Problems. Section 13.

1.

* Ratio test.

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| , \quad \rho = \lim_{n\to\infty} \rho_n$$

$\Rightarrow$ If
- $\rho < 1$, the series converges
- $\rho = 1$, use a different test
- $\rho > 1$, the series diverges.

* binomial series.

$$(1+x)^p = \sum_{n=0}^{\infty} \binom{p}{n} x^n = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots$$

$$a_{n+1} = \binom{p}{n+1} x^{n+1} = \frac{p(p-1)\cdots(p-n)}{(n+1)!} x^{n+1}$$

$$a_n = \binom{p}{n} x^n = \frac{p(p-1)\cdots(p-n+1)}{n!} x^n$$

$$\rho_n = \left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{p-n}{n+1} x\right| , \quad \rho = \lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \left|\frac{p-n}{n+1} x\right| = |x|$$

$\therefore \rho = |x| < 1$: the series converges.

2. $$\binom{-1}{n} = \frac{(-1)(-2)\cdots(-1-n+1)}{n!} = \frac{(-1)(-2)\cdots(-n)}{n\cdot(n-1)\cdots 2\cdot 1} = (-1)^n$$

3. When $n > p$, $\binom{p}{n} = \dfrac{p(p-1)\cdots(p-p)\cdots(p-n+1)}{n!}$

$$= \frac{p(p-1)\cdots 0 \cdots (p-n+1)}{n!}$$

$$= 0.$$

4. $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} x^n = 1 - \frac{1}{2}x + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}x^2 + \dfrac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})}{3!}x^3 \cdots$

For $n \geq 2$, $\binom{-\frac{1}{2}}{n} = \dfrac{(-\frac{1}{2})(-\frac{3}{2})\cdots(-\frac{1}{2}-n+1)}{n!} = \dfrac{(-1)^n \cdot 3 \cdot 5 \cdots (2n-1)}{n! \cdot 2^n}$

$$= \frac{(-1)^n \cdot (2n-1)!!}{(2n)!!}$$

5. $x^2 \ln(1-x) = x^2 \cdot \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1} \cdot (-x)^n}{n} = -\sum_{n=1}^{\infty} \dfrac{x^{n+2}}{n}$

$$= -\frac{x^3}{1} - \frac{x^4}{2} - \frac{x^5}{3} - \cdots$$

6. $x\sqrt{1+x} = x(1+x)^{\frac{1}{2}} = x \cdot \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^{n+1}$

$$= x + \frac{1}{2}x^2 - \frac{1}{8}x^3 + \frac{1}{16}x^4 - \frac{5}{128}x^5 + \cdots$$

7. $\frac{1}{x}\sin x = \frac{1}{x} \cdot \sum_{n=0}^{\infty} \dfrac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \dfrac{(-1)^n \cdot x^{2n}}{(2n+1)!}$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots$$

8. $\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-x^2)^n = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-1)^n \cdot x^{2n}$

For $n=0$, $\binom{-\frac{1}{2}}{0} = 1$, $n=1$, $\binom{-\frac{1}{2}}{1} = -\frac{1}{2}$.

For $n \geq 2$, $\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\cdots(-\frac{2n-1}{2})}{n!} = \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}$

$= \frac{(-1)^n \cdot (2n-1)!!}{(2n)!!}$

Thus, $\frac{1}{\sqrt{1-x^2}} = \frac{(2n-1)!!}{(2n)!!} \cdot x^{2n}$

9. $\frac{1+x}{1-x} = -1 + \frac{2}{1-x} = -1 + 2(1-x)^{-1}$

$(1-x)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} (-x)^n \Leftarrow \frac{(-1)(-2)\cdots(-n)}{n!} (-x)^n = (-1)^n \cdot (-1)^n \cdot x^n = x^n$

Thus,

$\frac{1+x}{1-x} = -1 + 2\sum_{n=0}^{\infty} x^n = 1 + 2 \cdot \sum_{n=1}^{\infty} x^n$

10. $\sin x^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+2}}{(2n+1)!}$

$= x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \cdots$

11. $\frac{\sin\sqrt{x}}{\sqrt{x}}$, $x > 0$

$\sin\sqrt{x} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (x^{\frac{1}{2}})^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+\frac{1}{2}}}{(2n+1)!}$

Thus. $\frac{\sin\sqrt{x}}{\sqrt{x}} = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^n}{(2n+1)!} = 1 - \frac{x}{3!} + \frac{x^2}{5!} - \cdots$

12. $\int_0^x \cos t^2 dt$

$$\cos t^2 = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (t^2)^{2n}}{(2n)!} = 1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \cdots$$

$$\int_0^x \cos t^2 dt = \int_0^x \left(1 - \frac{t^4}{2!} + \frac{t^8}{4!} - \frac{t^{12}}{6!} + \cdots\right) dt$$

$$= x - \frac{x^5}{2!\cdot 5} + \frac{x^9}{4!\cdot 9} - \frac{x^{13}}{6!\cdot 13} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{4n+1}}{(2n)!\cdot(4n+1)}$$

13. $\int_0^x e^{-t^2} dt$

$$e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = 1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots$$

$$\int_0^x e^{-t^2} dt = \int_0^x \left(1 - t^2 + \frac{t^4}{2!} - \frac{t^6}{3!} + \cdots\right) dt = x - \frac{x^3}{3} + \frac{x^5}{2!\cdot 5} - \frac{x^7}{3!\cdot 7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{n!\cdot(2n+1)}$$

14. $\ln\sqrt{\frac{1+x}{1-x}} = \int_0^x \frac{dt}{1-t^2}$

$$\frac{1}{1-t^2} = (1-t^2)^{-1} = \sum_{n=0}^{\infty} \binom{-1}{n} (-t^2)^n = 1 + (-1)(-t^2) + \frac{(-1)(-2)}{2!}(-t^2)^2$$

$$+ \frac{(-1)(-2)(-3)}{3!}(-t^2)^3 + \cdots$$

$$= 1 + t^2 + t^4 + t^6 + \cdots$$

$$= \sum_{n=0}^{\infty} t^{2n}$$

$$\int_0^x \frac{1}{1-t^2}\,dt = \int_0^x (1+t^2+t^4+t^6+\cdots)\,dt = x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)}$$

15. $\arcsin x = \int_0^x \frac{dt}{\sqrt{1-t^2}}$

$$\frac{1}{\sqrt{1-t^2}} = (1-t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} (-t^2)^n = 1 + \left(-\frac{1}{2}\right)(-t^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\cdot(-t^2)^2$$

$$+ \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\cdot(-t^2)^3 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\, t^{2n}$$

$$= 1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots$$

$$\int_0^x \frac{dt}{\sqrt{1-t^2}} = \int_0^x \left(1 + \frac{1}{2}t^2 + \frac{3}{8}t^4 + \cdots\right)dt = x + \frac{1}{6}x^3 + \frac{3}{40}t^5 + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!\cdot(2n+1)}\, x^{2n+1}$$

16. $\cosh x = \frac{e^x + e^{-x}}{2}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

17. $\ln \frac{1+x}{1-x}$

$$\ln \frac{1+x}{1-x} = \ln(1+x) - \ln(1-x)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\ln(1-x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-x)^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\therefore \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) = 2\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} = 2\sum_{n\ \text{odd}} \frac{x^n}{n}$$

18. $\int_0^x \frac{\sin t \, dt}{t}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\frac{\sin t}{t} = 1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots$$

$$\int_0^x \frac{\sin t}{t} dt = \int_0^x \left(1 - \frac{t^2}{3!} + \frac{t^4}{5!} - \frac{t^6}{7!} + \cdots\right) dt = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots$$

$$\therefore \int_0^x \frac{\sin t \, dt}{t} = x - \frac{x^3}{3 \cdot 3!} + \frac{x^5}{5 \cdot 5!} - \frac{x^7}{7 \cdot 7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)(2n+1)!}$$

19. $\ln(x + \sqrt{1+x^2}) = \int_0^x \frac{dt}{\sqrt{1+t^2}}$

Problem 4 & Problem 8.

$$\frac{1}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n-1)!!}{(2n)!!} t^{2n} = 1 - \frac{1}{2}t^2 + \frac{1 \cdot 3}{2 \cdot 4} t^4 - \cdots$$

$$\int_0^x \frac{dt}{\sqrt{1+t^2}} = \int_0^x \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} t^{2n} \, dt = \sum_{n=0}^{\infty} \binom{-1/2}{n} \frac{x^{2n+1}}{2n+1}$$

20. $e^x \cdot \sin x$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \quad \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^x \cdot \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$$

$$+ x^2 - \frac{x^4}{3!} + \frac{x^6}{5!}$$

$$+ \frac{x^3}{2!} - \frac{x^5}{2!\cdot 3!} + \frac{x^7}{2!\cdot 5!} + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^4}{6} - \frac{3x^5}{40} + \cdots \quad (\times)$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \frac{x^6}{90} + \cdots \quad (\circ)$$

21. $\tan^2 x$

$$\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\tan^2 x = (\tan x)^2 = \left(1 + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right)\left(1 + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots\right)$$

$$= x^2 + \frac{2x^4}{3} + \frac{17x^6}{45} + \cancel{\frac{4x^8}{45}} + \cancel{\frac{4x^{10}}{255}} + \cdots$$

$$\frac{62}{315}x^8$$

22. $\dfrac{e^x}{1-x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\frac{e^x}{1-x} = 1 + 2x + \frac{5x^2}{2} + \frac{8x^3}{3} + \cdots$$

23. $\dfrac{1}{1+x+x^2} = 1 - x + x^3 - x^4 + \cdots$ Just divide it by 1.

24. $\sec x = \dfrac{1}{\cos x}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\frac{1}{\cos x} = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots$$

25. $\dfrac{2x}{e^{2x}-1}$

$$e^{2x} = \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \cdots$$

$$\frac{2x}{e^{2x}-1} = \frac{1}{1 + \frac{(2x)}{2!} + \frac{(2x)^2}{3!} + \frac{(2x)^3}{4!} + \cdots}$$

$$= 1 - x + \frac{x^2}{3} - \frac{x^4}{45} + \cdots$$

26. $\dfrac{1}{\sqrt{\cos x}}$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

Since $\dfrac{1}{\sqrt{\cos x}} = \sqrt{\dfrac{1}{\cos x}} = \sqrt{\sec x}$; Used problem 23.

$$\frac{1}{\sqrt{\cos x}} = \sqrt{\sec x} = \sqrt{1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \cdots}$$

$$= 1 + \frac{x^2}{4} + \frac{7x^4}{96} + \frac{139x^6}{5760} + \cdots$$

27. $e^{\sin x}$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

Thus, $e^{\sin x} = 1 + \sin x + \dfrac{\sin^2 x}{2!} + \dfrac{\sin^3 x}{3!} + \dfrac{\sin^4 x}{4!} + \cdots$

$$= 1 + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right) + \frac{1}{2!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)^2$$

$$+ \frac{1}{3!}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)^3 + \cdots$$

$$= 1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \cdots$$

28. $\sin[\ln(1+x)]$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\therefore \sin[\ln(1+x)] = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right) - \frac{1}{3!}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)^3$$

$$+ \frac{1}{5!}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)^5 - \frac{1}{7!}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \right)^7 + \cdots$$

$$= x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^6}{8} + \cdots$$

29. $\sqrt{1+\ln(1+x)}$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \frac{7x^5}{256} + \cdots$$

$$\therefore \sqrt{1+\ln(1+x)} = 1 + \frac{1}{2}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)$$

$$- \frac{1}{8}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)^2 + \frac{1}{16}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)^3$$

$$- \frac{5}{128}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)^4 + \frac{7}{256}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots\right)^5 + \cdots$$

$$= 1 + \frac{x}{2} - \frac{3x^2}{8} + \frac{17x^3}{48} - \frac{143 \cdot x^4}{384} + \frac{1609 \cdot x^5}{3840} + \cdots$$

30. $\sqrt{\dfrac{1-x}{1+x}} = \dfrac{\sqrt{1-x}}{\sqrt{1+x}}$

$$\sqrt{1-x} = \sum_{n=0}^{\infty} \binom{1/2}{n} \cdot (-x)^n = 1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} \cdot x^n = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$$\therefore \sqrt{\frac{1-x}{1+x}} = \frac{1 - \frac{1}{2}x - \frac{x^2}{8} - \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}{1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots}$$

$$= 1 - x + \frac{x^2}{2} - \frac{x^3}{2} + \frac{3x^4}{8} + \cdots$$

31. $\cos(e^x - 1)$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$e^x - 1 = \left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots$$

$$\therefore \cos(e^x - 1) = 1 - \frac{1}{2!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^2$$

$$+ \frac{1}{4!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^4$$

$$- \frac{1}{6!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^6$$

$$+ \frac{1}{8!}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots\right)^8 - \cdots$$

$$= 1 - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{24} + \cdots$$

32. $\ln(1+x\cdot e^x)$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad x\cdot e^x = x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots$$

$$\therefore \ln(1+x\cdot e^x) = \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right) - \frac{1}{2}\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right)^2$$

$$+ \frac{1}{3}\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right)^3 - \frac{1}{4}\left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \frac{x^5}{4!} + \cdots\right)^4 + \cdots$$

$$= x + \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^4}{12} + \cdots$$

33. $\dfrac{1-\sin x}{1-x}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

$$1 - \sin x = 1 - x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots$$

$$\therefore \frac{1-\sin x}{1-x} = 1 + \frac{x^3}{6} + \frac{x^4}{6} + \frac{19x^5}{120} + \cdots$$

34. $\ln(2-e^{-x}) = \ln(1+(1-e^{-x}))$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots$$

$$1 - e^{-x} = +x - \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} + \frac{x^5}{5!} - \cdots$$

$$\therefore \ln(1+(1-e^{-x})) = \left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)$$

$$-\frac{1}{2}\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)^2+\frac{1}{3}\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)^3$$

$$-\frac{1}{4}\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)^4+\frac{1}{5}\left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots\right)^5+\cdots$$

$$= x-x^2+x^3-\frac{13x^4}{12}+\frac{5x^5}{4}+\cdots$$

35. $\dfrac{x}{\sin x}$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$$

$$\therefore \frac{x}{\sin x} = \frac{x}{x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots}$$

$$= 1+\frac{x^2}{6}+\frac{7x^4}{360}+\cdots$$

36. $\displaystyle\int_0^u \frac{\sin x\,dx}{\sqrt{1-x^2}}$

$$\sin x = \sum_{n=0}^{\infty}\frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!} = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots$$

$$\frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty}\binom{-1/2}{n}(-x^2)^n = 1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\cdots$$

$$\frac{\sin x}{\sqrt{1-x^2}} = \left(x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\cdots\right)\cdot\left(1+\frac{1}{2}x^2+\frac{3}{8}x^4+\frac{5}{16}x^6+\cdots\right)$$

$$= x+\frac{x^3}{3}+\frac{3x^5}{10}+\frac{16x^7}{63}+\cdots$$

$$\therefore \int_0^u \frac{\sin x \cdot dx}{\sqrt{1-x^2}} = \int_0^u \left(x + \frac{x^3}{3} + \frac{3x^5}{10} + \frac{16x^7}{63} + \cdots \right) dx$$

$$= \frac{1}{2}u^2 + \frac{1}{12}u^4 + \frac{1}{20}u^6 + \frac{2}{63}u^8 + \cdots$$

37. $\ln \cos x$

<Method 1>

$$\ln \cos x = \ln(1 + (\cos x - 1))$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots$$

$$\therefore \ln\left(1 + (\cos x - 1)\right) = \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right)$$

$$-\frac{1}{2}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right)^2 + \frac{1}{3}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right)^3$$

$$-\frac{1}{4}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right)^4 + \frac{1}{5}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots \right)^5 - \cdots$$

$$= -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots$$

<Method 2>

$$\ln \cos x = -\int_0^x \tan u \, du$$

From the series of Example 2 in method B,

$$\tan u = u + \frac{u^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\therefore -\int_0^x \tan u \, du = -\int_0^x \left(u + \frac{u^3}{3} + \frac{2x^5}{15} + \cdots \right) du = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots$$

38. $e^{\cos x}$

$$e^{\cos x} = e \cdot e^{\cos x - 1}$$

From the previous problem (problem 37),

$$\cos x - 1 = -\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

$$e^{\cos x - 1} = 1 + \left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right) + \frac{1}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)^2$$

$$+ \frac{1}{3!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)^3 + \frac{1}{4!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots\right)^4 + \cdots$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{6} + \cdots$$

$$\therefore e^{\cos x} = e \cdot e^{\cos x - 1} = e \cdot \left(1 - \frac{x^2}{2} + \frac{x^4}{6} + \cdots\right)$$

39. $f(x) = \sin x$, $a = \pi/2$ $\quad (\because \sin(\frac{\pi}{2} - x) = \cos x)$

$$\sin x = \sin\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = \cos\left(x - \frac{\pi}{2}\right)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$\therefore f(x) = \sin x$ at $\pi/2$

$$= \sin\left[\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right] = \cos\left(x - \frac{\pi}{2}\right)$$

$$= 1 - \frac{1}{2!}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{4!}\left(x - \frac{\pi}{2}\right)^4 - \frac{1}{6!}\left(x - \frac{\pi}{2}\right)^6 + \frac{1}{8!}\left(x - \frac{\pi}{2}\right)^8 - \cdots$$

40. $f(x)=\frac{1}{x}$, $a=1$

$$\frac{1}{x}=\frac{1}{1+(x-1)}$$

$$(1+x)^{-1}=\sum_{n=0}^{\infty}\binom{-1}{n}x^n=1-x+x^2-x^3+x^4-x^5+\cdots$$

$\therefore f(x)=\frac{1}{x}$, $a=1$

$$=\frac{1}{1+(x-1)}=1-(x-1)+(x-1)^2-(x-1)^3+(x-1)^4-(x-1)^5+\cdots$$

41. $f(x)=e^x$, $a=3$.

$$e^x=e^{3+(x-3)}=e^3\cdot e^{x-3}$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$$

$\therefore f(x)=e^x$, $a=3$

$$=e^{3+(x-3)}=e^3\cdot e^{x-3}$$

$$=e^3\cdot\left[1+(x-3)+\frac{(x-3)^2}{2!}+\frac{(x-3)^3}{3!}+\frac{(x-3)^4}{4!}+\cdots\right]$$

42. $f(x)=\cos x$, $a=\pi$

$$\cos x=\cos(\pi+(x-\pi))=-\cos(x-\pi)$$

$$\cos x=\sum_{n=0}^{\infty}\frac{(-1)^n\cdot x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\frac{x^8}{8!}-\cdots$$

$\therefore f(x) = \cos x$ at $x=\pi$

$$= \cos(\pi + (x-\pi)) = -\cos(x-\pi)$$

$$= -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \frac{(x-\pi)^8}{8!} + \cdots$$

43. $f(x) = \cot x$, $a = \pi/2$

$$\cot x = \cot\left(\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

Since $\tan x = \frac{\sin x}{\cos x} = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$

$\therefore f(x) = \cot x$, $a = \pi/2$

$$= \cot\left(\frac{\pi}{2} + \left(x - \frac{\pi}{2}\right)\right) = -\tan\left(x - \frac{\pi}{2}\right)$$

$$= -\left(x - \frac{\pi}{2}\right) - \frac{1}{3}\left(x - \frac{\pi}{2}\right)^3 - \frac{2}{15}\left(x - \frac{\pi}{2}\right)^5 - \cdots$$

44. $f(x) = \sqrt{x}$ $a = 25$

$$\sqrt{x} = \sqrt{25 + (x-25)} = \sqrt{1 + (x+24)}$$

$$= 5 \cdot \sqrt{1 + \frac{x-25}{25}}$$

$$\sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} \cdot x^n = 1 + \frac{1}{2}x - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots$$

$\therefore f(x) = \sqrt{x}$, $a = 25$

$$= \sqrt{25 + (x-25)} = 5 \cdot \sqrt{1 + \left(\frac{x-25}{25}\right)}$$

$$= 5 \cdot \left(1 + \frac{1}{2}\left(\frac{x-25}{25}\right) - \frac{1}{8}\left(\frac{x-25}{25}\right)^2 + \frac{1}{16}\left(\frac{x-25}{25}\right)^3 + \cdots \right)$$

$$= 5 + \frac{x-25}{10} - \frac{(x-25)^2}{10^3} + \frac{(x-25)^3}{5 \cdot 10^4} - \cdots$$