VJEROJATNOST I STATISTIKA

Drugi jesenski ispitni rok (9. 9. 2020.)

- RJEŠENJA ZADATAKA -

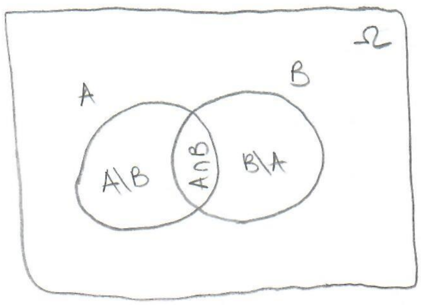
1. (a) Preslikavanje $\mathbb{P} : \mathcal{F} \to [0,1]$ koje ima svojstva:

(i) $\mathbb{P}(\Omega) = 1$, $\mathbb{P}(\emptyset) = 0$, (normiranost)

(ii) ako su $A, B \in \mathcal{F}$ takvi da $A \subseteq B$, onda $\mathbb{P}(A) \leq \mathbb{P}(B)$, (monotonost)

(iii) ako su $A, B \in \mathcal{F}$ takvi da $A \cap B = \emptyset$, onda $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, (aditivnost)

zovemo VJEROJATNOST (na $\mathcal{F}$).

(b) 

Uočimo da vrijedi:

$$A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A),$$

$$A = (A \setminus B) \cup (A \cap B),$$

$$B = (B \setminus A) \cup (A \cap B),$$

pri čemu su skupovi $A \setminus B$, $A \cap B$, $B \setminus A$ u parovima disjunktni.

Zbog aditivnosti vjerojatnosti:

$$\mathbb{P}(A \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B) + \mathbb{P}(B \setminus A)$$

$$= \big(\mathbb{P}(A \setminus B) + \mathbb{P}(A \cap B)\big) + \big(\mathbb{P}(B \setminus A) + \mathbb{P}(A \cap B)\big) - \mathbb{P}(A \cap B)$$

$$= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B).$$

(c) Označimo događaje

$A = \{$izvučena je barem jedna karta pik boje$\}$,

$B = \{$izvučen je barem jedan as$\}$.

Tražimo

$$\mathbb{P}(A \cap B) = 1 - \mathbb{P}(\overline{A \cap B})$$

$$= 1 - \mathbb{P}(\overline{A} \cup \overline{B})$$

$$\stackrel{(6)}{=} 1 - \left(\mathbb{P}(\overline{A}) + \mathbb{P}(\overline{B}) - \mathbb{P}(\overline{A} \cap \overline{B})\right)$$

$$= 1 - \frac{\binom{39}{5}}{\binom{52}{5}} - \frac{\binom{48}{5}}{\binom{52}{5}} + \frac{\binom{36}{5}}{\binom{52}{5}}$$

$$= \frac{229297}{866320} \approx 0.264679$$

2. Definirajmo potpun sustav događaja

$$H_0 = \{\text{poslan je znak } 0\},$$

$$H_1 = \{\text{poslan je znak } 1\},$$

te događaj

$$A = \{\text{primljen je znak } 1\}.$$

Prema uvjetima zadatka

$$\mathbb{P}(H_0) = 0.6, \quad \mathbb{P}(A \mid H_0) = 0.01,$$

$$\mathbb{P}(H_1) = 0.4, \quad \mathbb{P}(A \mid H_1) = 0.99.$$

Sada prema Bayesovoj formuli slijedi

$$\mathbb{P}(H_1 \mid A) = \frac{\mathbb{P}(A \mid H_1)\mathbb{P}(H_1)}{\mathbb{P}(A \mid H_0)\mathbb{P}(H_0) + \mathbb{P}(A \mid H_1)\mathbb{P}(H_1)}$$

$$= \frac{66}{67} \approx 0.985075.$$

Budući da se svaki znak emitira neovisno o drugima, tražena je vjerojatnost jednaka

$$\mathbb{P}(H_1 \mid A)^4 = \left(\frac{66}{67}\right)^4 \approx 0.941622.$$

3. (a) Neka je $\lambda > 0$. Kažemo da slučajna varijabla X ima POISSONOVU RAZDIOBU s parametrom λ $(X \sim \mathcal{P}(\lambda))$ ako je njen zakon razdiobe

$$\mathbb{P}(X=k) = \frac{\lambda^k}{k!} e^{-\lambda},$$

za sve $k \in \mathbb{N}_0$.

Za $X \sim \mathcal{P}(\lambda)$ imamo

$$\vartheta(t) = \mathbb{E}\left(e^{itX}\right) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} \cdot e^{itk} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^{it})^k}{k!}$$

$$= e^{-\lambda} \cdot e^{\lambda e^{it}} = e^{\lambda(e^{it}-1)}.$$

(b) Imamo

$$\vartheta'(t) = e^{\lambda(e^{it}-1)} \cdot \lambda e^{it} \cdot i,$$

odakle slijedi

$$\mathbb{E}X = \frac{1}{i} \vartheta'(0) = \frac{1}{i} e^{\lambda(1-1)} \cdot \lambda \cdot 1 \cdot i = \lambda.$$

(c) Neka je X broj primljenih SMS-ova u jednom danu. Tada $X \sim \mathcal{P}(\lambda)$ i iz uvjeta zadatka

$$30\lambda = 110 \implies \lambda = \frac{11}{3}.$$

(i) $$\mathbb{P}(X < \underbrace{\mathbb{E}X}_{=\lambda}) = \mathbb{P}\left(X < \frac{11}{3}\right) = \mathbb{P}(X=0) + \mathbb{P}(X=1) + \mathbb{P}(X=2) + \mathbb{P}(X=3)$$

$$= e^{-\frac{11}{3}} \left(1 + \frac{11}{3} + \frac{1}{2} \cdot \frac{121}{9} + \frac{1}{6} \cdot \frac{1331}{27}\right)$$

$$= \frac{1588}{81} e^{-\frac{11}{3}} \approx 0.501132$$

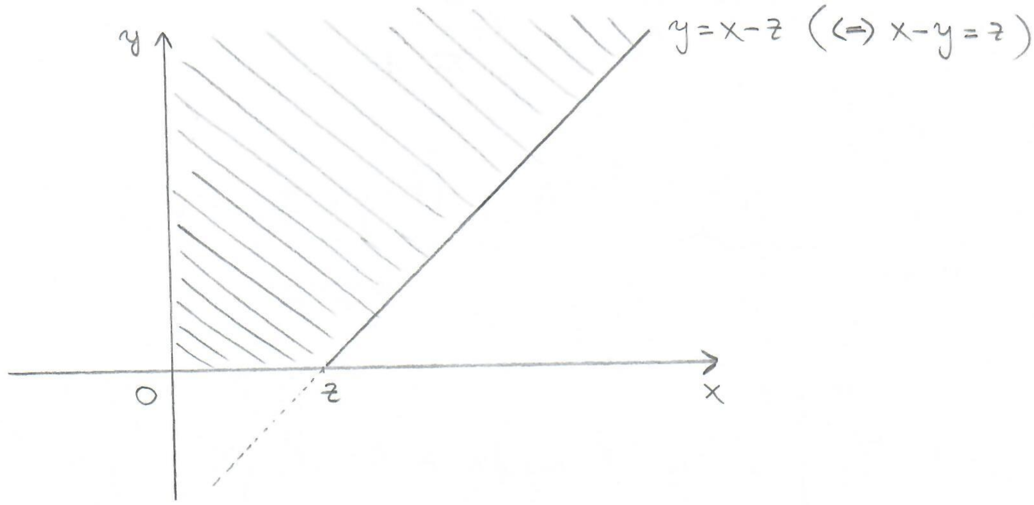
(ii) $$\mathbb{P}(X=0) = e^{-\frac{11}{3}} \approx 0.0255615$$

4. Neka su X i Y redom vremena kašnjenja Josipa i Renata. Tada su $X, Y \sim Exp(\lambda)$ nezavisne slučajne varijable i $Z = X - Y$.

Odredimo funkciju razdiobe od Z:

$$F_Z(z) = \mathbb{P}(X - Y \leq z), \quad z \in \mathbb{R}.$$

1° $z > 0$



$$F_Z(z) = \iint_{\{x-y \leq z\}} \underbrace{f_{X,Y}(x,y)}_{=f_X(x)f_Y(y)} \, dx\, dy = \int_0^\infty \int_0^{y+z} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} \, dx\, dy$$

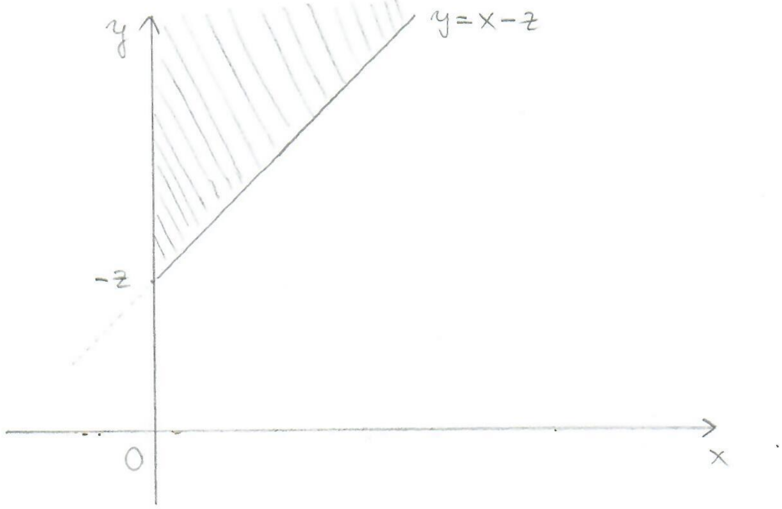
$$= \lambda^2 \int_0^\infty e^{-\lambda y} \left(\int_0^{y+z} e^{-\lambda x} \, dx \right) dy = \lambda^{\cancel{2}} \int_0^\infty e^{-\lambda y} \left(-\frac{1}{\cancel{\lambda}} e^{-\lambda x} \right) \Big|_0^{y+z} dy$$

$$= \lambda \int_0^\infty e^{-\lambda y} \left(1 - e^{-\lambda z - \lambda y} \right) dy = \lambda \int_0^\infty \left(e^{-\lambda y} - e^{-\lambda z - 2\lambda y} \right) dy$$

$$= \lambda \left(-\frac{1}{\lambda} e^{-\lambda y} + \frac{1}{2\lambda} e^{-\lambda z} \cdot e^{-2\lambda y} \right) \Big|_0^\infty = \lambda \left(\frac{1}{\lambda} - 0 + 0 - \frac{1}{2\lambda} e^{-\lambda z} \right)$$

$$= 1 - \frac{1}{2} e^{-\lambda z}$$

$2^\circ \quad z \le 0$



$$F_Z(z) = \iint\limits_{\{x-y\le z\}} \underbrace{f_{X,Y}(x,y)}_{=f_X(x)f_Y(y)}\,dx\,dy = 1 - \iint\limits_{\{x-y>z\}} f_X(x)f_Y(y)\,dx\,dy$$

$$= 1-\lambda^2\int_0^\infty\int_0^{x-z} e^{-\lambda x}e^{-\lambda y}\,dy\,dx = 1-\lambda^2\int_0^\infty e^{-\lambda x}\left(\int_0^{x-z}e^{-\lambda y}\,dy\right)dx$$

$$= 1-\cancel{\lambda^2}\int_0^\infty e^{-\lambda x}\left(-\frac{1}{\cancel{\lambda}}e^{-\lambda y}\right)\Bigg|_0^{x-z}dx = 1-\lambda\int_0^\infty e^{-\lambda x}\left(1-e^{\lambda(z-x)}\right)dx$$

$$= 1-\lambda\int_0^\infty\left(e^{-\lambda x}-e^{\lambda z-2\lambda x}\right)dx = 1-\lambda\left(-\frac{1}{\lambda}e^{-\lambda x}+\frac{1}{2\lambda}e^{\lambda z}\cdot e^{-2\lambda x}\right)\Bigg|_0^\infty$$

$$= 1-\lambda\left(\frac{1}{\lambda}-0+0-\frac{1}{2\lambda}e^{\lambda z}\right) = 1-\left(1-\frac{1}{2}e^{\lambda z}\right) = \frac{1}{2}e^{\lambda z}$$

$$\Rightarrow F_Z(z) = \begin{cases} 1-\frac{1}{2}e^{-\lambda z}, & z>0 \\ \frac{1}{2}e^{\lambda z}, & z\le 0 \end{cases} \qquad \Rightarrow f_Z(z) = \begin{cases} \frac{\lambda}{2}e^{-\lambda z}, & z>0 \\ \frac{\lambda}{2}e^{\lambda z}, & z\le 0 \end{cases}$$

Sada tražimo

$$\mathbb{P}(X<Y) = \mathbb{P}(Z<0) = \int_{-\infty}^0 f_Z(z)\,dz = \frac{\lambda}{2}\int_{-\infty}^0 e^{\lambda z}\,dz = \frac{\cancel{\lambda}}{2}\cdot\left(\frac{1}{\cancel{\lambda}}e^{\lambda z}\right)\Bigg|_{-\infty}^0 = \frac{1}{2}$$

5. (a) Teorem.

Ako je $(X_n)_{n\in\mathbb{N}}$ niz nezavisnih i jednako distribuiranih slučajnih varijabli s očekivanjem m i disperzijom σ^2, onda

$$\frac{\sum_{k=1}^{n}(X_k - m)}{\sigma\sqrt{n}} \xrightarrow{\mathcal{D}} \mathcal{N}(0,1).$$

(b) Za svaki $i \in \{1,2,\dots,100\}$ imamo

$$m = \mathbb{E}(X_i) = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^1 x^2\,dx + \int_1^2 (2x - x^2)\,dx$$

$$= \left(\frac{1}{3}x^3\right)\Big|_0^1 + \left(x^2 - \frac{1}{3}x^3\right)\Big|_1^2 = 1,$$

$$\sigma^2 = \mathbb{D}(X_i) = \int_{-\infty}^{\infty}(x-1)^2 f(x)\,dx = \int_0^1 (x-1)^2 x\,dx + \int_1^2 (x-1)^2(2-x)\,dx$$

$$= \int_0^1 (x^3 - 2x^2 + x)\,dx + \int_1^2 (-x^3 + 4x^2 - 5x + 2)\,dx$$

$$= \left(\frac{1}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 + \left(-\frac{1}{4}x^4 + \frac{4}{3}x^3 - \frac{5}{2}x^2 + 2x\right)\Big|_1^2 = \frac{1}{6}.$$

Sada imamo

$$\mathbb{P}(1-a < \overline{X} < 1+a) = \mathbb{P}\left(\frac{-a}{\frac{1}{6}\cdot\frac{1}{\sqrt{100}}} < \frac{\overline{X}-1}{\frac{1}{6}\cdot\frac{1}{\sqrt{100}}} < \frac{a}{\frac{1}{6}\cdot\frac{1}{\sqrt{100}}}\right)$$

$$\approx [\text{CGT}] \approx \Phi^*(60a) = 0.99 \implies 60a = 2.58 \implies a = 0.043$$

Alternativno, koristeći tablicu kvantila jedinične normalne razdiobe:

$$\Phi^*(60a) = 2\Phi(60a) - 1 = 0.99 \implies \Phi(60a) = 0.995$$

$$\implies 60a = 2.57583 \implies a = 0.0429305$$

6. (a) Funkcija izglednosti je

$$L(\theta, x_1, \dots, x_n) = f(\theta, x_1)\, f(\theta, x_2) \cdot \ldots \cdot f(\theta, x_n).$$

(b) Funkcija izglednosti glasi

$$L(\lambda, x_1, \dots, x_n) = \lambda x_1^{\lambda-1} \cdot \lambda x_2^{\lambda-1} \cdot \ldots \cdot \lambda x_n^{\lambda-1} = \lambda^n (x_1 x_2 \cdot \ldots \cdot x_n)^{\lambda-1}.$$

Umjesto funkcije izglednosti odredit ćemo stacionarne točke log-izglednosti:

$$\begin{aligned} \ell(\lambda, x_1, \dots, x_n) &= \ln L(\lambda, x_1, \dots, x_n) \\ &= n \ln \lambda + (\lambda - 1) \ln (x_1 x_2 \cdot \ldots \cdot x_n) \end{aligned}$$

$$\Rightarrow \frac{\partial \ell}{\partial \lambda}(\lambda, x_1, \dots, x_n) = \frac{n}{\lambda} + \ln(x_1 x_2 \cdot \ldots \cdot x_n) = 0$$

$$\Rightarrow \lambda = -\frac{n}{\ln(x_1 x_2 \cdot \ldots \cdot x_n)} = -\frac{n}{\sum_{i=1}^{n} \ln x_i}.$$

Dakle, traženi procjenitelj najveće izglednosti za λ je

$$\lambda = -\frac{n}{\sum_{i=1}^{n} \ln X_i} = -\frac{1}{\overline{\ln X}}.$$