VJEROJATNOST I STATISTIKA

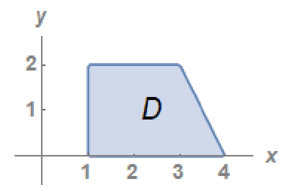
Dodatni zadaci za 10. tjedan (19. i 20. predavanje)

08 Funkcije slučajnih vektora

1. Neka je slučajni vektor (X, Y) zadan funkcijom gustoće $f(x, y) = Cy$ na četverokutu s vrhovima $(1, 0)$, $(4, 0)$, $(1, 2)$ i $(3, 2)$. Odredite konstantu C i funkciju razdiobe slučajne varijable $Z = Y - X$. Potom izračunajte vjerojatnost $P(Z \geq 0)$.

Rješenje. Najprije računamo konstantu C. Vrijedi:

$$\begin{aligned}&\iint_D f(x, y)\,\mathrm{d}x\,\mathrm{d}y = 1\\ &\Leftrightarrow C\int_0^2 y\,\mathrm{d}y\int_1^{4-\frac{y}{2}}\mathrm{d}x = C\int_0^2\left(3y - \frac{y^2}{2}\right)\mathrm{d}y = 1\\ &\Leftrightarrow C = \frac{3}{14}.\end{aligned}$$



1. način. Za funkciju gustoće $f_Z(z)$ slučajne varijable $Z = \psi(X, Y)$ vrijedi

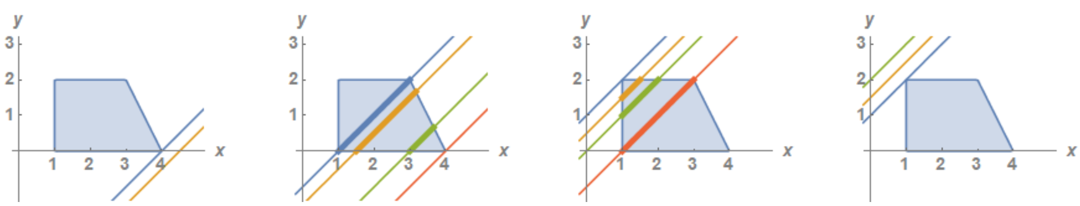
$$f_Z(z) = \int_{-\infty}^{\infty} f(x, y)\left|\frac{\partial y}{\partial z}\right|\mathrm{d}x. \tag{1}$$

U našem slučaju je

$$z = \psi(x, y) = y - x, \quad y = x + z, \quad \left|\frac{\partial y}{\partial z}\right| = 1.$$

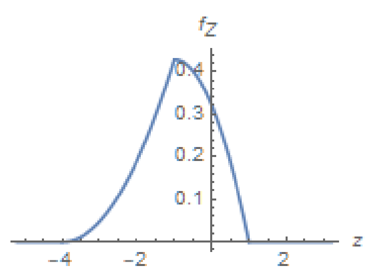
Uvrštavanjem u formulu (1) dobivamo

$$f_Z(z) = \int_{-\infty}^{\infty} f(x, x + z)\,\mathrm{d}x.$$



Vidimo da će granice u integralu biti drugačije za različite vrijednosti varijable z.

$$\begin{aligned}f_Z(z) &= \begin{cases}0, & z \leq -4,\\ \frac{3}{14}\int_{-z}^{\frac{8-z}{3}}(x+z)\,\mathrm{d}x, & z \in \langle -4, -1]\\ \frac{3}{14}\int_{1}^{2-z}(x+z)\,\mathrm{d}x, & z \in \langle -1, 1]\\ 0, & z > 1\end{cases}\\ &= \begin{cases}\frac{1}{21}(z+4)^2, & z \in \langle -4, -1]\\ -\frac{3}{28}(z^2 + 2z - 3), & z \in \langle -1, 1]\end{cases}\end{aligned}$$



Sada računamo funkciju razdiobe $F_Z(z)$ slučajne varijable Z:

$$F_Z(z) = \int_{-\infty}^{z} f_Z(u)\,\mathrm{d}u = \begin{cases} \int_{-\infty}^{z} 0\,\mathrm{d}u, & z \le -4 \\ \frac{1}{21}\int_{-4}^{z}(u+4)^2\,\mathrm{d}u & z \in \langle -4,-1] \\ \frac{1}{21}\int_{-4}^{-1}(u+4)^2\,\mathrm{d}u - \frac{3}{28}\int_{-1}^{z}(u^2+2u-3)\,\mathrm{d}u & z \in \langle -1,1] \\ \frac{1}{21}\int_{-4}^{-1}(u+4)^2\,\mathrm{d}u - \frac{3}{28}\int_{-1}^{z}(u^2+2u-3) & z > 1 \end{cases}$$

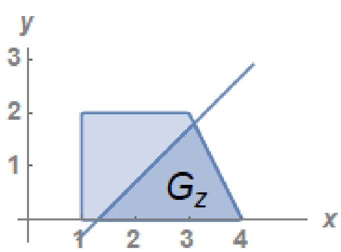
$$= \begin{cases} 0, & z \le -4 \\ \frac{64}{63} + \frac{16}{21}z + \frac{4}{21}z^2 + \frac{1}{63}z^3, & z \in \langle -4,-1] \\ \frac{23}{28} + \frac{9}{28}z - \frac{3}{28}z^2 - \frac{1}{28}z^3 & z \in \langle -1,1] \\ 1, & z > 1 \end{cases}$$

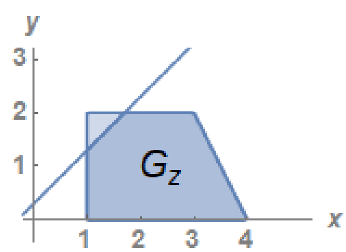
2. način. Ovaj zadatak je moguće riješiti i tako da zaobiđemo eksplicitno računanje funkcije gustoće slučajne varijable Z. Primjetimo da je

$$F_Z(z) = P\left(Z < z\right) = P((X,Y) \in G_z) = \iint_{G_z} f(x,y)\,\mathrm{d}x\,\mathrm{d}y,$$

gdje je područje G_z definirano s

$$G_z = \{(x,y) \colon y - x < z\} = \{(x,y) \colon y < x + z\}.$$

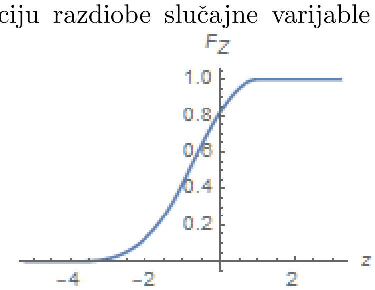




Funkcija F_Z je netrivijalna za $z \in \langle -4, 1\rangle$, te ćemo ovaj interval morati podijeliti na dva dijela u ovisnosti o obliku područja G_z. Računamo funkciju razdiobe slučajne varijable Z:

$$F_Z(z) = \begin{cases} \frac{3}{14}\int_0^{\frac{8+2z}{3}} y\,\mathrm{d}y \int_{y-z}^{\frac{8-y}{2}} \mathrm{d}x, & z \in \langle -4,-1] \\ 1 - \frac{3}{14}\int_1^{2-z}\mathrm{d}x \int_{z+x}^{2} y\,\mathrm{d}y, & z \in \langle -1,1] \end{cases}$$

$$= \begin{cases} \frac{64}{63} + \frac{16}{21}z + \frac{4}{21}z^2 + \frac{1}{63}z^3, & z \in \langle -4,-1] \\ \frac{23}{28} + \frac{9}{28}z - \frac{3}{28}z^2 - \frac{1}{28}z^3, & z \in \langle -1,1] \end{cases}$$



Kod računanja vrijednosti funkcije $F_Z(z)$ za $z \in \langle -1, 1]$ koristili smo vjerojatnost suprotnog događaja.

Konačno dobivamo:

$$P\left(Z \ge 0\right) = 1 - P\left(Z < 0\right) = 1 - F\left(0\right) = 1 - \frac{23}{28} = \frac{5}{28}.$$

2. Slučajni vektor (X, Y) zadan je funkcijom gustoće

$$f(x,y) = x + y, \qquad \text{za} \;\; 0 \le x \le 1, \;\; 0 \le y \le 1.$$

Jesu li komponente X i Y nezavisne? Odgovor obrazložite. Izračunajte funckiju gustoće slučajne varijable $Z = XY$.

Rješenje.

Da bismo provjerili jesu li slučajne varijable X i Y nezavisne trebamo ispitati vrijedi li jednakost

$$f\left(x,y\right)=f_X\left(x\right)f_Y\left(y\right). \tag{2}$$

Računamo marginalne gustoće:

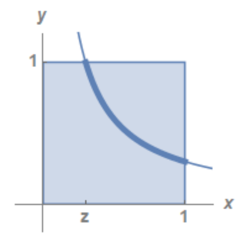
$$f_X(x)=\int_0^1 (x+y)\,\mathrm{d}y=x+\frac{1}{2},\ x\in[0,1],$$

$$f_Y(y)=\int_0^1 (x+y)\,\mathrm{d}x=y+\frac{1}{2},\ y\in[0,1].$$

Zaključujemo da jednakost (2) ne vrijedi odnosno da slučajne varijalbe X i Y nisu nezavisne.

Funkciju gustoće slučajne varijable Z računamo po formuli (1). Znamo da vrijedi $Y=\frac{Z}{X}$ te zaključujemo:

$$\begin{aligned}f_Z\left(z\right)&=\int_{-\infty}^{\infty}f\left(x,y\right)\left|\frac{\partial y}{\partial z}\right|\mathrm{d}x=\int_z^1\left(x+\frac{z}{x}\right)\frac{1}{x}\,\mathrm{d}x\\&=\int_z^1\left(1+\frac{z}{x^2}\right)\mathrm{d}x=2-2z,\ z\in[0,1].\end{aligned}$$



3. Slučajni vektor (X,Y) zadan je funkcijom gustoće

$$f(x,y)=Cx,\qquad \text{za}\ \ 0\le y\le x\le 1.$$

Izračunajte konstantu C, marginalne gustoće $f_X(x)$ i $f_Y(y)$, te gustoću slučajne varijable $Z=X-Y$.

Rješenje.

Prvo računamo konstantu C. Vrijedi:

$$\iint_D f\left(x,y\right)\mathrm{d}x\,\mathrm{d}y=1\Leftrightarrow C\int_0^1 x\,\mathrm{d}x\int_0^x \mathrm{d}y=1\Leftrightarrow C=3.$$

Zatim računamo marginalne gustoće. Vrijedi:

$$f_X(x)=\int_0^x 3x\,\mathrm{d}y=3x^2,\ x\in[0,1],$$

$$f_Y(y)=\int_y^1 3x\,\mathrm{d}x=\frac{3}{2}\left(1-y^2\right),\ y\in[0,1].$$

Funkciju gustoće slučajne varijable Z računamo po formuli (1) uz

$$z=\psi(x,y)=x-y,\quad y=x-z,\quad f\left(x,y\right)=3x,\quad \left|\frac{\partial y}{\partial z}\right|=1.$$

Konačno dobivamo

$$f_Z\left(z\right)=\int_{-\infty}^{\infty}f\left(x,y\right)\left|\frac{\partial y}{\partial z}\right|\mathrm{d}x=\int_z^1\left(3x\right)\mathrm{d}x=\frac{3}{2}\left(1-z^2\right),\ z\in[0,1].$$

4. Slučajni vektor (X, Y) zadan je funkcijom gustoće

$$f(x,y) = C(x+y), \quad \text{za} \ \ 0 \le x \le 2, \ 0 \le y \le 2.$$

Izračunajte konstantu C te gustoću slučajne varijable $Z = \frac{Y}{X}$.

Rješenje.

Računamo konstantu C. Vrijedi:

$$\iint_D f(x,y) \, \mathrm{d}x \, \mathrm{d}y = 1 \Leftrightarrow C \int_0^2 \mathrm{d}x \int_0^2 (x+y) \, \mathrm{d}y = 1 \Leftrightarrow C = \frac{1}{8}.$$

Traženu funkciju gustoće f_Z računamo po formuli (1) uz

$$z = \psi(x,y) = \frac{y}{x}, \quad y = zx, \quad f(x,y) = \frac{1}{8}(x+zx), \quad \left| \frac{\partial y}{\partial z} \right| = |x|.$$

Razlikujemo dva slučaja u ovisnosti o vrijednosti varijable z:

$$f_Z(z) = \begin{cases} \frac{1}{8}\int_0^2 (x+zx)|x| \, \mathrm{d}x, & z \in [0,1] \\ \frac{1}{8}\int_0^{\frac{2}{z}} (x+zx)|x| \, \mathrm{d}x, & z \in \langle 1, \infty \rangle \end{cases} = \begin{cases} \frac{1}{3}(z+1), & z \in [0,1] \\ \frac{1}{3z^2}(z+1), & z \in \langle 1, \infty \rangle \end{cases}$$

5. Nezavisne slučajne varijable X i Y imaju eksponencijalne razdiobe s očekivanjem 2. Odredite funkciju razdiobe slučajne varijable $Z = Y - X$.

Rješenje.

Funkcija gustoće slučajne varijable s eksponencijalnom razdiobom je

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0,$$

pri čemu je očekivanje

$$E(X) = \frac{1}{\lambda}.$$

Marginalne gustoće slučajnih varijabli X i Y su

$$f_X(x) = f_Y(x) = f(x).$$

Budući da su slučajne varijable X i Y nezavisne vrijedi

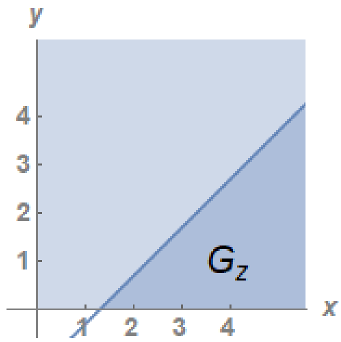
$$f(x,y) = f_X(x) f_Y(y) = f(x) f(y) = \frac{1}{4} e^{-\frac{1}{2}(x+y)}, \quad x > 0, \ y > 0.$$

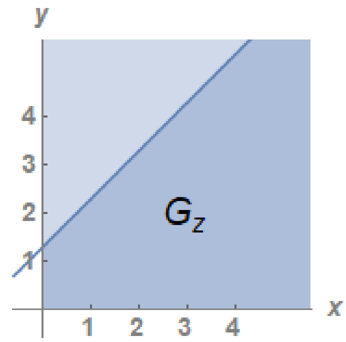
Odredimo funkciju razdiobe F_Z slučajne varijable Z:

$$F_Z(z) = P(Z < z) = P(Y - X < z) = P((X,Y) \in G_z) = \iint_{G_z} f(x,y) \, \mathrm{d}x \, \mathrm{d}y,$$

gdje je područje G_z definirano s

$$G_z = \{(x,y) \colon y - x < z\} = \{(x,y) \colon y < x + z\}.$$





Interval $\langle -\infty, \infty\rangle \ni z$ morat ćemo podijeliti na dva dijela u ovisnosti o vrijednosti koju poprima Z: $z \in \langle -\infty, 0] \cup \langle 0, \infty\rangle$. Sada je

$$F_Z(z) = \begin{cases} \int_{-z}^{\infty} \frac{1}{4} e^{\frac{-x}{2}}\,\mathrm{d}x \int_0^{x+z} e^{\frac{-y}{2}}\,\mathrm{d}y = \frac{1}{2}e^{\frac{z}{2}}, & z \le 0, \\ \int_0^{\infty} \frac{1}{4} e^{\frac{-x}{2}}\,\mathrm{d}x \int_0^{x+z} e^{\frac{-y}{2}}\,\mathrm{d}y = 1 - \frac{1}{2}e^{\frac{-z}{2}}, & z > 0. \end{cases}$$

Primijetimo da smo funkciju razdiobe F_Z slučajne varijable Z mogli dobiti i koristeći uvjetnu vjerojatnost:

$$\begin{aligned} F_Z(z) &= P(Z < z) = P(Y - X < z) = P(Y < X + z) \\ &= \int_{-\infty}^{\infty} P(Y < X + z \mid X = x) f_X(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} P(Y < x + z) f_X(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} F_Y(x+z) f_X(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} (1 - e^{-\lambda(x+z)})\mathbf{1}_{\langle 0,\infty\rangle}(x+z)\lambda e^{-\lambda x}\mathbf{1}_{\langle 0,\infty\rangle}(x)\,\mathrm{d}x \\ &= \begin{cases} \lambda \int_{-z}^{\infty} (e^{-\lambda x} - e^{-\lambda(2x+z)})\,\mathrm{d}x, & z \le 0 \\ \lambda \int_0^{\infty} (e^{-\lambda x} - e^{-\lambda(2x+z)})\,\mathrm{d}x, & z > 0 \end{cases} \\ &= \begin{cases} \frac{1}{2}e^{\frac{z}{2}}, & z \le 0 \\ 1 - \frac{1}{2}e^{-\frac{z}{2}}, & z > 0 \end{cases} \end{aligned}$$