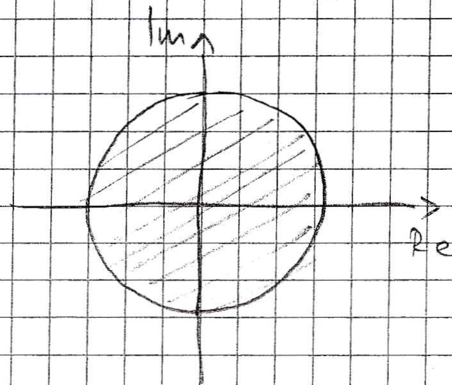
ZADATAK 1:

Unutrašnja stabilnost vremenski diskretnog sustava određuje se tako da se gleda položaj polova u z-ravnini. Nalaze li se svi polovi unutar jedinične kružnice, sustav je stabilan.



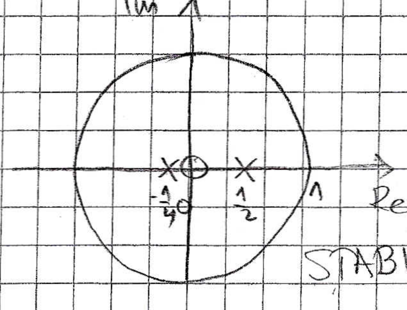
a) $8y(n) - 2y(n-1) - y(n-2) = u(n)$

$8Y(z) - 2z^{-1}Y(z) - z^{-2}Y(z) = U(z)$

$H(z) = \dfrac{Y(z)}{U(z)} = \dfrac{1}{8 - 2z^{-1} - z^{-2}} = \dfrac{z^2}{8z^2 - 2z - 1}$

b) **nule**

$n_{1/2} = 0$



STABILAN!

polovi

$8z^2 - 2z - 1 = 0$

$p_{1/2} = \dfrac{2 \pm \sqrt{4+32}}{16} = \dfrac{2 \pm 6}{16} = \dfrac{1 \pm 3}{8}$

$p_1 = \dfrac{1}{2}$

$p_2 = -\dfrac{1}{4}$

c) $H(z) \circ\!\!-\!\!\circ\; h(n)$

$$H(z) = \frac{z^2}{8z^2 - 2z - 1}$$

$$\frac{H(z)}{z} = \frac{z}{8 \cdot \left(z - \frac{1}{2}\right)\left(z + \frac{1}{4}\right)} = \frac{1}{8} \cdot \left(\frac{A}{z - \frac{1}{2}} + \frac{B}{z + \frac{1}{4}} \right) =$$

$$= \frac{1}{8} \cdot \left(\frac{Az + \frac{A}{4} + Bz - \frac{B}{2}}{\left(z - \frac{1}{2}\right) \cdot \left(z + \frac{1}{4}\right)} \right) = \frac{1}{24} \cdot \left(\frac{2}{z - \frac{1}{2}} + \frac{1}{z + \frac{1}{4}} \right)$$

$A + B = 1$

$\frac{A}{4} - \frac{B}{2} = 0 \quad / \; 4 \qquad A = 2B$

$\boxed{B = \frac{1}{3}} \qquad \boxed{A = \frac{2}{3}}$

$$H(z) = \frac{1}{24} \cdot \left(\frac{2z}{z - \frac{1}{2}} + \frac{z}{z + \frac{1}{4}} \right) = \frac{z}{12\left(z - \frac{1}{2}\right)} + \frac{z}{24 \cdot \left(z + \frac{1}{4}\right)}$$

$$h(n) = \left[\frac{1}{12} \cdot \left(\frac{1}{2}\right)^n + \frac{1}{24} \cdot \left(-\frac{1}{4}\right)^n \right] u(n)$$

ZADATAK 1: (II DIO)

d) $Y(z) = H(z) \cdot U(z)$

$$H(z) = \frac{1}{12} \cdot \frac{z}{z-\frac{1}{2}} + \frac{1}{24} \frac{z}{z+\frac{1}{4}}$$

$$U(z) = \frac{z}{z-1}$$

$$Y(z) = \left(\frac{1}{12} \cdot \frac{z}{z-\frac{1}{2}} + \frac{1}{24} \cdot \frac{z}{z+\frac{1}{4}} \right) \cdot \frac{z}{z-1}$$

$$\frac{Y(z)}{z} = \frac{1}{12} \cdot \frac{z}{\left(z-\frac{1}{2}\right)(z-1)} + \frac{1}{24} \cdot \frac{z}{\left(z+\frac{1}{4}\right)\cdot(z-1)}$$

$$\frac{z}{\left(z-\frac{1}{2}\right)\cdot(z-1)} = \frac{A}{z-\frac{1}{2}} + \frac{B}{z-1} = \frac{A(z-1)+B\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\cdot(z-1)} = \frac{z(A+B) - A - \frac{B}{2}}{\left(z-\frac{1}{2}\right)(z-1)}$$

$A+B=1$

$-A-\frac{B}{2}=0 \quad / \cdot 2 \qquad B=-2A$

$\boxed{B=2} \qquad \boxed{A=-1}$

$$\frac{z}{\left(z+\frac{1}{4}\right)(z-1)} = \frac{C}{z+\frac{1}{4}} + \frac{D}{z-1} = \frac{C(z-1)+D\left(z+\frac{1}{4}\right)}{\left(z+\frac{1}{4}\right)(z-1)} = \frac{z(C+D)-A+\frac{B}{4}}{\left(z+\frac{1}{4}\right)(z-1)}$$

$A+B=1$

$-A+\frac{B}{4}=0 \quad / \cdot 4 \qquad B=4A$

$\boxed{B=\frac{4}{5}} \qquad \boxed{A=\frac{1}{5}}$

$$\frac{Y(z)}{z} = \frac{1}{12}\cdot\left(\frac{-1}{z-\frac{1}{2}} + \frac{2}{z-1}\right) + \frac{1}{24}\left(\frac{1}{5}\cdot\frac{1}{z+\frac{1}{4}} + \frac{4}{5}\cdot\frac{1}{z-1}\right)$$

$$Y(z) = -\frac{1}{12}\cdot\frac{z}{z-\frac{1}{2}} + \frac{1}{6}\cdot\frac{z}{z-1} + \frac{1}{120}\cdot\frac{z}{z+\frac{1}{4}} + \frac{1}{30}\cdot\frac{z}{z-1}$$

$$y(n) = \left(-\frac{1}{12}\cdot\left(\frac{1}{2}\right)^n + \frac{1}{6} + \frac{1}{120}\cdot\left(-\frac{1}{4}\right)^n + \frac{1}{30}\right)\mu(n)$$

$$\underline{\underline{y(n) = \left(-\frac{1}{12}\cdot\left(\frac{1}{2}\right)^n + \frac{1}{120}\cdot\left(-\frac{1}{4}\right)^n + \frac{1}{5}\right)\mu(n)}}$$

e)

```
B = [1 0 0];
A = [8 -2 -1];
S = tf(B, A, 1);
S
```

Sampling time: 1

f)

```
pzmap(S);
```

g)

```
impulse(S);
```

h)

```
step(S);
```

PRETPOSTAVLJENO PARTIKULARNO RJEŠENJE

$u(n) = z^n$

$Y_p = Kz^n$

Kada se vrijednost konstante z poklopi s polom sustava partikularno rješenje ima oblik $Y_p = Kn^M z^n$ gdje je M broj polova. Ta pojava zove se REZONANCIJA

ZADATAK 1: (III DIO)

i) $u(n) = 2^{-n}\mu(n) \Rightarrow U(z) = \dfrac{z}{z-\frac{1}{2}}$

$$H(z) = \frac{z^2}{8\cdot\left(z-\frac{1}{2}\right)\left(z+\frac{1}{4}\right)}$$

$$Y(z) = H(z)\cdot U(z) = \frac{z^3}{8\left(z-\frac{1}{2}\right)^2\left(z+\frac{1}{4}\right)}$$

$$\frac{Y(z)}{z} = \frac{1}{8}\cdot\frac{z^2}{8\cdot\left(z-\frac{1}{2}\right)^2\left(z+\frac{1}{4}\right)} = \frac{1}{8}\left(\frac{A}{z-\frac{1}{2}} + \frac{B}{\left(z-\frac{1}{2}\right)^2} + \frac{C}{z+\frac{1}{4}}\right)$$

$$z^2 = A\left(z-\frac{1}{2}\right)\cdot\left(z+\frac{1}{4}\right) + B\cdot\left(z+\frac{1}{4}\right) + C\cdot\left(z-\frac{1}{2}\right)^2$$

$$z^2 = A\cdot\left(z^2-\frac{z}{4}-\frac{1}{8}\right) + B\cdot\left(z+\frac{1}{4}\right) + C\cdot\left(z^2-z+\frac{1}{4}\right)$$

$$z^2 = z^2\left(A+C\right) + z\left(-\frac{A}{4}+B-C\right) - \frac{A}{8} + \frac{B}{4} + \frac{C}{4}$$

$A + C = 1$

$-\frac{A}{4} + B - C = 0 \quad /\cdot 4 \Rightarrow \quad -A + 4B - 4C = 0$

$-\frac{A}{8} + \frac{B}{4} + \frac{C}{4} = 0 \quad /\cdot 8 \Rightarrow \quad -A + 2B + 2C = 0 \quad /\cdot(-2)$ $+$

$A - 8C = 0$
$A + C = 1$ $-$

$\boxed{C = \frac{1}{9}}$

$\boxed{A = \frac{8}{9}}$ $\boxed{B = \frac{1}{3}}$

$$Y(z) = \frac{1}{8}\cdot\left(\frac{8}{9}\cdot\frac{z}{z-\frac{1}{2}} + \frac{1}{3}\cdot\frac{z}{\left(z-\frac{1}{2}\right)^2} + \frac{1}{9}\cdot\frac{z}{z+\frac{1}{4}}\right)$$

$$Y(z) = \frac{1}{9}\cdot\frac{z}{z-\frac{1}{2}} + \frac{1}{24}\cdot\frac{1}{z}\cdot\frac{z^2}{\left(z-\frac{1}{2}\right)^2} + \frac{1}{72}\,\frac{z}{z+\frac{1}{4}}$$

$$y(n) = \left[\frac{1}{9} \cdot \left(\frac{1}{2}\right)^n + \frac{1}{24} \cdot (n+1-1) \left(\frac{1}{2}\right)^{n-1} + \frac{1}{72} \left(-\frac{1}{4}\right)^n \right] \mu(n)$$

$$\underline{\underline{y(n) = \left[\frac{1}{9} \cdot \left(\frac{1}{2}\right)^n + \frac{n}{24} \left(\frac{1}{2}\right)^{n-1} + \frac{1}{72} \left(-\frac{1}{4}\right)^n \right] \mu(n)}}$$

j)

```
n = 0 : 1 : 150;
u = 2 ^ (-n);
y = fielter (B, A, u);
stem (n, y);
```

ZADATAK 2:

a) ŠTEDNJA

$$y(n) = 1{,}05\, u(n) + 1{,}05\, y(n-1)$$

POTROŠNJA

$$y(n) = -1{,}05\, u(n) + 1{,}05\, y(n-1)$$

$u(n)$ - ulaz

$y(n)$ - izlaz

b) $y(n) - 1{,}05\, y(n-1) = 1{,}05\, u(n)$

$Y(z) - 1{,}05 z^{-1} Y(z) = 1{,}05\, U(z)$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1{,}05}{1 - 1{,}05 z^{-1}} = \frac{1{,}05 z}{z - 1{,}05}$$

nule: $z = 0$

polovi: $z = 1{,}05 \Rightarrow |z| > 1$ SUSTAV JE NESTABILAN

Sustav je nestabilan za bilo koju kamatnu stopu (pozitivnu) osim kada nema kamate, tada je sustav na granici stabilnosti. Sustav bi bio stabilan kada bi kamatna stopa bila negativna.

c) $Y(z) = H(z)\,U(z)$

$$Y(z) = \frac{1{,}05\,z}{z-1{,}05} \cdot \frac{z}{z-1} \cdot 15000 = \frac{15\,750\,z^2}{(z-1{,}05)\cdot(z-1)}$$

$$\frac{Y(z)}{z} = \frac{15\,750\,z}{(z-1{,}05)(z-1)} = \frac{A}{z-1{,}05} + \frac{B}{z-1} = \frac{Az - A + Bz - B\cdot 1{,}05}{(z-1{,}05)\cdot(z-1)}$$

$A + B = 15750$

$-A - B\cdot 1{,}05 = 0$

$A = -B\,1{,}05 \quad \Rightarrow \quad \boxed{B = -315\,000}$

$\boxed{A = 330\,750}$

$$\frac{Y(z)}{z} = \frac{330\,750}{z-1{,}05} - \frac{315\,000}{z-1}$$

$$Y(z) = \frac{330\,750\,z}{z-1{,}05} - \frac{315\,000\,z}{z-1}$$

$$y(n) = \left(330\,750\cdot(1{,}05)^n - 315\,000\right)\mu(n)$$

$y(30) = 1.114.482{,}44 = 1\,114\,482$

POTROŠNJA

$y(0) = 1\,114\,482$

$y(n) = -1{,}05\,u(n) + 1{,}05\,y(n-1) \quad ; \quad n \geq 1$

HOMOGENO

$\gamma - 1{,}05 = 0$

$\gamma = 1{,}05$

$y_h = C\cdot(1{,}05)^n$

PARTIKULARNO

$y_p = K$

$K - 1{,}05K = -15\,750$

$K = 315\,000$

$y_p = 315\,000$

$$y_t = C\cdot(1{,}05)^n + 315\,000$$

ZADATAK 2: (II DIO)

$$Y(0) = 1\,114\,482$$

$$Y_t = C(1{,}05)^n + 315\,000$$

$$1\,114\,482 = C + 315\,000$$

$$C = 799\,482$$

$$\underline{Y_t(n) = 799\,482(1{,}05)^n + 315\,000}$$

$$\left.\begin{array}{l} Y_t(0) = 1\,114\,482 \\ Y_t(1) = 1\,154\,456 \end{array}\right\} Y_t(1) > Y_t(0) \Rightarrow$$ ukupna štednja raste iako podižemo novce stopa možemo podizati 15 000 kn svake godine unedogled

Razlog tome je činjenica da je godišnja kamata veća od 15 000 kn kojih podižemo.

$$(Y(0) - X) \cdot 0{,}05 = X$$

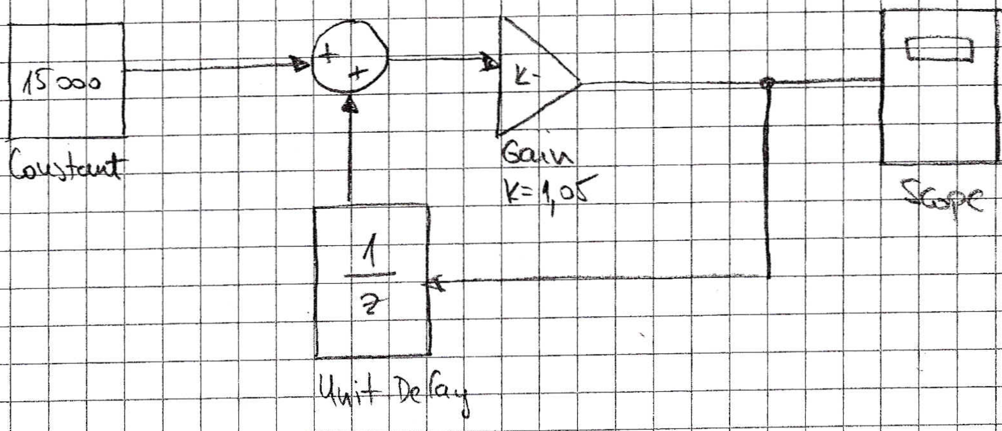
$$0{,}05\,Y(0) = 1{,}05X$$

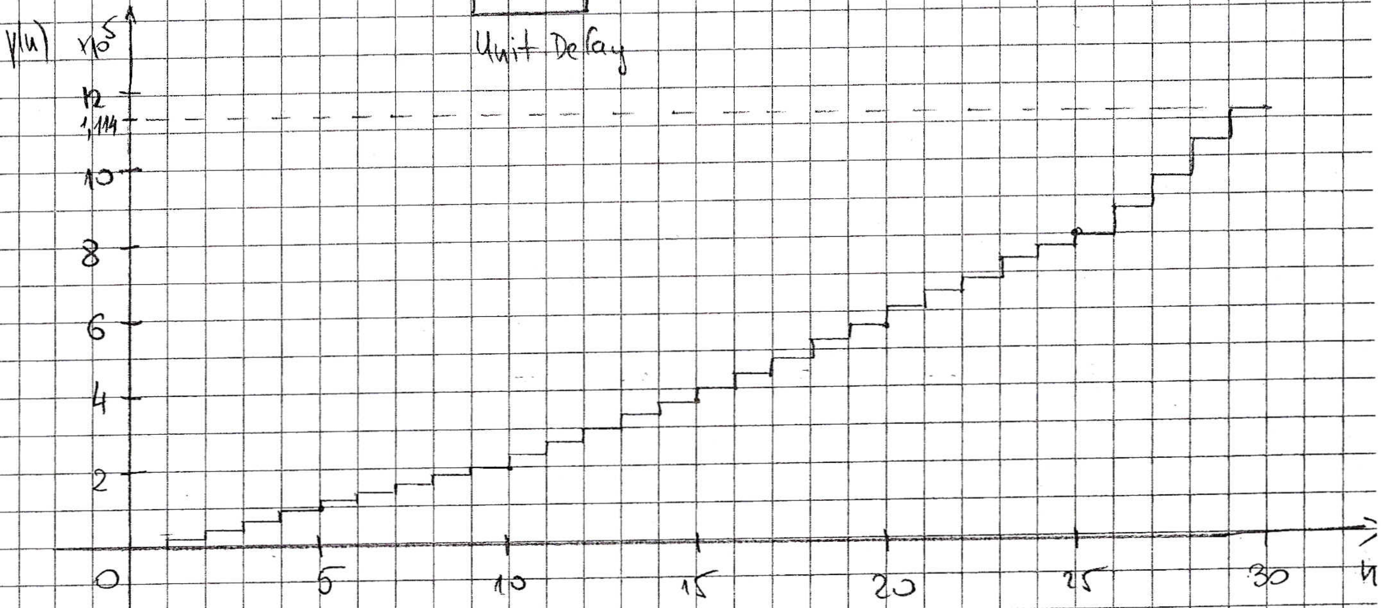
$$X = Y(0) \cdot \frac{0{,}05}{1{,}05} = 53\,070{,}57 \text{ kn}$$

X - najveći iznos koji možemo podizati unedogled nakon 30 g. štednje

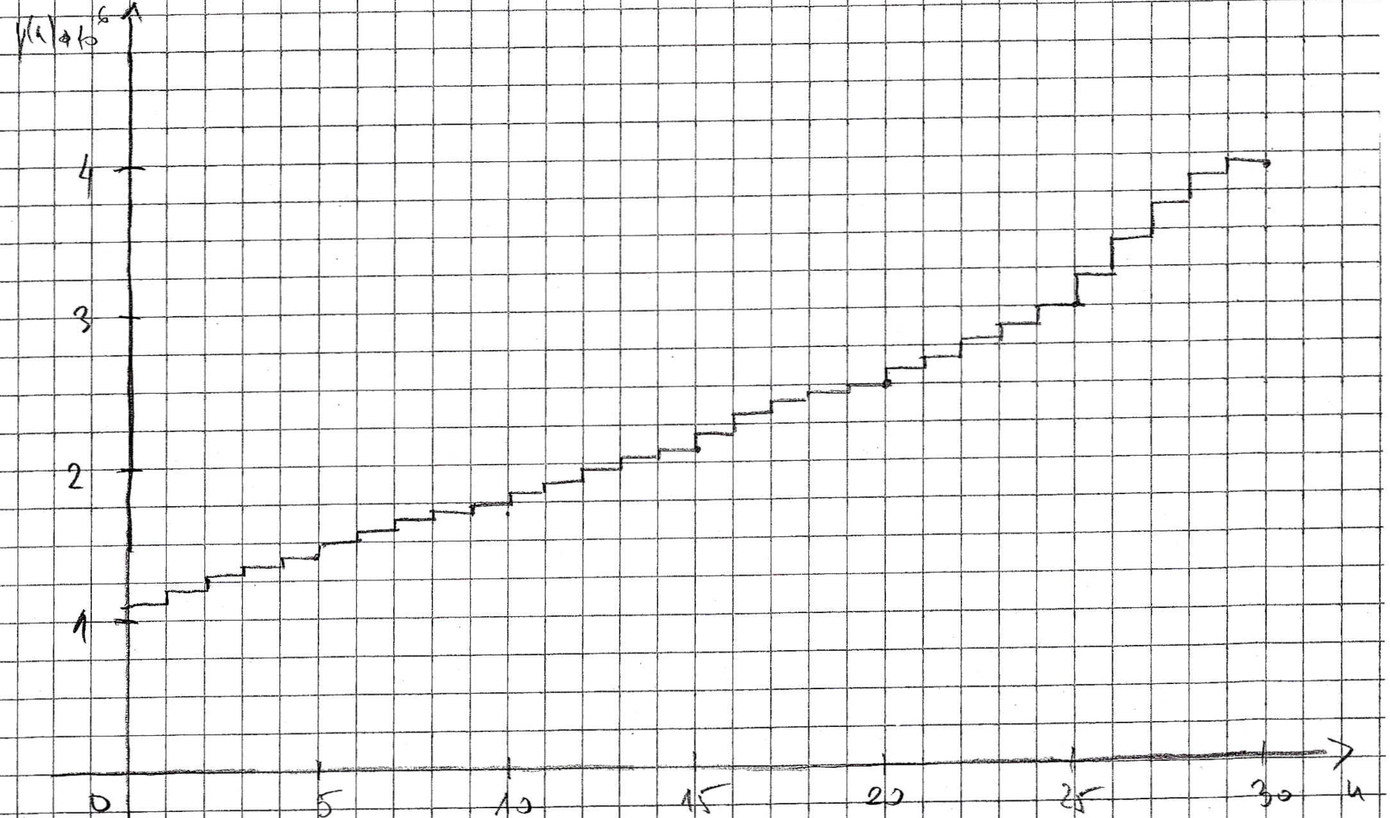
SVOJSTVO STABILNOSTI (GRANIČNE) je ključno za takvo ponašanje sustava

d)
e)
f)

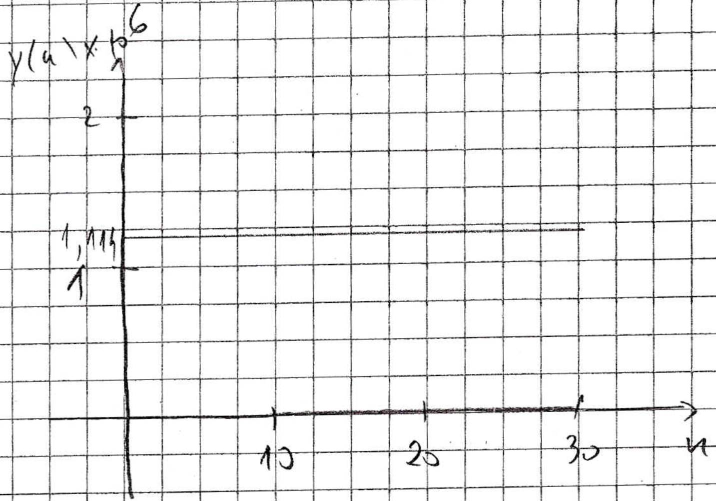




$y(30) = 1,114 \cdot 10^5$

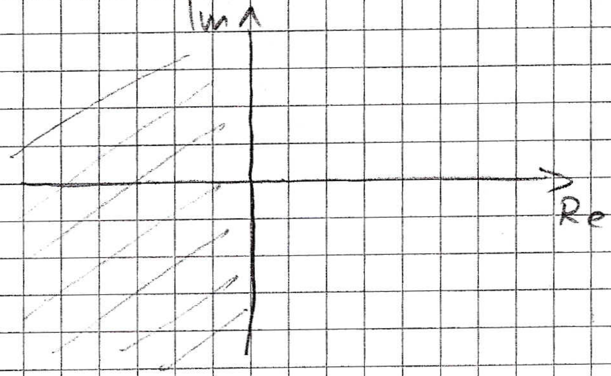


ZADATAK 2: (III DIO)



ZADATAK 3:

Unutrašnja stabilnost vremenski kontinuiranog sustava određuje se tako da se gleda položaj polova u s-ravnini. Nalaze li se svi polovi u lijevoj poluravnini, sustav je stabilan



a) $y''(t)+4y'(t)+5y(t)=u(t)$

$$s^2Y(s)+4sY(s)+5Y(s)=U(s)$$

$$H(s)=\frac{Y(s)}{U(s)}=\frac{1}{s^2+4s+5}$$

b) nule

$$n_{1,2}=\infty$$

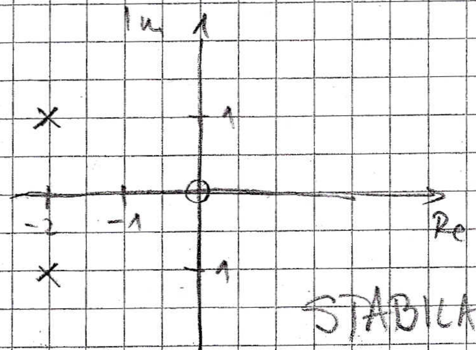
polovi

$$s^2+4s+5=0$$

$$s_{1,2}=\frac{-4\pm\sqrt{16-20}}{2}=\frac{-4\pm 2j}{2}=-2\pm j$$

$$s_1=-2+j$$

$$s_2=-2-j$$



STABILAN!

c) $H(s) = \dfrac{1}{s^2+4s+5} = \dfrac{1}{(s+2)^2+1}$

$$h(t) = e^{-2t} \sin t \, \mu(t)$$

d) $u(t) = \mu(t)$

$y'' + 4y' + 5y = 0$

$s_1 = -2 + j$

$s_2 = -2 - j$

$y_h = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t}$

$y_p = K$

$5K = 1$

$\boxed{K = \frac{1}{5}}$

$y_t = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t} + \frac{1}{5}$

$y(0^+) = 0$

$y'(0^+) = 0$

$C_1 + C_2 = -\frac{1}{5}$

$(-2+j) e^{(-2+j)t} \cdot C_1 + (-2-j) \cdot e^{(-2-j)t} \cdot C_2 = 0 \Big|_{t=0}$

$(-2+j) \cdot C_1 + (-2-j) C_2 = 0$

$(-2+j)\left(-\frac{1}{5} - C_2\right) - (2+j) C_2 = 0$

$\frac{2}{5} - \frac{j}{5} + \cancel{2C_2} - jC_2 - \cancel{2C_2} - jC_2 = 0 \Rightarrow -2jC_2 = -\frac{2}{5} + \frac{j}{5}$

$\boxed{C_1 = -\frac{1}{10} + \frac{j}{5}}$

$\boxed{C_2 = \frac{1}{5j} - \frac{1}{10} = -\frac{1}{10} - \frac{j}{5}}$

ZADATAK 3 (II DIO)

$$y_t(t) = \left[\left(-\frac{1}{10}+\frac{j}{5}\right)e^{(-2+j)t} + \left(-\frac{1}{10}-\frac{j}{5}\right)e^{(-2-j)t} + \frac{1}{5}\right]\mu(t)$$

$$y_t(t) = \left[-\frac{1}{5}e^{-2t}\cos t - \frac{2}{5}e^{-2t}\sin t + \frac{1}{5}\right]\mu(t)$$

e)

```
B=[0 0 1];
A=[1 4 5];
s=tf(B,A);
s
```

f) `pzmap(s)`

g) `impulse(s)`

h) `step(s)`

PRETPOSTAVLJENO PARTIKULARNO RJEŠENJE

$u(t) = e^{st}$

$y_p(t) = Ke^{st}$

Kada se vrijednost konstante s poklopi s polom sustava partikularno rješenje ima oblik $y_p(t) = Kt^M e^{st}$ gdje je M broj polova

Ta pojava se zove REZONANCIJA

i) $u(t) = e^{-2t}\cos t\,\mu(t) = e^{-2t}\left(\frac{e^{jt}+e^{-jt}}{2}\right)\mu(t) = \frac{1}{2}e^{(-2+j)t} + \frac{1}{2}e^{(-2-j)t}$

$y_h = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t}$

$y_{p_1}(t) = K_1 t e^{(-2+j)t}$

$y_{p_2}(t) = K_2 t e^{(-2-j)t}$

$y_{p_1}'(t) = K_1 e^{(-2+j)t} + (-2+j)K_1 t e^{(-2+j)t} = K_1 e^{(-2+j)t}(1-2t+jt)$

$y_{p_1}''(t) = (-2+j)K_1 e^{(-2+j)t} + (-2+j)K_1 e^{(-2+j)t} + (-2+j)^2 K_1 t e^{(-2+j)t}$

$= K_1 e^{(-2+j)t}(3t - 4tj - 4 + 2j)$

$K_1 \cancel{e^{(-2+j)t}}(3t-4tj-4+4j) + 4K_1 \cancel{e^{(-2+j)t}}(1-2t+jt) +$

$5K_1 t \cancel{e^{(-2+j)t}} = \frac{1}{2}\cancel{e^{(-2+j)t}}$

$3tK_1 - 4tK_1 j - 4K_1 + 2K_1 j + 4K_1 - 8K_1 t + 4K_1 tj + 5K_1 t = \frac{1}{2}$

$2K_1 j = \frac{1}{2} \Rightarrow K_1 = -\frac{1}{4}j$

$K_2 \cancel{e^{(-2-j)t}}(3t+4jt-4-2j) + 4K_2 \cancel{e^{(-2-j)t}}(1-2t-jt) +$

$5K_2 t \cancel{e^{(-2-j)t}} = \frac{1}{2}\cancel{e^{(-2-j)t}}$

$-2K_2 j = \frac{1}{2} \Rightarrow K_2 = \frac{1}{4}j$

$y_p(t) = y_{p_1}(t) + y_{p_2}(t) = -\frac{1}{4}j\,t e^{(-2+j)t} + \frac{1}{4}j\,t e^{(-2-j)t} = \frac{1}{2}te^{-2t}\sin t \quad t \geq 0$

$y(0^-) = y'(0^-) = 0 \Rightarrow$ MIRNI SUSTAV

$y(0^+) = y(0^-) = 0$

$y'(0^+) = y'(0^-) = 0$

$y_t(t) = y_h + y_p = C_1 e^{(-2+j)t} + C_2 e^{(-2-j)t} - \frac{1}{4}j\,t e^{(-2+j)t} + \frac{1}{4}j\,t e^{(-2-j)t}$

ZADATAK 3: (III DIO)

$$y'(t) = (-2+j)\,c_1 e^{(-2+j)t} - (2+j)\,c_2 e^{(-2-j)t} - \frac{1}{4}j e^{(-2+j)t} +$$

$$- \frac{1}{4}(-2+j)\,j\,t e^{(-2+j)t} + \frac{1}{4}j e^{(-2-j)t} - \frac{1}{4}(2+j)\,j\,t e^{(-2-j)t}$$

$$y(0^+) = 0 = c_1 + c_2$$

$$y'(0^+) = 0 = (-2+j)\,c_1 - (2+j)\,c_2 - \frac{1}{4}j + \frac{1}{4}j = -2c_1 + jc_1 - 2c_2 - jc_2$$

$$c_1 + c_2 = 0 \quad \Rightarrow \quad c_1 = -c_2$$

$$-2c_1 - 2c_2 + jc_1 - jc_2 = 0$$

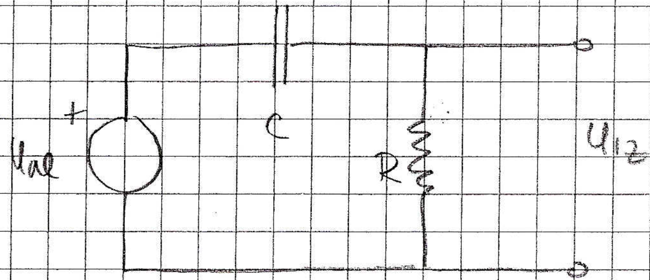
$$\cancel{2c_2} - \cancel{2c_2} - jc_2 - jc_2 = 0 \quad \Rightarrow \quad c_2 = 0 \qquad c_1 = 0$$

$$y_m(t) = y_p(t) = \underline{\underline{\frac{1}{2} t e^{-2t} \sin t}} \qquad t \geq 0$$

j)

```
t = 0:0.001:10;
u = exp(-2*t).*cos(t);
y = lsim(S, u, t);
plot(t, y)
```

ZADATAK 4:



$R = 1\,k\Omega = 10^3\,\Omega$

$C = 10\,\mu F = 10^{-5}\,F$

$\frac{1}{RC} = 100$

$$u(t) = \begin{cases} \sin t, & t \in \langle -\infty, 0 \rangle \\ 1, & t \in [0, \infty \rangle \end{cases}$$

$$\frac{d}{dt} u_{iz}(t) + \frac{1}{RC} u_{iz}(t) = \frac{d}{dt} u_{ul}(t)$$

a) $u(0^-) = 0$

$t < 0$ $u(0^+) = 1$

homogeno rješenje istitra pa do $t = 0^-$ ostane

$y_p = K_1 \cos t + K_2 \sin t$

$y_p' = -K_1 \sin t + K_2 \cos t$

$-K_1 \sin t + K_2 \cos t + 100 K_1 \cos t + 100 K_2 \sin t = \cos t$

$\sin t \,(100 K_2 - K_1) + \cos t\,(K_2 + 100 K_1) = \cos t$

$100 K_2 - K_1 = 0 \Rightarrow K_1 = 100 K_2$

$K_2 + 100 K_1 = 1$

$10\,001 K_2 = 1$

$\boxed{K_2 = \frac{1}{10\,001}} \Rightarrow \boxed{K_1 = \frac{100}{10\,001}}$

$$y_p(t) = \frac{100}{10\,001} \cos t + \frac{1}{10\,001} \sin t$$

$$y(0^-) = \frac{100}{10\,001}$$

$$y(0^+) - y(0^-) = b_0\, u(0^+) \qquad b_0 = 1$$

$$y(0^+) = y(0^-) + u(0^+) = \frac{100}{10\,001} + 1 = \frac{10\,101}{10\,001}$$

$$\underline{\underline{y(0^+) = 1{,}01 = u_c(0)}}$$

$t \geq 0$

$$y' + 100\, y = u'$$

HOMOGENO $\quad y' + 100\, y = 0 \qquad s = -100 \qquad u = 1$

$$y_h = c\, e^{-100t}$$

PARTICULARNO $\quad y_p = k$

$$y_p' = 0$$

$$0 + 100\, k = 0$$

$$\boxed{k = 0}$$

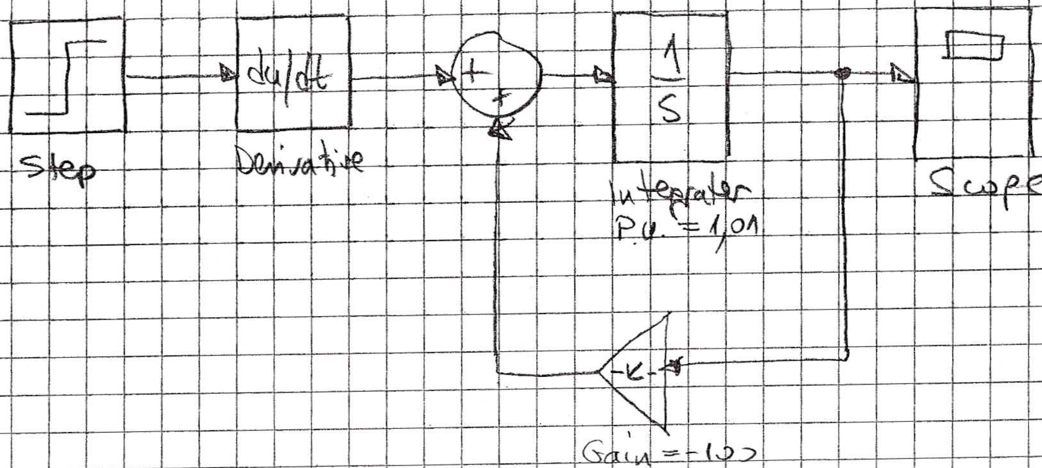
$$y = c\, e^{-100t}$$

$$y(0^+) = c\, e^{-100 \cdot 0} \Rightarrow c = 1{,}01$$

$$\underline{\underline{y = 1{,}01\, e^{-100t}\, u(t)}}$$

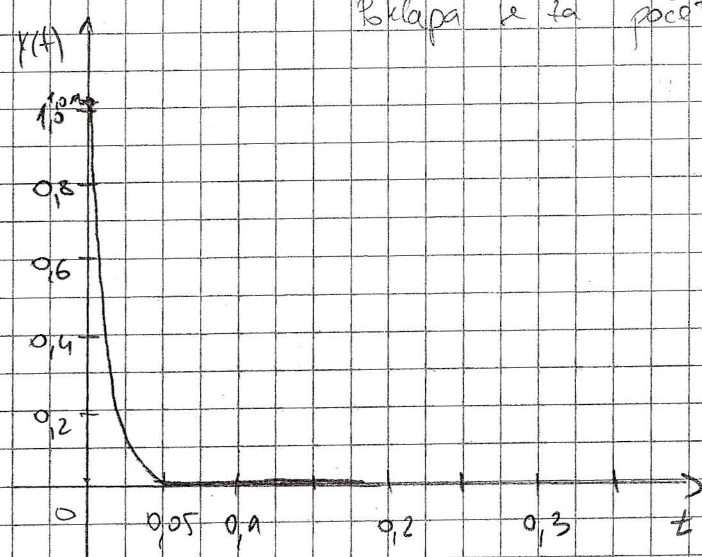
ZADATAK 4: (II DIO)

b)



Potrebno je upisati početne uvjete za $t=0^+$

Poklapa se za početne uvjete u $t=0^+$



c)

$$u(t) = \begin{cases} \sin t, & t \in \langle -\infty, 34\pi \rangle \\ 1, & t \in [34\pi, \infty \rangle \end{cases}$$

$$y_p(t) = \frac{100}{10\,001} \cos t + \frac{1}{10\,001} \sin t$$

$$y_p(34\pi^-) = \frac{100}{10\,001} = 0{,}01$$

- stanje u $t = 34\pi^-$ se poklapa sa stanjem u $t = 0^-$ iz podzadatka b)
- razlog tome je činjenica da je odziv u trenutku $t = 34\pi$ isti odzivu u trenutku $t = 0$ iz podzadatka b)