VIS Z1 Rješenja

1) $f(x) = \begin{cases} Ce^{-2(x-1)} & , x>0 \\ 0 & \text{inače} \end{cases}$

$$1 = \int_0^\infty Ce^{-2(x-1)} = \int_0^\infty Ce^2 e^{-2x} dx = Ce^2 \left(\frac{1}{2}e^{-2x}\right)\Big|_0^\infty = \frac{Ce^2}{2} \Rightarrow C = 2e^{-2}$$

$$\Rightarrow f(x) = 2e^{-2}e^{-2x}e^{2} = 2e^{-2x}$$

$\Rightarrow X \sim \mathcal{E}(\lambda), \lambda = 2.$ $\mathbb{E}X = \frac{1}{\lambda} = \frac{1}{2}$, $\mathrm{Var}X = \frac{1}{2^2} = \frac{1}{4}$.

$\mathbb{P}(X<2 \mid X>1) = (\text{svojstvo zaboravljanja}) = \mathbb{P}(X<1) = F_X(1) = 1-e^{-2\cdot 1} = 1-e^{-2}$

$$g(x) = Ce^{-(x^2-2x)} = Ce^{-(x^2-2x+1)} \cdot e$$
$$= Ce \cdot e^{-(x-1)^2} = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$\Rightarrow \mu = 1,\ 2\sigma^2 = 1 \Rightarrow \sigma = \frac{1}{\sqrt{2}}$

$Ce = \frac{1}{\sqrt{\pi}} \Rightarrow \quad \Rightarrow C = \frac{1}{e\sqrt{\pi}} \quad \Rightarrow g(x) = \frac{1}{e\sqrt{\pi}} e^{-(x-1)^2}$

$Y \sim \mathcal{N}(1, \frac{1}{2})$, $\mathbb{E}(Y) = 1$, $\mathrm{Var}Y = \frac{1}{2}$

$$\mathbb{P}(Y<2 \mid Y>1) = \frac{\mathbb{P}(Y \in \langle 1,2 \rangle)}{\mathbb{P}(Y>1)} = \mathbb{P}\left(\frac{Y-1}{1/\sqrt{2}} \in \langle 0, \sqrt{2} \rangle\right)$$
$$= \Phi(\sqrt{2}) - \frac{1}{2} = 0,841$$

2) $X_1, X_2 \sim \mathcal{E}(\lambda) > 0$

$$X_1 = \ln Y_2$$
$$X_2 = Y_1 - X_1$$

$$Y_1 = X_1 + X_2 \qquad \Psi(X_1, X_2) = (x_1 + x_2, e^{x_1}) = (Y_1, Y_2)$$
$$Y_2 = e^{X_1}$$
$$(X_1, X_2) = \Psi^{-1}(Y_1, Y_2) = (\ln Y_2, Y_1 - \ln Y_2)$$

$$J_{\Psi^{-1}} = \begin{bmatrix} 0 & 1 \\ \frac{1}{Y_2} & -\frac{1}{Y_2} \end{bmatrix}, \quad |\det J_{\Psi^{-1}}| = \left| -\frac{1}{Y_2} \right| = \frac{1}{Y_2}, \text{ jer je } Y_2 = e^{X_1} > 1$$

$$\Rightarrow f_{Y_1 Y_2} = f_{X_1 X_2}(\Psi^{-1}(Y_1, Y_2)) \cdot |J_{\Psi^{-1}}(Y_1, Y_2)| = \left[f_{X_1 X_2}(x_1, x_2) = f_{X_1}(x_1) \cdot f_{X_2}(x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)} \right]$$

nezavisnost

$$= \lambda^2 e^{-\lambda(\ln Y_2 + Y_1 - \ln Y_2)} \cdot \frac{1}{Y_2}$$

$$= \boxed{\lambda^2 e^{-\lambda Y_1} \cdot \frac{1}{Y_2}}$$

Uvjeti: $\ln Y_2 = X_1 > 0 \Rightarrow Y_2 > 1$

$Y_1 - \ln Y_2 = X_2 > 0 \Rightarrow \boxed{Y_1 > \ln Y_2 \quad Y_2 < e^{Y_1}}$

$$f_{Y_1 Y_2}(y_1, y_2) = \begin{cases} \lambda^2 e^{-\lambda y_1} \cdot \frac{1}{y_2}, & 1 < y_2 < e^{y_1} \\ 0 & \text{inače} \end{cases}$$

$$f_{Y_1}(y_1) = \int_1^{e^{y_1}} \lambda^2 e^{-\lambda y_1} \cdot \frac{1}{y_2} \, dy_2 = \lambda^2 e^{-\lambda y_1} \cdot \ln y_2 \Big|_1^{e^{y_1}} = \lambda^2 y_1 e^{-\lambda y_1}, \quad y_1 \geq 0$$

$$f_{Y_2}(y_2) = \int_{\ln y_2}^{\infty} \lambda^2 e^{-\lambda y_1} \cdot \frac{1}{y_2} \, dy_1 = \lambda^2 \cdot \frac{1}{y_2} \cdot \frac{1}{-\lambda} e^{-\lambda y_1} \Big|_{\ln y_2}^{\infty} = \frac{\lambda}{y_2} e^{-\lambda \ln y_2} = \frac{\lambda}{y_2} y_2^{-\lambda} = \lambda y_2^{-(\lambda+1)}, \quad y_2 > 1$$

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b) $$f_{Y_1 | Y_2 = y_2}(y_1) = \frac{f_{Y_1 Y_2}(y_1, y_2)}{f_{Y_2}(y_2)} = \frac{\lambda^2 e^{-\lambda y_1} \cdot \frac{1}{y_2}}{\lambda y_2^{-(\lambda+1)}} = \lambda e^{-\lambda y_1} y_2^{\lambda}, \quad 1 < y_2 < e^{y_1}$$

c) Nisu jer $f_{Y_1 | Y_2 = y_2}$ ovisi o y_2.

d) $$\mathbb{P}(Y_1 < 2 \mid Y_2 = e) = \int_0^2 f_{Y_1 | Y_2 = e}(y_1) \, dy_1 = \lambda e^{\lambda} \int_0^2 e^{-\lambda y_1} \, dy_1 = \dots = 1 - e^{-\lambda}$$

3) a) X - vrijeme koje je Josipu potrebno do prepliva 25 m.

$\mathbb{E}X = 35$

$Var(X) = 2^2 = 4$

$$P(30 < X < 40) = P(-5 < X-35 < 5) = P(|X-35| < 5) \overset{\text{Čebišev}}{\geq} 1 - \frac{Var X}{5^2}$$

$$= 1 - \frac{4}{25} = \frac{21}{25} = 0{,}84.$$

b) $$P(30 < X < 40) = P\left(\frac{30-35}{2} < \frac{X-35}{2} < \frac{40-35}{2}\right) \qquad \frac{X-\mu}{\sigma} \sim N(0,1)$$

$$= P\left(-\frac{5}{2} < \frac{X-35}{2} < \frac{5}{2}\right) = \Phi^*\left(\frac{5}{2}\right) = 0{,}9876$$

c) $$P\left(\sum_{i=1}^{80} X_i < 45 \cdot 60\right) = P\left(\frac{\sum_{i=1}^{80} X_i - 80 \cdot 35}{2 \cdot \sqrt{80}} < \frac{45 \cdot 60 - 80 \cdot 35}{2 \cdot \sqrt{80}}\right)$$

$$\overset{CGT}{\approx} \Phi\left(\frac{45 \cdot 60 - 80 \cdot 35}{2 \cdot \sqrt{80}}\right) = \Phi\left(\frac{-100}{17{,}83}\right)$$

$$= \Phi(-5{,}59) = 0.$$

4) $X \sim \mathcal{N}(\mu, \sigma^2)$

Neka je $n \in \mathbb{N}$, $X_1, \dots, X_n \sim X$ n.j.d.

Definiramo $F(x_1, \dots, x_n, \mu) := f(x_1) f(x_2) \dots f(x_n)$, $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$$= \frac{1}{\sigma^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \cdot e^{-\frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2}$$

$$\ln F(x_1, \dots, x_n, \mu) = \ln\left(\frac{1}{\sigma^n} \frac{1}{(2\pi)^{\frac{n}{2}}}\right) - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \quad / \frac{\partial}{\partial \mu}$$

$$0 = \sum_{i=1}^n (x_i - \mu) \Rightarrow n\mu = \sum x_i \rightarrow \mu = \frac{1}{n} \sum_{i=1}^n x_i$$

b) $\mathbb{E}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n} \sum_{i=1}^n \mathbb{E} X_i = \frac{1}{n} \sum \mu = \frac{n}{n}\mu = \mu$

c) $\mathrm{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) = \frac{1}{n^2} \mathrm{Var}\left(\sum_{i=1}^n X_i\right) = \frac{1}{n^2} \sum \mathrm{Var} X_i = \frac{1}{n} \sigma^2$

$\mathrm{Var}\left(\frac{1}{n} \sum_{i=1}^n X_i\right) \to 0$ kada $n \to \infty$.

Iz tog razloga je $\lim_n \mathbb{P}\left(\left|\frac{1}{n} \sum_{i=1}^n X_i - \mu\right| > \varepsilon\right) = 0$

d) Procjenitelj za μ^2 je $\left(\frac{1}{n} \sum x_i\right)^2$. Računamo $\mathbb{E}(\bar{X}^2)$

Znamo da je $0 < \frac{\sigma^2}{n} = \mathrm{Var}(\bar{X}) = \mathbb{E}(\bar{X}^2) - \underbrace{(\mathbb{E}(\bar{X}))^2}_{\mu^2}$

$\Rightarrow 0 < \mathbb{E}(X^2) - \mu^2 \rightarrow \boxed{\mathbb{E}(X^2) > \mu^2}$

nije nepristran procjenitelj