SIGNALI I SUSTAVI – MATERIJALI ZA PRIPREMU ZAVRŠNOG ISPITA

Sadržaj:

(prazna stranica)

GRUPA A

Signali i sustavi

Završni ispit – 26. lipnja 2007.

1. Kontinuirani kauzalan LTI sustav opisan diferencijalnom jednadžbom $y'(t) + 4y(t) = 2u'(t) + u(t)$ pobuđen je signalom $u(t) = 2\,\mu(t)$. Početni uvjet je $y(0^-) = 1$.

a) Izračunajte koliki je početni uvjet u $t = 0^+$!

b) Odredite odziv sustava na zadanu pobudu rješavanjem jednadžbe u vremenskoj domeni.

c) Odredite odziv sustava na zadanu pobudu korištenjem Laplaceove transformacije.

d) Odredite prijenosnu funkciju sustava. Je li sustav stabilan?

2. Diskretan kauzalan LTI sustav opisan je jednadžbom $4y(n) + 4y(n-1) + y(n-2) = u(n)$. Sustav je pobuđen signalom $u(n) = 5\,\mu(n)$. Početni uvjeti su jednaki nuli.

a) Odredite impulsni odziv sustava i prijenosnu funkciju sustava.

b) Je li sustav stabilan?

c) Odredite odziv sustava na zadanu pobudu korištenjem $\mathcal{Z}$ transformacije.

3. Prijenosna funkcija kontinuiranog LTI sustava je

$$H(s) = \frac{s^2 + 3s + 2}{(s-1)(s-2)(s-3)}.$$

Odredite matrice **A**, **B**, **C** i **D** paralelne realizacije.

4. Odredite rastav u Fourierov red signala

$$x(t) = 10\cos(50\pi t) + 5\sin(100\pi t) + \sin(150\pi t + 2\pi/3) + \cos(200\pi t + \pi/4)$$

te skicirajte dobiveni amplitudni i fazni spektar. Ako signal $x(t)$ otipkamo s periodom otipkavanja $T_s = 0{,}02$ je li došlo do preklapanja spektra?

5. Prijenosna funkcija nekog diskretnog LTI sustava je $H(z) = \dfrac{1}{4 - z^{-1}}$, no nije poznato područje konvergencije prijenosne funkcije.

a) Koliko ima različitih mogućih područja konvergencije?

b) Za svako od područja konvergencije odredite impulsni odziv sustava.

c) Nacrtajte amplitudnu i faznu karakteristiku sustava čije područje konvergencije obuhvaća beskonačnost.

GRUPA B

Signali i sustavi

Završni ispit – 26. lipnja 2007.

1. Kontinuirani kauzalan LTI sustav opisan diferencijalnom jednadžbom $y'(t) + 4y(t) = 2u'(t) + u(t)$ pobuđen je signalom $u(t) = \mu(t)$. Početni uvjet je $y(0^-) = 2$.

a) Izračunajte koliki je početni uvjet u $t = 0^+$!

b) Odredite odziv sustava na zadanu pobudu rješavanjem jednadžbe u vremenskoj domeni.

c) Odredite odziv sustava na zadanu pobudu korištenjem Laplaceove transformacije.

d) Odredite prijenosnu funkciju sustava. Je li sustav stabilan?

2. Diskretan kauzalan LTI sustav opisan je jednadžbom $4y(n) - 4y(n-1) + y(n-2) = u(n)$. Sustav je pobuđen signalom $u(n) = 5\,\mu(n)$. Početni uvjeti su jednaki nuli.

a) Odredite impulsni odziv sustava i prijenosnu funkciju sustava.

b) Je li sustav stabilan?

c) Odredite odziv sustava na zadanu pobudu korištenjem $\mathcal{Z}$ transformacije.

3. Prijenosna funkcija kontinuiranog LTI sustava je

$$H(s) = \frac{s^2 - 3s + 2}{(s+1)(s+2)(s+3)}.$$

Odredite matrice **A**, **B**, **C** i **D** paralelne realizacije.

4. Odredite rastav u Fourierov red signala

$$x(t) = 10\cos(50\pi t) + 5\sin(100\pi t) + \sin(150\pi t + 2\pi/3) + \cos(200\pi t + \pi/4)$$

te skicirajte dobiveni amplitudni i fazni spektar. Ako signal $x(t)$ otipkamo s periodom otipkavanja $T_s = 0{,}01$ je li došlo do preklapanja spektra?

5. Prijenosna funkcija nekog diskretnog LTI sustava je $H(z) = \dfrac{1}{4 + z^{-1}}$, no nije poznato područje konvergencije prijenosne funkcije.

a) Koliko ima različitih mogućih područja konvergencije?

b) Za svako od područja konvergencije odredite impulsni odziv sustava.

c) Nacrtajte amplitudnu i faznu karakteristiku sustava čije područje konvergencije obuhvaća beskonačnost.

ZADATAK 1 $\quad y'(t) + 4y(t) = 2u'(t) + u(t)$ $\quad$ 1

$$u(t) = A \cdot \mu(t) \qquad y(0^-) = y_0$$

Ovisno o grupi $\boxed{\begin{matrix} A = 2 \\ y_{\bar{0}} = 1 \end{matrix}}$ ili $\boxed{\begin{matrix} A = 1 \\ y_{\bar{0}} = 2 \end{matrix}}$

U nastavku će rješenje biti određeno za općenit A i y_0

a) Odredite poč. uvj. $y(t)$ za $t = 0^+$

... prema šalabahteru:

$$\Delta y = y(0^+) - y(0^-) = b_0 u(0^+)$$

koliko iznosi b_0 ?

$$y'(t) + a_1 y(t) = b_0 u'(t) + b_1 u(t)$$

$$\Rightarrow a_1 = 4 \quad \boxed{b_0 = 2} \quad b_1 = 1$$

Dakle

$$\begin{aligned} y(0^+) &= y(0^-) + \Delta y \\ &= y(0^-) + b_0 u(0^+) \\ &= y_{\bar{0}} + 2 \cdot A \end{aligned}$$

pobuda je
$u(t) = A \cdot \mu(t)$
$u(0^+) = A$

za $A=2$, $y_{\bar{0}}=1$ imamo $y(0^+) = 1 + 4 = 5$

za $A=1$, $y_{\bar{0}}=2$ ... $y(0^+) = 2 + 2 = 4$

u nastavku ovaj poč uvjet biti će označen sa y_{0^+}

b) pronađimo odziv rješavanjem dif. jednadžbe u vremenskoj domeni:

Pronađimo prvo partikularno rješenje $y_p(t)$

00.A.503.A.gp

Obzirom da je sve-vremenska pobuda oblika konstante A, sve-vremensko partikularno rješenje pretpostavljamo u istom obliku ②

$$y_p(t) = K$$

uvrštavanjem u dif. jed. dobivamo:

$$y_p'(t) + 4y_p(t) = 2\cdot(A)' + A$$

$$4K = A \Rightarrow K = \frac{A}{4}$$

za grupu sa $A=2$ ... $K = \frac{1}{2}$

— " — sa $A=1$ ... $K = 1/4$

kauzalno partikularno rješenje je dakle:

$$y_p(t) = K\cdot\mu(t)$$

Nađimo sada totalno rj.

$$y_{tot}(t) = y_h(t) + y_p(t)$$

Da odredimo $y_h(t)$ potrebno je odrediti polove homogen.

$$y_h(t) = C_1\cdot e^{s_1 t}$$ ... uvrštavanjem u dif. jed.

$$C_1 s_1\cdot e^{s_1 t} + 4C_1\cdot e^{s_1 t} = \emptyset$$

$$C_1\cdot e^{s_1 t}\underbrace{(s_1+4)}_{\text{karakteristični polinom}} = \emptyset$$

$s_1 = -4$ ← pol sustava

$$y_h(t) = C_1\cdot e^{-4t}$$

sve-vremensko homogeno rj.

③

Konstantu homogenog rj. C_1 nalazimo iz poč uvjeta u $t=0^+$

sve-vremensko rj.

$$y_{tot_{sv}}(t) = C_1 \cdot e^{-4t} + K$$

odnosno kauzalno rj.

$$y_{tot}(t) = C_1 \cdot e^{-4t} \cdot \mu(t) + K \cdot \mu(t)$$

Poznato je da $y_{tot}(0^+) = y_{0^+}$

Odredimo dakle C_1

$$C_1 \cdot \underbrace{e^{-4 \cdot 0^+}}_{1} \cdot \underbrace{\mu(0^+)}_{1} + K \cdot \underbrace{\mu(0^+)}_{1} = y_{0^+}$$

$$C_1 + K = y_0^+$$

$$C_1 = y_{0^+} - K$$

$$= y_{0^-} + 2A - \frac{A}{4}$$

$$= y_{0^-} + \frac{7}{4} A$$

za grupu $A=2$, $y_{0^-}=1$ $\Rightarrow$ $C_1 = 1 + \frac{7}{2} = \frac{9}{2}$

— " — $A=1$, $y_{0^-}=2$ $\Rightarrow$ $C_1 = 2 + \frac{7}{4} = \frac{15}{4}$

Dakle kauzalno totalno rješenje je:

$$y_{tot} = \left(\frac{9}{2} e^{-4t} + \frac{1}{2} \right) \cdot \mu(t)$$ za grupu $A=2$, $y_{0^-}=1$

$$y_{tot} = \left(\frac{15}{4} \cdot e^{-4t} + \frac{1}{4} \right) \cdot \mu(t)$$ — " — $A=1$ $y_{0^-}=2$

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a) Sada određujemo odziv pomoću $\mathcal{L}$ transformacije

$$y'(t) + 4y(t) = 2u'(t) + u(t)$$

$\mathcal{L}$ $\quad u(0^-) = \phi$

$$sY(s) - y(0^-) + 4Y(s) = 2sU(s) - 2u(0^-) + U(s)$$

$$(s+4)Y(s) = (2s+1)U(s) + y(0^-)$$

$$Y(s) = \frac{2s+1}{s+4}U(s) + \frac{y(0^-)}{s+4}$$

$$Y(s) = \underbrace{H(s)\cdot U(s)}_{Y_m(s)} + Y_0(s)$$

odziv mirnog sustava

odziv nepobuđenog sustava na poč. stanje

$\Rightarrow$ Prepoznajemo da prijenosna funkcija $H(s)$ glasi

$$H(s) = \frac{2s+1}{s+4}$$ ... što se traži u d)

Promotrimo sada $U(s)$?

Prema tablici: $A\mu(t) \overset{\mathcal{L}}{\frown} \frac{A}{s}$

Dakle $$Y(s) = \frac{2s+1}{s+4}\cdot\frac{A}{s} + \frac{y_{0^-}}{s+4}$$

$$= \frac{A(2s+1) + y_{0^-}\cdot s}{s(s+4)} = \frac{s(2A+y_{0^-}) + A}{s(s+4)}$$

Radi određivanja inv. $\mathcal{L}$ transformacije moramo $Y(s)$ rastaviti u parc. razlomke:

(polovi u nazivniku su jednostruki ...)

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$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s+4} \quad / \cdot \text{naz}$$

$$s(2A + y_{0^-})^{+A} = C_1(s+4) + C_2 \cdot s$$

$$= \underbrace{s(C_1+C_2)} + \underbrace{C_1 \cdot 4}$$

$$C_1 + C_2 = (2A + y_{0^-}) \qquad C_1 \cdot 4 = A$$

$$C_1 = \frac{A}{4}$$

$$C_2 = 2A + y_{0^-} - C_1$$

$$= 2A - \frac{A}{4} + y_{0^-}$$

$$= \frac{7}{4}A + y_{0^-}$$

$$Y(s) = \frac{A/4}{s} + \frac{\frac{7}{4}A + y_{0^-}}{s+4}$$

Inverznu $\mathcal{L}$ transf. radimo po tablici:

$$y(t) = \frac{A}{4} \cdot \mu(t) + \left(\frac{7}{4}A + y_{0^-}\right) \cdot e^{-4t} \cdot \mu(t)$$

što je naravno potpuno jednako rješenju određenom u vrem. domeni

za grupu $A=2$, $y_{0^-}=1$ ... $y(t) = \left(\frac{1}{2} + \frac{9}{2} \cdot e^{-4t}\right)\mu(t)$

—‖— $A=1$ $y_{0^-}=2$ .. $y(t) = \left(\frac{1}{4} + \frac{15}{4} \cdot e^{-4t}\right)\mu(t)$

d) konačno, obzirom da smo $H(s)$ već pronašli

$$H(s) = \frac{2s+1}{s+4}$$

i obzirom da je zadano da se radi o kauzalnom sustavu područje stabilnosti je lijeva poluravnina $\sigma < 0$

$\uparrow$ $\sigma = Re(s_1)$

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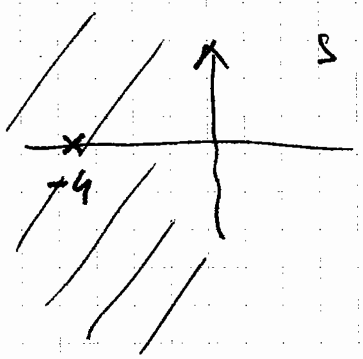
Nađimo pol sustava s_1 ?

karakteristični polinom $A(s)$ jednak je nazivniku $H(s)$

$$A(s) = s + 4 = \emptyset$$

$$\Longrightarrow s_1 = -4$$

$$Re(s_1) = -4 < \emptyset$$



sustav je stabilan ☺

(za obje grupe)

ZADATAK

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2. $4y(n) \pm 4y(n-1) + y(n-2) = u(n)$ $u(n) = 5\mu(n)$

↑ ovisno o grupi

a) Krećemo od $H(z)$. Obzirom da se radi o mirnom sustavu $H(z)$ pišemo direktno iz jedn. dif.

$$4Y(z) \pm 4Y(z)\cdot z^{-1} + Y(z)\cdot z^{-2} = U(z)$$

$$Y(z)(4 \pm z^{-1}4 + z^{-2}) = U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{4 \pm 4z^{-1} + z^{-2}} = \frac{1}{A(z)}$$

Radi određivanja rastava u parc. razlomke moramo odrediti polove sustava

$$A(z) = \emptyset \Rightarrow 4 \pm 4z^{-1} + z^{-2} = \emptyset$$

$$z_{1,2} = \frac{\mp 4 \pm \sqrt{16-16}}{2\cdot 4} = \frac{\mp 4}{2\cdot 4} = \mp\frac{1}{2}$$

Za grupu sa + ... $z_1 = z_2 = -1/2$
—"— − ... $z_1 = z_2 = 1/2$
Dvostruki korijen !

$$H(z) = \frac{1}{4(1 \pm z^{-1} + 1/4)} = \frac{\frac{1}{4}}{(1-z_1z^{-1})(1-z_2z^{-1})} =$$

$$= \frac{\frac{1}{4}}{\left(1-\left(\mp\frac{1}{2}\right)z^{-1}\right)^2} = \frac{\frac{1}{4}}{\left(1 \pm \frac{1}{2}z^{-1}\right)^2} = \frac{\frac{1}{4}z^2}{\left(z \pm \frac{1}{2}\right)^2}$$

$$H_1(z) = H(z)\cdot z^{-1} = \frac{\frac{1}{4}z}{\left(z \pm \frac{1}{2}\right)^2} = \frac{C_{11}}{z \pm 1/2} + \frac{C_{12}}{(z \pm 1/2)^2} \Big/ \cdot naz$$

$$\left.\begin{aligned} \frac{1}{4}z &= C_{11}\left(z \pm \frac{1}{2}\right) + C_{12} \\ \frac{1}{4}z &= C_{11}z + \left(C_{12} \pm \frac{1}{2}C_{11}\right) \end{aligned}\right\} \Rightarrow \begin{aligned} C_{11} &= \frac{1}{4} \\ C_{12} &= \mp\frac{1}{2}C_{11} \\ &= \mp\frac{1}{8} \end{aligned}$$

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$$H(z) = z \cdot H_1(z) = c_{11} \cdot \frac{z}{z \pm 1/2} + c_{12} \cdot \frac{z}{(z \pm 1/2)^2}$$

Određujemo $h[n]$ inverznom $\mathcal{Z}$ transformacijom $H(z)$

U tablici čitamo slijedeće parove:

$$\frac{az}{(z-a)^2} \xleftrightarrow{\mathcal{Z}} n \cdot a^n \mu(n) \qquad \frac{z}{z-a} \xleftrightarrow{\mathcal{Z}} a^n \cdot \mu(n)$$

$$H(z) = c_{11} \cdot \frac{z}{z \pm 1/2} + c_{12} \frac{\left(\mp \frac{1}{2}\right) \cdot (\mp 2)\, z}{\left(z \pm \frac{1}{2}\right)^2}$$

$$= c_{11} \frac{z}{z \pm 1/2} \mp 2 c_{12} \cdot \frac{\mp \frac{1}{2} z}{\left(z - \left(\mp \frac{1}{2}\right)\right)^2}$$

$$h[n] = c_{11} \cdot \left(\mp \frac{1}{2}\right)^n \mu(n) \mp 2 c_{12} \cdot n \cdot \left(\mp \frac{1}{2}\right)^n \cdot \mu(n)$$

$$= (c_{11} \mp 2 c_{12} n) \left(\mp \frac{1}{2}\right)^n \cdot \mu(n)$$

Uvrstimo određene koef. c_{11}, c_{12}

$$h[n] = \left(\frac{1}{4} \mp 2\left(\mp \frac{1}{8}\right) n\right) \left(\mp \frac{1}{2}\right)^n \mu(n)$$

$$= \frac{1}{4} (1+n) \left(\mp \frac{1}{2}\right)^n \mu(n)$$

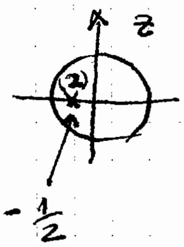
b) stabilnost ?

$$|z_i| < 1 \quad \forall i$$

za grupu sa $(+)$ $\quad z_1 = z_2 = -\frac{1}{2}$

$$|z_1| = |z_2| = \frac{1}{2} < 1$$

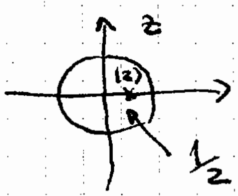
$\Rightarrow$ stabilan



za grupu sa $(-)$ $\quad z_1 = z_2 = \frac{1}{2}$

$$|z_1| = |z_2| = \frac{1}{2} < 1$$

$\Rightarrow$ stabilan



(9)

c) Odziv na $u(n) = 5\cdot\mu(n)$

Kako se radi o mirnom sustavu:

$$y_{tot}[n] = y_m[n],$$

a $y_m[n]$ možemo odrediti inv. Z transf. $Y_m(z)$

$$Y_m(z) = H(z)\cdot U(z)$$

Po tablici $U(z)$ nalazimo kao:

$$\mathcal{Z}\{5\mu(n)\} = \frac{5z}{z-1}$$

$$Y_m(z) = \frac{\frac{1}{4}z^2}{(z\pm\frac{1}{2})^2}\cdot\frac{5z}{z-1} = \frac{\frac{5}{4}z^3}{(z\pm\frac{1}{2})^2(z-1)}$$

Radi određivanja $y_m[n]$ moramo $Y_m(z)$ rastaviti u parc. razlomke:

$$Y_{m1}(z) = Y_m(z)\cdot z^{-1} = \frac{\frac{5}{4}z^2}{(z\pm\frac{1}{2})^2(z-1)} = \frac{C_{11}}{z\pm\frac{1}{2}} + \frac{C_{12}}{(z\pm\frac{1}{2})^2} + \frac{C_2}{z-1} \Big/ \text{naz}$$

$$\frac{5}{4}z^2 = C_{11}\left(z\pm\frac{1}{2}\right)\cdot(z-1) + C_{12}(z-1) + C_2\left(z\pm\frac{1}{2}\right)^2$$

$$\frac{5}{4}z^2 = C_{11}\left(z^2 - z \pm\frac{1}{2}z \mp\frac{1}{2}\right) + C_{12}(z-1) + C_2\left(z^2 \pm z + \frac{1}{4}\right)$$

$$\frac{5}{4}z^2 = z^2(C_{11}+C_2) + z\left(-C_{11} \pm\frac{1}{2}C_{11} + C_{12} \pm C_2\right) + \left(\mp\frac{1}{2}C_{11} - C_{12} + \frac{1}{4}C_2\right)$$

Radi nalaženja rješenja koje je zajedničko za obje grupe uvedimo α

$$\alpha = \begin{cases} 1 & \text{za grupu sa } (+) \\ -1 & \text{—"— } (-) \end{cases}$$

$$\frac{5}{4}z^2 = z^2\underbrace{(C_{11}+C_2)}_{=\frac{5}{4}} + z\underbrace{\left(\left(\frac{1}{2}\alpha - 1\right)C_{11} + C_{12} + \alpha C_2\right)}_{=\emptyset} + \underbrace{\left(-\alpha\frac{1}{2}C_{11} - C_{12} + \frac{1}{4}C_2\right)}_{=\emptyset}$$

JEDN. I JEDN. II JEDN. III

Zbrajanjem jedn. II i III eliminira se c_{12} (10)

$$\begin{aligned} II + III \ldots \quad & -c_{11} + c_2\left(\frac{1}{4}+\alpha\right) = 0 \\ I \ldots \quad & c_{11} + c_2 = \frac{5}{4} \end{aligned} \Big> +$$

$$c_2\left(\frac{5}{4}+\alpha\right) = \frac{5}{4} \qquad c_2 = \frac{\frac{5}{4}}{\frac{5}{4}+\alpha} = \frac{5}{5+4\alpha}$$

Iz jedn. I

$$c_{11} = \frac{5}{4} - c_2 = \frac{5}{4} - \frac{5}{5+4\alpha} = \frac{25+20\alpha-20}{20+16\alpha}$$

$$c_{11} = \frac{5+20\alpha}{20+16\alpha}$$

Iz jedn. III

$$c_{12} = \frac{1}{4}c_2 - \frac{\alpha}{2}c_{11} = \frac{5}{20+16\alpha} - \frac{\frac{\alpha}{2}\cdot 5 + \frac{20}{2}\alpha^2}{20+16\alpha}$$

$$c_{12} = \frac{5\left(1-\frac{\alpha}{2}\right)-10}{20+16\alpha}$$

Nadimo sada rješenje za oba slučaja $\alpha = 1$, $\alpha = -1$

$\alpha = 1$	$\alpha = -1$
$c_2 = \frac{5}{5+4} = \frac{5}{9}$	$c_2 = \frac{5}{5-4} = 5$
$c_{11} = \frac{5+20}{20+16} = \frac{25}{36}$	$c_{11} = \frac{5-20}{20-16} = \frac{-15}{4}$
$c_{12} = \frac{\frac{5}{2}-10}{36} = \frac{-15}{72}$	$c_{12} = \frac{\frac{15}{2}-10}{20-16} = \frac{-5}{8}$

$$Y_m(z) = z \cdot Y_{m1} = c_{11}\cdot\frac{z}{z\pm\frac{1}{2}} + c_{12}(\mp 2)\frac{\left(\mp\frac{1}{2}\right)z}{\left(z-\left(\mp\frac{1}{2}\right)\right)^2} + c_2\frac{z}{z-1}$$

$$y_m(n) = c_{11}\left(\mp\frac{1}{2}\right)^n\cdot\mu(n) \mp 2c_{12}\cdot n\cdot\left(\mp\frac{1}{2}\right)^n\mu(n) + c_2\cdot\mu(n)$$

$$= \left[\left(c_{11}\mp 2c_{12}\cdot n\right)\left(\mp\frac{1}{2}\right)^n + c_2\right]\mu(n)$$

Za slučaj sa (+), tj $\alpha = 1$ imamo: (11)

$$y_m(n) = \left[\left(\frac{25}{36} - 2\cdot\left(\frac{-15}{72}\right)\cdot n\right)\left(-\frac{1}{2}\right)^n + \frac{5}{9} \right] \mu(n)$$

$$= \frac{5}{9}\left[\left(\frac{5}{4} + \frac{3}{4}n\right)\left(-\frac{1}{2}\right)^n + 1 \right] \mu(n)$$

$$= \frac{5}{36}\left[(3n+5)\left(-\frac{1}{2}\right)^n + 4 \right] \mu(n)$$

Za slučaj sa (−), $\alpha = -1$ imamo:

$$y_m(n) = \left[\left(-\frac{15}{4} + 2\cdot\left(-\frac{5}{8}\right)\cdot n\right)\left(\frac{1}{2}\right)^n + 5 \right] \mu(n)$$

$$= \left[5 - \frac{5}{4}(n+3)\left(\frac{1}{2}\right)^n \right] \cdot \mu(n)$$

$$= 5\cdot\left[1 - \frac{n+3}{4}\left(\frac{1}{2}\right)^n \right] \cdot \mu(n)$$

$$= 5\cdot\left[1 - (n+3)\left(\frac{1}{2}\right)^{n+2} \right] \mu(n)$$

$$= 5\cdot\left[1 - (n+3)\cdot 2^{-(n+2)} \right] \mu(n) \qquad (-)$$

Analogno možemo srediti i gornji slučaj $\alpha = 1$

$$y_m(n) = \frac{5}{9}\left[(3n+5)(-2)^{-(n+2)} + 1 \right] \mu(n) \qquad (+)$$

3. zadatak: (12)

$$H(s) = \frac{s^2+3s+2}{(s-1)(s-2)(s-3)}$$

Za paralelnu realizaciju moramo H(s) rastaviti u parcijalne razlomke. Iz nazivnika je očito da sustav ima tri jednostruka realna pola, pa paralelna realizacija ima članove prvog reda (nema potrebe za uparivanjem konj. kompl. parova)

$$H(s) = \frac{s^2+3s+2}{(s-1)(s-2)(s-3)} = \frac{c_1}{s-1} + \frac{c_2}{s-2} + \frac{c_3}{s-3} + c_0$$

$$c_1 = \lim_{s\to 1}\left(H(s)\cdot(s-1)\right) = \left.\frac{s^2+3s+2}{(s-2)(s-3)}\right|_{s=1} = \frac{1+3+2}{(-1)(-2)} = \frac{6}{2} = 3$$

$$c_2 = \lim_{s\to 2}\left(H(s)\cdot(s-2)\right) = \left.\frac{s^2+3s+2}{(s-1)(s-3)}\right|_{s=2} = \frac{4+6+2}{1\cdot(-1)} = \frac{12}{-1} = -12$$

$$c_3 = \lim_{s\to 3}\left(H(s)\cdot(s-3)\right) = \left.\frac{s^2+3s+2}{(s-1)(s-2)}\right|_{s=3} = \frac{9+9+2}{2\cdot 1} = 10$$

$c_0 = \emptyset$ jer je red brojnika manji od reda nazivnika !

Provjera:

$$H(s) = \frac{3}{s-1} - \frac{12}{s-2} + \frac{10}{s-3} = \frac{3(s-2)(s-3)-12(s-1)(s-3)+10(s-1)(s-2)}{(s-1)(s-2)(s-3)}$$

$$= \frac{3(s^2-5s+6)-12(s^2-4s+3)+10(s^2-3s+2)}{(s-1)(s-2)(s-3)} = \frac{s^2(3-12+10)+s(-15+48-30)+(18-36+20)}{naz}$$

$$= \frac{s^2(1)+s(3)+(2)}{naz}$$ ✓ OK

Alternativno neki studenti su mogli umjesto limesa koef. odrediti pomoću sustava jednadžbi:

$$s^2+3s+2 = c_1(s^2-5s+6)+c_2(s^2-4s+3)+c_3(s^2-3s+2)$$

$$= s^2\underbrace{(c_1+c_2+c_3)}_{==1} + s\underbrace{(-5c_1-4c_2-3c_3)}_{==3} + \underbrace{(6c_1+3c_2+2c_3)}_{==2}$$

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Dobivamo sustav jednadžbi:

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ 5c_1 + 4c_2 + 3c_3 &= -3 \\ 6c_1 + 3c_2 + 2c_3 &= 2 \end{aligned}$$

$$c_1 = 1 - c_2 - c_3 = 1 + 12 - 10 = 3 \qquad \boxed{c_1 = 3}$$

$$\begin{aligned} 5 - 5c_2 - 5c_3 + 4c_2 + 3c_3 &= -3 \\ 6 - 6c_2 - 6c_3 + 3c_2 + 2c_3 &= 2 \end{aligned}$$

$$\begin{aligned} -c_2 - 2c_3 &= -8 \\ -3c_2 - 4c_3 &= -4 \end{aligned}$$

$$c_2 + 2c_3 = 8 \qquad c_2 = 8 - 2c_3 = 8 - 20 = -12 \qquad \boxed{c_2 = -12}$$

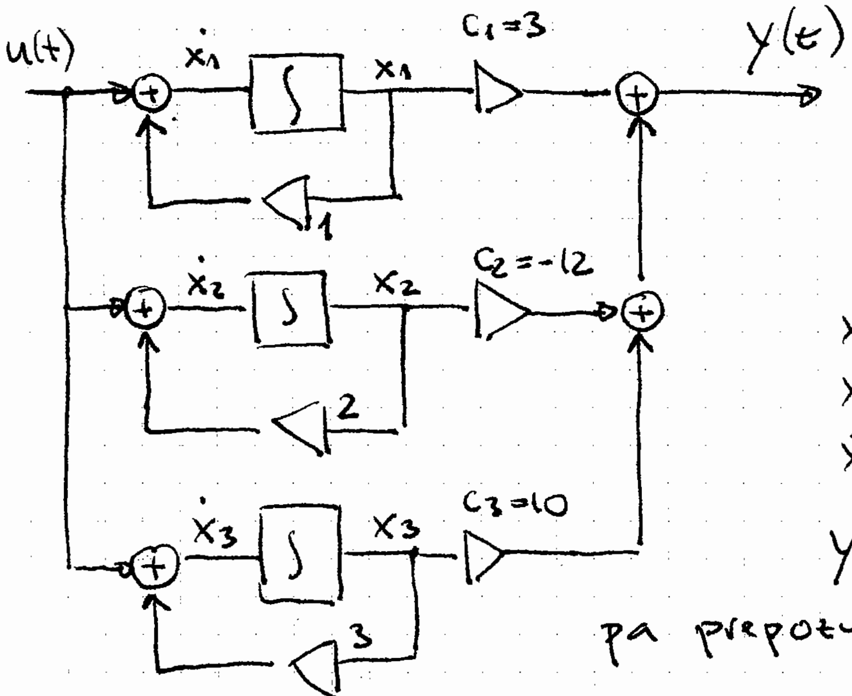
$$3c_2 + 4c_3 = 4$$

$$24 - 6c_3 + 4c_3 = 4$$

$$-2c_3 = -20$$

$$\boxed{c_3 = 10}$$

Paralelna realizacija sustava ima slijedeći oblik:



Izlaze integratora odabiremo kao varijable stanja x_1, x_2 i x_3. Iz strukture čitamo:

$$\begin{aligned} \dot{x}_1 &= 1 \cdot x_1 + u \\ \dot{x}_2 &= 2 \cdot x_2 + u \\ \dot{x}_3 &= 3x_3 + u \end{aligned}$$

$$y = 3x_1 - 12x_2 + 10x_3 + 0 \cdot u,$$

pa prepoznajemo:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{A} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B} \cdot u$$

$$y = \underbrace{\begin{bmatrix} 3 & -12 & 10 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{[0]}_{D} \cdot u$$

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Konačno rješenje je:

$$A=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\quad B=\begin{bmatrix}1\\1\\1\end{bmatrix}\quad C=\begin{bmatrix}3&-12&10\end{bmatrix}\quad D=[\varnothing]$$

Za drugu grupu prijenosna funkcija je:

$$H(s)=\frac{s^2-3s+2}{(s+1)(s+2)(s+3)}$$

$\Rightarrow$ polovi su realni i jednostruki, red brojnika je opet niži od reda nazivnika .... dakle imamo tri paralelne sekcije prvog reda bez direktne veze ulaza i izlaza

$$H(s)=\frac{c_1}{s+1}+\frac{c_2}{s+2}+\frac{c_3}{s+3}$$

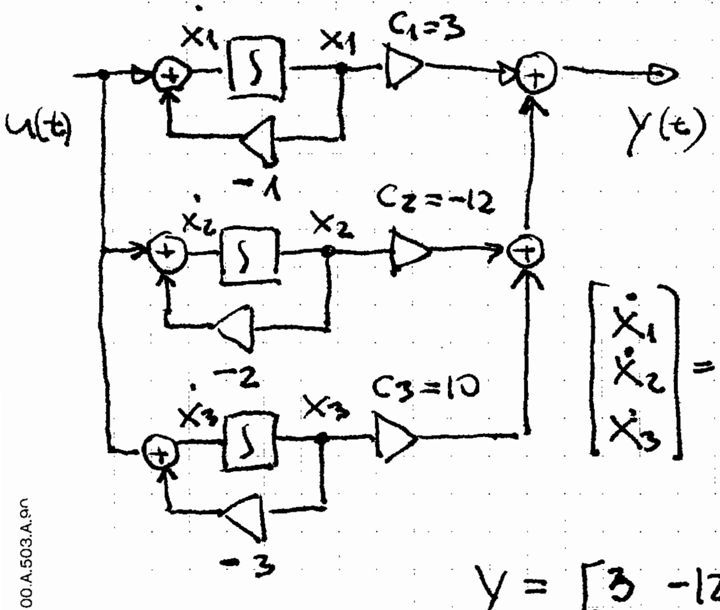
$$c_1=\lim_{s\to-1}\left\{(s+1)H(s)\right\}=$$

$$=\left.\frac{s^2-3s+2}{(s+2)(s+3)}\right|_{s=-1}=\frac{1+3+2}{1\cdot 2}=\frac{6}{2}=3$$

$$c_2=\lim_{s\to-2}\left\{(s+2)H(s)\right\}=\frac{s^2-3s+2}{(s+1)(s+3)}=\frac{4+\ 6+2}{(-1)\cdot(1)}=-12$$

$$c_3=\lim_{s\to-3}\left\{(s+3)H(s)\right\}=\frac{s^2-3s+2}{(s+1)(s+2)}=\frac{9+9+2}{(-2)(-1)}=\frac{20}{2}=10$$

Dakle $H(s)=\dfrac{3}{s+1}-\dfrac{12}{s+2}+\dfrac{10}{s+3}$



$$\dot{x}_1=-x_1+u$$
$$\dot{x}_2=-2x_2+u$$
$$\dot{x}_3=-3x_3+u$$

$$y=3x_1-12x_2+10x_3$$

$$\begin{bmatrix}\dot{x}_1\\\dot{x}_2\\\dot{x}_3\end{bmatrix}=\underbrace{\begin{bmatrix}-1&0&0\\0&-2&0\\0&0&-3\end{bmatrix}}_{A}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\underbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}}_{B}\cdot u$$

$$y=\underbrace{\begin{bmatrix}3&-12&10\end{bmatrix}}_{C}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\underbrace{[\varnothing]}_{D}\cdot u$$

4. ZADATAK (15)

$$\left.\begin{aligned} x(t) = & \ 10\cos(50\pi t) \\ & + 5\sin(100\pi t) \\ & + \sin(150\pi t + 2\pi/3) \\ & + \cos(200\pi t + \pi/4) \end{aligned}\right\} \text{FR ?}$$

Moramo odrediti period ovog signala T_0

Frekvencije njegovih komponenti su:

$\omega_1 = 50\pi$ rad/s $\Rightarrow T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{50\pi} = \frac{1}{25}$ [s]

$\omega_2 = 100\pi$ rad/s $\Rightarrow T_2 = \frac{2\pi}{\omega_2} = \frac{1}{50}$ [s]

$\omega_3 = 150\pi$ rad/s $\Rightarrow T_3 = \frac{2\pi}{\omega_3} = \frac{1}{75}$ [s]

$\omega_4 = 200\pi$ rad/s $\Rightarrow T_4 = \frac{2\pi}{\omega_4} = \frac{1}{100}$ [s]

Zajednički period ovih signala T_0, jednak je periodu najsporije komponente T_1 jer.

$$T_1 = T_0 = \frac{1}{25} \text{ [s]}$$

$$T_2 = \frac{T_0}{2}, \quad T_3 = \frac{T_0}{3} \quad T_4 = \frac{T_0}{4}$$

Dakle, osnovni period T_0 pri razvoju ovog signala jednak je $T_0 = \frac{1}{25}$ [s], a osnovna kružna frekvencija je $\frac{2\pi}{T_0} = \Omega_0 = 50\pi$ rad/s

Frekvencije komponenata signala su stoga:

$$\omega_1 = 1\cdot\Omega_0, \quad \omega_2 = 2\Omega_0, \quad \omega_3 = 3\Omega_0, \quad \omega_4 = 4\Omega_0$$

Odredimo sada koeficijente razvoja u FR

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot e^{jk\Omega_0 t}, \quad \text{gdje:}$$

$$X_k = \frac{1}{T_0}\int_{T_0} x(t)\cdot e^{-jk\Omega_0 t}\,dt$$

Koef. Xk uopće nije potrebno određivati korišlenjem formulez izraza, jer je očito da svaka komponenta signala generira jedan par kompl. exponencijala.

Ako cos raspišemo kao sumu exp. imamo:

$$10\cdot\cos(50\pi t) = 10\cdot\cos(\omega_1 t) = 10\cdot\cos(1\cdot\Omega_0 t)$$
$$= \frac{10}{2}\left(e^{j1\cdot\Omega_0 t} + e^{-j1\cdot\Omega_0 t}\right)$$
$$= \left(\frac{10}{2}\cdot e^{j\cdot\emptyset}\right)\cdot e^{j1\cdot\Omega_0 t} + \left(\frac{10}{2}\cdot e^{j\emptyset}\right)\cdot e^{-j1\cdot\Omega_0 t}$$
$$= X_1\cdot e^{j1\cdot\Omega_0 t} + X_{-1}\cdot e^{-j1\cdot\Omega_0 t}$$

Prepoznajemo da prva komponenta signala se u razvoju u FR vidi na koeficijentima X_1 i X_{-1} koji iznose $X_1 = 5\cdot e^{j\emptyset}$ $X_{-1} = X_1^* = 5\cdot e^{j\emptyset}$

Analogno radimo i za preostale 3 komponente signala:

2. komp.
$$5\cdot\sin(100\pi t) = 5\cdot\sin(\omega_2 t) = 5\cdot\sin(2\cdot\Omega_0 t)$$
$$= 5\cdot\cos\left(2\Omega_0 t - \frac{\pi}{2}\right) = \frac{5}{2}\left(e^{j(2\Omega_0 t-\frac{\pi}{2})} + e^{-j(2\Omega_0 t-\frac{\pi}{2})}\right)$$
$$= \left(\frac{5}{2}\cdot e^{-j\frac{\pi}{2}}\right)\cdot e^{j2\Omega_0 t} + \left(\frac{5}{2}e^{j\frac{\pi}{2}}\right)e^{-j2\Omega_0 t}$$
$$= X_2\cdot e^{j2\Omega_0 t} + X_{-2}\cdot e^{-j2\Omega_0 t}$$

$$\Rightarrow X_2 = \frac{5}{2}\cdot e^{-j\frac{\pi}{2}} \qquad X_{-2} = X_2^* = \frac{5}{2}\cdot e^{j\frac{\pi}{2}}$$

3 komp:
$$\sin\left(150\pi t + \frac{2\pi}{3}\right) = \sin\left(\omega_3 t + \frac{2\pi}{3}\right) = \sin\left(3\Omega_0 t + \frac{2\pi}{3}\right)$$
$$= \cos\left(3\Omega_0 t - \frac{\pi}{2} + \frac{2\pi}{3}\right) = \cos\left(3\Omega_0 t + \frac{-3+4}{6}\pi\right) = \cos\left(3\Omega_0 t + \frac{\pi}{6}\right)$$
$$= \frac{1}{2}\left(e^{j(3\Omega_0 t+\frac{\pi}{6})} + e^{-j(3\Omega_0 t+\frac{\pi}{6})}\right) = \frac{1}{2}\cdot e^{j\frac{\pi}{6}}\cdot e^{j3\Omega_0 t} + \frac{1}{2}e^{-j\frac{\pi}{6}}e^{-j3\Omega_0 t}$$
$$= X_3 e^{j3\Omega_0 t} + X_{-3}e^{-j3\Omega_0 t} \Rightarrow X_3 = \frac{1}{2}\cdot e^{j\frac{\pi}{6}},\ X_{-3} = X_3^* = \frac{1}{2}e^{-j\frac{\pi}{6}}$$

Konačno i zadnja komponenta:

$$\cos(200\pi t + \pi/4) = \cos(\omega_4 t + \pi/4) = \cos(4\Omega_0 t + \pi/4)$$

$$= \frac{1}{2}\cdot\left[e^{j(4\Omega_0 t + \pi/4)} + e^{-j(4\Omega_0 t+\pi/4)}\right] =$$

$$= \frac{1}{2}\cdot e^{j\pi/4}\cdot e^{j4\Omega_0 t} + \frac{1}{2}\cdot e^{-j\pi/4}\cdot e^{-j4\Omega_0 t}$$

$$= X_4\cdot e^{j4\Omega_0 t} + X_{-4}\cdot e^{-j4\Omega_0 t}$$

$$\Rightarrow X_4 = \tfrac{1}{2} e^{j\pi/4} \qquad X_{-4} = X_4^* = \frac{1}{2} e^{-j\pi/4}$$

Zaključujemo FR se sastoji od 8 članova za $|k| \in [1,2,3,4]$, dok su svi ostali koef. X_k jednaki $\emptyset$

$$X_k = \begin{cases} 5\cdot e^{j0} \text{ za } k=1, & 5\cdot e^{j\emptyset} \text{ za } k=-1 \\ 5/2\cdot e^{-j\pi/2} \text{ za } k=2 & 5/2\cdot e^{j\pi/2} \text{ za } k=-2 \\ 1/2\cdot e^{j\pi/6} \text{ za } k=3 & 1/2\cdot e^{-j\pi/6} \text{ za } k=-3 \\ 1/2\cdot e^{j\pi/4} \text{ za } k=4 & 1/2\cdot e^{-j\pi/4} \text{ za } k=-4 \\ \emptyset \text{ za sve ostale } k \end{cases}$$

Ovo se može zapisati i pomoću Kroneckerovog deltla impulsa kao:

$$\begin{aligned} X_k = \; & 5\cdot e^{j\emptyset}\cdot\delta[k-1] + 5\cdot e^{j\emptyset}\,\delta[k+1] + \\ & \tfrac{5}{2}\cdot e^{-j\pi/2}\cdot\delta[k-2] + \tfrac{5}{2}\, e^{j\pi/2}\,\delta[k+2] + \\ & \tfrac{1}{2}\cdot e^{j\pi/6}\,\delta[k-3] + \tfrac{1}{2}\cdot e^{-j\pi/6}\,\delta[k+3] + \\ & \tfrac{1}{2}\cdot e^{j\pi/4}\,\delta[k-4] + \tfrac{1}{2}\cdot e^{-j\pi/4}\,\delta[k+4] \end{aligned}$$

Koeficijente X_k smo odredili „prepoznavanjem" koeficijenata uz članove razvoja u FR. Pokažimo da smo do istog rezultata mogli doći i primjenom direktnog izraza za X_k, ... za ilustraciju uzmimo samo prvu komponentu

$$x_1(t) = 10\cdot\cos(50\pi t) = 5\cdot e^{j\Omega_0 t} + 5\cdot e^{-j\Omega_0 t}$$

Ovaj signal $x_1(t)$ bi dao:

$$X_k = \frac{1}{T_0}\int_{T_0} x_1(t)\cdot e^{-jk\Omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0}\left[5 e^{j\Omega_0 t} + 5\cdot e^{-j\Omega_0 t}\right]\cdot e^{-jk\Omega_0 t}dt$$

$$= \frac{1}{T_0}\int_0^{T_0} 5\cdot e^{j(1-k)\Omega_0 t}dt + \frac{1}{T_0}\int_0^{T_0} 5\cdot e^{-j(1+k)\Omega_0 t}dt$$

$$= \frac{5}{T_0}\cdot\frac{1}{j(1-k)\Omega_0}\cdot \underbrace{e^{j(1-k)\Omega_0 t}\Big|_0^{T_0}} + \frac{5}{T_0}\,\frac{1}{-j(1+k)\Omega_0}\cdot e^{-j(1+k)\Omega_0 t}\Big|_0^{T_0}$$

za $k\neq 1$ ovaj integral je jednak nuli jer:

$e^{j(1-k)\Omega_0 T_0} - e^{j\emptyset} =$

$e^{j2\pi(1-k)} - 1 =$

$1 - 1 = \emptyset$, jer $1-k \in Z$, a nazivnik $j(1-k)\Omega_0$ je $\neq\emptyset$

Specijalno za $k=1$ integral ima slijedeći oblik:

$$\frac{1}{T_0}\int_0^{T_0} 5\cdot e^{j\emptyset\Omega_0 t}dt = \frac{1}{T_0}\int_0^{T_0} 5\,dt = 5$$

Zaključujemo da je prvi član jednak 5 za $k=1$, a jednak nuli za sve ostale k

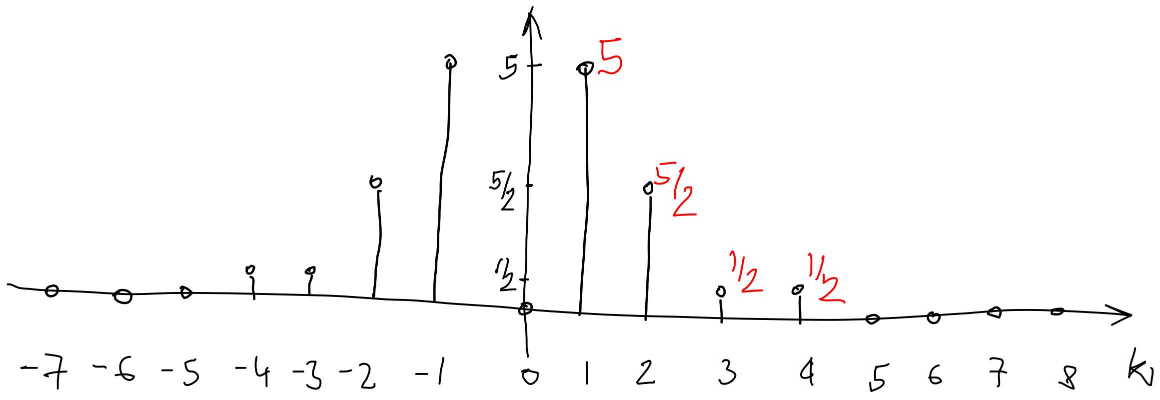
Po analogiji ovaj integral je jednak 5 za $k=-1$, a jednak nuli za sve ostale k

Vidimo da prva komponenta signala $x_1(t)$ ima u razvoju u FR dvije komponente X_1 i X_{-1}, a koeficijenti iznose $X_1 = 5$, $X_{-1} = 5$, tj.

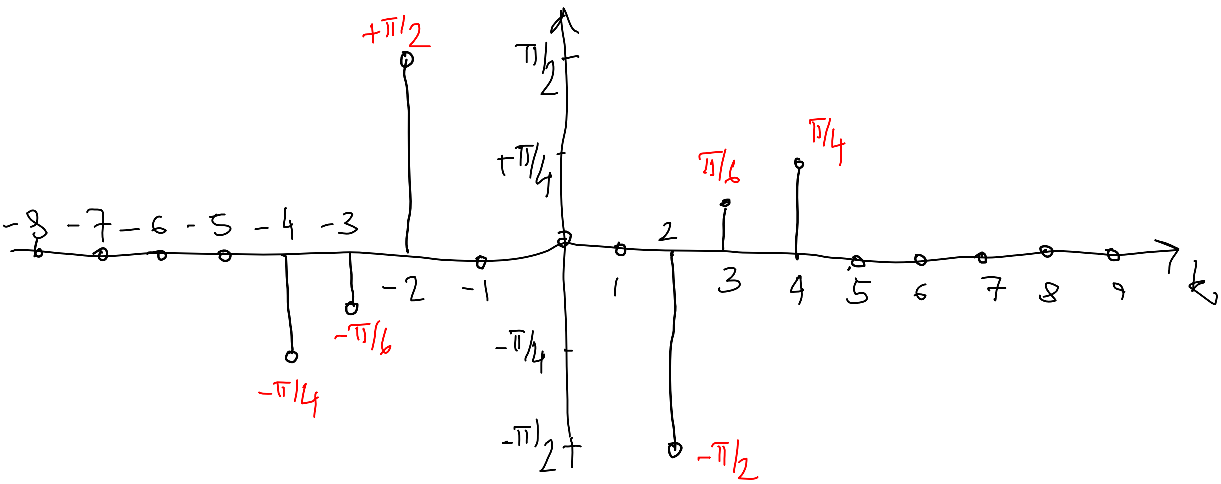
$$X_k = 5\cdot\delta[k-1] + 5\cdot\delta[k+1]$$

Slično možemo napraviti za preostale 3 komp. signala.

AMPLITUDNI SPEKTAR $|X(k)|$



FAZNI SPEKTAR $\measuredangle X(k)$



Da prilikom otipkavanja ne dođe do pojave preklapanja spektra frekvencija otipkavanja f_s mora biti barem 2puta viša od najviše frekvencije signala f_{max}

U našem primjeru $f_{max} = \frac{\omega_{max}}{2\pi} = \frac{\omega_4}{2\pi} =$

$= \frac{200\pi}{2\pi} = 100\,Hz$ .... frekvencija 4. komponente

$f_s > 2 \cdot f_{max} = 2 \cdot 100 = 200\,Hz$ ... da nema preklapanja

U jednoj grupi $f_s = \frac{1}{T_s} = \frac{1}{0.02} = 50\,Hz$

U drugoj grupi $f_s = \frac{1}{T_s} = \frac{1}{0.01} = 100\,Hz$

Vidimo da je za obje grupe $f_s < 2 \cdot f_{max}$

pa zaključujemo da dolazi do pojave preklapanja spektra, jer je frekvencija otipkavanja nedovoljno visoka.

ZADATAK 5.

$$H(z) = \frac{1}{4+\alpha z^{-1}}$$

$\alpha = 1$ ... za jednu grupu

$\alpha = -1$ za drugu.

a) Sjetimo se .... kod dvostrane z-transformacije ulazni sustav je mogao biti kauzalan ili anti-kauzalan, pa da za oba slučaja dobijemo jednaku prijenosnu funkciju ali komplementarno područje konvergencije

Razmotrimo prvo kauzalni slučaj ....
kauzalni impulsni odziv $h(n)$ dobivamo običnom jednostranom inverznom z-transf.

Iz tablice čitamo par:

$$\frac{K}{1-z_1 z^{-1}} \Rightarrow K \cdot z_1^n \cdot \mu(n)$$

za naš primjer

$$K = \frac{1}{4}, \quad z_1 = -\frac{\alpha}{4}$$ pol sustava

$$H(z) = \frac{\frac{1}{4}}{1-\left(-\frac{\alpha}{4}\right)z^{-1}} \quad h[n] = \frac{1}{4}\cdot\left(-\frac{\alpha}{4}\right)^n \cdot \mu(n)$$

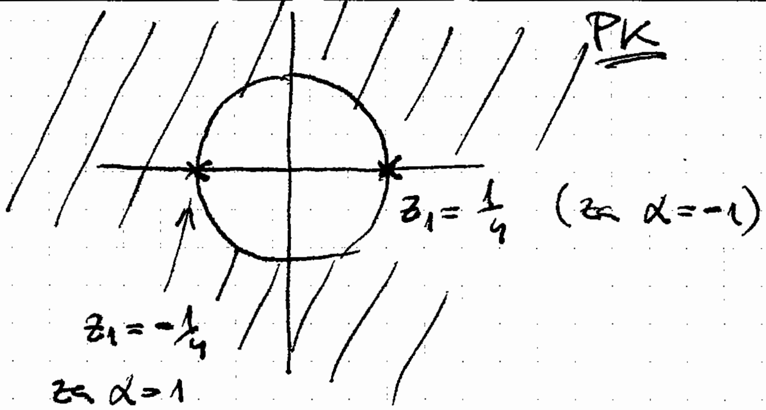
Ovisno o grupi $z_1 = -\frac{1}{4}$ (za $\alpha=1$)

$z_1 = \frac{1}{4}$ (za $\alpha=-1$)

Za kauzalni sustav područje konvergencije je

$$|z| > |z_1| = \frac{1}{4}$$

Dakle za obje grupe područje konvergencije kauzalnog sustava je $|z| > \frac{1}{4}$



Druga mogućnost je da sustav ima anti-kauzalni impulsni odziv oblika:

$$h(n) = -(z_1)^n \cdot \mu(-n-1)$$

jer dvostrana Z-transf. ovakvog anti-kauzalnog imp. odziva daje isti (zadani) oblik $H(z)$

Pokažimo to:

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) \cdot z^{-n} = \sum_{n=-\infty}^{\infty} -(z_1)^n \cdot \underbrace{\mu(-n-1)}_{=\emptyset \text{ za } n \geq 0} z^{-n} =$$

$$= \sum_{n=-\infty}^{-1} -\left(\frac{z_1}{z}\right)^n = -\underbrace{\sum_{n'=1}^{\infty}}_{\text{geom. red.}} \left(\frac{z}{z_1}\right)^n = -\frac{q}{1-q}\Bigg|_{q=\frac{z}{z_1}}$$

$|q| < 1$ ... uvjet na konvergenciju geom. reda.

$$H(z) = -\frac{\frac{z}{z_1}}{1-\frac{z}{z_1}} \cdot \frac{\left(-\frac{z_1}{z}\right)}{\left(-\frac{z_1}{z}\right)}$$

$$H(z) = \frac{1}{1-z_1 z^{-1}}$$

U našem primjeru još imamo i konstantni član $K = \frac{1}{4}$, pa dakle imamo anti-kauzalni par

$$\frac{\frac{1}{4}}{1-z_1 z^{-1}} \xleftrightarrow[]{\mathcal{Z} \leftarrow \text{dvostrana z-transf.}} -\frac{1}{4}(z_1)^n \cdot \mu(-n-1)$$

z_1 ovisi o grupi

za grupu sa $\alpha = 1$ $z_1 = -1/4$ pa imp. odziv glasi:

$$h_{ac}(n) = -\frac{1}{4}\left(-\frac{1}{4}\right)^n \cdot \mu(-n-1)$$

anti-kauzalni

$$= \left(-\frac{1}{4}\right)^{n+1} \cdot \mu(-n-1)$$

Odnosno za grupu sa $\alpha = -1$ $z_1 = \frac{1}{4}$, ...

$$h_{ac}(n) = \frac{-1}{4} \cdot \left(\frac{1}{4}\right)^n \cdot \mu(-n-1)$$

$$= -\left(\frac{1}{4}\right)^{n+1} \cdot \mu(-n-1)$$

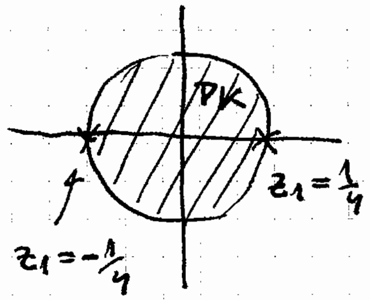
Područje konvergencije za anti-kauzalni slučaj ... pokazali smo:

$$|q| = \left|\frac{z}{z_1}\right| = \frac{|z|}{|z_1|} < 1$$

$$\Rightarrow |z| < |z_1| = \frac{1}{4}$$

Dakle neovisno o grupi PK za anti-kauzalni slučaj je:

$$|z| < \frac{1}{4}$$



c) Odrediti: $H(e^{j\omega})$ za sustav čije PK uključuje beskonačnost ... očito daje to slučaj za kauzalni imp. odziv.

$$H(e^{j\omega}) = H(z)\Big|_{z=e^{j\omega}} = \frac{1}{4 + \alpha \cdot e^{-j\omega}} =$$

$$\frac{1}{4 + \alpha\cos\omega - j\alpha\sin\omega}$$

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$$\left|H(e^{j\omega})\right| = \frac{1}{\sqrt{(4+\alpha\cos\omega)^2 + \alpha^2\sin^2\omega}} =$$

$$= \frac{1}{\sqrt{16 + 8\alpha\cos\omega + \underbrace{\alpha^2\cos^2\omega + \alpha^2\sin^2\omega}_{\alpha^2 = 1}}}$$

$$= \frac{1}{\sqrt{17 + 8\alpha\cos\omega}}$$

... ovisno o grupi $\alpha = 1$ ili $\alpha = -1$

Odredimo $|H(e^{j\omega})|$ u nekoliko karakt. točaka

$\alpha = 1$

$$\left|H(e^{j0})\right|_{\omega=0} = \frac{1}{\sqrt{17+8}} = \frac{1}{5}$$

$$\left|H(e^{j\pi})\right|_{\omega=\pi} = \frac{1}{\sqrt{17-8}} = \frac{1}{3}$$

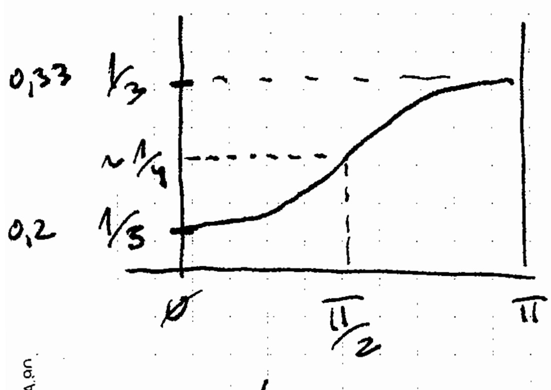
$$\left|H(e^{j\pi/2})\right|_{\omega=\pi/2} = \frac{1}{\sqrt{17+0}} \cong \frac{1}{4}$$

$\alpha = -1$

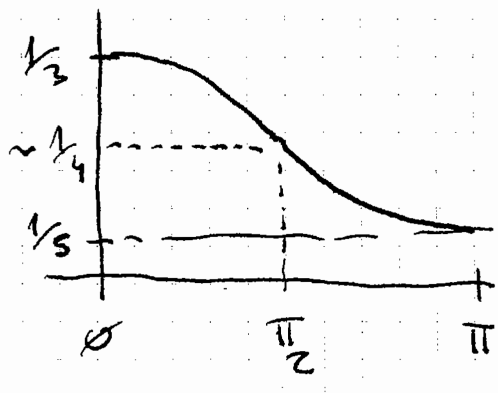
$$\left|H(e^{j0})\right| = \frac{1}{3}$$

$$\left|H(e^{j\pi})\right| = \frac{1}{5}$$

$$\left|H(e^{j\pi/2})\right| \cong \frac{1}{4}$$



$\alpha = 1$



$\alpha = -1$

Slično nalazimo i fazno-frekvencijsku karak.

$$\sphericalangle H(e^{j\omega}) = \sphericalangle \frac{1}{(4+\alpha\cos\omega) - j\alpha\sin\omega}$$

$$= -\sphericalangle\left((4+\alpha\cos\omega) - j\alpha\sin\omega\right)$$

$$= -\mathrm{atan}_2(\mathrm{Im}, \mathrm{Re})$$

$$= -\mathrm{atan}_2(-\alpha\sin\omega, 4+\alpha\cos\omega)$$

$$= \mathrm{atan}_2(\alpha\sin\omega, 4+\alpha\cos\omega)$$

Za par karakterističnih frekvencija $(\omega = 0, \frac{\pi}{2}, \pi)$

$\alpha = 1$

$$\sphericalangle H(e^{j0}) = \mathrm{atan}_2(\sin 0, 4+\cos 0) = \mathrm{atan}_2(0, 5) = 0$$

$$\sphericalangle H(e^{j\pi}) = \mathrm{atan}_2(\sin\pi, 4+\cos\pi) = \mathrm{atan}_2(0, 3) = 0$$

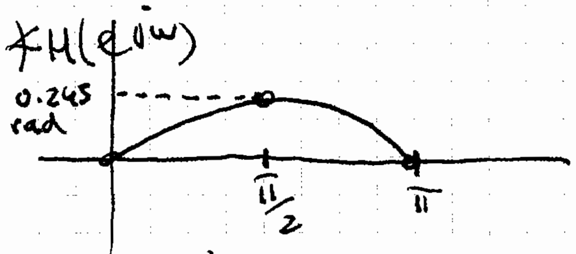
$$\sphericalangle H(e^{j\frac{\pi}{2}}) = \mathrm{atan}_2\left(\sin\tfrac{\pi}{2}, 4+\cos\tfrac{\pi}{2}\right) = \mathrm{atan}_2(1, 4) = 0.245 \text{ rad}$$

$\alpha = -1$

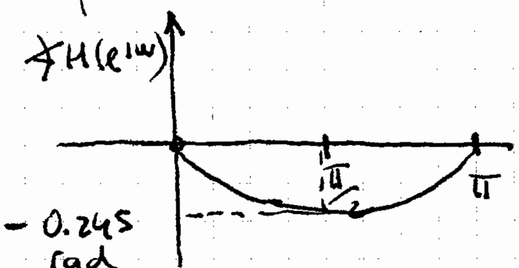
$$\sphericalangle H(e^{j0}) = \mathrm{atan}_2(-\sin 0, 4-\cos 0) = \mathrm{atan}_2(0, 3) = 0$$

$$\sphericalangle H(e^{j\pi}) = \mathrm{atan}_2(-\sin\pi, 4-\cos\pi) = \mathrm{atan}_2(0, 5) = 0$$

$$\sphericalangle H(e^{j\frac{\pi}{2}}) = \mathrm{atan}_2\left(-\sin\tfrac{\pi}{2}, 4-\cos\tfrac{\pi}{2}\right) = \mathrm{atan}_2(-1, 4) = -0.245 \text{ rad}$$



$\alpha = 1$



$\alpha = -1$

Signali i sustavi

Ponovljeni završni ispit – 3. srpnja 2007.

1. Diskretni kauzalni LTI sustav opisan je jednadžbom $y(n) - 0{,}5y(n-1) = u(n)$.

a) Odredite prisilni odziv ako je pobuda $u(n) = 2\sin(n\pi/2) + 3\sin(n\pi + 0{,}1\pi)$.

b) Izračunajte i skicirajte amplitudnu i faznu karakteristiku sustava.

2. Prijenosna funkcija kauzalnog diskretnog LTI sustava je

$$H(z) = \frac{z^3}{(z-\frac{1}{2})(z+\frac{1}{2})(z-\frac{3}{4})}.$$

Odredite matrice **A**, **B**, **C** i **D** kaskadne realizacije. Redoslijed sekcija kaskade neka bude prema redoslijedu polova, $z_1 = \frac{1}{2}$, $z_2 = -\frac{1}{2}$ i $z_3 = \frac{3}{4}$.

3. Kontinuirani sustav s više ulaza i izlaza (MIMO) opisan je matricama

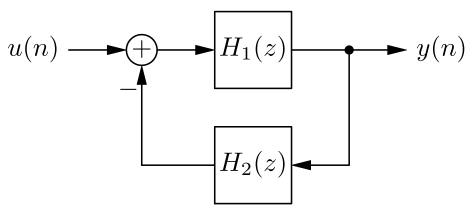
$$\mathbf{A} = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{i} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

a) Odredite prijenosnu matricu sustava.

b) Odredite matricu impulsnog odziva sustava.

c) Odredite odziv na pobudu $\mathbf{u}(t) = \begin{bmatrix} 4\,\mu(t) \\ \delta(t) \end{bmatrix}$

4. Na slici je prikazan složeni diskretni kauzalni LTI sustav. Ako je odziv cijelog sustava na jedinični skok $\mu(n)$ signal $y(n) = \{\underline{\frac{1}{2}}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \ldots\}$ i ako znate da je impulsni odziv drugog sustava $h_2(n) = \frac{1}{2}(-\frac{1}{2})^n + \frac{1}{2}(\frac{1}{2})^n$, $n \geq 0$ odredite prijenosnu funkciju prvog podsustava $H_1(z)$!



5. Zadan je signal $x(t) = e^{2t}\,\mu(-t)$.

a) Nađite Fourierovu transformaciju zadanog signala.

b) Nacrtajte amplitudni i fazni spektar.

c) Odredite energiju signala u vremenskoj domeni.

d) Odredite energiju signala u frekvencijskoj domeni korištenjem Parsevalove jednakosti.

Signali i sustavi

Ponovljeni završni ispit – 3. srpnja 2007.

1. Diskretni kauzalni LTI sustav opisan je jednadžbom $y(n) - 0{,}5y(n-1) = u(n)$.

a) Odredite prisilni odziv ako je pobuda $u(n) = 3\sin(n\pi/2 + 0{,}1\pi) + 2\sin(n\pi)$.

b) Izračunajte i skicirajte amplitudnu i faznu karakteristiku sustava.

2. Prijenosna funkcija kauzalnog diskretnog LTI sustava je

$$H(z) = \frac{z^3}{(z+\frac{1}{2})(z-\frac{1}{2})(z+\frac{3}{4})}.$$

Odredite matrice $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ i $\mathbf{D}$ kaskadne realizacije. Redoslijed sekcija kaskade neka bude prema redoslijedu polova, $z_1 = -\frac{1}{2}$, $z_2 = \frac{1}{2}$ i $z_3 = -\frac{3}{4}$.

3. Kontinuirani sustav s više ulaza i izlaza (MIMO) opisan je matricama

$$\mathbf{A} = \begin{bmatrix} 0 & -5 \\ -5 & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{i} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

a) Odredite prijenosnu matricu sustava.

b) Odredite matricu impulsnog odziva sustava.

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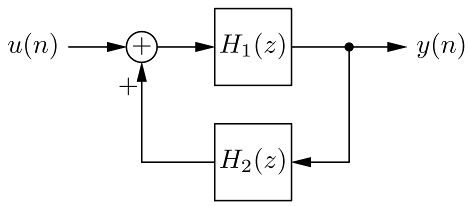
4. Na slici je prikazan složeni diskretni kauzalni LTI sustav. Ako je odziv cijelog sustava na jedinični skok $\mu(n)$ signal $y(n) = \{\underline{\frac{1}{2}}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \ldots\}$ i ako znate da je impulsni odziv drugog sustava $h_2(n) = \frac{1}{2}(-\frac{1}{2})^n + \frac{1}{2}(\frac{1}{2})^n$, $n \geq 0$ odredite prijenosnu funkciju prvog podsustava $H_1(z)$!



5. Zadan je signal $x(t) = e^{4t}\,\mu(-t)$.

a) Nađite Fourierovu transformaciju zadanog signala.

b) Nacrtajte amplitudni i fazni spektar.

c) Odredite energiju signala u vremenskoj domeni.

d) Odredite energiju signala u frekvencijskoj domeni korištenjem Parsevalove jednakosti.

Zadatak 1.

Zadan je sustav:

$$y[n] - 0.5\,y[n-1] = u[n]$$

Traži se prisilni odziv na harmonijsku pobudu oblika:

$$u[n] = A_1 \cdot \cos(\omega_1 n + \Theta_1) + A_2 \cdot \cos(\omega_2 n + \Theta_2)$$

Prisilni odziv mirnog kauzalnog LTI sustava na harmonijsku pobudu najlakše se nalazi primjenom frekvencijske karakteristike sustava $H(e^{j\omega})$. Prisilni odziv mora biti slijedećeg oblika:

$$y[n] = Y_1 \cdot \cos(\omega_1 n + \Psi_1) + Y_2 \cdot \cos(\omega_2 n + \Psi_2)$$

gdje

$$\left.\begin{aligned} Y_i &= A_i \cdot |H(e^{j\omega_i})| \\ \Psi_i &= \Theta_i + \sphericalangle H(e^{j\omega_i}) \end{aligned}\right\} \quad i = 1, 2$$

Dakle dovoljno je odrediti $H(e^{j\omega})$ i evaluirati je za $\omega = \omega_1$, $\omega = \omega_2$.

Zbog toga krećemo od pitanja B)

$$H(e^{j\omega}) = H(z)\Big|_{z = e^{j\omega}}$$

$$y[n] - 0.5\,y[n-1] = u[n] \xrightarrow{\mathcal{Z}} Y(z)(1 - 0.5z^{-1}) = U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - 0.5z^{-1}}$$

$$H(e^{j\omega}) = \frac{1}{1 - 0.5 \cdot e^{-j\omega}}$$

$$= \frac{1}{1 - 0.5\cos\omega + j0.5\sin\omega}$$

Nađimo amplitudno-frekv. karakt.

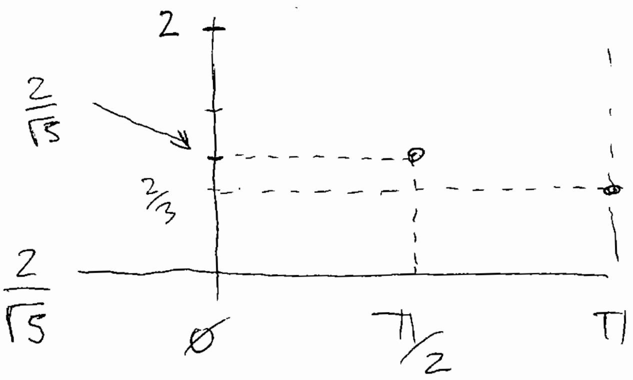
$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-\frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}}$$

$$= \frac{1}{\sqrt{1-\cos\omega + \underbrace{\frac{1}{4}\cos^2\omega + \frac{1}{4}\sin^2\omega}_{1/4}}} =$$

$$= \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}}$$

$$\measuredangle H(e^{j\omega}) = -\mathrm{atan}_2\left(\frac{1}{2}\sin\omega,\ 1-\frac{1}{2}\cos\omega\right)$$
$$= \mathrm{atan}_2\left(-\frac{1}{2}\sin\omega,\ 1-\frac{1}{2}\cos\omega\right)$$
$$= \mathrm{atan}_2(-\sin\omega,\ 2-\cos\omega)$$

Skicirajmo A-F i F-F karakteristiku u nekoliko točaka. Obzirom da nam za A) dio zadatka treba $H(e^{j\omega_1})$ & $H(e^{j\omega_2})$ za $\omega_1 = \frac{\pi}{2}$ & $\omega_2 = \pi$, neka dvije frekvencije budu upravo te, a dodajmo i treću za npr. $\omega_0 = \emptyset$.



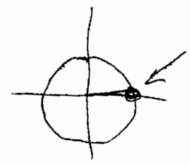
$$|H(e^{j\emptyset})| = \frac{1}{\sqrt{\frac{5}{4}-\frac{4}{4}}} = 2$$

$$|H(e^{j\omega_1})| = |H(e^{j\frac{\pi}{2}})| = \frac{1}{\sqrt{\frac{5}{4}-\emptyset}} = \frac{2}{\sqrt{5}}$$

$$|H(e^{j\omega_2})| = |H(e^{j\pi})| = \frac{1}{\sqrt{\frac{5}{4}+\frac{4}{4}}} = \frac{2}{3}$$

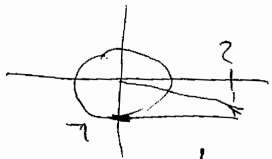
Slično određujemo F-F karakteristiku:

$$\measuredangle H(e^{j0}) = \text{atan}_2(-\sin 0,\ 2-\cos 0)$$
$$= \text{atan}_2(0,\ 1)$$
$$= 0$$



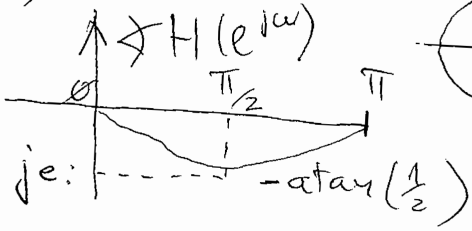
Im = 0
Re = 1
ρ = 0

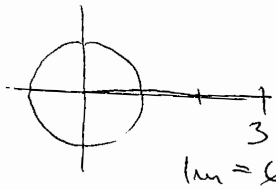
$$\measuredangle H(e^{j\frac{\pi}{2}}) = \text{atan}_2\left(-\sin\frac{\pi}{2},\ 2-\cos\frac{\pi}{2}\right)$$
$$= \text{atan}_2(-1,\ 2)$$
$$= -\text{atan}(1/2)$$



Im = -1
Re = 2
⇒ IV kvadrant

$$\measuredangle H(e^{j\pi}) = \text{atan}_2(-\sin\pi,\ 2-\cos\pi)$$
$$= \text{atan}_2(0,\ 3)$$
$$= 0$$





Im = 0
Re = 3
ρ = 0

Dakle prisilni odziv je:

$$y_{pris}[n] = A_1 \cdot |H(e^{j\omega_1})| \cos(\omega_1 n + \Theta_1 + \measuredangle H(e^{j\omega_1}))$$
$$+ A_2 |H(e^{j\omega_2})| \cdot \cos(\omega_2 n + \Theta_2 + \measuredangle H(e^{j\omega_2}))$$
$$= A_1 \cdot \frac{2}{\sqrt{5}} \cdot \cos(\omega_1 n + \Theta_1 + (-\text{atan}(1/2))) +$$
$$+ A_2 \cdot \frac{2}{3} \cdot \cos(\omega_2 n + \Theta_2 + 0)$$

U obje grupe je $\omega_1 = \pi/2$, $\omega_2 = \pi$, a konstante A_1, A_2, Θ_1 i Θ_2 izmore ovisno o grupi:

GRUPA A

$A_1 = 2$ $\quad \Theta_1 = -\frac{\pi}{2}$

$A_2 = 3$ $\quad \Theta_2 = \frac{\pi}{10} - \frac{\pi}{2}$

GRUPA B

$A_1 = 3$ $\quad \Theta_1 = \frac{\pi}{10} - \frac{\pi}{2}$

$A_2 = 2$ $\quad \Theta_2 = -\frac{\pi}{2}$

jer:

$$\sin\alpha = \cos\left(\alpha - \frac{\pi}{2}\right)$$

Dakle konačno rješenje pišemo:

za grupu A.

$$y_{pris}[n] = \frac{4}{\sqrt{5}} \cdot \cos\left(\frac{\pi}{2}n - \frac{\pi}{2} - \operatorname{atan}\left(\frac{1}{2}\right)\right) + 2\cos\left(\pi n + \frac{\pi}{10} - \frac{\pi}{2} + \phi\right)$$

$$= \frac{4}{\sqrt{5}} \cdot \cos\left(\frac{\pi}{2}n - \left(\frac{\pi}{2} + \operatorname{atan}\frac{1}{2}\right)\right) + 2\cos\left(\pi n - \frac{2\pi}{5}\right)$$

ili pomoću sinusa

$$= \frac{4}{\sqrt{5}} \cdot \sin\left(\frac{\pi}{2}n - \operatorname{atan}\left(\frac{1}{2}\right)\right) + 2\sin\left(\pi n + \frac{\pi}{10}\right)$$

Za grupu B.

$$y_{pris}[n] = \frac{6}{\sqrt{5}}\cos\left(\frac{\pi}{2}n + \frac{\pi}{10} - \frac{\pi}{2} - \operatorname{atan}\frac{1}{2}\right) + \frac{4}{3}\cos\left(\pi n - \frac{\pi}{2} + \phi\right)$$

$$= \frac{6}{\sqrt{5}} \cdot \cos\left(\frac{\pi}{2}n - \frac{2\pi}{5} - \operatorname{atan}\frac{1}{2}\right) + \frac{4}{3}\cos\left(\pi n - \frac{\pi}{2}\right)$$

$$= \frac{6}{\sqrt{5}} \sin\left(\frac{\pi}{2}n + \frac{\pi}{10} - \operatorname{atan}\frac{1}{2}\right) + \frac{4}{3}\sin\left(\pi n\right)$$

ZADATAK 2.

GRUPA A	GRUPA B
$z_1 = \frac{1}{2}$	$z_1 = -\frac{1}{2}$
$z_2 = -\frac{1}{2}$	$z_2 = \frac{1}{2}$
$z_3 = \frac{3}{4}$	$z_3 = -\frac{3}{4}$

5

Sustav

$$H(z) = \frac{z^3}{(z-z_1)(z-z_2)(z-z_3)}$$

realizirati kaskadnom realizacijom, te odrediti matrice A, B, C, D

Sustav razbijamo na kaskadu tri sustava prvog reda (zbog realnih polova)

$$H(z) = \underbrace{\frac{z}{z-z_1}}_{H_1(z)} \cdot \underbrace{\frac{z}{z-z_2}}_{H_2(z)} \cdot \underbrace{\frac{z}{z-z_3}}_{H_3(z)} = H_1(z) \cdot H_2(z) \cdot H_3(z)$$

ili sa negativnim potencijama varijable z:

$$H(z) = \frac{1}{1-z_1 z^{-1}} \cdot \frac{1}{1-z_2 z^{-1}} \cdot \frac{1}{1-z_3 z^{-1}} = H_1(z) \cdot H_2(z) \cdot H_3(z)$$

Sve tri sekcije imaju identičan oblik:

$$H_i(z) = \frac{1}{1-z_i z^{-1}} \qquad i = 1, 2, 3$$

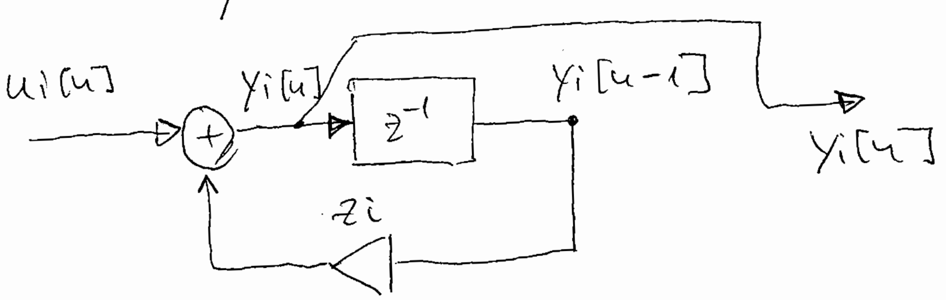
Odredimo realizaciju općenite sekcije $H_i(z)$

$$H_i(z) = \frac{Y_i(z)}{U_i(z)} = \frac{1}{1-z_i z^{-1}} \quad / \cdot U_i(z)(1-z_i z^{-1})$$

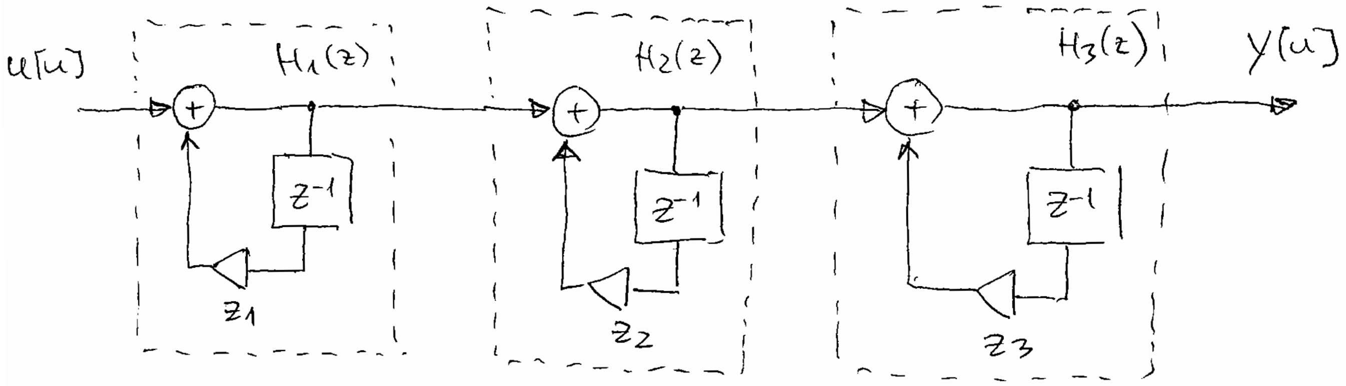
$$Y_i(z)(1-z_i z^{-1}) = U_i(z) \quad \Big) z^{-1}$$

$$y_i[n] - z_i y_i[n-1] = u_i[n]$$

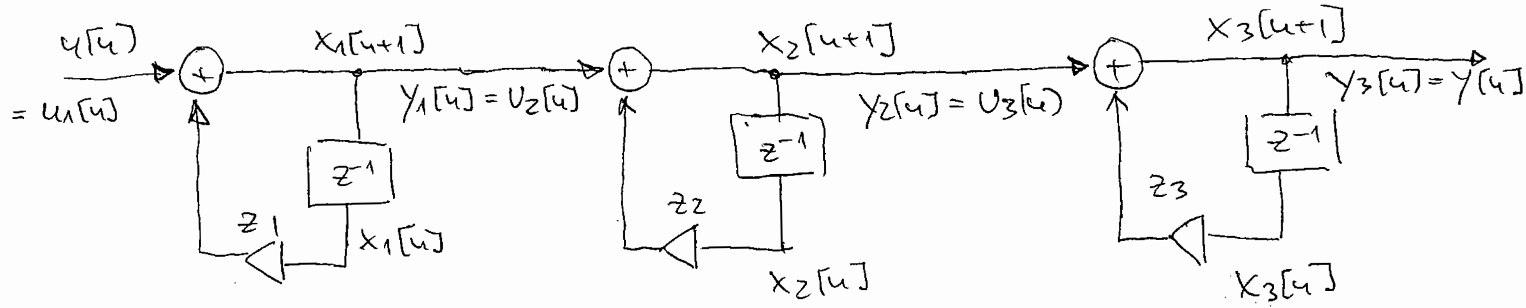
$$y_i[n] = u_i[n] + z_i \cdot y_i[n-1]$$



Cijeli sustav $H(z)$ je kaskada H_1, H_2 & H_3 6



Prema konvenciji sa predavanja (19. cjelina) kao varijable stanja odabiremo izlaze elemenata za kašnjenje, gdje indeks varijable stanja x_i odgovara indeksu sekcije 1. reda H_i



Pišemo jednadžbe:

$$x_i[n+1] = u_i[n] + z_i \cdot x_i[n] \qquad i = 1, 2, 3$$

$$x_1[n+1] = u[n] + z_1 \cdot x_1[n]$$

$$x_2[n+1] = u_2[n] + z_2 \, x_2[n] \qquad u_2[n] = y_1[n] = x_1[n+1]$$

$$= u[n] + z_1 x_1[n] + x_2[n] \cdot z_2$$

$$x_3[n+1] = u_3[n] + z_3 x_3[n] \qquad u_3[n] = y_2[n] = x_2[n+1]$$

$$= u[n] + z_1 x_1[n] + z_2 x_2[n] + z_3 x_3[n]$$

ili matrično:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \\ x_3[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 & 0 & 0 \\ z_1 & z_2 & 0 \\ z_1 & z_2 & z_3 \end{bmatrix}}_{A} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{B} \cdot u[n]$$

Napišimo i izlazne jednadžbe:

$$y[n] = y_3[n] = x_3[n+1]$$
$$= u[n] + z_1 x_1[n] + z_2 x_2[n] + z_3 x_3[n]$$

ili matrično:

$$\begin{bmatrix} y[n] \end{bmatrix} = \underbrace{\begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix}}_{C} \begin{bmatrix} x_1[n] \\ x_2[n] \\ x_3[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} \cdot u[n]$$

Dakle matrice koje opisuju nacrtanu realizaciju sustava su:

$$A = \begin{bmatrix} z_1 & 0 & 0 \\ z_1 & z_2 & 0 \\ z_1 & z_2 & z_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} z_1 & z_2 & z_3 \end{bmatrix} \quad D = \begin{bmatrix} 1 \end{bmatrix}$$

Uvrstimo vrijednosti polova

Grupa A

$$A = \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1/2 & -1/2 & 3/4 \end{bmatrix} \qquad C = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}$$

Grupa B

$$A = \begin{bmatrix} -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 \\ -1/2 & 1/2 & -3/4 \end{bmatrix} \qquad C = \begin{bmatrix} -1/2 & \frac{1}{2} & -\frac{3}{4} \end{bmatrix}$$

3. A)

$$A = \begin{bmatrix} 0 & -4 \\ -4 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

a) $H(s) = C(sI-A)^{-1}B + D$

$$(sI-A)^{-1} = \begin{bmatrix} s & 4 \\ 4 & s \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s & -4 \\ -4 & s \end{bmatrix}}{s^2-16} = \begin{bmatrix} \frac{s}{s^2-16} & \frac{-4}{s^2-16} \\ \frac{-4}{s^2-16} & \frac{s}{s^2-16} \end{bmatrix}$$

$$H(s) = \begin{bmatrix} \frac{s}{s^2-16} & \frac{-4}{s^2-16} \\ \frac{-4}{s^2-16} & \frac{s}{s^2-16} \end{bmatrix}$$

b) $h(t) = ?$

$$\frac{s}{s^2-16} = \frac{A}{s-4} + \frac{B}{s+4} \qquad\qquad \frac{-4}{s^2-16} = \frac{A}{s-4} + \frac{B}{s+4}$$

$A+B = 1$ $\qquad\qquad$ $A+B=0 \quad A=-B$

$-4B+4A=0$ $\qquad\qquad$ $-4B+4A=-4$

$A=B=\frac{1}{2}$ $\qquad\qquad$ $-8B=-4$

$\qquad\qquad$ $B=\frac{1}{2} \qquad A=-\frac{1}{2}$

$$H(s) = \begin{bmatrix} \frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} & \frac{-\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} \\ \frac{-\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} & \frac{\frac{1}{2}}{s-4} + \frac{\frac{1}{2}}{s+4} \end{bmatrix}$$

$$h(t) = \begin{bmatrix} \left(\frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}\right)\mu(t) & \left(-\frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}\right)\mu(t) \\ \left(-\frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}\right)\mu(t) & \left(\frac{1}{2}e^{4t} + \frac{1}{2}e^{-4t}\right)\mu(t) \end{bmatrix}$$

c) $u(t) = \begin{bmatrix} 4\mu(t) \\ \delta(t) \end{bmatrix}$

$$U(s) = \begin{bmatrix} \frac{4}{s} \\ 1 \end{bmatrix}$$

$$y(s) = H(s)\cdot U(s) = \begin{bmatrix} \frac{4}{s^2-16} - \frac{4}{s^2-16} \\ \frac{-16}{s(s^2-16)} + \frac{s}{s^2-16} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\cancel{-16+s^2}}{s\cancel{(s^2-16)}} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

$$y(t) = \begin{bmatrix} 0 \\ \mu(t) \end{bmatrix}$$

B 3.

$$A=\begin{bmatrix}0 & -5\\ -5 & 0\end{bmatrix}\qquad B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\qquad D=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

a)

$$\dot{x}=Ax+Bu \;\rightarrow\; sX=Ax+Bu$$
$$y=Cx+Du \qquad (sI-A)X=BU$$
$$X=(sI-A)^{-1}BU$$
$$y=(C(sI-A)^{-1}B+D)U$$

$$H(s)=C(sI-A)^{-1}B+D$$

$$sI-A=\begin{bmatrix}s & 5\\ 5 & s\end{bmatrix}$$

$$(sI-A)^{-1}=\left(\begin{array}{cc|cc}s & 5 & 1 & 0\\ 5 & s & 0 & 1\end{array}\right)\sim\left[\begin{array}{cc|cc}1 & \frac{5}{s} & \frac{1}{s} & 0\\ 5 & s & 0 & 1\end{array}\right]\sim\left[\begin{array}{cc|cc}1 & \frac{5}{s} & \frac{1}{s} & 0\\ 0 & s-\frac{25}{s} & -\frac{5}{s} & 1\end{array}\right]$$

$$\sim\left[\begin{array}{cc|cc}1 & \frac{5}{s} & \frac{1}{s} & 0\\ 0 & 1 & -\frac{5}{\cancel{s}}\cdot\frac{\cancel{s}}{-s^2+25} & \frac{s}{s^2-25}\end{array}\right]\sim\left[\begin{array}{cc|cc}1 & 0 & \frac{1}{s}+\frac{-25}{(25-s^2)s} & \frac{-5s}{s(s^2-25)}\\ 0 & 1 & \frac{5}{25-s^2} & \frac{s}{s^2-25}\end{array}\right]$$

$$\sim\left[\begin{array}{cc|cc}1 & 0 & \frac{s^2-25+25}{\cancel{s}(s^2-25)} & \frac{-5}{(s^2-25)}\\ 0 & 1 & \frac{5}{25-s^2} & \frac{s}{s^2-25}\end{array}\right]$$

$$C(sI-A)^{-1}B=\begin{bmatrix}\frac{s}{s^2-25} & \frac{-5}{s^2-25}\\ \frac{5}{25-s^2} & \frac{s}{s^2-25}\end{bmatrix}=H(s)$$

b)

ili

$$(sI-A)^{-1}=\frac{\operatorname{adj}(sI-A)}{\det(sI-A)}=\frac{1}{s^2-25}\cdot\begin{bmatrix}s & -5\\ -5 & s\end{bmatrix}=\begin{bmatrix}\frac{s}{s^2-25} & \frac{-5}{s^2-25}\\ \frac{-5}{s^2-25} & \frac{s}{s^2-25}\end{bmatrix}$$

$$\frac{s}{s^2-25}=\frac{A}{s-5}+\frac{B}{s+5}$$
$$A+B=1$$
$$5A-5B=0$$
$$A=B=\frac{1}{2}$$

$$\frac{-5}{s^2-25}=\frac{A}{s-5}+\frac{B}{s+5}$$
$$A+B=0 \qquad A=-B$$
$$-5B+5A=-5$$
$$-5B-5B=-5$$
$$B=\frac{1}{2}$$
$$A=-\frac{1}{2}$$

$$H(s)=\begin{bmatrix}\frac{\frac{1}{2}}{s-5}+\frac{\frac{1}{2}}{s+5} & \frac{-\frac{1}{2}}{s-5}+\frac{\frac{1}{2}}{s+5}\\ \frac{-\frac{1}{2}}{s-5}+\frac{\frac{1}{2}}{s+5} & \frac{\frac{1}{2}}{s-5}+\frac{1/2}{s+5}\end{bmatrix}$$

$$h(t)=\begin{bmatrix}\left(\frac{1}{2}e^{5t}+\frac{1}{2}e^{-5t}\right)\mu(t) & \left(-\frac{1}{2}e^{5t}+\frac{1}{2}e^{-5t}\right)\mu(t)\\ \left(-\frac{1}{2}e^{5t}+\frac{1}{2}e^{-5t}\right)\mu(t) & \left(\frac{1}{2}e^{5t}+\frac{1}{2}e^{-5t}\right)\mu(t)\end{bmatrix}$$

c) $u(t) = \begin{bmatrix} 5\mu(t) \\ \delta(t) \end{bmatrix}$

$U(s) = \begin{bmatrix} \frac{5}{s} \\ 1 \end{bmatrix}$

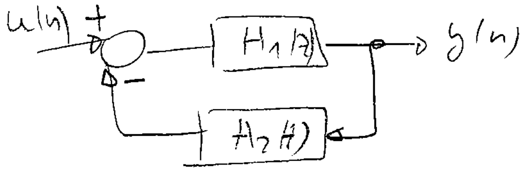
$$y(s) = H(s)\,U(s)$$

$$= \begin{bmatrix} \frac{s}{s^2-25} & \frac{-5}{s^2-25} \\ \frac{-5}{s^2-25} & \frac{s}{s^2-25} \end{bmatrix} \begin{bmatrix} \frac{5}{s} \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{s^2-25} - \frac{5}{s^2-25} \\ \frac{-25}{s(s^2-25)} + \frac{s}{s^2-25} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\cancel{-25+s^2}}{s(\cancel{s^2-25})} \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix}$$

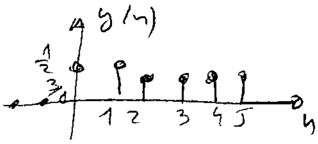
$y(t) = \begin{bmatrix} 0 \\ \mu(t) \end{bmatrix}$

4. A



$$u(n)=\mu(n) \longrightarrow U(z)=\frac{z}{z-1}$$

$$y(n)=\left\{\frac{1}{2},\frac{1}{2},\frac{3}{8},\frac{3}{8},\dots\right\}$$



$$Y(z)=\frac{1}{2}\cdot z^{0}+\frac{1}{2}z^{-1}+\frac{3}{8}\left(z^{-2}+z^{-3}+z^{-4}+\dots\right)$$

$$=\frac{1}{2}\left(1+\frac{1}{z}\right)+\frac{3}{8}z^{-2}\left(1+z^{-1}+z^{-2}+\dots\right)$$

$$\sum_{n=0}^{\infty} z^{-n}=\frac{1}{1-z^{-1}}=\frac{z}{z-1}$$

$$=\frac{1}{2}\cdot\frac{1+z}{z}+\frac{3}{8}\frac{\cancel{z}}{(z-1)z^{\cancel{2}}}$$

$$=\frac{4(z-1)(z+1)+3}{8z(z-1)}=\boxed{\frac{4z^{2}-1}{8z(z-1)}=\frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z(z-1)}=\frac{z^{2}-\frac{1}{4}}{2z(z-1)}}$$

$$H(z)=\frac{Y(z)}{U(z)}=\frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z(\cancel{z-1})}\cdot\frac{\cancel{z-1}}{z}=\boxed{\frac{z^{2}-\frac{1}{4}}{2z^{2}}=\frac{4z^{2}-1}{8z^{2}}}$$

$$H_2(n)=\frac{1}{2}\left(-\frac{1}{2}\right)^{n}+\frac{1}{2}\left(\frac{1}{2}\right)^{n}$$

$$=\boxed{\frac{1}{2}\frac{z}{z+\frac{1}{2}}+\frac{1}{2}\frac{z}{z-\frac{1}{2}}}=\boxed{\frac{z^{2}-\cancel{\frac{1}{2}z}+z^{2}+\cancel{\frac{1}{2}z}}{2(z+\frac{1}{2})(z-\frac{1}{2})}=\frac{z^{2}}{(z-\frac{1}{2})(z+\frac{1}{2})}}$$

$$\frac{\cancel{2}z^{2}}{\cancel{2}(z^{2}-\frac{1}{4})}=\frac{4z^{2}}{4z^{2}-1}$$

$$H(z)=\frac{Y(z)}{U(z)}$$

$$\left(U(z)-H_2(z)Y(z)\right)H_1(z)=Y(z)$$

$$H_1(z)U(z)-H_1(z)H_2(z)Y(z)=Y(z)$$

$$U(z)H_1(z)=Y(z)\left(1+H_1(z)H_2(z)\right)$$

$$\frac{Y(z)}{U(z)}=H=\frac{H_1(z)}{1+H_1(z)H_2(z)} \Rightarrow H(z)+H_1(z)H_2(z)H(z)=H_1(z)$$

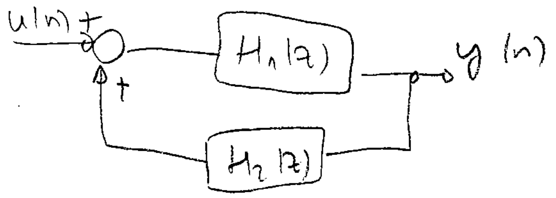
$$H_1(z)\left[-H_2H+1\right]=+H$$

$$H_1(z)=\frac{H(z)}{1-H(z)H_2(z)}$$

$$=\frac{\frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z^{2}}}{1-\frac{\cancel{(z-\frac{1}{2})}\cancel{(z+\frac{1}{2})}}{2\cancel{z^{2}}}\cdot\frac{\cancel{z^{2}}}{\cancel{(z-\frac{1}{2})}\cancel{(z+\frac{1}{2})}}}=\frac{\frac{z^{2}-\frac{1}{4}}{\cancel{2}z^{2}}}{\frac{1}{\cancel{2}}}=\frac{z^{2}-\frac{1}{4}}{z^{2}}$$

$$\boxed{H_1(z)=\frac{z^{2}-\frac{1}{4}}{z^{2}}=\frac{4z^{2}-1}{4z^{2}}}$$

4.



$u(n) = \mu(n)$

$U(z) = \frac{z}{z-1}$

$y(n) = \{\frac{1}{2}, \frac{1}{2}, \frac{3}{8}, \frac{3}{8}, \frac{3}{8}, \ldots\}$

$y(z) = \frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z(z-1)}$

$H(z) = \frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z^2}$

$H_2(z) = \frac{z^2}{(z-\frac{1}{2})(z+\frac{1}{2})}$

$[U(z) + H_2(z)y(z)]H_1(z) = y(z)$

$U \cdot H_1 + H_1 H_2 y = y$

$U H_1 = y(1 - H_1 H_2)$

$\frac{y(z)}{U(z)} = \frac{H_1(z)}{1 - H_1(z)H_2(z)} = H(z)$

$H_1 = H - H_1 H_2 H$

$H_1(1 + H_2 H) = H$

$H_1 = \frac{H}{1+H_2 H} = \frac{\frac{(z-\frac{1}{2})(z+\frac{1}{2})}{2z^2}}{1 + \frac{\cancel{z^2}}{\cancel{(z-\frac{1}{2})(z+\frac{1}{2})}} \cdot \frac{\cancel{(z-\frac{1}{2})(z+\frac{1}{2})}}{2\cancel{z^2}}}$

$H_1(z) = \frac{\frac{z^2-\frac{1}{4}}{\cancel{2}z^2}}{\frac{3}{\cancel{2}}} = \frac{(z^2-\frac{1}{4})}{3z^2}$

5. $x(t) = e^{2t}\mu(-t)$

a) Fourierova transformacija

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} e^{2t}\mu(-t)\, e^{-j\omega t}\, dt$$

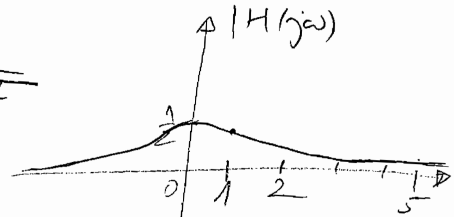
$$= \int_{-\infty}^{0} e^{2t - j\omega t}\, dt$$

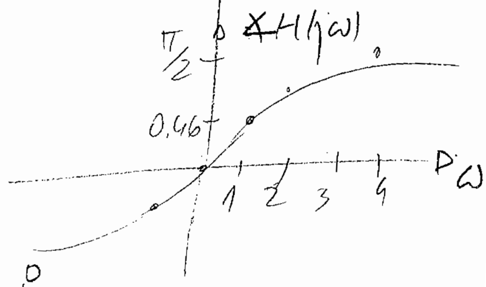
$$= \frac{e^{(2-j\omega)t}}{2-j\omega}\Bigg|_{-\infty}^{0} = \frac{1}{2-j\omega} = \frac{1}{2-j\omega}\cdot\frac{2+j\omega}{2+j\omega} = \frac{2+j\omega}{4+\omega^2}$$

b)

$$|H(j\omega)| = \sqrt{\frac{2^2+\omega^2}{(4+\omega^2)^2}} = \sqrt{\frac{\cancel{2^2+\omega^2}}{(4+\omega^2)^{\cancel{2}}}} = \frac{1}{\sqrt{4+\omega^2}}$$

$$\measuredangle H(j\omega) = \operatorname{arctg} \frac{\frac{\omega}{\cancel{4+\omega^2}}}{\frac{2}{\cancel{4+\omega^2}}} = \operatorname{arctg} \frac{\omega}{2}$$





c)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$= \int_{-\infty}^{\infty} |e^{2t}\mu(t)|^2 dt = \int_{-\infty}^{0} e^{4t} dt = \frac{e^{4t}}{4}\Bigg|_{-\infty}^{0} = \frac{1}{4}$$

d)

$$E_x = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(e^{j\omega})|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{1}{2-j\omega}\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{4+\omega^2} d\omega =$$

$$= \frac{1}{2\pi}\frac{1}{2} \operatorname{arctg} \frac{1}{2}\omega \Bigg|_{-\infty}^{\infty} = \frac{1}{2}\frac{1}{2\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{2}\cdot\frac{1}{2\pi} = \frac{1}{4}$$

5. B

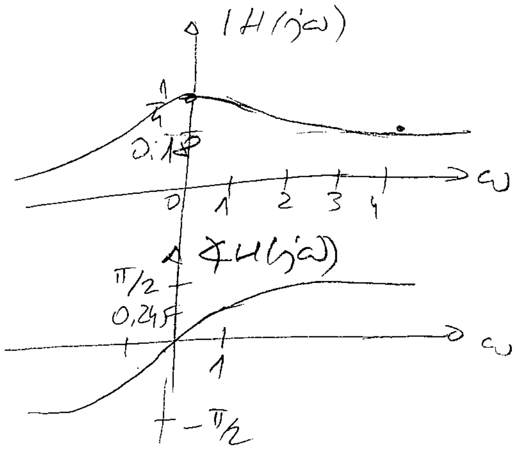
$x(t) = e^{4t}\mu(-t)$

a) Fourierova transformacija

$$X(j\omega) = \int_{-\infty}^{\infty} e^{4t}\mu(-t)\,e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{0} e^{4t}\,e^{-j\omega t}\,dt$$
$$= \int_{-\infty}^{0} e^{(4-j\omega)t}\,dt$$
$$= \frac{e^{(4-j\omega)t}}{4-j\omega}\Bigg|_{-\infty}^{0} = \frac{1}{4-j\omega} = \frac{4+j\omega}{16+\omega^2}$$

b) $$|H(j\omega)| = \sqrt{\frac{1}{16+\omega^2}}$$

$$\measuredangle H(j\omega) = \mathrm{arctg}\frac{\frac{\omega}{16+\omega^2}}{\frac{4}{16+\omega^2}} = \mathrm{arctg}\,\frac{\omega}{4}$$



c) $$E_x = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$
$$= \int_{-\infty}^{\infty} |e^{4t}\mu(-t)|^2\,dt$$
$$= \int_{-\infty}^{0} e^{8t}\,dt = \frac{e^{8t}}{8}\Bigg|_{-\infty}^{0} = \frac{1}{8}$$

d) $$E_x = \frac{1}{2\pi}\int_{+\infty}^{\infty} |X(j\omega)|^2\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{4-j\omega}\right|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{16+\omega^2}\,d\omega$$
$$= \frac{1}{2\pi}\,\frac{1}{4}\,\mathrm{arctg}\,\frac{\omega}{4}\Bigg|_{-\infty}^{\infty} = \frac{1}{2\pi}\cdot\frac{1}{4}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{1}{8}$$

Signali i sustavi

Završni ispit – 24. lipnja 2008.

1. Zadan je kontinuirani sustav $y''(t) + 5y'(t) + 6y(t) = u(t)$. Pronađite:

a) odziv sustava u vremenskoj domeni ako je sustav pobuđen signalom $u(t) = (12t + 16)\,\mu(t)$ te ako su početni uvjeti $y(0^-) = 3$, $y'(0^-) = -8$. Rješavati bez korištenja Laplaceove transformacije,

b) amplitudnu i faznu karakteristiku sustava,

c) impulsni odziv sustava. Riješiti pomoću Laplaceove transformacije.

2. Zadan je diskretni sustav

$$y(n) - \frac{1}{2}y(n-1) = u(n).$$

Ako je sustav pobuđen signalom $u(n) = 2\sin\left(\frac{\pi}{3}n + \frac{\pi}{6}\right)$ te ako je početni uvjet $y(-1) = 1$, odredite:

a) prijenosnu funkciju sustava,

b) je li sustav stabilan i obrazložite zašto,

c) frekvencijsku karakteristiku sustava i pomoću nje prisilni odziv sustava,

d) totalni odziv sustava.

3. Diskretan kauzalan LTI sustav zadan je prijenosnom funkcijom

$$H(z) = \frac{z^{-1} + z^{-2} + z^{-3}}{(1 + 2z^{-1} + 3z^{-2})\,(1 - 2z^{-1})}.$$

Pronađite odziv mirnog sustava na kauzalnu pobudu $u(n) = \{\underline{1}, 2, 3, 1, 2, 3, \ldots\}$. Uočite da se $(1, 2, 3)$ ponavlja.

4. Vremenski kontinuiran sustav zadan je sustavom diferencijalnih jednadžbi:

$$\begin{aligned} y_1'(t) - 2y_2(t) &= u(t) \\ y_2'(t) + 3y_1(t) + 5y_2(t) &= u(t) \end{aligned}$$

Odredite:

a) matrice sustava **A, B, C, D** uz varijable stanja $x_1 = y_1, x_2 = y_2$,

b) prijenosnu matricu sustava,

c) impulsni odziv sustava,

d) odziv sustava na pobudu $u(t) = \delta(t) + 6e^{3t}\,\mu(t)$ uz početne uvjete jednake nuli.

5. Impulsni odziv diskretnog LTI sustava je $h(n) = \{\ldots, 0, 0, \underline{1}, 2, 3, 2, 1, 0, 0, \ldots\}$. Uočite da je niz različit od nule samo u pet točaka. Odredite:

a) vremenski diskretnu Fourierovu transformaciju (DTFT) impulsnog odziva $h(n)$,

b) diskretnu Fourierovu transformaciju (DFT) u 5 točaka za točke različite od nule,

c) energiju signala,

d) koja od izračunatih transformacija odgovara frekvencijskoj karakteristici zadanog sustava.

A

1. $y''(t) + 5y'(t) + 6y(t) = u(t)$

a) $u(t) = (12t + 16)\mu(t)$

$y(0^-) = 3 \rightarrow y(0^+) = 3$

$y'(0^-) = -8 \rightarrow y'(0^+) = -8$

$s^2 + 5s + 6 = 0 \qquad (s+2)(s+3) = 0$

$s_1 = -2$

$s_2 = -3$

$y_h(t) = C_1 e^{-2t} + C_2 e^{-3t}$

$$\left.\begin{aligned} y_p &= A + Bt \\ y_p'(t) &= B \\ y_p''(t) &= 0 \end{aligned}\right\} \quad 5B + 6A + 6Bt = 12t + 16$$

$6B = 12 \qquad 5B + 6A = 16$

$B = 2 \qquad 6A = 6$

$A = 1$

$y_p(t) = 1 + 2t$

$y_t(t) = C_1 e^{-2t} + C_2 e^{-3t} + 1 + 2t$

$y_t'(t) = -2C_1 e^{-2t} - 3C_2 e^{-3t} + 2$

$y_t(0^+) = C_1 + C_2 + 1 = 3 \qquad C_1 + C_2 = 2$

$y_t'(0^+) = -2C_1 - 3C_2 + 2 = -8 \qquad -2C_1 - 3C_2 = -10$

$-C_2 = -6$

$\boxed{y_t(t) = (-4e^{-2t} + 6e^{-3t} + 1 + 2t)\mu(t)} \qquad C_2 = 6 \qquad C_1 = -4$

b) $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2 + 5s + 6}$

$U(s) = 1$

$H(j\Omega) = \dfrac{1}{(j\Omega)^2 + 5j\Omega + 6} = \dfrac{1}{6 - \Omega^2 + j \cdot 5\Omega}$

$|H(j\Omega)| = \dfrac{1}{\sqrt{(6-\Omega^2)^2 + 25\Omega^2}} = \dfrac{1}{\sqrt{36 - 12\Omega^2 + \Omega^4 + 25\Omega^2}}$

$= \dfrac{1}{\sqrt{36 + 13\Omega^2 + \Omega^4}}$

$\measuredangle H(j\Omega) = -\operatorname{arctg} \dfrac{5\Omega}{6 - \Omega^2}$

c) $u(t) = \delta(t)$

$U(s) = 1$

$Y(s) = \dfrac{1}{s^2 + 5s + 6} = \dfrac{A}{s+2} + \dfrac{B}{s+3}$

$A + B = 0 \qquad A = -B$

$3A + 2B = 1 \rightarrow -3B + 2B = 1$

$B = -1$

$A = 1$

$= \dfrac{1}{s+2} - \dfrac{1}{s+3}$

$y(t) = (e^{-2t} - e^{-3t})\mu(t)$

A

2. $y(n) - \frac{1}{2} y(n-1) = u(n)$

a) $Y(z) - \frac{1}{2} z^{-1} Y(z) = U(z)$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1 - \frac{1}{2} z^{-1}} = \frac{z}{z - \frac{1}{2}}$$

b) $z = \frac{1}{2}$ POL

$|z| < 1 \rightarrow$ STABILAN SUSTAV

c)
$$H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2} e^{-j\omega}} = \frac{1}{1 - \frac{1}{2}\cos\omega + \frac{1}{2} j \sin\omega}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} = \frac{1}{\sqrt{1 - \cos\omega + \underbrace{\frac{1}{4}\cos^2\omega + \frac{1}{4}\sin^2\omega}_{\frac{1}{4}}}}$$

$$= \frac{1}{\sqrt{\frac{5}{4} - \cos\omega}}$$

$$\measuredangle H(e^{j\omega}) = -\operatorname{arctg} \frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}$$

pobuda:

$$u(n) = 2 \sin\left(\frac{\pi}{3} n + \frac{\pi}{6}\right)$$

$$y_p(n) = U |H(e^{j\omega})| \sin(\omega n + \varphi + \measuredangle H(e^{j\omega}))$$

$$|H(e^{j\frac{\pi}{3}})| = \frac{1}{\sqrt{\frac{5}{4} - \cos\frac{\pi}{3}}} = \frac{1}{\sqrt{\frac{5}{4} - \frac{1}{2}}} = \frac{1}{\sqrt{\frac{3}{4}}} = \frac{2}{\sqrt{3}}$$

$$\measuredangle H(e^{j\frac{\pi}{3}}) = -\operatorname{arctg} \frac{\frac{1}{2} \cdot \frac{\sqrt{3}}{2}}{1 - \frac{1}{2} \cdot \frac{1}{2}} = -\operatorname{arctg} \frac{\frac{\sqrt{3}}{\cancel{4}}}{\frac{3}{\cancel{4}}} = -\operatorname{arctg} \frac{\sqrt{3}}{3} = -\frac{\pi}{6}$$

$$y_p(n) = 2 \cdot \frac{2}{\sqrt{3}} \sin\left(\frac{\pi}{3} n + \frac{\pi}{6} - \frac{\pi}{6}\right)$$
$$= \frac{4}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right)$$

d) $y_{TOT}(n) = y_h(n) + y_p(n)$

$y_h(n) = \left(\frac{1}{2}\right)^n \cdot C_1$

$y_{TOT}(n) = C_1 \left(\frac{1}{2}\right)^n + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right)$

$y_{TOT}(-1) = 1$

$$y_{TOT}(-1) = 2C_1 + \frac{4}{\sqrt{3}} \sin\left(-\frac{\pi}{3}\right) = 2C_1 + \frac{\overset{2}{\cancel{4}}}{\cancel{\sqrt{3}}} \cdot \frac{-\cancel{\sqrt{3}}}{\cancel{2}} = 2C_1 - 2 = 1$$

$$2C_1 = 3$$

$$C_1 = \frac{3}{2}$$

$$y_{TOT}(n) = \frac{3}{2} \left(\frac{1}{2}\right)^n + \frac{4}{\sqrt{3}} \sin\left(\frac{\pi}{3} n\right)$$

A

3. $H(z)=\dfrac{z^{-1}+z^{-2}+z^{-3}}{(1+2z^{-1}+3z^{-2})(1-2z^{-1})}=\dfrac{z^{-3}(z^2+z+1)}{z^{-2}\cancel{z^{-1}}(z^2+2z+3)(z-2)}$

$=\dfrac{z^2+z+1}{(z^2+2z+3)(z-2)}$

$u(n)=\{\underline{1},2,3,1,2,3,\dots\}$

$U(z)=z^0+2z^{-1}+3z^{-2}+z^{-3}+2z^{-4}+3z^{-5}+\dots$

$=z^0(1+z^{-3}+z^{-6}+\dots)+2z^{-1}(1+z^{-3}+z^{-6})+3z^{-2}(1+z^{-3}+z^{-6}+\dots)$

$=(1+z^{-3}+z^{-6}+\dots)(1+2z^{-1}+3z^{-2})$

$=\dfrac{z^2+2z+3}{z^2}\cdot\sum_{n=0}^{\infty}(z^{-3})^n=$

$=\dfrac{z^2+2z+3}{z^2}\;\dfrac{1}{1-z^{-3}}=\dfrac{z^2+2z+3}{\cancel{z^2}}\cdot\dfrac{z^{\cancel{3}}}{z^3-1}$

$=\dfrac{(z^2+2z+3)z}{(z-1)(z^2+z+1)}$

$H(z)=\dfrac{Y(z)}{U(z)}$

$Y(z)=H(z)\cdot U(z)$

$=\dfrac{\cancel{z^2+z+1}}{(\cancel{z^2+2z+3})(z-2)}\cdot\dfrac{(\cancel{z^2+2z+3})z}{(z-1)(\cancel{z^2+z+1})}=\dfrac{z}{(z-2)(z-1)}$

$\dfrac{Y(z)}{z}=\dfrac{A}{z-2}+\dfrac{B}{z-1}=\dfrac{1}{(z-2)(z-1)}$

$z^1:\quad A+B=0$

$z^0:\quad -A-2B=1$

$-B=1 \qquad A=1$

$B=-1$

$Y(z)=\dfrac{z}{z-2}-\dfrac{z}{z-1}$

$y(n)=2^n\mu(n)-\mu(n)$

A

4. $y_1' - 2y_2 = u$
$y_2' + 3y_1 + 5y_2 = u$

a) $x_1 = y_1$
$x_2 = y_2$
$\dot{x}_1 = u + 2y_2$
$= u + 2x_2$
$\dot{x}_2 = u - 3y_1 - 5y_2$
$= u - 3x_1 - 5x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

b) $H(s) = C(sI - A)^{-1} B + D$

$$(sI - A)^{-1} = \begin{bmatrix} s & -2 \\ 3 & s+5 \end{bmatrix}^{-1} = \frac{1}{s^2+5s+6} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} s+5 & 2 \\ -3 & s \end{bmatrix} \frac{1}{s^2+5s+6} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+7}{s^2+5s+6} \\ \frac{s-3}{s^2+5s+6} \end{bmatrix}$$

c) $u(t) = \delta(t)$
$U(s) = 1$
$Y(s) = H(s)U(s)$

$$= \begin{bmatrix} \frac{s+7}{s^2+5s+6} \\ \frac{s-3}{s^2+5s+6} \end{bmatrix} = \begin{bmatrix} \frac{A_1}{s+2} + \frac{B_1}{s+3} \\ \frac{A_2}{s+2} + \frac{B_2}{s+3} \end{bmatrix}$$

$A_1 + B_1 = 1$
$3A_1 + 2B_1 = 7$
$-2A_1 - 2B_1 = -2$
$A_1 = 5$
$B_1 = -4$

$A_2 + B_2 = 1$
$3A_2 + 2B_2 = -3$
$-2A_2 - 2B_2 = -2$
$A_2 = -5$
$B_2 = 6$

$$= \begin{bmatrix} \frac{5}{s+2} - \frac{4}{s+3} \\ \frac{-5}{s+2} + \frac{6}{s+3} \end{bmatrix} \longrightarrow y(t) = \begin{matrix} (5e^{-2t} - 4e^{-3t})\mu(t) \\ (-5e^{-2t} + 6e^{-3t})\mu(t) \end{matrix}$$

d) $u(t) = \delta(t) + 6e^{3t}\mu(t)$
$U(s) = 1 + 6\frac{1}{s-3} = \frac{s+3}{s-3}$

$$Y(s) = \begin{bmatrix} \frac{s+7}{(s+2)\cancel{(s+3)}} \cdot \frac{\cancel{s+3}}{s-3} \\ \frac{\cancel{s-3}}{(s+2)\cancel{(s+3)}} \cdot \frac{\cancel{s+3}}{\cancel{s-3}} \end{bmatrix} = \begin{bmatrix} \frac{s+7}{(s+2)(s-3)} \\ \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{A}{s+2} + \frac{B}{s-3} \\ \frac{1}{s+2} \end{bmatrix}$$

$A + B = 1$
$-3A + 2B = 7$
$-2A - 2B = -2$
$-5A = 5$
$A = -1$
$B = 2$

$$= \begin{bmatrix} \frac{-1}{s+2} + \frac{2}{s-3} \\ \frac{1}{s+2} \end{bmatrix}$$

$$y(t) = \begin{bmatrix} (-e^{-2t} + 2e^{+3t})\mu(t) \\ (e^{-2t})\mu(t) \end{bmatrix}$$

5. $h(n) = \{\dots 0, \underline{1}, 2, 3, 2, 1, 0, \dots\}$

A

a) DTFT

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-i\omega n}$$

$$= 1e^{-i\omega\cdot 0} + 2e^{-i\omega 1} + 3e^{-i\omega\cdot 2} + 2e^{-i\omega\cdot 3} + e^{-i\omega\cdot 4}$$

$$= e^{-i\omega\cdot 2}\left[e^{j\omega 2} + e^{-i\omega\cdot 2} + 2\left(e^{+i\omega\cdot 1} + e^{-i\omega\cdot 1}\right) + 3e^{-i\omega\cdot 0}\right]$$

$$= e^{-j\omega\cdot 2}\left[2\cos 2\omega + 4\cos\omega + 3\right]$$

b) DFT

$$H(k) = \sum_{n=0}^{N-1} h(n)\, e^{-\frac{2\pi}{N}\cdot nk\cdot j}$$

$$= \sum_{n=0}^{4} h(n)\, e^{-j\cdot\frac{2\pi}{5}\cdot n\cdot k}$$

$$= e^{-i\frac{2\pi}{5}\cdot k\cdot 0} + 2e^{-i\frac{2\pi}{5}\cdot k\cdot 1} + 3e^{-i\frac{2\pi}{5}k\cdot 2} + 2e^{-i\frac{2\pi}{5}k\cdot 3} + e^{-i\frac{2\pi}{5}\cdot k\cdot 4} + \dots$$

$$= e^{-j\frac{4\pi}{5}k}\left(e^{j\frac{4\pi}{5}k} + 2\left(e^{j\frac{2\pi}{5}k} + e^{-i\frac{2\pi}{5}k}\right) + 3 + e^{-i\frac{4\pi}{5}k}\right]$$

$$= e^{-j\frac{4\pi}{5}k}\left[2\cos\frac{4\pi}{5}k + 4\cos\frac{2\pi}{5}k + 3\right]$$

c) $$E = \sum_{n=-\infty}^{\infty} |h(n)|^2$$

$$= 1^2 + 2^2 + 3^2 + 2^2 + 1^2 = 19$$

d) DTFT $H(e^{i\omega})$

$$H(z) = z^0 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4} + 0\dots$$

$$z = e^{i\omega}$$

$$H(e^{i\omega}) = e^{j\omega\cdot 0} + 2e^{-i\omega 1} + 3e^{-2i\omega} + 2e^{-i\omega\cdot 3} + e^{-i\omega\cdot 4}$$

$\downarrow$

frekvencijska karakteristika sustava odgovara DTFT transformaciji.

Signali i sustavi

Završni ispit – 24. lipnja 2008.

1. Zadan je kontinuirani sustav $y''(t) + 6y'(t) + 8y(t) = u(t)$. Pronađite:

a) odziv sustava u vremenskoj domeni ako je sustav pobuđen signalom $u(t) = (8t + 22)\,\mu(t)$ te ako su početni uvjeti $y(0^-) = -5$, $y'(0^-) = 3$. Rješavati bez korištenja Laplaceove transformacije,

b) amplitudnu i faznu karakteristiku sustava,

c) impulsni odziv sustava. Riješiti pomoću Laplaceove transformacije.

2. Zadan je diskretni sustav

$$y(n) - \frac{1}{\sqrt{2}}y(n-1) = u(n).$$

Ako je sustav pobuđen signalom $u(n) = 2\sin\left(\frac{\pi}{4}n + \frac{\pi}{4}\right)$ te ako je početni uvjet $y(-1) = 2$, odredite:

a) prijenosnu funkciju sustava,

b) je li sustav stabilan i obrazložite zašto,

c) frekvencijsku karakteristiku sustava i pomoću nje prisilni odziv sustava,

d) totalni odziv sustava.

3. Diskretan kauzalan LTI sustav zadan je prijenosnom funkcijom

$$H(z) = \frac{z^{-1} + z^{-2} + z^{-3}}{(3 + 2z^{-1} + z^{-2})\,(1 - 4z^{-1})}.$$

Pronađite odziv mirnog sustava na kauzalnu pobudu $u(n) = \{\underline{3}, 2, 1, 3, 2, 1, \ldots\}$. Uočite da se $(3, 2, 1)$ ponavlja.

4. Vremenski kontinuiran sustav zadan je sustavom diferencijalnih jednadžbi:

$$\begin{aligned} y_1'(t) - 2y_2(t) &= u(t) \\ y_2'(t) + 4y_1(t) + 6y_2(t) &= u(t) \end{aligned}$$

Odredite:

a) matrice sustava **A, B, C, D** uz varijable stanja $x_1 = y_1, x_2 = y_2$,

b) prijenosnu matricu sustava,

c) impulsni odziv sustava,

d) odziv sustava na pobudu $u(t) = \delta(t) + 8e^{4t}\,\mu(t)$ uz početne uvjete jednake nuli.

5. Impulsni odziv diskretnog LTI sustava je $h(n) = \{\ldots, 0, 0, \underline{3}, 2, 1, 2, 3, 0, 0, \ldots\}$. Uočite da je niz različit od nule samo u pet točaka. Odredite:

a) vremenski diskretnu Fourierovu transformaciju (DTFT) impulsnog odziva $h(n)$,

b) diskretnu Fourierovu transformaciju (DFT) u 5 točaka za točke različite od nule,

c) energiju signala,

d) koja od izračunatih transformacija odgovara frekvencijskoj karakteristici zadanog sustava.

B

1. $y''(t) + 6y'(t) + 8y(t) = u(t)$

a) homogeno:

$s^2 + 6s + 8 = 0$ $\quad (s+2)(s+4) = 0$

$s_1 = -2$

$s_2 = -4$

$y_h(t) = c_1 e^{-2t} + c_2 e^{-4t}$

partikularno

$u(t) = (8t + 22)\mu(t)$

$$\left.\begin{aligned} y_p(t) &= A + Bt \\ y_p'(t) &= B \\ y_p''(t) &= 0 \end{aligned}\right\} \quad 6B + 8A + 8Bt = 8t + 22$$

$8B = 8$ $\qquad 6B + 8A = 22$

$B = 1$ $\qquad 8A = 22 - 6 = 16$

$A = 2$

$y_p(t) = 2 + t$

totalno

$y(t) = (c_1 e^{-2t} + c_2 e^{-4t} + 2 + t)\mu(t)$

konstante

$y(0^-) = -5 \to y(0^+) = -5$

$y'(0^-) = 3 \to y'(0^+) = 3$

$y'(t) = -2c_1 e^{-2t} + (-4) c_2 e^{-4t} + 1$

$y(0^+) = c_1 + c_2 + 2 = -5$ $\qquad c_1 + c_2 = -7$

$y'(0^+) = -2c_1 - 4c_2 + 1 = 3$ $\qquad -2c_1 - 4c_2 = 2$

$2c_1 = -26$

$c_1 = -13$ $\qquad c_2 = 6$

$$\boxed{y(t) = (-13e^{-2t} + 6e^{-4t} + 2 + t)\mu(t)}$$

b) $H(s) = \dfrac{Y(s)}{U(s)} = \dfrac{1}{s^2 + 6s + 8}$

$H(j\Omega) = \dfrac{1}{(j\Omega)^2 + 6j\Omega + 8} = \dfrac{1}{8 - \Omega^2 + j \cdot 6\Omega}$

$|H(j\Omega)| = \dfrac{1}{\sqrt{(8-\Omega^2)^2 + 36\Omega^2}}$

$= \dfrac{1}{\sqrt{64 - 16\Omega^2 + \Omega^4 + 36\Omega^2}}$

$$\boxed{\begin{aligned} |H(j\Omega)| &= \frac{1}{\sqrt{64 + 20\Omega^2 + \Omega^4}} \\ \measuredangle H(j\Omega) &= -\operatorname{arctg} \frac{6\Omega}{8 - \Omega^2} \end{aligned}}$$

c) $U(s) = 1 \leftarrow u(t) = \delta(t)$

$Y(s) = \dfrac{1}{s^2 + 6s + 8} = \dfrac{1}{(s+2)(s+4)}$

$= \dfrac{A}{s+2} + \dfrac{B}{s+4} \longrightarrow (s+4)A + (s+2)B = 1$

$A + B = 0 \to A = -B$

$4A + 2B = 1 \to -4B + 2B = 1$

$-2B = 1 \to B = -\frac{1}{2}$

$A = \frac{1}{2}$

$= \frac{1}{2} \cdot \dfrac{1}{s+2} - \frac{1}{2} \dfrac{1}{s+4}$

$$\boxed{y(t) = \left(\tfrac{1}{2} e^{-2t} - \tfrac{1}{2} e^{-4t}\right)\mu(t)}$$

B

2. $y(n) - \frac{1}{\sqrt{2}} y(n-1) = u(n)$

a) Prijenosna funkcija

$$Y(z) - \frac{1}{\sqrt{2}} z^{-1} Y(z) = U(z)$$

$$H(z) = \frac{Y(z)}{U(z)} = \frac{1}{1-\frac{1}{\sqrt{2}}z^{-1}} = \boxed{\frac{z}{z-\frac{1}{\sqrt{2}}}}$$

b) $z - \frac{1}{\sqrt{2}} = 0$

$z = \frac{1}{\sqrt{2}} \to$ POL

$|z| < 1 \to$ MANJI JE OD 1 PO APS. VRIJEDNOSTI $\to$ STABILAN SUSTAV

c) $$H(e^{j\omega}) = \frac{1}{1-\frac{1}{\sqrt{2}}e^{-j\omega}} = \frac{1}{1-\frac{1}{\sqrt{2}}\cos\omega + j\frac{1}{\sqrt{2}}\sin\omega}$$

$$|H(e^{j\omega})| = \frac{1}{\sqrt{(1-\frac{1}{\sqrt{2}}\cos\omega)^2 + (\frac{1}{\sqrt{2}}\sin\omega)^2}} = \frac{1}{\sqrt{1-\frac{2}{\sqrt{2}}\cos\omega + \underbrace{\frac{1}{2}\cos^2\omega + \frac{1}{2}\sin^2\omega}_{\frac{1}{2}}}}$$

$$\boxed{|H(e^{j\omega})| = \frac{1}{\sqrt{\frac{3}{2} - \frac{2}{\sqrt{2}}\cos\omega}}}$$

$$\boxed{\measuredangle H(e^{j\omega}) = -\operatorname{arctg}\frac{\frac{1}{\sqrt{2}}\sin\omega}{1-\frac{1}{\sqrt{2}}\cos\omega}}$$

pobuda $u(n) = 2\sin(\frac{\pi}{4}n - \frac{\pi}{4})$

$\omega = \frac{\pi}{4}$

$$|H(e^{j\frac{\pi}{4}})| = \frac{1}{\sqrt{\frac{3}{2} - \frac{\cancel{2}}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\cancel{2}}}} = \frac{1}{\sqrt{\frac{1}{2}}} = \sqrt{2}$$

$$\measuredangle H(e^{j\frac{\pi}{4}}) = -\operatorname{arctg}\frac{\frac{1}{\sqrt{2}}\cdot\frac{\sqrt{2}}{2}}{1-\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{2}} = -\operatorname{arctg}\frac{\frac{1}{2}}{\frac{1}{2}} = -\operatorname{arctg} 1 = -\frac{\pi}{4}$$

$$y_p(n) = U\cdot|H(e^{j\omega})|\sin(\omega n + \varphi + \measuredangle H(e^{j\omega}))$$

$$= 2\cdot\sqrt{2}\sin(\frac{\pi}{4}n + \frac{\pi}{4} + (-\frac{\pi}{4}))$$

$$\boxed{y_p(n) = 2\sqrt{2}\sin(\frac{\pi}{4}n)}$$

d) $y_h(n) = C_1 g^n$

$= C_1 (\frac{1}{\sqrt{2}})^n$

$y_{TOT}(n) = C_1(\frac{1}{\sqrt{2}})^n + 2\sqrt{2}\sin\frac{\pi}{4}n$

$y_{TOT}(-1) = 2$

$$y_{TOT}(-1) = C_1\cdot\sqrt{2} + 2\sqrt{2}\sin(-\frac{\pi}{4}) = \sqrt{2}C_1 + \cancel{2}\sqrt{2}\cdot\frac{-\sqrt{2}}{\cancel{2}} = \sqrt{2}C_1 - 2 = 2$$

$$C_1 = \frac{4\sqrt{2}}{\sqrt{2}\sqrt{2}} = 2\sqrt{2}$$

$$\boxed{y_{TOT}(n) = 2\sqrt{2}(\frac{1}{\sqrt{2}})^n + 2\sqrt{2}\sin\frac{\pi}{4}n}$$

3. $H(z) = \dfrac{z^{-1} + z^{-2} + z^{-3}}{(3 + 2z^{-1} + z^{-2})(1 - 4z^{-1})}$ **B**

$$= \frac{\cancel{z^{-3}}(z^2 + z + 1)}{z^{-2} \cdot \cancel{z^{-1}}(3z^2 + 2z + 1)(z - 4)} = \frac{z^2 + z + 1}{(3z^2 + 2z + 1)(z - 4)}$$

$u(n) = \{\underline{3}, 2, 1, 3, 2, 1, \dots\}$

$$U(z) = 3z^0 + 2z^{-1} + z^{-2} + 3z^{-3} + 2z^{-4} + z^{-5} + \dots$$

$$= 3z^0(1 + z^{-3} + z^{-6} + \dots) + 2z^{-1}(1 + z^{-3} + z^{-6} + \dots) + z^{-2}(1 + z^{-3} + z^{-6} + \dots)$$

$$= (3z^0 + 2z^{-1} + z^{-2})(1 + z^{-3} + z^{-6} + \dots)$$

$$= \frac{3z^2 + 2z + 1}{z^2} \cdot \sum_{n=0}^{\infty} (z^{-3})^n$$

$$= \frac{3z^2 + 2z + 1}{z^2} \cdot \frac{1}{1 - z^{-3}}$$

$$= \frac{3z^2 + 2z + 1}{\cancel{z^2}} \cdot \frac{z^{\cancel{3}}}{z^3 - 1}$$

$$= \frac{(3z^2 + 2z + 1) \cdot z}{(z - 1)(z^2 + z + 1)}$$

$$H(z) = \frac{Y(z)}{U(z)}$$

$$Y(z) = H(z) \cdot U(z)$$

$$= \frac{\cancel{(3z^2 + 2z + 1)}z}{(z - 1)\cancel{(z^2 + z + 1)}} \cdot \frac{\cancel{z^2 + z + 1}}{(z - 4)\cancel{(3z^2 + 2z + 1)}}$$

$$= \frac{z}{(z - 1)(z - 4)}$$

$$\frac{Y(z)}{z} = \frac{1}{(z - 1)(z - 4)} = \frac{A}{z - 1} + \frac{B}{z - 4}$$

z^1: $A + B = 0$

z^0: $-4A - B = 1$

$-3A = 1$

$A = -\frac{1}{3}$ $\quad B = \frac{1}{3}$

$$Y(z) = -\frac{1}{3} \frac{z}{z - 1} + \frac{1}{3} \frac{z}{z - 4}$$

$$y(n) = -\frac{1}{3}\mu(n) + \frac{1}{3}(4)^n \mu(n)$$

B

4. $y_1' - 2y_2 = u$

$y_2' + 4y_1 + 6y_2 = u$

a) $x_1 = y_1$

$x_2 = y_2$

$\dot{x}_1 = u + 2y_2$

$= u + 2x_2$

$\dot{x}_2 = u - 4y_1 - 6y_2$

$= u - 4x_1 - 6x_2$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -4 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

b) $\dot{x} = Ax + Bu$

$y = Cx + Du$

$(sI - A)x = BU$

$x = (sI - A)^{-1} BU$

$y = (C(sI-A)^{-1}B + D)U$

$H(s) = \dfrac{Y(s)}{U(s)} = C(sI-A)^{-1}B + D$

$sI - A = \begin{bmatrix} s & -2 \\ 4 & s+6 \end{bmatrix}$

$(sI-A)^{-1} = \dfrac{1}{s^2+6s+8} \cdot \begin{bmatrix} s+6 & 2 \\ -4 & s \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+6}{s^2+6s+8} & \frac{2}{s^2+6s+8} \\ \frac{-4}{s^2+6s+8} & \frac{s}{s^2+6s+8} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+8}{s^2+6s+8} \\ \frac{s-4}{s^2+6s+8} \end{bmatrix}$$

c) Impulsni odziv $\to u(t) = \delta(t)$

$U(s) = 1$

$$H(s) = \begin{bmatrix} \frac{A_1}{(s+2)} + \frac{B_1}{(s+4)} \\ \frac{A_2}{s+2} + \frac{B_2}{s+4} \end{bmatrix}$$

$A_1 + B_1 = 1$

$4A_1 + 2B_1 = 8$

$-4A_1 - 4B_1 = -4$

$-2B_1 = 4$

$B_1 = -2$

$A_1 = 3$

$A_2 + B_2 = 1$

$4A_2 + 2B_2 = -4$

$-4A_2 - 4B_2 = -4$

$-2B_2 = -8$

$B_2 = 4$

$A_2 = -3$

$$= \begin{bmatrix} \frac{3}{s+2} - \frac{2}{s+4} \\ \frac{-3}{s+2} + \frac{4}{s+4} \end{bmatrix} \to h(t) = \begin{bmatrix} (3e^{-2t} - 2e^{-4t})\mu(t) \\ (-3e^{-2t} + 4e^{-4t})\mu(t) \end{bmatrix}$$

d) $u(t) = \delta(t) + 8e^{4t}\mu(t)$

$U(s) = 1 + 8\dfrac{1}{s-4} = \dfrac{s+4}{s-4}$

$$Y(s) = H(s)\cdot U(s) = \begin{bmatrix} \frac{(s+8)\cancel{(s+4)}}{(s+2)\cancel{(s+4)}(s-4)} \\ \frac{\cancel{(s-4)}\cancel{(s+4)}}{(s+2)\cancel{(s+4)}\cancel{(s-4)}} \end{bmatrix} = \begin{bmatrix} \frac{s+8}{(s+2)(s-4)} \\ \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{A_3}{s+2} + \frac{B_3}{s-4} \\ \frac{1}{s+2} \end{bmatrix}$$

$A_3 + B_3 = 1$

$\to -4A_3 + 2B_3 = 8$

$-2A_3 - 2B_3 = -2$

$-6A_3 = 6$

$A_3 = -1$

$B_3 = 2$

$$Y(s) = \begin{bmatrix} \frac{-1}{s+2} + \frac{2}{s-4} \\ \frac{1}{s+2} \end{bmatrix} \to y(t) = \begin{bmatrix} (-e^{-2t} + 2e^{4t})\mu(t) \\ e^{-2t}\mu(t) \end{bmatrix}$$

B

5. $h(n) = \{\ldots 0, \underline{3}, 2, 1, 2, 3, 0, \ldots\}$

a) DTFT

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= 3e^{-j\omega\cdot 0} + 2e^{-j\omega\cdot 1} + 1e^{-j\omega\cdot 2} + 2e^{-j\omega\cdot 3} + 3e^{-j\omega\cdot 4} + 0\ldots$$

$$= e^{-j\omega 2}(3\cdot 2\cos 2\omega + 2\cdot 2\cos\omega + 1)$$

$$= e^{-j\omega\cdot 2}(6\cos 2\omega + 4\cos\omega + 1)$$

b) DFT

$$H(k) = \sum_{n=0}^{N-1} h(n) e^{-\frac{2\pi}{N} nk\cdot j}$$

$$= 3e^{-\frac{2\pi}{5} 0 k j} + 2e^{-\frac{2\pi}{5}\cdot 1 k\cdot j} + e^{-\frac{2\pi}{5}\cdot 2kj} + 2e^{-\frac{2\pi}{5}\cdot 3kj} + 3e^{-\frac{2\pi}{5}\cdot 4kj}$$

$$= e^{-j\frac{4\pi}{5}\cdot k}\left[3\cdot 2\cos\left(\frac{4\pi}{5}k\right) + 2\cdot 2\cos\left(\frac{2\pi}{5}k\right) + 1\right]$$

$$= e^{-j\frac{4\pi}{5}k}\left[6\cos\frac{4\pi}{5}k + 4\cos\frac{2\pi}{5}k + 1\right)$$

c) $$E = \sum_{n=-\infty}^{\infty} |h(n)|^2$$

$$= 3^2 + 2^2 + 1^2 + 2^2 + 3^2$$

$$= 9 + 5 + 4 + 9 = 27$$

d)

$$H(z) = 3z^{-0} + 2z^{-1} + 1z^{-2} + 2z^{-3} + 3z^{-4} + \ldots$$

$$z = e^{+j\omega}$$

$$H(e^{j\omega}) = 3 + 3e^{-j\omega} + e^{-2j\omega} + 2e^{-j\omega\cdot 3} + 3e^{-j\omega\cdot 4}$$

što odgovara DTFT transformaciji

Signali i sustavi

Ponovljeni završni ispit – 3. srpnja 2008.

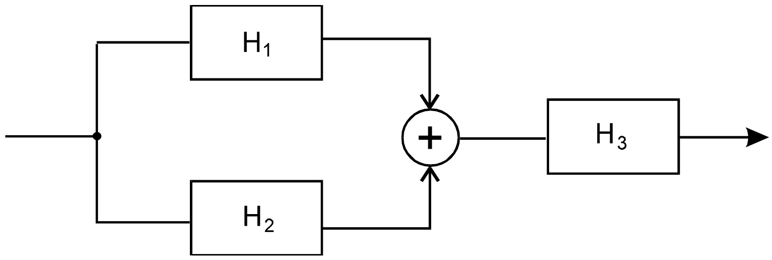
1. Zadan je impulsni odziv kontinuiranog LTI sustava $h(t) = 2te^{-t}\,\mu(t)$. Pronađite:

a) prijenosnu funkciju sustava,

b) amplitudnu i faznu karakteristiku sustava (ne treba crtati),

c) odziv sustava, ako je sustav pobuđen signalom $u(t) = 2\,\mu(t)$ te ako su početni uvjeti $y(0^-) = 2, y'(0^-) = 0$.

2. Zadan je složeni diskretni sustav prema slici. Nađite odziv na jediničnu stepenicu $\mu(n)$, trećeg podsustava ako je poznat impulsni odziv prvog podsustava $h_1(n) = \{\underline{1}, 0, 1, 0, 1, 0, \ldots\}$, odziv na jediničnu stepenicu drugog podsustava $y_2(n) = \frac{1}{4} + \frac{1}{2}n - \frac{1}{4}(-1)^n$, $n \geq 0$, te impulsni odziv cijelog sustava $h(n) = \{\underline{1}, 1\}$



3. Diskretan kauzalan LTI sustav zadan je jednadžbom diferencija:

$$y(n) - \frac{1}{9}y(n-2) = u(n).$$

Odredite:

a) odziv sustava, ako je sustav pobuđen signalom $u(n) = 80 \cdot 3^n\,\mu(n)$ te ako su početni uvjeti $y(-1) = 18, y(-2) = 0$,

b) je li sustav stabilan. Objasnite.

4. Vremenski kontinuiran sustav zadan je matricama **A, B, C, D**:

$$\mathbf{A} = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 & 4 \end{bmatrix}, \quad \text{i} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

a) Koliko ovaj sustav ima ulaza, a koliko izlaza,

b) Pronađite prijenosnu matricu sustava,

c) Odredite impulsni odziv.

5. Zadan je signal $x(t) = e^{2t}\,\mu(-t) + e^{-2t}\,\mu(t)$. Odredite:

a) vremenski kontinuiranu Fourierovu transformaciju (CTFT) ovog signala te skicirajte amplitudni spektar,

b) energiju signala.

1. $h(t) = 2t e^{-t} \mu(t)$

a) $\boxed{H(s) = \frac{2}{(s+1)^2}}$

b) $H(j\omega) = \frac{2}{(j\omega+1)^2} = \frac{2}{1-\omega^2+2j\omega}$

$|H(j\omega)| = \frac{2}{\sqrt{(1-\omega^2)^2+4\omega^2}} = \frac{2}{\sqrt{1+2\omega^2+\omega^4}} = \frac{2}{\sqrt{(1+\omega^2)^2}}$

$$\boxed{|H(j\omega)| = \frac{2}{|1+\omega^2|}}$$

$$\boxed{\measuredangle H(j\omega) = -\operatorname{arctg} \frac{2\omega}{1-\omega^2}}$$

c) diferencijalna jednadžba:

$\frac{Y(s)}{U(s)} = \frac{2}{(s+1)^2}$

$(s+1)^2 Y(s) = 2U(s)$

$y''(t) + 2y'(t) + y(t) = 2u(t)$

homogeno rješenje

$s^2 + 2s + 1 = 0$

$s_{1,2} = -1$

$$\boxed{y_h(t) = (C_1 + C_2 t) e^{-t}}$$

partikularno rješenje

$y''(t) + 2y'(t) + y(t) = 4\mu(t)$

$y_p(t) = K$

$K = 4$

$$\boxed{y_p(t) = 4}$$

početni uvjeti

$$\boxed{\begin{aligned} y(0^-) &= y(0^+) = 2 \\ y'(0^-) &= y'(0^+) = 0 \end{aligned}}$$

totalno rješenje

$y_{tot}(t) = (C_1 + C_2 t) e^{-t} + 4$

$y'_{tot}(t) = -C_1 e^{-t} - C_2 t e^{-t} + C_2 e^{-t}$

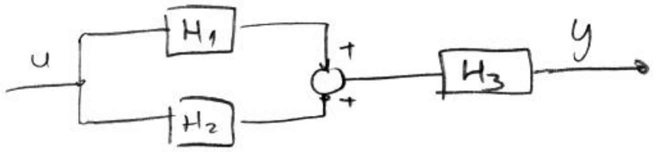
$y_{tot}(0^+) = C_1 + 4 = 2 \qquad C_1 = 2-4 = -2$

$y'_{tot}(0^+) = -C_1 + C_2 = 0 \qquad C_2 = C_1 = -2$

$$\boxed{y_{tot}(t) = \left[(-2-2t)e^{-t} + 4\right] \mu(t)}$$

A

2.



$$H(z) = (H_1 + H_2)\, H_3$$

$$\boxed{H_3(z) = \frac{H(z)}{H_1(z) + H_2(z)}}$$

$h_1(n) = \{\underline{1}, 0, 1, 0, \dots\}$

$$H_1(z) = z^0 + z^{-2} + z^{-4} + \dots = \sum_{n=0}^{\infty} (z^{-2})^n = \frac{1}{1 - z^{-2}} = \frac{z^2}{z^2 - 1}$$

$y_2(n) = \frac{1}{4} + \frac{1}{2} n - \frac{1}{4}(-1)^n \quad , n \geq 0$

$$Y_2(z) = \frac{1}{4}\frac{z}{z-1} + \frac{1}{2}\frac{z}{(z-1)^2} - \frac{1}{4}\frac{z}{z+1}$$

$$= \frac{z(z^2-1) + 2z(z+1) - z(z-1)^2}{4(z-1)^2(z+1)} = \frac{\cancel{z^3} - \cancel{z} + 2z^2 + \cancel{2z} - \cancel{z^3} + 2z^2 - \cancel{z}}{4(z-1)^2(z+1)}$$

$$= \frac{z^2}{(z-1)^2(z+1)}$$

$u_2(n) = \mu(n)$

$$U_2(z) = \frac{z}{z-1}$$

$$H_2(z) = \frac{Y_2(z)}{U_2(z)} = \frac{z^{\cancel{2}}}{(z-1)^{\cancel{2}}(z+1)} \cdot \frac{\cancel{z-1}}{\cancel{z}} = \frac{z}{(z-1)(z+1)}$$

$h(n) = \{\underline{1}, 1\}$

$$H(z) = z^0 + z^{-1} = 1 + z^{-1} = \frac{1+z}{z}$$

$$H_3(z) = \frac{\frac{1+z}{z}}{\frac{z^2}{z^2-1} + \frac{z}{z^2-1}} = \frac{1+z}{z} \cdot \frac{1}{\frac{z(z+1)}{z^2-1}} = \frac{1+z}{z} \cdot \frac{(z-1)\cancel{(z+1)}}{z\cancel{(z+1)}} = \frac{z^2-1}{z^2}$$

$u_3(n) = \mu(n)$

$$U_3(z) = \frac{z}{z-1}$$

$$Y_3(z) = H_3(z) \cdot U_3(z) = \frac{\cancel{(z-1)}(z+1)}{z^{\cancel{2}}} \cdot \frac{\cancel{z}}{\cancel{z-1}} = \frac{z+1}{z} = 1 + z^{-1}$$

$$y_3(n) = \delta(n) + \delta(n-1) = \{\underline{1}, 1\}$$

3. A

$$y(n) - \frac{1}{9} y(n-2) = u(n)$$

a) homogeno:

$$q^2 - \frac{1}{9} = 0$$

$$q^2 = \frac{1}{9}$$

$$q_1 = \frac{1}{3} \qquad q_2 = -\frac{1}{3}$$

$$y_h = C_1 \left(\frac{1}{3}\right)^n + C_2 \left(-\frac{1}{3}\right)^n$$

partikularno

$$y(n) - \frac{1}{9} y(n-2) = 80 \cdot 3^n \mu(n)$$

$$y_p(n) = K \cdot 3^n$$

$$K \cdot \cancel{3^n} - \frac{1}{9} \cdot K \cdot \cancel{3^n} \cdot 3^{-2} = 80 \cdot \cancel{3^n}$$

$$K - \frac{1}{81} K = 80$$

$$\frac{80}{81} K = 80$$

$$K = 81$$

$$y_p = 81 \cdot 3^n$$

totalno:

$$y_{tot}(n) = C_1 \left(\frac{1}{3}\right)^n + C_2 \left(-\frac{1}{3}\right)^n + 81 \cdot 3^n$$

$$y_{tot}(-1) = 3C_1 - 3C_2 + 81 \frac{1}{3} = 3C_1 - 3C_2 + 27 = 18$$

$$y_{tot}(-2) = 9C_1 + 9C_2 + 81 \cdot \frac{1}{9} = 9C_1 + 9C_2 + 9 = 0$$

$$3C_1 - 3C_2 = -9 \qquad C_1 - C_2 = -3$$

$$9C_1 + 9C_2 = -9 \qquad C_1 + C_2 = -1$$

$$2C_1 = -4 \qquad C_2 = -1 - C_1$$

$$C_1 = -2 \qquad = -1 + 2$$

$$= 1$$

$$y_{tot}(n) = \left[-2 \cdot \left(\frac{1}{3}\right)^n + \left(-\frac{1}{3}\right)^n + 81 \cdot 3^n\right] \mu(n)$$

b) $\left. \begin{array}{l} |q_1| = \frac{1}{3} < 1 \\ |q_2| = \frac{1}{3} < 1 \end{array} \right\}$ stabilan sustav

A

4.

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -5 \end{bmatrix} \qquad B = \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix} \qquad C = [1 \quad 4] \qquad D = [0 \quad 0]$$

a) $D = [0 \quad 0]$

1 redak → 1 izlaz
2 stupca → 2 ulaza

b)

$$H(s) = C(sI - A)^{-1} B + D$$

$$(sI - A)^{-1} = \begin{bmatrix} s+4 & 0 \\ 0 & s+5 \end{bmatrix}^{-1} = \frac{1}{(s+4)(s+5)} \begin{bmatrix} s+5 & 0 \\ 0 & s+4 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+4} & 0 \\ 0 & \frac{1}{s+5} \end{bmatrix}$$

$$H(s) = [1 \quad 4] \begin{bmatrix} \frac{1}{s+4} & 0 \\ 0 & \frac{1}{s+5} \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix} + [0 \quad 0]$$

$$= \begin{bmatrix} \frac{1}{s+4} & \frac{4}{s+5} \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix} + [0 \quad 0]$$

$$= \begin{bmatrix} \frac{-3}{s+4} + \frac{4}{s+5} & \frac{-2}{s+4} + \frac{8}{s+5} \end{bmatrix}$$

c) $h(t) = \left[\left(-3e^{-4t} + 4e^{-5t}\right)\mu(t) \quad \left(-2e^{-4t} + 8e^{-5t}\right)\mu(t) \right]$

A

5. $x(t) = e^{2t}\mu(-t) + e^{-2t}\mu(t)$

a) CTFT

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[e^{2t}\mu(-t) + e^{-2t}\mu(t)\right] e^{-j\Omega t}\, dt$$

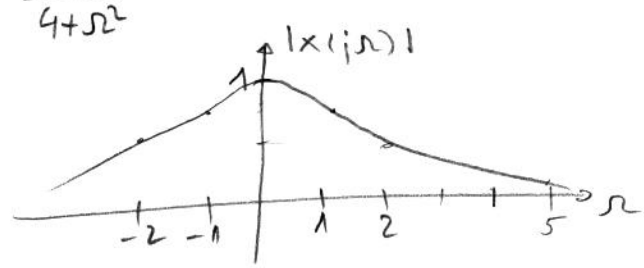
$$= \int_{-\infty}^{\infty} e^{(2-j\Omega)t}\mu(-t)\, dt + \int_{-\infty}^{\infty} e^{-(2+j\Omega)t}\mu(t)\, dt$$

$$= \int_{-\infty}^{0} e^{(2-j\Omega)t}\, dt + \int_{0}^{\infty} e^{-(2+j\Omega)t}\, dt$$

$$= \frac{e^{(2-j\Omega)t}}{2-j\Omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(2+j\Omega)t}}{-(2+j\Omega)}\Bigg|_{0}^{\infty} =$$

$$= \frac{1}{2-j\Omega} - \frac{1}{-(2+j\Omega)} = \frac{4}{4+\Omega^2}$$

$$|X(j\Omega)| = \frac{4}{4+\Omega^2}$$



b) $E_x = ?$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \left(e^{2t}\mu(-t) + e^{-2t}\mu(t)\right)^2 dt$$

$$= \int_{-\infty}^{\infty} e^{4t}\mu^2(-t)\, dt + \underbrace{\int_{-\infty}^{\infty} 2e^{2t}e^{-2t}\mu(-t)\mu(t)}_{\emptyset} + \int_{-\infty}^{\infty} e^{-4t}\mu^2(t)\, dt$$

$$= \int_{-\infty}^{0} e^{4t}\, dt + \int_{0}^{\infty} e^{-4t}\, dt$$

$$= \frac{e^{4t}}{4}\Bigg|_{-\infty}^{0} + \frac{e^{-4t}}{-4}\Bigg|_{0}^{\infty} =$$

$$= \frac{1}{4} - \frac{1}{-4} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Signali i sustavi

Ponovljeni završni ispit – 3. srpnja 2008.

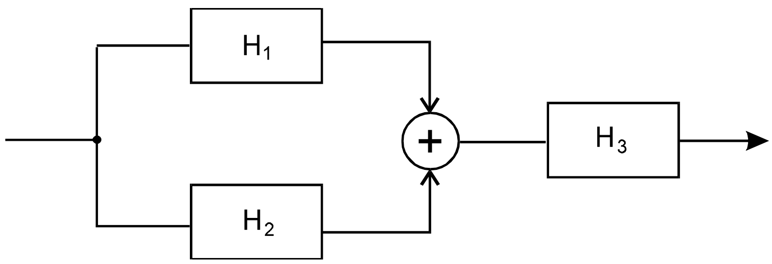
1. Zadan je impulsni odziv kontinuiranog LTI sustava $h(t) = 3te^{-t}\,\mu(t)$. Pronađite:

a) prijenosnu funkciju sustava,

b) amplitudnu i faznu karakteristiku sustava (ne treba crtati),

c) odziv sustava, ako je sustav pobuđen signalom $u(t) = 3\,\mu(t)$ te ako su početni uvjeti $y(0^-) = 3, y'(0^-) = 0$.

2. Zadan je složeni diskretni sustav prema slici. Nađite odziv na jediničnu stepenicu $\mu(n)$, trećeg podsustava ako je poznat impulsni odziv prvog podsustava $h_1(n) = \{\underline{1}, 0, 1, 0, 1, 0, \ldots\}$, odziv na jediničnu stepenicu drugog podsustava $y_2(n) = \frac{1}{4} + \frac{1}{2}n - \frac{1}{4}(-1)^n$, $n \geq 0$, te impulsni odziv cijelog sustava $h(n) = \{\underline{0}, 1\}$



3. Diskretan kauzalan LTI sustav zadan je jednadžbom diferencija:

$$y(n) - \frac{1}{4}y(n-2) = u(n).$$

Odredite:

a) odziv sustava, ako je sustav pobuđen signalom $u(n) = 15 \cdot 2^n\,\mu(n)$ te ako su početni uvjeti $y(-1) = 0, y(-2) = 12$,

b) je li sustav stabilan. Objasnite.

4. Vremenski kontinuiran sustav zadan je matricama **A, B, C, D**:

$$\mathbf{A} = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 5 & 1 \end{bmatrix}, \quad \text{i} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

a) Koliko ovaj sustav ima ulaza, a koliko izlaza,

b) Pronađite prijenosnu matricu sustava,

c) Odredite impulsni odziv.

5. Zadan je signal $x(t) = e^{3t}\,\mu(-t) + e^{-3t}\,\mu(t)$. Odredite:

a) vremenski kontinuiranu Fourierovu transformaciju (CTFT) ovog signala te skicirajte amplitudni spektar,

b) energiju signala.

B

1. $h(t) = 3te^{-t}\mu(t)$

a) $H(s) = \frac{3}{(s+1)^2}$

b) $H(j\omega) = \frac{3}{(j\omega+1)^2} = \frac{3}{1-\omega^2+2j\omega}$

$|H(j\omega)| = \frac{3}{\sqrt{(1-\omega^2)^2+4\omega^2}} = \frac{3}{\sqrt{1-2\omega^2+\omega^4+4\omega^2}} = \frac{3}{\sqrt{(1+\omega^2)^2}}$

$= \frac{3}{|1+\omega^2|}$

$\sphericalangle H(j\omega) = -\text{arctg}\,\frac{2\omega}{1-\omega^2}$

c) početni uvjeti:

$y(0^+) = y(0^-) = 3$

$y'(0^+) = y'(0^-) = 0$

diferencijalna jednadžba:

$\frac{Y(s)}{U(s)} = \frac{3}{(s+1)^2}$ $\quad y''(t) + 2y'(t) + y(t) = 3u(t)$

pobuda: $u(t) = 3\mu(t)$

$y''(t) + 2y'(t) + y(t) = 9\mu(t)$

homogeno

$s^2+2s+1=0$

$s_{1,2} = -1$

$y_h(t) = (C_1 + C_2 t)e^{-t}$

partikularno

$y_p(t) = K$

$K = 9$

totalno

$y_{tot}(t) = (C_1 + C_2 t)e^{-t} + 9$

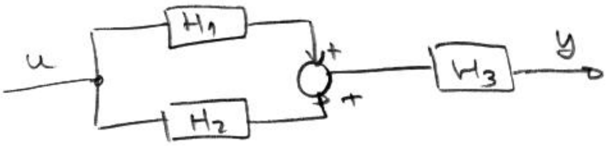
$y'_{tot}(t) = -C_1 e^{-t} - C_2 t e^{-t} + C_2 e^{-t}$

$y_{tot}(0^+) = C_1 + 9 = 3$ $\quad C_1 = 3 - 9 = -6$

$y'_{tot}(0^+) = -C_1 + C_2 = 0$ $\quad C_2 = C_1 = -6$

$$\boxed{y_{tot}(t) = [(-6-6t)e^{-t} + 9]\,\mu(t)}$$

2. B



$h_1(n) = \{\underline{1}, 0, 1, 0, 1, 0, \dots\}$

$$H_1(z) = z^0 + z^{-2} + z^{-4} + \dots$$
$$= \sum_{n=0}^{\infty} (z^{-2})^n = \frac{1}{1-z^{-2}} = \boxed{\frac{z^2}{z^2-1}}$$

$y_2(n) = \frac{1}{4} + \frac{1}{2} n - \frac{1}{4}(-1)^n \qquad n \geq 0$

$u_2(n) = \mu(n)$

$$y_2(z) = \frac{1}{4} \cdot \frac{z}{z-1} + \frac{1}{2} \frac{z}{(z-1)^2} - \frac{1}{4} \frac{z}{z+1}$$
$$= \frac{z(z^2-1) + 2z(z+1) - z(z-1)^2}{4(z-1)^2(z+1)} = \frac{\cancel{z^3} - \cancel{z} + 2z^2 + \cancel{2z} - \cancel{z^3} + 2z^2 - \cancel{z}}{4(z-1)^2(z+1)}$$
$$= \frac{\cancel{4}z^2}{\cancel{4}(z-1)^2(z+1)} = \boxed{\frac{z^2}{(z-1)^2(z+1)}}$$

$$U_2(z) = \frac{z}{z-1}$$
$$H_2(z) = \frac{y_2(z)}{U_2(z)} = \frac{z^{\cancel{2}}}{(z-1)^{\cancel{2}}(z+1)} \cdot \frac{\cancel{z-1}}{\cancel{z}} = \boxed{\frac{z}{(z-1)(z+1)}}$$

$$H(z) = \left[H_1(z) + H_2(z)\right] \cdot H_3(z) \longrightarrow H_3(z) = \frac{H(z)}{H_1(z) + H_2(z)}$$

$h(n) = \{\underline{0}, 1\}$

$$H(z) = 0 \cdot z^0 + 1 \cdot z^{-1} = \boxed{\frac{1}{z}}$$

$$H_3(z) = \frac{1}{z} \cdot \frac{1}{\frac{z^2}{z^2-1} + \frac{z}{z^2-1}} = \frac{1}{z} \cdot \frac{1}{\frac{z(z+1)}{z^2-1}} = \frac{1}{z} \cdot \frac{(z-1)\cancel{(z+1)}}{z\cancel{(z+1)}} = \boxed{\frac{z-1}{z^2}}$$

$U_3(n) = \mu(n)$

$$U_3(z) = \frac{z}{z-1}$$

$$y_3(z) = H_3(z) \cdot U_3(z)$$
$$= \frac{\cancel{(z-1)}}{z^{\cancel{2}}} \cdot \frac{\cancel{z}}{\cancel{z-1}} = \boxed{\frac{1}{z}}$$

$$\boxed{\begin{aligned} y_3(n) &= \delta(n-1) \\ &= \{\underline{0}, 1\} \end{aligned}}$$

B

3. $y(n) - \frac{1}{4} y(n-2) = u(n)$

a) $u(n) = 15 \cdot 2^n \mu(n)$

$y(-1) = 0$

$y(-2) = 12$

homogeno

$q^2 - \frac{1}{4} = 0$

$q^2 = \frac{1}{4}$

$q_1 = \frac{1}{2} \quad q_2 = -\frac{1}{2}$

$$\boxed{y_h = C_1 \cdot \left(\frac{1}{2}\right)^n + C_2 \left(-\frac{1}{2}\right)^n}$$

partikularno

$y_p = K \cdot 2^n$

$K \cdot 2^n - \frac{1}{4} \cdot K \cdot 2^{n-2} = 15 \cdot 2^n$

$K - \frac{1}{4} \cdot K \cdot \frac{1}{4} = 15$

$\frac{15}{16} K = 15$

$\boxed{K = 16} \rightarrow \boxed{y_p = 16 \cdot 2^n}$

totalno

$y_{tot}(n) = C_1 \cdot \left(\frac{1}{2}\right)^n + C_2 \left(-\frac{1}{2}\right)^n + 16 \cdot 2^n$

$y_{tot}(-1) = 2C_1 - 2C_2 + 8 = 0 \qquad 2C_1 - 2C_2 = -8$

$y_{tot}(-2) = 4C_1 + 4C_2 + 4 = 12 \qquad 2C_1 + 2C_2 = 4$

$4C_1 = -4 \qquad 2C_2 = 4 - 2C_1$

$C_1 = -1 \qquad = 4 + 2 = 6$

$C_2 = 3$

$$\boxed{y_{tot}(n) = \left[-\left(\frac{1}{2}\right)^n + 3\left(-\frac{1}{2}\right)^n + 16 \cdot 2^n\right] \mu(n)}$$

b) $|q_1| = \frac{1}{2} < 1$

$|q_2| = \frac{1}{2} < 1$

$\Big\}$ sustav je stabilan.

B

4. $A = \begin{bmatrix} -3 & 0 \\ 0 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix}$ $C = [5 \quad 1]$ $D = [0 \quad 0]$

a) $D = [0 \quad 0]$

$\downarrow$

1 redak = 1 izlaz

2 stupca = 2 ulaza

b) $H(s) = C \cdot (sI - A)^{-1} \cdot B + D$

$$= [5 \quad 1] \begin{bmatrix} s+3 & 0 \\ 0 & s+4 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix} + [0 \quad 0]$$

$$\begin{bmatrix} s+3 & 0 \\ 0 & s+4 \end{bmatrix}^{-1} = \frac{1}{(s+3)(s+4)} \begin{bmatrix} s+4 & 0 \\ 0 & s+3 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+4} \end{bmatrix}$$

$$H(s) = [5 \quad 1] \begin{bmatrix} \frac{1}{s+3} & 0 \\ 0 & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{5}{s+3} & \frac{1}{s+4} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -2 & -3 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{10}{s+3} - \frac{2}{s+4} & \frac{5}{s+3} - \frac{3}{s+4} \end{bmatrix}}$$

c) $h(t) = ?$

$$\boxed{h(t) = \left[(10 e^{-3t} - 2 e^{-4t}) \mu(t) \quad (5 e^{-3t} - 3 e^{-4t}) \mu(t) \right]}$$

B

5. $x(t) = e^{3t}\mu(-t) + e^{-3t}\mu(t)$

a) CTFT

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

$$= \int_{-\infty}^{\infty} \left[e^{3t}\mu(-t) + e^{-3t}\mu(t)\right] e^{-j\Omega t}\, dt$$

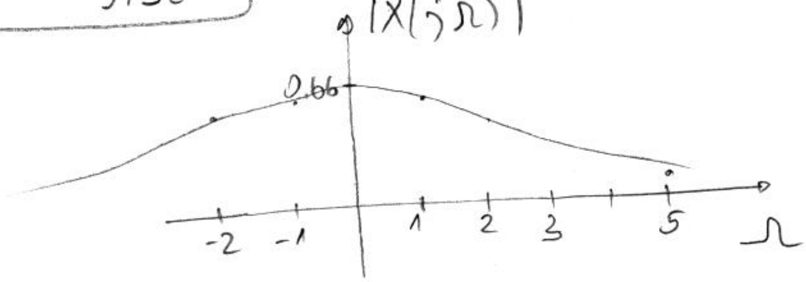
$$= \int_{-\infty}^{\infty} e^{(3-j\Omega)t}\mu(-t)\, dt + \int_{-\infty}^{\infty} e^{-(3+j\Omega)t}\mu(t)\, dt$$

$$= \int_{-\infty}^{0} e^{(3-j\Omega)t}\, dt + \int_{0}^{\infty} e^{-(3+j\Omega)t}\, dt$$

$$= \frac{e^{(3-j\Omega)t}}{3-j\Omega}\Bigg|_{-\infty}^{0} + \frac{e^{-(3+j\Omega)t}}{-(3+j\Omega)}\Bigg|_{0}^{\infty}$$

$$= \frac{1}{3-j\Omega} + \frac{1}{3+j\Omega} = \boxed{\frac{6}{9+\Omega^2}}$$

$$\boxed{|X(j\Omega)| = \frac{6}{9+\Omega^2}}$$



b) $E_x = ?$

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} \left[e^{3t}\mu(-t) + e^{-3t}\mu(t)\right]^2 dt$$

$$= \int_{-\infty}^{\infty} \Big[e^{6t}\mu^2(-t) + \underbrace{2e^{0t}\mu(-t)\mu(t)}_{0} + e^{-6t}\mu^2(t)\Big]\, dt$$

$$= \int_{-\infty}^{\infty} e^{6t}\mu(-t)\, dt + \int_{-\infty}^{\infty} e^{-6t}\mu(t)\, dt$$

$$= \frac{e^{6t}}{6}\Bigg|_{-\infty}^{0} + \frac{e^{-6t}}{-6}\Bigg|_{0}^{\infty}$$

$$\boxed{E_x = \frac{1}{6} - \frac{1}{-6} = \frac{1}{3}}$$

Sadržaj

1. Diferencijske jednadžbe

1.1. Tipovi sustava

Imamo dva tipa sustava:

- **Nepobuđeni** ($u(n) = 0 \rightarrow$ rješavamo samo homogenu)
- **Pobuđeni** ($u(n) \neq 0 \rightarrow$ rješavamo homogenu pa partikularnu)

NAPOMENA 1: Ako vam kažu odredite odziv sustava u 5. (petom) koraku, onda to riješavate uvrštavanjem. Uz to, ponekad je lakše i cijeli sustav riješiti uvrštavanjem.

NAPOMENA 2:

- Totalni odziv = odziv mirnog + odziv nepobuđenog sustava
  ali i također
  Totalni odziv = prisilni odziv + prirodni odziv
- Odziv nepobuđenog sustava se dobije iz homogene jednadžbe uz zadane početne uvjete.
- Odziv mirnog sustava je zbroj partikularnog rješenja i rješenja homogene jednadžbe, a pripadne konstante c1, c2 itd. određujemo uz početne uvjete jednake nuli.
- Rješenje homogene jednadžbe je prirodni odziv sustava.
- Partikularno rješenje je prisilni odziv sustava.
- Pogledajte: http://www.rasip.fer.hr/predmet/sis2?@=g23o#news_11484

NAPOMENA 3: Tutorial ne uključuje rješavanje preko odziva mirnog i nepobuđenog sustava (to pogledajte u scanu sa massovnih).

1.2. Nepobuđeni sustav

Uzmimo primjer:

$$y[n+2] - y[n+1] - 6y[n] = 0$$

$$y[0] = 1$$

$$y[1] = 0$$

Rješenje sustava je rješenje homogene (jer je sustav nepobuđen).

Uvodimo zamjenu:

$$y[n] = Cq^n$$

$$C, q \neq 0$$

Imamo:

$$Cq^{(n+2)} - Cq^{(n+1)} - 6Cq^n = 0$$
$$q^{(n+2)} - q^{(n+1)} - 6 \cdot q^n = 0$$

Izvlačimo q na najmanju potenciju:

$$q^n \left(q^2 - q - 6\right) = 0$$

$$q^2 - q - 6 = 0$$

To je **karakteristična jednadžba** (vidi dio *Vrste rješenja karakteristične jednadžbe*).
NAPOMENA: Možete dobiti n različiti rješenja (za karakterističnu jednadžbu stupnja n), neka rješenja više kratnosti (dvostruka ili trostruka rješenja), kompleksno konjugirana rješenja. Svaki tip rješenja ima poseban zapis (vidi *Određivanje karakteristične jednadžbe*).
Za gore navedenu jednadžbu $q^2 - q - 6 = 0$ rješenja su:

$$q_1 = -2$$

$$q_2 = 3$$

Pa je rješenje homogene:

$$y[n] = c_1(q_1)^n + c_2(q_2)^n$$
$$y[n] = c_1(-2)^n + c_2(3)^n$$

Uvrstimo početne uvjete i izračunajmo koeficijente c_1 i c_2 (ako početni uvjeti **nisu** zadani rješenje ostavljamo u ovom obliku - $y[n] = c_1(-2)^n + c_2(3)^n$).

$$n = 0 \Rightarrow y[0] = c_1(-2)^0 + c_2(3)^0$$

$$1 = c_1 + c_2$$

$$n = 1 \Rightarrow y[1] = c_1(-2)^1 + c_2(3)^1$$

$$0 = -2c_1 + 3c_2$$

To je sustav od dvije jednadžbe s dvije nepoznanice i rješenja su:

$$c_1 = \frac{3}{5}$$

$$c_2 = \frac{2}{5}$$

Rješenje homogene je:

$$y[n] = \frac{3}{5}(-2)^n + \frac{2}{5}(3)^n$$

A to je ujedno i rješenje (nepobuđenog) sustava.

1.2.1. Vrste rješenja karakteristične jednadžbe

- **Mogu sva rješenja biti različita**
- **Možemo imati dvostruka ili trostruka rješenja**
- **Kompleksno konjugirana**

Sva rješenja su različita

Rješavamo po postupku kojim smo rješavali prošli primjer.

Imamo višestruka rješenja

PRIMJER 1. Recimo da imamo homogenu jednadžbu s karakterističnim polinomom stupnja 3 s rješenjima:

$$q_1 = q_2 = 4$$
$$q_3 = 5$$

Rješenje homogene je:

$$y_h[n] = c_1(q_1)^n + c_2 n(q_1)^n + c_3(q_3)^n$$

PRIMJER 2.
Da smo imali homogenu jednadžbu:

$$y[n+3] - 6y[n+2] + 12y[n] - 8 = 0$$
$$x^3 - 6x^2 + 12x - 8 = 0$$
$$(x-2)^3 = 0$$

Rješenja karakteristične jednadžbe su:

$$q_1 = q_2 = q_3 = 2$$

Rješenje homogene je:

$$y_h[n] = c_1(q_1)^n + c_2 n(q_1)^n + c_3 n^2(q_1)^n$$

Itd.

Imamo kompleksno konjugirana rješenja

Uzmimo primjer homogene jednadžbe:

$$y[n+3] - 13y[n+2] + 59y[n] - 87 = 0$$
$$x^3 - 13^2 + 59x - 87 = 0$$

Rješenja su:

$$q_1 = 5 + 2j$$
$$q_2 = 5 - 2j$$
$$q_3 = 3$$

(**NAPOMENA:** kompleksna riješenja uvijek dolaze u paru kompleksno-konjugiranih brojeva!)
Imamo:

$$y_h[n] = c_1(q_1)^n + c_2(q_2)^n + c_3(q_3)^n$$

Prebacimo kompleksna rješenja u eksponencijalni zapis. Postupak:

$$q_1 = a + jb$$
$$q_2 = a - jb$$
$$q_1 = \left| \begin{array}{c} r = \sqrt{a^2+b^2} \\ \varphi = \arctan\frac{b}{a} \end{array} \right| = re^{j\varphi}, \varphi \in \langle -\pi, \pi \rangle$$
$$q_2 = q_1^* = re^{-j\varphi}$$

$$\begin{aligned} y(n) &= c_1(q_1)^n + c_2(q_2)^n = c_1(re^{j\varphi})^n + c_2(re^{-j\varphi})^n \\ &= r^n\left(c_1 e^{j\varphi n} + c_2 e^{-j\varphi n}\right) = r^n\left(c_1 e^{j\varphi n} + c_2 e^{-j\varphi n}\right) \\ &= r^n\left(c_1\left(\cos(\varphi n) + j\sin(\varphi n)\right) + c_2\left(\cos(\varphi n) - j\sin(\varphi n)\right)\right) \\ &= r^n\left((c_1+c_2)\cos(\varphi n) + j\,(c_1-c_2)\sin(\varphi n)\right) = \left| \begin{array}{c} A = c_1 + c_2 \\ B = j(c_1 - c_2) \end{array} \right| \\ &= r^n\left(A\cos(\varphi n) + B\sin(\varphi n)\right) \end{aligned}$$

U našem primjeru:

$$y_h[n] = \sqrt{29}^n \left(A\cos(21.8014n) + B\sin(21.8014n)\right) + c_3(3)^n$$

Pri tome su A, B i c_3 koeficijenti koje treba odrediti iz početnih uvjeta.

Ako imate problema sa kompleksnim brojevima, **tutorial**:
http://materijali.fer2.net/file.988.aspx Ako netko ima problema sa trigonometrijskom kružnicom:
http://zd-mioc.hr/doc/trigonometrijska_kruznica.pdf

1.2.2. Određivanje karakteristične jednadžbe

Na par primjera (zanemario sam konstante uz q_n):
1)

$$y[n+2] + 3y[n+1] + 5y[n] = 0$$
$$q^{(n+2)} + 3q^{(n+1)} + 5q^n = 0$$
$$q^n\left(q^2 + 3q + 5\right) = 0$$
$$q^2 + 3q + 5 = 0$$

2)

$$y[n+2] - 10y[n] = 0$$
$$q^{(n+2)} - 10q^n = 0$$
$$q^n\left(q^2 - 10\right) = 0$$
$$q^2 - 10 = 0$$

3)

$$y[n-2] + 5y[n-1] + 44y[n] = 0$$
$$q^{(n-2)} + 5q^{(n-1)} + 44q^n = 0$$
$$q^{(n-2)}\left(1 + 5q + 44q^2\right) = 0$$
$$1 + 5q + 44q^2 = 0$$

4)

$$y[n-1] + y[n] + 12y[n+1] = 0$$
$$q^{(n-1)} + q^n + 12q^{(n+1)} = 0$$
$$q^{(n-1)}\left(1 + q + 12q^2\right) = 0$$
$$1 + q + 12q^2 = 0$$

1.3. Pobuđeni sustav

Uzmimo primjer (primjer je sličan gornjem, samo smo dodali pobudu ($u(n)$, a to je neka funkcija od n)):

$$y[n+2] - y[n+1] - 6y[n] = u[n]$$
$$y[0] = 1$$
$$y[1] = 0$$
$$u[n] = 4\left(\frac{1}{2}\right)^n$$

Rješenje sustava je ZBROJ *homogene i partikularne.*

1.3.1. Homogena

Umjesto u(n) stavimo 0 i imamo:

$$y[n+2] - y[n+1] - 6y[n] = 0$$

To riješimo po gornjem postupku. Rješenje homogene je: $y_h[n] = c_1(-2)^n + c_2(3)^n$

NAPOMENA: NE uvrštavamo početne uvjete u rješenje homogene!

1.3.2. Partikularna

Rješenje partikularne ovisi o obliku funkcije u(n), pa ga očitamo iz tablice.
Tablica (cjelina 12, slide 41.):

pobuda $u(n)$	partikularno rješenje $y_p(n)$
A (konstanta)	K
Ar^n, $r \neq q_i (i = 1, 2, \ldots, N)$	Kr^n
Ar^n, $r = q_i$	Knr^n
An^M	$K_0 + K_1 n + \ldots + K_M n^M$
$r^n n^M$	$r^n \left(K_0 + K_1 n + \ldots + K_M n^M\right)$
$A\cos(\omega_0 n)$	$K_1 \cos(\omega_0 n) + K_2 \sin(\omega_0 n)$
$A\sin(\omega_0 n)$	$K_1 \cos(\omega_0 n) + K_2 \sin(\omega_0 n)$

Funkcija pobude je:

$$u(n) = 4\left(\frac{1}{2}\right)^n$$

To je oblik:

$$u(n) = Ar^n$$

Iz tablice čitamo partikularno rješenje:

$$y_p[n] = Kr^n$$

NAPOMENA: Da je *r* jednak nekom od rješenja rješenja karakterističnog polinoma homogene jednadžbe, oblik bi bio: Knr^n.
Uvrstimo y_p u početnu jednadžbu:

$$K\left(\frac{1}{2}\right)^{(n+2)} - K\left(\frac{1}{2}\right)^{(n+1)} - 6K\left(\frac{1}{2}\right)^n = 4\left(\frac{1}{2}\right)^n$$

Sve podjelimo sa $\frac{1}{2}^n$:

$$\frac{1}{4}K - \frac{1}{2} - 6K = 4$$

$$K = -\frac{16}{25}$$

Imamo:

$$y[n] = y_h[n] + y_p[n]$$

$$y[n] = c_1(-2)^n + c_2(3)^n - \frac{16}{25} \cdot \frac{1}{2}^n$$

Uvrstimo početne uvjete, te dobijemo:

$$1 = c_1 + c_2 - \frac{16}{25}$$

$$0 = -2c_1 + 3c_2 - \frac{16}{50}$$

Riješimo sustav:

$$c_1 = \frac{191}{175}$$

$$c_2 = \frac{18}{35}$$

Konačno rješenje je:

$$y[n] = \frac{191}{175}(-2)^n + \frac{18}{35}(3)^n - \frac{16}{25} \cdot \frac{1}{2}^n$$

Gotovo :)

Zadatak

Primjer sustava drugog reda opisanog jednadžbom diferencija:
y(n)+a1*y(n-1)+a2*y(n-2) = b0*u(n)

a1= -1/2, a2=1/4, b0=1

uz početne uvjete y(-1)=2, y(-2)=8

pobuđen sa u(n)=cos(2*pi/3*n + pi/3)

Potrebno je analitički odrediti :

a) odziv nepobuđenog sustava na početna stanja,
b) prisilni odziv sustava,
c) odziv mirnog sustava vlastitim frekvencijama,
d) ukupni odziv mirnog sustava,
e) prirodni odziv sustava,
f) ukupni odziv sustava,

Napisati Matalb program koji racuna i prikazuje sve analiticki izvedene odzive za n=0,1,2,...30

Napisati Matlab program koji odredjuje odziv metodom korak po korak.

Usporediti analitička i numerička rješenja.

Općenit oblik jed. dif. za sustav II reda (1)

$$y(n+2)+a_1y(n+1)+a_2y(n)=b_0u(n+2)+b_1u(n+1)+b_2u(n)$$

$n'=n+2$

$$y(n')+a_1y(n'-1)+a_2y(n'-2)=b_0u(n')+b_1u(n'-1)+b_2u(n'-2)$$

homogeni dio rješenja $\quad u(n')=\emptyset \rightarrow \forall n'$

$$y(n')+a_1y(n'-1)+a_2y(n'-2)=\emptyset$$

Pretpostavimo rj. oblika $\quad c\cdot q^{n'}=y(n')$

$$c\cdot q^{n'}+a_1\cdot c\cdot q^{n'-1}+a_2\cdot c\cdot q^{n'-2}=\emptyset$$

$$c\cdot q^{n-2}\underbrace{\left(q^2+a_1q+a_2\right)}_{\text{karakteristični polinom}}=\emptyset$$

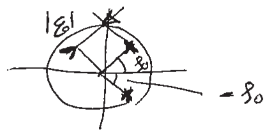
$$\underbrace{q^2+a_1q+a_2=0}_{\text{karakteristična jednadžba}}$$

Potražimo rješenje:

$$q_{12}=\frac{-a_1\pm\sqrt{a_1^2-4a_2}}{2}$$

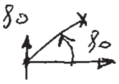
Odaberimo primjer sa konj. kompl. parom $q_1=q_0$, $q_2=q_0^*$

$q_1=|q_0|\cdot e^{j\vartheta_0}\qquad q_2=|q_0|\cdot e^{-j\vartheta_0}$



radijus $|q_0|$

kut ϑ_0



Kako glase koeficijenti ~~karakter.~~ polinoma za ovakav par karakterističnih frekvencija?

Prikažimo polinom kao produkt korijenskih faktora

$$(q-q_1)(q-q_2)=\left(q-|q_0|\cdot e^{j\vartheta_0}\right)\left(q-|q_0|\cdot e^{-j\vartheta_0}\right)$$

$$=q^2-q|q_0|\cdot\underbrace{\left(e^{j\vartheta_0}+e^{-j\vartheta_0}\right)}_{2\cos\vartheta_0}+|q_0|^2\cdot\underbrace{e^{j\vartheta_0}\cdot e^{-j\vartheta_0}}_{1}$$

$$=q^2\underbrace{-2|q_0|\cdot\cos\vartheta_0\cdot q}_{a_1}+\underbrace{|q_0|^2}_{a_2}$$

2

Npr. odaberimo $|q_0| = \frac{1}{2}$, $\vartheta_0 = \frac{\pi}{3} \Rightarrow \cos\vartheta_0 = \frac{1}{2}$

$$a_1 = -2|q_0| \cdot \cos\vartheta_0 = -2 \cdot \frac{1}{2} \cdot \frac{1}{2} = -\frac{1}{2}$$

$$a_2 = |q_0|^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Dakle karakteristični polinom (jednadžba) glasi:

$$q^2 - \frac{1}{2} q + \frac{1}{4} = \emptyset$$

Za provjeru nađimo rješenje:

par konjug.-kompl. karakterističnih frekvencija

$$q_{12} = \frac{\frac{1}{2} \pm \sqrt{\frac{1}{4} - 1}}{2} = \frac{\frac{1}{2} \pm \frac{1}{2} j\sqrt{3}}{2} = \underbrace{\frac{1}{2}}_{|q_0|} \underbrace{\left(\frac{1}{2} \pm \frac{\sqrt{3}}{2} j\right)}_{e^{j\pm\pi/3}}$$

$$e^{j\pm\pi/3} = \left(\cos\left(\frac{\pi}{3}\right) \pm j\sin\left(\frac{\pi}{3}\right)\right)$$

$$\frac{1}{2} \pm j\frac{\sqrt{3}}{2} \quad \text{OK}$$

Kada smo odredili karakteristične (ili vlastite) frekvencije sustava q_1, q_2 možemo odrediti odziv nepobuđenog sustava (početna stanja $\neq \emptyset$, ali $u(n) = \emptyset \ \forall n$)

Pretpostavimo rješenje oblika

$$y_H(n) = c_1 \cdot q_1^n + c_2 \cdot q_2^n$$

$$y(n) = \underbrace{-a_1 y(n-1) - a_2 y(n-2)}$$

Moramo poznavati početne uvjete

$$\left.\begin{matrix} y(n-1) \\ y(n-2) \end{matrix}\right\}$$ npr. za $n=\emptyset$

$$y(\emptyset) = -a_1 y(-1) - a_2 y(-2)$$

$$y(1) = -a_1 y(\emptyset) - a_2 y(-1)$$

npr. poznamo $\left.\begin{matrix} y(-1) \\ y(-2) \end{matrix}\right\}$ treba nam $y(\emptyset)$ i $y(1)$

$$y(\emptyset) = -a_1 y(-1) - a_2 y(-2)$$

$$y(1) = a_1^2 y(-1) + a_1 a_2 y(-2) - a_2 y(-1)$$

$$= (a_1^2 - a_2) y(-1) + a_1 a_2 y(-2)$$

Iz homogenog rješenja

Za homogeno rješenje mogli smo odabrati i bilo koji drugi par n-ova uključujući $n=-1$, $n=-2$ → svi daju isto rješenje za c_1/c_2

$$c_1 q_1^n + c_2 q_2^n = /n=\emptyset/ = c_1 + c_2 = y(\emptyset)$$

$$= /n=1/ = c_1 q_1 + c_2 q_2 = y(1)$$

Dobivamo 2 jedn. sa 2 nepoz. (c_1, c_2)

$$c_1 + c_2 = -a_1 y(-1) - a_2 y(-2)$$

$$c_1 q_1 + c_2 q_2 = (a_1^2 - a_2) y(-1) + a_1 a_2 y(-2)$$

$a_1 = -\frac{1}{2}$

$a_2 = \frac{1}{4}$

$$c_1 + c_2 = \frac{1}{2} y(-1) - \frac{1}{4} y(-2)$$

$$c_1 q_1 + c_2 q_2 = \left(\frac{1}{4} - \frac{1}{4}\right) y(-1) - \frac{1}{2} \cdot \frac{1}{4} y(-2) = -\frac{1}{8} y(-2)$$

Npr. odaberemo početne uvjete $y(-1) = 2$, $y(-2) = 8$

$$c_1 + c_2 = \frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 8 = 1 - 2 = -1 \qquad c_1 = -c_2 - 1$$

$$c_1 q_1 + c_2 q_2 = -\frac{1}{8} y(-2) = -1$$

$$q_1(-c_2 - 1) + c_2 q_2 = -1$$

$$c_2(q_2 - q_1) = q_1 - 1$$

$$c_2 = \frac{q_1 - 1}{q_2 - q_1}$$

$$q_1 = \frac{1}{4} + \frac{\sqrt{3}}{4} j$$

$$q_2 = \frac{1}{4} - \frac{\sqrt{3}}{4} j$$

$$q_2 - q_1 = -\frac{\sqrt{3}}{2} j$$

$$c_2 = \frac{-\frac{3}{4} + \frac{\sqrt{3}}{4} j}{-\frac{\sqrt{3}}{2} j} \cdot \frac{\frac{2}{\sqrt{3}} j}{\frac{2}{\sqrt{3}} j} = -\frac{6}{4\sqrt{3}} j - \frac{1}{2}$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} j$$

$$c_1 = -c_2 - 1 = \frac{1}{2} + \frac{\sqrt{3}}{2} j - 1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} j = c_2^*$$

Provjera

$$c_1 q_1 + c_2 q_2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} j\right)\left(\frac{1}{4} + \frac{\sqrt{3}}{4} j\right) + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} j\right)\left(\frac{1}{4} - \frac{\sqrt{3}}{4} j\right)$$

$$= \left(-\frac{1}{8} - \frac{3}{8} + j\left(\cancel{\frac{\sqrt{3}}{8}} - \cancel{\frac{\sqrt{3}}{8}}\right)\right) + \left(-\frac{1}{8} - \frac{3}{8} + j\left(\cancel{-\frac{\sqrt{3}}{8}} + \cancel{\frac{\sqrt{3}}{8}}\right)\right) = -\frac{1}{2} - \frac{1}{2} = -1$$

Dakle: rješenje glasi:

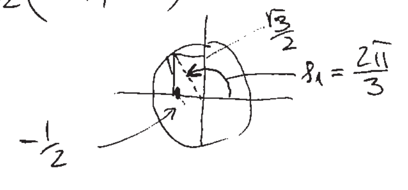
$$y_H(n) = c_1 q_1^n + c_2 q_2^n = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2} j\right)\left(\frac{1}{4} + \frac{\sqrt{3}}{4} j\right)^n + \left(-\frac{1}{2} - \frac{\sqrt{3}}{2} j\right)\left(\frac{1}{4} - \frac{\sqrt{3}}{4} j\right)^n$$

ili u polarnom obliku:

$$|c_1| = |c_2| = \frac{1}{4} + \frac{3}{4} = 1 \qquad c_1 = |c_1| \cdot e^{j\theta_1} \qquad c_2 = |c_2| \cdot e^{j\theta_2}$$

$$c_1 = 1 \cdot e^{j\frac{2\pi}{3}} \qquad c_2 = c_1^* = 1 \cdot e^{-j\frac{2\pi}{3}}$$

$$\theta_1 = \mathrm{atan}_2(\mathrm{Im}, \mathrm{Re})$$



$$q_1 = |q_0| \cdot e^{j\theta_0} \qquad q_2 = q_1^* = |q_0| \cdot e^{-j\theta_0}$$

$$|q_0| = \frac{1}{2}, \quad \theta_0 = \frac{\pi}{3}$$

Dalje u polarnom obliku:

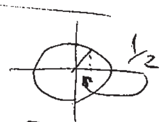
$$y_H(n) = e^{j\frac{2\pi}{3}}\left(\frac{1}{2}e^{j\frac{\pi}{3}}\right)^n + e^{-j\frac{2\pi}{3}}\left(\frac{1}{2}\cdot e^{-j\frac{\pi}{3}}\right)^n$$

$$= \left(\frac{1}{2}\right)^n \left[e^{j\left(n\cdot\frac{\pi}{3}+\frac{2\pi}{3}\right)} + e^{-j\left(n\cdot\frac{\pi}{3}+\frac{2\pi}{3}\right)}\right]$$

$$= \left(\frac{1}{2}\right)^n \cdot 2\cos\left(n\frac{\pi}{3}+\frac{2\pi}{3}\right)$$

$$= 2^{(1-n)}\cdot\cos\left(n\frac{\pi}{3}+\frac{2\pi}{3}\right)$$

Provjeravamo početne uvjete:



$$y_H(-1) = 2^{(1-(-1))}\cdot\cos\left(-\frac{\pi}{3}+\frac{2\pi}{3}\right) = 4\cdot\cos\left(\frac{\pi}{3}\right) = 4\cdot\frac{1}{2} = 2 \quad ✓$$

$$y_H(-2) = 2^{(1+2)}\cdot\cos\left(-\frac{2\pi}{3}+\frac{2\pi}{3}\right) = 8\cdot\cos(\emptyset) = 8 \quad ✓$$

Metoda korak po korak ... nađimo $y(\emptyset)$, $y(1)$

$$y(\emptyset) = -a_1 y(-1) - a_2 y(-2) = \Big/ \begin{matrix} a_1 = -\frac{1}{2} \\ a_2 = \frac{1}{4} \end{matrix} \Big/ = \frac{1}{2}y(-1) - \frac{1}{4}y(-2)$$

$$= \frac{1}{2}\cdot 2 - \frac{1}{4}\cdot 8 = -1$$

$$y(1) = -a_1 y(\emptyset) - a_2 y(-1) = \Big/ -\|- \Big/ = \frac{1}{2}\cdot(-1) - \frac{1}{4}\cdot 2$$

$$= -\frac{1}{2} - \frac{1}{2} = -1$$

Provjeravamo opće rješenje za $n = \emptyset, 1$

$$y_H(\emptyset) = 2^1\cdot\underbrace{\cos\left(\frac{2\pi}{3}\right)}_{-\frac{1}{2}} = -1 \quad ✓ \text{ ok}$$

$$y_H(1) = 2^{\emptyset}\cdot\cos\left(\frac{\pi}{3}+\frac{2\pi}{3}\right) = 1\cdot(-1) = -1 \quad ✓ \text{ ok}$$

Pobudimo ovaj sustav drugog reda sa pobudom harmonijskog oblika

$$u[n] = A\cos\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right)$$ npr. neka $A=1$

$$\omega_0 = \frac{2\pi}{3} \qquad \Theta = \frac{\pi}{3}$$

Općenito:

$$u[n] = A\cdot\cos(\omega_0 n + \Theta)$$

možemo je prikazati i u exp. obliku (suma dvije kompl. exp.)

$$u[n] = \frac{A}{2}\cdot e^{j\Theta}\cdot e^{+j\omega n} + \frac{A}{2}\cdot e^{-j\Theta} e^{-j\omega n}$$, jer

$$= \frac{A}{\cancel{2}}\Big(\underbrace{e^{j(\Theta+\omega n)} + e^{-j(\Theta+\omega n)}}_{\cancel{2}\cos(\omega n+\Theta)}\Big) = A\cdot\cos(\omega n+\Theta)$$

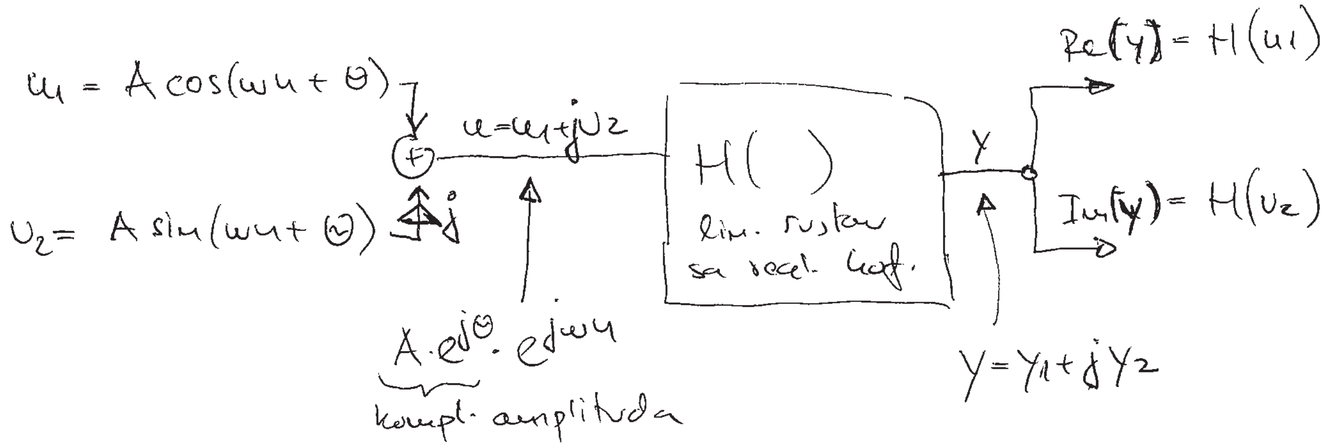
ili

$$u[n] = Re\left(A\cdot e^{j\Theta}\cdot e^{j\omega n}\right) =$$

$$= Re\left(A\cdot e^{j(\omega n+\Theta)}\right) =$$

$$= Re\left(A\cdot\cos(\omega n+\Theta) + jA\sin(\omega n+\Theta)\right)$$

$$= A\cdot\cos(\omega n+\Theta)$$



Odziv na pobudu $H(u_1 + ju_2) = y_1 + jy_2$

Zbog linearnosti ako je u_1 realan i u_2 realan i H realan tada je $Re(y) = y_1$ a $Im(y) = y_2$

7

Pri određivanju partikularnog rješenja metodom neodređenog koef. uz pobudu oblika:

$$u[n] = \underbrace{A}_{} \cdot e^{j\omega_0 n}$$

A... kompl. amplituda

moramo pretpostaviti rješenje istog <u>oblika</u> ali druge kompl. amplitude K

$$y_p[n] = K \cdot e^{j\omega_0 n}$$

K... obuhvaća i realnu amplitudu $|K|$ i fazni kut (početni kut)

Uvrstimo pretpostavljeno rj. u jedn. dif.

$$y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 u(n)$$

upr. $b_1 = b_2 = \emptyset$

$a_1 = -\frac{1}{2}$ $a_2 = \frac{1}{4}$

neka je $b_0 = 1$

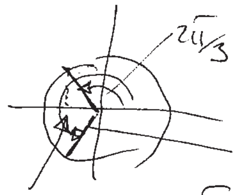
$$y(n) - \frac{1}{2} y(n-1) + \frac{1}{4} y(n-2) = u(n)$$

$$K \cdot e^{j\omega_0 n} - \frac{1}{2} K \cdot e^{j\omega_0(n-1)} + \frac{1}{4} K e^{j\omega_0(n-2)} = A \cdot e^{j\omega_0 n}$$

$$K \cdot \cancel{e^{j\omega_0 n}} \left(1 - \frac{1}{2} \cdot e^{-j\omega_0} + \frac{1}{4} \cdot e^{-2j\omega_0}\right) = A \cdot \cancel{e^{j\omega_0 n}}$$

$$K = \frac{A}{\left(1 - \frac{1}{2} e^{-j\omega_0} + \frac{1}{4} \cdot e^{-2j\omega_0}\right)}$$

Drugi primjer... za $\omega_0 = 2\pi/3$, $A = 1\cdot e^{j\pi/3}$



$$K = \frac{A}{\left(1 - \frac{1}{2}\cdot e^{-j\omega_0} + \frac{1}{4} e^{-2j\omega_0}\right)}$$

$$e^{-j\frac{4\pi}{3}} = \cos\left(\frac{4\pi}{3}\right) - j\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$e^{-j\frac{2\pi}{3}} = \cos\left(\frac{2\pi}{3}\right) - j\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} - j\cdot\frac{\sqrt{3}}{2}$$

$$K = \frac{e^{j\pi/3}}{1 - \frac{1}{2}\left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)}$$

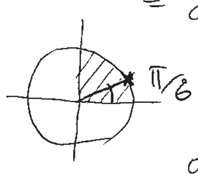
$$K = \frac{e^{j\pi/3}}{1 + \frac{1}{4} + j\frac{\sqrt{3}}{4} - \frac{1}{8} + j\frac{\sqrt{3}}{8}} = \frac{e^{j\pi/3}}{\underbrace{\frac{9}{8} + j\frac{3\sqrt{3}}{8}}_{\frac{3\sqrt{3}}{4}\cdot e^{j\pi/6}}} \quad \text{ok}$$

$$|\,|^2 = \frac{9^2}{8^2} + \frac{3^2\cdot 3}{8^2} = \frac{81+27}{64} = \frac{108}{64} = \frac{54}{32} = \frac{27}{16}$$

$$|\,| = \frac{3\sqrt{3}}{4}$$

$$\arg = \text{atan}_2(\text{im}, \text{re}) = \text{atan}_2(\underbrace{3\sqrt{3}}_{>0}, \underbrace{9}_{>0})$$

I kvadrant



$$\text{atan}\left(\frac{3\sqrt{3}}{9}\right) = \text{atan}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$K = \frac{4}{3\sqrt{3}} e^{j\pi/3}\cdot e^{-j\pi/6} = \frac{4}{3\sqrt{3}}\cdot e^{j\frac{\pi}{6}} = \frac{4\sqrt{3}}{9}\cdot e^{j\frac{\pi}{6}}$$

Dakle uz pretpostavljeno rješenje oblika

$$y_p[n] = K\cdot e^{j\omega_0 n}$$

dobivamo

$$y_p[n] = \frac{4\sqrt{3}}{9}\, e^{j\frac{\pi}{6}}\cdot e^{j\frac{2\pi}{3}n}$$

modul   početna faza (za $n=0$)   frekvencija odziva

na pobudu oblika:

$$u[n] = 1\cdot e^{j\pi/3}\cdot e^{j\frac{2\pi}{3}n}$$

frekvencija pobudnog signala

moraju biti jednake zbog linearnosti sustava!

$$y_p[n] = \text{Re}[y_p[n]] + j\,\text{Im}(y_p[n]) = y_{pr}[n] + j\,y_{pi}[n]$$

su odziv na

$$u[n] = \text{Re}[u[n]] + j\,\text{Im}(u[n]) = u_r[n] + j\,u_i[n]$$

pri tome $y_{pr}[n]$ je odziv na u_r

a $y_{pi}[n]$ je odziv na $u_i[n]$

$$u_r[n] = \cos\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right)$$

$$y_{pr}[n] = \mathrm{Re}\left(\frac{4\sqrt{3}}{9} \cdot e^{j\pi/6} \cdot e^{j\frac{2\pi}{3}n}\right)$$

$$= \frac{4\sqrt{3}}{9} \cdot \mathrm{Re}\left(e^{j\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)}\right)$$

$$= \frac{4\sqrt{3}}{9} \cdot \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)$$

Iako nas je interesirao odziv na $u_r[n]$, primjenom kompleksne eksp. automatski smo dobili i odziv na $u_i[n] = \sin\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right)$ kao imaginarni dio odziva $\mathrm{Im}(y_p[n])$

$$y_{pi}[n] = \mathrm{Im}\left(\frac{4\sqrt{3}}{9} \cdot e^{j\pi/6} \cdot e^{j\frac{2\pi}{3}n}\right)$$

$$= \frac{4\sqrt{3}}{9} \cdot \sin\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)$$

Alternativan način rješavanja kod kojeg izbjegavamo rad sa kompleksnim brojevima je da pretpostavimo realno rješenje u obliku:

$$y_{pr}[n] = K_1 \cdot \cos(\omega_0 n) + K_2 \cdot \sin(\omega_0 n)$$

$\Rightarrow$ u konstantama K_1, K_2 sadržana je informacija o realnoj amplitudi i početnoj fazi izlazne kosinusoide:

$$y_{pr}[n] = \underbrace{A_y}_{\text{realna ampl.}} \cdot \cos(\omega_0 n + \underbrace{\Theta_0}_{\text{početna faza}})$$

$$y_{pr}[n] = A_y \left(\cos(\omega_0 n)\cos(\Theta_0) - \sin(\omega_0 n) \cdot \sin(\Theta_0)\right)$$

$$= \underbrace{\left(A_y \cdot \cos(\Theta_0)\right)}_{K_1} \cdot \cos(\omega_0 n) + \underbrace{\left(-A_y \cdot \sin(\Theta_0)\right)}_{K_2} \cdot \sin(\omega_0 n)$$

$$K_1 = A_y \cdot \cos(\Theta_0)$$

$$K_2 = -A_y \cdot \sin(\Theta_0)$$

kako odrediti A_y i Θ_0 iz k_1 & k_2 ?

$$k_1^2 + k_2^2 = A_y^2 \cdot \cos^2(\Theta_0) + A_y^2 \cdot \sin^2(\Theta_0)$$
$$= A_y^2(\cos^2\Theta_0 + \sin^2\Theta_0)$$
$$= A_y^2$$
$$\Rightarrow A_y = \sqrt{k_1^2 + k_2^2} \qquad A_y \in R$$

kada znamo A_y treba odrediti Θ_0 :

$$\cos(\Theta_0) = \frac{k_1}{A_y} \quad , \quad \sin(\Theta_0) = -\frac{k_2}{A_y}$$

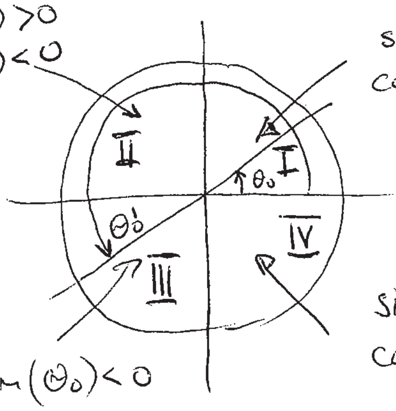
$$tg(\Theta_0) = \frac{\sin(\Theta_0)}{\cos(\Theta_0)} \Rightarrow \Theta_0 = atan\left(\frac{\sin(\Theta_0)}{\cos(\Theta_0)}\right)$$

Moramo koristiti četvero-kvadrantnu atan funkciju.

Obična atan daje samo dvo-kvadrantno rješenje

$\sin(\Theta_0) > 0$
$\cos(\Theta_0) < 0$

$\sin(\Theta_0) > 0$
$\cos(\Theta_0) > \emptyset$



$\sin(\Theta_0) < 0$
$\cos(\Theta_0) > 0$

$\sin(\Theta_0) < 0$
$\cos(\Theta_0) < 0$

Za obična atan kutevi Θ_0 u I kvadrantu i Θ_0' u III kvadrantu su ekvivalentni

$$\Theta_0' = \Theta_0 + \pi \qquad \Theta_0 \in (0, \tfrac{\pi}{2})$$

$$tg(\Theta_0') = \frac{\sin(\Theta_0 + \pi)}{\cos(\Theta_0 + \pi)} = \frac{\sin(\Theta_0)\cdot \underbrace{\cos(\pi)}_{-1} + \cos(\Theta_0)\cdot \underbrace{\sin(\pi)}_{\emptyset}}{\cos(\Theta_0)\cdot \underbrace{\cos(\pi)}_{-1} - \sin(\Theta_0)\cdot \underbrace{\sin(\pi)}_{\emptyset}}$$

$$= \frac{-\sin(\Theta_0)}{-\cos(\Theta_0)} = tg(\Theta_0)$$

Jednaka "ekvivalencija" vrijedi za kuteve u II i IV kvadrantu za $\Theta_0 \in [\emptyset, -\frac{\pi}{2})$ i $\Theta' = \Theta_0 + \pi$

lako su predznaci sin & cos suprotni imamo istu vrijednost tg funkcije

Kako odrediti četvero-kvadrantno rješenje ako znamo vrijednost sinusa i kosinusa? (11)

1. Izračunamo obični $\operatorname{atan}\left(\frac{\sin}{\cos}\right) = \Phi_{I,IV}$
2. Uz pretpostavku da je kodomena običnog atan-a $[-\pi/2, \pi/2]$ tada automatski dobivamo traženo rješenje u slučaju da je Θ_0 u I ili IV kvadrantu.
3. Međutim, ako je $\cos < \emptyset$ tada je sigurno rješenje u II ili III kvadrantu ... dakle pravo rješenje je tada

$$\Phi_{\pm} \begin{cases} \Phi_{I,IV} & \text{ako } \cos \geq \emptyset \\ \Phi_{I,IV} + \pi & \text{inače} \end{cases}$$

4. Ako želimo da konačna kodomena četvero-kvadratnog atan-a bude $[\emptyset, 2\pi)$ tada je još potrebno rješenja iz IV kvadranta pretvoriti u pozitivna dodavanjem 2π

$$\Phi = \begin{cases} \Phi & \text{ako } \Phi \geq \emptyset \\ 2\pi + \Phi & \text{ako } \Phi < 0 \end{cases}$$

Uvrstimo sada pretpostavljeno rješenje u jed. dif. (12)

$$y_{pr}[n] = k_1 \cdot \cos(\omega_0 n) + k_2 \sin(\omega_0 n) \qquad \omega_0 = \frac{2\pi}{3}$$

$$u_r[n] = \cos\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right)$$

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = u_r[n]$$

$$= \cos\frac{2\pi}{3}n \cos\frac{\pi}{3} - \sin\frac{2\pi}{3}n \sin\frac{\pi}{3}$$

$$= \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}n\right)$$

$$k_1 \cdot \cos(\omega_0 n) + k_2 \cdot \sin(\omega_0 n) +$$

$$a_1 k_1 \cos(\omega_0(n-1)) + a_1 k_2 \sin(\omega_0(n-1)) +$$

$$a_2 k_1 \cos(\omega_0(n-2)) + a_2 k_2 \sin(\omega_0(n-2)) = \frac{1}{2}\cos\left(\frac{2\pi}{3}n\right) - \frac{\sqrt{3}}{2}\sin\left(\frac{2\pi}{3}n\right)$$

$$\left.\begin{aligned}\cos(\omega_0(n-k)) &= \cos(\omega_0 n)\cos(k\omega_0) + \sin(\omega_0 n)\sin(k\omega_0)\\ \sin(\omega_0(n-k)) &= \sin(\omega_0 n)\cos(k\omega_0) - \cos(\omega_0 n)\sin(k\omega_0)\end{aligned}\right\} \text{ za } \omega_0 = \frac{2\pi}{3}$$

$k=1$ $\cos(\omega_0(n-1)) = \cos(\omega_0 n) \cdot \left(-\frac{1}{2}\right) + \sin(\omega_0 n) \cdot \frac{\sqrt{3}}{2}$

$k=2$ $\cos(\omega_0(n-2)) = \cos(\omega_0 n) \cdot \left(-\frac{1}{2}\right) + \sin(\omega_0 n)\left(-\frac{\sqrt{3}}{2}\right)$

$k=1$ $\sin(\omega_0(n-1)) = \sin(\omega_0 n) \cdot \left(-\frac{1}{2}\right) - \cos(\omega_0 n)\frac{\sqrt{3}}{2}$

$k=2$ $\sin(\omega_0(n-2)) = \sin(\omega_0 n) \cdot \left(-\frac{1}{2}\right) - \cos(\omega_0 n) \cdot \left(-\frac{\sqrt{3}}{2}\right)$

Grupirajmo sve članove sa lijeve strane koji sadrže titranje istog oblika, ali različitih amplituda (... dakle posebno $\cos(\omega_0 n)$, posebno $\sin \omega_0 n$)

$$\left(k_1 + a_1 k_1\left(-\frac{1}{2}\right) + a_1 k_2\left(-\frac{\sqrt{3}}{2}\right) + a_2 k_1 \left(-\frac{1}{2}\right) + a_2 k_2\left(\frac{\sqrt{3}}{2}\right)\right)\cos(\omega_0 n) +$$

$$\left(k_2 + a_1 k_1\left(\frac{\sqrt{3}}{2}\right) + a_1 k_2\left(-\frac{1}{2}\right) + a_2 k_1 \cdot \left(-\frac{\sqrt{3}}{2}\right) + a_2 k_2\left(-\frac{1}{2}\right)\right)\sin(\omega_0 n) =$$

$$\left/\begin{matrix} a_1 = -1/2 \\ a_2 = 1/4 \end{matrix}\right/ = \left(k_1 + k_1 \cdot \frac{1}{4} + k_2 \cdot \frac{\sqrt{3}}{4} + k_1\left(-\frac{1}{8}\right) + k_2\frac{\sqrt{3}}{8}\right) \cdot \cos(\omega_0 n) +$$

$$\left(k_2 - k_1 \cdot \frac{\sqrt{3}}{4} + k_2 \cdot \frac{1}{4} + k_1\left(-\frac{\sqrt{3}}{8}\right) + k_2\left(-\frac{1}{8}\right)\right) \cdot \sin(\omega_0 n) =$$

$$= \underbrace{\frac{1}{2}\cos\left(\overset{\omega_0}{\frac{2\pi}{3}}n\right) - \frac{\sqrt{3}}{2}\sin\left(\overset{\omega_0}{\frac{2\pi}{3}}n\right)}_{\text{desna strana}} \Rightarrow$$

Dobivamo sustav od dvije jednadžbe sa dvije nepoz.

$$K_1\left(1+\frac{1}{4}-\frac{1}{8}\right)+K_2\left(\frac{\sqrt{3}}{4}+\frac{\sqrt{3}}{8}\right)=\frac{1}{2} \quad /\cdot 8$$

$$K_1\left(-\frac{\sqrt{3}}{4}-\frac{\sqrt{3}}{8}\right)+K_2\left(1+\frac{1}{4}-\frac{1}{8}\right)=-\frac{\sqrt{3}}{2} \quad /\cdot 8$$

$$9K_1+3\sqrt{3}K_2=4 \Rightarrow K_1=\frac{4}{9}-\frac{3\sqrt{3}}{9}\cdot K_2$$

$$-3\sqrt{3}K_1+9K_2=-4\sqrt{3}$$

$$-\frac{12\sqrt{3}}{9}+\frac{27}{9}K_2+9K_2=-4\sqrt{3}$$

$$12K_2=-4\sqrt{3}+\frac{4}{3}\sqrt{3}=-\frac{8}{3}\sqrt{3}$$

$$K_2=-\frac{2}{9}\sqrt{3}$$

$$K_1=\frac{4}{9}+\frac{3\sqrt{3}}{\cancel{9}}\cdot\frac{8}{3}\sqrt{3}\cdot\frac{1}{12}$$

$$K_1=\frac{4}{9}+\frac{2}{9}=\frac{6}{9}=\frac{2}{3}$$

Dakle odziv glasi:

$$y_{pr}(t)=\frac{2}{3}\cos(\omega_0 t)-\frac{2\sqrt{3}}{9}\cdot\sin(\omega_0 t)$$

$$=A_y\cdot\cos(\omega_0 t+\Theta_0)$$

$$A_y=\sqrt{\left(\frac{2}{3}\right)^2+\left(\frac{2\sqrt{3}}{9}\right)^2}=\sqrt{\frac{4}{9}+\frac{4\cdot 3}{81}}=\sqrt{\frac{4\cdot 9+4\cdot 3}{81}}=\sqrt{\frac{4\cdot 4\cdot 3}{81}}=\frac{4}{9}\sqrt{3}$$

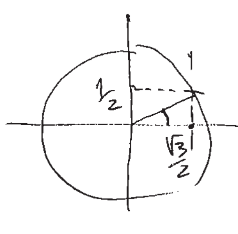
Odredimo početni kut odziva Θ_0

$$K_1=\frac{2}{3}=A_y\cdot\cos(\Theta_0) \Rightarrow \cos(\Theta_0)=\frac{K_1}{A_y}=\frac{\frac{2}{3}}{\frac{4}{9}\sqrt{3}}=\frac{3}{2\sqrt{3}}=\frac{\sqrt{3}}{2}$$

$$K_2=-\frac{2\sqrt{3}}{9}=-A_y\cdot\sin(\Theta_0) \Rightarrow \sin(\Theta_0)=\frac{K_2}{-A_y}=\frac{-\frac{2\sqrt{3}}{9}}{-\frac{4\sqrt{3}}{9}}=\frac{1}{2}$$

$$\left.\begin{aligned}\cos\Theta_0&=\frac{\sqrt{3}}{2}>0\\ \sin\Theta_0&=\frac{1}{2}>0\end{aligned}\right\}\text{ prvi kvadrant}$$

$$\Theta_0=\operatorname{atan}\left(\frac{\sin\Theta_0}{\cos(\Theta_0)}\right)=\operatorname{atan}\left(\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)=$$



$\Theta_0=\pi/6$

$$=\operatorname{atan}\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$$

Dakle rješenje glasi:

$$\boxed{\begin{aligned} y_{p0}[n] &= A_{y0} \cdot \cos(\omega_0 n + \theta_0) \\ &= \frac{4}{9}\sqrt{3} \cdot \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right) \end{aligned}}$$

Dobili smo isto rj. kao i metodom sa kompleksnom eksponencijalom, ali postupkom koji je nešto duži i složeniji jer zahtjeva raspisivanje sinusa i kosinusa, kao i rješavanje sustava jed. sa dvije nepoznanice k_1, k_2, te konačno pretvorbu rješenja $k_1 \cos + k_2 \sin$ u $A \cdot \cos(\omega_0 n + \theta_0)$

Sada uz poznato partikularno rj. možemo odrediti odziv mirnog sustava (onog čija su početna stanja $= 0$)

Taj odziv $y_m[n]$ mora sadržavati i titranje vlastitim frekvencijama q_1, q_2 jer sustav „nije spreman" na pobudu koja se pojavila u trenutku $n = 0$, jer su njegova stanja jednaka nuli. Da je pobuda djelovala od $n = -\infty$ na dalje, tada bi stanja stabilnog sustava u trenutku $n = 0$ bila sukladna pobudi, jer bi prijelazna pojava davno već utrnula (nestala) te bi se sustav nastavio se odazivati frekvencijama pobude.

Međutim kako pobuda djeluje tek od $n = 0$ na dalje.. jer $u[n] = (\ )\cdot\mu(n)$ dolazi do prijelazne pojave, pa pišemo

$$y_m[n] = c_3 q_1^n + c_4 q_2^n + \underbrace{y_{p}[n]}_{\text{ovo smo upravo odredili}}$$

Zbog činjenice da je sustav miran znamo da je

$y_m[n] = 0$ za $n < 0$, a zbog kauzalnosti pobudnog signala znamo da je $u[n] = 0$ također za $n < 0$.

Partikularno rješenje $y_p[n]$ vrijedi samo za $n \geq 0$, pa stoga u svrhu određivanja nepoznatih koef. c_3 i c_4 moramo odabrati bilo koja dva uzorka odziva $y_m[n]$ za $n \geq 0$, npr. $n = 0$ i $n = 1$. ili npr. $n = 3$ i $n = 7$.

Međutim $y_m[n]$ nije poznat za te n-ove, te ga trebamo odrediti metodom korak po korak.

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Znamo da je $y_m(-2) = y_m(-1) = 0$ $y_{pr}[n] = \frac{4\sqrt{3}}{9}\cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)$

$u_r[n] = \cos\left(\frac{2\pi}{3}n + \frac{\pi}{3}\right)$

Iz jed. dif.

$u_r(0) = \cos\frac{\pi}{3} = \frac{1}{2}$

$u_r(1) = \cos(\pi) = -1$

$$\boxed{y_m(0)} = -a_1 y_m(-1) - a_2 y(-2) + u_r(0)$$
$$= \frac{1}{2} y_m(-1) - \frac{1}{4} y_m(-2) + \frac{1}{2}$$
$$= \frac{1}{2}\cdot 0 - \frac{1}{4}\cdot 0 + \frac{1}{2} = \frac{1}{2}$$

$$\boxed{y_m(1)} = \frac{1}{2} y_m(0) - \frac{1}{4} y_m(-1) + u_r(1)$$
$$= \frac{1}{2}\cdot\frac{1}{2} - \frac{1}{4}\cdot 0 + (-1) = -\frac{3}{4}$$

Iz pretpostavljenog općeg rješenja za $y_m(n)$ za $n = 0$ & 1 slijedi:

$y_{pr}[0] = \frac{4\sqrt{3}}{9}\cdot\frac{\sqrt{3}}{2} = \frac{2}{3}$

za $n=0$ $\quad y_m(0) = c_3\cdot q_1^0 + c_4\cdot q_2^0 + y_{pr}(0)$

za $n=1$ $\quad y_m(1) = c_3\cdot q_1^1 + c_4 q_2^1 + y_{pr}(1)$

$y_{pr}[1] = \frac{4\sqrt{3}}{9}\cdot\cos\left(\frac{5\pi}{6}\right) = -\frac{2}{3}$

$$\left.\begin{aligned} y_m(0) &= \frac{1}{2} = c_3 + c_4 + \frac{2}{3} \\ y_m(1) &= -\frac{3}{4} = c_3 q_1 + c_4 q_2 - \frac{2}{3}\end{aligned}\right\}$$

$c_3 + c_4 = -\frac{1}{6}$

$c_3 q_1 + c_4 q_2 = -\frac{1}{12}$

Dvije jedn. sa dvije nep.

$c_3 = -c_4 - \frac{1}{6}$

$-c_4 q_1 - \frac{1}{6} q_1 + c_4 q_2 = -\frac{1}{12}$

$c_4(q_2 - q_1) = -\frac{1}{12} + \frac{1}{6} q_1$

$$\left.\begin{aligned} 2\cdot q_1 &= \frac{1}{2} + j\frac{\sqrt{3}}{2} \\ 2\cdot q_2 &= \frac{1}{2} - j\frac{\sqrt{3}}{2}\end{aligned}\right\}$$

$q_2 - q_1 = -j\sqrt{3}/2$

$-1 + 2q_1 = \cancel{-1} + \cancel{\frac{1}{2}} + j\frac{\sqrt{3}}{2}$

$= j\frac{\sqrt{3}}{2} - \frac{1}{2}$

$c_4 = \frac{1}{12}\cdot\frac{(-1 + 2q_1)}{q_2 - q_1} =$

$= \frac{1}{12}\cdot\frac{-\frac{1}{2} + j\frac{\sqrt{3}}{2}}{-j\frac{\sqrt{3}}{2}}\cdot\frac{\frac{2}{\sqrt{3}}j}{\frac{2}{\sqrt{3}}j} =$

$= \frac{1}{12}\left(-\frac{1}{\sqrt{3}}j - 1\right)$

$= -\frac{1}{12}\cdot\frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}j\right)$

$= -\frac{\sqrt{3}}{18}\cdot e^{j\frac{\pi}{6}}$

$c_3 = c_4^* = -\frac{\sqrt{3}}{18}\cdot e^{-j\frac{\pi}{6}}$

Provjera:

$$q_1 = \frac{1}{2} \cdot e^{j\pi/3} \qquad c_4 = -\frac{\sqrt{3}}{18} \cdot e^{j\pi/6}$$

$$q_2 = \frac{1}{2} \cdot e^{-j\pi/3} \qquad c_3 = -\frac{\sqrt{3}}{18} \cdot e^{-j\pi/6}$$

$$q_1 \cdot c_3 + q_2 \cdot c_4 = \frac{1}{2} \cdot \left(-\frac{\sqrt{3}}{18}\right) \cdot \left(e^{j\frac{\pi}{3}} \cdot e^{-j\pi/6} + e^{-j\frac{\pi}{3}} \cdot e^{j\pi/6}\right)$$

$$= \frac{1}{2}\left(\frac{-\sqrt{3}}{18}\right) \cdot \left(e^{j\pi/6} + e^{-j\pi/6}\right) = \left(-\frac{\sqrt{3}}{18}\right) \cdot \cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{\cancel{18}_{6}} \cdot \frac{\sqrt{3}}{2} = -\frac{1}{12} \text{ ok}$$

$$c_3 + c_4 = -\frac{\sqrt{3}}{18}\left(e^{j\pi/6} + e^{-j\pi/6}\right) = -\frac{\sqrt{3}}{18} \cdot 2\cos\left(\frac{\pi}{6}\right) = -\frac{\sqrt{3}}{\cancel{18}_{6}} \cdot \cancel{2} \cdot \frac{\sqrt{3}}{\cancel{2}} = -\frac{1}{6} \text{ ok}$$

Dobivamo odziv mirnog sustava vlastitim frekvencijama:

$$y_{m\text{-}vlast}[n] = c_3 \cdot q_1^n + c_4 \cdot q_2^n = -\frac{\sqrt{3}}{18} \cdot |q_0|^n \cdot \left(e^{-j\pi/6} \cdot e^{j\frac{\pi}{3} \cdot n} + e^{j\frac{\pi}{6}} \cdot e^{-j\frac{\pi}{3} n}\right)$$

$$= -\frac{\sqrt{3}}{9} \cdot |q_0|^n \cdot \cos\left(\frac{\pi}{3}n - \frac{\pi}{6}\right) \text{ ok}$$

$$= \frac{\sqrt{3}}{9} \cdot |q_0|^n \cdot \cos\left(\frac{\pi}{3}n + \frac{5\pi}{6}\right)$$

$$= \frac{\sqrt{3}}{9} \cdot 2^{-n} \cdot \cos\left(\frac{\pi}{3}n + \frac{5\pi}{6}\right)$$

Za vježbu odredite početna stanja $y(-1)$ & $y(-2)$ takva da odziv mirnog sustava vlastitim frekvencijama bude jednak odzivu nepobuđenog sustava, ali suprotnog predznaka tj:

$$y_{m\text{-}vlast}[n] = -y_0[n]$$

$$\Rightarrow \quad c_1 = -c_3$$
$$c_2 = -c_4$$

Konačno dobivamo izraz za odziv mirnog sustava (18)

$$y_m[n] = \underbrace{\frac{\sqrt{3}}{9} 2^{-n} \cdot \cos\left(\frac{\pi}{3}n + \frac{5\pi}{6}\right)} + \underbrace{\frac{4\sqrt{3}}{9} \cdot \cos\left(\frac{2\pi}{3}n + \frac{\pi}{6}\right)}$$

titranje vlastitim frekvencijama zbog nesklada početnog stanja sustava (nul-stanja) i stanja koje bi odgovaralo sve-vremenskoj pobudi koja djeluje od $-\infty$

titranje frekvencijama pobude

$\boxed{\Rightarrow \text{prisilni odziv}}$

Odziv nepobuđenog sustava sa početnim stanjima: $y(-1)=2$ & $y(-2)=8$ bio je:

$$y_0[n] = 2^{1-n} \cdot \cos\left(n \cdot \frac{\pi}{3} + \frac{2\pi}{3}\right)$$

ista frekvencija kao i dli druga početna faza i druga amplituda !!

Označimo ovaj dio odziva sa $y_{m\_vlast}[n]$

$y_{m\_vlast}[n]$ i $y_0[n]$ se mogu „udružiti" u jedan signal frekvencije $\Omega_0 = \pi/3$, a neke nove amplitude i početne faze. Ovo udruživanje je najlakše provesti zbrajanjem c-koeficijenata

$\boxed{\text{prirodni odziv}}$

$$y_{m\_vlast}[n] + y_0[n] = (c_1+c_3)\, z_1^n + (c_2+c_4)\, z_2^n = y_{prir}[n]$$

$$\left.\begin{array}{l} c_1 = c_2^* \\ c_3 = c_4^* \end{array}\right\} \Rightarrow (c_1+c_3) = (c_2+c_4)^*$$

od prije sa strane 4 znamo da je

$$c_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} j$$

a na strani 16 imamo c_3 & c_4

$$c_3 = -\frac{\sqrt{3}}{18} \cdot e^{-j\pi/6} = -\frac{\sqrt{3}}{18}\left(\frac{\sqrt{3}}{2} - j\frac{1}{2}\right) = -\frac{1}{12} + j\frac{\sqrt{3}}{36}$$

Dakle $c_1 + c_3 = -\frac{1}{2} + \frac{\sqrt{3}}{2} j - \frac{1}{12} + j\frac{\sqrt{3}}{36} = \frac{-7}{12} + \frac{19}{36} j\sqrt{3}$ w ok

$$|c_1+c_3| = \sqrt{\frac{49}{144} + \frac{19\cdot 19\cdot \cancel{3}}{\underbrace{36\cdot \cancel{36}_{12}}_{3\cdot 144}}} = \sqrt{\frac{3\cdot 49 + 19\cdot 19}{3\cdot 144}} = \sqrt{\frac{508}{3\cdot 144}} = \frac{2\sqrt{127}}{12\cdot\sqrt{3}} = \frac{1}{6}\sqrt{\frac{127}{3}}$$

$$\measuredangle(c_1+c_3) = \operatorname{atan2}(Im, Re) = \operatorname{atan2}\left(\frac{19\sqrt{3}}{36}, -\frac{7}{12}\right)$$

$$= 0.6808\,\pi$$

$$|c_2+c_4| = |c_1+c_3| = \frac{1}{6}\cdot\sqrt{\frac{127}{3}}$$

$$\measuredangle(c_2+c_4) = -\measuredangle(c_1+c_3)$$

$$y_{prir}(n) = |c_1+c_3|\cdot e^{j\measuredangle(c_1+c_3)}\cdot|z_0|^n\cdot e^{j\theta_0 n} + |c_2+c_4|\cdot e^{j\measuredangle(c_2+c_4)}|z_0|^n\cdot e^{-j\theta_0 n}$$

$$= \frac{1}{\underset{3}{\cancel{6}}}\sqrt{\frac{127}{3}}\cdot 2^{-n}\cdot \cancel{2}\cos\left(\theta_0 n + \measuredangle(c_1+c_3)\right)$$

$$\underbrace{= \frac{1}{3}\sqrt{\frac{127}{3}}\cdot 2^{-n}\cos\left(\theta_0 n + \operatorname{atan2}\left(\frac{19\sqrt{3}}{36}, \frac{-7}{12}\right)\right)}$$ ok w

ukupno titranje vlastitim frekvencijama

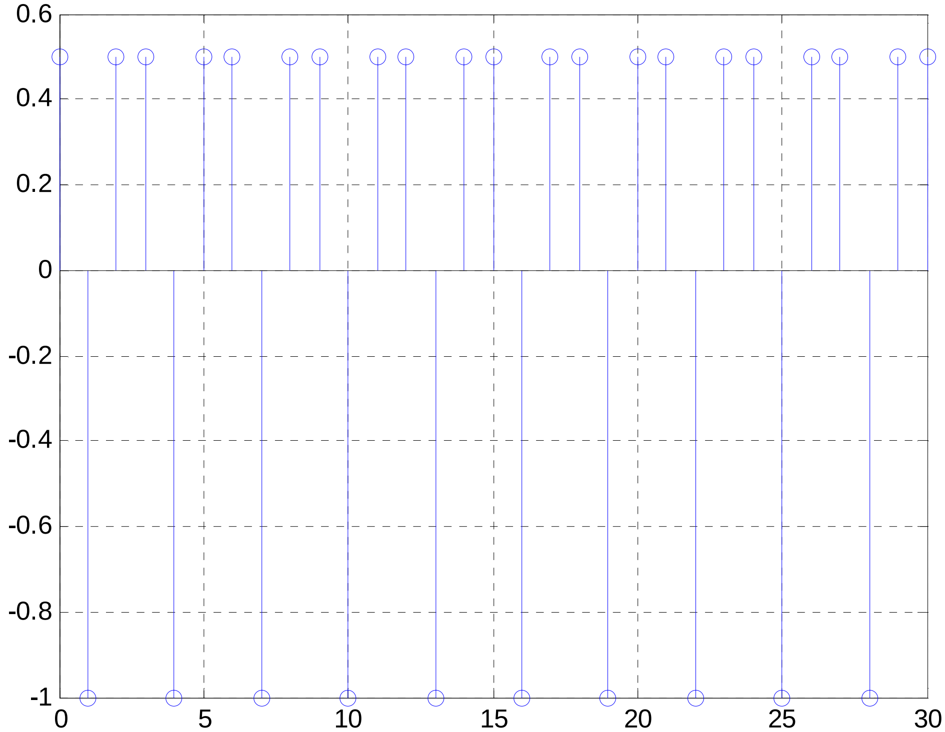
ili prirodni odziv sustava

Traženi Matlab program

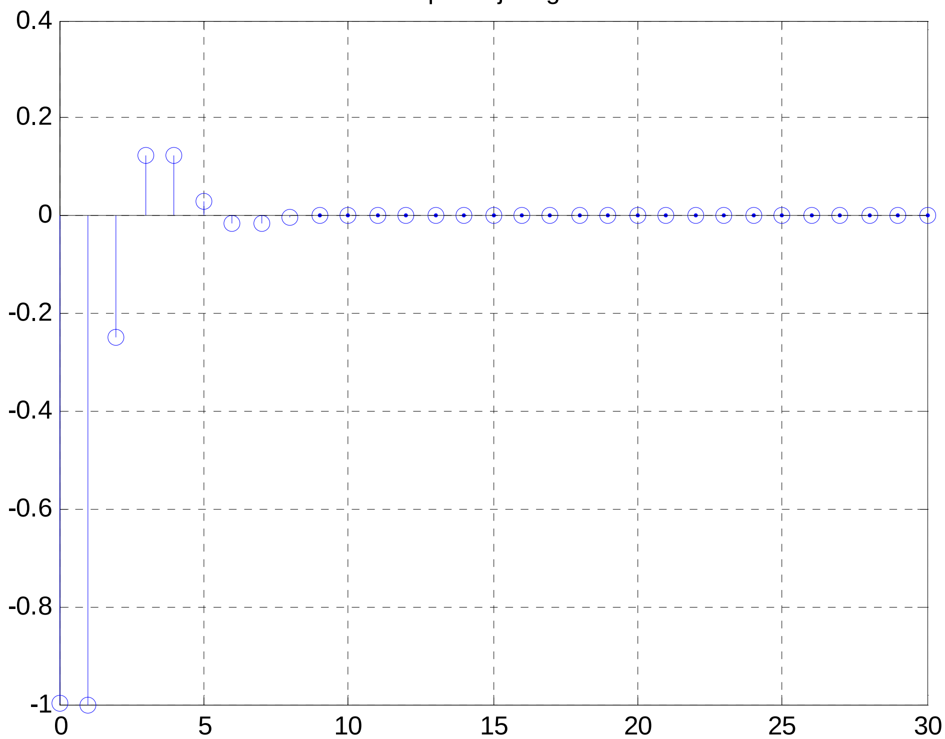
```
clear;
n_max=30;
% vremenska os
n=[0:n_max];

% Odziv mirnog sustava vlastitim frekvencijama
ym_vlast=sqrt(3)/9*(2.^(-n)).*cos(pi/3*n+5*pi/6);

% Odziv nepobudjenog sustava vlastitim frekvencijama
y0=(2.^(1-n)).*cos(pi/3*n+2*pi/3);

% Pocetna faza ukupnog titranja vlastitim frekvencijama
an=atan2(19*sqrt(3)/36,-7/12);

% Ukupno titranje vlastitim frekvencijama
y_prir=sqrt(127/3)/6*(2.^(1-n)).*cos(pi/3*n+an);

% pokazi da je y_prir jednak sumi ym_vlast i y0
disp(max(abs(y_prir-(y0+ym_vlast))))

% partikularno rjesenje ... titranje frekvencijama pobude
y_par=4*sqrt(3)/9*cos(2*pi/3*n+pi/6);

% totalni odziv (suma titranja vlastitim frekvencijama i
% prisilnog odziva)
ytot=y_prir+y_par;

% Ukupni odziv mirnog sustava
y_mir=ym_vlast+y_par;

% Pobuda sustava
u=1*cos(2*pi/3*n+pi/3);

% Inicijaliziraj vektor odziva za metodu korak po korak
y=0*u;

% Odredi odziv metodom korak po korak
y_nm_2=8;     % pocetna vrijednost y(n-2) za n=0 ... y(-2)
y_nm_1=2;     % pocetna vrijednost y(n-1) za n=0 ... y(-1)

% koeficijenti jedn. dif. uz y(n-1) i y(n-2)
a1=-1/2;
a2=1/4;

% koef. jedn. dif. uz u(n)
b0=1;

for nn=0:n_max,
   % Zbog Matlaba koji ne pozna indekse polja koji su manji od
   % jedan ... vremenski indeks n=0 pretvaramo u indeks polja 1
   nM=nn+1;
   % Jednadzba diferencije ...
   ykk(nM)= -a1*y_nm_1 -a2*y_nm_2 +b0*u(nM);

   % za novi prolaz petlje (n+1)
   y_nm_2=y_nm_1;     % y((n+1)-2) = y(n-1)
   y_nm_1=ykk(nM);    % y((n+1)-1) = y(n)

end;

% Usporedi analiticko rjesenje i ono dobiveno metodom korak po korak
disp(max(abs(ykk-ytot)))
```

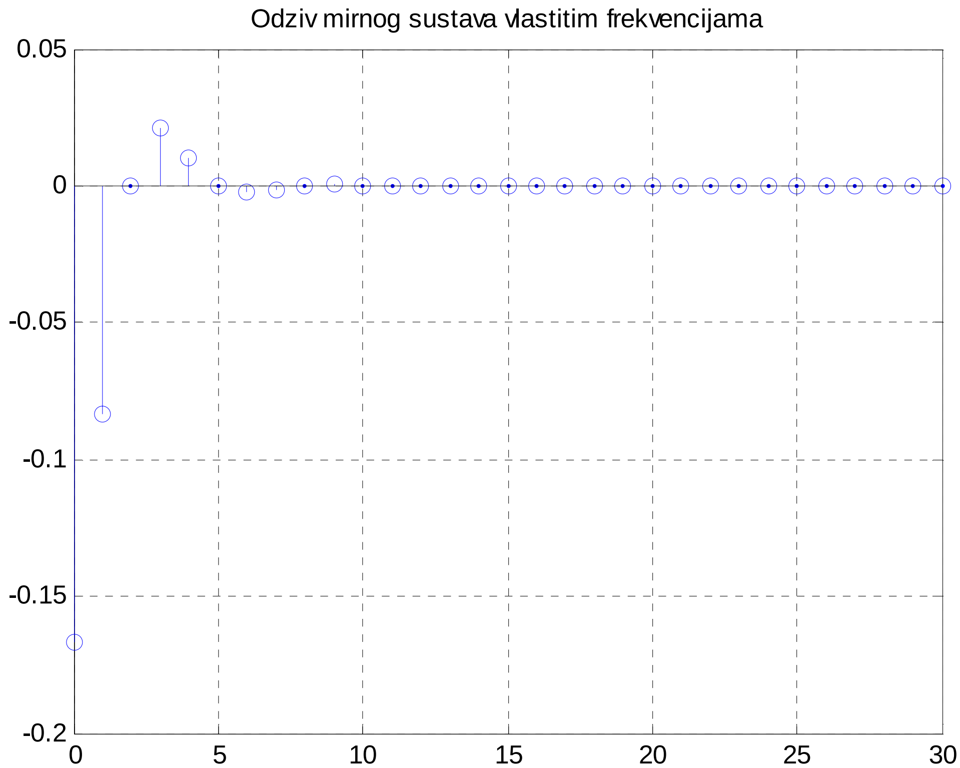
```
figure(1);
stem(n,u)
title('Pobuda sustava');
grid;

figure(2);
stem(n,y0)
title('Odziv nepobudjenog sustava');
grid;

figure(3);
stem(n,ym_vlast)
title('Odziv mirnog sustava vlastitim frekvencijama');
grid;

figure(4);
stem(n,y_par)
title('Prisilni odziv sustava');
grid;

figure(5);
stem(n,y_prir)
title('Prirodni odziv sustava');
grid;

figure(6);
stem(n,y_mir)
title('Ukupni odziv mirnog sustava');
grid;

figure(7);
stem(n,ytot)
title('Ukupni odziv sustava - analiticko rjesenje');
grid;

figure(8);
stem(n,ykk)
title('Ukupni odziv sustava - korak po korak');
grid;
```

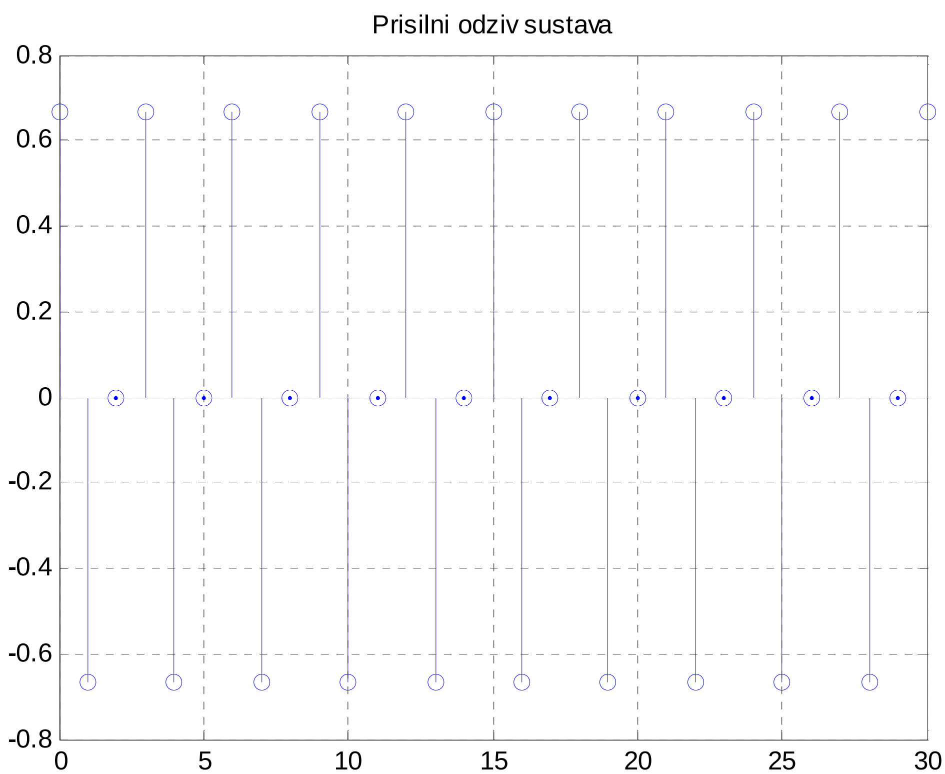
Pobuda sustava

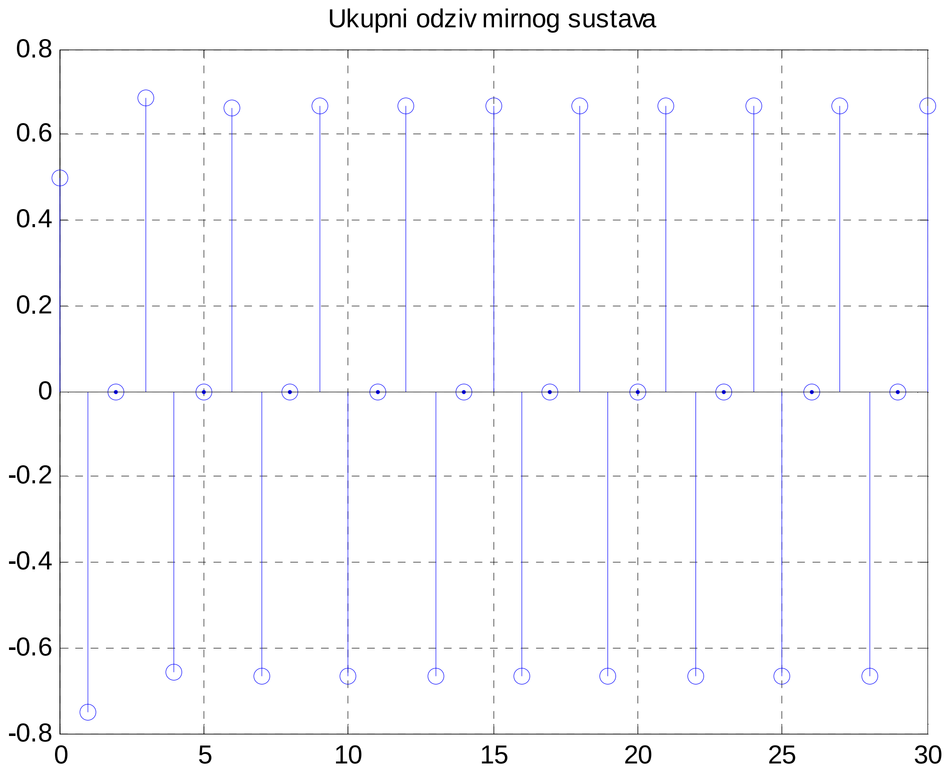


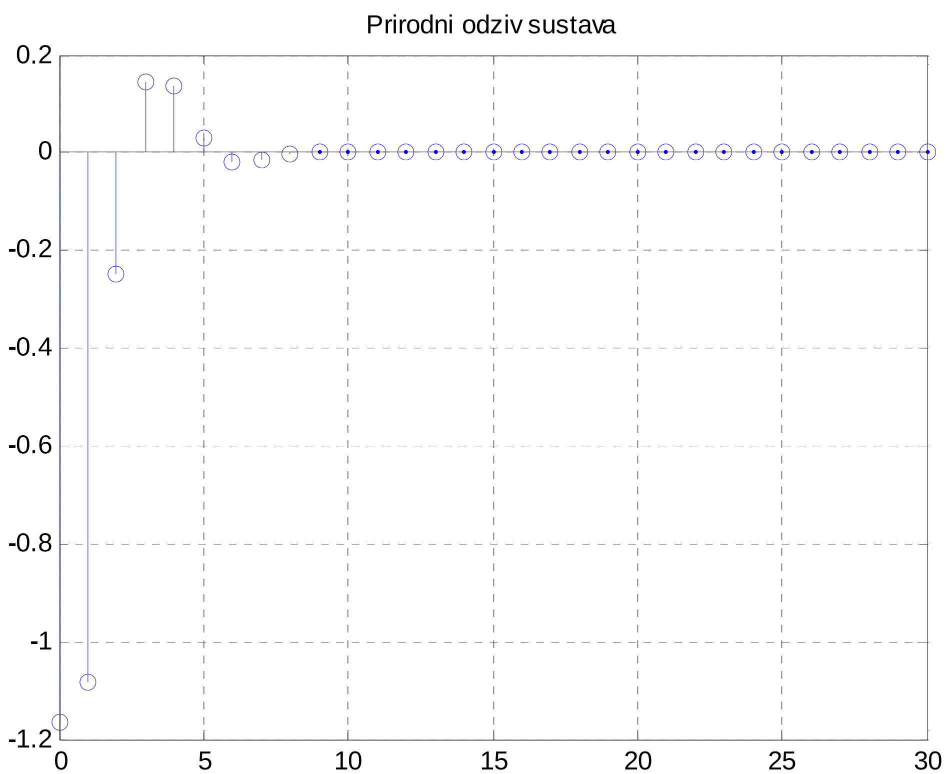
Odziv nepobudjenog sustava

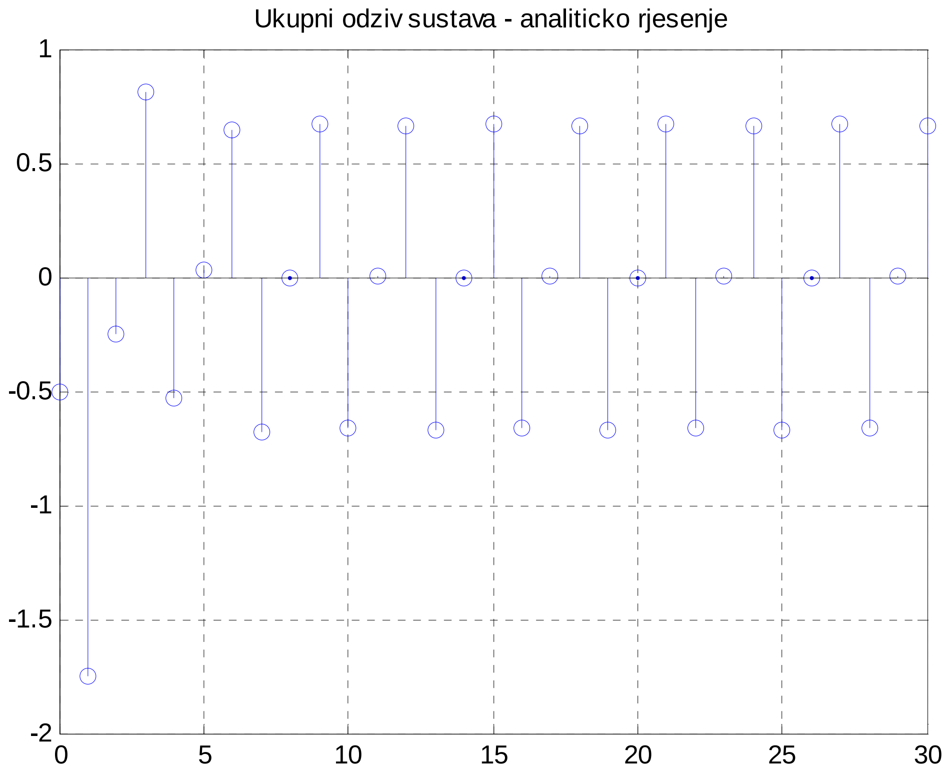


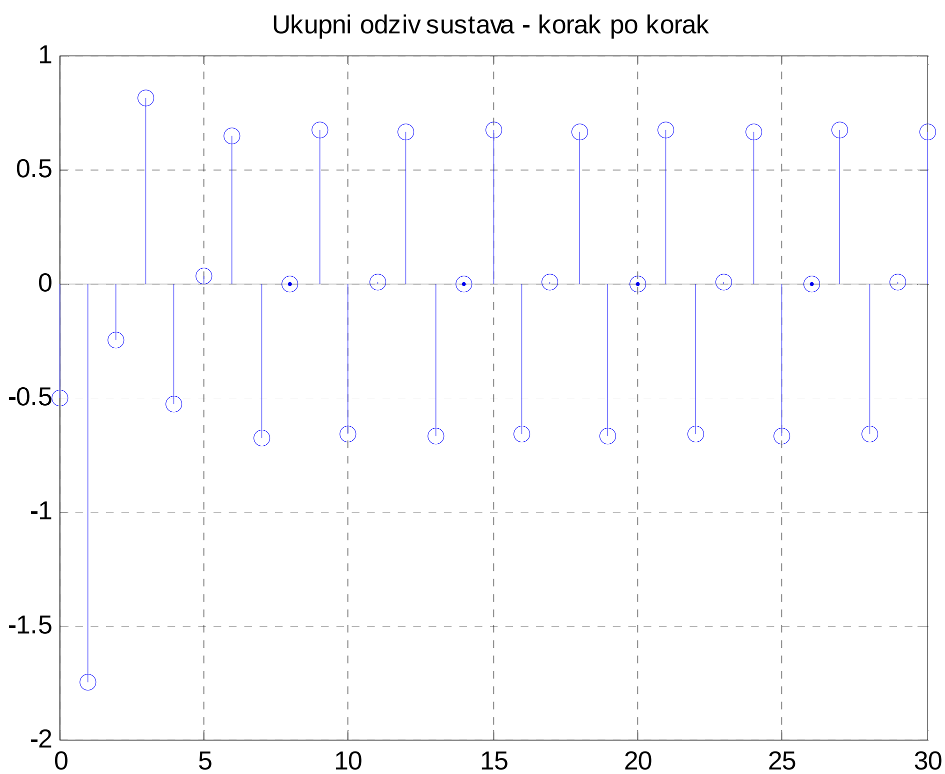

Odziv mirnog sustava vlastitim frekvencijama
0.05
0
-0.05
-0.1
-0.15
-0.2
0
5
10
15
20
25
30


Prisilni odziv sustava
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30


Ukupni odziv mirnog sustava
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
0
5
10
15
20
25
30


Prirodni odziv sustava
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1.2
0
5
10
15
20
25
30


Ukupni odziv sustava - analiticko rjesenje
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30


Ukupni odziv sustava - korak po korak
1
0.5
0
-0.5
-1
-1.5
-2
0
5
10
15
20
25
30

PRIMJER RJEŠAVANJA DIFERENCIJALNIH JEDNADŽBI U VREMENSKOJ DOMENI

- za sustav 1. reda oblika

$$y' + a_1 y = b_0 u' + b_1 u$$

za pobude oblika: $u(t) = A_0 \sin(\omega_0 t) \cdot \mu(t)$

ili $u(t) = A_0 \cos(\omega_0 t + \delta_0) \cdot \mu(t)$

ili $u(t) = A_0 \cdot \mu(t)$

- postupak rješavanja za pobude bez $\delta(t)$ i za pobude koje sadrže $\delta(t)$

- nalaženje impulsnog odziva

Problem prebacivanja poč uvjeta iz 0^- u 0^+

Za dif jedn. oblika

$$y^{(N)} + a_1 y^{(N-1)} + a_2 y^{(N-2)} + \dots + a_{N-1} y' + a_N y =$$

$$b_0 u^{(N)} + b_1 u^{(N-1)} + \dots + b_{N-1} u' + b_N u$$

zadani su poč. uvj. u $t = 0^-$

$$y(0^-),\ y'(0^-),\ y''(0^-) \dots y^{(N-2)}(0^-),\ y^{(N-1)}(0^-)$$

Radi određivanja koeficijenata uz članove odziva vlastitim frekvenc. ($c_1 \dots c_n$) potrebno je odrediti poč. uv. za $t = 0^+$

Na predavanjima je pokazano da se to radi rješavanjem sustava $N \times N$ oblika:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_1 & 1 & 0 & & 0 \\ a_2 & a_1 & 1 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 0 \\ a_{N-1} & a_{N-2} & a_{N-3} \ \dots & a_1 & 1 \end{bmatrix}}_{A} \cdot \begin{bmatrix} \Delta y \\ \Delta y' \\ \Delta y'' \\ \vdots \\ \Delta y^{(N-1)} \end{bmatrix} = \underbrace{\begin{bmatrix} b_0 & 0 & 0 & \cdots & 0 \\ b_1 & b_0 & 0 & & 0 \\ b_2 & b_1 & b_0 & \ddots & 0 \\ \vdots & & & \ddots & \vdots \\ & & & & 0 \\ b_{N-1} & b_{N-2} & b_{N-3} \ \dots & b_1 & b_0 \end{bmatrix}}_{B} \cdot \begin{bmatrix} u(0^+) \\ u'(0^+) \\ u''(0^+) \\ \vdots \\ u^{(N-1)}(0^+) \end{bmatrix}$$

Gdje je traženi stupac:
$$\begin{bmatrix} \Delta y \\ \Delta y' \\ \vdots \\ \Delta y^{(N-1)} \end{bmatrix} = \begin{bmatrix} y(0^+) - y(0^-) \\ y'(0^+) - y'(0^-) \\ \vdots \\ y^{(N-1)}(0^+) - y^{(N-1)}(0^-) \end{bmatrix}$$

dakle traženi vektor poč. uv. se nalazi kao:

$$\begin{bmatrix} y(0^+) \\ y'(0^+) \\ \vdots \\ y^{(N-1)}(0^+) \end{bmatrix} = \begin{bmatrix} y(0^-) \\ y'(0^-) \\ \vdots \\ y^{(N-1)}(0^-) \end{bmatrix} + \begin{bmatrix} \Delta y \\ \Delta y' \\ \vdots \\ \Delta y^{(N-1)} \end{bmatrix}$$

Sustav jednadžbi je donjeg trokutastog oblika pa se rješenje nalazi jednostavnom unaprijednom supstitucijom počevši od Δy, pa $\Delta y'$, ... sve do $\Delta y^{(N-1)}$.

Uoči da sustav jed. ne sadrži koeficijente a_N i b_N iz diferencijalne jednadžbe !!

Ako desna strana jed. degenerira u slijedeći oblik

$$(\text{lijeva}) = b_N \cdot u$$

to znači da su svi koef. uz derivacije pobude $(b_{N-1}, \ldots b_0)$ jednaki nuli. Zbog toga desna strana matrične jednadžbe postaje jednaka nul-vektoru i to neovisno o konačnim vrijednostima pobude i njenih derivacija u $\emptyset$, jer matrica B postaje nul-matrica:

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \Delta y \\ \Delta y' \\ \vdots \\ \Delta y^{(N-1)} \end{bmatrix} = \begin{bmatrix} 0 \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix}$$

Determinanta matrice A jednaka je 1, $\det(A) = 1$, neovisno o vrijednostima a-koef. dif. jednadžbe, pa stoga slijedi da je konačno rješenje $\begin{bmatrix} \Delta y \\ \Delta y' \\ \vdots \\ \Delta y^{(N-1)} \end{bmatrix} = \begin{bmatrix} \emptyset \\ \emptyset \\ \vdots \\ \emptyset \end{bmatrix}$

Dakle pokazali smo da će početni uvjeti biti jednaki u $t = 0^-$ i $t = \emptyset^+$ ako desna strana diferencijalne jednadžbe sadrži samo zadnji član $(b_N \cdot u(t))$. $\Rightarrow$ $\boxed{y^{(i)}(0^+) = y^{(i)}(0^-)}$

Prema tome, ako se desna strana prevede u ovu formu, tada nije uopće potrebno provoditi preračunavanje rubnih uvjeta !!

Kako to napraviti ... JEDNOSTAVNIM deriviranjem pobude kako to traži desna strana

Umjesto zadane pobude $u(t)$ formiramo novi pobudni signal $u_n(t)$ kao:

$$u_n(t) = b_0 \cdot u^{(N)}(t) + b_1 u^{(N-1)}(t) + \ldots + b_{N-1} u'(t) + b_N u(t)$$

te formiramo novu diferencijalnu jednadžbu:

$$(\text{lijeva}) = b_{N_{novo}} \cdot u_n(t) \quad ,$$

gdje je $b_{N_{novo}} = 1$

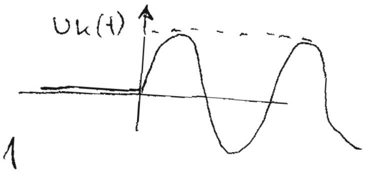
Dodatna prednost deriviranja pobude jest u svrhu provjere da li se u $u_n(t)$ javljaju $\delta(t)$ funkcije ili njihove derivacije, jer ako je to slučaj tada se dif. jednadžba ne može rješavati uobičajenim postupkom sa metodom neodređenog koeficijenta i pretpostavkom sve-vremenske pobude i sve-vremenskog partikularnog rješenja, nego se mora koristiti pristup temeljen na određivanju impulsnog odziva, odnosno njegovih derivacija.

Ovi postupci će biti ilustrirani na primjeru sustava prvog reda.

Promotrimo prvo pobudu sustava i desnu stranu dif. jed. 2

$u_s(t) = A_0 \sin(\omega_0 t)$ sve-vremenska pobuda

kauzalna pobuda $u_k(t) = u_s(t) \cdot \mu(t)$
$= A \sin(\omega_0 t) \cdot \mu(t)$



Desna strana diferenc. jed. sustava prvog reda za $M = N = 1$

(lijeva) $= b_0 u'(t) + b_1 u(t)$

Za kauzalnu pobudu $u_k(t)$ uvrštavanje u desnu stranu daje:

$$b_0 \cdot \left(A_0 \sin(\omega_0 t) \cdot \mu(t)\right)' + b_1 A_0 \sin(\omega_0 t) \cdot \mu(t) =$$

$$= b_0 \left(A_0 \sin(\omega_0 t) \cdot \underbrace{\delta(t)}_{\neq 0 \text{ za } t=0} + A_0 \omega_0 \cos(\omega_0 t)\, \mu(t) \right) + b_1 A_0 \sin(\omega_0 t) \cdot \mu(t)$$

$A \sin(\omega_0 t)$ za $t = 0$ iznosi 0 ... dakle ovaj član je $0 \cdot \delta(t)$, pa stoji desna strana jest:

$$= \left(b_0 A_0 \omega_0 \cos(\omega_0 t) + b_1 A_0 \sin(\omega_0 t) \right) \cdot \mu(t)$$

Uočimo da desna strana ne sadrži $\delta(t)$ funkcije ili pak njihove derivacije. Dakle problem se može rješavati postupcima opisanim u 11. predavanju i početku 12. predavanja.

Cijela desna strana diferencijalne jednadžbe može se zamijeniti novom pobudom $u_n(t)$ i novim koef. desne strane: b_{0n} i b_{1n},

$$(\text{lijeva}) = \underbrace{0}_{b_{0n}} \cdot u_n'(t) + \underbrace{1}_{b_{1n}} \cdot u_n(t)$$

gdje je $u_n(t) = \left(A \cos(\omega_0 t) + B \sin(\omega_0 t)\right) \cdot \mu(t)$

a konstante A & B su:

$$A = b_0 A_0 \omega_0$$
$$B = b_1 A_0$$

ALTERNATIVNI PRIKAZ HARMONIJSKE POBUDE

Pobude oblika
$A\cos(\omega_0 t) + B\sin(\omega_0 t)$
ili $|C|\cdot\cos(\omega_0 t+\gamma)$
pogodno je prikazati u obliku kompleksne exponenc.
$Re(C\cdot e^{j\omega_0 t})$

$$= Re(C\cdot e^{j\omega_0 t}) \qquad C = |C|\cdot e^{j\gamma}$$

$$= Re(|C|\cdot e^{j\gamma}\cdot e^{j\omega_0 t})$$

$$= Re(|C|\cdot e^{j(\omega_0 t+\gamma)})$$

$$= |C|\cdot\cos(\omega_0 t+\gamma)$$

$$= |C|\cdot\cos(\omega_0 t)\cos\gamma - |C|\cdot\sin(\omega_0 t)\sin\gamma$$

$$= (|C|\cdot\cos\gamma)\cdot\cos(\omega_0 t) + (-|C|\cdot\sin\gamma)\cdot\sin(\omega_0 t)$$

$$= A\cdot\cos(\omega_0 t) + B\cdot\sin(\omega_0 t)$$

---

kako naći $|C|$ i γ iz $\boxed{A \text{ i } B}$?

$$A = |C|\cdot\cos\gamma \quad /^2$$
$$B = -|C|\cdot\sin\gamma \quad /^2 \quad +$$

$$A^2+B^2 = |C|^2\cdot(\cos^2\gamma+\sin^2\gamma) = |C|^2$$

$$\Rightarrow |C| = \sqrt{A^2+B^2}$$

$$\Rightarrow \cos\gamma = \frac{A}{|C|} \qquad \sin\gamma = -\frac{B}{|C|}$$

$$\gamma = \mathrm{atan2}(\sin\gamma, \cos\gamma)$$
$$= \mathrm{atan2}\left(-\frac{B}{|C|}, \frac{A}{|C|}\right)$$

Za odabrani primjer sustava prvog reda $A = b_0 A_0 \omega_0$, $B = b_1 A_0$

pa stoga: $|C| = \sqrt{(b_0 A_0\omega_0)^2 + (b_1 A_0)^2} = A_0\sqrt{b_0^2\omega_0^2+b_1^2}$

$$\cos\gamma = \frac{b_0\cancel{A_0}\omega_0}{\cancel{A_0}\sqrt{b_0^2\omega_0^2+b_1^2}} = \frac{b_0\omega_0}{\sqrt{b_0^2\omega_0^2+b_1^2}} \qquad \sin\gamma = -\frac{b_1\cancel{A_0}}{\cancel{A_0}\sqrt{b_0^2\omega_0^2+b_1^2}} = \frac{-b_1}{\sqrt{b_0^2\omega_0^2+b_1^2}}$$

$$\gamma = \mathrm{atan2}(\sin\gamma, \cos\gamma) = \mathrm{atan2}\left(\frac{b_0\omega_0}{\sqrt{b_0^2\omega_0^2+b_1^2}}, \frac{-b_1}{\sqrt{b_0^2\omega_0^2+b_1^2}}\right) = \mathrm{atan2}(b_0\omega_0, -b_1$$

pozitivni nazivnik ne utječe na kut

4

Dakle polazni sustav:

$$(\text{lijeva}) = b_0 u'(t) + b_1 u(t)$$

uz pobudu $u_k(t) = A_0 \sin(\omega_0 t) \cdot \mu(t)$

sveli smo na novi ekvivalentni sustav:

$$(\text{lijeva}) = \emptyset \cdot u_n'(t) + 1 \cdot u_n(t)$$

uz novu pobudu $u_n(t) = \mathrm{Re}(C e^{j\omega_0 t}) \cdot \mu(t)$

gdje je C kompleksna amplituda kompl. eksponencijale

$$C = |C| \cdot e^{j\gamma}, \quad |C| = A_0 \sqrt{b_0^2 \omega_0^2 + b_1^2}, \quad \gamma = \mathrm{atan2}(b_0 \omega_0, -b_1)$$

Obzirom da nova pobuda ne sadrži $\delta(t)$ možemo diferencijalnu jed. rješavati uz pretpostavku sve-vremenskih signala pobude i odziva, pa tek na samom kraju nakon određivanja rješenja na osnovu početnih uvjeta pomnožiti sve-vremensko rješenje odziva sa $\mu(t)$.

Dakle sve-vremensko partikularno rješenje možemo pretpostaviti u obliku:

$$y_{pc} = K \cdot e^{j\omega_0 t}$$

Uvrštavanjem ovakvog pretpostavljenog rj. u dif. jed. odredit ćemo kompleksni koeficijent K. Zbog činjenice da sustav ima realne koef. imaginarni dio odziva je odziv na imaginarni dio pobude $\mathrm{Im}(Ce^{j\omega_0 t})$, a realni dio odziva je odziv na realni dio pobude, jednako kao što je opisano kod vrem. diskretnih sustava. Dakle, nakon što nađemo K i y_{pc} realni dio rješenja predstavlja željeni partikularni dio rješenja

$$y_p(t) = \mathrm{Re}(y_{pc}(t)) = \mathrm{Re}(K \cdot e^{j\omega_0 t}),$$

uz $K = |K| \cdot e^{j\delta} \Rightarrow \quad y_p(t) = |K| \cdot \cos(\omega_0 t + \delta)$

Ilustrirajmo ovo na stvarnom primjeru:

$$a_0 y'(t) + a_1 y(t) = b_{0n} u_n'(t) + b_{1n} \cdot u_n(t)$$

gdje $b_{0n} = \emptyset$, $b_{1n} = 1$, $a_0 = 1$ (zbog normalizacije, ...

tj uvijek možemo lijevu i desnu stranu dif. jednadžbe podijeliti sa a_0, ako je on $\neq \emptyset$)

Dakle dif jed. glasi:

$$y'(t) + a_1 y(t) = u_n(t)$$

Uz pretpostavljenu sve-vremensku kompleksnu pobudu oblika:

$$u_{n_c}(t) = C \cdot e^{j\omega_0 t}, \quad \text{i odziv oblika } y_{pc}(t) = K \cdot e^{j\omega_0 t}$$

dobivamo uvrštavanjem u dif. jed.:

$$K \cdot j\omega_0 e^{j\omega_0 t} + a_1 K e^{j\omega_0 t} = C \cdot e^{j\omega_0 t} \quad / \cdot e^{-j\omega_0 t}$$

$$jK\omega_0 + a_1 K = C$$

$$K(j\omega_0 + a_1) = C \Rightarrow K = \frac{C}{j\omega_0 + a_1} = |K| \cdot e^{j\delta}$$

$$|K| = \frac{|C|}{\sqrt{a_1^2 + \omega_0^2}}$$

$$e^{j\delta} = e^{j\gamma} \cdot e^{-j\arg(\text{nazivnik})}$$
$$= e^{j\gamma} \cdot e^{-j\,\mathrm{atan2}(\omega_0, a_1)}$$
$$= e^{j(\gamma - \mathrm{atan2}(\omega_0, a_1))}$$
$$\Rightarrow \delta = \gamma - \mathrm{atan2}(\omega_0, a_1)$$

Dakle sve-vremensko partikularno rješenje je

$$y_p(t) = \mathrm{Re}(y_{pc}(t)) = \mathrm{Re}\left(|K| \cdot e^{j\delta} \cdot e^{j\omega_0 t}\right) =$$

$$= |K| \cdot \cos(\omega_0 t + \delta)$$

$$= \frac{|C|}{\sqrt{a_1^2 + \omega_0^2}} \cdot \cos(\omega_0 t + (\gamma - \mathrm{atan2}(\omega_0, a_1)))$$

predstavlja odziv na $\mathrm{Re}(u_{n_c}(t))$

Promotrimo sada lijevu stranu jednadžbe koja određuje titranje vlastitim frekvencijama. Neka je zadan početni uvjet $y(0^-) = y_{poc}$ u trenutku $t = 0^-$.

Nađimo prvo karakteristične frekvencije iz homogene jed.

$$y'(t) + a_1 y(t) = 0$$

Pretpostavimo sve-vremensko rješenje oblika

$$y_H(t) = C_1 \cdot e^{s_1 t}$$, te uvrstimo ga u dif. jed.

$$y_H'(t) + a_1 y_H(t) = C_1 s_1 e^{s_1 t} + a_1 C_1 e^{s_1 t} = C_1 e^{s_1 t} \underbrace{(s_1 + a_1)}_{\text{ne trivijalno rj.}} = 0$$

$s_1 + a_1 = 0$ ... karakteristična jednadžba

$s_1 = -a_1$ ... karakteristična frekv. (realna, jednostruka)

Dakle: $$y_H(t) = C_1 \cdot e^{-a_1 t}$$

Ako želimo odrediti odziv nepobuđenog sustava $y_0(t)$ dovoljno je odrediti konstantu C_1 iz zadanog rubnog uvjeta $\underbrace{y(0) = y(0^-) = y(0^+)}$ $= y_{poc}$

Ova jednakost uvijek vrijedi za odziv nepobuđenog sustava

Pa pišemo:

$$y_0(t) = y_H(t) = C_1 \cdot e^{-a_1 t}$$

za $t = 0$ $y_0(0) = C_1 \cdot e^0 = C_1$, a zadano je $y(0^-) = y_{poc}$, a obzirom da vrijedi , pišemo:

$$\boxed{y_{poc} = C_1}$$

Dakle odziv nepobuđenog sustava glasi:

$$y_0(t) = y_{poc} \cdot e^{-a_1 t}$$

Odredimo sada odziv mirnog sustava na traženu pobudu. Prije ovog, potvrdimo da je frekvencija pobude $\xi = j\omega_0$, a vlastita frekvencija sustava $s_1 = -a_1 \in \mathbb{R}$. Vidimo da $\xi \neq s_1$ osim za slučaj $\omega_0 = \emptyset$ (istosmjerni signal, $\cos(\emptyset t) = 1$). Dakle, osim ovog specijalnog slučaja kada je $a_1 = \emptyset$ & $\omega_0 = \emptyset$ vlastita frekvencija sustava se razlikuje od frekvencije pobude. Tada se partikularni dio rješenja smije pretpostaviti u obliku pobude $u_n(t)$

Dakle odziv mirnog sustava pretpostavljamo u obliku

$$y_m(t) = \underbrace{C_{1m} \cdot e^{s_1 t}}_{\substack{\text{titranje} \\ \text{vlastitim frek.} \\ \text{uslijed neskladа početnog} \\ \text{i stacionarnog stanja}}} + \underbrace{y_p(t)}_{\text{partikularno rj.}}$$

Konstantu C_{1m} moramo odrediti na osnovu početnog uvjeta $y_m(0^+)$, a ova se mora odrediti iz $y_m(0^-)$ koja je jednaka nuli, jer se radi o odzivu mirnog sustava na kauzalnu pobudu.

Prijenos početnog uvjeta iz 0^- u 0^+ radimo postupkom opisanom u 11. predavanju. (SLIDE 29) ... za sustav prvog reda

$$y_m(0^+) - y_m(0^-) = b_{0n} \cdot u_n(0^+)$$

Obzirom da je $b_{0n} = \emptyset$ neovisno o $u_n(0^+)$, slijedi da su početni uvjeti u 0^+ i 0^- jednaki: $y_m(0^+) = y_m(0^-)$, iako

$$u_n(0^+) = \mathrm{Re}(c \cdot e^{j\omega_0 \cdot 0^+}) = \mathrm{Re}(c) = |c| \cdot \cos\gamma,$$ što je općenito $\neq \emptyset$

ako su $b_0 = b_1 = \ldots = b_{N-1} = \emptyset$ ovo se ne mora raditi jer $y^{(i)}(0^+) = y^{(i)}(0^-)$

Da odredimo c_{1m}, evaluiramo rješenje $y_m(t)$ za $t = \emptyset^+$, te izjednačimo to rješenje sa početnim uvjetom $y_m(\emptyset^+)$, za koji smo pokazali da je jednak $y_m(0^-)$, a koji je jednak nuli.

$$y_m(0^+) = c_{1m} \cdot \underbrace{e^{s_1 \cdot 0^+}}_{1} + y_p(0^+) = \emptyset$$

Prije smo izveli $y_p(t) = \frac{|C|}{\sqrt{a_1^2 + w_0^2}} \cdot \cos(w_0 t + (\gamma - \mathrm{atan2}(w_0, a_1)))$

za $t = \emptyset^+$ imamo

$$y_p(0^+) = \frac{|C|}{\sqrt{a_1^2 + w_0^2}} \cdot \cos(\underbrace{\gamma - \mathrm{atan2}(w_0, a_1)}_{\delta})$$

Dakle obzirom da sve konstante $|C|, \gamma, a_1, w_0$ poznamo, možemo odrediti (izračunati) iznos $y_p(0^+)$, a tražena konstanta $c_{1m} = -y_p(\emptyset^+)$

Konačno dobivamo sve-vremenski odziv mirnog sustava!

$$y_m(t) = c_{1m} \cdot e^{s_1 t} + y_p(t)$$

$$= -\frac{|C|}{\sqrt{a_1^2 + w_0^2}} \cos(\delta) \cdot e^{-a_1 t} + \frac{|C|}{\sqrt{a_1^2 + w_0^2}} \cdot \cos(w_0 t + \delta)$$

Za provjeru uvjerimo se da uvrštavanjem $t = \emptyset$ u gornji izraz dobivamo $y_m(\emptyset) = \emptyset$, što predstavlja traženi početni uvjet ovog mirnog sustava.

$$y_m(t) = \frac{|C|}{\sqrt{a_1^2 + w_0^2}} \Big(\underbrace{\cos(w_0 t + \delta)}_{\text{titranje frekv. pobude } (jw_0)} - \cos(\delta) \cdot \underbrace{e^{-a_1 t}}_{\text{titranje vlastitim frekvencijama}} \Big)$$

Kauzalni odziv na kauzalnu pobudu $u_n(t) \cdot \mu(t)$ dobivamo množenjem dobivenog sve-vremenskog rješenja $y_m(t)$ sa $\mu(t)$

$$y_{m_{kauz}}(t) = y_m(t) \cdot \mu(t)$$

Da dobijemo totalni odziv sustava moramo zbrojiti odziv nepobuđenog sustava $y_0(t)$ i odziv mirnog sustava. Sve-vremensko rješenje je stoga:

$$y_{tot}(t) = y_0(t) + y_m(t)$$

$$= y_{poc}\cdot e^{-a_1 t} + \frac{|c|}{\sqrt{a_1^2+w_0^2}}\left(\cos(w_0 t+\delta) - \cos(\delta)\cdot e^{-a_1 t}\right), \quad \text{ili}$$

zbrajanjem dijelova koji opisuju titranje vlastitim frekvencijama imamo:

$$= \underbrace{\left(y_{poc} - \frac{|c|}{\sqrt{a_1^2+w_0^2}}\cos(\delta)\right)e^{-a_1 t}}_{y_{prir}(t)} + \underbrace{\frac{|c|}{\sqrt{a_1^2+w_0^2}}\cdot\cos(w_0 t+\delta)}_{y_{pris}(t)}$$

prirodni ili prijelazni odziv sustava = ukupno titranje vlastitim frekvencijama

prisilni odziv sustava

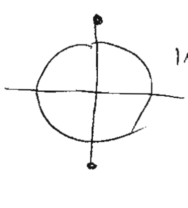
kauzalni totalni odziv

$$\boxed{y_{tot\_kauz}(t) = y_{tot}(t)\cdot \mu(t)}$$

Uočimo da za $t=0$ $y_{tot}(0) = y_{poc}$, a za $t\nearrow\infty$, ako je $Re(s_1)<0$ tada $|e^{+s_1 t}|$ teži $\searrow 0$. U našem slučaju $Re(s_1) = -a_1 < 0$ ako $a_1>0$. Inače za rubni slučaj $a_1=0$ odziv postaje:

$$y_{tot}(t)\Big|_{a_1=0} = \left(y_{poc} - \frac{|c|}{|w_0|}\cdot\cos(\delta)\right)\cdot \underset{\uparrow\,1}{e^0} + \frac{|c|}{|w_0|}\cdot\cos(w_0 t+\delta)$$

obzirom da je $\delta = \gamma - atan_2(\underset{\uparrow\,Im}{w_0}, \underset{\uparrow\,Re=0}{a_1}) = \gamma - sign(w_0)\cdot\frac{\pi}{2}$



Ako promatramo samo pozitivnu frekv. pobude $\omega_0 > 0$ tada je $\delta = \gamma - \frac{\pi}{2}$, pa odziv postaje:

$$y_{tot}(t)\Big|_{a_1=\emptyset} = y_{poc} + \frac{|c|}{\omega_0}\left(\cos(\omega_0 t + \gamma - \tfrac{\pi}{2}) - \cos(\gamma - \tfrac{\pi}{2})\right)$$

$$= y_{poc} + \underbrace{\frac{|c|}{\omega_0}\left(\sin(\omega_0 t + \gamma) - \sin(\gamma)\right)}_{\int_\emptyset^t u_n(\tau)\,d\tau}$$

Provjerimo ovo rješenje.....

Za $a_1 = \emptyset$ diferencijalna jednadžba degenerira u slijedeći oblik:

$$y'(t) = b_{1n} \cdot u_n(t), \quad \text{uz } b_{1n} = 1 \quad \Big/ \int_\emptyset^t$$

Integriramo obje strane jednakosti od nula do t

$$\int_\emptyset^t y'(\tau)\,d\tau = \int_\emptyset^t u_n(\tau)\cdot d\tau$$

$$y(t) - y(\emptyset) = \int_0^t u_n(\tau)\,d\tau,$$

$$y(t) = y(\emptyset) + \int_0^t |c|\cdot\cos(\omega_0\tau + \gamma)\,d\tau$$

$$= y(\emptyset) + |c|\cdot\frac{1}{\omega_0}\cdot\sin(\omega_0\tau + \gamma)\Big|_0^t$$

$$= y(\emptyset) + \underbrace{\frac{|c|}{\omega_0}\left(\sin(\omega_0 t + \gamma) - \sin(\gamma)\right)}_{\int_0^t u_n(\tau)\,d\tau}$$

OK ✓

Dakle za navedeni rubni slučaj kada se vlastita frekvencija s_1 nalazi na jw osi ($s_1 = -a_1 = \emptyset$) odziv vlastitim frekvencijama je nepriguŝen pa stoga početni uvjet y_{poc} ostaje trajno prisutan u odzivu sustava, tj. nikada se ne „istitra".

Kada totalni odziv postaje jednak prisilnom odzivu ? (11)

⇒ Nikada, ali mu asimptotski teži ako su vlastite frekvencije sustava takve <u>da vrijedi $Re(s_i) < 0$</u>.

Tada prirodni odziv sustava $y_{prir}(t)$ trne eksponencijalno kako vrijeme teži $\nearrow \infty$

$$y_{tot}(t) = y_{prir}(t) + y_{pris}(t)$$

$$|y_{prir}(t)| \searrow 0 \text{ kada } t \nearrow \infty \text{, dakle}$$

$$y_{tot}(t) \nearrow y_{pris}(t) \text{ kada } t \nearrow \infty$$

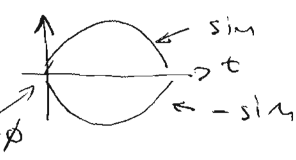
, pa kažemo da prisilni odziv sustava predstavlja stacionarno stanje totalnog odziva. Još ga nazivamo "ravnotežnim" stanjem.

Promotrimo sada isti sustav, ali uz drugačiju pobudu

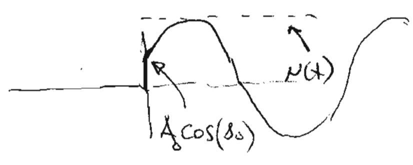
$$u_k(t) = A_0 \cos(\omega_0 t + \vartheta_0) \cdot \mu(t)$$

Do sada razmatran primjer odgovara specijalnom slučaju za $\vartheta_0 = -\pi/2$ kada se dobiva $A \sin(\omega_0 t) \cdot \mu(t)$

Za taj slučaj (ili za drugi slučaj kada $\vartheta_0 = \pi/2$) pobuda počinje iz 0 u $t=0$ kao sin (ili $-$sin)



Međutim za općenit (proizvoljan) ϑ_0 kauzalna pobuda ima skok u $t=0$ zbog množenja sa $\mu(t)$



Uvrstimo ovu novu pobudu u desnu stranu polazne diferencijalne jednadžbe:

$$(\text{lijeva}) = b_0 u'(t) + b_1 u(t)$$

$u(t) = u_k(t) = \underbrace{A_0 \cos(\omega_0 t + \delta_0)\cdot \mu(t)}$

produkt dvije funkcije koji deriviramo po pravilima deriviranja produkta

$$(\text{lijeva}) = b_0\big(A_0\omega_0\cdot(-\sin(\omega_0 t+\delta_0))\mu(t) +$$
$$A_0\cos(\omega_0 t+\delta_0)\cdot\delta(t)\big) +$$
$$+ b_1\cdot(A_0\cos(\omega_0 t+\delta_0)\cdot\mu(t))$$

$\neq \emptyset$ samo za $t=0$

$A_0\cos(\omega_0 t+\delta_0)$ za $t=\emptyset$ iznosi $A_0\cos(\delta_0)$

$$= -b_0A_0\omega_0(\sin(\omega_0 t)\cos\delta_0 + \cos(\omega_0 t)\cdot\sin\delta_0)\cdot\mu(t)$$
$$+ b_1A_0(\cos(\omega_0 t)\cdot\cos\delta_0 - \sin(\omega_0 t)\cdot\sin\delta_0)\cdot\mu(t)$$
$$+ b_0A_0\cos(\delta_0)\cdot\delta(t)$$

$$= [\cos(\omega_0 t)\cdot\overbrace{(-b_0A_0\omega_0\sin\delta_0 + b_1A_0\cos\delta_0)}^{A} +$$
$$\sin(\omega_0 t)\cdot\underbrace{(-b_0A_0\omega_0\cos\delta_0 - b_1A_0\sin\delta_0)}_{B}]\mu(t) + b_0A_0\cos(\delta_0)\cdot\delta(t)$$

$$= (A\cos(\omega_0 t) + B\sin(\omega_0 t))\cdot\mu(t) + b_0A_0\cos(\delta_0)\cdot\delta(t)$$

$$A = -b_0A_0\omega_0\sin\delta_0 + b_1A_0\cos\delta_0$$
$$B = -b_0A_0\omega_0\cos\delta_0 - b_1A_0\sin\delta_0$$

Provjerimo da li ovaj općeniti oblik vodi na konstante A i B za prvi primjer kod kojeg je $\delta_0 = -\pi/2$

$$A\Big|_{\delta_0=-\pi/2} = -b_0A_0\omega_0(-1) + b_1A_0\cdot\emptyset = b_0A_0\omega_0 \quad \checkmark$$
$$B\Big|_{\delta_0=-\pi/2} = -b_0A_0\omega_0\cdot\emptyset - b_1A_0(-1) = b_1A_0 \quad \checkmark$$

OK

Dakle u ovom slučaju novu pobudu s desne strane diferencijalne jednadžbe možemo rastaviti u dva dijela

$$u_n(t) = u_{n1}(t) + u_{n2}(t) \rightarrow u_{n2}(t) = b_0A_0\cos(\delta_0)\cdot\delta(t)$$

$$\downarrow$$

$$u_{n1}(t) = (A\cos(\omega_0 t) + B\sin(\omega_0 t))\cdot\mu(t)$$

Prvi dio pobude $u_{n1}(t)$ jednak je po obliku prvom (13)
primjeru, a jedino se razlikuju konstante A & B uz cos i
sin. Zbog linearnosti sustava odziv na $u_n(t) = u_{n1}(t) + u_{n2}(t)$
možemo odrediti kao sumu odziva $H(u_{n1}(t)) = y_1(t)$ i
$H(u_{n2}(t)) = y_2(t)$. Dakle, odziv $y_1(t)$ nalazimo istim
postupkom opisanom za prvi primjer, dok odziv
na $u_{n2}(t) = b_0 A_0 \cos(\vartheta_0) \cdot \delta(t)$ zapravo predstavlja
skalirani impulsni odziv $h(t)$ sustava opisanog
diferencijalnom jednadžbom:

$$h'(t) + a_1 h(t) = \delta(t)$$

$$y_2(t) = b_0 A_0 \cos(\vartheta_0) \cdot h(t)$$

Određivanje $y_2(t)$ se stoga svodi na nalaženje
impulsnog odziva sustava čija je desna strana
diferencijalne jednadžbe $= u(t)$

$b_1 = 1\ ,\ b_0 = \emptyset$

Promatramo ovaj impulsni odziv postupkom opisanim
na 12. predavanju.

Jednadžba $h'(t) + a_1 h(t) = \delta(t)$ postaje
homogena za $t > \emptyset$, pa je dovoljno pretpostaviti rješenje oblika:

$$h(t) = C_h \cdot e^{-a_1 t}$$

gdje konstantu C_h nalazimo na osnovu početnih uvjeta u $t = 0^+$
Obzirom da se radi o sustavu 1. reda postoji samo jedan
početni uvjet, a to je $h(\emptyset^+)$, koji u skladu sa diskusijom
na 12. predavanju mora biti jednak 1.

Evaluacijom izraza $h(t)$ za $t=\emptyset^+$ i izjednačavanjem sa očekivanim početnim uvjetom slijedi: (14)

$$h(\emptyset^+) = C_h \cdot e^{-a_1 \cdot 0^+} = C_h \cdot 1 = \underline{\underline{1}}$$

Dakle vidimo da je $C_h = 1$

Prema tome impulsni odziv promatranog sustava je:

$$h(t) = e^{-a_1 t}$$

Naglasimo da je to izraz za sve-vremenski impulsni odziv. Obzirom da se radi o kauzalnom sustavu, impulsni odziv mora biti kauzalna funkcija, pa je dovoljno dobiveni sve-vremenski $h(t)$ pomnožiti sa $\mu(t)$:

$$h_{kauz}(t) = h(t) \cdot \mu(t) = e^{-a_1 t} \cdot \mu(t)$$

Za provjeru uvrstimo rješenje za kauzalni impulsni odziv u diferencijalnu jednadžbu iz koje smo ga izveli:

$$h'_{kauz}(t) + a_1 h_{kauz}(t) = \delta(t)$$

$$\cancel{-a_1 \cdot e^{-a_1 t} \cdot \mu(t)} + e^{-a_1 t} \cdot \underbrace{\delta(t)}_{t=\emptyset} + \cancel{a_1 \cdot e^{-a_1 t} \mu(t)} = \delta(t)$$

$$\underbrace{e^{-a_1 \cdot \emptyset}}_{1} \cdot \delta(t) = \delta(t)$$

$$\underline{\delta(t) = \delta(t)}$$ ok ... vidimo da zadovoljava.

Dakle drugi dio odziva $y_2(t)$ se nalazi skaliranjem dobivenog $h(t)$:

$$y_2(t) = b_0 A_0 \cos(\vartheta_0) \cdot e^{-a_1 t} \cdot \mu(t)$$

Očito je ~~da~~ za $|\vartheta_0| = \pi/2$, $\cos(\vartheta_0) = \emptyset$, pa je $y_2(t) = \emptyset$, što je bio slučaj za prvi primjer, sa pobudom $A_0 \sin(\omega_0 t)$

Sustav prvog reda: (isti kao do sada)

$$\overset{=1}{a_0} \cdot y'(t) + a_1 y(t) = b_0 u'(t) + b_1 u(t)$$

Riješimo prvo slučaj kada je $b_0 = 0$, a $b_1 = 1$

Dakle $$y'(t) + a_1 y(t) = u(t)$$

Pretpostavimo rj. homogene jed. za sve-vremenski slučaj

$$y_h(t) = C_1 \cdot e^{s_1 t}$$

$$\boxed{\text{PRIMJER SA POBUDOM } \mu(t)}$$

Uvrstimo u jednadžbu:

$$y_h'(t) + a_1 y_h(t) = 0$$

$$C_1 s_1 e^{s_1 t} + a_1 C_1 e^{s_1 t} = 0$$

$$C_1 e^{s_1 t} \underbrace{(s_1 + a_1)} = 0$$

netrivijalno rješenje $s_1 = -a_1$

Dakle sve-vremenski odziv moramo pretpostaviti u obliku

$$y_h(t) = C_1 \cdot e^{-a_1 t}$$

Nađimo odziv sustava na $\overset{A}{1} \cdot \mu(t)$ „jedinični step"
... to je specijalni slučaj sve-vremenske pobude sa konstantom $A = 1$
... pretpostavimo sve-vremensko rješenje oblika konstante K ... $y_p(t) = K$

Uvrstimo u dif. jed. i dobivamo:

$$y_p'(t) + a_1 y_p(t) = A$$

$$0 + a_1 \cdot K = A \Rightarrow K = A/a_1$$

Dakle $$y_p(t) = \frac{A}{a_1}$$

Ukupno sve-vremensko rješenje se dakle sastoji od:

$$y_{tot}(t) = C_1 \cdot e^{-a_1 t} + \frac{A}{a_1}$$

Neka je početni uvjet definiran u $t=0^-$

$$y(0^-) = y_{poc}$$

Ovaj početni uvjet možemo prebaciti u $t=0^+$ korištenjem sustava jednadžbi na SLIDE 29, PREDAVANJE 11.

$$y(0^+) - y(0^-) = b_0 u(0^+)$$

Za odabrani primjer $b_0 = 0$, pa stoji.

$$y(0^+) = y(0^-) = y_{poc}$$

Radi određivanja konstante c_1 moramo evaluirati opće rješenje $y_{tot}(t)$ za $t=0^+$, te ga usporediti (izjednačiti) sa $y(0^+) = y_{poc}$, pa dobivamo:

$$y(0^+) = y_{poc} = c_1 \cdot e^{-a_1 \cdot 0^+} + \frac{A}{a_1}$$
$$= c_1 \cdot 1 + \frac{A}{a_1}$$

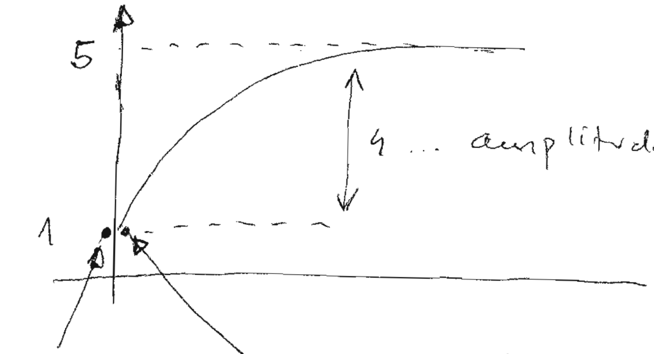
$$\Rightarrow c_1 = y_{poc} - \frac{A}{a_1}$$

Dakle ukupno rješenje sustava prvog reda u odzivu na step je

$$y_{tot}(t) = \underbrace{\left[\underbrace{\left(y_{poc} - \frac{A}{a_1}\right) e^{-a_1 t} + \frac{A}{a_1}}_{\text{sve-vremensko rješenje}}\right] \cdot \mu(t)}_{\text{kauzalno rj. na kauzalnu pobudu}}$$

Uzmimo sada konkretni primjer za $a_1 = 0.2$ & $y_{poc} = 1$ i amplitudu pobudnog stepa $A=1$

$$y_{tot}(t) = \left[\left(1 - \frac{1}{0{,}2}\right) e^{-0{,}2t} + \frac{1}{0{,}2}\right] \cdot \mu(t)$$
$$= \left[-4 e^{-0{,}2t} + 5\right] \cdot \mu(t)$$



(17)

$y(0^-) = y_{poc} = 1$ $\quad y(0^+) = y(0^-) = y_{poc} = 1$

za $t \nearrow \infty$ $\quad y_{tot}(t) = [-4 \cdot \emptyset + 5] \cdot 1$

$= 5$ ... odziv u stacionarnom stanju

Sada dozvolimo da postoji i član $u'(t)$ sa desne strane, tj $b_0 \neq \emptyset$.

Sve-vremensko rješenje homogene jednadžbe ostaje jednako kao i prije jer ne ovisi o desnoj strani. Dakle

$$y_h(t) = c_1 \cdot e^{-a_1 t}$$

Uvrštavanjem pobude $x_1(t)$ u desnu stranu dif. jednadžbe dobivamo novu pobudu $u_n(t)$

Ovaj slučaj rješavamo jednako kao onaj sa harmonijskom pobudom proizvoljne početne faze kod koje postoji skok u $t=\emptyset$

$$u_n(t) = A \cdot b_0 \delta(t) + A b_1 \mu(t)$$

rastavimo na dva dijela $u_n(t) = u_1(t) + u_2(t)$

$u_1(t) = A \cdot b_0 \cdot \delta(t)$ ... $y_1 = H(u_1) = A \cdot b_0 \cdot h(t)$

$u_2(t) = A \cdot b_1 \mu(t)$ ... $y_2 = H(u_2)$ ...

↳ ako je b_1 i dalje $b_1 = 1$ tada y_2 smo već odredili u prošlom primjeru.

Dakle sve što moramo napraviti jest da odzivu prošlog primjera dodamo $A \cdot b_0 \cdot h(t)$ ... skalirani imp. odziv

U skladu sa kratkim predavanjima nadimo impulsni odziv mirnog sustava

$$a_0^{=1} \cdot y'(t) + a_1 y(t) = u(t)$$

$u(t) = \delta(t)$, $y(t) = h(t)$

$$h'(t) + a_1 h(t) = \delta(t)$$

za $t > 0$ prelazi u homogenu jednadžbu.

$$h(t) = C_1 \cdot e^{-a_1 t}$$

Obzirom da smo početni uvjet već uključili u odzivu vezanom za u_2, pretpostavljamo da je sustav mirni $h(0^-) = 0$

Potrebno je odrediti $h(0^+)$

Radi se o sustavu 1. reda $(N=1)$ dakle

$$\underline{h(0^+) = 1} \quad \text{jer} \quad h^{(N-1)}(0^+) = 1$$

$N=1\ldots$

Napišimo opis oblika impulsnog odziva i evaluirajmo ga za $t = 0^+$

$$h(0^+) = C_1 \cdot e^{-a_1 0^+} = 1 \Rightarrow C_1 = 1$$

Dakle, konačni impulsni odziv glasi

$$h(t) = e^{-a_1 t} \cdot \mu(t)$$

---

zaključimo $y_1(t) = A \cdot b_0 \cdot h(t) = A \cdot b_0 \cdot e^{-a_1 t} \cdot \mu(t)$

$$y_2(t) = \left[\left(y_{poc} - \frac{b_1 A}{a_1}\right) e^{-a_1 t} + \frac{b_1 A}{a_1}\right] \cdot \mu(t)$$

za općenit b_0, b_1

$$y_{tot}(t) = y_1(t) + y_2(t) = \left[\left(A b_0 + y_{poc} - \frac{A b_1}{a_1}\right) e^{-a_1 t} + \frac{b_1 A}{a_1}\right] \mu(t)$$

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Provjerimo pomoću Laplace-a (OVO TEK BUDEMO UČILI)

$$y'(t) + a_1 y(t) = b_0 u'(t) + b_1 u(t)$$

$$sY(s) - y(0^-) + a_1 Y(s) = s b_0 U(s) - b_0 u(0^-) + b_1 U(s)$$

$$Y(s)(s+a_1) = y(0^-) + U(s)(s b_0 + b_1) - \underbrace{b_0 u(0^-)}_{=\emptyset}$$

zbog kauzalnosti pobude

$U(s) = \frac{A}{s}$

$$Y(s) = \frac{y(0^-)}{s+a_1} + \underbrace{\frac{A}{s}\,\frac{s b_0 + b_1}{s+a_1}}$$

$y(0^-) = y_{poc}$

$$y(t) = y_{poc} \cdot e^{-a_1 t} \cdot \mu(t)$$

$$\frac{A b_0}{s+a_1} + \frac{A b_1}{s(s+a_1)}$$

$$A b_0 \cdot e^{-a_1 t} \mu(t) \qquad \frac{A b_1}{a_1}\left(1 - e^{-a_1 t}\right)\mu(t)$$

$$y(t) = \left(\left(y_{poc} + A b_0 - \frac{A b_1}{a_1}\right) e^{-a_1 t} + \frac{A b_1}{a_1}\right)\mu(t)$$

OK

Isto rješenje kao i superpozicijom normalnog odziva i impulsnog odziva.

Konačno pokažimo da se do rješenja za odziv mirnog sustava može doći i na drugi način bez određivanja impulsnog odziva. Osnovna ideja je odrediti opći oblik totalnog rješenja za mirni sustav i to kauzalnog, ali sa neodređenim koef. Takvo rješenje uvrštavamo u dif. jednadžbu i u jednom koraku nalazimo sve tražene koeficijente. Jedini problem ovakvog načina određivanja odziva je u nemogućnosti postavljanja početnih uvjeta koji su $\neq \emptyset$. Za odziv mirnog sustava, to je ok, ali za općenit odziv treba na $y_m(t)$ dodati $y_0(t)$ $\Downarrow$

Sada dozvolimo da desna strana sadrži i derivacije pobude. Obzirom da tražimo odziv na step, derivacija stepa je $\delta(t)$. Dakle u prisilnom odzivu na $A_0\mu(t)$ moramo pretpostaviti odziv oblika

$$y_p(t) = K_1 \cdot \delta(t) + K_2\mu(t)$$

kako ovaj tip pobude i odziva nije nije sve-vremenski već je kauzalan, moramo i homogeni dio rješenja pretpostaviti u kauzalnom obliku

$$y_h = C_1 \cdot e^{-a_1 t} \cdot \mu(t)$$

ALTERNATIVNI NAČIN ODREĐIVANJA RJ. ODZIVA MIRNOG SUSTAVA ČIJA POBUDA SADRŽI $\delta(t)$ ILI NJEGOVE DERIVACIJE

Dakle totalno rješenje je

$$y_{tot}(t) = y_h(t) + y_p(t)$$
$$= C_1 \cdot e^{-a_1 t} \cdot \mu(t) + K_1\delta(t) + K_2\mu(t)$$

Ovakvo totalno rješenje mora zadovoljiti dif. jed.

$$y' + a_1 y = b_0 u' + b_1 u$$

$$y'_{tot}(t) = \underline{C_1 \cdot e^{-a_1 t} \cdot (-a_1) \cdot \mu(t)} + \underline{\overbrace{C_1 e^{-a_1 t}}^{C_1 \; t=0} \cdot \delta(t)} + \underline{K_1\delta'(t)} + \underline{K_2\delta(t)}$$

$$a_1 y(t) = \underline{a_1 C_1 e^{-a_1 t}\mu(t)} + \underline{a_1 K_1 \delta(t)} + a_1 K_2 \mu(t)$$

$$b_0 u' + b_1 u = \underline{A b_0 \delta(t)} + A b_1 \mu(t)$$

$$C_1 e^{-a_1 t}\underbrace{(-a_1 + a_1)}_{\emptyset}\mu(t) = \emptyset \cdot \mu(t) \cdot e^{-a_1 t} \text{ u on homogeni dio}$$

$$C_1\delta(t) + K_2\delta(t) + a_1 K_1 \delta(t) = A b_0 \cdot \delta(t)$$

$$(C_1 + K_2 + a_1 K_1) = A b_0$$

$$K_1\delta'(t) = \emptyset \cdot \delta'(t) \Rightarrow K_1 = \emptyset$$

IZJEDNAČAVANJEM ISTIH OBLIKA FUNKC. SA LIJEVE I DESNE STRANE DIF. JED.

$$a_1 k_2 \mu(t) = A b_1 \mu(t)$$

$$k_2 = \frac{A b_1}{a_1} \qquad k_1 = \emptyset$$

$$c_1 + k_2 + a_1 \cancel{k_1} = A b_0$$

$$c_1 = A b_0 - k_2$$
$$= A b_0 - \frac{A b_1}{a_1}$$

Dakle totalno rješenje bi bilo

$$y_{tot}(t) = y_h(t) + y_p(t)$$
$$= c_1 e^{-a_1 t} \mu t + k_1 \delta(t) + k_2 \mu(t)$$
$$= \left(A b_0 - \frac{A b_1}{a_1}\right) e^{-a_1 t} \mu(t) + \emptyset \cdot \delta(t) + \frac{A b_1}{a_1} \cdot \mu(t)$$

$$= \left(\left(A b_0 - \frac{A b_1}{a_1}\right) e^{-a_1 t} + \frac{A b_1}{a_1} \right) \mu(t)$$

Ovo što je dobiveno predstavlja odziv mirnog sustava uz $y_{poc} = \emptyset$. Uočimo da su konstante c_1 & (k_1 & k_2) nađene u jednom koraku uvrštavanjem pretpostavljenog rješenja mirnog sustava u dif. jedn. (a ne na osnovu početnih uvjeta)!... koji su naravno nula u tom pogledu radi o mirnom sustavu

1. Računanje inverza matrice.

Općenito za kvadratne matrice 2x2 vrijedi:

$$\text{Za } A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \text{ inverz glasi: } \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \text{ uz } \det A = a_{11}a_{22}-a_{12}a_{21}.$$

Općenito za kvadratne matrice 3x3 vrijedi:

$$\text{Za A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \text{ inverz glasi: } \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^{-1} = \frac{1}{\det A}\begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}^{T},$$

$$\text{uz } \det A = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} =$$

$$= a_{11}(a_{22}a_{33}-a_{23}a_{32}) - a_{12}(a_{21}a_{33}-a_{23}a_{31}) + a_{13}(a_{21}a_{32}-a_{22}a_{31})$$

Sada se jasnije vidi što se misli kada se kaže:

Adjugirana matrica je transponirana matrica kofaktora.

Adjugirana matrica jest:

$$adj(A) = \begin{pmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{pmatrix}^{T}$$

Na ispitu će vam najvjerojatnije doći samo matrica 2x2. Primjer za matricu 3x3 je napravljen radi kompletnosti.

2. Izravno računanje $\phi(s)$ ili $\phi(z)$ iz matrice A:

$$\phi(s) = (sI - A)^{-1}$$
$$H(s) = C\phi(s)B + D$$

Za kontinuirane sustave vrijedi:

$\dot{x}(t) = Ax(t) + Bu(t),$

$y(t) = Cx(t) + Du(t).$

Za diskretne sustave vrijedi:

$x(n+1) = Ax(n) + Bu(n),$

$y(n) = Cx(n) + Du(n).$

Općenito za kvadratnu matricu A oblika $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ vrijedi:

$$\phi(s) = \frac{1}{\det(sI - A)} \begin{pmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{pmatrix},$$

$\det(sI - A) = (s - a_{22})(s - a_{11}) - a_{12}a_{21} = s^2 - (a_{11} + a_{22})s + a_{11}a_{22} - a_{12}a_{21}.$

Konačno dobijemo:

$$\phi(s) = \begin{pmatrix} \dfrac{s - a_{22}}{(s - a_{22})(s - a_{11}) - a_{12}a_{21}} & \dfrac{a_{12}}{(s - a_{22})(s - a_{11}) - a_{12}a_{21}} \\ \dfrac{a_{21}}{(s - a_{22})(s - a_{11}) - a_{12}a_{21}} & \dfrac{s - a_{11}}{(s - a_{22})(s - a_{11}) - a_{12}a_{21}} \end{pmatrix}$$

Za diskretne sustave, jedina promjena jest z umjesto s-a.

3. Množenje matrica:

Općenito vrijedi:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{pmatrix}$$

Posebni slučaj (za brzo rješavanje):

Ako je neka od matrica jedinična (*I*), onda dobijemo slijedeće:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Vidimo da dobijemo onu matricu koja nije jedinična.