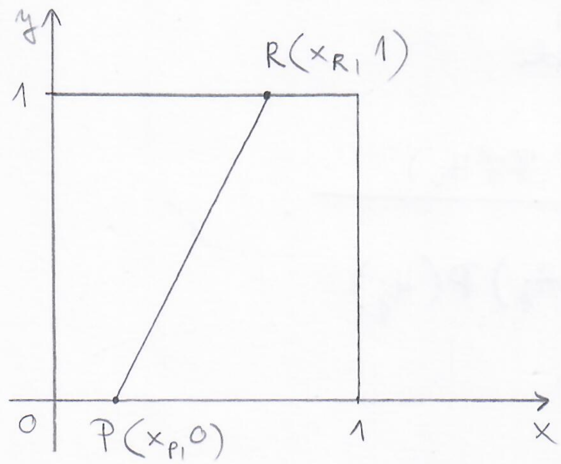
VJEROJATNOST I STATISTIKA

Zimski ispitni rok (9.2.2020.)

RJEŠENJA ZADATAKA

1.

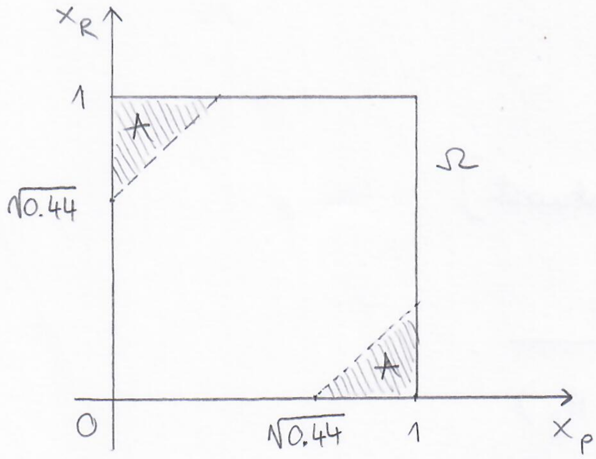


Postavimo kvadrat u koordinatni sustav kao na slici i označimo sa x_P, x_R redom apscise točaka P i R.

Imamo

$$|PR| > 1.2 \iff (x_P - x_R)^2 + (0-1)^2 > 1.44$$

$$\iff |x_P - x_R| > \sqrt{0.44} \qquad (*)$$



Budući da su $x_P, x_R \in [0,1]$, imamo

$\Omega = [0,1]^2$.

Nadalje, skup svih točaka (x_P, x_R) za koje vrijedi $(*)$ dan je osjenčanim područjem A na slici pa slijedi

$$\mathbb{P}(A) = \frac{m(A)}{m(\Omega)} = \frac{2 \cdot \frac{1}{2} \cdot (1-\sqrt{0.44})^2}{1^2} = 0.11335$$

2. (a) Teorem. (Bayesova formula)

Neka je $\{H_1, H_2, \dots, H_n\}$ potpun sustav događaja u vjerojatnosnom prostoru Ω te $A \subseteq \Omega$ događaj takav da $\mathbb{P}(A) > 0$.

Tada za svaki $i = 1, 2, \dots, n$ vrijedi

$$\mathbb{P}(H_i | A) = \frac{\mathbb{P}(A|H_i)\mathbb{P}(H_i)}{\sum_{j=1}^{n} \mathbb{P}(A|H_j)\mathbb{P}(H_j)}.$$

Dokaz.

Prema definiciji uvjetne vjerojatnosti

$$\mathbb{P}(H_i | A) = \frac{\mathbb{P}(H_i \cap A)}{\mathbb{P}(A)}$$

$$= \frac{\mathbb{P}(A|H_i)\mathbb{P}(H_i)}{\mathbb{P}(A)}$$

= [formula potpune vjerojatnosti]

$$= \frac{\mathbb{P}(A|H_i)\mathbb{P}(H_i)}{\sum_{j=1}^{n} \mathbb{P}(A|H_j)\mathbb{P}(H_j)}.$$

Q.E.D.

(b) Definiramo potpun sustav događaja

$$H_i = \{\text{Bartol je metu pogodio točno } i \text{ puta}\}, \quad i = 0, 1, 2, 3, 4,$$

te događaj

$$A = \{\text{meta je pogođena dvaput}\}.$$

Imamo:

$$\mathbb{P}(H_i) = \binom{4}{i}(0.45)^i (0.55)^{4-i}, \quad i = 0, 1, 2, 3, 4,$$

$$\mathbb{P}(A|H_0) = \binom{4}{2}(0.25)^2(0.75)^2 \quad \text{(Antun je pogodio metu točno dvaput)}$$

$$\mathbb{P}(A|H_1) = \binom{4}{1}(0.25)^1(0.75)^3 \quad \text{(Antun je pogodio metu točno jednom)}$$

$$\mathbb{P}(A|H_2) = (0.75)^4 \quad \text{(Antun je svaki put promašio)}$$

$$\mathbb{P}(A|H_3) = \mathbb{P}(A|H_4) = 0$$

Prema Bayesovoj formuli slijedi

$$\mathbb{P}(H_2|A) = \frac{\mathbb{P}(A|H_2)\mathbb{P}(H_2)}{\sum_{j=1}^{4} \mathbb{P}(A|H_j)\mathbb{P}(H_j)}$$

$$= \frac{(0.75)^4 \cdot 6 \cdot (0.45)^2 (0.55)^2}{6 \cdot (0.25)^2 (0.75)^2 (0.55)^4 + 4 \cdot 0.25 \cdot (0.75)^3 \cdot 4 \cdot 0.45 \cdot (0.55)^3 + (0.75)^4 \cdot 6 \cdot (0.45)^2 (0.55)^2}$$

$$= \frac{729}{1642} = 0.44397$$

3. Vjerojatnost da se na 4 igraće kocke pojave barem 2 šestice:

$$p = 1 - \underbrace{\left(\frac{5}{6}\right)^4}_{\substack{\text{vjerojatnost} \\ \text{da se pojavi} \\ \text{0 šestica}}} - \underbrace{\frac{4 \cdot 1 \cdot 5^3}{6^4}}_{\substack{\text{vjerojatnost} \\ \text{da se pojavi} \\ \text{točno 1 šestica}}} = \frac{19}{144}$$

$$\Rightarrow X \sim G\left(\frac{19}{144}\right)$$

Zakon razdiobe od X:

$$\mathbb{P}(X = n) = \left(\frac{125}{144}\right)^{n-1} \cdot \frac{19}{144}, \quad n \in \mathbb{N}.$$

$$\mathbb{E}X = \sum_{n=1}^{\infty} n \left(\frac{125}{144}\right)^{n-1} \cdot \frac{19}{144}$$

$$= \left[\begin{array}{l} \sum_{n=0}^{\infty} q^n = \frac{1}{1-q} \Rightarrow \sum_{n=1}^{\infty} n q^{n-1} = \frac{1}{(1-q)^2} = \frac{1}{p^2} \\ \Rightarrow \sum_{n=1}^{\infty} n q^{n-1} p = \frac{1}{p} \end{array}\right] = \frac{144}{19}$$

$$\mathbb{P}(X > \mathbb{E}X) = \mathbb{P}\left(X > \frac{144}{19}\right) = \mathbb{P}(X \geq 8) = \sum_{n=8}^{\infty} \mathbb{P}(X = 8)$$

$$= \sum_{n=8}^{\infty} \left(\frac{125}{144}\right)^{n-1} \cdot \frac{19}{144} = \frac{19}{144} \cdot \left(\frac{125}{144}\right)^7 \sum_{n=0}^{\infty} \left(\frac{125}{144}\right)^n$$

$$= \frac{\cancel{19}}{\cancel{144}} \cdot \left(\frac{125}{144}\right)^7 \cdot \frac{1}{1 - \frac{125}{\cancel{144}}} = \left(\frac{125}{144}\right)^7$$

$$= 0.371392$$

4. (a) Kažemo da slučajna varijabla X ima eksponencijalnu razdiobu s parametrom $\lambda > 0$ ako joj je funkcija gustoće

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

(b)
$$\mathbb{E}X = \int_{-\infty}^{\infty} x f(x)\,dx = \lambda \int_{0}^{\infty} x e^{-\lambda x}\,dx = \left[\begin{array}{ll} u = x & \Rightarrow du = dx \\ dv = e^{-\lambda x}dx & \Rightarrow v = -\frac{1}{\lambda}e^{-\lambda x} \end{array}\right]$$

$$= \underbrace{-x e^{-\lambda x}\Big|_0^{\infty}}_{=0-0=0} + \int_0^{\infty} e^{-\lambda x}\,dx = -\frac{1}{\lambda} e^{-\lambda x}\Big|_0^{\infty}$$

$$= \frac{1}{\lambda}\cdot 1 - 0 = \frac{1}{\lambda}$$

(c)
$$\mathbb{P}(X \leq 9 \mid X > 7) = \left[\begin{array}{l} \text{odsutstvo pamćenja} \\ \text{eksponencijalne razdiobe} \end{array}\right]$$

$$= \mathbb{P}(X \leq 2) = \left[\begin{array}{l} \frac{1}{\lambda} = 10 \Rightarrow \lambda = 0.1 \\ \Rightarrow F(x) = 1 - e^{-0.1x}, \quad x > 0 \end{array}\right]$$

$$= 1 - e^{-0.2} = 0.181269$$

(d) $X_1, \dots, X_{500} \sim \mathrm{Exp}(10)$ $\qquad \overline{X}_{500} = \dfrac{X_1 + \dots + X_{500}}{500} = \dfrac{S_{500}}{500}$

$$\mathbb{P}(\overline{X}_{500} > 11) = \mathbb{P}(S_{500} > 5500)$$

$$= \mathbb{P}\left(\frac{S_{500} - 500\cdot 10}{10\sqrt{500}} > \frac{5500 - 500\cdot 10}{10\sqrt{500}}\right)$$

$$\approx \left[\begin{array}{c} \text{centralni granični} \\ \text{teorem} \end{array}\right] \approx \frac{1}{2}\left[1 - \Phi^*(2.24)\right]$$

$$= \frac{1}{2}\left[1 - 0.97491\right] = 0.012545$$

5. $1 = \int_{-\infty}^{\infty} f(x)\,dx = c \int_0^1 x\,dx = c \cdot \frac{1}{2} \Longrightarrow c = 2$

$\Longrightarrow f(x) = 2x, \quad x \in [0,1]$

(a) $f(x,y) = f_X(x) f_Y(y) = 4xy, \quad (x,y) \in [0,1]^2$

(b) $$\mathbb{P}(X^2+Y^2>1) = \iint_{\{x^2+y^2>1\}} f(x,y)\,dx\,dy$$

$$= 4 \int_0^1 \int_{\sqrt{1-x^2}}^1 xy\,dy\,dx = 4 \int_0^1 x \left(\frac{1}{2} y^2 \Big|_{\sqrt{1-x^2}}^1 \right) dx$$

$$= 2 \int_0^1 x\left(1-(1-x^2)\right) dx = 2 \int_0^1 x^3\,dx = \frac{1}{2}$$

(c) $$\mathbb{P}\left(X^2+Y^2>1 \,\middle|\, X>\tfrac{1}{2}\right) = \frac{\mathbb{P}\left(\{X^2+Y^2>1\} \cap \{X>\tfrac{1}{2}\}\right)}{\mathbb{P}(X>\tfrac{1}{2})}$$

$$= \frac{4 \int_{\frac{1}{2}}^1 \int_{\sqrt{1-x^2}}^1 xy\,dy\,dx}{2 \int_{\frac{1}{2}}^1 x\,dx} = \frac{2 \int_{\frac{1}{2}}^1 x^3\,dx}{\frac{3}{4}}$$

$$= \frac{\frac{15}{32}}{\frac{3}{4}} = \frac{5}{8}$$

6. (a) Kažemo da je $\Theta_n = \Theta(X_1, \dots, X_n)$ valjan procjenitelj za ϑ ako za svaki $\varepsilon > 0$ vrijedi

$$\lim_{n \to \infty} \mathbb{P}(|\Theta_n - \vartheta| < \varepsilon) = 1$$

(tj. Θ_n konvergira po vjerojatnosti prema ϑ).

(b) Teorem.

Da bi nepristran procjenitelj Θ_n za ϑ bio i valjan, dovoljno je da za njegovu disperziju vrijedi $\lim_{n\to\infty} \mathbb{D}(\Theta_n) = 0$.

Dokaz.

Za svaki $\varepsilon > 0$ prema Čebiševljevoj nejednakosti imamo

$$\mathbb{P}(|\Theta_n - \underbrace{\vartheta}_{\substack{\| \\ \mathbb{E}\Theta_n \\ \text{(zbog nepristranosti)}}}| < \varepsilon) \geq 1 - \frac{\mathbb{D}(\Theta_n)}{\varepsilon^2}.$$

Zbog $\lim_{n\to\infty} \mathbb{D}(\Theta_n) = 0$ prema teoremu o sendviču slijedi

$$\lim_{n \to \infty} \mathbb{P}(|\Theta_n - \vartheta| < \varepsilon) = 1$$

za svaki $\varepsilon > 0$, pa je Θ_n po definiciji valjan procjenitelj za ϑ.

Q.E.D.

(c) Funkcija izglednosti:

$$L(\lambda, x_1, \ldots, x_n) = \prod_{i=1}^{n} f_\lambda(x_i) = 2^n \lambda^{2n} \left(\prod_{i=1}^{n} x_i \right) e^{-\lambda^2 \sum_{i=1}^{n} x_i^2}$$

$$\Rightarrow \ln L(\lambda, x_1, \ldots, x_n) = n \ln 2 + 2n \ln \lambda + \sum_{i=1}^{n} \ln x_i - \lambda^2 \sum_{i=1}^{n} x_i^2 \quad \Big| \frac{\partial}{\partial \lambda}$$

$$\Rightarrow \frac{\frac{\partial L}{\partial \lambda}(\lambda, x_1, \ldots, x_n)}{L(\lambda, x_1, \ldots, x_n)} = \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2$$

$$\Rightarrow \frac{2n}{\lambda} - 2\lambda \sum_{i=1}^{n} x_i^2 = 0$$

$$\Rightarrow \lambda^2 \sum_{i=1}^{n} x_i^2 = n$$

$$\Rightarrow \hat{\lambda} = \sqrt{\frac{n}{\sum_{i=1}^{n} x_i^2}} = \frac{1}{\sqrt{\overline{x^2}}}$$