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Factorisation of Algebraic expressions



Let's recall.

In the previous standard we have learnt to factorise the expressions of the form $ax + ay$ and $a^2 - b^2$

For example,

$$(1) \quad 4xy + 8xy^2 = 4xy(1 + 2y)$$

$$(2) \quad p^2 - 9q^2 = (p)^2 - (3q)^2 = (p + 3q)(p - 3q)$$



Let's learn.

Factors of a quadratic trinomial

An expression of the form $ax^2 + bx + c$ is called a quadratic trinomial.

We know that $(x + a)(x + b) = x^2 + (a + b)x + ab$

\therefore the factors of $x^2 + (a + b)x + ab$ are $(x + a)$ and $(x + b)$.

To find the factors of $x^2 + 5x + 6$, by comparing it with $x^2 + (a + b)x + ab$ we get, $a + b = 5$ and $ab = 6$. So, let us find the factors of 6 whose sum is 5. Then writing the trinomial in the form $x^2 + (a + b)x + ab$, find its factors.

$$\begin{aligned} x^2 + 5x + 6 &= x^2 + (3 + 2)x + 3 \times 2 \quad \dots\dots\dots x^2 + (a + b)x + ab \\ &= \underline{x^2 + 3x} + \underline{2x + 2 \times 3} \quad \dots\dots\dots \text{multiply } (3 + 2) \text{ by } x, \text{ make two} \\ &\hspace{15em} \text{groups of the four terms obtained.} \\ &= x(x + 3) + 2(x + 3) = (x + 3)(x + 2) \end{aligned}$$

Study the following examples to know how a given trinomial is factorised.

Ex. (1) Factorise : $2x^2 - 9x + 9$.

Solution: First we find the product of the coefficient of the square term and the constant term. Here the product is $2 \times 9 = 18$.

Now, find factors of 18 whose sum is -9, that is equal to the coefficient of the middle term.

$$18 = (-6) \times (-3) ; (-6) + (-3) = -9$$

Write the term $-9x$ as $-6x - 3x$

$$\begin{aligned} 2x^2 - 9x + 9 &= 2x^2 - 6x - 3x + 9 \\ &= 2x(x - 3) - 3(x - 3) \\ &= (x - 3)(2x - 3) \end{aligned}$$

$$\therefore 2x^2 - 9x + 9 = (x - 3)(2x - 3)$$

Ex. (2) Factorise : $2x^2 + 5x - 18$.

Solution : $2x^2 + 5x - 18$

$$= \frac{2x^2 + 9x - 4x - 18}{+9 \quad -4}$$

$$= x(2x + 9) - 2(2x + 9)$$

$$= (2x + 9)(x - 2)$$

Ex. (3) Factorise : $x^2 - 10x + 21$.

solution: $x^2 - 10x + 21$

$$= \frac{x^2 - 7x - 3x + 21}{-7 \quad -3}$$

$$= x(x - 7) - 3(x - 7)$$

$$= (x - 7)(x - 3)$$

Ex. (4) Find the factors of $2y^2 - 4y - 30$.

Solution : $2y^2 - 4y - 30$

$$= 2(y^2 - 2y - 15) \quad \text{..... taking out the common factor 2}$$

$$= 2(\underline{y^2 - 5y} + \underline{3y - 15}) \quad \text{.....}$$

$$= 2[y(y - 5) + 3(y - 5)] \quad \begin{matrix} -15 \\ -5 \quad +3 \end{matrix}$$

$$= 2(y - 5)(y + 3)$$

Practice Set 6.1

1. Factorise.

(1) $x^2 + 9x + 18$

(2) $x^2 - 10x + 9$

(3) $y^2 + 24y + 144$

(4) $5y^2 + 5y - 10$

(5) $p^2 - 2p - 35$

(6) $p^2 - 7p - 44$

(7) $m^2 - 23m + 120$

(8) $m^2 - 25m + 100$

(9) $3x^2 + 14x + 15$

(10) $2x^2 + x - 45$

(11) $20x^2 - 26x + 8$

(12) $44x^2 - x - 3$



Let's learn.

Factors of $a^3 + b^3$

We know that, $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, which we can write as

$$(a + b)^3 = a^3 + b^3 + 3ab(a + b)$$

Now, $a^3 + b^3 + 3ab(a + b) = (a + b)^3$ interchanging the sides.

$$\begin{aligned} \therefore a^3 + b^3 &= (a + b)^3 - 3ab(a + b) = [(a + b)(a + b)^2] - 3ab(a + b) \\ &= (a + b)[(a + b)^2 - 3ab] = (a + b)(a^2 + 2ab + b^2 - 3ab) \\ &= (a + b)(a^2 - ab + b^2) \end{aligned}$$

$$\therefore \boxed{a^3 + b^3 = (a + b)(a^2 - ab + b^2)}$$

Lets us solve some examples using the above formula for factorising the addition of two cubes.

$$\begin{aligned}\text{Ex. (1)} \quad x^3 + 27y^3 &= x^3 + (3y)^3 \\ &= (x + 3y) [x^2 - x(3y) + (3y)^2] \\ &= (x + 3y) [x^2 - 3xy + 9y^2]\end{aligned}$$

$$\begin{aligned}\text{Ex. (2)} \quad 8p^3 + 125q^3 &= (2p)^3 + (5q)^3 = (2p + 5q) [(2p)^2 - 2p \times 5q + (5q)^2] \\ &= (2p + 5q) (4p^2 - 10pq + 25q^2)\end{aligned}$$

$$\begin{aligned}\text{Ex. (3)} \quad m^3 + \frac{1}{64m^3} &= m^3 + \left(\frac{1}{4m}\right)^3 = \left(m + \frac{1}{4m}\right) \left[m^2 - m \times \frac{1}{4m} + \left(\frac{1}{4m}\right)^2\right] \\ &= \left(m + \frac{1}{4m}\right) \left(m^2 - \frac{1}{4} + \frac{1}{16m^2}\right)\end{aligned}$$

$$\begin{aligned}\text{Ex. (4)} \quad 250p^3 + 432q^3 &= 2(125p^3 + 216q^3) \\ &= 2[(5p)^3 + (6q)^3] = 2(5p + 6q)(25p^2 - 30pq + 36q^2)\end{aligned}$$

Practice Set 6.2

1. Factorise. (1) $x^3 + 64y^3$ (2) $125p^3 + q^3$ (3) $125k^3 + 27m^3$ (4) $2l^3 + 432m^3$
(5) $24a^3 + 81b^3$ (6) $y^3 + \frac{1}{8y^3}$ (7) $a^3 + \frac{8}{a^3}$ (8) $1 + \frac{q^3}{125}$



Factors of $a^3 - b^3$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = a^3 - b^3 - 3ab(a - b)$$

$$\text{Now, } a^3 - b^3 - 3ab(a - b) = (a - b)^3$$

$$\begin{aligned}\therefore a^3 - b^3 &= (a - b)^3 + 3ab(a - b) \\ &= [(a - b)(a - b)^2 + 3ab(a - b)] \\ &= (a - b) [(a - b)^2 + 3ab] \\ &= (a - b) (a^2 - 2ab + b^2 + 3ab) \\ &= (a - b) (a^2 + ab + b^2)\end{aligned}$$

$$\therefore \boxed{a^3 - b^3 = (a - b)(a^2 + ab + b^2)}$$

Lets us solve some examples using the above formula for factorising the difference of two cubes.

$$\begin{aligned}\text{Ex. (1)} \quad x^3 - 8y^3 &= x^3 - (2y)^3 \\ \therefore x^3 - 8y^3 &= x^3 - (2y)^3 \\ &= (x - 2y) (x^2 + 2xy + 4y^2)\end{aligned}$$

$$\text{Ex. (2)} \quad 27p^3 - 125q^3 = (3p)^3 - (5q)^3 = (3p - 5q) (9p^2 + 15pq + 25q^2)$$

$$\begin{aligned}\text{Ex. (3)} \quad 54p^3 - 250q^3 &= 2 [27p^3 - 125q^3] = 2 [(3p)^3 - (5q)^3] \\ &= 2(3p - 5q)(9p^2 + 15pq + 25q^2)\end{aligned}$$

$$\text{Ex. (4)} \quad a^3 - \frac{1}{a^3} = \left(a - \frac{1}{a}\right) \left(a^2 + 1 + \frac{1}{a^2}\right)$$

$$\text{Ex. (5)} \quad \text{Simplify : } (a - b)^3 - (a^3 - b^3)$$

$$\text{Solution : } (a - b)^3 - (a^3 - b^3) = a^3 - 3a^2b + 3ab^2 - b^3 - a^3 + b^3 = -3a^2b + 3ab^2$$

$$\text{Ex. (6)} \quad \text{Simplify : } (2x + 3y)^3 - (2x - 3y)^3$$

$$\text{Solution : Using the formula } a^3 - b^3 = (a - b) (a^2 + ab + b^2)$$

$$\begin{aligned}\therefore (2x + 3y)^3 - (2x - 3y)^3 &= [(2x + 3y) - (2x - 3y)] [(2x + 3y)^2 + (2x + 3y)(2x - 3y) + (2x - 3y)^2] \\ &= [2x + 3y - 2x + 3y] [4x^2 + 12xy + 9y^2 + 4x^2 - 9y^2 + 4x^2 - 12xy + 9y^2] \\ &= 6y (12x^2 + 9y^2) = 72x^2y + 54y^3\end{aligned}$$



Now I know.

$$(i) a^3 + b^3 = (a + b) (a^2 - ab + b^2) \quad (ii) a^3 - b^3 = (a - b) (a^2 + ab + b^2)$$

Practice Set 6.3

$$1. \text{ Factorise : } (1) y^3 - 27 \quad (2) x^3 - 64y^3 \quad (3) 27m^3 - 216n^3 \quad (4) 125y^3 - 1$$

$$(5) 8p^3 - \frac{27}{p^3} \quad (6) 343a^3 - 512b^3 \quad (7) 64x^3 - 729y^3 \quad (8) 16a^3 - \frac{128}{b^3}$$

$$2. \text{ Simplify : } (1) (x + y)^3 - (x - y)^3 \quad (2) (3a + 5b)^3 - (3a - 5b)^3$$

$$(3) (a + b)^3 - a^3 - b^3 \quad (4) p^3 - (p + 1)^3$$

$$(5) (3xy - 2ab)^3 - (3xy + 2ab)^3$$



Let's learn.

Rational algebraic expressions

If A and B are two algebraic expressions then $\frac{A}{B}$ is called a rational algebraic expression. While simplifying a rational algebraic expression, we have to perform operations of addition, subtraction, multiplication and division. They are similar to those performed on rational numbers.

Note that, the denominators or the divisors of algebraic expressions are non-zero.

Ex. (1) Simplify : $\frac{a^2+5a+6}{a^2-a-12} \times \frac{a-4}{a^2-4}$

Solution:
$$\begin{aligned} & \frac{a^2+5a+6}{a^2-a-12} \times \frac{a-4}{a^2-4} \\ &= \frac{(a+3)(a+2)}{(a-4)(a+3)} \times \frac{(a-4)}{(a+2)(a-2)} \\ &= \frac{1}{a-2} \end{aligned}$$

Ex. (2) $\frac{7x^2+18x+8}{49x^2-16} \times \frac{14x-8}{x+2}$

Solution :
$$\begin{aligned} & \frac{7x^2+18x+8}{49x^2-16} \times \frac{14x-8}{x+2} \\ &= \frac{(7x+4)(x+2)}{(7x+4)(7x-4)} \times \frac{2(7x-4)}{(x+2)} \\ &= 2 \end{aligned}$$

Ex. (3) Simplify : $\frac{x^2-9y^2}{x^3-27y^3}$

Solution:
$$\frac{x^2-9y^2}{x^3-27y^3} = \frac{(x+3y)(x-3y)}{(x-3y)(x^2+3xy+9y^2)} = \frac{x+3y}{x^2+3xy+9y^2}$$

Practice Set 6.4

1. Simplify :

(1) $\frac{m^2-n^2}{(m+n)^2} \times \frac{m^2+mn+n^2}{m^3-n^3}$

(2) $\frac{a^2+10a+21}{a^2+6a-7} \times \frac{a^2-1}{a+3}$

(3) $\frac{8x^3-27y^3}{4x^2-9y^2}$

(4) $\frac{x^2-5x-24}{(x+3)(x+8)} \times \frac{x^2-64}{(x-8)^2}$

(5) $\frac{3x^2-x-2}{x^2-7x+12} \div \frac{3x^2-7x-6}{x^2-4}$

(6) $\frac{4x^2-11x+6}{16x^2-9}$

(7) $\frac{a^3-27}{5a^2-16a+3} \div \frac{a^2+3a+9}{25a^2-1}$

(8) $\frac{1-2x+x^2}{1-x^3} \times \frac{1+x+x^2}{1+x}$



Answers

Practice Set 6.1

1. (1) $(x + 6)(x + 3)$ (2) $(x - 9)(x - 1)$ (3) $(y + 12)(y + 12)$
 (4) $5(y + 2)(y - 1)$ (5) $(p - 7)(p + 5)$ (6) $(p + 4)(p - 11)$
 (7) $(m - 15)(m - 8)$ (8) $(m - 20)(m - 5)$ (9) $(x + 3)(3x + 5)$
 (10) $(x + 5)(2x - 9)$ (11) $2(5x - 4)(2x - 1)$ (12) $(11x - 3)(4x + 1)$

Practice Set 6.2

1. (1) $(x + 4y)(x^2 - 4xy + 16y^2)$ (2) $(5p + q)(25p^2 - 5pq + q^2)$
 (3) $(5k + 3m)(25k^2 - 15km + 9m^2)$ (4) $2(l + 6m)(l^2 - 6lm + 36m^2)$
 (5) $3(2a + 3b)(4a^2 - 6ab + 9b^2)$ (6) $\left(y + \frac{1}{2y}\right)\left(y^2 - \frac{1}{2} + \frac{1}{4y^2}\right)$
 (7) $\left(a + \frac{2}{a}\right)\left(a^2 - 2 + \frac{4}{a^2}\right)$ (8) $\left(1 + \frac{q}{5}\right)\left(1 - \frac{q}{5} + \frac{q^2}{25}\right)$

Practice Set 6.3

1. (1) $(y - 3)(y^2 + 3y + 9)$ (2) $(x - 4y)(x^2 + 4xy + 16y^2)$
 (3) $27(m - 2n)(m^2 + 2mn + 4n^2)$ (4) $(5y - 1)(25y^2 + 5y + 1)$
 (5) $\left(2p - \frac{3}{p}\right)\left(4p^2 + 6 + \frac{9}{p^2}\right)$ (6) $(7a - 8b)(49a^2 + 56ab + 64b^2)$
 (7) $(4x - 9y)(16x^2 + 36xy + 81y^2)$ (8) $16\left(a - \frac{2}{b}\right)\left(a^2 + \frac{2a}{b} + \frac{4}{b^2}\right)$
2. (1) $6x^2y + 2y^3$ (2) $270a^2b + 250b^3$ (3) $3a^2b + 3ab^2$
 (4) $-3p^2 - 3p - 1$ (5) $-108x^2y^2ab - 16a^3b^3$

Practice Set 6.4

1. (1) $\frac{1}{m+n}$ (2) $a + 1$ (3) $\frac{4x^2 + 6xy + 9y^2}{2x + 3y}$
 (4) 1 (5) $\frac{(x-1)(x-2)(x+2)}{(x-3)^2(x-4)}$
 (6) $\frac{x-2}{4x+3}$ (7) $5a + 1$ (8) $\frac{1-x}{1+x}$

