Zadatak 1.

52 karte, na sreću 3 karte

a) $P(A) = \frac{\binom{4}{1}\binom{48}{2}}{\binom{52}{3}} = \frac{4512}{22100} \approx 0.2042$

$P(B) = \frac{\binom{13}{1}\binom{4}{3}}{\binom{52}{3}} = \frac{1}{425} \approx 0.0024$

$A \cap B$ = „točno jedan as i 3 karte su iste jačine" $= \emptyset$

$\Rightarrow P(A \cup B) = P(A) + P(B) \approx 0.2066$

$P(C) = \frac{\binom{4}{3}\binom{13}{1}\binom{13}{1}\binom{13}{1}}{\binom{52}{3}} \approx 0.3976$

$P(D) = \frac{\binom{13}{3}\binom{4}{1}\binom{4}{1}\binom{4}{1}}{\binom{52}{3}} \approx 0.8282$

$C \cap D$ = „nema karata iste boje ni iste jačine"

$\rightarrow P(C \cap D) = \frac{\binom{4}{3}\binom{13}{3}\, 3!}{\binom{52}{3}} \approx 0.3106$

$P(C \cup D) = P(C) + P(D) - P(C \cap D) \approx 0.9152$

b) $P(A \cap B) = 0 \neq P(A)P(B)$ $\rightarrow$ nisu nezavisni

c) $P(C \cap D) = 0.3106 \neq 0.3976 \cdot 0.8282 = P(C)P(D)$ $\rightarrow$ nisu nezavisni

2)

H_0 {izabrali smo pravi novčić} $\quad \mathbb{P}(H_0)=\frac{1}{2}$

H_1 = {izabrali smo lažni novčić} $\quad \mathbb{P}(H_1)=\frac{1}{2}$ $\quad$ 1

B = { novčić će u 6. bacanju pasti na glavu }

A = { novčić je 5 put za redom pao na glavu }

Traži se $\mathbb{P}(B|A)$, $\mathbb{P}(H_1|A)$

---

a) $\mathbb{P}(B|A) = \frac{\mathbb{P}(B\cap A)}{\mathbb{P}(A)}$ 1 $= \frac{\mathbb{P}(\text{novčić je 6 put pao na glavu})}{\mathbb{P}(\text{novčić je 5 put pao na glavu})}$

$\mathbb{P}(B\cap A) = \mathbb{P}(B\cap A|H_0)\cdot\mathbb{P}(H_0) + \mathbb{P}(B\cap A|H_1)\cdot\mathbb{P}(H_1)$ 1 1

$\mathbb{P}(B\cap A|H_0) = \frac{1}{2}^6$

$= \frac{1}{2}^6\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 0{,}50781$ 1

$\mathbb{P}(B\cap A|H_1) = 1$

1

$\mathbb{P}(A) = \mathbb{P}(A|H_0)\cdot\mathbb{P}(H_0) + \mathbb{P}(A|H_1)\cdot\mathbb{P}(H_1)$

$= \frac{1}{2}^5\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 0{,}51562$ 1

BACANJA NISU NEZAVISNA

$\mathbb{P}(B|A) = 0{,}984848\ldots$ 1

b) $\mathbb{P}(H_1|A) = \frac{\mathbb{P}(A|H_1)\cdot\mathbb{P}(H_1)}{\mathbb{P}(A)}$ 1 $= \frac{1\cdot\frac{1}{2}}{0{,}51562} = 0{,}96970$

1

Zadatak 3.

a) Neka je pri izvođenju nekog pokusa vjerojatnost realizacije događaja A jednaka p. Ponavljamo taj pokus u nepromijenjenim uvjetima do prve realizacije tog događaja. Za sl. varijablu X koja mjeri broj pokusa u kojem se realizirao događaj A kažemo da ima geometrijsku razdiobu s parametrom p ; $X \sim G(p)$ i vrijedi $P(X=k) = p(1-p)^{k-1}$, $k = 1,2,\ldots$

b) $X \sim G(p)$

$$E\left[\frac{1}{2^X}\right] = \sum_{k=1}^{\infty} \frac{1}{2^k} P(X=k) = \sum_{k=1}^{\infty} \frac{1}{2^k} p(1-p)^{k-1} =$$

$$= \frac{p}{2} \sum_{k=1}^{\infty} \left(\frac{1-p}{2}\right)^{k-1} = \frac{p}{2} \sum_{k=0}^{\infty} \left(\frac{1-p}{2}\right)^{k} = \frac{p}{2} \cdot \frac{1}{1 - \frac{1-p}{2}} =$$

$$= \frac{p}{2} \cdot \frac{1}{\frac{1+p}{2}} = \frac{p}{1+p}$$

c) $p = 0.01$

$$P(X=9) = 0.01 \cdot 0.99^8 = 0.0092$$

Zadatak 4.

a) $X \sim U(0,1)$

$F(x) = x \quad , \quad x \in [0,1]$

b) $Z = \dfrac{X}{X+Y} \quad , \quad X, Y \sim \mathcal{E}(1) \quad \left(f_X(x) = e^{-x} , x > 0 \right)$

$$F_Z(z) = P(Z < z) = P\left(\frac{X}{X+Y} < z\right) = P(X < zX + zY) =$$

$$= P(z \cdot Y > (1-z)X) \underset{\substack{z>0 \\ \text{jer } x>0 \\ \text{i } y>0}}{=} P\left(Y > \frac{1-z}{z} X\right) =$$

$$= \int_0^{+\infty} \left(1 - F_Y\left(\frac{1-z}{z}x\right)\right) f_X(x)\, dx =$$

$$= \begin{cases} \int_0^{+\infty} e^{-\frac{1-z}{z}x} e^{-x}\, dx = \int_0^{+\infty} e^{-\frac{1}{z}x}\, dx = z & , \ z \in \langle 0, 1] \\ \int_0^{+\infty} f_X(x)\, dx = 1 & , \ z \in \langle 1, +\infty \rangle \\ 0 & , \ z \in \langle -\infty, 0] \end{cases}$$

$\Rightarrow F_Z(z) = z , \quad z \in [0,1]$

$\rightarrow Z$ ima uniformnu razdiobu na intervalu $[0,1]$

Zadatak 5

$$X_k \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} \sim X$$

(a)

$$\varphi_X(t) = \sum_{k=1}^{4} \frac{1}{4} e^{itx_k} = \frac{1}{4}\sum_{k=1}^{4} e^{kit}$$

b)

$$\varphi_S(t) = \prod_{k=1}^{n} \frac{1}{4} \sum_{m=1}^{4} e^{mit} = \left(\frac{1}{4}\right)^n \left(\sum_{k=1}^{4} e^{kit}\right)^n$$

c) $\mathbb{E}X = \frac{1}{4}(1+2+3+4) = \frac{10}{4} = 2{,}5$

$S \sim \mathcal{N}(n\mathbb{E}X, n\mathbb{D}X)$

$(\mathbb{E}X)^2 = 6{,}25$

$\mathbb{E}X^2 = \frac{1}{4}(1+4+9+16) = 7{,}5$

$\mathbb{D}X = \mathbb{E}X^2 - (\mathbb{E}X)^2 = 1{,}25$

$\mu_S = 2{,}5 \cdot n$

$\sigma_S = \sqrt{n} \cdot \sqrt{1{,}25} = \sqrt{n} \cdot 1{,}118$

d) $\mathbb{P}(S > 220) = \mathbb{P}\left(\frac{S-\mu_S}{\sigma_S} > \frac{220-\mu_S}{\sigma_S}\right)$

$\frac{S-\mu_S}{\sigma_S} \sim \mathcal{N}(0,1)$

$n = 100$

$$= \mathbb{P}\left(\frac{S-\mu_S}{\sigma_S} > \frac{220-250}{11{,}18}\right)$$

$$= 1 - \Phi(-2{,}68) = \Phi(2{,}68)$$

$$= \frac{1}{2} + \frac{1}{2}\Phi^*(2{,}68) = \frac{1}{2} + \frac{1}{2} \cdot 0{,}992638$$

$$= 0{,}996319$$

6) a) $X_1, \ldots, X_n \sim X$, Θ nepoznati parametar za populaciju X.

Statistika $T = g(X_1, \ldots, X_n)$ zove se nepristrani procjenitelj za Θ ukoliko je

$$\mathbb{E}T = \Theta.$$

b)

$$\mathbb{E}\bar{X} = \mathbb{E}\left(\frac{X_1 + \ldots + X_n}{n}\right), \quad \mathbb{E}X = a, \quad X_i \sim X$$

$$= \frac{1}{n}\left[\mathbb{E}X_1 + \ldots + \mathbb{E}X_n\right] = \frac{1}{n}(a + \ldots + a) = a$$

$$\mathbb{E}(S^2) = \mathbb{E}\left(\frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2\right) =$$

$$= \frac{1}{n-1}\sum_{n=1}^{n}\mathbb{E}(X_k - \bar{X})^2 = \frac{1}{n-1}\sum_{k=1}^{n}D(X_i - \bar{X})$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}D\left(X_i - \frac{1}{n}\sum_{j=1}^{n}X_j\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}D\left(\frac{n-1}{n}X_i - \frac{1}{n}\sum_{j\neq i}X_j\right)$$

$$= \frac{1}{n-1}\sum_{i=1}^{n}\left[\left(\frac{n-1}{n}\right)^2 DX_i + \frac{1}{n^2}\sum_{j\neq i}D(X_j)\right]$$

$$= \frac{1}{n-1}\left[\left(\frac{n-1}{n}\right)^2 \cdot n \cdot \sigma^2 + n \cdot \frac{1}{n^2}(n-1)\cdot\sigma^2\right]$$

$$= \sigma^2$$

$$D(X_i - \bar{X})$$

$$= \mathbb{E}(X_i - \bar{X})^2 - \cancel{(\mathbb{E}(X_i - \bar{X}))^2}$$

$$= \mathbb{E}(X_i - \bar{X})^2$$