z - \left(\frac{1}{2} + \frac{j}{2}\right)\right)\left(z - \left(\frac{1}{2} - \frac{j}{2}\right)\right)}$$

$$H(e^{j\Omega}) = \frac{(e^{j\Omega})^{2}}{(e^{j\Omega})^{2} - e^{j\Omega} + \frac{1}{2}} = \frac{\cos(2\Omega) + j\sin(2\Omega)}{\cos(2\Omega) + j\sin(2\Omega) - \cos(\Omega) - j\sin(\Omega) + \frac{1}{2}}$$

$$|H(e^{j\Omega})| = \frac{1}{\sqrt{\left(\cos(2\Omega) - \cos(\Omega) + \frac{1}{2}\right)^{2} + \left(\sin(2\Omega) - \sin(\Omega)\right)^{2}}}$$

$$\measuredangle H(e^{j\Omega}) = \operatorname{arctg}\frac{\sin(2\Omega)}{\cos(2\Omega)} - \operatorname{arctg}\frac{\sin(2\Omega) - \sin(\Omega)}{\cos(2\Omega) - \cos(\Omega) + \frac{1}{2}}$$

$$u(n) = 2\cos\left(\frac{\pi}{6} n\right) \mu(n)$$

$$y_p(n) = 2 \cdot |H(e^{j\Omega})| \cdot \cos\left(\frac{\pi}{6} n + \measuredangle H(e^{j\Omega})\right)$$

$$H(e^{j\frac{\pi}{6}}) = \frac{\cos\left(2 \cdot \frac{\pi}{6}\right) + j\sin\left(2\frac{\pi}{6}\right)}{\cos\left(2 \cdot \frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right) + \frac{1}{2} + j\sin\left(2 \cdot \frac{\pi}{6}\right) - j\sin\left(\frac{\pi}{6}\right)} = \frac{1}{0{,}384 + j0{,}067}$$

$$|H(e^{j\frac{\pi}{6}})| = 2{,}56$$

$$\measuredangle H(e^{j\frac{\pi}{6}}) = -0{,}173$$

$$y_p(n) = 2 \cdot 2{,}56 \cos\left(\frac{\pi}{6} n - 0{,}173\right) \mu(n)$$

Auditorne vježbe

③

$$y''(t) + 2y'(t) + 5y(t) = u(t)$$

$$H(s) = \frac{Y}{U} = \frac{1}{s^2+2s+5} \rightarrow \text{polovi su } -1 \pm 2j \rightarrow$$

Sustav je stabilan jer su realni dijelovi polova manji od ϕ

$s = j\omega \rightarrow$ za frekvencijsku karakteristiku

$$H(j\omega) = \frac{1}{(j\omega)^2 + 2j\omega + 5} = \frac{1}{-\omega^2 + 2j\omega + 5} = \frac{1}{5-\omega^2 + 2\omega j}$$

$$|H(j\omega)| = \left|\frac{1}{5-\omega^2+2\omega j}\right| = \frac{|1|}{|5-\omega^2+2\omega j|} = \frac{1}{\sqrt{(5-\omega^2)^2+(2\omega)^2}} = \frac{1}{\sqrt{25-10\omega^2+\omega^4+4\omega^2}} =$$

$$= \frac{1}{\sqrt{25-6\omega^2+\omega^4}} \text{ - amplitudno-frekvencijska karakteristika}$$

$$\measuredangle H(j\omega) = \operatorname{arctg}\frac{Im(H(j\omega))}{Re(H(j\omega))} = -\operatorname{arctg}\frac{2\omega}{5-\omega^2} \text{ - fazno-frekvencijska karakt.}$$

$t < 0$

$u(t) = \sin(t) \Rightarrow \omega = 1$

$$y_p(t) = |H(j\omega)| \cdot U \cdot \sin(\omega t + \measuredangle H(j\omega))$$

$$|H(j\cdot 1)| = \frac{1}{\sqrt{25-6+1}} = \frac{1}{\sqrt{20}} = \frac{1}{2\sqrt{5}}$$

$$\measuredangle H(j\cdot 1) = -\operatorname{arctg}\frac{2}{5-1} = -\operatorname{arctg}\frac{1}{2} = -26{,}56^\circ$$

$$y_p(t) = \frac{1}{2\sqrt{5}} \cdot 1 \cdot \sin(t - 26{,}56^\circ)$$

$t > 0$

$u(t) = 2\sin(2t)$

$$|H(j\cdot 2)| = \frac{1}{\sqrt{25-24+16}} = \frac{1}{\sqrt{17}}$$

$$\measuredangle H(j\cdot 2) = -\operatorname{arctg}\frac{4}{5-4} = -75{,}96^\circ$$

$$y_p(t) = \frac{1}{\sqrt{17}}\cdot 2\cdot \sin\left(2t - 75{,}96^\circ\right)$$

Homogeno rješenje:

$$y''(t) + 2y'(t) + 5y(t) = 0$$

$$s^2 + 2s + 5 = 0$$

$$s_{1,2} = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4j}{2} = -1 \pm 2j$$

$$y_h(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t}$$

(Totalni) Odziv prije nule je samo partikularno rješenje jer se homogeno rješenje istitralo u $-\infty$.

$$y_p(0^-) = \frac{1}{2\sqrt{5}}\sin(0 - 26{,}56) = -0{,}1$$

$$y_p'(t) = \frac{1}{2\sqrt{5}}\cos(t - 26{,}56)$$

$$y_p'(0^-) = \frac{1}{2\sqrt{5}}\cos(0 - 26{,}56) = 0{,}2$$

$y_p(0^+) = -0{,}1$ $\qquad y_T(t) = C_1 e^{(-1+2j)t} + C_2 e^{(-1-2j)t} + \frac{2}{\sqrt{17}}\sin(2t - 75{,}96^\circ)$

$y_p(0^+) = 0{,}2$ $\qquad y_T'(t) = (-1+2j)C_1 e^{(-1+2j)t} + C_2(-1-2j)e^{(-1-2j)t} + \frac{4}{\sqrt{17}}\cos(2t - 75{,}96)$

$y(0^+) = C_1 + C_2 + \frac{2}{\sqrt{17}} \sin(-75,96) = -0,1$

$y'(0^+) = (-1+2j) C_1 + (-1-2j) C_2 + \frac{4}{\sqrt{17}} \cos(-75,96) = 0,2$

$C_1 = 0,19 + 0,08 j$

$C_2 = 0,19 - 0,08 j$

$t>0 \quad y_T(t) = (0,19 + 0,08j) e^{(-1+2j)t} + (0,19 - 0,08j) e^{(-1-2j)t} + \frac{2}{\sqrt{17}} \sin(2t - 75,96^\circ)$

$t<0 \quad y_T(t) = \frac{1}{2\sqrt{5}} \sin(t - 26,56^\circ)$

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$y'(t) + a_1 y(t) = b_1 u(t)$

$u_1(t) = e^{-3t} \longrightarrow y_1(t) = 3e^{-3t}$

$u_2(t) = e^{-4t} \longrightarrow y_2(t) = 4e^{-4t}$

$H(s) = \frac{b_1}{s + a_1}$

$H(s) = \left. \frac{y_p(t)}{u(t)} \right|_{u(t) = Ue^{st}}$

$H(-3) = \frac{3e^{-3t}}{e^{-3t}} = 3$

$H(-4) = \frac{4e^{-4t}}{e^{-4t}} = 4$

$\left. \begin{aligned} \frac{b_1}{-3 + a_1} &= 3 \\ \frac{b_1}{-4 + a_1} &= 4 \end{aligned} \right\} \quad \begin{aligned} a_1 &= 7 \\ b_1 &= 12 \end{aligned}$

$H(s) = \frac{12}{s + 7}$

$y'(t) + 7y(t) = 12 u(t)$

$u_3(t) = e^{-5t}$

Ovaj postupak se može raditi ako je pobuda svevremenska

$H(-5) = \frac{12}{-5+7} = 6$

$y_3(t) = 6e^{-5t}$

(5) $y(n) + 0{,}5y(n-1) = u(n)$

$H(z) = \frac{1}{1+0{,}5z^{-1}} = \frac{z}{z+0{,}5}$

nule: $z_1 = 0$

pol: $z = -0{,}5$

$|z| < 1$ ⇓

Sustav je stabilan ako su apsolutne vrijednosti polova manje od 1

$z = e^{j\Omega}$

$$H(e^{j\Omega}) = \frac{1}{1+0{,}5e^{-j\Omega}} = \frac{1}{1+0{,}5[\cos\Omega - j\sin\Omega]} = \frac{1}{1+0{,}5\cos\Omega - j0{,}5\sin\Omega}$$

$$|H(e^{j\Omega})| = \frac{1}{\sqrt{(1+0{,}5\cos\Omega)^2 + (0{,}5\sin\Omega)^2}} = \frac{1}{\sqrt{1+\cos\Omega + 0{,}5^2\cos^2\Omega + 0{,}5^2\sin^2\Omega}} =$$

$$= \frac{1}{\sqrt{1{,}25 + \cos\Omega}}$$

$$\measuredangle H(e^{j\Omega}) = -\operatorname{arctg}\frac{-0{,}5\sin\Omega}{1+0{,}5\cos\Omega}$$

$u_1(n) = \cos\left(\frac{\pi n}{2} + \frac{\pi}{5}\right) \rightarrow \Omega = \frac{\pi}{2}$

$$y_{p1}(n) = |H(e^{j\Omega})|\, U \cos\left(\frac{\pi}{2}n + \frac{\pi}{5} + \measuredangle H(e^{j\Omega})\right)$$

$$|H(e^{j\frac{\pi}{2}})| = \frac{2}{\sqrt{5}}$$

$$\measuredangle H(e^{j\frac{\pi}{2}}) = -\operatorname{arctg}\frac{-0{,}5\sin\frac{\pi}{2}}{1+0{,}5\cos\frac{\pi}{2}} = 0{,}46$$

$$y_{p1}(n) = \frac{2}{\sqrt{5}}\cos\left(\frac{\pi}{2}n + \frac{\pi}{5} + 0{,}46\right)$$

$$y(n) + 0{,}5\,y(n-1) = 0$$

$$1 + 0{,}5\,g^{-1} = 0$$

$$g = -0{,}5$$

$$y_h(n) = c\,(-0{,}5)^n$$

Nastavak predavanja

Z-transformacija

$$H(z) = \sum_{n=-\infty}^{+\infty} h(n)\, z^{-n}$$

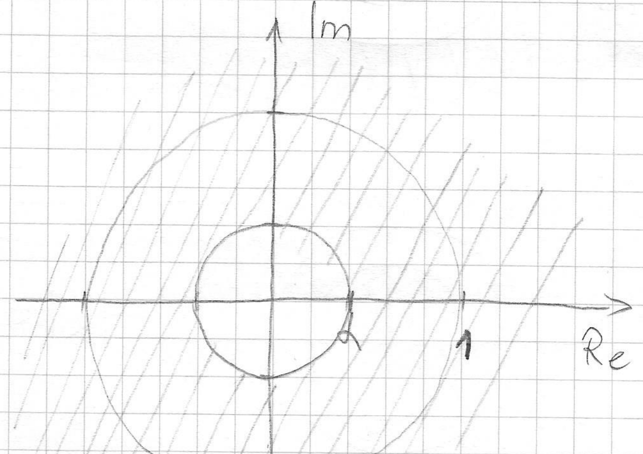
$$h(n) = \left\{ \begin{array}{cccccc} n=0 & n=1 & n=2 & n=3 & n=4 & n=5 \\ \underline{1}, & 2, & 3, & 3, & 2, & 1 \end{array} \right\}$$

$$H(z) = 1\cdot z^{0} + 2\cdot z^{-1} + 3\cdot z^{-2} + 3\cdot z^{-3} + 2\cdot z^{-4} + 1\cdot z^{-5} =$$

$$= \frac{z^5 + 2\cdot z^4 + 3z^3 + 3z^2 + 2z + 1}{z^5}$$

2) $x(n) = \alpha^n \mu(n)$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^n \mu(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n = \frac{1}{1-\frac{\alpha}{z}} = \frac{z}{z-\alpha}$$



$$\left|\frac{\alpha}{z}\right| < 1$$

$$|\alpha| < |z|$$

$$|z| > |\alpha|$$

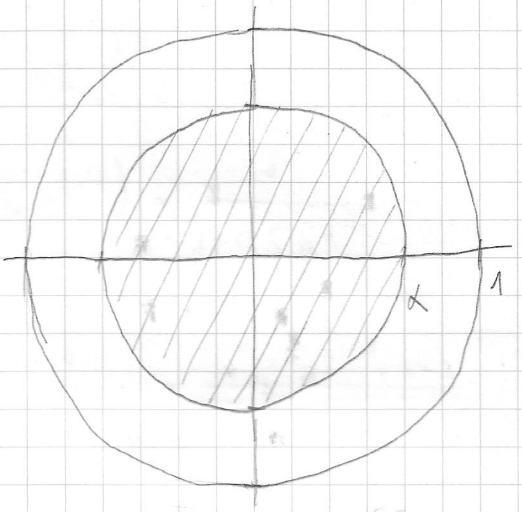
$$x(n) = -\alpha^n \mu(-n-1)$$

$$X(z) = \sum_{n=-\infty}^{\infty} -\alpha^n \mu(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} -\alpha^n z^{-n} = -\sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n = -\sum_{n=1}^{\infty} \left(\frac{\alpha}{z}\right)^{-n} =$$

$$= -\left[\sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n - \left(\frac{z}{\alpha}\right)^0\right] = -\sum_{n=0}^{\infty} \left(\frac{z}{\alpha}\right)^n + 1 = \frac{-1}{1-\frac{z}{\alpha}} + 1 =$$

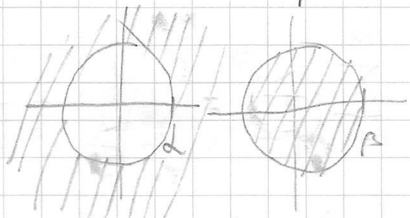
$$= -\frac{\alpha}{\alpha - z} + 1 = \frac{\alpha - z - \alpha}{\alpha - z} = \frac{z}{z-\alpha}, \quad \left|\frac{z}{\alpha}\right| < 1$$

$$|z| < |\alpha|$$

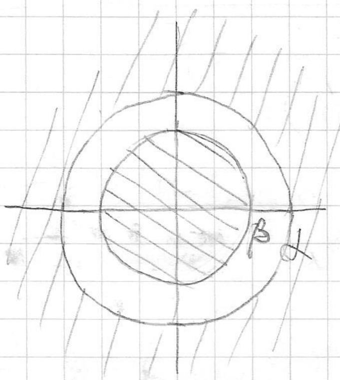


$$x(n) = \alpha^n \mu(n) + \beta^n \mu(-n-1)$$

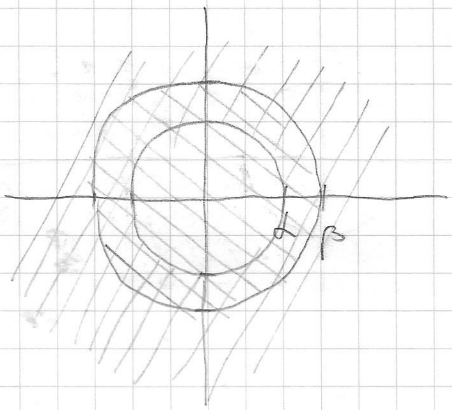
$$X(z) = \frac{z}{z-\alpha} - \frac{z}{z-\beta}$$



→ Tražimo presjek



$|\beta| < |\alpha|$ → nema z-transformacije (nema presjeka)



$$|\alpha| < |z| < |\beta|$$

$$\delta(n) \xrightarrow{z} 1 \qquad \sum_{n=0}^{\infty} \delta(n) z^{-n} = 1$$

$$\delta(n-m) \xleftrightarrow{z} z^{-m} \qquad \sum_{n=0}^{\infty} \delta(n-m) z^{-n} = z^{-m}$$

$$\mu(n) \xleftrightarrow{z} \frac{z}{z-1} \qquad \sum_{n=0}^{\infty} \mu(n) z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}} = \frac{z}{z-1}$$

$$a^n \mu(n) \xleftrightarrow{z} \frac{z}{z-a}$$

Z-transformacija – kašnjenje za korak p

$$x(n) \xleftrightarrow{z} X(z)$$

$$x(n-p) \xleftrightarrow{z} ?$$

$$Z\big(x(n-p)\big) = \sum_{n=0}^{\infty} x(n-p) z^{-n} = \left| \begin{matrix} n-p=m \\ n=m+p \end{matrix} \right| = \sum_{m=-p}^{\infty} x(m) z^{-m-p} =$$

$$= \sum_{m=0}^{\infty} x(m) z^{-m-p} + \sum_{m=-p}^{-1} x(m) z^{-m-p} =$$

$$= z^{-p} \sum_{m=0}^{\infty} x(m) z^{-m} + z^{-p} \sum_{m=-p}^{-1} x(m) z^{-m} =$$

$$= z^{-p} X(z) + z^{-p} \sum_{m=-p}^{-1} x(m) z^{-m}$$

$$x(n-1) \longleftrightarrow z^{-1} X(z) + z^{-1} \sum_{n=-1}^{-1} x(m) z^{-m} = z^{-1} X(z) + x(-1)$$

$$z^{-1}\left[x(-1) z\right] = x(-1)$$

$$x(n-2) \longleftrightarrow z^{-2} X(z) + z^{-2} \sum_{m=-2}^{-1} x(m) z^{-m} = z^{-2} X(z) + z^{-2}\left[x(-2) z^{2} + x(-1) z\right]$$

$$= z^{-2} X(z) + x(-2) + x(-1) z^{-1}$$

Primjena z-transformacije u analizi linearnih sustava
- prijenosna funkcija

$v(n) = -9 \cdot (-0{,}9)^n \mu(n)$

Tražimo odziv mirnog sustava

$$y(n) - 0{,}9\,y(n-1) + 0{,}2\,y(n-2) = v(n) + v(n-1) + v(n-2)$$

$$Y(z) - 0{,}9\left[z^{-1}Y(z) + y(-1)\right] - 0{,}2\left[z^{-2}Y(z) + y(-2) + y(-1)z^{-1}\right] =$$

$$= U(z) + z^{-1}U(z) + v(-1) + z^{-2}U(z) + v(-2) + v(-1)z^{-1}$$

$$Y(z)\left[1 - 0{,}9z^{-1} - 0{,}2z^{-2}\right] + \left[-0{,}9y(-1) + 0{,}2y(-2) + 0{,}2z^{-1}y(-1)\right] =$$

$$= U(z)\left[1 + z^{-1} + z^{-2}\right] + \left[v(-1) + v(-2) + v(-1)z^{-1}\right]$$

$$Y(z) = \underbrace{\frac{1 + z^{-1} + z^{-2}}{1 - 0{,}9z^{-1} - 0{,}2z^{-2}}}_{H(z)\text{ - prijenosna f-ja}} U(z) - \underbrace{\frac{-0{,}9y(-1) + 0{,}2y(-2) + 0{,}2z^{-1}y(-1)}{1 - 0{,}9z^{-1} + 0{,}2z^{-2}}}_{=\phi \text{ jer je sustav miran pa } y(-1) = y(-2) = \phi}$$

$v(n) = -9(-0{,}9)^n \mu(n)$

$$U(z) = -9\,\frac{z}{z+0{,}9}$$

$$Y(z) = \frac{1 + z^{-1} + z^{-2}}{1 - 0{,}9z^{-1} - 0{,}2z^{-2}} \cdot \frac{-9z}{z+0{,}9} = \frac{z^2 + z + 1}{z^2 - 0{,}9z + 0{,}2} \cdot \frac{-9z}{z+0{,}9}$$

Inverzna z-transformacija

1) $$Y_1(z) = \frac{Y(z)}{z} = \frac{z^2 + z + 1}{z^2 - 0{,}9z + 0{,}2} \cdot \frac{-9}{z+0{,}9} = \frac{-9z^2 - 9z - 9}{(z-0{,}5)(z-0{,}4)(z+0{,}9)} =$$

$$= \frac{A}{z-0{,}5} + \frac{B}{z-0{,}4} + \frac{C}{z+0{,}9}$$

1. način rješavanja - svedemo na zajednički nazivnik

$$\frac{A(z-0,4)(z+0,9)+B(z-0,5)(z+0,9)+C(z-0,5)(z-0,4)}{(z-0,5)(z-0,4)(z+0,9)} =$$

$$= \frac{A(z^2+0,9z-0,4z-0,36)+B(z^2+0,9z-0,5z-0,45)+C(z^2-0,4z-0,5z+0,2)}{(z-0,5)(z-0,4)(z+0,9)}$$

brojnik izjednačimo sa $-9z^2-9z-9$

$$\left.\begin{aligned} A+B+C&=-9\\ 0,5A+0,4B-0,9C&=-9\\ -0,36-0,45+0,2&=-9\end{aligned}\right\} \qquad \begin{aligned}A&=-112,5\\ B&=108\\ C&=-4,5\end{aligned}$$

$$Y_1(z)=\frac{-112,5}{z-0,5}+\frac{108}{z-0,4}+\frac{-4,5}{z+0,9}$$

(to još treba pomnožiti sa z jer smo ranije podijelili sa z)

$$y_m(z)=\frac{-112,5z}{z-0,5}+\frac{108z}{z-0,4}+\frac{-4,5z}{z+0,9}$$

$$y_m(n)=-112,5(0,5)^n+108(0,4)^n-4,5(-0,9)^n, \quad n\geq 0$$

2. način rješavanja:

$$A=\lim_{z\to 0,5}\left[\cancel{(z-0,5)}\frac{-9z^2-9z-9}{\cancel{(z-0,5)}(z-0,4)(z+0,9)}\right]=$$

$$=\frac{-9[0,5^2+0,5+1]}{(0,5-0,4)(0,5+0,9)}=-112,5$$

$$B=\lim_{z\to 0,4}\left[\cancel{(z-0,4)}\frac{-9(z^2+z+1)}{(z-0,5)\cancel{(z-0,4)}(z+0,9)}\right]=\frac{-9(0,4^2+0,4+1)}{(0,4-0,5)(0,4+0,9)}=108$$

$$B=\lim_{z\to -0,9}\left[\cancel{(z+0,9)}\frac{-9(z^2+z+1)}{(z-0,5)(z-0,4)\cancel{(z+0,9)}}\right]=\frac{-9((-0,9)^2-0,9+1}{(-0,9-0,5)(-0,9-0,4)}=-4,5$$

$$y_m(z)=\frac{-112,5z}{z-0,5}+\frac{108z}{z-0,4}+\frac{-4,5z}{z+0,9}$$

Impulsni odziv u vremenu je prijenosna f-ja u z-domeni

Odziv nepobuđenog sustava

$y(-1)=6 \quad y(-2)=-3$

$$Y_n(z) = -\frac{-0{,}9y(-1)+0{,}2y(-2)+0{,}2z^{-1}y(-1)}{1-0{,}9z^{-1}+0{,}2z^{-2}} = +\frac{0{,}9\cdot 6-0{,}2\cdot(-3)-0{,}2z^{-1}\cdot 6}{1-0{,}9z^{-1}+0{,}2z^{-2}} =$$

$$= \frac{6-1{,}2z^{-1}}{1-0{,}9z^{-1}+0{,}2z^{-2}} = \frac{6z^2-1{,}2z}{z^2-0{,}9z+0{,}2}$$

$$\frac{Y_n(z)}{z} = \frac{A}{(z-0{,}5)} + \frac{B}{(z-0{,}4)} = \frac{6z-1{,}2}{(z-0{,}5)(z-0{,}4)}$$

$$A(z-0{,}4)+B(z-0{,}5)=6z-1{,}2$$

$$\left.\begin{aligned} A+B&=6 \\ -0{,}4A-0{,}5B&=-1{,}2 \end{aligned}\right\} \quad \begin{aligned} A&=18 \\ B&=-12 \end{aligned}$$

$$\frac{Y_n(z)}{z} = \frac{18}{z-0{,}5} + \frac{-12}{z-0{,}4}$$

$$Y_n(z) = \frac{18z}{z-0{,}5} - \frac{12z}{z-0{,}4}$$

$$y_n(n) = 18(0{,}5)^n - 12(0{,}4)^n, \quad n \geq 0$$

Jednostrana Laplace transformacija

$$X(s) = \int_{0^-}^{\infty} x(t) e^{-st} dt$$

$\mathcal{L}$ - transformacija osnovnih signala

$$\delta(t) \xrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \delta(t) e^{-st} dt = 1$$

$$\delta(t-t_0) \xrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \delta(t-t_0) e^{-st} dt = e^{-st_0}$$

$$\mu(t) \xrightarrow{\mathcal{L}} \int_{0^-}^{\infty} \mu(t) e^{-st} dt = \int_{0}^{\infty} e^{-st} dt = \frac{e^{-st}}{-s} \Big|_0^{\infty} = \frac{1}{s}$$

$$e^{-at}\mu(t) \xrightarrow{\mathcal{L}} \int_{0^-}^{\infty} e^{-at}\mu(t) e^{-st} dt = \int_{0}^{\infty} e^{-(a+s)t} dt = \frac{e^{-(a+s)t}}{-(a+s)} \Big|_0^{\infty} = \frac{1}{s+a}$$

Frekvencijski pomak

$$x(t) e^{s_0 t} \xleftrightarrow{\mathcal{L}} \int_{0^-}^{\infty} x(t) e^{s_0 t} e^{-st} dt = \int_{0^-}^{\infty} x(t) e^{-(s-s_0)t} dt = X(s-s_0)$$

$$\sin(bt)\mu(t) \circ\!\!-\!\!\bullet \frac{b}{s^2+b^2}$$

$$e^{-at}\sin(bt)\mu(t) \circ\!\!-\!\!\bullet \frac{b}{(s+a)^2+b^2}$$

$$\frac{d}{dt}x(t) \circ\!\!-\!\!\bullet \int_{0^-}^{\infty} \frac{d}{dt}x(t) e^{-st} dt = \int_{0^-}^{\infty} e^{-st} dx(t) = \begin{vmatrix} v = e^{-st} & dx = dw \\ dv = -se^{-st} dt & x(t) = w \end{vmatrix}$$

$$= x(t) e^{-st} \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t) e^{-st} dt = -x(0^-) + sX(s)$$

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$$y''(t) + 2y'(t) + 5y(t) = u(t)$$

$$u(t) = 5\mu(t)$$

$$y(0^-) = -1 \quad y'(0^-) = -1$$

$$U(s) = \frac{5}{s}$$

$$s^2 Y(s) - s y(0^-) - y'(0^-) + 2\left[s Y(s) - y(0^-)\right] + 5Y(s) = U(s)$$

$$Y(s)\left[s^2 + 2s + 5\right] - s y(0^-) - y'(0^-) - 2y(0^-) = U(s)$$

$$Y(s) = \underbrace{\frac{1}{s^2+2s+5}}_{H(s)} U(s) + \frac{s y(0^-) + y'(0^-) + 2y(0^-)}{s^2+2s+5}$$

Odziv mirnog sustava:

$$Y_m(s) = \frac{1}{s^2+2s+5} \cdot \frac{5}{s} = \frac{A}{s} + \frac{Bs + C}{s^2+2s+5}$$

$$A\left[s^2 + 2s + 5\right] + Bs^2 + Cs = 5$$

$$\left.\begin{aligned} A + B &= 0 \\ 2A + C &= 0 \\ 5A &= 5 \end{aligned}\right\} \quad \begin{aligned} A &= 1 \\ B &= -1 \\ C &= -2 \end{aligned}$$

$$Y_m(s) = \frac{1}{s} + \frac{-s-2}{\underbrace{s^2+2s+5}_{(s+1)^2+2^2}} = \frac{1}{s} - \frac{s+1+1}{(s+1)^2+2^2} = \frac{1}{s} - \frac{s+1}{(s+1)^2+2^2} - \frac{1 \cdot 2}{(s+1)^2+2^2} \cdot \frac{1}{2}$$

$$y_m(t) = \mu(t) - e^{-t}\cos(2t) - \frac{1}{2}e^{-t}\sin(2t), \quad t \geq 0$$

Odziv nepobuđenog sustava

$$Y_0(s) = \frac{s\cdot(-1) - 1 + 2\cdot(-1)}{s^2+2s+5} = -\frac{s+3}{s^2+2s+5} = -\frac{(s+1)+2}{(s+1)^2+2^2} =$$

$$= -\frac{s+1}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+2^2}$$

$$y_0(t) = -e^{-t}\cos(2t) - e^{-t}\sin(2t), \quad t \geq 0$$

Auditorne vježbe

(5) $y(n) - y(n-2) = u(n)$

$$Y(z) - z^{-2}Y(z) = U(z)$$

$$Y(z) = \frac{1}{1-z^{-2}}U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1-z^{-2}} = \frac{z^2}{z^2-1}$$

nule: $z_{1,2} = 0$

polovi: $z_{1,2} = \pm 1$

$$u(n) = \mu(t) \circ\!\!\longrightarrow \frac{z}{z-1}$$

$$Y(z) = \frac{z^2}{z^2-1}\cdot\frac{z}{z-1} = \frac{z^3}{(z-1)^2(z+1)} \quad \Bigg| \quad y(0), y(\infty) = ?$$

$$y(0) = \lim_{z\to\infty} Y(z) = \lim_{z\to\infty}\frac{z^3}{(z+1)(z-1)^2} = 1$$

$$\lim_{n\to\infty} y(n) = \lim_{z\to 1}(1-z^{-1})Y(z) = \lim_{z\to 1}\left[\frac{z-1}{\cancel{z}}\,\frac{z^{\cancel{3}}}{(z-1)^2(z+1)}\right] =$$

$$\lim_{z\to 1}\frac{z^2}{(z-1)(z+1)} = \frac{1^2}{(1-1)(1+1)} = \infty$$

$$Y(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$Y_1(z) = \frac{Y(z)}{z} = \frac{z^2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2} =$$

$$= \frac{A(z-1)^2 + B(z+1)(z-1) + C(z+1)}{(z+1)(z-1)^2}$$

$$A[z^2 - 2z + 1] + B[z^2 - 1] + C[z+1] = z^2$$

$$\left.\begin{aligned} A + B &= 1 \\ -2A + C &= 0 \\ A - B + C &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= \frac{1}{4} \\ B &= \frac{3}{4} \\ C &= \frac{1}{2} \end{aligned}$$

$$Y_1(z) = \frac{1}{4}\frac{1}{z+1} + \frac{3}{4}\frac{1}{z-1} + \frac{1}{2}\frac{1}{(z-1)^2}$$

$$Y(z) = \frac{1}{4}\frac{z}{z+1} + \frac{3}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2}$$

$$y(n) = \left[\frac{1}{4}(-1)^n + \frac{3}{4}(1)^n + \frac{1}{2}n\right]\mu(n)$$

Drugi način računanja koeficijenata:

$$A = \lim_{z \to -1}\left[\frac{z^2}{(z-1)^2}\right] = \frac{(-1)^2}{(-1-1)^2} = \frac{1}{4}$$

$$B = \frac{1}{(2-1)!}\lim_{z \to 1}\left[\frac{d^{2-1}}{dt^{2-1}}\left(\cancel{(z-1)^2}\frac{z^2}{(z+1)\cancel{(z-1)^2}}\right)\right] = \lim_{z \to 1}\left[\frac{d}{dt}\frac{z^2}{z+1}\right] =$$

$$= \lim_{z \to 1}\left[\frac{2z(z+1) - z^2}{(z+1)^2}\right] = \frac{2+2-1}{4} = \frac{3}{4}$$

$$C = \frac{1}{(2-2)!}\lim_{z \to 1}\left[\frac{d^{2-2}}{dt^{2-2}}\left(\cancel{(z-1)^2}\frac{z^2}{(z+1)\cancel{(z-1)^2}}\right)\right] = \lim_{z \to 1}\frac{z^2}{z+1} = \frac{1}{2}$$