DEKANSKI ROK, 15.9.2020.
RJEŠENJA

1. 10 C, 10 P

a) Ω ... svi mogući poretci

$$|\Omega| = \frac{20!}{10!\,10!} = 184756$$

$A = \{$ Sve crvene su u prvih 14 $\}$

$$\Rightarrow \quad |A| = \binom{14}{10} = 1001$$

$$\Rightarrow \quad P(A) = \frac{1001}{184756} \approx 0.0054$$

b) $B = \{$ točno 8 crvenih je u prvih 14 $\}$

$$|B| = \binom{14}{8}\binom{6}{2} = 45045$$

$$\Rightarrow \quad P(B) = \frac{45045}{184756} = 0.2438$$

c) $C = \{$ najviše 8 crvenih u prvih 14 $\}$

$\overline{C} = A_1 \cup A_2$, $A_1 = \{$ točno 9 $\}$, $A_2 = \{$ točno 10 $\} = A$ (iz a) dijela)

$A_1 \cap A_2 = \emptyset \quad \Rightarrow \quad |A_1 \cup A_2| = |A_1| + |A_2|$

$$|A_1| = \binom{14}{9}\binom{6}{1} = 12012$$

$$\Rightarrow P(C) = 1 - P(\overline{C}) = 1 - \frac{|A_1 \cup A_2|}{|\Omega|}$$

$$P(C) \approx 0.92956$$

(2.) $\Omega \neq \emptyset$, $B \in \mathcal{F}$ t.d. $P(B) > 0$

a) Neka je $B \in \mathcal{F}$ t.d. $P(B) > 0$.

Uvjetna vjerojatnost uz uvjet B je funkcija

$P_B : \mathcal{F} \to [0,1]$ definirana formulom:

$$P_B(A) = \frac{P(AB)}{P(B)}, \quad \forall A \in \mathcal{F}$$

b) i) Normiranost:

$$P_B(\Omega) = \frac{P(B\Omega)}{P(B)} \overset{B \subseteq \Omega}{=} \frac{P(B)}{P(B)} = 1$$

$$P_B(\emptyset) = \frac{P(B \cap \emptyset)}{P(B)} \overset{\emptyset \subseteq B}{=} \frac{P(\emptyset)}{P(B)} = 0$$

ii) Monotonost:

$A_1 \subseteq A_2 \Rightarrow A_1 \cap B \subseteq A_2 \cap B$ i P je vjerojatnost $\Rightarrow P(A_1 \cap B) \leq P(A_2 \cap B)$

$$A_1 \subseteq A_2 ; \quad P_B(A_1) = \frac{P(A_1 \cap B)}{P(B)} \leq \frac{P(A_2 \cap B)}{P(B)} = P_B(A_2)$$

iii) Aditivnost:

A_1 i A_2 disjunktni događaji

A_1 i A_2 disj. $\Rightarrow A_1 \cap B$ i $A_2 \cap B$ su disj. + P je vjeroj.

$$P_B(A_1 \cup A_2) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P((A_1 \cap B) \cup (A_2 \cap B))}{P(B)} =$$

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} = \frac{P(A_1 \cap B)}{P(B)} + \frac{P(A_2 \cap B)}{P(B)} = P_B(A_1) + P_B(A_2).$$

$\Rightarrow P_B$ je vjerojatnost na $\mathcal{F}$.

3. a) X diskretna slučajna varijabla, neka je njen zakon razdiobe dan s: $X \sim \begin{pmatrix} x_1 & x_2 & x_3 \cdots \\ p_1 & p_2 & p_3 \cdots \end{pmatrix}$

Tada je $\mathbb{E}(X) := \sum_k x_k p_k$.

b) X diskr. sl. var., $\lambda \in \mathbb{R}$

D: $\mathbb{E}[\alpha X] = \alpha \mathbb{E}[X]$

$$E[\alpha X] = \sum_k (\alpha x_k) p_k = \alpha \sum_k x_k p_k = \alpha \cdot \mathbb{E}[X]$$

c) 8 kuglica $(1, 2, 3, \ldots, 8)$

X = „veći izvučeni broj od dvije na sreću izv. kugl."

Zakon razdiobe:

$$\begin{array}{cc} X \sim \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \frac{1}{28} & \frac{2}{28} & \frac{3}{28} & \frac{4}{28} & \frac{5}{28} & \frac{6}{28} & \frac{7}{28} \end{pmatrix} & \begin{array}{c} x_i \\ p_i \end{array} \end{array}$$

$P(X=1)=0$, $P(X=2) = \frac{1}{8} \cdot \frac{1}{7} \cdot 2 = \frac{1}{28}$, $P(X=3) = \frac{1}{8} \cdot \frac{2}{7} \cdot 2 = \frac{2}{28}$

izl. 1 i 2 (na 2 načina)

1 i 3 ili 2 i 3

$P(X=4) = \frac{1}{8} \cdot \frac{3}{7} \cdot 2 = \frac{3}{28}$ $P(X=5) = \frac{1}{8} \cdot \frac{4}{7} \cdot 2 = \frac{4}{28}$

$P(X=6) = \frac{5}{28}$, $P(X=7) = \frac{6}{28}$, $P(X=8) = \frac{7}{28}$

$$\Rightarrow E[X] = \sum x_i p_i = 2 \cdot \frac{1}{28} + 3 \cdot \frac{2}{28} + \ldots + 8 \cdot \frac{7}{28} = 6$$

(4.) a) Neka je dana slučajna varijabla $X : \Omega \to \mathbb{R}$.

Funkcija razdiobe sl. var. X je funkcija $F : \mathbb{R} \to [0,1]$ definirana formulom:

$$F(x) := P(\{X < x\})$$

(gdje je $\{X < x\} = \{\omega \in \Omega : X(\omega) < x\}$)

b) i) $\lim\limits_{x \to -\infty} F(x) =$ [Neka je (x_n) padajući niz realnih brojeva t.d. $\lim\limits_n x_n = -\infty$; Označimo $A_n = \{X < x_n\}$ $\Rightarrow A_n$ su padajući skupovi: $A_1 \supset A_2 \supset \ldots$; $\bigcap\limits_n A_n = \emptyset$] $=$

$$= \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} P(A_n) = P\left(\bigcap_n A_n\right) = P(\emptyset) = 0$$

ii) $\lim\limits_{x \to +\infty} F(x) =$ [Neka je (x_n) rastući niz realnih brojeva t.d. $\lim\limits_n x_n = +\infty$; označimo $A_n = \{X < x_n\}$ $\Rightarrow A_n$ su rastući skupovi: $A_1 \subset A_2 \subset \ldots$ $\bigcup\limits_n A_n = \{X < \infty\} = \Omega$] $=$

$$= \lim_{n \to \infty} F(x_n) = \lim_{n \to \infty} P(A_n) = P\left(\bigcup_n A_n\right) = P(\Omega) = 1$$

$\Rightarrow$ Postoje (i izračunati su)

c) $F(x) = \begin{cases} 0, & x \le 0 \\ Cx^2, & x \in \langle 0, 2 \rangle \\ D, & x \ge 2 \end{cases}$; $F(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{4}x^2, & x \in \langle 0, 2 \rangle \\ 1, & x \ge 2 \end{cases}$

$F(+\infty) = 1 \Rightarrow D = 1$

F je neprekinuta slijeva; uzmimo točku $x = 2$

$$1 = F(2) = \lim_{\varepsilon \searrow 0} F(2 - \varepsilon) = \lim_{\varepsilon \searrow 0} C(2-\varepsilon)^2 = C \cdot 2^2 =$$

$$= 4C \Rightarrow C = \frac{1}{4}$$

$f(x) = \begin{cases} 0, & x \le 0 \\ \frac{1}{2}x, & x \in \langle 0, 2 \rangle \\ 0, & x \ge 2 \end{cases} \Rightarrow \mathbb{E}X = \displaystyle\int_0^2 x \cdot \frac{1}{2} x \, dx = \frac{1}{2} \cdot \frac{x^3}{3} \Big|_0^2 = \frac{4}{3}$

(5.) $X_1, \dots, X_n$ nez. eksp. sl. vel. ; $\sigma^2 = 100$

$$\overline{X_n} = \frac{X_1 + \dots + X_n}{n}, \quad n \in \mathbb{N}$$

$$100 = \sigma^2 = \frac{1}{\lambda^2} \Rightarrow \lambda = \frac{1}{10} \Rightarrow \mu = 10$$

Po CGT:

$$P(9 \le \overline{X_n} \le 11) = P\left(\frac{9-10}{10} \cdot \sqrt{n} \le \underbrace{\frac{\overline{X_n} - 10}{10}\sqrt{n}}_{\sim N(0,1)} \le \frac{11-10}{10}\sqrt{n}\right) =$$

$$= P\left(\frac{-\sqrt{n}}{10} \le Z \le \frac{\sqrt{n}}{10}\right) = \Phi^*\left(\frac{\sqrt{n}}{10}\right) \ge 0.95$$

$$\Rightarrow \frac{\sqrt{n}}{10} \geq 1.96 \Rightarrow \underline{n \ge 385}$$

6. $\hat{\mu} \overset{MLE}{=} \frac{1}{5} \sum_{i=1}^{5} x_i = 4.18$

$\hat{\sigma}^2 \overset{MLE}{=} \frac{1}{n} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 = \frac{1}{5} \sum_{i=1}^{5} (x_i - 4.18)^2 =$

$= 1.3696$

6. $\hat{p} = \frac{m}{n} = \frac{15}{1000} = 0.015$ ... p - postotak neispravnih vijaka

a) 90% p.i., $u_{1-\alpha/2} = u_{0.95} = 1.64$, $n = 1000$

$$p_{1,2} = \hat{p} \mp u_{1-\alpha/2} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.015 \mp 0.006304$$

$\Rightarrow$ 90% interval pouzdanosti : $[0.008696, 0.021304]$

b) $\alpha = 0.05$

$H_0 : p = 0.01$ (ili $p \leq 0.01$)
$H_1 : p > 0.01$

$$u_{1-\alpha} = u_{0.95} = 1.64$$

$$\hat{u} = (\hat{p} - p_0)\sqrt{\frac{n}{p_0 q_0}} = (0.015 - 0.01)\sqrt{\frac{1000}{0.01 \cdot 0.99}} =$$

$$= 0.005 \cdot \sqrt{101\,010.101} = 1.589$$

$\Rightarrow \hat{u} < u_{1-\alpha} \Rightarrow$ Na razini značajnosti $\alpha = 0.05$ ne odbacujemo da je neispravnih vijaka najviše 1%.