...nih vrijednosti možemo očitati vjerojatnosti pojavljivanja sim-
...sustava kanala, koje iznose:

$$\boxed{\ldots 02 \quad 0{,}112 \quad 0{,}528 \quad 0{,}114 \quad 0{,}080 \quad 0{,}056 \quad 0{,}004 \quad 0{,}004]}$$

...ija u sustavu kanala $I(X;Y)$ jednaka je:

$$I(X;Y) = H(X) + H(Y) - H(X,Y)$$

...ropija ulaznog skupa simbola, $H(Y)$ entropija skupa simbola,
...na entropija.

...e izračunanih vrijednosti u formule za dane izraze, dobit ćemo:

$H(X) = 0{,}971$ bit/simbol

$H(Y) = 2{,}1214$ bit/simbol

$H(X,Y) = 2{,}9981$ bit/simbol

$\boxed{I(X;Y) = 0{,}0943 \text{ bit/simbol}}$

1.3. Zadaci za vježbu: Teorija informacije, kapacitet diskretnog komunikacijskog kanala, Markovljevi lanci

1.1. Diskretno bezmemorijsko izvorište generira simbole iz skupa simbola $X = \{x_1, x_2, x_3, x_4\}$. Vjerojatnosti pojavljivanja simbola su sljedeće: $p(x_1) = 0{,}4$, $p(x_2) = 0{,}3$, $p(x_3) = 0{,}2$ i $p(x_4) = 0{,}1$. Izračunajte količinu informacije koja se prenosi u poruci $x_1x_2x_1x_3$.

Rješenje: 6,70 bit/ poruka

1.2. Neka je Z slučajna varijabla koja poprima vrijednosti iz skupa $\{0, 1\}$ i neka je

$$p(Z = z) = \begin{cases} p, & z = 0 \\ 1 - p, & z = 1 \end{cases}$$

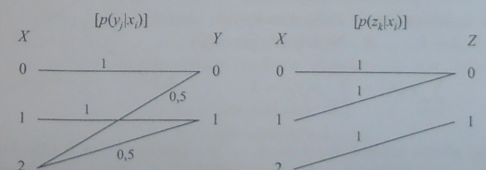
Neka je p slučajna varijabla koja poprima, s jednakom vjerojatnosti, vrijednosti iz skupa $\{0;\ 0{,}5;\ 1\}$. Kolika je očekivana vrijednost entropije slučajne varijable Z?

Rješenje: 1/ 3 bit/simbol

1.3. Slučajni vektor $(X_1X_2X_3)$ poprima sljedeće vrijednosti: (000), (001), (011), (101) i (111), i to svaku s vjerojatnosti 1/ 5. Odredite sljedeće entropije: $H(X_1)$, $H(X_2)$, $H(X_3)$, $H(X_1, X_2)$, $H(X_1, X_2, X_3)$, $H(X_2|X_1)$, $H(X_2|X_1 = 0)$, $H(X_2|X_1 = 1)$ i $H(X_3|X_1, X_2)$.

Rješenje: 0,971-; 0,971-; 0,722-; 1,922-; 2,322-; 0,9509-; 0,918-; 1-; 0,4-bit/simbol

1.4. Diskretno bezmemorijsko izvorište generira simbole iz skupa simbola $X = \{0, 1, 2\}$ s vjerojatnostima pojavljivanja $p(0) = 0{,}25$, $p(1) = 0{,}25$ i $p(2) = 0{,}5$. Svaki od izvorišnih simbola istodobno se šalje dvama diskretnim komunikacijskim kanalima (slika!) čiji su izlazi y_j, $j = 1, 2$, odnosno z_k, $k = 1, 2$.



$H(X)$, $H(Y)$, $H(Z)$, $H(Y,Z)$, $I(X;Y)$ i $I(X;Z)$.

Odredite: $H(X)$, $H(Y)$, $H(Z)$, $H(Y,Z)$, $I(X;Y)$ i $I(X;Z)$.

Rješenje: 1,5-; 1-; 1-; 2-; 0,5-; 1-bit/simbol

1.5. Neka su X i Y slučajne varijable koje poprimaju vrijednosti iz skupova $x \in \{0,1\}$ i $y \in \{0,1,2\}$, slijedno gledano. Neka je $p(X{=}x, Y{=}y) = K \cdot (x+y)$, $K \in \mathbf{R}^+$.

Odredite:

i) K; **ii)** $H(X)$; **iii)** $H(Y)$; **iv)** $I(X;Y)$.

Rješenje: **i)** 1/9; **ii)** 0,9183 bit/simbol; **iii)** 1,3516 bit/simbol; **iv)** 0,0728 bit/simbol

1.6. Dana je diskretna slučajna varijabla X koja poprima vrijednosti 0 i 1 s vjerojatnošću 1/4 i vrijednost 2 s vjerojatnošću 1/2. Slučajne varijable Y i Z definirane su na sljedeći način: ako je $X=0$, tada je $Y=Z=0$; ako je $X=1$, tada je $Y=1$ i $Z=0$; ako je $X=2$, tada je $Z=1$ dok Y slučajno poprima jednu od vrijednosti $\{0,1\}$ s jednakom vjerojatnošću. Odredite: $H(X)$, $H(Y)$, $H(Z)$, $H(Y|X)$, $H(X,Y)$, $H(X|Y)$, $H(X,Z)$, $H(X|Z)$, $H(Y,Z)$, $H(Z|Y)$.

Rješenje: 1,5-; 1-; 1-; 0,5-; 2-; 1-; 1,5-; 0,5-; 2-; 1-bit/simbol

1.7. Neka su X i Y diskretne slučajne varijable koje poprimaju vrijednosti iz diskretnih skupova $\mathcal{X}$ i $\mathcal{Y}$, slijedno gledano. Neka je $H(X) = 11$ bit/simbol i neka je $H(Y|X) = H(X|Y)$. Odredite najmanji mogući broj elemenata skupa $\mathcal{Y}$.

$H(X)-H(X|Y)=H(Y)-H(Y|X)$

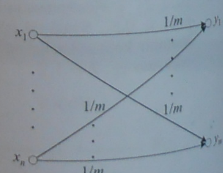
zašto se uzima $p(x)=\frac{1}{n}$

Rješenje: 2048

1.8. Razmatrajte diskretni informacijski kanal s međusobno neovisnim ulazima i izlazima, prikazan na slici.

Vjerojatnosti pojave simbola na ulazu kanala zadane su kao $p(x_i)$, $i = 1,\ldots,n$. Vjerojatnosti pojave simbola na izlazu kanala zadane su kao $p(y_j)$, $j = 1,\ldots,m$. Nadalje, vrijedi:

$$\sum_{i=1}^{n} p(x_i) = \sum_{j=1}^{m} p(y_j) = 1$$



Svaki ulazni simbol x_i, $i = 1,\ldots,n$, preslikava se u bilo koji od izlaznih simbola y_j, $j = 1,\ldots,m$, s jednakom vjerojatnošću: $p(y_j|x_i) =$

$1/m$, $\forall i,j$. Kanal je dodatno definiran matricom združenih vjerojatnosti $[P(X,Y)]$ koja ima m identičnih stupaca:

$$[P(X,Y)] = \begin{bmatrix} p_1 & p_1 & \ldots & p_1 \\ p_2 & p_2 & \ldots & p_2 \\ \vdots & \vdots & \vdots & \vdots \\ p_n & p_n & \ldots & p_n \end{bmatrix}$$

U potpitanjima i), ii), iii) i iv) odredite tražene veličine kao funkcije isključivo varijabli m i p_i:

i) Izraz za $H(X)$; **ii)** Izraz za $H(Y)$;
iii) Izraz za $H(X|Y)$; **iv)** Izraz za $H(Y|X)$.

Rješenje: **i) i iii)** $H(X) = H(X|Y) = -m \cdot \sum_{i=1}^{k(p_i)} p_i \cdot \log_2(m \cdot p_i)$

$$= -\log_2(m) - m \cdot \sum_{i=1}^{k(p_i)} p_i \cdot \log_2 p_i \ \frac{\text{bit}}{\text{simbol}}$$

ii) i iv) $H(Y) = H(Y|X) = \log_2 m \ \frac{\text{bit}}{\text{simbol}}$

Napomena: $k(p_i)$ – kardinalni broj skupa

1.9. Diskretno bezmemorijsko izvorište X generira simbole iz skupa simbola od m elemenata s vjerojatnostima pojavljivanja $p_1, p_2, \ldots, p_m$, $p_i \geqslant 0$, $i = 1,\ldots,m$ i $\sum_{i=1}^{m} p_i = 1$. Neka je $\mathbf{q}$ neka druga razdioba vjerojatnosti pojavljivanja $m-1$ elemenata, i neka je $q_1 = p_1$, $q_2 = p_2, \ldots, q_{m-2} = p_{m-2}$ i $q_{m-1} = p_{m-1} + p_m$. Odredite $H(X)$ u ovisnosti o $H(\mathbf{q})$, p_m, p_{m-1}, te entropiji $H\left(\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right)$ gdje je općenito gledano $H(a,b) = -a\log_2 a - b\log_2 b$.

Rješenje: $H(\mathbf{q}) + (p_{m-1}+p_m)H\left(\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right)$ bit/simbol

1.10. Neka su X_1 i X_2 diskretne slučajne varijable koje poprimaju vrijednosti iz skupova $\{1,2,\ldots,m\}$, odnosno $\{m+1, m+2, \ldots, m+n\}$, $m,n \in \mathbf{N}$, te neka su njihove pripadajuće razdiobe vjerojatnosti p_{X_1}, odnosno p_{X_2}. Neka je

$$X = \begin{cases} X_1 \text{ s vjerojatnošću } \alpha \\ X_2 \text{ s vjerojatnošću } 1-\alpha \end{cases}$$

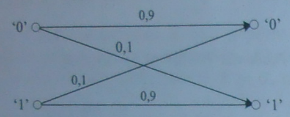
i) Odredite $H(X)$ kao funkciju od α, $H(X_1)$ i $H(X_2)$.

ii) Odredite maksimalnu vrijednost entropije $H(X)$ u ovisnosti o parametru α.

Rješenje: i) $-\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha) + \alpha H(X_1) + (1-\alpha)H(X_2)$ $\frac{\text{bit}}{\text{simbol}}$

ii) $H(X) \leqslant \log_2\left(2^{H(X_1)} + 2^{H(X_2)}\right)$

1.11. Četiri poruke, generirane iz skupa od četiriju jednako vjerojatnih simbola $X = \{x_1, \ldots, x_4\}$, kodirane binarnim kodom (x_1 – '00'; x_2 – '01'; x_3 – '10'; x_4 – '11'), prenose se binarnim simetričnim kanalom (slika). Izračunajte transinformaciju u kanalu ako se u prijenosu kao zaštita poruka uvede jedan paritetni bit (parni paritet!).



Rješenje: 1,3012 bit/simbol

1.12. Na ulazu diskretnog binarnog komunikacijskog kanala pojavljuju se dva simbola $X = \{x_1, x_2\}$ s vjerojatnostima $1-u$ i u, slijedno gledano. Matrica uvjetnih vjerojatnosti prijelaza u kanalu je

$$[p(y_j|x_i)] = \begin{bmatrix} 1 & 0 \\ f & g \end{bmatrix}, \quad f + g = 1.$$

Odredite $u = F(f,g)$ za koje se ostvaruje maksimum $I(X;Y)$.

Napomena: F je funkcija.

Rješenje: $u = \dfrac{f^{f/g}}{1 + g \cdot f^{f/g}}$

1.13. Mjerni uređaj mjeri napon čija je funkcija gustoće vjerojatnosti zadana jednadžbom

$$f(u) = a \cdot u \cdot (3-u), \quad u \in [0,3], \quad a \in \mathbf{R}$$
$$f(u) = 0, \quad u \notin [0,3]$$

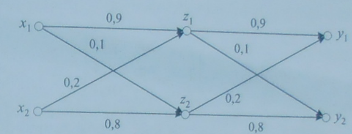
Uređaj može prikazati samo cijele brojeve i polovine, koji su zaokruženi na prvu veću vrijednost (npr. 1,2 V se zaokružuje na 1,5 V, a 1,9 V se zaokružuje na 2,0 V). Ako se napon uzorkuje svakih 10 ms, koliki je srednji sadržaj informacije generiran za jednu minutu?

Rješenje: 1,83 kbyte

1.14. Neka je k binarnih simetričnih kanala (BSK), svaki s vjerojatnošću pogrešnog prijenosa p, vezano u seriju. Odredite opći izraz za vjerojatnost pogrešnog prijenosa ekvivalentnog BSK-a.

Rješenje: $\frac{1}{2}(1-(1-2p)^k)$

1.15. Dva binarna kanala serijski su povezana kako je to predočeno na slici. Odredite srednji uzajamni sadržaj informacije ($I(X;Y)$) u sustavu kanala ako je $p(x_1) = p(x_2) = 0{,}5$.



Rješenje: 0,1878 bit/simbol

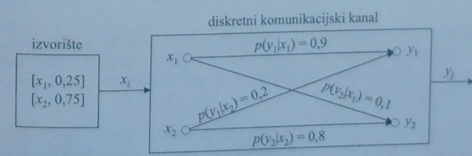
1.16. Dana je diskretna slučajna varijabla Z koja poprima vrijednosti 0 i 1 s vjerojatnostima $1-p$ i p, slijedno gledano. Neka slučajna varijabla X, neovisna od Z, poprima vrijednosti $1, 2, \ldots, n$ s vjerojatnostima $\mathbf{q} = [q_1, q_2, \ldots, q_n]$ i neka je $Y = XZ$. Odredite:

i) $H(Y)$ u ovisnosti o $H(X)$ i $H(Z)$;

ii) p i $\mathbf{q}$ uz uvjet da je $H(Y)$ maksimalno.

Rješenje: i) $H(Y) = H(Z) + pH(X)$;
ii) $p = n/(n+1)$, $\mathbf{q} = [1/n, \ldots, 1/n]$

1.17. Diskretno bezmemorijsko izvorište generira simbole iz skupa simbola $X = \{x_1, x_2\}$ s vjerojatnostima pojavljivanja $p(x_1) = 0{,}25$ i $p(x_2) = 0{,}75$. Diskretni komunikacijski kanal je modeliran kao na slici:



Odredite:

i) vjerojatnost pojave pogrešnog simbola na izlazu kanala;

ii) matricu združenih vjerojatnosti $[p(x_i, y_j)]$;

iii) iznos korisne informacije koja se pojavljuje na izlazu kanala.

Rješenje: i) 0,175; **ii)** $[p(x_i,y_j)] = [0{,}225 \quad 0{,}025; \quad 0{,}15 \quad 0{,}6]$; **iii)** $\approx 0{,}2958$ bit/simbol

1.18. Na ulazu diskretnog komunikacijskog kanala, sa smetnjama, pojavljuju se tri simbola $X = \{x_1, x_2, x_3\}$. Na izlazu istog kanala pojavljuju se simboli $Y = \{y_1, y_2, y_3\}$. Statističke veze između ulaznog i izlaznog skupa simbola dane su preko matrice združenih vjerojatnosti – $[p(x_i, y_j)]$:

$$[p(x_i,y_j)] = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}$$

Odredite vrijednosti $p_{11}, \ldots, p_{33}$ za koje se postiže maksimum $H(X,Y)$. Također, $p(x_1) = 1/2$; $p(x_2) = 1/4$; $p(y_1) = 1/3$ i $p(y_2) = 1/6$.

Rješenje: $[p(x_i,y_j)] = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{24} & \frac{1}{8} \\ \frac{1}{12} & \frac{1}{24} & \frac{1}{8} \end{bmatrix}$

1.19. Dana je diskretna slučajna varijabla X koja poprima vrijednosti iz skupa $\{a, b, c\}$. Promatrajmo dvije razdiobe dane slučajne varijable:

Simbol	$p(x)$	$q(x)$
a	0,5	1/3
b	0,25	1/3
c	0,25	1/3

i) Odredite: $H(p)$, $H(q)$, $D(p\|q)$, $D(q\|p)$. Uočite da vrijedi: $D(p\|q) \neq D(q\|p)$.

ii) Kako je pokazano u **i)**, u općem slučaju $D(p\|q) \neq D(q\|p)$. Međutim, pronađite p i q za slučaj u kojem diskretna slučajna varijabla X poprima vrijednosti iz skupa vrijednosti $\{0, 1\}$ tako da je $D(p\|q) = D(q\|p)$.

Rješenje: **i)** 1,5-; 1,585-; 0,085-; 0,082-bit/simbol; **ii)** općenito za $p = 1 - q$ jednakost je zadovoljena

1.20. Komunikacijskim kanalom prenose se četiri poruke generirane iz skupa četiriju simbola $X = \{x_1, \ldots, x_4\}$. Vjerojatnosti pojavljivanja simbola su

sljedeće: $p/2$, $p/2$, $(1-p)/2$ i $(1-p)/2$, slijedno gledano ($p \in (0,1)$). Matrica uvjetnih vjerojatnosti prijelaza u kanalu je:

$$[p(y_j|x_i)] = \begin{bmatrix} 1-f & f & 0 & 0 \\ f & 1-f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{uz } 0 \leqslant f \leqslant 1.$$

Odredite općeniti izraz za varijablu p koji osigurava maksimalnu količinu informacije po simbolu što se u prosjeku može prenijeti danim kanalom.

Napomena: $H(f) = f \log_2 \frac{1}{f} + (1-f) \log_2 \frac{1}{1-f}$.

provjeriti

Rješenje: $p = \dfrac{1}{1 + 2^{H(f)}}$

1.21. Na ulazu diskretnog binarnog komunikacijskog kanala pojavljuju se dva simbola $X = \{x_1, x_2\}$. Odredite vjerojatnosti pojavljivanja ulaznog skupa simbola za koje se postiže maksimum transinformacije te nakon toga odredite kapacitet danog kanala. Matrica uvjetnih vjerojatnosti prijelaza u kanalu je

le preko deriviranja

$$[p(y_j|x_i)] = \begin{bmatrix} 1 & 0 \\ 0{,}5 & 0{,}5 \end{bmatrix}.$$

Rješenje: 3/5; 2/5; 0,322 bit/simbol

1.22. Na ulazu diskretnog komunikacijskog kanala pojavljuju se simboli $X = \{0, 1, 2, 3, 4\}$. Uvjetne vjerojatnosti prijelaza u kanalu dane su sljedećim izrazom:

$$p(y_j|x_i) = \begin{cases} 0{,}5, & y_j = (x_i \pm 1) \mod 5 \\ 0, & \text{inače} \end{cases}$$

Odredite kapacitet danog kanala.

Rješenje: 1,322 bit/simbol

1.23. Dan je diskretni komunikacijski kanal s matricom uvjetnih vjerojatnosti prijelaza, i to:

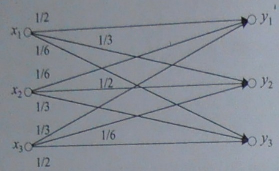
$$[p(y_j|x_i)] = \begin{bmatrix} 1-\beta & \beta & 0 & 0 \\ \beta & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\gamma & \gamma \\ 0 & 0 & \gamma & 1-\gamma \end{bmatrix}$$

Odredite kapacitet danog kanala ako je:

i) $\gamma = 0$ i $\beta = 1$; **ii)** $\gamma = \beta$.

Rješenje: i) 2 bit/simbol; ii) $2 - H(\beta)$ bit/simbol

1.32. Odredite kapacitet diskretnog bezmemorijskog kanala sa slike:



Rješenje: 0,126 bit/simbol

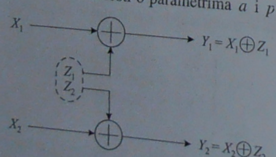
1.33. Dana je diskretna slučajna varijabla Z koja poprima vrijednosti 0 i 1 s vjerojatnostima $1-p$ i p, slijedno gledano. Neka slučajna varijabla X, neovisna od Z, poprima vrijednosti $1,2,\ldots,n$ s vjerojatnostima $\mathbf{q}=[q_1,q_2,\ldots,q_n]$ i neka je $Y=XZ$. Nadalje, neka X i Y predstavljaju ulaz, odnosno izlaz diskretnog bezmemorijskog kanala. Odredite kapacitet danog kanala.

Rješenje: $p\log_2 n$

1.34. Dana su dva diskretna binarna komunikacijska kanala kao na slici. Neka za $i\in\{1,2\}$, X_i, Y_i, i Z_i predstavljaju ulaz, izlaz i šum na i-tom kanalu, slijedno gledano. Također, X_i, Y_i i Z_i poprimaju vrijednosti iz skupa $\{0,1\}$. Združena razdioba šuma, tj. para (Z_1,Z_2) dana je kao:

$p(Z_1{=}z_1,Z_2{=}z_2)$	$z_2=0$	$z_2=1$
$z_1=0$	$1+a-p$	$p-a$
$z_1=1$	$p-a$	a

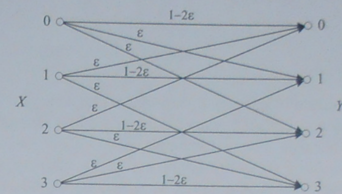
gdje je a parametar za koji vrijedi $0\leqslant a\leqslant p\leqslant 0{,}5$. Odredite kapacitet danog sustava kanala u ovisnosti o parametrima a i p.



Napomena: Znak $\oplus$ predstavlja zbrajanje u logici modulo-2. XOR

Rješenje: $C=2-H(1+a-2p,\ p-a,\ p-a,\ a)$ bit/ simbol

1.35. Dan je diskretni komunikacijski kanal.



i) Odredite kapacitet danog kanala u ovisnosti o parametru ε.

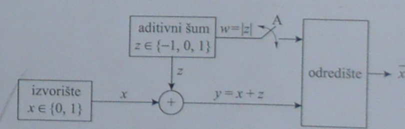
ii) Koliko iznosi minimalna i maksimalna vrijednost kapaciteta danog kanala?

ε

Rješenje: **i)** $2+2\varepsilon\log_2\varepsilon+(1-2\varepsilon)\log_2(1-2\varepsilon)$ [bit/simbol]; **ii)** $2-\log_2 3\leqslant C\leqslant 2$

Napomena: Također, dani model kanala može se iskoristiti za analizu modulacijske tehnike 4-QAM (više o QAM-u čitatelj može naći u [22]).

1.36. Zadan je diskretni komunikacijski sustav kao na slici. Izvorište (opisano slučajnom varijablom X) generira simbole iz skupa simbola $\{0,1\}$. Vjerojatnosti pojavljivanja izvorišnih simbola su $p(x=0)=p_0$, odnosno $p(x=1)=p_1$ i $p_0+p_1=1$. Simboli se potom prenose preko bezmemorijskog kanala uz djelovanje aditivnog šuma Z. Neka je Z slučajna varijabla (neovisna o X) koja poprima vrijednosti iz skupa $\{-1,0,1\}$ s jednakom vjerojatnošću te neka se na odredištu pojavljuju simboli $y=x+z$. Također, preko preklopke A moguće je dobiti informaciju o apsolutnom iznosu aditivnog šuma na kanalu, tj. $w=|z|$.



i) Odredite kapacitet danog kanala kada je preklopka A otvorena, tj. kada odredište nema informaciju o apsolutnom iznosu aditivnog šuma.

ii) Odredite kapacitet danog kanala kada je preklopka A zatvorena, tj. kada odredište ima informaciju o apsolutnom iznosu aditivnog šuma.

Napomena: Ukupnu transinformaciju računajte prema izrazu $I(X;Y|W) = \sum_{w} I(X;Y|W=w)p(W=w)$.

Rješenje: **i)** 1/3 bit/simbol; **ii)** 1 bit/simbol

1.37. Dano je n diskretnih bezmemorijskih kanala s kapacitetima $C_1, C_2, \ldots, C_n$, slijedno gledano. Ulazni i izlazni skupovi simbola za različite su kanale disjunktni. Neka je ukupni (sumarni) kanal za svih n kanala definiran kao kanal koji ima na raspolaganju svih n kanala ali samo jedan kanal može koristiti za prijenos u danom vremenskom trenutku. Dokažite da je kapacitet ukupnog (sumarnog) kanala jednak

$$C = \log_2 \sum_{i=1}^{n} 2^{C_i}$$

Rješenje: Dokaz provedite sami.

$\boxed{15}$ $x \in \{0,1\}$ $\quad y \in \{0,1,2\}$

Zadaci

1

Zad 1.5 $p(X=x, Y=y) = k \cdot (x+y) \quad , \; k \in \mathbb{R}^+$

a) $k=?$

$$\begin{array}{c|ccc} x \backslash y & 0 & 1 & 2 \\ \hline 0 & 0 & k & 2k \\ 1 & k & 2k & 3k \end{array}$$

$$[p(x_i y_j)] = \begin{bmatrix} 0 & k & 2k \\ k & 2k & 3k \end{bmatrix} = \begin{bmatrix} 0 & 0.111 & 0.222 \\ 0.111 & 0.222 & 0.333 \end{bmatrix}$$

$$p(y_j) = [k \quad 3k \quad 5k] = \left[\tfrac{1}{9} \quad \tfrac{3}{9} \quad \tfrac{5}{9}\right]$$

$$9k = 1$$

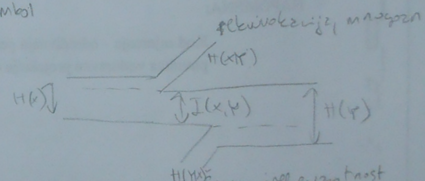
$$\boxed{k = \tfrac{1}{9}}$$

$$p(x_i) = [3k \quad 6k] = \left[\tfrac{3}{9} \quad \tfrac{6}{9}\right]$$

b) $H(x) = -\sum_{i=1}^{2} p(x_i) \log p(x_i) = 0.9183$ bit/simbol

c) $H(y) = -\sum_{j=1}^{3} p(y_j) \log p(y_j) = 1{,}3516 \; \frac{\text{bit}}{\text{simbol}}$

d) $I(x,Y) = ?$



$I(X,Y) = H(X) + H(Y) - H(X,Y)$

$H(X,Y) = -\sum\sum p(x_i, y_j) \log_2 p(x_i, y_j) = 2{,}1964 \; \frac{\text{bit}}{\text{simbol}}$

$I(X,Y) = 0.9183 + 1.3516 - 2{,}1964 = 0{,}0735$ bit/simb.

1.6

1.6

$$X = \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{bmatrix} 0.25 & 0.25 & 0.5 \end{bmatrix} \end{array}$$

$X=0 \rightarrow Y=Z=0$

$X=1 \longrightarrow Y=1 \quad Z=0$

$X=2 \longrightarrow Z=1 \quad Y = \begin{array}{c} P=[\begin{array}{cc} 0.5 & 0.5 \end{array}] \\ \begin{Bmatrix} 0 & 1 \end{Bmatrix} \end{array}$

a) $H(X) = -\sum p(x_i) \log_2 p(x_i) = 1.5$ bit/simbol

b)

$$[p(Y|X)] = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0.5 & 0.5 \end{array}$$

$$[p(x,y)] = [p(x_i) \cdot p(y_j|x_i)] = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$$

$p(y_j) = [0.5 \quad 0.5]$

$H(Y) = 1$ bit/simbol

c)

$$p(z|x) = \begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{array}$$

$$p(x,z) = \begin{bmatrix} 0.25 & 0 \\ 0.25 & 0 \\ 0 & 0.5 \end{bmatrix} \qquad p(z) = [0.5 \quad 0.5]$$

$H(Z) = 1$ bit/simbolu

d) $H(X,Y) = 2$

$H(Y|X) = H(X,Y) - H(X) = 0.5$

e) $H(X|Y) = H(X,Y) - H(Y) = 1$

$H(X) = 11$ bit/simb.

1.7 $H(Y|X) = H(X|Y)$

min br. elemenata u Y

$H(X) - H(X|Y) = H(Y) - H(Y|X)$

$H(X) = H(Y) = 11$

$H(Y) = 11 = \cancel{11}\, \frac{1}{n} \log_2 \left(\frac{1}{n}\right)^{-1} \Big|_2$

vjerojatnost pri kojoj se dobiva najmanji broj elemenata skupa y

$2^{11} = \left(\frac{1}{n}\right)^{-1}$

$\boxed{n = 2048}$

1.8 $p(x_i)$

$p(y_j)$

$p(y_j|x_i) = \frac{1}{m}$

$\boxed{m_y\ p_i}$

$$[p(x_i y_j)] = \begin{array}{c} \\ i=1 \\ i=2 \\ \vdots \\ i=n \end{array} \overbrace{\begin{bmatrix} p_1 & p_1 & p_1 & \dots & p_1 \\ p_2 & p_2 & p_2 & \dots & p_2 \\ \vdots & & \vdots & & \vdots \\ p_n & p_n & p_n & & p_n \end{bmatrix}}^{m} n \qquad [p(y_j)] = \underbrace{\left[\sum_{i=1}^{n} p_i \quad \sum_{i=1}^{n} p_i \quad \dots \quad \sum_{i=1}^{n} p_i\right]}_{m}$$

$p(y_j)$

$[p(x_i)] = [m p_1 \quad m p_2 \quad m p_3 \dots m p_n]$

$H(X) = ?$

a) $H(X) = -\sum_{i=1}^{n} p(x_i) \log_2 p(x_i) = -m \sum_{i=1}^{n = k(p_i)} p_i \log(m p_i)$

b) $H(Y) = -m \cdot \sum_{i=1}^{k(p_i)} p_i \cdot \log_2 \sum_{i=1}^{k(p_i)} p_i = -\log_2 \underbrace{\sum_{i=1}^{k(p_i)} p_i}_{\frac{1}{m}} = \log_2 m$ bit/simb

$m \underbrace{\sum_{i=1}^{k(p_i)} p_i}_{} = 1$

$\sum_{i=1}^{k(p_i)} p_i = \frac{1}{m}$

1.8

c) $$H(X|Y) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j) \log_2 p(x_i|y_j) =$$

$$p(x_i|y_j) = \frac{p(x_i, y_j)}{p(y_j)} = m\, p(x_i, y_j) =$$

$$p(y_j) = \frac{1}{m}$$

$$H(X|Y) = -\sum\sum p(x_i y_j) \cdot \log_2 \underbrace{m \cdot p(x_i, y_j)}_{m} =$$

$$= -\left[\begin{array}{l} p_1 \log_2 mp_1 + p_1 \log_2 mp_1 + \dots + p_1 \log_2 mp_1 + \\ \vdots \\ p_n \log_2 mp_n + p_n \log_2 mp_n + \dots + p_n \log_2 mp_n \end{array}\right] \Big\} n$$

$$= -\left[m \cdot \sum_{i=1}^{n} p_i \log_2 mp_i\right] = -m \sum_{i=1}^{H(p_i)} p_i \log_2 mp_i \quad \text{bit/simb.}$$

d) $$H(Y|X) = -\sum_{i=1}^{n}\sum_{j=1}^{m} p(x_i, y_j) \log_2 p(y_j|x_i) =$$

$$p(y_j|x_i) = \begin{bmatrix} \frac{1}{m} & \cdots & \frac{1}{m} \\ \vdots & \ddots & \vdots \\ \frac{1}{m} & \cdots & \frac{1}{m} \end{bmatrix}$$

$$H(Y|X) = -\log_2 \frac{1}{m} \sum_i\sum_j p(x_i, y_j) = \log_2 m \cdot 1 = \log_2 m$$

1.9 X generira simbole iz skupa od m elem. s vjerojatnost. $p_1, p_2 \dots p_m \geq 0$

g druga razdioba za pojavlj. $m-1$ element

$$g_1 = p_1,\ g_2 = p_2, \dots, g_{m-2} = p_{m-2},\ g_{m-1} = p_{m-1} + p_m$$

$$H(x) = f\left[H(g), p_m, p_{m-1}, H\left(\frac{p_{m-1}}{p_{m-1}+p_m}, \frac{p_m}{p_{m-1}+p_m}\right)\right]$$

$$H(a,b) = -a\log_2 a - b\log_2 b$$

Rj.

$$H(x) = -\sum_{i=1}^{m} p_i \log_2 p_i =$$

$$H(g) = -\sum_{i=1}^{m-1} p_i \log_2 p_1 = \underbrace{-\sum_{i=1}^{m-2} p_i \log_2 p_i}_{} - (p_{m-1}+p_m)\log_2(p_{m-1}+p_m)$$

$$H(x) = -\sum_{i=1}^{m-1} p_i \log_2 p_i - p_m \log_2 p_m =$$

$$= \underbrace{-\sum_{i=1}^{m-2} p_i \log_2 p_1}_{H(g) + (p_{m-1}+p_m)\log_2(p_{m-1}+p_m)} - p_{m-1}\log_2 p_{m-1} - p_m \log_2 p_m$$

$$H(x) = H(g) + (p_{m-1}+p_m)\log_2(p_{m-1}+p_m) - p_{m-1}\log_2 p_{m-1} - p_m\log_2 p_m$$

$$= H(g) + p_{m-1}\log_2(p_{m-1}+p_m) + p_m\log_2(p_{m-1}+p_m) - p_{m-1}\log_2 p_{m-1} - p_m\log_2 p_m =$$

$$= H(g) + p_{m-1}\log_2\frac{p_{m-1}+p_m}{p_{m-1}} + p_m\log_2\frac{p_{m-1}+p_m}{p_m} =$$

$$= H(g) - p_{m-1}\log_2\frac{p_{m-1}}{p_{m-1}+p_m} - p_m\log_2\frac{p_m}{p_{m-1}+p_m}$$

3) $H(x) = H(g) - (p_{m-1}+p_m)\dfrac{p_{m-1}}{p_{m-1}+p_m}\log_2\dfrac{p_{m-1}}{p_{m-1}+p_m} - (p_{m-1}+p_m)\dfrac{p_m}{p_{m-1}+p_m}\log_2\dfrac{p_m}{p_{m-1}+p_m}$

$= H(g) + (p_{m-1}+p_m)\, H\left(\dfrac{p_{m-1}}{p_{m-1}+p_m}, \dfrac{p_{m-}}{p_{m-1}+p_m}\right)$

$\boxed{1.10}$ $X_1 \{1, 2, \dots, m\}$ $\quad p_{x_1}$

$X_2 \{m+1, m+2, \dots, m+n\}$ $\quad p_{x_2}$

$m, n \in \mathbb{N}$

$X = \begin{cases} X_1 & \text{s vjerog. } \alpha \\ X_2, & \text{s vjerog. } 1-\alpha \end{cases}$

a) $H(X) = ? = f(\alpha, H(X_1), H(X_2))$

$X = \begin{cases} 1, 2, 3, \dots, m, & p(x) = \alpha \\ m+1, m+2, \dots, m+n, & p(x) = 1-\alpha \end{cases}$

$$H(X) = -\sum_{i=1}^{m+n} p(x_i)\log_2 p(x_i) = -\sum_{i=1}^{m} \alpha\log_2\alpha - \sum_{i=1}^{n}(1-\alpha)\log_2(1-\alpha) =$$

$$= -\alpha\log_2\alpha \cdot m - (1-\alpha)\log_2(1-\alpha)\cdot n$$

$$H(X_1) = -\sum_{i=1}^{m} p_1\log_2 p_1 = -p_1\log p_1 \cdot m$$

$$m = -\frac{H(X_1)}{p_1 \log p_1}$$

1.11

$X = \{x_1, x_2, x_3, x_4\}$

$x_1 = 00$
$x_2 = 01$
$x_3 = 10$
$x_4 = 11$

$I(X;Y) = ?$
paritet

x_1 00 0 $\xrightarrow{0.9^3}$ 00 0 y_1
$\xrightarrow{0.9 \cdot 0.1^2}$ 01 1 y_2
$\xrightarrow{0.9 \cdot 0.1^2}$ 10 1 y_3
$\xrightarrow{0.9 \cdot 0.1^2}$ 11 0 y_4
$\longrightarrow \varepsilon$ (001, 010, 100, 111)

x_2 01 1
x_3 10 1
x_4 11 0

$$p(y_j|x_i) = \begin{array}{c|ccccc} x_i \backslash y_j & 000 & 011 & 101 & 110 & \varepsilon \\ \hline 000 & 0.729 & 0.009 & 0.009 & 0.009 & 0.244 \\ 011 & 0.009 & 0.729 & 0.009 & 0.009 & 0.244 \\ 101 & 0.009 & 0.009 & 0.729 & 0.009 & 0.244 \\ 110 & 0.009 & 0.009 & 0.009 & 0.729 & 0.244 \end{array}$$

$$p(x_i) = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$$

$$p(x_i, y_j) = \begin{bmatrix} 0.18225 & 0.00225 & 0.00225 & 0.00225 & 0.061 \\ 0.00225 & & & & 0.061 \\ 0.00225 & & & & 0.061 \\ 0.00225 & & & & 0.061 \end{bmatrix}$$

$$p(y_j) = [0.189 \quad 0.189 \quad 0.189 \quad 0.189] \; 0.244]$$

$H(X) = 2$ bit/simb.

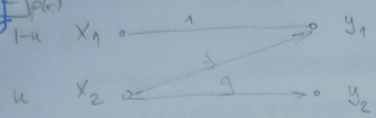
$H(Y) = 2{,}314$ bit/simb

$I(X;Y) = H(X) + H(Y) - H(X,Y) = 1{,}3015$ bit/simb.

$H(X,Y) = 3{,}0125$ bit/simb.

$\boxed{1.12}$

1.12 $p(x_i)$



$\log xy = \log x + \log y$

$\log \frac{x}{y} = \log x - \log y$

$u = F(f,g) = ?$

$I(X,Y) = \max \quad = H(Y) - H(Y|X)$

$[p(Y|X)] = \begin{bmatrix} 1 & 0 \\ f & g \end{bmatrix}$ $\qquad p(X) = [$

$p(x) = [1-u \quad u] \Rightarrow H(x) = -(1-u)\log_2(1-u) - u\log_2 u$

$[p(x_i y_j)] = \begin{bmatrix} 1-u & 0 \\ fu & gu \end{bmatrix} \Rightarrow H(X,Y) = -(1-u)\log_2(1-u) - fu\log_2 fu - gu\log_2$

$p(y_j) = [(1-u)+fu \quad gu] \Rightarrow H(Y) = -((1-u)+fu)\log_2((1-u)+fu) - gu\log_2 gu$

$I(X;Y) = H(Y) - H(Y|X)$

$H(Y|X) = -fu\log_2 f - gu\log_2 g$

$I = -(fu-fu^2)\log_2(fu-fu^2) - gu\log_2 u + fu\log_2 f \quad \Big|' \frac{d}{du}$

$g = 1-f$

$fu + g(1-u) = 1-u$

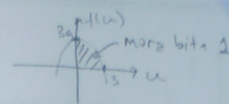
$p(x_i)fu + g - gu = 1-u$

$u(f-g)+g = 1-u$

$u[f-g+1] = 1-g$

$u = \frac{1-g}{f-g+1} = \frac{1-1+f}{f-1+f+1} = \frac{1}{2}$

13) $f(u) = a \cdot u \cdot (3-u)$, $u \in [0,3]$, $a \in \mathbb{R}$
$f(u) = 0$, $u \notin [0,3]$



$$F(u) = \int_0^u a u (3-u)\, du = a \int_0^u 3u - u^2 \, du = a\left[\frac{3u^2}{2} - \frac{u^3}{3}\right] =$$

$$f(u) = a\left[\frac{3u^2}{2} - \frac{u^3}{3}\right]$$

I=?
H(u)=?
svakih 10ms   u   1 min = 60s = $60 \cdot 10^3$ ms
N=60 uzoraka

$$F(u) = 1 = \int_0^3 a u (3-u)\, du = a\left[\frac{3 \cdot 9}{2} - \frac{3 \cdot 27}{3}\right] = 1$$

$$1 = a\,\frac{9}{2}$$

$$\boxed{a = \frac{2}{9}} \leftarrow \text{KRIVO}$$

$$\boxed{f(u) = \frac{2}{9} \cdot u \cdot (3-u)\ , \ u \in [0,3]}$$

$P(x=0) = \int_0^{0.5} \frac{2}{9} u(3-u)\,du = 0.074074$

$p(x=1) = \int_{0.5}^{1} \dots = 0.185185$

$p(x=1.5) = \int_{1}^{1.5} \dots = 0.24074$

$p(x=2) = \int_{1.5}^{2} \dots = 0.24074$

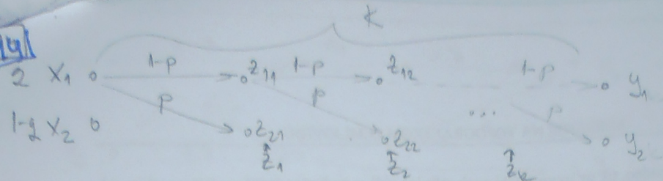
$p(x=2.5) = \int_{2}^{2.5} = 0.185185$

$p(x=3) = 0.074074$

$H(x) = 2{,}44655$ bit/simb.

$M = \frac{60\,s}{10ms} \cdot H(x) = 1{,}46$

1.14



$$p(y_2|x_1) = ? \qquad q = \frac{1}{2}$$

$$p(z_1|x_1) = \begin{array}{c|cc} & z_{11} & z_{12} \\ \hline x_1 & 1-p & p \\ x_2 & p & 1-p \end{array}$$

$$p(x, z_1) = \begin{bmatrix} \frac{1}{2}(1-p) & \frac{p}{2} \\ \frac{p}{2} & \frac{1}{2}(1-p) \end{bmatrix}$$

$$p(z_1) = \left[\frac{p}{2} + \frac{1}{2}(1-p) \quad \frac{p}{2} + \frac{1}{2}(1-p)\right]$$

$$p(z_2|z_1) = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$$

$$p(z_1, z_2) = \begin{bmatrix} \frac{p}{2}(1-p) + \frac{1}{2}(1-p)^2 & \frac{p^2}{2} + \frac{p}{2}(1-p) \\ \frac{p^2}{2} + \frac{1}{2}p(1-p) & \frac{p}{2}(1-p) + \frac{1}{2}(1-p)^2 \end{bmatrix}$$

$$p(z_2) = \left[\frac{p}{2}(1-p) + \frac{1}{2}(1-p)^2 + \frac{p^2}{2} + \frac{1}{2}p(1-p) \qquad -\|- \right]$$

$$p(z_k) = \frac{1}{2}\left(p^{k-1}(1-p)^{k-1} + (1-p)^k + p^k + p^{k-1}(1-p)^{k-1}\right) =$$

$$= \frac{1}{2}\left(2p^{k-1}(1-p)^{k-1} + \right.$$

1.15

$p(x_i) = [0.5 \quad 0.5] \Rightarrow H(X) = 1$

$p(z_k|x_i) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$

$p(x,z) = \begin{bmatrix} 0.45 & 0.05 \\ 0.1 & 0.4 \end{bmatrix}$

$p(z) = [0.65 \quad 0.45]$

$p(y|z) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$ $\quad I(X;Y) = 0.365$ bit/simb.

$p(y,z) = \begin{bmatrix} 0.495 & 0.055 \\ 0.09 & 0.36 \end{bmatrix}$

$p(y) = [0.585 \quad 0.415] \Rightarrow H(Y) = 0.979$

$I(X;Y) = H(X) + H(Y) - H(X,Y) =$ $\quad p\left(\frac{y|z}{x\ y}\right) = p(x|y) = \frac{p(x,y)}{p(y)}$

$H(X,Y) = \sum\sum p(x,y) \cdot \log p(x,y) =$ $\quad = \frac{p(y,z)}{p(z)}$

$p(x,y) = p(x)\, p(y|x) =$

$[p(Y|X)] = [p(Z|X)] \cdot [p(y|z)] = \begin{bmatrix} 0.83 & 0.17 \\ 0.36 & 0.66 \end{bmatrix}$

$p(x,y) = \begin{bmatrix} 0.415 & 0.085 \\ 0.18 & 0.33 \end{bmatrix}$ $\quad H(X;Y) = 1{,}754$ bit/s

1.16

$$Z \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \qquad Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n \\ (1-p)\cdot\cancel{\sum_{i=1}^{n} g_i} & g_1 p & g_2 p & g_3 p & & g_n p \end{pmatrix}$$

$$X \sim \begin{pmatrix} 1 & 2 & \dots & n \\ g_1 & g_2 & \dots & g_n \end{pmatrix} \qquad \sum_{i=1}^{n} g_i = 1$$

$$Y = X \cdot Z$$

a) $H(Y) = ? = f(H(X), H(Z))$

$$H(X) = -\sum_{i=1}^{n} g_i \log_2 g_i$$

$$H(Z) = -p\log p - (1-p)\log(1-p) \Rightarrow p\log p = -H(Z) - (1-p)\log(1-p)$$

$$H(Y) = -\left[(1-p)\log_2(1-p) + g_1 p \log g_1 \dots + g_n p \log_2 g_n p\right] =$$

$$= -\left[(1-p)\log_2(1-p) + g_1 p \log_2 g_1 + g_1 p \log p + g_2 p \log_2 g_2 + g_2 p \log_2 p + \dots\right.$$

$$\left.\dots g_n p \log g_n + g_n p \log p\right] =$$

$$= -\left[\cancel{1}\,\overset{\sum_{i=1}^{n} g_i}{(1-p)}\log_2(1-p)\sum_{i=1}^{n} g_i + \underbrace{p\log p}\sum_{i=1}^{n} g_i + p\underbrace{\sum_{i=1}^{n} g_i \log_2 g_i}_{-H(X)}\right]$$

$$= -\left[\cancel{\sum_{i=1}^{n} g_i}^{1} \cdot (1-p)\log_2\left[(1-p)\cancel{\sum_{i=1}^{p} g_i}^{1}\right] - H(Z)\sum_{i=1}^{n} g_i - (1-p)\log(1-p) - pH(X)\right] =$$

$$= -\left[\cancel{(1-p)\log_2(1-p)} - H(Z) - \cancel{(1-p)\log(1-p)} - p\,H(X)\right]$$

$$= H(Z) + p\,H(X) \checkmark$$

1.16 b) $H(Y)$ max

$p, q = ?$

$I = H(X) + H(Y) - H(X,Y)$

$I = H(Y) - H(Y|X)$

Da $H(Y)$ bude max $\quad H(X)$ mora biti $\boxed{p(x_i) = \frac{1}{n} = q_i}$

$$p(y) = \begin{bmatrix} 1-p & \frac{p}{n} & \frac{p}{n} & \cdots & \frac{p}{n} \end{bmatrix}$$

$$H(Y) = -(1-p)\log(1-p) - \frac{p}{n}\log\frac{p}{n} - \frac{p}{n}\log\frac{p}{n} - \dots - \frac{p}{n}\log\frac{p}{n} =$$

$$H(y) = -(1-p)\log(1-p) - p\log_2 p + p\log_2 n \quad |'$$

$$H'(y) = 0 = \log(1-p) + \frac{\cancel{(1-p)}\cdot 1}{\cancel{(1-p)}\ln 2} - \log_2 p + \frac{\cancel{p}}{\cancel{p}\ln 2} + \log_2 n = 0$$

$$\log\left[\frac{(1-p)}{p}\cdot n\right] = 0$$

$$(1-p)\,n = p$$

$$n - np = p$$

$$p(1+n) = n$$

$$\boxed{p = \frac{n}{1+n}}$$

1.17 $X = \{x_1, x_2\}$

$[P(x_i)] = [0.25 \quad 0.75]$ 0.175

a)

$$P(y|x) = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P(x,y) = \begin{bmatrix} 0.225 & 0.025 \\ 0.150 & 0.600 \end{bmatrix}$$

$$P(y) = [0.375 \quad 0.625]$$

vjerojatnost pojave pogrešnog simbola: poslana x_1 primlj. y_2 + poslan x_2 primljen y_1

$$P_{pogr.} = 0.150 + 0.025 = 0.175$$

c) $H(Y) = 0.9544$ bit/simbol

$H(X) = 0.8113$ bit/simbol

$H(x,y) = 1.46997$ bit/simb.

$I(X,Y) = 0.2957$ bit/simbol

količina korisne informacije tzv. transinformacija

1.18

1.18 $[P(x_i, y_j)] = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$ $P_{11} \cdots P_{33} = ?$ da $H(X,Y)$ bude max

$p(x) = [0.5 \quad 0.25 \quad 0.25]$

$p(y) = [\frac{1}{3} \quad \frac{1}{6} \quad \frac{1}{2}]$

X i Y su nezavisne

$H(X) = 1.5$ bit/simbol

$H(Y) = 1.458$ bit/simbol

$\Rightarrow P_{11} = p(x_1) \cdot p(y_1)$

$P_{12} = p(x_1) \cdot p(y_2)$

...

$$P(x_i, y_j) = \begin{bmatrix} \frac{1}{6} & \frac{1}{12} & \frac{1}{4} \\ \frac{1}{12} & \frac{1}{24} & \frac{1}{8} \\ \frac{1}{12} & \frac{1}{24} & \frac{1}{8} \end{bmatrix}$$

$H(X,Y) = H(X) + H(Y) - \underbrace{I(X,Y)}_{\text{minimalan}}$ da bi $H(X,Y)$ bio max.

da bi $I(X;Y)$ bio minimalan

$$I(X;Y) = \sum\sum p(x,y) \log_2 \underbrace{\frac{p(x,y)}{p(x_i)p(y_j)}}_{=1} = 0$$

$$p(x,y) = p(x_i)\,p(y_j)$$

12 + 1 jmd5 → 4

1.19

1.19

Simbol	$p(x)$	$q(x)$
a	0.5	0.33
b	0.25	0.33
c	0.25	0.33

a) $H(p) = 1.5$ bit/simbol

$H(q) = 1.5864$ bit/simbol

$$D(p\|q) = -\sum_{i=1}^{3} p(x_i) \log_2 \frac{p(x_i)}{q(x_i)} = 0.085 \text{ bit/simbol}$$

$$D(q\|p) = \frac{1}{3}\left(\log 0.5 \cdot 0.25 \cdot 0.25 \cdot 3 \cdot 3 \cdot 3\right) = 0.0817 \text{ bit/simbol}$$

b) naći p, q za koje je $D(q\|p) = D(p\|q)$ $\quad X = \{0, 1\}$

$$\begin{array}{cc} p & 1-p \\ q & 1-q \end{array}$$

$$p \log \frac{p}{q} + (1-p) \log \frac{1-p}{1-q} = -q \log \frac{p}{q} - (1-q) \log \frac{1-p}{1-q}$$

$$(p+q) \log \frac{p}{q} = \frac{q-1}{1-p} \log \frac{1-p}{1-q}$$

$$\left(\frac{p}{q}\right)^{p+q} = \left(\frac{1-p}{1-q}\right)^{-\frac{1-q}{1-p}}$$

$$\frac{p}{q} = \frac{1-p}{1-q} \qquad p+q = -\frac{1-q}{1-p}$$

$$p - \cancel{pq} = q - \cancel{pq} \qquad p - p^2 + \cancel{q} - pq = -1 + \cancel{q}$$

$p = q$

1 $(2+1) \bmod 5 \to 4$

$\boxed{1.21}$

$$[p(y_j|x_i)] = \begin{bmatrix} 1 & 0 \\ 0.5 & 0.5 \end{bmatrix}$$

$$X = \{x_1, x_2\} \qquad p(x) = [p \quad 1-p]$$

$$\max I = H(Y) - H(Y|X)$$

$$p(x_i y_j) = \begin{bmatrix} p & 0 \\ \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix}$$

PREKO DERIVACIJE DO C-a

$$p(y_j) = \begin{bmatrix} \frac{1+p}{2} & \frac{1-p}{2} \end{bmatrix} \Rightarrow H(Y) = -\frac{1+p}{2}\log_2\frac{1+p}{2} + \frac{1-p}{2}\log_2\frac{1-p}{2}$$

$$H(Y|X) = -\frac{1-p}{2}\log\frac{1}{2} - \frac{1-p}{2}\log\frac{1}{2} = \frac{1-p}{2} + \frac{1-p}{2} = 1-p$$

$$\max I = -\frac{1+p}{2}\log_2\frac{1+p}{2} - \frac{1-p}{2}\log\frac{1-p}{2} - 1 + p \quad /'$$

$$0 = -\left[\frac{1}{2}\log_2\frac{1+p}{2} + \frac{\cancel{\frac{1+p}{2}}}{\cancel{\frac{1+p}{2}}\ln 2}\cdot\frac{1}{2} - \frac{1}{2}\log\frac{1-p}{2} - \frac{\cancel{\frac{1-p}{2}}}{\cancel{\frac{1-p}{2}}\ln 2}\cdot\frac{1}{2}\right] + 1$$

$$1 = \frac{1}{2}\log_2\frac{1+p}{2} - \frac{1}{2}\log\frac{1-p}{2}$$

$$2 = \log_2\frac{1+p}{2}\cdot\frac{2}{1-p}$$

$$4 = \frac{1+p}{1-p}$$

$$4 - 4p = 1 + p$$

$$5p = 3$$

$$p = \frac{3}{5} \qquad 1-p = \frac{2}{5}$$

$$C = \max I = -\frac{8}{5\cdot 2}\log\frac{8}{10} - \frac{2}{10}\log\frac{2}{10} - 1 + \frac{3}{5} = 0.322 \text{ bit/simbol}$$

1.22

$X = \{0, 1, 2, 3, 4\}$

$(0 \pm 1) \bmod 5 \rightarrow 1, -1$ $\quad (3 \pm 1) \bmod 5 \rightarrow 4, 2$

$(1 \pm 1) \bmod 5 \rightarrow 0, 2$ $\quad (4 \pm 1) \bmod 5 \rightarrow 0, 3$

$(2 \pm 1) \bmod 5 \rightarrow 3, 1$

$$p(y_j|x_i) = \begin{cases} 0.5, & y_j = (x_i \pm 1) \bmod 5 \\ 0, & \text{inače} \end{cases}$$

$$p(y_j|x_i) = \begin{array}{c|cccccc} & -1 & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 \\ 1 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 2 & 0 & 0 & 0.5 & 0 & 0.5 & 0 \\ 3 & 0 & 0 & 0 & 0.5 & 0 & 0.5 \\ 4 & 0 & 0.5 & 0 & 0 & 0.5 & 0 \end{array}$$

Kapacitet kanala odgovara transinform. koja bi bila postignuta u sluč. IDEALNE RAZDIOBE vjerojatn. pojave simb. na ULAZU

$$C = \max_{\{p(x_i)\}} I(X;Y)$$

$I(X;Y) = \underbrace{H(X)}\; - \cancel{H(X|Y)} \rightarrow 0$

$I(X;Y) = H(Y) - H(Y|X)$

$H(Y|X) = 1$

$H(X) = -\frac{1}{n} \cdot n \log \frac{1}{n} = 2{,}32193$ bit/simb

$$p(x_i, y_j) = \begin{bmatrix} 0.1 & 0 & 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.1 & 0 & 0.1 & 0 \\ 0 & 0 & 0 & 0.1 & 0 & 0.1 \\ 0 & 0.1 & 0 & 0 & 0.1 & 0 \end{bmatrix}$$

$$p(y_j) = \begin{bmatrix} 0.1 & 0.2 & 0.2 & 0.2 & 0.2 & 0.1 \end{bmatrix}$$

$H(Y) = 2{,}52$ bit/simb.

1.23 $\left[p(y_j|x_i)\right] = \begin{bmatrix} 1-\beta & \beta & 0 & 0 \\ \beta & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\sigma & \sigma \\ 0 & 0 & \sigma & 1-\sigma \end{bmatrix}$

a) $\sigma = 0$, $\beta = 1$ $\quad C = ?$

$p(x) = [0.25 \quad 0.25 \quad 0.25 \quad 0.25] \Rightarrow H(x) = 2$ bit/simbolu

$\left[p(x_i, y_j)\right] = \begin{bmatrix} 0 & 0.25 & 0 & 0 \\ 0.25 & 0 & 0 & 0 \\ 0 & 0 & 0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{bmatrix}$

$H(X,Y) = 2$

$C = \max_{\{p(x_i)\}} I(X;Y) = H(X) = 2$ bit/simb

b) $\sigma = \beta \Rightarrow \left[p(y_j|x_i)\right] = \begin{bmatrix} 1-\beta & \beta & 0 & 0 \\ \beta & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ 0 & 0 & \beta & 1-\beta \end{bmatrix}$

$p(x_i) = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$

$C = \max_{\{p(x_i)\}} I(X;Y) = \max\left(2 - H(X|Y)\right)$

1.24

1.24

a) $p(y_j|x_i) = \begin{bmatrix} 1 & \cdots & & 0 \\ 0 & 1 & & \vdots \\ \vdots & & 1 & \vdots \\ 0 & \cdots & \ddots & 1 \end{bmatrix}$ → ovdje po zad. nema šumova u kanalu,

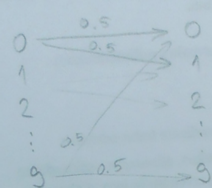
→ $p(y)=p(x)$ jer se info ne mogu izgubiti

→ ide se preko $H(Y)-H(Y|X)$, jer je zadana matrica $p(y|x)$

$p(x) = p(y) = [p_0 \ p_1 \ p_2 \cdots p_9]$ → $p_0 = p_1 = p_2 \cdots = p_9$ jer kapacitet mora biti max

$C = \max I = \max(H(Y) - H(Y|X)) = -10 \cdot \frac{1}{10} \log_2 \frac{1}{10} = 3{,}32 \ \frac{bit}{simbol}$

b)



$$p(y_j|x_i) = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & \cdots & & 9 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ \vdots \\ 9 \end{matrix} & \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & \cdots & 0 \\ 0 & 0.5 & 0.5 & 0 & \cdots & 0 \\ 0 & 0 & 0.5 & 0.5 & \cdots & 0 \\ \vdots & & & & 0.5 & 0.5 \\ 0.5 & 0 & \cdots & & \cdots & 0.5 \end{bmatrix} \end{array}$$

$$p(x_i, y_j) = \begin{bmatrix} \frac{p_0}{2} & \frac{p_0}{2} & 0 & 0 & \cdots & 0 \\ 0 & \frac{p_1}{2} & \frac{p_1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{p_2}{2} & \frac{p_2}{2} & \cdots & 0 \\ \vdots & & & & & \\ \frac{p_9}{2} & 0 & 0 & \cdots & & \frac{p_9}{2} \end{bmatrix}$$

$$p(y_j) = \left[\frac{p_0+p_9}{2} \quad \frac{p_0+p_1}{2} \quad \frac{p_1+p_2}{2} \quad \cdots \quad \frac{p_8+p_9}{2}\right]$$

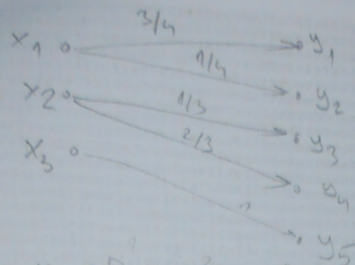
da bi $H(Y)$ bila max $\quad p_0 = p_1 = p_2 = \ldots = p_9 = \frac{1}{10}$

$H(Y|X) = -\frac{1}{20} \cdot 2 \cdot 10 \log \frac{1}{2} = 1$ bit/simb.

$H(Y) = -\log_2 \frac{1}{10}$

$C = -1 + \log_2 10$ bit/simb.

1.26



$$[P(y_j|x_i)] = \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ p_1 & 0.75 & 0.25 & 0 & 0 & 0 \\ p_2 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ p_3 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$p(x) = [p_1 \quad p_2 \quad p_3]$

$p_1 + p_2 + p_3 = 1$

$$C = \max_{\{p(x_i)\}} I(X;Y) = \max_{\{p(x_i)\}} \left(H(Y) - H(Y|X)\right)$$

$$p(x_i, y_j) = \begin{bmatrix} 0.75p_1 & 0.25p_1 & 0 & 0 & 0 \\ 0 & 0 & \frac{p_2}{3} & \frac{2p_2}{3} & 0 \\ 0 & 0 & 0 & 0 & p_3 \end{bmatrix}$$

$$p(y_j) = \left[0.75p_1 \quad 0.25p_1 \quad \frac{p_2}{3} \quad \frac{2}{3}p_2 \quad p_3\right]$$

$p_1 = p_2 = p_3 = \frac{1}{3}$

$$p(y_j) = \left[0.25 \quad \frac{1}{12} \quad \frac{1}{9} \quad \frac{2}{9} \quad \frac{1}{3}\right]$$

$H(Y) = 2,16149$ bit/simbol

$H(Y|X) = 0.5765$ bit/simbol

$C = H(Y) - H(Y|X) = 1,58499$ bit/simbol

1.27

$$p(y_j|x_i) = \begin{array}{c} \\ x_1 \\ x_2 \end{array}\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$p(x_i) = [0.5 \quad 0.5]$$

$$C = \max_{\{p(x)\}} I(X;Y) = \max_{\{p(x_i)\}} \left(H(Y) - H(Y|X) \right)$$

$$p(x_i, y_j) = \begin{bmatrix} 0.25 & 0.25 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$p(y_j) = [0.25 \quad 0.25 \quad 0.5]$$

$H(Y) = 1.5$ bit/simbol

$H(Y|X) = 0.5$ bit/simbol

$C = 1$ bit/simbol

1.28 $C = ?$

$$p(y_j|x_i) = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 2/3 & 2/3 \end{bmatrix}$$

$$p(x_i y_j) = \begin{bmatrix} \frac{2}{5} & \frac{1}{5} & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \\ 0 & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$p(y_j) = \left[\frac{3}{5} \quad \frac{3}{5} \quad \frac{3}{5}\right]$$

$H(Y|X) = 1{,}14052$ bit/simb

$H(Y) = 1{,}58496 = 0.$

$C = 0{,}444$

1.29

$$P(y_j|x_i) = \begin{array}{c} 1 \\ 0 \end{array}\begin{bmatrix} p_1 & p_2 & p_3 \\ p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$p(x_i) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$p(x,y) = \begin{bmatrix} \frac{p_1}{2} & \frac{p_2}{2} & \frac{p_3}{2} \\ \frac{p_1}{2} & \frac{p_2}{2} & \frac{p_3}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

$$p(y_j) = \begin{bmatrix} p_1 & p_2 & p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$H(Y|X) = 1,58496$ bit/simb.

$H(Y) = 1,58496$

$C = 0$

1.30

$$p(y_j|x_i) = \begin{bmatrix} p_1 & 1-p_1 & 0 \\ p_2 & 1-p_2 & 0 \\ 0 & p_3 & 1-p_3 \\ 0 & p_4 & 1-p_4 \end{bmatrix}$$

$$p(x) = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix}$$

$$p(x,y) = \begin{bmatrix} 0.25p_1 & 0.25(1-p_1) & 0 \\ 0.25p_2 & 0.25(1-p_2) & 0 \\ 0 & 0.25p_3 & 0.25(1-p_3) \\ 0 & 0.25p_4 & 0.25(1-p_4) \end{bmatrix}$$

$$p(y_j) = \begin{bmatrix} 0.25(p_1+p_2) & 0.25(2-p_1-p_2+p_3+p_4) & 0.25(2-p_3-p_4) \end{bmatrix}$$

$p_1 = p_2 = p_3 = p_4 = p$

$$p(y_j) = \begin{bmatrix} 0.5p & 0.5 & 0.25(2-2p) \end{bmatrix} =$$

22 $p = 1$

$p(y_j) = [0.5 \quad 0.5 \quad 0] \Rightarrow H(Y) = 1$ bit/simbol

1.32

$$p(y_j|x_i) = \begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline x_1 & 0.5 & \frac{1}{3} & \frac{1}{6} \\ x_2 & \frac{1}{6} & 0.5 & \frac{1}{3} \\ x_3 & \frac{1}{3} & \frac{1}{6} & 0.5 \end{array}$$

$$p(x) = [p_1 \quad p_2 \quad p_3]$$

$$p(x_i, y_j) = \begin{bmatrix} 0.5p_1 & \frac{1}{3}p_1 & \frac{1}{6}p_1 \\ \frac{1}{6}p_2 & 0.5p_2 & \frac{1}{3}p_2 \\ \frac{1}{3}p_3 & \frac{1}{6}p_3 & 0.5p_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{6} & \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{6} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{18} & \frac{1}{6} \end{bmatrix}$$

$$p(y_j) = \left[0.5p_1 + \frac{1}{6}p_2 + \frac{1}{3}p_3 \quad \frac{1}{3}p_1 + 0.5p_2 + \frac{1}{6}p_3 \quad \frac{1}{6}p_1 + \frac{1}{3}p_2 + 0.5p_3\right]$$

$$p_1 + p_2 + p_3 = 1 \Rightarrow p_1 = p_2 = p_3 = p \Rightarrow \boxed{p = \frac{1}{3}}$$

$H(Y) = 1,58496$ bit/simb.

$H(Y|X) = 1,45915$ bit/simbol

$\max I(X;Y) = 0.126$ bit/simb

$C = 0.126$ bit/simb

7.33

$$Z \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \qquad p(Y|X) = p(Z|X) \cdot p(Y|Z)$$

$$X \sim \begin{pmatrix} 1 & 2 & 3 \dots & n \\ q_1 & q_2 & q_3 \dots & q_n \end{pmatrix}$$

$$Y = XZ$$

$C = ?$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & \dots & n \\ 1-p & pq_1 & pq_2 & pq_3 & & pq_n \end{pmatrix} \quad p(y) = [1-p \;\; pq_1 \;\; pq_2 \;\; pq_3 \dots pq_n]$$

$$H(Y) = -\left[(1-p)\log(1-p) + pq_1 \log pq_1 + pq_2 \log_2 pq_2 + \dots + pq_n \log_2 pq_n\right] =$$

$$= -\left[(1-p)\log(1-p) + pq_1 \log p + pq_1 \log q_1 + pq_2 \log p + pq_2 \log q_2 \dots\right]$$

$$= -\left[(1-p)\log(1-p) + p\log p \cdot \underbrace{\sum_{i=1}^{n} q_i}_{n} + p \sum_{i=1}^{n} q_i \log q_i\right]$$

$$= -\left[(1-p)\log(1-p) + p\log p + p \cdot H(X)\right]$$

1.35

a) $$p(y|x_i) = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{bmatrix} 1-2\varepsilon & \varepsilon & \varepsilon & 0 \\ \varepsilon & 1-2\varepsilon & 0 & \varepsilon \\ \varepsilon & 0 & 1-2\varepsilon & \varepsilon \\ 0 & \varepsilon & \varepsilon & 1-2\varepsilon \end{bmatrix}$$

$$p(x) = [p_1 \quad p_2 \quad p_3 \quad p_4] = [0.25 \quad 0.25 \quad 0.25 \quad 0.25]$$

$$p(x_i y_j) = \begin{bmatrix} (1-2\varepsilon)p_1 & \varepsilon p_1 & \varepsilon p_1 & 0 \\ \varepsilon p_2 & (1-2\varepsilon)p_2 & 0 & \varepsilon p_2 \\ \varepsilon p_3 & 0 & (1-2\varepsilon)p_3 & \varepsilon p_3 \\ 0 & \varepsilon p_4 & \varepsilon p_4 & (1-2\varepsilon)p_4 \end{bmatrix}$$

$$p(y_j) = [(1-2\varepsilon)p_1 + \varepsilon(p_2+p_3) \quad (1-2\varepsilon)p_2 + \varepsilon(p_1+p_4) \quad \dots \quad 0.5\varepsilon + (1-2\varepsilon)\cdot 0.25]$$

$$H(Y) = -\left[\left(\frac{(1-2\varepsilon)}{4} + \frac{\varepsilon}{2}\right)\log_2\left(\frac{1-2\varepsilon}{4} + \frac{\varepsilon}{2}\right) \cdot 4\right] =$$

$$= -\left[(1-2\varepsilon)+2\varepsilon \quad \log_2\left(\frac{1-2\varepsilon+2\varepsilon}{4}\right)\right] =$$

$$= -\left[\log_2 \frac{1}{4}\right] = 2 \text{ bit/simb.}$$

$$H(Y|X) = -\left[\frac{(1-2\varepsilon)}{\cancel{4}}\log_2(1-2\varepsilon) + \frac{\varepsilon}{\cancel{4}}\log\varepsilon + \frac{\varepsilon}{\cancel{4}}\log\varepsilon\right] \cdot \cancel{4}$$

$$= -\left[(1-2\varepsilon)\log_2(1-2\varepsilon) + 2\varepsilon\log\varepsilon\right]$$ u tom simbolu je greška ili u rj. ili meni, p. rj. je −

$$\max I(X;Y) = 2 + (1-2\varepsilon)\log_2(1-2\varepsilon) - 2\varepsilon\log\varepsilon$$

$$C = 2 + (1-2\varepsilon)\log_2(1-2\varepsilon) - 2\varepsilon\log\varepsilon \text{ bit/simbol}$$

b) max $C = 2$ bit/simb

$$(1-2\varepsilon)\log_2(1-2\varepsilon) - 2\varepsilon\log\varepsilon = 0 \quad \Big| \frac{d}{d\varepsilon}$$

1.35

$$-2\log(1-2\varepsilon) + \cancel{\frac{(1-2\varepsilon)}{(1-2\varepsilon)\ln 2}\cdot(-2)} + 2\left(\log\varepsilon + \cancel{\frac{\varepsilon}{\varepsilon \ln 2}}\right) = 0$$

$$-\cancel{2}\log(1-2\varepsilon) + \cancel{2}\log\varepsilon = 0$$

$$\log\frac{\varepsilon}{1-2\varepsilon} = 0$$

$$1 = \frac{\varepsilon}{1-2\varepsilon}$$

$$1-2\varepsilon = \varepsilon$$

$$\varepsilon = \frac{1}{3}$$

$$I = 2 + \left(1-\frac{2}{3}\right)\log_2\left(1-\frac{2}{3}\right) + \frac{2}{3}\log\frac{1}{3} =$$

$$= 2 - \frac{1}{3}\log_2 3 - \frac{2}{3}\log 3 = 2 - \log_2 3$$ ← minimum

$$2-\log_2 3 \leq C \leq 2$$

$X = \{0, 1\}$

$\begin{matrix} p_0 & p_1 \end{matrix} \Rightarrow \begin{matrix} \overset{p_0}{p}, & \overset{p_1}{1-p} \end{matrix}$

1.36

$Z = \{-1, 0, 1\}$

$p(z) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$

$y = x + z$

$z = p = 1 \Rightarrow C = \max$

a)

x, z	y	$p(y)$
0 −1	−1	$\frac{p_0}{3}$
0 0	0	$\frac{p_0}{3}$
0 1	1	$\frac{p_0}{3}$
1 −1	0	$\frac{p_1}{3}$
1 0	1	$\frac{p_1}{3}$
1 1	2	$\frac{p_1}{3}$

$$\begin{array}{cccc} -1 & 0 & 1 & 2 \end{array}$$

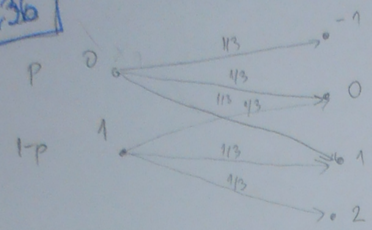
$$p(y_j) = \begin{bmatrix} \frac{p_0}{3} & \frac{p_0+p_1}{3} & \frac{p_0+p_1}{3} & \frac{p_1}{3} \end{bmatrix}$$

$$p(y_j) = \begin{bmatrix} \frac{p}{3} & \frac{1}{3} & \frac{1}{3} & \frac{p}{3} \end{bmatrix}$$

$$H(Y) = -\frac{2}{3} p \log_2 \frac{p}{3} - \frac{2}{3} \log_2 \frac{1}{3} = -\frac{4}{3} \log_2 \frac{1}{3} = 2{,}1133 \frac{bit}{sib}$$

$$p(y|x) = p(z|x) \cdot p(y|z)$$

2 $\boxed{1.36}$



$$H(x) = -p\log p - (1-p)\log(1-p)$$

$$[p(x_i)] = [p \quad 1-p]$$

$$[p(y_j|x_i)] = \begin{array}{c|cccc} & -1 & 0 & 1 & 2 \\ \hline 0 & 1/3 & 1/3 & 1/3 & 0 \\ 1 & 0 & 1/3 & 1/3 & 1/3 \end{array}$$

$$[p(x_i, y_j)] = \begin{array}{c|cccc} & -1 & 0 & 1 & 2 \\ \hline 0 & \frac{p}{3} & \frac{p}{3} & \frac{p}{3} & 0 \\ 1 & 0 & \frac{1-p}{3} & \frac{1-p}{3} & \frac{1-p}{3} \end{array}$$

$$[p(y_j)] = \left[\frac{p}{3} \quad \frac{1}{3} \quad \frac{1}{3} \quad \frac{1-p}{3}\right]$$

$$H(Y) = -\frac{p}{3}\log\frac{p}{3} - \frac{2}{3}\log\frac{1}{3} - \frac{1-p}{3}\log\frac{1-p}{3}$$

$$H(Y|X) = -\left[\frac{p}{3}\cdot 3\log_2\frac{1}{3} + \frac{1-p}{3}\cdot 3\log\frac{1}{3}\right] =$$

$$= -\left[\log\frac{1}{3}\right] = \log 3 = -\log\frac{1}{3}$$

$H(X)$

$$H(Y) = \frac{1}{3}\left[-p\log p - p\log\frac{1}{3} - (1-p)\log(1-p) - (1-p)\log\frac{1}{3}\right] - \frac{2}{3}\log\frac{1}{3} =$$

$$= \frac{1}{3}\left[H(x) - \log\frac{1}{3}\right] - \frac{2}{3}\log\frac{1}{3} =$$

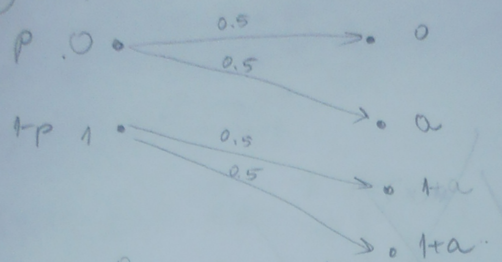
$$= \frac{1}{3}H(x) - \frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{1}{3} =$$

$$= \frac{1}{3}H(x) - \log\frac{1}{3}$$

max za $p = \frac{1}{2}$

$$I_{max} = \frac{1}{3}H(x) - \cancel{\log\frac{1}{3}} + \cancel{\log\frac{1}{3}} = \frac{1}{3}H(x) = -\frac{1}{3}\,2\,\frac{1}{2}\log\frac{1}{2} = \frac{1}{3}$$

$\boxed{1.25}$ (i) $a \neq 0, \pm 1$



p(x_i

$$[p(y_j|x_i)] = \begin{array}{c|cccc} & 0 & a & 1 & 1+a \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 1 & 0 & 0 & 0.5 & 0.5 \end{array}$$

$$[p(x_i)] = [\,p \quad 1-p\,]$$

$$[p(x_i, y_j)] = \begin{array}{c|cccc} & 0 & a & 1 & 1+a \\ 0 & \frac{p}{2} & \frac{p}{2} & 0 & 0 \\ 1 & 0 & 0 & \frac{1-p}{2} & \frac{1-p}{2} \end{array}$$

$$H(Y|X) = -\left[2 \cdot \frac{p}{2} \log_2 \frac{1}{2} + 2 \cdot \frac{1-p}{2} \log_2 \frac{1}{2}\right] = -\log \frac{1}{2}$$

$$H(Y) = -\left[2 \cdot \frac{p}{2} \log \frac{p}{2} + 2 \cdot \frac{(1-p)}{2} \log \frac{1-p}{2}\right] =$$

$$= -\left[p\log p + p\log\frac{1}{2} + (1-p)\log(1-p) + (1-p)\log\frac{1}{2}\right] =$$

$$= H(X) - \log\frac{1}{2}$$

$$I(X) = H(X) - \log\frac{1}{2} + \log\frac{1}{2} = H(X)$$

$$\max I = -\log\frac{1}{2} = \underline{\underline{1 \text{ bit}|\text{simb}}}$$

$\boxed{125}$ $a=-1$

$$\begin{array}{ll} p & 0 \xrightarrow{1} -1 \\ 1-p & 1 \xrightarrow{1} 0 \end{array}$$

$$[p(y_j|x_i)] = \begin{array}{c|cc} & -1 & 0 \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array}$$

$$[p(x_i, y_j)] = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$$

$$I = H(Y) - H(Y|X)$$

$$= -p\log p - (1-p)\log(1-p) - (\cancel{p\log 1} + \cancel{(1-p)\log(1-p)}) =$$

$$= 2\, H(X) = 2 \left[\tfrac{1}{2}\right.$$

$H(X) =$

$\max I = 2$ $H(X)$ max $\Rightarrow p = \frac{1}{2}$

$$\max I = -2\left(\frac{1}{2}\log\frac{1}{2} + \frac{1}{2}\log\frac{1}{2}\right) = 1 \text{ bit/simbol}$$

?

$a=0$

$$\begin{array}{ll} p & 0 \xrightarrow{1} 0 \\ 1-p & 1 \xrightarrow{1} 1 \end{array}$$

$$[p(y|x)] = \begin{array}{c|cc} & & \\ \hline 0 & 1 & 0 \\ 1 & 0 & 1 \end{array} \qquad p(y) = [p \quad 1-p]$$

$$[p(x,y)] = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix}$$

$$H(Y) = -(p\log p + (1-p)\log(1-p)) = H(X)$$

$$H(Y|X) = -(p\log 1 + (1-p)\log 1) = 0$$

$$C = \max I = H(X) = 1 \text{ bit/simbol}$$

$a=1$

$$\begin{array}{ll} p & 0 \xrightarrow{0.5} 0,\quad 0 \xrightarrow{0.5} 1 \\ 1-p & 1 \xrightarrow{0.5} 1,\quad 1 \xrightarrow{0.5} 2 \end{array}$$

$$[p(y|x)] = \begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & 0.5 & 0.5 & 0 \\ 1 & 0 & 0.5 & 0.5 \end{array}$$

$$p(x,y) = \begin{bmatrix} \frac{p}{2} & \frac{p}{2} & 0 \\ 0 & \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix} \Rightarrow [p(y)] = \begin{bmatrix} \frac{p}{2} & \frac{1}{2} & \frac{1-p}{2} \end{bmatrix}$$

$$H(Y|X) = -\left[\frac{p}{\cancel{2}} \cdot \cancel{2}\log\frac{1}{2} + \frac{1-p}{\cancel{2}} \cdot \cancel{2}\log\frac{1}{2}\right] = -\log\frac{1}{2}$$

$$H(Y) = -\left[\frac{p}{2}\log\frac{p}{2} + \frac{1}{2}\log\frac{1}{2} + \frac{1-p}{2}\log\frac{1-p}{2}\right] =$$

$$= -\frac{1}{2}\left[p\log p + \cancel{p\log\frac{1}{2}} + \log\frac{1}{2} + (1-p)\log(1-p) + (1-\cancel{p})\log\frac{1}{2}\right] =$$

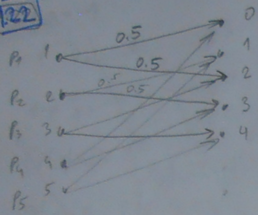
$$= -\frac{1}{2}\left[-H(X) + 2\log\frac{1}{2}\right] = \frac{1}{2}H(X) - \log\frac{1}{2}$$

$$I(X;Y) = \frac{1}{2}H(X) - \cancel{\log\frac{1}{2}} + \cancel{\log\frac{1}{2}} = \frac{1}{2}H(X)$$

$$C = \max I(X;Y) = \frac{1}{2}\left[-\frac{1}{\cancel{2}} \cdot \cancel{2}\log\frac{1}{2}\right) = \frac{1}{2} \text{ bit/simb}$$

//

1.22



$P_1 \neq$

$$(1 \pm 1) \bmod 5 \to 2, \; 0$$

$$(2 \pm 1) \bmod 5 \to 3, \; 1$$

$$(3 \pm 1) \bmod 5 \to 4, \; 2$$

$$(4 \pm 1) \bmod 5 \to 0, \; 3$$

$$(5 \pm 1) \bmod 5 \to 1, \; 4$$

$$[p(y_j|x_i)] = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 1 & 0.5 & 0 & 0.5 & 0 & 0 \\ 2 & 0 & 0.5 & 0 & 0.5 & 0 \\ 3 & 0 & 0 & 0.5 & 0 & 0.5 \\ 4 & 0.5 & 0 & 0 & 0.5 & 0 \\ 5 & 0 & 0.5 & 0 & 0 & 0.5 \end{array}$$

$$[p(x_i, y_j)] = \begin{array}{c|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 1 & \frac{P_1}{2} & 0 & \frac{P_1}{2} & 0 & 0 \\ 2 & 0 & \frac{P_2}{2} & 0 & \frac{P_2}{2} & 0 \\ 3 & 0 & 0 & \frac{P_3}{2} & 0 & \frac{P_3}{2} \\ 4 & \frac{P_4}{2} & 0 & 0 & \frac{P_4}{2} & 0 \\ 5 & 0 & \frac{P_5}{2} & 0 & 0 & \frac{P_5}{2} \end{array}$$

$$[p(y_j)] = \left[\frac{P_1+P_4}{5} \quad \frac{P_2+P_5}{2} \quad \frac{P_1+P_3}{2} \quad \frac{P_2+P_4}{2} \quad \frac{P_3+P_5}{2} \right]$$

$$H(Y|X) = -\left[P_1 \log \frac{1}{2} + P_2 \log \frac{1}{2} + P_3 \log \frac{1}{2} + P_4 \log \frac{1}{2} + P_5 \log \frac{1}{2} \right] = -\log \frac{1}{2}$$

$$P_1 = P_2 = P_3 = P_4 = P_5 \Rightarrow P = \frac{1}{5}$$

$$H(Y) = H(X) = 5 \cdot \frac{1}{5} \log \frac{1}{5} = 2{,}3219 \text{ bit/sims.}$$

$$I = 2{,}3219 + 1 = 3{,}3219 \text{ bit/sims.}$$

zad 1

$H(X) = 11$ bit/simbol

$H(Y|X) = H(X|Y)$

min. br. elem. u Y

$H(X) - H(X|Y) = H(Y) - \cancel{H(Y|X)}$

$H(X) = H(Y)$

$H(Y) = 11$

$H(Y) = -\sum_j p_j \log p_j = 11$

$n \frac{1}{n} \log \frac{1}{n} = -11$

$\log n = 11$

$n = 2^{11} = 2048$

Zad 1.10

$X_1 \sim \begin{pmatrix} 1 & 2 & 3 & \dots & m \\ p_1 & p_1 & p_1 & & p_1 \end{pmatrix}$ $\quad X_2 \sim \begin{pmatrix} m+1 & m+2 & m+3 & \dots & m+n \\ p_2 & p_2 & p_2 & & p_2 \end{pmatrix}$

$X \sim \begin{pmatrix} 1 & 2 & 3 & \dots & m & m+1 & m+2 & \dots & m+n \\ \alpha p_1 & \alpha p_1 & \alpha p_1 & \dots & \alpha p_1 & (1-\alpha)p_2 & (1-\alpha)p_2 & \dots & (1-\alpha)p_2 \end{pmatrix}$

a) $H(X) = -\sum_{i=1}^{m} \alpha \underbrace{p_i \log_2 \alpha p_i} - \sum_{j=1}^{n} (1-\alpha) p_j \log_2 (1-\alpha) p_j =$

$= \overset{-\alpha \log_2 \alpha +}{\vee} \alpha H(X_1) - (1-\alpha) \underbrace{\sum_{j=1}^{n} p_j \log_2 p_j}_{H(Y)} - (1-\alpha) \underbrace{\sum_{j=1}^{n} p_j}_{1} \log_2 (1-\alpha) =$

$= -\alpha \log_2 \alpha + \alpha H(X_1) + (1-\alpha) H(Y) - (1-\alpha) \log_2 (1-\alpha)$ ✓

b) $H(X) = ?$ max

$H'(X) = -\log_2 \alpha - \frac{\cancel{\alpha}}{\cancel{\alpha} \ln 2} + H(X_1) + \left[-H(X_2)\right] - \left[-\log_2 (1-\alpha) + \cancel{\frac{(1-\alpha)(-1)}{(1-\alpha)\ln 2}}\right] = 0$

$-\log_2 \alpha + H(X_1) - H(X_2) + \log_2 (1-\alpha) = 0$

$\log_2 \frac{1-\alpha}{\alpha} = H(X_2) - H(X_1) \quad | 2^{\cdot}$

$\frac{1-\alpha}{\alpha} = 2^{H(X_2) - H(X_1)} \Rightarrow \alpha\left(2^{H(X_2) - H(X_1)} + 1\right) = 1$

$\alpha = \frac{1}{2^{H(X_2) - H(X_1)}}$ //

1.20

$$X \sim \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ \frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} \end{pmatrix}$$

$$[p(y_j|x_i)] = \begin{array}{c} \frac{p}{2} \\ \frac{p}{2} \\ \frac{1-p}{2} \\ \frac{1-p}{2} \end{array} \begin{bmatrix} 1-f & f & 0 & 0 \\ f & 1-f & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$H(f) = f \log_2 \frac{1}{f} + (1-f) \log_2 \frac{1}{1-f}$$

$$[p(x_i, y_j)] = \begin{bmatrix} \frac{p(1-f)}{2} & \frac{fp}{2} & 0 & 0 \\ \frac{fp}{2} & \frac{(1-f)p}{2} & 0 & 0 \\ 0 & 0 & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & \frac{1-p}{2} \end{bmatrix}$$

$$[p(y_j)] = \begin{bmatrix} \frac{p}{2} & \frac{p}{2} & \frac{1-p}{2} & \frac{1-p}{2} \end{bmatrix}$$

$$H(Y) = -\sum p_j \log_2 p_j = -2 \cdot \frac{p}{2} \log_2 \frac{p}{2} - (1-p) \log_2 \left(\frac{1-p}{2}\right) =$$

$$\boxed{H(Y) = -p \log_2 \frac{p}{2} - (1-p) \log_2 \left(\frac{1-p}{2}\right)} \checkmark$$

$$H(Y|X) = -p(1-f) \log_2 (1-f) - fp \log_2 f - \cancel{(1-p) \log_2 1}$$

$$\boxed{H(Y|X) = p H(f)}$$

$$I(X;Y) = H(Y) - H(Y|X) =$$

$$= -p \log_2 \frac{p}{2} - (1-p) \log_2 \frac{1-p}{2} + p H(f) \quad |'$$

$$-\left(\log_2 \frac{p}{2} + \cancel{p} \cdot \cancel{\frac{1}{\frac{p}{2} \cdot \ln 2}} \cdot \frac{1}{2}\right) - \left(-\log_2 \frac{1-p}{2} + \cancel{(1-p) \frac{1}{\frac{1-p}{2} \cdot \ln 2}} \cdot \frac{1}{2}\right) + H(f)$$

$$-\log_2 \frac{p}{2} + \log_2 \frac{1-p}{2} + H(f) = 0$$

$$H(f) = -\log_2 \frac{1-p}{2} + \log_2 \frac{p}{2}$$

$$H(f) = \log_2 \frac{\frac{p}{2}}{\frac{1-p}{2}} = \log_2 \frac{p}{1-p} \quad \Big| 2^{\wedge}$$

$$2^{H(f)} = \frac{p}{1-p}$$

$$2^{H(f)} - 2^{H(f)} \cdot p = p$$

$$p\left(1 + 2^{H(f)}\right) = 2^{H(f)}$$

$$\boxed{p = \frac{2^{H(f)}}{1 + 2^{H(f)}}}$$

1.30 $[p(x_i)] = [\,p_1 \quad p_2 \quad p_3 \quad p_4\,]$

$$p(y_j|x_i) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} y_1 & y_2 & y_3 \end{array} \\ \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix} \end{array} \qquad [p(x_i y_j)] = \begin{bmatrix} 0.5p_1 & 0.5p_1 & 0 \\ 0.5p_2 & 0.5p_2 & 0 \\ 0 & 0.5p_3 & 0.5p_3 \\ 0 & 0.5p_4 & 0.5p_4 \end{bmatrix}$$

$$[p(y_j)] = \left[\frac{p_1+p_2}{2} \quad \frac{1}{2} \quad \frac{p_3+p_4}{2}\right]$$

$$C = \max_{\{p(x_i)\}} I(X;Y) = \max H(Y) - H(Y|X)$$

$$H(Y) = -\frac{p_1+p_2}{2}\log_2\frac{p_1+p_2}{2} + \frac{1}{2} - \frac{p_3+p_4}{2}\log_2\frac{p_3+p_4}{2}$$

$$H(Y|X) = +\Big[+0.5p_1 + 0.5p_2 + 0.5\underbrace{(p_1+p_2+p_3+p_4)}_{1} + 0.5p_3 + 0.5p_4\Big]$$

$$= 0.5 + 0.5 = 1 \text{ bit/s}$$

$I(X;Y) = H(Y) - H(Y) =$

← konačni broj $\Rightarrow$ ne treba deriv. jer $H(Y)$ mora biti max a to je kada su $p_1, p_2, p_3, p_4 = \frac{1}{4}$

$p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

$$H(Y) = -\frac{1}{4}\log_2\frac{1}{4} + \frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} = \frac{3}{2}$$

$$I(X;Y) = \frac{3}{2} - 1 = \frac{1}{2} \text{ bit/s}$$

1.23

$$[p(y_j|x_i)] = \begin{bmatrix} 1-\beta & \beta & 0 & 0 \\ \beta & 1-\beta & 0 & 0 \\ 0 & 0 & 1-\beta & \beta \\ 0 & 0 & \beta & 1-\beta \end{bmatrix}$$

$C = ?$ ako je $\sigma = \beta$

$[p(x_i)] = [p_1 \quad p_2 \quad p_3 \quad p_4]$ $\qquad \sum_{i=1}^{4} p_i = 1$

$$p(x_i, y_j) = \begin{bmatrix} p_1(1-\beta) & p_1\beta & 0 & 0 \\ p_2(\beta) & p_2(1-\beta) & 0 & 0 \\ 0 & 0 & p_3(1-\beta) & p_3\beta \\ 0 & 0 & p_4\beta & p_4(1-\beta) \end{bmatrix}$$

$[p(y_j)] = [p_1(1-\beta)+p_2(\beta) \quad p_1\beta+p_2(1-\beta) \quad p_3(1-\beta)+p_4\beta \quad p_3\beta+p_4(1-\beta)]$

$H(\beta) = \beta \log \frac{1}{\beta} + (1-\beta)\log\frac{1}{1-\beta}$?! nisam sigurna da li vrijedi, ali u zad 1.20 vrijedi

$H(Y|X) = -[p_1(1-\beta)\log(1-\beta) + p_2\beta\log_2\beta + p_1\beta\log_2\beta + p_2(1-\beta)\log_2(1-\beta) +$

$+ p_3 H(\beta) + p_4 H(\beta)]$

$= -[H(\beta)] = -H(\beta)$

$H(Y|X) = -H(\beta)$

?!: pošto β ne ovisi o p_1, p_2, p_3, p_4 može se uzet

$p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

$H(Y) = -\left[\frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4} + \frac{1}{4}\log\frac{1}{4}\right] = 2$

$C = \max I = 2 - H(\beta)$

$\boxed{7.28}$

$$[p(y_j|x_i)] = \begin{bmatrix} 2/3 & 1/3 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/3 & 2/3 \end{bmatrix}$$

$$p(x) = [p_1 \quad p_2 \quad p_3]$$

$$[p(x_i, y_j)] = \begin{bmatrix} \frac{2}{3}p_1 & \frac{1}{3}p_1 & 0 \\ \frac{1}{3}p_2 & \frac{1}{3}p_2 & \frac{1}{3}p_2 \\ 0 & \frac{1}{3}p_3 & \frac{2}{3}p_3 \end{bmatrix}$$

$$[p(y_j)] = \left[\frac{1}{3}(2p_1+p_2) \quad \frac{1}{3} \quad \frac{1}{3}(p_2+2p_3)\right]$$

$$H(Y|X) = -\left[\frac{2}{3}p_1 \log_2\frac{2}{3} + \frac{1}{3}p_1 \log_2\frac{1}{3} + \frac{1}{3}p_2 \log_2\frac{1}{3} + \frac{1}{3}p_2 \log\frac{1}{3} + \frac{1}{3}p_2 \log\frac{1}{3} + \frac{1}{3}p_3 \log\frac{1}{3} + \frac{2}{3}p_3 \log\frac{2}{3}\right.$$

$$= -\left[\frac{1}{3}\log\frac{1}{3} + \frac{2}{3}\log\frac{2}{3}\right] = 0.9182 \quad \Rightarrow p_1 = p_2 = p_3 = \frac{1}{3}$$

$$H(Y) = 3 \cdot \frac{1}{3}\log\frac{1}{3} = 1.58496$$

$$C = 1.58496 - 0.9182 = 0.666 \text{ bit/simb.}$$

7.33

$$Z \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix} \qquad Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & \cdots & n \\ 1-p & pq_1 & pq_2 & pq_3 & \cdots & pq_n \end{pmatrix}$$

$$X \sim \begin{pmatrix} 1 & 2 & \cdots & n \\ q_1 & q_2 & \cdots & q_n \end{pmatrix} \qquad [p(z)] = [1-p \quad p]$$

$$Y = X \cdot Z \qquad [p(x)] = [q_1 \; q_2 \cdots q_n]$$

$$C = ? \qquad [p(y)] = [1-p \;\; pq_1 \;\; pq_2 \cdots pq_n]$$

$$C = \max_{\{p(x_i)\}} I(X;Y) = \max_{\{p(x_i)\}} H(Y) - H(Y|X) - H(Y|Z)$$

?

$$p(x_i y_j) = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & \cdots & n \\ \hline 1 & (1-p)q_1 & pq_1 & 0 & 0 & \cdots & 0 \\ 2 & (1-p)q_2 & 0 & pq_2 & 0 & \cdots & 0 \\ 3 & (1-p)q_3 & 0 & 0 & pq_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \ddots & \\ n & (1-p)q_n & 0 & \cdots & & & pq_n \end{array}$$

$$[p(y_j|x_i)] = \frac{[p(x_i,y_j)]}{[p(x_i)]} = \begin{bmatrix} 1-p & p & 0 & 0 & & 0 \\ 1-p & 0 & p & 0 & & \\ 1-p & \vdots & & p & \ddots & \vdots \\ 1-p & & & \vdots & & \\ 1-p & 0 & 0 & 0 & & p \end{bmatrix}$$

$$H(Y|X) = -\sum p(x_i,y_j)\log_2 p(y_j|x_i) =$$

$$= -\left[(1-p)q_1\log_2(1-p) + pq_1\log_2 p + (1-p)q_2\log(1-p) + pq_2\log_2 p + \ldots + pq_n\log_2 q_n\right]$$

$$= -\left[(1-p)\log(1-p) + p\log p\right] = \boxed{H(Z) = H(Y|X)}$$

$$H(Y) = -\left[(1-p)\log(1-p) + pq_1\log_2 pq_1 + pq_2\log_2 pq_2 + \ldots + pq_n\log_2 pq_n\right] =$$

$$= -\left[(1-p)\log(1-p) + pq_1\log_2 p + pq_1\log_2 q_1 + pq_2\log_2 p + pq_2\log_2 q_2 \ldots\right] =$$

$$= -\left[(1-p)\log(1-p) + p\underbrace{\left(\sum_{i=1}^{n} q_i\log q_i\right)}_{-H(X)} + p\log_2 p\right] =$$

$$H(Y) = -\left[-H(z) - H(x)\cdot p\right] = H(z) + pH(x)$$

$$C = H(Y) - H(Y|X) - H(Y|Z) = \cancel{H(z)} + pH(x) - \cancel{H(z)} - H(Y|Z)$$

$$C = pH(x) - H(Y|Z)$$

$$p(z_i, y_j) = \begin{array}{c|c} & \begin{array}{cccccc} 0 & 1 & 2 & 3 & \cdots & n \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} (1-p) & 0 & 0 & 0 & \cdots & 0 \\ 0 & pq_1 & pq_2 & pq_3 & \cdots & pq_n \end{bmatrix} \end{array}$$

$$p(y_j|z_i) = \frac{[p(z_i, y_j)]}{[p(z)]} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & q_1 & q_2 & \cdots & q_n \end{bmatrix}$$

$$H(Y|Z) = -\left[\cancel{(1-p)\log_2 1} + pq_1 \log q_1 + pq_2 \log q_2 + \cdots pq_n \log q_n\right] =$$

$$= -\left[p \cdot H(x)\right]$$

$$C = p(H(x) - p$$