Datum

SIS - MASS 2 - DIO II

11.5.08 (1)

DIFERENCIJSKE JEDNADŽBE

- diskretni sustavi

• $y(n) = u(n) + a\,y(n-N)$ ⟸ ULAZNO-IZLAZNI OBLIK

$y(n)$ → trenutni izlaz, $u(n)$ → trenutni ulaz, $y(n-N)$ → prošli izlaz

$N=4$, $a=0{,}6$

$y(n) = u(n) + 0.6\,y(n-4)$,

ITERATIVNI POSTUPAK RJEŠAVANJA

početni uvjeti zadani: $y(-1)=y(-2)=y(-3)=y(-4)=0$

$u(n)=\delta(n)$

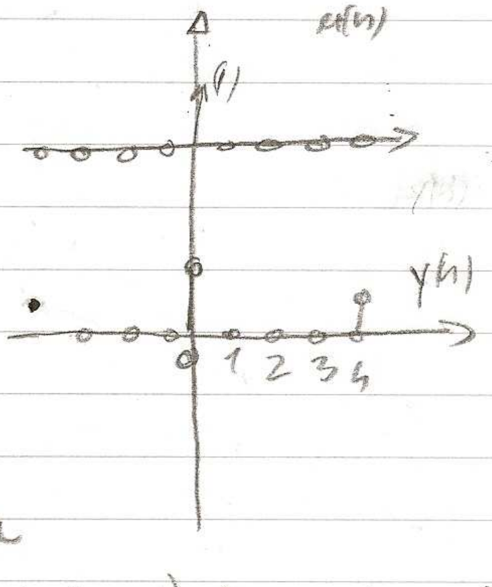
$y(0)= u(0)+0.6\,y(-4) = 1$

$y(1) = u(1) + 0{,}6\,y(1-4)=0$

$y(2) = u(2) + 0{,}6\,y(-2) = 0$

$y(3) = u(3) + 0{,}6\,y(-1) = 0$

$y(4) = u(4) + 0{,}6\,y(0) = 0{,}6$



→ DIFERENCIJSKA J. N-tog stupnja

$$y(n)+a_1 y(n-1) + a_2 y(n-2) + \dots + a_N y(n-N) = b_0 u(n) + b_1 u(n-1) + \dots + b_N u(n-N)$$

$a_i \in \mathbb{R}$, $b_i \in \mathbb{R}$

$y(n) + 0\cdot y(n-1) + 0\cdot y(n-2) \dots$

Stupanj određuje najveća razlika pomaka!

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- **Pr.** $a_0 y(n) = u(n) + 2u(n-1) \quad | : a_0$

$$y(n) = \underbrace{\frac{1}{a_0}}_{b_0} u(n) + \underbrace{\frac{2}{a_0}}_{b_1} u(n-1)$$

→ U ispit neće doć sustavi veći od 2. reda!

→ Ako slučajno dođe onda je neki trik u pitanju!

- $y(n) + a_1 y(n-1) + a_2 y(n-2) = 0$ → NEPOBUĐENI SUST.

Rješenje je oblika $C \cdot q^n$

$$Cq^n + a_1 C q^{n-1} + a_2 C q^{n-2} = 0$$

$$Cq^n (1 + a_1 q^{-1} + a_2 q^{-2}) = 0 \;/ : Cq^n, \quad q \neq 0, C \neq 0$$

$$1 + a_1 q^{-1} + a_2 q^{-2} = 0 \;/ \cdot q^2$$

$$q^2 + a_1 q + a_2 = 0 \quad \text{karakteristična j.}$$

$$q_{1,2} = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2}$$ → Linearna kombinacija rješenja $y(n) = C_1 (q_1)^n + C_2 (q_2)^n$

- $y(n) - a_1 y(n-1) = 0$

$q - a_1 = 0$ karakt. jed.

Lin. komb. rj. $y(n) = Cq^n$

početni uvjeti

$y(1) = P_1 \quad y(-1) = C_1 q^{-1} + C_2 q^{-1} = P_1$

$y(2) = P_2 \quad y(-2) = C_1 q^{-2} + C_2 q^{-2} = P_2$

2 jed. sa 2 nep.

- 2.MI.

$$y(n+3)+5y(n+2)+11y(n+1)+6y(n)=u(n)$$

$$y(0)=y(1)=y(2)=0$$

$$u(n)=0$$

$$q^3+5q^2+11q+6=0$$

$$y(n)=C_1(q_1)^n+C_2(q_2)^n+C_3(q_3)^n$$

Nema odziva

Ako su početni uvjeti 0 onda je sustav mrtav!
Onda nema energije!

- Vranković str. 63

$$y(n)-4y(n-2)=0 \quad , \quad y(-1)=1$$
$$y(-2)=-1$$

$$q^2-4=0$$

$$(q-2)(q+2)=0$$

$q_1=2$, $q_2=-2$ ⟹ korijeni jednadžbe

$$y(n)=C_1\cdot(2)^n+C_2(-2)^n$$

$$y(-1)=$$

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• $y(n+2) + y(n) = 0$

$$g^2 + 1 = 0$$

$$g^2 = -1$$

$g_{1,2} = \pm i$ – Kompleksno rješenje karakt. j. (korijen)

$$g_{1,2} = e^{\pm j\frac{\pi}{2}} \qquad g_1 = \overset{*}{g_2}$$

$$y(n) = C_1 a_1^n + C_2 a_2^n$$

$$y(n) = C_1 e^{j\frac{\pi}{2}n} + C_2 e^{-j\frac{\pi}{2}n} \qquad \left\{ e^{j\varphi} = \cos\varphi + i\sin\varphi \right\}$$

$$y(n) = C_1\left(\cos\tfrac{\pi}{2}n + j\sin\tfrac{\pi}{2}n\right) + C_2\left(\cos\tfrac{\pi}{2}n - j\sin\tfrac{\pi}{2}n\right)$$

$$y(n) = \cos\left(\tfrac{\pi}{2}n\right)\underbrace{\left(C_1 + C_2\right)}_{A} + j\sin\left(\tfrac{\pi}{2}n\right)\underbrace{\left(C_1 - C_2\right)}_{B}$$

$$C_1 = a + jb$$

$$C_2 = a - jb$$

$$C_1 + C_2 = a + jb + a - jb = 2a$$

$$j(C_1 - C_2) = j(a + jb - a + jb) = -2b$$

$$y(n) = A\cos\left(\tfrac{\pi}{2}n\right) + B\cos\left(\tfrac{\pi}{2}n\right)$$

i sad početni uvjeti, 2 jed s 2 nepozn. itd

- Ako su korijeni kompleksni

$$q_{1,2} = a \pm jb = r \cdot e^{\pm j\varphi}$$

$$r = \sqrt{a^2+b^2}$$

$$\varphi = \operatorname{arctg}\frac{b}{a}$$ PAZI NA KVADRANT!!

$$y(n) = r^n \left(A\cos\varphi n + B\sin\varphi n\right)$$

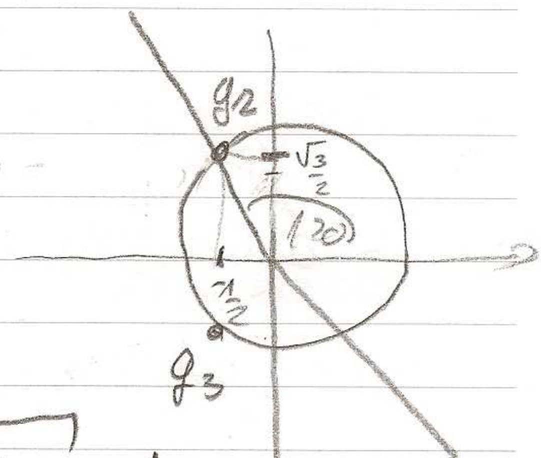
- Tj. 11. 3

$$y(n+3) - y(n) = 0 \quad , y(0)=0, \; y(1)=0, \; y(2)=1$$

$$q^3 - 1 = 0$$

$$(q-1)(q^2+q+1) = 0$$

$$q_1 = 1$$



$$q^2 + q + 1 = 0$$

$$q_{1,2} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm j\sqrt{3}}{2}$$

$$q_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j \quad , \quad q_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2}j$$

$$a = -\frac{1}{2}, \quad b = \frac{\sqrt{3}}{2} \qquad r = \sqrt{\frac{1}{4}+\frac{3}{4}} = 1$$

$$\varphi = \operatorname{arctg}\frac{b}{a} = \operatorname{arctg}\frac{\frac{\sqrt{3}}{2}}{-\frac{1}{2}} = \operatorname{arc}(-\sqrt{3}) = \frac{2\pi}{3}$$

$$q_2 = 1 \cdot e^{j\frac{2\pi}{3}} \quad , \quad q_3 = 1 \cdot e^{-j\frac{2\pi}{3}}$$

$\longrightarrow$

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$$y(n) = \underbrace{C_1(1)^n}_{q_1} + \underbrace{(1)^n\left(A\cos\left(\frac{2\pi}{3}n\right) + B\sin\left(\frac{2\pi}{3}n\right)\right)}_{q_2,\, q_3}$$

$$y(n) = C_1 + A\cos\frac{2\pi}{3}n + B\sin\frac{2\pi}{3}n$$

$y(0) = 0 \Rightarrow C_1 + A = 0$ Uvrštavamo n!

$y(1) = 0 \Rightarrow C_1 - \frac{1}{2}A + \frac{\sqrt{3}}{2}B = 0$

$y(2) = 1 \Rightarrow C_1 - \frac{1}{2}A - \frac{\sqrt{3}}{2}B = 1$

KALKUL.: $C_1 = \frac{1}{3}$ $A = -\frac{1}{3}$ $B = -0{,}5773502692$

$$\frac{1}{3} + \frac{1}{6} - \frac{\sqrt{3}}{2}B = 1 \Rightarrow B = -\frac{\sqrt{3}}{3}$$

Rj: $y(n) = \frac{1}{3} - \frac{1}{3}\cos\left(\frac{2\pi}{3}n\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}n\right)$ ✓

- $y(n+2) + 2y(n+1) + y(n) = 0$

$q^2 + 2q + 1 = 0$

$(q+1)^2 = 0 \Rightarrow q_{1/2} = -1$

$\cancel{y(n) = C_1(-1)^n + C_2(-1)^n = C(-1)^n}$ ← NE!

SAMO ROKAJ: $y(n) = C_1(-1)^n + C_2 \underline{\underline{n}} \cdot (-1)^n + \left\{ C_3 \underbrace{\underline{\underline{n^2}}(-1)^n \ldots}_{\text{OPĆENITO}} \right\}$

DA!

FREKVENCIJE SUSTAVA = KORIJENI KARAK. JED

Datum

- $y(n-1) + y(n) - y(n+1) = 0$

$y(n) = q^n$

$q^{n-1} + q^n - q^{n+1} = 0$

$q^{n-1}(1 + q^1 - q^2) = 0 \;/ : q^{n-1}$

$1 + q^1 - q^2 = 0$

- $y(n) + a_1 y(n-1) + a_2 y(n-2) = u(n) + b_1 u(n-1) + \ldots$

Nije homogena jer ima "smetnju"
Treba rastaviti rješenje na 2 dijela

Ukupni odziv sustava : homogenog dijela + partikularnog
$y(n) = y_h(n) + y_p(n)$
Homogeni - normalno (kao dosada)

- Pr. 3.2.9 - Vrankić

$y(n) - \frac{2}{3} y(n-1) + \frac{1}{9} y(n-2) = u(n)$

$u(n) = \left(\frac{1}{3}\right)^n \mu(n)$, Odredi opće rješenje !

$y(n) = y_h(n) + y_p(n)$

1) y homogeni

$q^2 - \frac{2}{3} q + \frac{1}{9} = 0$

$\longrightarrow$

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$$q_{1,2} = \frac{\frac{2}{3} \pm \sqrt{\frac{4}{9} - \frac{4}{9}}}{2} = \frac{1}{3}$$

$$\underline{y_h(n)} = C_1\left(\frac{1}{3}\right)^n + C_2 n \cdot \left(\frac{1}{3}\right)^n$$

2) Y PARTIKULARNO – Objašnjeno na sl. 41 (tablica)

$\mu(n) = \left(\frac{1}{3}\right)^n \mu(n)$ → s ovim se pozabavimo kasnije

→ $A r^n \Rightarrow K r^n \Rightarrow$ opći $K n^i r^n$, i-broj korijena koji se poklapaju s "r"

$$y_p(n) = K\left(\frac{1}{3}\right)^n$$

3) $y = y_h + y_p$

$$\boxed{y(n) = C_1\left(\frac{1}{3}\right)^n + C_2 n \cdot \left(\frac{1}{3}\right)^n + K n^2 \left(\frac{1}{3}\right)^n}$$

$\mu(n)$	$y_p(n)$
$A r^n$	$K n^i r^n$
$A_N n^m + \dots A_2 n^2 + A_1 n + A_0$	$K_N n^m + \dots K_1 n + K_0$
5	K_0
$2n^2 - 2$	$K_2 n^2 + K_0$
$r^n (A_m n^m + \dots + A_1 n + A_0)$	$r^n (K_m n^m + \dots K_1 n + K_0)$
$A\cos(\omega_0 n)$, $A\sin(\omega_0 n)$	$K_1 \cos(\omega_0 n) + K_2 \sin(\omega_0 n)$

↑ ↑ ↑ ↑ ↑ ↑ ↑ ↑

Tablica za partikularna rješenja

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• 3.2.6 – Vrankić

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + \frac{1}{2}u(n-1)$$

$y(-1) = 14$

$y(-2) = 52$

$$u(n) = 3\left(\frac{1}{8}\right)^n \mu(n)$$

$$y_H(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n \qquad Ar^n \Rightarrow y_p(n) = K\left(\frac{1}{8}\right)^n$$

1. Prvo treba riješiti konstante partikularnog:

$$K\left(\frac{1}{8}\right)^n - \frac{3}{4}K\left(\frac{1}{8}\right)^{n-1} + \frac{1}{8}\left(\frac{1}{8}\right)^{n-2} = 3\cdot\left(\frac{1}{8}\right)^n + \frac{1}{2}3\cdot\left(\frac{1}{8}\right)^{n-1} \Big/ :\left(\frac{1}{8}\right)^n$$

$$K - 6K + 8K = 3 + 12 \Rightarrow K = 5$$

$$y_p(n) = 5\left(\frac{1}{8}\right)^n$$

2. Sad pišemo ukupni odziv

$$y(n) = y_H(n) + y_p(n)$$

$$y(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n + 5\left(\frac{1}{8}\right)^n$$

Zbog $\mu(n)$ u pobudi nema pobude prije nule! Zato treba pomaknuti početne uvjete

$\begin{matrix} y(-1) \\ y(-2) \end{matrix} \Rightarrow \begin{matrix} y(1) \\ y(0) \end{matrix}$ Uvrštavanje u početnu jed. ⊛ $\Rightarrow \begin{matrix} y(0) = 7 \\ y(1) = \frac{43}{8} \end{matrix}$ $\Rightarrow$

$\Rightarrow$

$$y(0) = 7 = C_1\left(\frac{1}{2}\right)^0 + C_2\left(\frac{1}{4}\right)^0 + 5\left(\frac{1}{2}\right)^0$$

$$C_1 + C_2 = 2$$

$$y(1) = \frac{43}{8} = C_1 \frac{1}{2} + C_2 \frac{1}{4} + 5 \cdot \frac{1}{2}$$

$$\frac{1}{2}C_1 + \frac{1}{4}C_2 = \frac{38}{8}$$

$$C_1 = 17 \quad , \quad C_2 = -15$$

$$\boxed{Rj: \; y(n) = 17\left(\frac{1}{2}\right)^n - 15\left(\frac{1}{4}\right)^n + 5\left(\frac{1}{2}\right)^n}$$

$$* \quad y(0) - \frac{3}{4}\, y(-1) + \frac{1}{8}\, y(-2) = u(0) + \frac{1}{2}(u-1)$$

$$y(0) - \frac{3}{4} \cdot 14 + \frac{1}{8} \cdot 32 = 3 + 0$$

$$y(0) = 7$$

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LTI SUSTAVI

→ naredno stanje

$$x(n+1) = A\,x(n) + B\,u(n)$$

$$y(n) = C\,x(n) + D\,u(n)$$

↳ pobuda

• $x(n+1)$

3 naredna stanja

$$x_1(n+1) = 3x_1(n) - 2x_2(n) + u_1(n) + u_2(n)$$
$$x_2(n+1) = 4x_1(n) + 2u_1(n)$$
$$x_3(n+1) = 3x_2(n) - x_3(n) + u_2(n)$$

$$x(n+1) = \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -2 & 0 \\ 4 & 0 & 0 \\ 0 & 3 & -1 \end{bmatrix}}_{A} \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} +$$

$$+ \overset{B}{\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} u_1(n) \\ u_2(n) \end{bmatrix}$$

$$y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \\ y_3(n) \\ y_4(n) \end{bmatrix} = \underset{C}{\begin{bmatrix} \quad \end{bmatrix}} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} + \underset{D}{\begin{bmatrix} \text{izlazi} \end{bmatrix}} \begin{bmatrix} \text{ulazi} \end{bmatrix}$$

npr. 4 izlaza, 3 ulaza, 5 varijabli st.

fundamentalna

$A = 5 \times 5$

$B \quad 5 \times 3$

$C \quad 4 \times 5$

$D \quad 4 \times 3$

Odziv stanja sustava

$$x(n) = A^n x(0) + \sum_{m=0}^{n-1} A^{n-1-m} B u(m), \forall n \geq 0$$

Odziv sustava

$$y(n) = \begin{cases} 0, & n < 0 \\ C x(0) + D u(0), & n = 0 \\ C A^n x(0) + \sum_{m=0}^{n-1} C A^{n-1-m} B u(m) + D u(n), & n > 0 \end{cases}$$

MIRNI SUSTAV

$$\boxed{x(0) = 0}$$

NEPOBUĐENI SUSTAV

$u(n) = 0 \quad , \forall n$

IMPULSNI ODZIV

$u(n) = \delta(n)$

Npr. Impulsni odziv mirnog sustava

$u(n) = \delta(n)$

$x(0) = 0$

$$y(n) = \begin{cases} 0, & n < 0 \\ D, & n = 0 \\ CA^{n-1}B, & n > 0 \end{cases}$$

$$\sum_{m=0}^{n-1} CA^{n-1-m} B \delta(m) = CA^{n-1}B$$

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⇒ Kvadriranje matrice $A^2 = ?$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} {}^{1.}4 & \boxed{2.} & \boxed{3.} \\ \boxed{} & {}^{5.}0 & \boxed{} \\ \boxed{} & {}^{8.}2 & \boxed{} \end{bmatrix}$$

$3 \times \boxed{3 \qquad 3} \times 3$

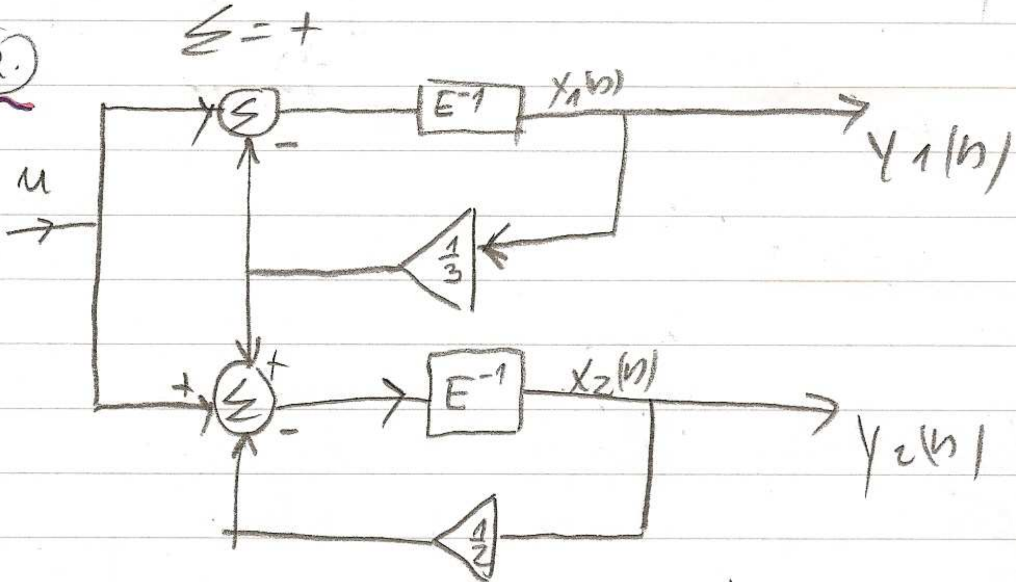
↓

moraju bit isti !

(1.) $1 \cdot 1 + 2 \cdot 0 + 3 \cdot 1 = 4$, (5.) $0 \cdot 2 + 0 \cdot 0 + 1 \cdot 0 = 0$

(8.) $1 \cdot 2 + 0 \cdot 0 \cdot 0 \cdot 0 = 2$

• (2.)



$E^{-1} x(n) = x(n-1)$

$E^{-5} x(n+1) = x(n+1-5)$

$$x_1(n) = E^{-1}\left(u(n) - \frac{1}{3} x_1(n)\right) \Big/ \cdot E^{+1}$$

$$E^{+1} x_1(n) = u(n) - \frac{1}{3} x_1(n)$$

$$\underline{x_1(n+1) = u(n) - \frac{1}{3} x_1(n)}$$

→

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$$x_2(n) = E^{-1}\left(u(n) + \frac{1}{3}x_1(n) - \frac{1}{2}x_2(n)\right)$$

$$x_2(n+1) = u(n) + \frac{1}{3}x_1(n) - \frac{1}{2}x_2(n)$$

Popisujemo matrice

$$x(n+1) = \begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{1}{3} & 0 \\ \frac{1}{3} & -\frac{1}{2} \end{bmatrix}}_{A} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} \begin{bmatrix} u(n) \end{bmatrix}$$

$$y_1(n) = x_1(n)$$

$$y_2(n) = x_2(n)$$

$$y(n) = \begin{bmatrix} y_1(n) \\ y_2(n) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{D=0} \begin{bmatrix} u(n) \end{bmatrix}$$

4 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$x(2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$u(0) = ?$, $u(1) = ?$

$$x(2) = A\,x(1) + B\,u(1)$$
$$x(1) = A\,x(0) + B\,u(0)$$

$$x(2) = A\left(A\,x(0) + B\,u(0)\right) + B\,u(1)$$
$$x(2) = A \cdot A\,x(0) + A \cdot B\,u(0) + B\,u(1)$$

$\longrightarrow$

$$A \cdot A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A \cdot B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1)$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} u(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u(1) \end{bmatrix} = \begin{bmatrix} u(0) \\ u(1) \end{bmatrix} \Rightarrow \underline{\underline{\begin{matrix} u(0) = 1 \\ u(1) = 2 \end{matrix}}}$$

* ZAD s MI-d (1.-vog)

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 1 \end{bmatrix}$$

$$x[0] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad , \quad x[2] = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x(2) = A\,x(1) + B\,u(1)$$
$$x(1) = A(x(0)) + B\,u(0)$$

$$x(2) = A(A\,x(0) + B\,u(0)) + B\,u(1)$$

$$x(2) = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{2\times 1} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(0) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(1) =$$

$$= \begin{bmatrix} 2x_1 + x_2 \\ x_1 + x_2 \end{bmatrix} + \begin{bmatrix} u(0) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ u(1) \end{bmatrix}$$

Rj:

$$\begin{bmatrix} 2x_1 + x_2 + u(0) \\ x_1 + x_2 + u(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boxed{\begin{matrix} -2x_1 - x_2 = u(0) \\ -x_1 - x_2 = u(1) \end{matrix}}$$

- A, B, C, D

$$y(n) = \begin{cases} Cx(0) + Du(0), & n=0 \\ CA^n x(0) + \sum_{m=0}^{n-1} CA^{n-1-m} Bu(m) + Du(n), & n \geq 0 \end{cases}$$

- 18. (1. MI)

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D = [0]$$

$$u(n) = 0$$
$$x(0) = [1\ 1]^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(n) = CA^n x(0)$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad A^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \dots \quad A^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$$

$$y_n = \underset{1\times 2}{\begin{bmatrix} 1 & 0 \end{bmatrix}} \cdot \underset{2\times 2}{\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \underset{1\times 2}{\begin{bmatrix} 1 & n \end{bmatrix}} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1+n$$

DIFERENCIJSKE JEDNADŽBE

"Zaokruživanje"

- $y(n+2) + y(n) = 0$

$\boxed{\text{najveća} - \text{najmanja} = 2.\ \text{reda}}$

Imat ćemo 2 karakteristična korijena.
Trebat će nam 2 početna uvjeta

$y(100) = ?$, P.U. $y(0)=1$, $y(-1)=0$

$y(n) = g^n$

Kad je nepobuđeni sustav onda je rješenje homogene i ukupno rješenje

$g^{n+2} - g^n = 0$

$\underbrace{g^n}_{\text{trivijalno rj.}} \underbrace{(g^2-1)}_{\text{karakteristični polinom}} = 0$

$g^2 - 1 = 0$

$(g-1)(g+1) = 0 \qquad g_1 = 1,\ g_2 = -1$ - korijeni

$y(n) = C_1 g_1^n + C_2 g_2^n = C_1 1^n + C_2 (-1)^n \rightarrow$ homogeno rj.

Poč. uvj. $y(0)=1 \Rightarrow y(0) = C_1 + C_2 = 1 \Rightarrow C_1 = 1 - C_2$

$y(-1)=0 \Rightarrow y(-1) = C_1 - C_2 = 0 \Rightarrow C_1 = C_2$

$C_1 = \frac{1}{2},\ C = \frac{1}{2}$

$y(n) = \frac{1}{2} 1^n + \frac{1}{2}(-1)^n$

$y(100) = \frac{1}{2} 1^{100} + \frac{1}{2}(-1)^{100} = 1$ ✓

2 $y(n+2) - y(n) = u(n)$

$$y(n) = \underbrace{y_{homogeno}(n)}_{\text{prirodni odziv}} + \underbrace{y_{partikularno}(n)}_{\text{prisilni odziv}}$$

→ 2. reda

$u(n) = 2 + 3n$ → pobuda

1) homogeno

$$y(n) = C_1 + C_2 (-1)^n$$

→ uvjeti se smiju uvrštavat tek u ukupno rješenje

2) partikularno

$$y_p(n) = K_0 + K_1 n$$

$$y_p(n+2) = K_0 + K_1 (n+2)$$

$$1(K_0 + K_1(n+2)) - 1(K_0 + K_1(n)) = 2 + 3n$$

$$\cancel{K_0} + \cancel{K_1(n)} + 2K_1 - \cancel{K_0} - \cancel{K_1(n)} = 2 + 3n$$

$\{K_1 n\}$ $2K_1 = 2 + 3n$ $\quad 2K_1 = 2$

$K_1 = 1$, K_0

$$y_p = K_0 + K_1 n$$

$$y_p = K_0 + n$$

3) totalni odziv

$$y = y_h + y_p \qquad y = C_1 + C_2 (-1)^n + K_0 + n$$

Datum

$$y(n) = \underbrace{y_0(n)}_{\text{nepob.}} + \underbrace{y_m(n)}_{\text{mirni}} = y_h(n) + y_p(n)$$

Nepobuđeni sustav

$y(n) = u(n) = 0$, početni uvjet $\neq 0$

nema pobudu

Mirni sustav

$y_m(n) = \mu(n)$, početni uvjeti $= 0$

ima pobudu

- 3.2.6 - Vrankić

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{3}y(n-2) = u(n) + \frac{1}{2}u(n-1)$$

$y(-1) = 14$ $\qquad u(n) = 3\left(\frac{1}{8}\right)^n \mu(n)$

$y(-2) = 52$

1) računamo nepobuđeni sustav (kao da nema desne strane)

$y_0 = 0$

$q_1 = \frac{1}{2}$, $\quad q_2 = \frac{1}{4}$

$y_0(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n$ $\quad \Big| \quad \mu(n) = 0$, $y(-1) = 14$, $y(-2) = 52$

$y(-1) = C_1\left(\frac{1}{2}\right)^{-1} + C_2\left(\frac{1}{4}\right)^{-1} = 14$

$2C_1 + 4C_2 = 14$

$y(-2) = C_1\left(\frac{1}{2}\right)^{-2} + C_2\left(\frac{1}{4}\right)^{-2} = 52$

$4C_1 + 16C_2 = 52 \quad \Rightarrow C_1 = 1, \; C_2 = 3$

$y_0(n) = \left(\frac{1}{2}\right)^n + 3\left(\frac{1}{4}\right)^n$ $\longrightarrow$

Datum

2) računanje odziva mirnog sustava (imamo pobudu, početni uvj = 0)

$$y_m = y_h(n) + y_p(n) \quad \Big| \begin{matrix} u(n) \\ y(-1)=0 \\ y(-2)=0 \end{matrix}$$

$$y_m(n) = C_1'\left(\frac{1}{2}\right)^n + C_2'\left(\frac{1}{4}\right)^n + y_p(n) \quad \Big| \begin{matrix} u(n) \\ y(-1)=y(-2)=0 \end{matrix}$$

$$y_m(n) = C_1'\left(\frac{1}{2}\right)^n + C_2'\left(\frac{1}{4}\right)^n + \underbrace{5\left(\frac{1}{8}\right)^n}$$

↳ prije smo izračunali (str. 9.)

$$y(0) - \frac{3}{4}\cancel{y(-1)} + \frac{1}{8}\cancel{y(-2)} = u(0) + \frac{1}{2}\cancel{u(-1)} \quad , \quad y(-1)=y(-2)=0$$

$$y(0) = 3$$

$$y(1) - \frac{3}{4}\underbrace{y(0)}_{3} + \frac{1}{8}\cancel{y(-1)} = \underbrace{u(1)}_{\frac{3}{8}} + \frac{1}{2}\underbrace{u(0)}_{3}$$

$$y(1) = \frac{9}{4} + \frac{3}{8} + \frac{3}{2} = \frac{33}{8}$$

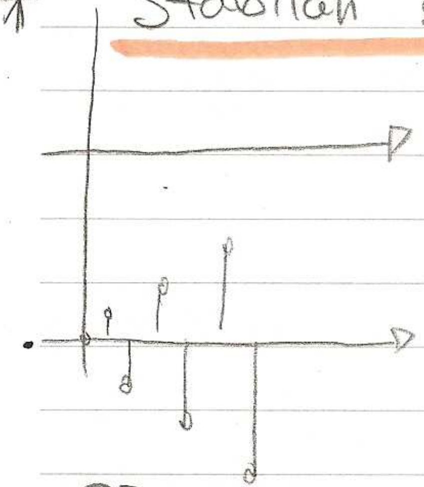
$$y_m(n) = 16\cdot\left(\frac{1}{2}\right)^n - 12\left(\frac{1}{4}\right)^n + 5\left(\frac{1}{8}\right)^n$$

$$y(n) = y_0(n) + y_m(n) = \underbrace{1\cdot\left(\frac{1}{2}\right)^n + 3\left(\frac{1}{4}\right)^n}_{y_0} + \underbrace{16\left(\frac{1}{2}\right)^n - 12\left(\frac{1}{4}\right)^n + 5\left(\frac{1}{8}\right)^n}_{y_m}$$

$$\boxed{y(n) = \underbrace{17\cdot\left(\frac{1}{2}\right)^n - 15\left(\frac{1}{4}\right)^n}_{\text{prirodni odziv}} + \underbrace{5\left(\frac{1}{8}\right)^n}_{\text{prisilni odziv}}}$$

Datum

* Stabilan sustav



$$y(n) = C_1 a_1^n + C_2 a_2^n + \cancel{y_p(n)}$$

$g_1 = \frac{1}{2}$

$g_2 = \frac{1}{4}$

$$y(n) = C_1\left(\frac{1}{2}\right)^n + C_2\left(\frac{1}{4}\right)^n$$

$n \to \infty$

$$y(n) = C_1 \cdot 0 + C_2 \cdot 0 = 0$$

STABILNI SUSTAV

Nakon nekog vremena će se smiriti.

STABILAN:

$$\forall g_i \; |g_i| < 1$$

* Nestabilan sustav:

$$\exists g_i \; |g_i| > 1$$

$$y(n) = C_1\left(\frac{1}{2}\right)^n + C_2(2)^n$$

$n \to \infty$

$$y(n) = C_1 \cdot 0 + C_2 \cdot \infty$$

Dovoljno je da samo jedna kar. frek. čija je apsolutna vrijednost veća od 1, onda je sustav Nestabilan

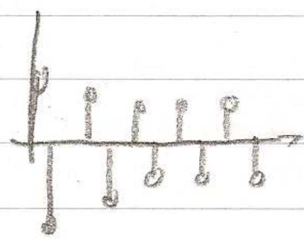
* GRANIČNO STABILAN

$$y(n) = C_1\left(\frac{1}{2}\right)^n + C_2(1)^n$$

$$y_2(n) = C_1\left(\frac{1}{2}\right)^n + C_2(-1)^n$$

$n \to \infty$

$$y(n) = 0 + C_2$$



$$\exists g_i \; |g_i| = 1$$

Datum

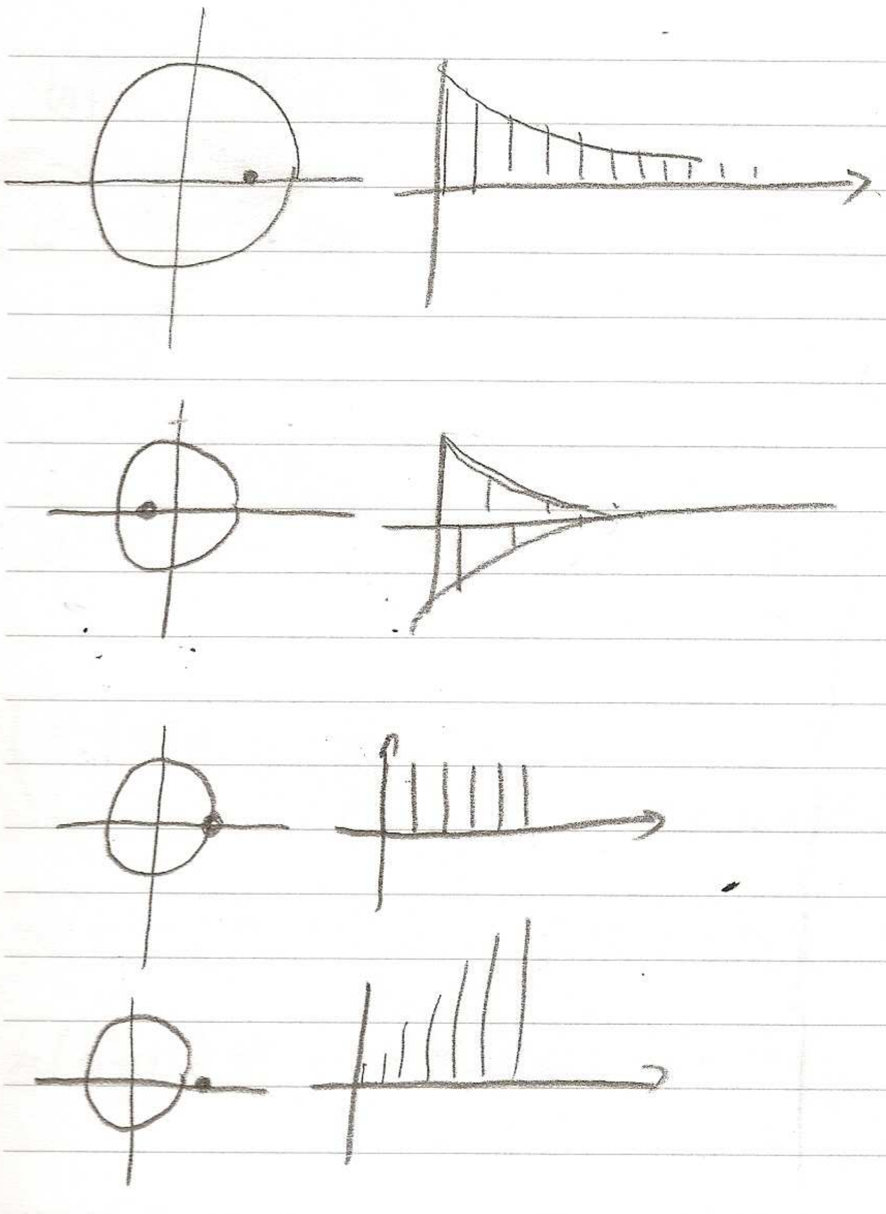
2:M1 - probni 15 Ispitaj stabilnost

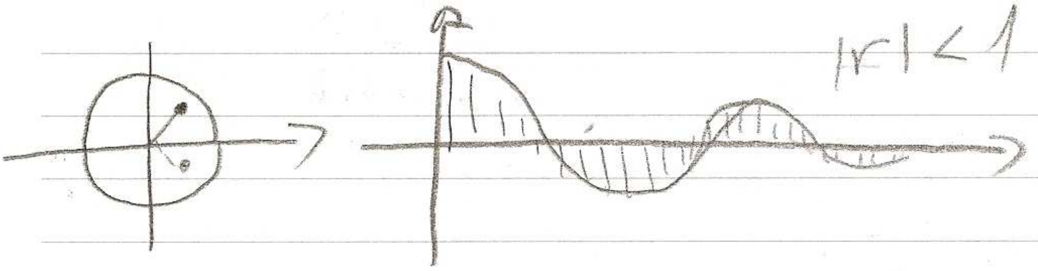
$$y(n+2) - \frac{8}{5}y(n+1) + \frac{32}{25}y(n) = u(n)$$

$$q^2 - \frac{8}{5}q - \frac{32}{25} = 0$$

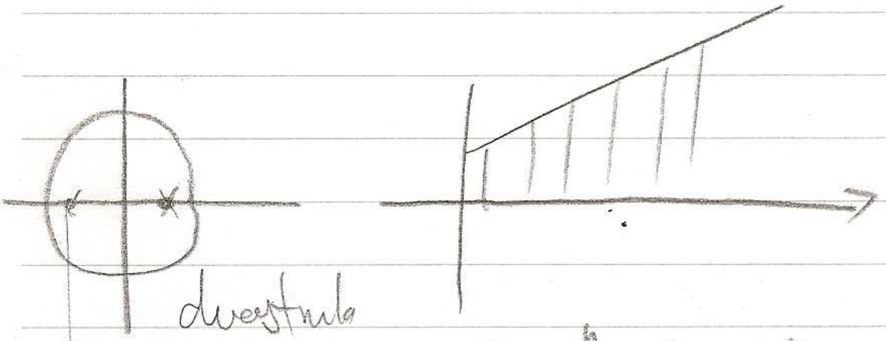
$$q_{1,2} = \underbrace{\frac{4}{5}}_{a} \pm j\underbrace{\frac{4}{5}}_{b}$$

$$|q_i| = \sqrt{a^2+b^2} = \sqrt{\frac{16}{25}+\frac{16}{25}} > 1$$ Nestabilan





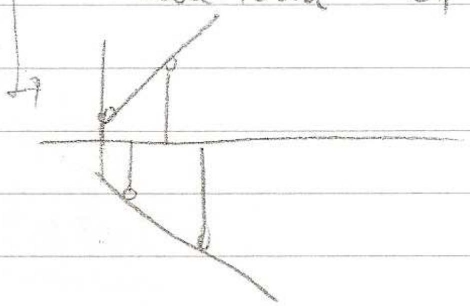
$|r| < 1$



linearno raste

dvostruka nul-točka

$C_1 g_1^n + C_2 n g_2^n$



Datum

Konvolucija

$$h(n) = \begin{cases} D, & n=0 \\ CA^{n+1}B, & n>0 \end{cases}$$

$$u(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$\underline{y(n) \doteq u(n) * h(n)}$$

$$u(n) = \delta(n)$$
$$y(n) = h(n)$$

$$h(n) = \delta(n) * h(n)$$

Definicija

$$h(n) * u(n) = \sum_{n=-\infty}^{\infty} h(\overbrace{n-m}^{k=n-m,\ m=n-k})\, u(m)$$, za diskretne sustave

$$u(n) * h(n) = \sum_{n=-\infty}^{\infty} h(k)\, u(n-k)$$ KOMUTATIVNOST

$$\left(h_1(n) * h_2(n)\right) * h_3(n) = h_1(n) * \left(h_2(n) * h_3(n)\right)$$
ASOCIJATIVNOST

$$u(n) * \left(h_1(n) + h_2(n)\right) = u(n) * h_1(n) + u(n) * h_2(n)$$ DISTRIB.

$$u(n-\underline{p}) * h(n-\underline{q}) = y(n-p-q)$$

$$K\begin{cases} u(n) = u(n)\,\mu(n) \\ h(m) = h(m)\,\mu(n) \end{cases}$$

$$\sum_{m=-\infty}^{\infty} \underbrace{h(n-m)\,\mu(n-m)\,u(m)\,\mu(m)}_{f(m)} = \quad , \; f(m)=0$$

$m<0$

$$= \sum_{m=0}^{m} h(n-m)\,u(m)$$

$\mu(m)=1$

$m \geq 0$

$f(m) \neq 0$

$\mu(n-m)=1$

$n-m \geq 0$

$m \leq n$

$$h(t) * u(t) = \int_{-\infty}^{\infty} h(t-\tau)\,\mu(\tau)\,d\tau$$

10. tj. 11

$$u(t) * \delta(t) = u(t) \qquad \{ f(t)\cdot\delta(t-t_0) = f(t_0)\,\delta(t-t_0) \}$$

$$\int_{-\infty}^{\infty} u(t-\tau)\,\mu(\tau)\,d\tau = \int_{-\infty}^{\infty} u(t-0)\,\delta(\tau)\,d\tau =$$

$$= u(t) \int_{-\infty}^{\infty} \delta(\tau)\,d\tau = u(t)\cdot 1 = \underline{\underline{u(t)}}$$

- $\left(\sin n * \delta(n+m)\right)\delta(n-m) =$

$$\left\{ u(t) * \delta(t-t_0) = u(t-t_0) \right\}$$

$$= \sin(n+m)\,\delta(n-\underset{n_0=m}{m}) = \sin(2m)\,\delta(n-m)$$

SVOJSTVA

$u(n) * \delta(n-n_0) = u(n-n_0)$

$u(n) \cdot \delta(n-n_0) = u(n_0)\,\delta(n-n_0)$

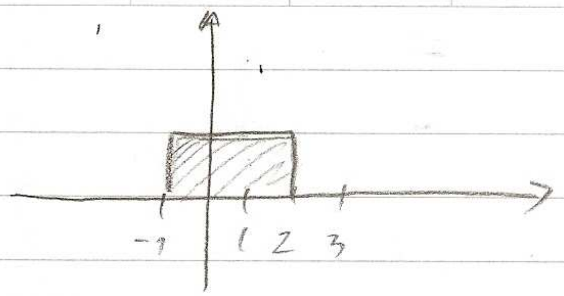
$$\sum_{n=-\infty}^{\infty} u(n)\cdot\delta(n-n_0) = u(n_0)$$

$$\int_{-\infty}^{\infty} u(t)\,\delta(t-t_0)\,dt = u(t_0)\cdot 1$$

- $u(n) * \delta(cn-d) = u(n) * \delta\left(n-\frac{d}{c}\right) = u\left(n-\frac{d}{c}\right)$
- $x(n) = (3n+2) * \delta(3n-6) = (3n+2) * \delta(n-2)$

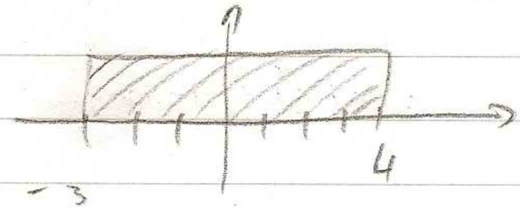
$x(n) = 3n$
$y(n) = 2$

$$(x(n)+y(n)) * \delta(n-2) = x(n) * \delta(n-2) + y(n) * \delta(n-2)$$
$$= x(n-2) + y(n-2) = 3(n-2) + 2 = 3n-6+2$$
$$= 3n-4$$



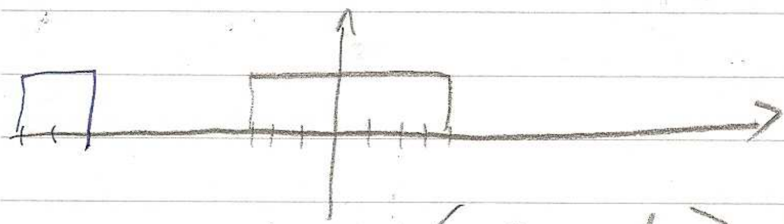
$$h(t) = \mu(t+1) - \mu(t-2)$$

$$\mu(t) = \mu(t+3) - \mu(t-4)$$



$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)\,\mu(\tau)\,d\tau$$

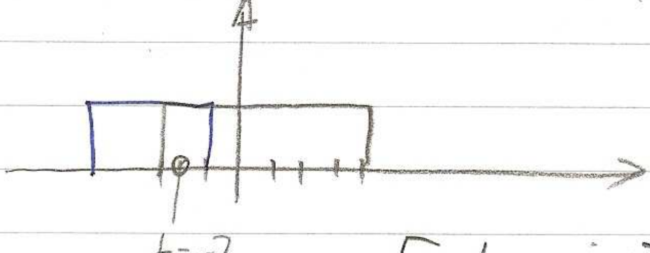
1) $h(t-\tau)$ $\mu(\tau)$



$\langle -\infty, -4 \rangle$ Ne preklapaju se

$y(t) = 0$

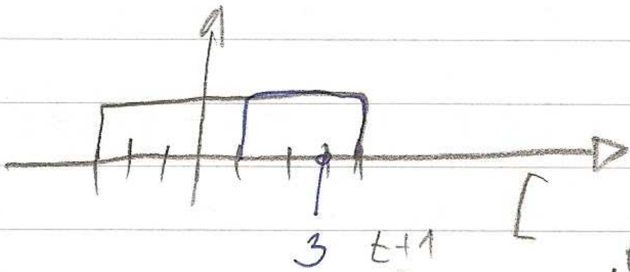
2)



$[-4, -1\rangle$ Djelomično preklapanje

$$y(t) = \int_{-3}^{t+1} d\tau = \tau \Big|_{-3}^{t+1} = t+4$$

3)



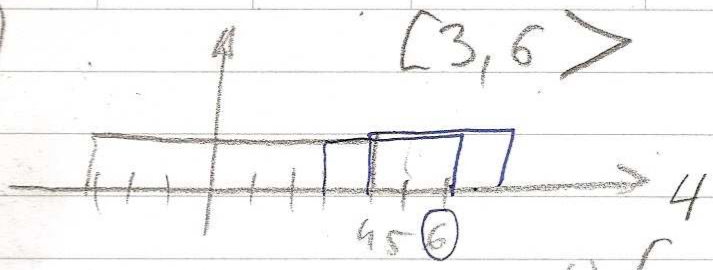
$[-1, 3\rangle$

$$y(t) = \int_{t-2}^{t+1} d\tau = \tau \Big|_{t-2}^{t+1} = t+1-t-2 = 3$$

Datum

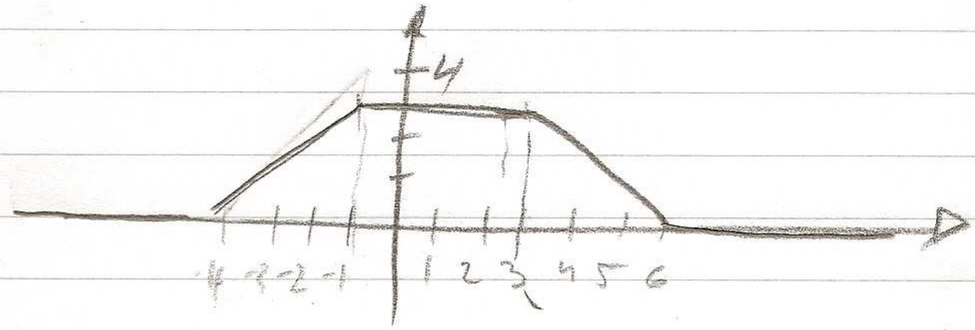
4)

$[3, 6\rangle$



$$y(t) = \int_{t-2}^{4} d\tau = \tau \Big|_{t-2}^{4}$$

$$= -t + 6$$



GOOD

Luck!