# VIS JIR 2022.

1) a) Neka su $A$ i $B$ proizvoljni događaji.

$$\mathbb{P}(\overline{A \cap B}) = \mathbb{P}(\bar{A} \cup \bar{B}) \quad (1)$$

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \quad (2)$$

$$\Rightarrow \mathbb{P}(\overline{A \cap B}) \overset{(1)}{=} \mathbb{P}(\bar{A} \cup \bar{B}) \overset{(2)}{=} \mathbb{P}(\bar{A}) + \mathbb{P}(\bar{B}) - \mathbb{P}(\bar{A} \cap \bar{B})$$

b) Definiramo događaje:

$A$ = "izvukli smo barem jedan ♥"

$B$ = "izvukli smo barem jednu crnu kartu"

$$\mathbb{P}(A \cap B) = ?$$

niti jedan ♥ — niti jedna crna

S druge strane znamo: $\mathbb{P}(\bar{A}) = \dfrac{\binom{39}{4}}{\binom{52}{4}}$, $\mathbb{P}(\bar{B}) = \dfrac{\binom{26}{4}}{\binom{52}{4}}$

$$\mathbb{P}(\bar{A} \cap \bar{B}) = \frac{\binom{12}{4}}{\binom{52}{4}}$$

samo smo izvukli ♦

Stoga je

$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}(\overline{A \cap B})$$

$$= 1 - \mathbb{P}(\bar{A}) - \mathbb{P}(\bar{B}) + \mathbb{P}(\bar{A} \cap \bar{B})$$

$$= 1 - \frac{\binom{39}{4}}{\binom{52}{4}} - \frac{\binom{26}{4}}{\binom{52}{4}} + \frac{\binom{12}{4}}{\binom{52}{4}} = \ldots$$

2) 10 studenata

| | |
|---|---|
| 3 | odlično |
| 4 | dobra |
| 2 | dovoljna |
| 1 | loš |

Nasumično je odabran student S.

$H_0 = \{ S = \text{odličan} \}$, $\mathbb{P}(H_0) = 3/10$

$H_1 = \{ S = \text{dobar} \}$, $\mathbb{P}(H_1) = 4/10$

$H_2 = \{ S = \text{dovoljan} \}$, $\mathbb{P}(H_2) = 2/10$

$H_3 = \{ S = \text{loš} \}$, $\mathbb{P}(H_3) = 1/10$

$A = \{ S \text{ odgovori točno na sva tri pitanja} \}$

$$\mathbb{P}(A \mid H_0) = 1$$

$$\mathbb{P}(A \mid H_1) = \frac{\binom{16}{3}}{\binom{20}{3}}$$

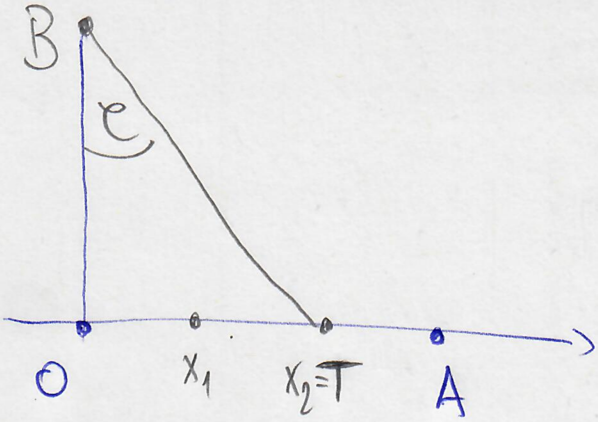
$$\mathbb{P}(A \mid H_2) = \frac{\binom{10}{3}}{\binom{20}{3}}$$

$$\mathbb{P}(A \mid H_3) = \frac{\binom{5}{3}}{\binom{20}{3}}$$

$\mathbb{P}(S = \text{loš} \mid S \text{ odgovori na sva pitanja točno})$

$$= \mathbb{P}(H_3 \mid A) = \frac{\mathbb{P}(A \mid H_3) \cdot \mathbb{P}(H_3)}{\sum_{i=1}^{4} \mathbb{P}(A \mid H_i)\, \mathbb{P}(H_i)} = \ldots$$

3)




uniformna razdioba za $X_1, X_2$

$$f_{X_i}(x) = 1$$

$$F_{X_i}(x) = x$$

a) $X = \max\{X_1, X_2\}$

$$F_X(x) = \mathbb{P}(X_1 \le x, X_2 \le x) = \mathbb{P}(X_1 \le x)\,\mathbb{P}(X_2 \le x)$$

$$= F_{X_1}(x)\,F_{X_2}(x) = x^2 \quad , \quad x \in [0,1]$$

$$f_X(x) = F_X'(x) = 2x \quad , \quad x \in [0,1]$$

b) $\Phi := \operatorname{arctg} X \quad , \quad X \in [0,1] \Rightarrow \Phi \in [0, \frac{\pi}{4}]$

$$\Rightarrow \psi(x) = \operatorname{arctg} x$$

$$g_\Phi(\varphi) = f_X(\psi^{-1}(\varphi)) \cdot \left| \frac{d}{d\varphi} \psi^{-1}(\varphi) \right|$$

$$= 2\operatorname{tg}(\varphi) \cdot \frac{1}{\cos^2\varphi} \quad , \quad \varphi \in [0, \frac{\pi}{4}]$$

$$= \frac{2\sin\varphi}{\cos^3\varphi} \quad ,$$

$$\mathbb{P}(\Phi > \frac{\pi}{6}) = 2\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{\sin\varphi}{\cos^3\varphi}\,d\varphi = \begin{bmatrix} u = \cos\varphi \\ du = -\sin\varphi\, d\varphi \end{bmatrix} = -2\int_{\sqrt{3}/2}^{\sqrt{2}/2} \frac{du}{u^3} = \frac{2}{3}$$

4) $\boxed{X_i \text{ nezavisne} \Rightarrow \mathbb{E}(X_iX_j) = \mathbb{E}(X_i)\mathbb{E}(X_j) \text{ za } i \neq j}$

a) Neka je $i \in \{1, \dots, n\}$

$$\mathbb{E}(X_i \cdot \overline{X}) = \mathbb{E}\left(X_i \cdot \frac{1}{n}\sum_{j=1}^{n} X_j\right) = \frac{1}{n}\sum_{j=1}^{n} \mathbb{E}(X_iX_j)$$

$$= \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbb{E}(X_i)\mathbb{E}(X_j) + \frac{1}{n}\mathbb{E}(X_i^2)$$

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S druge strane

$$\mathbb{E}(\overline{X}^2) = \mathbb{E}\left(\left(\frac{1}{n}\sum_{j=1}^{n} X_j\right)\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\right)$$

$$= \frac{1}{n^2}\mathbb{E}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} X_iX_j\right) = \frac{1}{n^2}\mathbb{E}\left(\sum_{j=1}^{n} X_j^2 + \sum_{j=1}^{n}\sum_{i\neq j} X_iX_j\right)$$

$$= \frac{1}{n^2}\sum_{j=1}^{n} \underbrace{\mathbb{E}(X_j^2)}_{\text{svi su isti kao } \mathbb{E}(X_i^2)} + \frac{1}{n^2}\sum_{j=1}^{n} \underbrace{\sum_{\substack{i=1 \\ i\neq j}}^{n} \mathbb{E}(X_i)\mathbb{E}(X_j)}_{\text{isto za svaki } j}$$

$$= \frac{1}{n}\mathbb{E}(X_i^2) + \frac{1}{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \mathbb{E}(X_i)\mathbb{E}(X_j)$$

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b) $\mathbb{E}\left(X_2 \frac{X_1+X_2+X_3}{3}\right) = \mathbb{E}\left(\left(\frac{X_1+X_2+X_3}{3}\right)^2\right) = \mathbb{E}(\overline{X}^2)$ | $\mathbb{E}X_i = a$, $\operatorname{Var}X_i = \sigma^2$

$\operatorname{Var}\overline{X} = \mathbb{E}(\overline{X}^2) - \mathbb{E}(\overline{X})^2$

$$\Rightarrow \mathbb{E}(\overline{X}^2) = \operatorname{Var}\overline{X} + \mathbb{E}(\overline{X})^2 = \operatorname{Var}\left(\frac{X_1+X_2+X_3}{3}\right) + \left(\mathbb{E}\left(\frac{X_1+X_2+X_3}{3}\right)\right)^2$$

$$= \frac{1}{9}\cdot 3\sigma^2 + a^2 = \frac{\sigma^2}{3} + a^2,$$ Preostaje procijeniti $\sigma^2, a^2$

5)

$X \sim$ masa jedne jabuke , $\mathbb{E}X = 170$ , $\sqrt{\mathrm{Var}X} = 15$

a) $\mathbb{P}\left(\sum_{i=1}^{117} X_i > 20\,000\right) = \mathbb{P}\left(\frac{\frac{1}{117}\sum_{i=1}^{117} X_i - 170}{15/\sqrt{117}} > \frac{\frac{20\,000}{117} - 170}{15/\sqrt{117}}\right)$

$X_1, \dots, X_n \sim X$ n.j.d.

$$= 1 - \Phi\left(\frac{0{,}940}{1{,}3867}\right) = 1 - \Phi(0{,}677)$$

$$= 1 - \left(\frac{1}{2} + \frac{1}{2}\Phi^*(0{,}677)\right) = \frac{1}{2} - \frac{1}{2}\Phi^*(0{,}677) =$$

$$= \frac{1}{2} - \frac{1}{2}\cdot 0{,}497 = 0{,}2515$$

b) $0{,}99 \le \mathbb{P}\left(\sum_{i=1}^{n} X_i > 20000\right)$

$$= \mathbb{P}\left(\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - 170}{15/\sqrt{n}} > \frac{\frac{20000}{n} - 170}{15/\sqrt{n}}\right) = \frac{1}{2} - \frac{1}{2}\cdot\Phi^*\left(\frac{\frac{20000}{n} - 170}{15/\sqrt{n}}\right)$$

$\Rightarrow \quad 0{,}98 \le -\Phi^*\left(\frac{\frac{20000}{n} - 170}{15/\sqrt{n}}\right) = \Phi^*\left(\frac{170 - \frac{20000}{n}}{15/\sqrt{n}}\right)$

$\Leftrightarrow$

$\frac{170 - \frac{20000}{n}}{15/\sqrt{n}} \ge 2{,}33 \Leftrightarrow 170 - \frac{20000}{n} \ge 34{,}95/\sqrt{n} \quad / \cdot n$

$\Leftrightarrow \boxed{170n - 34{,}95\sqrt{n} - 20000 \ge 0}$ $\qquad t = \sqrt{n}$

$$t_{1,2} = \frac{34{,}95 + 3687{,}9}{2\cdot 170} = 10{,}95$$

$\Rightarrow n \ge 119{,}898 \Rightarrow \boxed{n \ge 120}$

c) $\mathbb{P}(\text{u jabuci je crv}) = 0{,}1$. $\quad Y$ – broj crva na $n$ jabuka , $Y \sim B(n, 0{,}1)$

$$0{,}95 \le \mathbb{P}(n - Y \ge 100) = \mathbb{P}(Y \le n - 100) = \mathbb{P}\left(\frac{Y - 0{,}1n}{0{,}09\sqrt{n}} \le \frac{0{,}9n - 100}{0{,}09\sqrt{n}}\right) =$$

$= D.M.L \approx \Phi\left(\frac{0{,}9n - 100}{0{,}09\sqrt{n}}\right) = \frac{1}{2} + \frac{1}{2}\Phi^*\left(\frac{0{,}9n - 100}{0{,}09\sqrt{n}}\right)$

$\Rightarrow \quad 0{,}9 \leq \Phi^*\left(\frac{0{,}9n - 100}{0{,}09\sqrt{n}}\right)$

$\Leftrightarrow \quad 1{,}65 \leq \frac{0{,}9n - 100}{0{,}09\sqrt{n}}$

$$0{,}9n - 0{,}1485\sqrt{n} - 100 \geq 0$$

$\Rightarrow \sqrt{n} \geq 10{,}623 \quad , n \geq 113$

6.) $X_1, \dots, X_n \quad , X_i \sim \mathcal{N}(a, \sigma^2)$

a) Kyjitiw : $\left\langle \overline{X} - u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \leq a \leq \overline{X} + u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} \right\rangle \quad , \alpha = 1 - p$

b) $\sigma^2 = 0{,}0025 \quad , \sigma = 0{,}05 \quad , p = 0{,}95 , \quad \alpha = 0{,}05$

Duljina $= 0.02$

$$0.02 \geq 2 \cdot u_{1-\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = 2 \cdot 1{,}96 \cdot \frac{0{,}05}{\sqrt{n}}$$

$$\mathbb{P}\left(X \leq u_{1-\frac{\alpha}{2}}\right) = 1 - \frac{\alpha}{2}$$

$$\parallel$$

$$\Phi\left(u_{1-\frac{\alpha}{2}}\right) = \frac{1}{2} + \frac{1}{2} \Phi^*\left(u_{1-\frac{\alpha}{2}}\right)$$

$$1 - \alpha = \Phi^*\left(u_{1-\frac{\alpha}{2}}\right)$$

$$\parallel$$

$$0{,}95$$

pročitano iz tablice

$$u_{1-\frac{\alpha}{2}} = 1{,}96$$

$\Rightarrow \quad \sqrt{n} \geq 9{,}8 \quad \Rightarrow \quad n \geq 97$

c) može se odbaciti jer

je $\quad 0{,}5 \notin \left[0{,}55 - 0{,}02 , \; 0{,}55 + 0{,}02\right]$