1°) Rješenje:

$$\int y(1+y) = \int y + y^2 = \frac{y^2}{2} + \frac{y^3}{3} + C$$

a)
$$\mathbb{P}\left(\frac{1}{4} \le X < \frac{1}{2}, \frac{1}{3} \le Y < \frac{2}{3}\right) = \int_{\frac{1}{4}}^{\frac{1}{2}} \int_{\frac{1}{3}}^{\frac{2}{3}} \frac{12}{5} xy(1+y)\, dy\, dx =$$

$$= \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{12}{5} x \left(\frac{y^2}{2} + \frac{y^3}{3}\right)\Big/_{\frac{1}{3}}^{\frac{2}{3}} dx = \int_{\frac{1}{4}}^{\frac{1}{2}} \frac{12}{5} x \left(\frac{4}{2\cdot 9} + \frac{8}{3\cdot 27} - \frac{1}{2\cdot 9} - \frac{1}{3\cdot 27}\right) dx$$

$$= \frac{12}{5} \frac{41}{162} \cdot \int_{\frac{1}{4}}^{\frac{1}{2}} x\, dx = \frac{12}{5} \cdot \frac{41}{162} \cdot \frac{1}{2}\left(\frac{1}{4} - \frac{1}{16}\right) = \frac{41}{720}.$$

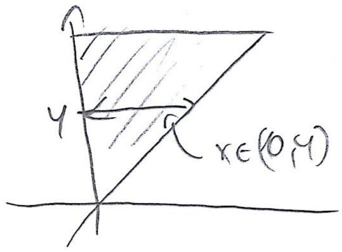
b)
$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(s,t)\, dt\, ds = \int_0^x \int_0^y \frac{12}{5} st(1+t)\, dt\, ds = \frac{12}{5} \cdot \frac{x^2}{2}\left(\frac{y^2}{2} + \frac{y^3}{3}\right)$$

$x, y \in [0,1]$

c) $F_X(x) = F_{X,Y}(x, \infty) = F_{X,Y}(x, 1) = x^2, \; x \in \langle 0, 1 \rangle$

d) Nezavisne su: $f_{XY}(x,y) = \underbrace{C_1 x}_{} \cdot \underbrace{C_2 y(1+y)}_{}$

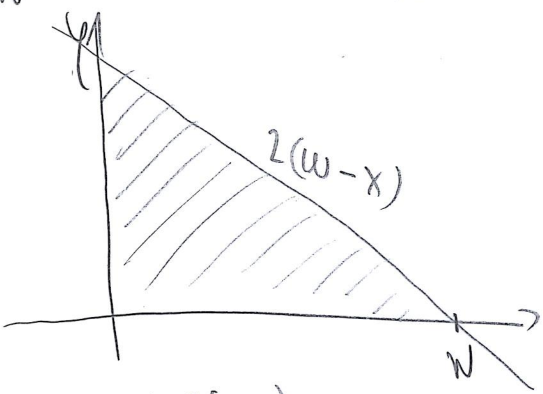
e)
$$\mathbb{P}(X < Y) = \int_0^1 \int_0^y \frac{12}{5} xy(1+y)\, dx\, dy = \frac{27}{50}$$



2°) $f_X(x) = \lambda e^{-\lambda x}$, $f_Y(y) = \lambda e^{-\lambda y}$, $F_X(x) = 1 - e^{-\lambda x}$, $F_Y(y) = 1 - e^{-\lambda y}$

$$F_Z(z) = \mathbb{P}(Z \le z) = \mathbb{P}(\max\{X, Y\} \le z)$$
$$= \mathbb{P}(X \le z, Y \le z) = \mathbb{P}(X \le z)\mathbb{P}(Y \le z) =$$
$$= \left(1 - e^{-\lambda z}\right)^2, \quad z \ge 0.$$

$$F_W(w) = \mathbb{P}\left(X + \tfrac{1}{2}Y < w\right) = \mathbb{P}\left(Y < 2(w - x)\right)$$



$$= \int_0^w \int_0^{2(w-x)} f_{XY}(x,y)\, dy\, dx$$

$$\int \lambda e^{-\lambda x} = -e^{-\lambda x}$$

$$= \lambda^2 \int_0^w \int_0^{2(w-x)} e^{-\lambda x} e^{-\lambda y}\, dy\, dx = -\lambda \int_0^w e^{-\lambda x} e^{-\lambda y} \Big|_0^{2(w-x)} dx$$

$$= -\lambda \int_0^w e^{-\lambda x}\left(e^{-2\lambda(w-x)} - 1\right) dx = \lambda \int_0^w e^{-\lambda x} - e^{-2\lambda w + \lambda x}\, dx$$

$$= 1 - e^{-\lambda w} - e^{-2\lambda w}\left(e^{\lambda w} - 1\right) = 1 - 2e^{-\lambda w} + e^{-2\lambda w} = \left(1 - e^{-\lambda w}\right)^2, \quad z \ge 0$$

3.) X_i - vrijeme koje student potroši na itom pitanju, $X_i \sim X$

$$\mathbb{E}X = \int_0^1 6x^2(1-x)\,dx = 6\left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{6}{12} = \frac{1}{2}$$

$$\mathbb{E}X^2 = \int_0^1 6x^3(1-x)\,dx = 6\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Big|_0^1 = 6 \cdot \frac{1}{20} = \frac{6}{20}$$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \frac{6}{20} - \frac{1}{4} = \frac{1}{20}$$

$n = 116$

ukupno vrijeme: $T = X_1 + \dots + X_n$, $\mathbb{E}T = n \cdot \mathbb{E}X = \frac{n}{2}$

$$Var T = n \cdot Var X = \frac{n}{20}$$

a) $$\mathbb{P}(T > 60) = \mathbb{P}\left(\frac{T - \frac{n}{2}}{\sqrt{\frac{n}{20}}} > \frac{60 - \frac{116}{2}}{\sqrt{\frac{116}{20}}}\right) = \frac{1}{2}\left(1 - \Phi^*(0{,}830)\right)$$

$$= 0{,}21186.$$

b) $\mathbb{P}(T \leq 60) = 0{,}98$

$$\frac{1}{2}\left(1 + \Phi^*(u)\right) = 0{,}98, \quad u = \frac{60 - \frac{n}{2}}{\sqrt{\frac{n}{20}}}$$

$$\Phi^*(u) = 1{,}86$$

$$u = 2{,}05$$

$$60 - \frac{n}{2} = \sqrt{\frac{n}{20}} \cdot 2{,}05 \Big/^2 \qquad 3600 - 60n + \frac{n^2}{4} = \frac{n}{20} \cdot (2{,}05)^2$$

$$\frac{n^2}{4} - \left(60 + \frac{2{,}05^2}{20}\right)n + 3600 = 0 \qquad \boxed{n = 110}$$

4) $f(x|a,\theta) = \theta a^{\theta} x^{-\theta-1}$, $x \geq a$, $\theta > 1$

a je zadano, $a > 0$, $x_1, \dots, x_n$ uzorak

$$L(\theta) = \prod_{i=1}^{n} f(x_i | a, \theta)$$

$$\ln L(\theta) = \sum_{i=1}^{n} \ln\left(f(x_i|a,\theta)\right)$$

$$= \sum_{i=1}^{n} \ln\left(\theta a^{\theta} x_i^{-\theta-1}\right)$$

$$= \sum_{i=1}^{n} \left[\ln(\theta) + \theta \ln a - (\theta+1)\ln(x_i)\right]$$

$$L(\theta) \to \max \iff \ln L(\theta) \to \max$$

$$\Rightarrow \frac{\partial}{\partial\theta} \ln L(\theta) = 0$$

$$0 = \frac{\partial}{\partial\theta} \ln L(\theta) = \frac{\partial}{\partial\theta}\left[n \cdot \ln(\theta) + n \cdot \theta \ln a - (\theta+1)\sum_{i=1}^{n} \ln(x_i)\right]$$

$$= \frac{n}{\theta} + n \ln a - \sum_{i=1}^{n} \ln(x_i) \quad \Bigg\} \quad \sum_{i=1}^{n} \ln(x_i) - n \ln a = \frac{n}{\theta}$$

$$\theta_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln(x_i) - n \ln a} = \frac{1}{\frac{\sum \ln(x_i)}{n} - \ln a}$$

$\blacksquare$