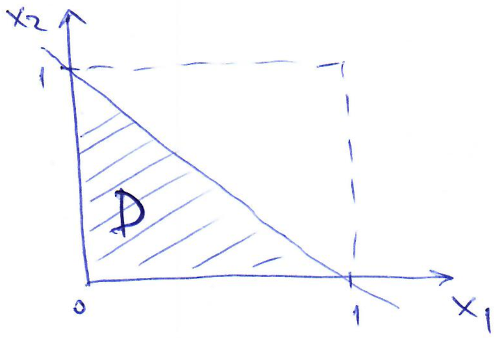
Zadatak 1

a) $1 = \int_0^1 cx\,dx \Rightarrow c = 2$ ; $P(X \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} 2x\,dx = x^2 \Big|_0^{\frac{1}{2}} = \frac{1}{4}$

b) $EX = \int_0^1 2x^2\,dx = \frac{2}{3}$ ; $Var\,X = EX^2 - (EX)^2 = \int_0^1 2x^3\,dx - \frac{4}{9} = \frac{1}{18}$

c) $X_1 + X_2 \leq 1$




$$P\left(\frac{X_1+X_2}{2} \leq \frac{1}{2}\right) = \iint_D f_X(x_1)\, f_X(x_2)\,dx_1\,dx_2 =$$

$$= \int_0^1 \int_0^{1-x_1} 4x_1x_2\,dx_2\,dx_1 = 2\int_0^1 x_1x_2^2 \Big|_0^{1-x} dx_1 = 2\int_0^1 x_1(1-x_1)^2\,dx_1 =$$

$$= 2\left(\frac{x^4}{4} - \frac{2}{3}x^3 + \frac{1}{2}x^2\right)\Big|_0^1 = \frac{1}{6}$$

d) $$P\left(\frac{X_1+\ldots+X_{100}}{100} \leq \frac{1}{2}\right) = P\left(\frac{\frac{1}{100}\sum X_i - \frac{2}{3}}{\sqrt{\frac{1}{18}}/10} \leq \frac{\frac{1}{2}-\frac{2}{3}}{\sqrt{\frac{1}{18}}/10}\right)$$

$$= (CGT) = \Phi\left(\frac{-1{,}666}{0{,}235}\right) = \Phi(-7) = 0.$$

Zadatak 2

a) $$1 = \int_0^1 c x^\theta \, dx = c \left. \frac{x^{\theta+1}}{\theta+1} \right|_0^1 = \frac{c}{\theta+1} \Rightarrow c = \theta+1.$$

b) $$L(\theta) = \prod_{i=1}^n f(x_i, \theta) = \prod_{i=1}^n (\theta+1) x_i^\theta = (\theta+1)^n \left( \prod_{i=1}^n x_i \right)^\theta$$

$$\ln L(\theta) = n \ln(\theta+1) + \theta \ln\left( \prod_{i=1}^n x_i \right)$$

$$\frac{\partial}{\partial \theta} \ln L = \frac{n}{\theta+1} + \sum_{i=1}^n \ln x_i$$

$$\frac{\partial}{\partial \theta} \ln L = 0 \iff \frac{n}{\theta+1} = -\sum \ln x_i$$

$$\iff \theta = -\frac{n}{\sum \ln x_i} - 1$$

$$\Rightarrow \hat{\theta} = -\frac{n}{\sum \ln x_i} - 1$$

c) $$E\hat{\lambda} = -\frac{1}{n} \sum_{i=1}^n E \ln X_i = -E \ln X_i = -\int_0^1 \ln x \, (\theta+1) x^\theta \, dx$$

$$= -(\theta+1) \int_0^1 x^\theta \ln x \, dx = \left| \begin{matrix} u = \ln x & v = \frac{x^{\theta+1}}{\theta+1} \\ du = \frac{1}{x} dx & dv = x^\theta dx \end{matrix} \right|$$

$$= -(\theta+1) \left( \left. \frac{x^{\theta+1}}{\theta+1} \ln x \right|_0^1 - \int_0^1 \frac{x^\theta}{\theta+1} dx \right)$$

$$= 0 + \left. \frac{x^{\theta+1}}{\theta+1} \right|_0^1 = \frac{1}{\theta+1} \neq \frac{1}{\theta}$$

$\Rightarrow \hat{\lambda}$ nije nepristrana.

Zadatak 3

a) $X \sim B(n,p)$ ako je $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$, $0 \leq k \leq n$,
$0<p<1$

b) $\left.\begin{array}{l} np = 80 \\ np(1-p) = 16 \end{array}\right\} \Rightarrow \begin{array}{l} n=100 \\ p=0{,}8 \end{array}$

c) $P(X>82) = P\left(\frac{X-80}{4} > \frac{82-80}{4}\right)$

$= (MLT) \doteq 1 - \Phi\left(\frac{1}{2}\right) = 0{,}3085$ .

d)
$P\left(\frac{1}{100}\sum X_i > 82\right) = P\left(\frac{\frac{1}{100}\sum X_i - 80}{4/10} > \frac{82-80}{4/10}\right)$

$= (CGT) \doteq 1 - \Phi(5) = 0$

Zadatak 4

Iz gustoća: $X \sim \mathcal{E}(3)$ i $Y \sim N(-4, 2^2)$

$\Rightarrow$ $EX = \frac{1}{3}$, $\text{Var}\, X = \frac{1}{9}$

$EY = -4$, $\text{Var}\, Y = 4$

$\Rightarrow$ $EX^2 = \text{Var}\, X + (EX)^2 = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$

$EY^2 = \text{Var}\, Y + (EY)^2 = 4 + (-4)^2 = 20$

$$EXY = EX \cdot EY + r(X,Y)\sqrt{\text{Var}X}\sqrt{\text{Var}Y}$$

$$= \frac{1}{3} \cdot (-4) + (-0{,}5)\sqrt{\frac{1}{9}} \cdot \sqrt{4}$$

$$= -\frac{5}{3}$$

$$EZ = EX^2 - 3EXY + 2EY^2$$

$$= \frac{2}{9} - 3\left(-\frac{5}{3}\right) + 2 \cdot 20$$

$$= \frac{407}{9}.$$

# Zadatak 5

a) $F_Z(z) = P(Z < z) = P(\max\{X_1, \dots, X_n\} \leq z) =$

$= P(X_1 \leq z) \cdot \ldots \cdot P(X_n \leq z) = \left(\frac{z-a}{b-a}\right)^n$

$f_Z(z) = \frac{d}{dz} F_z = \frac{n}{b-a} \left(\frac{z-a}{b-a}\right)^{n-1}$ , $z \in [a, b]$

$$EZ = \int_a^b \frac{nz}{b-a} \left(\frac{z-a}{b-a}\right)^{n-1} dz = \frac{n}{(b-a)^n} \int_a^b z(z-a)^{n-1} dz$$

$$= \frac{n}{(b-a)^n} \int_a^b (z-a)^n + a(z-a)^{n-1} dz =$$

$$= \frac{n}{(b-a)^n} \left( \frac{(b-a)^{n+1}}{n+1} + \frac{a(b-a)^n}{n} \right)$$

$$= \frac{n}{n+1}(b-a) + a$$

b) $L(b) = \frac{1}{(b-a)^n} \longrightarrow \max$

Potrebno je odabrati najmanji mogući $b$, a to je $\max\{x_1, \dots, x_n\}$.

Zadatak 6

a) $A \subseteq A \cup B \Rightarrow P(A) \leq P(A \cup B) \Rightarrow P(A \cup B) \geq 1$

$\Rightarrow P(A \cup B) = 1$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$1 = 1 + P(B) - P(A \cap B)$$

$\Rightarrow P(A \cap B) = P(B)$

b) $$P(A \cap B \mid C) = \frac{P(A \cap B \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(B \cap C)} \cdot \frac{P(B \cap C)}{P(C)}$$

$$= P(A \mid B \cap C) \cdot P(B \mid C) = P(A \mid C)\, P(B \mid C)$$

c) Dokažimo $P(A \cup B) \cdot P(A \cap B) - P(A) \cdot P(B) \leq 0$

$$\big(P(A) + P(B) - P(A \cap B)\big) P(A \cap B) - P(A) P(B) =$$

$$= P(A)\big(P(A \cap B) - P(B)\big) - P(A \cap B)\big(P(A \cap B) - P(B)\big) =$$

$= \big(P(A \cap B) - P(B)\big)\big(P(A) - P(A \cap B)\big) \leq 0$ jer je

$A \cap B \subseteq B \Rightarrow P(A \cap B) - P(B) \leq 0$

$A \cap B \subseteq A \Rightarrow P(A \cap B) - P(A) \geq 0$ .