DISCRETE MATHEMATICS

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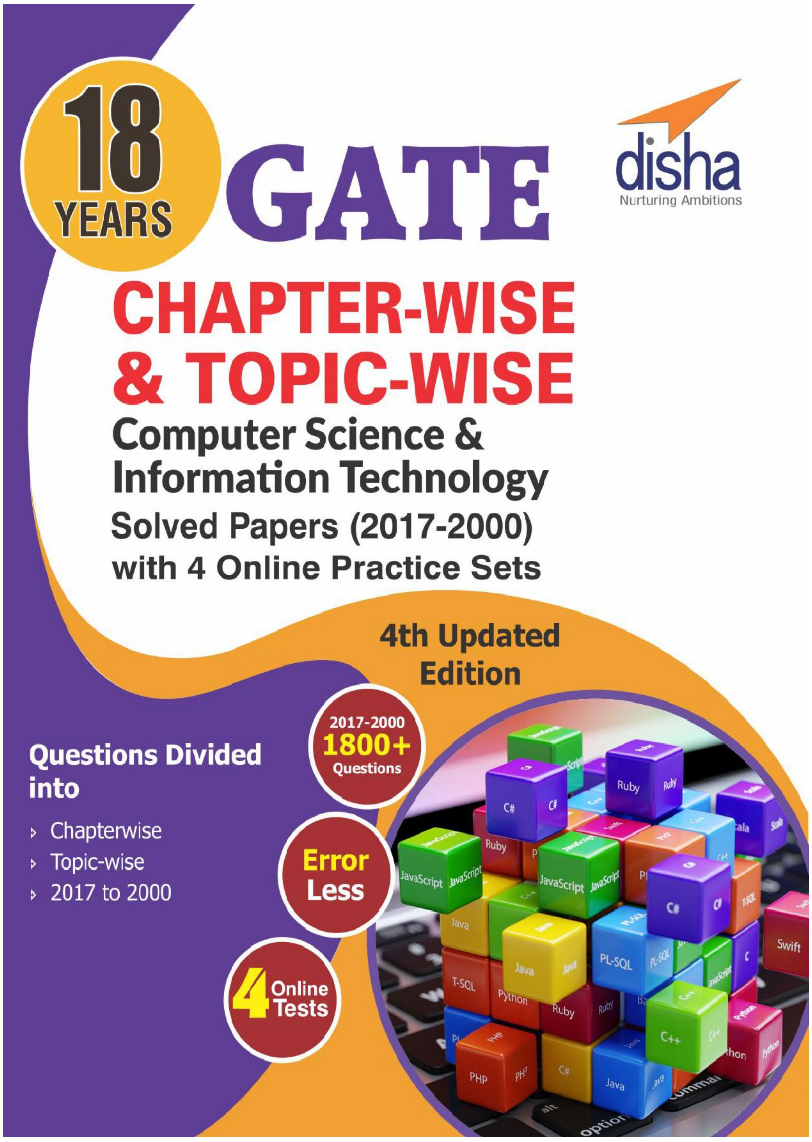

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DISCRETE MATHEMATICS

1

Quick Revision Material

MATHEMATICAL LOGIC

NOT $\rceil$

AND $\wedge$

OR $\vee$

IF ...THEN $\rightarrow$ or $\Rightarrow$

IF AND ONLY IF $\leftrightarrow$ or $\Rightarrow$

OR ($\vee$)

P	Q	$P \vee Q$
F	F	F
F	T	T
T	F	T
T	T	T

NOT ($\rceil$)

P	$\rceil P$
T	F
F	T

AND ($\wedge$)

P	Q	$P \wedge Q$
F	F	F
F	T	F
T	F	F
T	T	T

IMPLIES ($\rightarrow$)

P	Q	$P \rightarrow Q$
F	F	T
F	T	T
T	F	F
T	T	T

IF AND ONLY IF (P $\leftrightarrow$ Q)

P	Q	$P \leftrightarrow Q$
F	F	T
F	T	F
T	F	F
T	T	T

Converse and Contrapositive

For the proposition

(i) $P \rightarrow Q$, the $Q \rightarrow P$ is called its converse.

(ii) $\neg Q \rightarrow \neg P$ is called its contrapositive.

Tautologies and Contradictions

'A statement (or propositional function) which is true all possible truth values of its propositional variables is called a tautology.' A statement which is always false all possible truth values of its propositional variables is called a contradiction.

Reasoning

Logical reasoning is the process of drawing conclusion from premises using rules of inference.

List of Identities

1. $P \Leftrightarrow (P \vee P)$
   idempotence of $\vee$
2. $P \Leftrightarrow (P \wedge P)$
   idempotence of $\wedge$
3. $(P \vee Q) \Leftrightarrow (Q \vee P)$
   commutativity of $\vee$
4. $(P \wedge Q) \Leftrightarrow (Q \wedge P)$
   commutativity of $\wedge$
5. $[(P \vee Q) \vee R] \Leftrightarrow [P \vee (Q \vee R)]$
   associativity of $\vee$
6. $[(P \wedge Q) \wedge R] \Leftrightarrow [P \wedge (Q \wedge R)]$
   associativity of $\wedge$
7. $\neg (P \wedge Q) \Leftrightarrow [\neg P \vee \neg Q]$
   De-Morgan's law
8. $\neg (P \vee Q) \Leftrightarrow [\neg P \wedge \neg Q]$
   De-Morgan's law
9. $[P \wedge (Q \vee R] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$
   distributivity of $\wedge$ over $\vee$
10. $[P \vee (Q \wedge R] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$
    distributivity of $\vee$ over $\wedge$
11. $(P \vee \text{True}) \Leftrightarrow \text{True}$
12. $(P \wedge \text{False}) \Leftrightarrow \text{False}$
13. $(P \vee \text{False}) \Leftrightarrow P$
14. $(P \wedge \text{True}) \Leftrightarrow P$
15. $(P \vee \neg P) \Leftrightarrow \text{True}$
16. $(P \wedge \neg P) \Leftrightarrow \text{False}$
17. $P \Leftrightarrow \neg(\neg P)$
    double negation

18. $(P \to Q) \Leftrightarrow (\neg P \vee Q)$
implication
19. $(P \leftrightarrow Q) \Leftrightarrow (P \to Q) \wedge (Q \to P)$
equivalence
20. $[(P \wedge Q) \to R] \Leftrightarrow [P \to (Q \to R)]$
exportation
21. $[(P \to Q) \wedge [P \to \neg Q) \Leftrightarrow \neg P$
absurdity
22. $[P \to Q \Leftrightarrow (\neg Q \to \neg P)$
contrapositive

(1) Two formulae A and A* are said to be duals of each other if either one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.
If the formula A contains special variables 1 or 0, then its dual A* is obtained by replacing 1 by 0 and 0 by 1.
e.g., (i) Dual of $(p \vee q) \wedge r$ is $(p \wedge q) \vee r$
(ii) Dual of $(p \wedge q) \vee 0$ is $(p \vee q) \wedge 1$.

(2) **Tautology implications :** A statement A is said to tautologically imply a statement B if and only if $A \to B$ is a tautology which is read as "A implies B".
The implications listed below have important applications which can be proved by truth tables :

1.	$p \wedge q \Rightarrow p$	$p \Rightarrow p \vee q$
2.	$\sim p \Rightarrow p \to q$	$q \Rightarrow p \to q$
3.	$\sim (p \to q) \Rightarrow p$	$\sim (p \to q) \Rightarrow \sim q$
4.	$P \wedge (P \to q) \Rightarrow q$	$\sim p \wedge (p \vee q) \Rightarrow q$
5.	$(p \to q) \wedge (q \to r) \Rightarrow p \to r$	

Implications

The relationship between propositions that can be derived from the definitions of connectives.
List of implication
1. $P \Rightarrow (P \vee Q)$ addition
2. $(P \wedge Q) \Rightarrow P$ simplification
3. $[P \wedge (P \to Q)] \Rightarrow Q$ modus ponens
4. $[(P \to Q) \wedge \neg Q] \Rightarrow \neg P$ modus tollens
5. $[\neg P \wedge (P \vee Q)] \Rightarrow Q$ disjunctive syllogism
6. $[(P \to Q) \wedge (Q \to R)] \Rightarrow (P \to R)$ hypothetical syllogism
7. $(P \to Q) \Rightarrow [(Q \to R) \to (P \to R)]$
8. $(P \to Q) \wedge [(R \to S) \Rightarrow [(P \wedge R) \to (Q \wedge S)]$
9. $[(P \leftrightarrow Q) \wedge (Q \leftrightarrow R) \Rightarrow [(P \leftrightarrow R)$

Predicate Logic

A predicate is a verb phrase template that describes a property of objects or relationship among objects represented by the variables.

Set Theory & Algebra

ALGEBRAIC OPERATIONS ON SETS

Idempotent operation : For any set A, we have
(i) $A \cup A = A$ and
(ii) $A \cap A = A$

Identity operation : For any set A, we have
(i) $A \cup \phi = A$ and
(ii) $A \cap U = A$
i.e. ϕ and U are identity elements for union and intersection respectively.

Commutative operation : For any set A and B, we have
(i) $A \cup B = B \cup A$
(ii) $A \cap B = B \cap A$ and
(iii) $A \Delta B = B \Delta A$
i.e. union intersection and symmetric difference of two sets are commutative.

Note :

$A - B \neq B - A$ and $A \times B \neq B \times A$

LAWS OF SET THEORY

1. Commutative Law
$A \cup B = B \cup A$; $A \cap B = B \cap A$.
2. Associative Law
$A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$
3. Distributive Law
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
4. Complement Law
$A \cup A^{\circ} = U$; $A \cap A^c = \phi$.
5. Identity Law
$A \cup \phi = A = \phi \cup A$
$A \cap U = A = U \cap A$.
6. Absorption Law $A \cup (A \cap B) = A$; $A \cap (A \cup B) = A$
7. De Morgan's Law $(A \cup B)^c = A^c \cap B^c$; $(A \cap B)^c = A^c \cup B^c$
8. Involution Law $(A^c)^c = A$.

Obs. Using the distributive law, we can extend the above result for three sets A, B, C

$|A \cup B \cup C| = |(A \cup B) \cup C|$
$= |A \cup B| + |C| - |(A \cup B) \cap C|$
$= |A| + |B| - |A \cap B| + |C| - |(A \cap C) \cup (B \cap C)|$
$= |A| + |B| + |C| - |A \cap B| - [|A \cap C| + |B \cap C| - A \cap B \cap C|]$

hence follows the result

$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|]$

CARTESIAN PRODUCT OF SETS

Ordered Pair : A pair of objects, listed in a specific order, is called an ordered pair, for example (a, b) is an ordered pair of two elements a and b, a is called the FIRST ELEMENT and b the SECOND ELEMENT.
Two ordered pairs (a, b) and (c, d) are equal if any only if a = c and b = d.

Cartesian product of Sets : Let A and B are two non-empty sets. The set of all ordered pairs (a, b) of elements $a \in A$ and $b \in B$ is called the Cartesian Product of sets A and B and is denoted by $A \times B$. Thus $A \times B = \{(a, b) : a \in A, b \in B\}$

For example : If $A = \{1, 2, 3\}$ and $B = \{a, b\}$, then
(i) $A \times B = \{(1, a)\}, (1, b), (2, a), (2, b), (3, a), (3, b)\}$

RELATIONS

Let A and B be two non-empty sets. Then a relation (BINARY RELATION) R from A to B is a subset of A×B.
That is, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$
If $R \subseteq A \times A$, the R is said to be a relation on A.
If $(a, b) \in R$, then we write aRb and we say a is R related to b.
Thus, $(a, b) \in R \Leftrightarrow aRb$.
If $(a, b) \notin R$, then we write $a \not R\, b$ and say that *a* is not related to *b*.

DOMAIN AND RANGE OF A RELATION

Let A and B are two sets and R is a relation from A to B, i.e. $R \subseteq A \times B$

The set of all the first components of the ordered pairs of the relation R is called the DOMAIN of R. Thus

domain of R = $\{a \in A : (a, b) \in R$ for some $b \in B\}$

The set of all the second components of the ordered pairs of the relation R is called the RANGE of R. Thus,

range of R = $\{b \in B : (a, b) \in R$ for some $a \in A\}$

Clearly domain of $R \subseteq A$ and range of $R \subseteq B$

The set B is called the CO-DOMAIN of R

Example :

(i) If A = {1,2,3} and B = {a, b, c} let R = {(1,a) (1,c), (2, b)}
Then domain of R = {1, 2} range of R = {a, b, c}

Number of Relations : Let A contains m elements and B contains n element. Then A×B contains mn elements. Hence, A×B has 2^{mn} subsets. That is the total number of relations from A to B are 2^{mn}. The relations ϕ (called a VOID RELATION) and $A \times B$ (called an UNIVERSAL RELATION) are said to be TRIVIAL RELATIONS from A to B.

Inverse Relation : The inverse relation of a relation R is the set obtained by reversing each of the ordered pairs of R and is denoted by R^{-1}.

Example :

(i) Let A = {1, 2, 3} and B = {a, b, c}
If R = $\{(1, a), (2, a), (3, b), (3, c)\} \subseteq A \times B$
Then $R^{-1} = \{(a, 1), (a, 2), (b, 3), (c, 3)\} \subseteq B \times A$

TYPES OF RELATIONS ON THE SET A

Let A be a set and R is a relation on A, i.e. $R \subseteq A \times A$. Then we define

1. **Void Relation :** If R = ϕ, then R is called a void relation on A.
2. **Universal Relation :** If R = A×A, then R is called an universal relation on A.
3. **Identity Relation :** A relation R is defined as an identity relation if R = $\{(a, a) : a \in A\}$. Thus in an identity relation on A, every element of A is related to itself only. Identity relation on A is also denoted by I_A. Thus
$I_A = \{(a, a) : a \in A\}$
Example : If A = {1, 2, 3}, then I_A = {(1, 1), (2, 2), (3,3)}
4. **Reflexive Relation :** A relation R is said to be a reflexive relation on A if every element of A is related to itself.
Thus R is reflexive $\Leftrightarrow (a,a) \in R$, i.e. aRa $\forall\ a \in A$
[The symbol $\forall$ is read as "for every element"]
Example : Let A = {1, 2, 3} be a set.
Then R = {(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)} is a reflexive relation on A.
5. **Symmetric Relation :** A relation R on a set A is defined as a symmetric relation if $(a,b) \in R \Rightarrow (b, a) \in R$ That is, aRb $\Rightarrow$ bRa, where a, $b \in A$.
Example : Let A = {1, 2, 3, 4} and let R_1 be relation on A given by R_1 = {(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)} is a symmetric relation on A.
6. **Transitive relation :** A relation R on a set A is defined as a transitive relation if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$
That is, aRb and bRc $\Rightarrow$ aRc, where a, b, c, $\in$ A.
Example : Let L be the set of all straight line in a plane. Then the relation 'is parallel to on L is a transitive relation, because of any $\ell_1, \ell_2, \ell_3, \in L$.
$$\ell_1 \| \ell_2 \text{ and } \ell_2 \| \ell_3 \Rightarrow \ell_1 \| \ell_3$$
7. **Antisymmetric Relation :** A relation R on a set A is antisymmetric if $(a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$
If $(a, b) \in R$ and $(b, a) \notin R$, then still R is an antisymmetric relation.
Example : Let R be a relation on the set N of natural numbers defined by
xRY $\Leftrightarrow$ 'x divides y' for all x, $y \in N$
This relation is a antisymmetric relation on N.
Since for any two numbers a, $b \in N$.
a/b and b/a $\Rightarrow$ a = b i.e. aRb and bRa $\Rightarrow$ a = b

EQUIVALENCE RELATION

A relation R on a set A is an equivalence relation if and only if

(a) R is reflexive, i.e, aRa $\forall\ a \in A$
(b) R is symmetric, i.e., aRb $\Rightarrow$ bRa
(c) R is transitive, i.e., aRb and bRc $\Rightarrow$ aRc

Partial order relation :

A relation R on a set A is a partial order relation if and only if.

(a) R is reflexive, i.e. aRa $\forall\ a \in A$
(b) R is antisymmetric i.e., aRb and bRa $\Rightarrow$ a = b
(c) R is transitive, i.e., aRb and bRc $\Rightarrow$ aRc.

Relation of congruence modulo m :

Let m be a fixed positive integer. Two integers a and b are said to be "congruent modulo m" if a – b is divisible by m. We write $a \equiv b \pmod m$

Thus. $a \equiv b \pmod m$ [Read as " a is congruent to b modulo m"] iff a – b is divisible by m; a, $b \in$ **I**.

For example :

(i) $25 \equiv 5 \pmod 4$ because 25–5 = 20 is divisible by 4.
(ii) $23 \equiv 2 \pmod 3$ because 23 –2 = 21 is divisible by 3
(iii) $20 \not\equiv 3 \pmod 5$ because 20 – 3 = 17 is not divisible by 5
The relation " congruence modulo m" is an equivalence relation on **I**.

DOMAIN, CO-DOMAIN AND RANGE

If $f\ A \to B$ is a function, then A is called domain, B is called co-domain of f.

Range :

If $f : A \to B$ is a function, then set of all images of the elements of A is called range of f.

Range is a subset of the co-domain, i.e. $f(A) \subseteq B$.

e.g. If A = {1, 2, 3, 4}, B = {p, q, r} and f = (1, q), (2, q), (3, q), (4, q)}, then f is a function.

Domain = A, Co-domain = B.

Range of $f = f(A) = \{f(1), f(2), f(3), f(4)\}$
$= \{q, q, p, q\} = \{p, q\}$

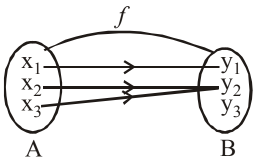
$\therefore\ f(A) \subseteq B$

KINDS OF FUNCTIONS

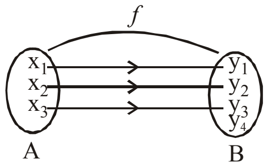
1. **One-to-one Function (or injective function)**
A function $f :\ A \to B$ is called one-one mapping if every distinct element of A has a distinct image in B.
Thus, a function $f :\ A \to B$ is one-one
2. **Many-one Function :** A function $f :\ A \to B$ is many-one if two or more different elements of A have the same image in B.
Thus, $f :\ A \to B$ is many-one if

3. **Onto or Surjective Function :** The function $f: A \to B$ is said to be an onto function if every element of B is image of at least one element of A.
For a surjective function f,
Range of f = co-domain (B)

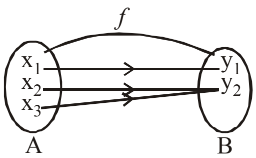
4. **Into Function :** If the function $f: A \to B$ is such that there is at least one element of B which is not the image of any element of A, then f is called an into function.
For an into function f.
Range of f $\subset$ co-domain (B)



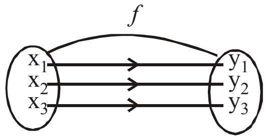
Many-one Into



One - One Into



Many-one Onto



One - One Onto

Bijective function : A function $f: A \to B$ is a bijective function if f is one-one as well as onto, i.e. f is injective and surjective both.

GROUP

A semigroup with identity and in which every element is invertible is called a group.

Definition 1

Let G be a non-empty set and * be a binary operation on G. Then algebraic system (G, *) is called a group if

I. $a * (b * c) = (a * b) * c, \ \forall \ a, b, c \in G$

II. $\exists e \in G \Rightarrow a * e = e * a = a, \ \forall a \in G$

III. $a \in G$

$\Rightarrow \exists b \in G \Rightarrow a * b = b * a = e$

- The element e in condition II of the definition of a group is called identity element of the group.
- The element b corresponding to a in condition III of the definition of a group is called inverse of a in the group.

Definition 3 :

A group (G, *) is said to be a non-abelian group, if (G, *) is not abelian.

PROPERTIES OF A GROUP.

Theorem 1 :

In a group, identity element is unique.

Theorem 2 :

In a group, inverse of every element is unique.

Theorem 3 :

If (G, .) be a group and $a \in G$, then $\left(a^{-1}\right)^{-1} = a$.

Theorem 4 :

If (G, .) be a group and a, b $\in$ G, then $(ab)^{-1} = b^{-1}a^{-1}$

Theorem 5 :

If (G, .) be a group and $a_1, a_2, ... a_n \in G$, then

$(a_1a_2.....a_n)^{-1} = a_n^{-1}a_{n-1}^{-1}........a_1^{-1}$

Theorem 6 :

Cancellation laws hold in a group, i.e. If (G, .) is a group then

I. a, b, c $\in$ G, $\quad ab = ac$
$\Rightarrow \quad b = c$

II. a, b, c, $\in$ G, $\quad ba = ca$
$\Rightarrow \quad b = c.$

Definition 4.

If (G, .) be a gorup and $a \in G, n \in z$, then a^n is defined as follows :

I. $a^0 = e$

II. If $n > 0$, then $\quad a^1 = a; a^{n+1} = a^n . a$

III. If $n < 0$, then $\quad a^n = (a^{-n})^{-1}$

Theorem 7.

Let (G, .) be a group and $a \in G$.

If m, n $\in$ Z, then

I. $a^m . a^n = a^{m+n} = a^na^m$

II. $\left(a^m\right)^n = a^{mn}$

Definition 5.

Let (G, .) be a group. An element $a \in G$ is called idempotent if $a^2 = a$.

Definition 6.

A group (G, .) is called finite group if G is a finite set The number of different elements in G is callled order of the fininte group (G, .). It is denoted by O(G).

Definition 7.

A group (G, .) is called infinite group if G is an infinite set. The order of an infinite group is defined to be ∞.

Definition 8.

If S is a finite set containing n elements, then group of all bijections on S is called a permutation group or symmetric group. It is denoted by P_n or S_n.

Note : $0(S_n) = n!$

LATTICE

It is a partially ordered set $(P, \leq)$ in which any two elements $\in$ P has single GLB and single LUB.

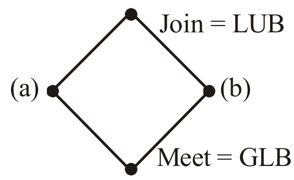


Fig. (a)

In the Fig. (a) GLB and LUB are shown, which are also called meet and join respectively.

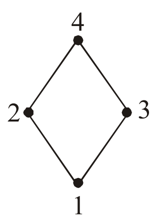
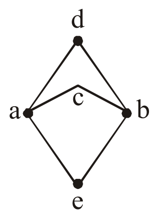


Fig. (b) Lattice Fig.



(c) Non-Lattice

The poset shown in Fig. (c) is not a lattice since a and b have two LUBs namely c and d.

From first lattice, following relations are obtained :

$1 \le 2$, $1 \le 3$, $2 \le 4$ and $3 \le 4$.

Here 2 and 3 are not compared.

If $(P, \le)$ is a lattice, then $(P, \ge)$ is also a lattice, we define as $\ge$ follows

$(x \le y) \Rightarrow y \ge x$.

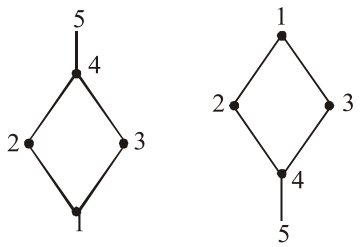
In other words, GLB and LUB are interchanged if we interchange relations $\le$ and $\ge$.

In terms of lattices, operations of meet and join on $(L, \le)$ become the operations of join and meet on $(L, \le)$.

DUAL LATTICE

For the lattice $(P, \le)$ the dual is $(P, \ge)$.

The duals are shown in the figure below. The diagram of $(P, \ge)$ can be obtained from that of $(P, \le)$ by simply turning it upside down.



$1 \le 2 \le 4 \le 5$ $\quad$ $5 \ge 4 \ge 2 \ge 1$

$1 \le 3 \le 4 \le 5$ $\quad$ $5 \ge 4 \ge 3 \ge 1$

Let $a * b$ = meet of a and b = GLB, and $a \oplus b$ = join of a and b = LUB

SUB-LATTICE.

Let $(L, *, \oplus)$ be a lattice and let $S \subseteq L$. The set $(S, *, \oplus)$ is called sublattice iff it is closed under * and $\oplus$. Sublattice is itself a lattice.

Closed Interval

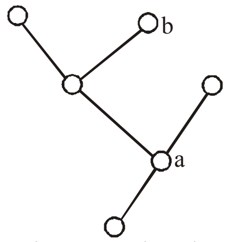


Fig. Interval [a, b]

Let $a \le b$.
Then closed interval of a and b is defined as
$[a, b] = [x / a \le x \text{ and } x \le b]$.
Clearly any closed interval is a chain.

Duality

The dual of a Boolean expression is obtained by interchanging Boolean sums and Boolean products and interchanging 0's and 1's.
e.g., The dual of $x \wedge (y \vee 0)$ is $x \vee (y \wedge 1)$

Boolean Algebra (Definition)

1. $\left.\begin{array}{l} x \vee 0 = x \\ x \wedge 1 = x \end{array}\right\}$ Identity laws
2. $\left.\begin{array}{l} x \vee x' = 1 \\ x \wedge x' = 0 \end{array}\right\}$ Domination laws
3. $\left.\begin{array}{l} (x \vee y) \vee z = x \vee (y \vee z) \\ (x \vee y) \wedge z) = x \wedge (y \wedge z) \end{array}\right\}$ Associative laws
4. $\left.\begin{array}{l} x \vee y = y \vee x \\ x \wedge y = y \wedge x \end{array}\right\}$ Commutative laws
5. $\left.\begin{array}{l} x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \end{array}\right\}$ Distributive laws

Boolean Algebra Homomorphism

If (A, +, ., ', 0, 1) and $(B, \wedge, \vee, -, 0', 1')$ are two Boolean algebras, a function $h : A \to B$ is called a Boolean algebra homomorphism if h preserves the two binary operation and the unary operations in the following since, for all $a, b \in A$

(a) $h(a + b) = h(a) \vee h(b)$

(b) $h(a \cdot b) = h(a) \wedge h(b)$

(c) $h(a') = h'(a)$

A Boolean homomorphism $h : A \to B$ is a Boolean isomorphism if h is one-to-one onto B.

Combinatorics

FUNDAMENTAL PRINCIPLES OF COUNTING

1. **Principle of Addition :** If an event can occur in 'm' ways and another event can occur in 'n' ways independent of the first event, then either of the two events can occur in (m + n) ways.
2. **Principle of Multiplication :** If an operation can be performed in 'm' ways and after it has been performed in any one of these ways, a second operation can be performed in 'n' ways, then the two operations in succession can be performed is $(m \times n)$ ways.

^nP_r and ^nC_r : If $n \in \mathbf{N}$ and 'r' is an integer such that $0 \le r \le n$, then we define the following symbols :

(i) $P(n, r)$ or $^nP_r = \dfrac{n!}{(n-r)!}$

(ii) $C(n, r)$ or $^nC_r = \dfrac{n!}{r!(n-r)!}$. The symbol ^nC_r is also deonoted by $\dbinom{n}{r}$.

IMPORTANT RESULTS

1. $^nP_0 = 1$, $^nP_n = n!$; $^nC_0 = 1$, $^nC_n = 1$
2. $^nP_r = r!\,^nC_r$

3. $^{n}P_{r} = r.^{n-1}P_{r-1} + ^{n-1}P_{r} = n.^{n-1}P_{r-1}$
4. If $r \le s \le n$, then $^{n}P_{s}$ is divisible by $^{n}P_{r}$.
5. $^{n}C_{r} = \frac{n}{r}\ ^{n-1}C_{r-1}$ that is, $r.^{n}C_{r} = n.^{n-1}C_{r-1}$ and $\frac{^{n}C_{r}}{r+1} = \frac{^{n+1}C_{r+1}}{n+1}$
6. $n.^{n-1}C_{r-1} = (n-r+1)\ ^{n}C_{r-1}$
7. $^{n}C_{r-1} + ^{n}C_{r} = ^{n+1}C_{r}$
8. $^{n}C_{r} = ^{n}C_{n-r}$
9. $^{n}C_{x} = ^{n}C_{y} \Leftrightarrow x = y$ or $x + y = n$
10. $^{n}C_{0} + ^{n}C_{1} + ^{n}C_{2} + ... + ^{n}C_{n} = 2^{n}$
11. $^{n}C_{0} + ^{n}C_{2} + ... = ^{n}C_{1} + ^{n}C_{3} + ... = 2^{n-1}$
12. $^{2n+1}C_{0} + ^{2n+1}C_{1} + ... ^{2n+1}C_{n} = 2^{2n}$

PERMUTATIONS

Each of the arrangements, which can be made by taking, some or all of a number of things is called a PERMUTATION.

RESULT - 1 : To find the number of permutations of 'n' things taken 'r' at a Time :

The number of ways of filling 'r' places with 'n' things

$= n(n-1)(n-2)....(n-r+1)$ ways $= \frac{n!}{[n-r]!}$ ways $= ^{n}P_{r}$ ways.

The above formula for $^{n}P_{r}$ involves following conditions :
1. All the things are distinct.
2. Repetition of things is not allowed in any of the arrangements.
3. No arrangement is repeated.
4. The arrangement is linear.

RESULT - 2 : The number of permutations of 'n' things taken all at a time.

This will be given by above formula after taking r = n.

Thus, required number of ways = $^{n}P_{n} = n!$

RESULT - 3 : To find the number of permutations of 'n' things taken all at a time, when 'p' are alike of one kind, 'q' are alike of Second, 'r' alike of Third, and so on :

Let 'x' be the required number of permutations.

If p alike things are replaced by p distinct things, which are also different from others, then without changing the positions of other things these new p-things can be arranged in p! ways.

Each of 'x' permutations will give p! permutations. Thus the total number of permutations now are x (p!)

With a similar argument for 'q' - alike and 'r' - alike things, we get that if all things are different the number of permutations would be x(p!) (q!) (r!)

But number of permutations of 'n' distinct things; taken all at a time = $^{n}P_{n} = n!$, thus, $(x)\, p!\, q!\, r! = n! \Rightarrow x = \frac{n!}{p!\, q!\, r!}$

RESULT - 4 : To find the number of Permutations of 'n' different things, taking 'r' at a time, when each thing can be repeated 'r' times:

In the problem we have to fill 'r' vacant places with 'n' things with repetition. Obviously, each place can be filled in 'n' ways, leaving again n ways for the other place.

Hence, the number of ways of filling r-places with 'n' things $= n \times n \times n \times .... \times n$ (r factors) $= n^{r}$

RESULT - 5 : Number of circular permutations of 'n' distinct objects :

The total number of circular permutations of 'n' distinct things is $(n-1)!$.

If no distinction is made between anti-clockwise and clockwise arrangements, then the number of permutations is $\frac{1}{2}(n-1)!$

CONDITIONAL PERMUTATIONS

1. Number of permutations of n things taking r at a time, when a particular object is to be always included in each $= r.^{n-1}P_{r-1}$.
2. Number of permutations of n things taking r at a time, when a particular object is never taken in any arrangement $= ^{n-1}P_{r}$.
3. Number of permutations of n different things taking all at a time, when m specified things always come together $= m!(n-m+1)!$.
4. Number of permutations of n different things taking all at a time, when m specified things never come together $= n! - m!(n-m+1)!$
5. Number of permutations of n different things taking r at a time, in which two specified objects always occur together $= 2!\,(r-1)\ ^{n-2}P_{r-2}$
6. Number of ways of arranging n objects on a circle taking r at a time

   $= \frac{^{n}P_{r}}{r}$, if clockwise and anticlockwise arrangements are distinct

   $= \frac{^{n}P_{r}}{2r}$, if clockwise and anticlockwise arrangements cannot be distinguished.

COMBINATIONS :

Each of the selections that can be made with a given number of objects taken some or all of them at a time is called a COMBINATION.

RESULT-1 : To find the number of combinations of 'n' dissimilar things taken 'r' at a time :

$$\frac{n!}{r!(n-r)!} = ^{n}C_{r}$$

Thus, the total number of combinations of 'n' dissimilar things taken 'r' at a time is $^{n}C_{r}$.

The number of combinations of 'n' dissimilar things taken all at a time $= ^{n}C_{n} = 1$.

RESULT-2 : To find the number of Combinations of 'n' different things taking some or all at a time :

$$^nC_1 + {^nC_2} + {^nC_3} + \ldots\ldots\ldots + {^nC_n} = 2^n - 1$$

RESULT-3 : The number of selections of some or all out of (p + q + r + ...) things out of which p are alike of one kind, q - alike of second kind and so on :

The total number of required ways

$$= (p + 1)(q + 1)(r + 1) \ldots\ldots - 1$$

RESULT-4 : The number of selections of one or more things from 'p' identical things of one kind, 'q' identical things of a second kind, 'r' identical things of a third kind and 'n' different things

The total number of required ways

$$= (p + 1)(q + 1)(r + 1)\, 2^n - 1$$

COUNTING

Basic Counting Rules.

There are two basic counting rules which can be used to solve many different counting problems.

(i) Sum Rule : If first task can be done in n_1 ways and a second task in n_2 ways, and if these tasks are such that they cannot be done at the same time, then there are $n_1 + n_2$ ways to do either task.

(ii) Product Rule : Whenever a procedure can be broken down into two tasks and then there are n_1 ways to do the first task and n_2 ways to do the next task after the first task has been done, then there are $n_1 n_2$ ways to do the procedure.

GENERATING FUNCTION

The generating function for the sequence $a_1, a_2, \ldots, a_r, \ldots$ of real numbers of a numeric functions $(a_0, a_1, a_2, \ldots a_r, \ldots)$ is the infinite series.

$$A(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + \ldots$$

$$A(z) = \sum_{r=1}^{n} a_r z^r$$

Some Results

1. Let a and b are any two discrete numeric functions and $b = \alpha a$, then

$$B(z) = \alpha A(z)$$

where B(z) is the generating function corresponding to numeric b and A(z) is that of a.

e.g., The generating function of the numeric function

$$a_r = 5 \cdot 2^r \quad r \geq 0$$

is $$A(z) = 5 \cdot \frac{1}{1 - 2z}$$

2. For any three discrete numeric functions a, b and c, if c = a + b, then C(z) = A(z) + B(z) is the generating function representation.

e.g., The generating function of the numeric function

$$a_r = 1^r + 2^r \ (r \geq 0)$$

is $$A(z) = \frac{1}{1-z} + \frac{1}{1-2z}$$

3. Let a be a numeric function and A(z) its generating function. Let b be a numeric function such that

$$b_r = \alpha^r a_r$$

Then, the generating function of *b* is

$$B(z) = \sum_{r=0}^{\infty} b_r z^r = \sum_{r=0}^{\infty} (\alpha^r a_r) z^r$$

$$= \sum_{r=0}^{\infty} a_r (\alpha z)^r = A(\alpha z)$$

$B(z) = A(\alpha z)$

e.g., The generating function of the numeric function

$a_r = 1, r \geq 0$ is $A(z) = \dfrac{1}{1-z}$

4. Let *A*(*z*) be the generating function of a. Then, z^1 A(z) is the generating function of $S^i a$ for any positive integer i.

5. Let *A*(*z*) be the generating function of a. Then,

$z^{-i}[A(z) - a_0 - a_1 z - a_2 z^2 - \ldots - a_{i-1} z^{i-1}]$

is the generating function of S^{-i} a.

e.g., The generating function of $a_r = 3^{r+2}, z \geq 0$ is

$$A(z) = z^{-2}\left(\frac{1}{1-3z} - 1 - 3z\right) = z^{-2}\left(\frac{9z^2}{1-3z}\right) = \frac{9}{1-3z}$$

6. For $b = \Delta a$, the generating function is given by

$$B(z) = \frac{1}{z}[A(z) - a_0] - A(z)$$

and for $b = \nabla a$

$$B(z) = A(z) - z A(z)$$

7. Let C = a*b, i.e., C is the convolution of two numeric functions and its generating function

$$C(z) = A(z) \cdot B(z)$$

$$C_r = a_0 b_r + a_1 b_{r-1} + a_2 b_{r-2} + \ldots + a_{r-1} b_1 + a_r b_0$$

is the coefficient of z^r in the product of

$$(a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + \ldots)(b_0 + b_1 z + b_2 z^2 + \ldots + b_r z^r + \ldots)$$

RECURRENCE RELATIONS

A recurrence relation for the sequence $\{a_r\}$ is an equations that expresses a_r in terms of one or more of the previous terms of the sequence.

A sequence is called a solution of a recurrence if its terms satisfy the recurrence relation.

e.g., The recurrence relation $a_r = a_{r-1} + 3$ with initial condition $a_1 = 2$ defines the sequence {2, 5, 8, 11...}.

Order of recurrence relation : The order of a recurrence relation (or difference equation) is the difference between the largest and smallest subscript appearing in the relation.

e.g., $a_r = a_{r-1} + a_{r-2}$ is a recurrence relation of order 2.

Degree of the recurrence relation : The degree of a recurrence relation is the highest power of a_r occuring in that relations.

Example $a_r^3 + 3a_{r-1}^2 + 6a_{r-2}^4 + 4a_{r-3}$ is a recurrence relation of degree 3.

Linear recurrence relation with constant coefficients : A recurrence relation of the form.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + C_3 a r_{r-3} + \ldots + C_k a_{r-k} = f(r) \quad \ldots\text{(i)}$$

Where, C_i's are constants is called a linear recurrence relation with constant coefficients of *k*th order, provided C_0 and C_k both are non-zero. f(r) is the function of the independent variable 'r' only.

e.g., $3a_r + 6a_{r-1} = 2^r$ is the first order linear recurrence relation with constant coefficients.

A recurrence relation is said to be linear if it's degree is one.

Homogeneous Solution of the Recurrence Relation

A homogeneous solution of a linear difference equation with constant coefficients is of the form $A\alpha_1^r$, where α_1 is called a characteristic root and A is a constant determined by the bounded conditions. Consider a recurrence relation in the form

$$C_0a_r + C_1a_{r-1} + C_2a_{r-2} + ... + ..C_ka_{r-k} = 0 \quad ...(i)$$

Since, right hand side of Eq. (i) is set to zero, we substitute $A\alpha^r$ for a_r. Eq. (i) become

$$C_0A\alpha^r + C_1A\alpha^{r-1} + C_2A\alpha^{r-2} + .. + C_kA\alpha^{r-k} = 0$$

$$A\alpha^{r-k}[C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + ... + C_k] = 0$$

$$C_0\alpha^k + C_1\alpha^{k-1} + C_2\alpha^{k-2} + ... + C_k = 0 \quad ...(ii)$$

Eq. (ii) is called characteristic equation. The solutions of this equaiton are called the characteristic roots of the recurrence relation.

A characteristic equation of kth degree has *k* characteristics roots. Two cases of the roots may arise.

(i) If roots are distinct and real. Then

$$a_r^{(h)} = A_1\alpha_1^r + A_2\alpha_2^r + ... + A_k\alpha_k^r$$

(ii) If the roots are multiple roots. Let α_1 be a root of multiplying m, then

$$(A_1r^{m-1} + A_2r^{m-2} + ... + A_{m-1}r + A_m)\alpha_1^r$$

is a homogeneous solution.

Note : Although, there is no question asked in past GATE exams from this chapter still, reading of this chapter would be helpful in future GATE exams.

Graph Theory

PATHS, CONNECTIVITY

Path and its length in a graph (multigraph) G : A path α in G with origin v_0 and end v_n is an alternating sequence of vertices and edges of the form.

$$v_0, e_1, v_1, e_2, v_2, ..., e_{n-1}, v_{n-1}, e_n, v_n$$

where each edge e_i is incident on vertices v_{i-1} and v_i. The number of edges, n is called length of α. When there is no ambiguity, we denote α by its sequence of edges, $\alpha = (e_1, e_2, ... e_n)$, or by its sequence of vertices $\alpha = (v_o, v_1, ..., v_n)$.

BRIDGE OR CUT EDGE

On removing an edge from a graph, the number of connected components of the graph either remains unchanged or it increases by exactly 1. So, an edge e of a graph G is called bridge or cut edge, if the subgraph G=e has more connected components than G has.

Theorem 1 :

Let *e* be an edge of the graph G and, as usual let G-e be the subgraph obtained by deleting e.

Then $\quad \omega(G) \le \omega(G-e) \le \omega(G)+1$

where ω(a) is number of connected components of G.

Theorem 2 :

An edge *e* of a grpah G is a bridge if *e* is not a part of any cycle in G.

Theorem 3 :

Let G be a graph with n vertices, and q edges and, let ω(G) denote number of connected components of G. Then G has at least n-ω(G) edges, i.e. $q \ge n-\omega(G)$

Corollary : A graph with *n* vertices less than (n – 1) edges can not be connected.

Theorem 4 :

Let G be a graph with *n* vertices, then following three statements are equivalent.

I. G is tree

II. G is an acyclic graph with (*n* – 1) edges.

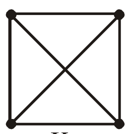
III. G is a connected graph with (*n* – 1) edges.

Spanning Trees and Connect or Problems

Let G be a graph. A subgraph H of G is called a spanning subgraph of G if vertex set of H is same as the vertex set of G. Similarly, a spanning tree of a graph G is a spanning subgraph of G, that is a tree.

Theorem.

A graph G is connected if it has a spanning tree.



K_4

Cayley's theorem

The complete graph K_n has n^{n-2} different spanning trees.

Kruskal's Algorithm

In this algorithm, choose an edge of G which has smallest weight among the edges of G which are not loops. This algorithm gives an acyclic subgraph T of G and theorem given below proves that T is a minimal spanning tree of G.

Following steps are required.

Step 1 : Choose e_1, an edge of G, such that weight of e_1, $\omega(e_1)$ is as small as possible and e_1 is not a loop.

Step 2 : If edges $e_1, e_2, ... e_i$ have been chosen, then choose an edge e_{i+1} not already chosen, such that

(i) induced subgraph G $[(e_1,.....e_{i+1})]$ is acyclic, and

(ii) $\omega(e_{i+1})$ is as small as possible.

Step 3 : If G has n vertices, stop after n-1 edges have been chosen. Otherwise repeat step 2.

Greedy Algorithms :

Greedy algorithms are essentially algorithms that proceed by selecting the choice that looks best at the moment.

Prim's Algorithm :

Another algorithm used for finding a minimal spanning tree is Prim's algorithm. It chooses a vertex first and chooses and edge with smallest weight incident on that vertex.

The algorithm involves following steps.

Step 1 : Choose any vertex v_1 of G.

Step 2 : Choose an edge $e_1=v_1v_2$ of G such that $v_2 \neq v_1$ and e_1 has smallest weight among the edges of G incident with v_1.

Step 3 : If edges $e_1, e_2, \ldots, e_i$ have been chosen involving end points $v_1, v_2, \ldots v_{i+1}$. Choose an edge $v_{i+1} = v_jv_k$ with $v_j \in \{v_i \ldots v_{i+1}\}$ and $v_k \notin [v_1, \ldots v_{i+1}]$ such that e_{i+1} has smallest weight among the edges of G with precisely one end in $[v_1, \ldots v_{i+1}]$.

Step 4 : Stop after n – 1 edges have been chosen. Otherwise go to step 3.

CUT VERTICES AND CONNECTIVITY

Cut vertex is analogue of a bridge. A vertex v of a graph is called a cut vertex of G if

$$\omega(G-v) > \omega(G),$$

Where ω(G) are number of components in graph G, i.e. a cut vertex breaks a graph into a subgraph having more connected components, then G has

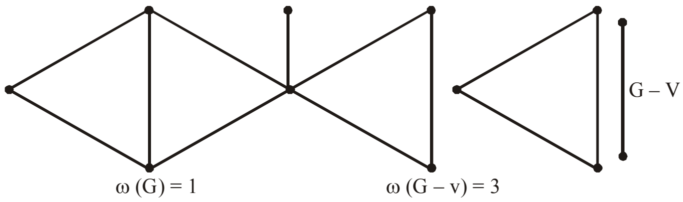


Fig. Graph after deleting cut vertex v

ISOMORPHISM OF GRAPHS

Let G(V, E) and G*(V*, E*) are graphs and $f: V \to V^*$ is a one-to-one correspondence beta sets of vertices such that [u, v] is an edge of G if and only if {f (u), f(v)} is an edge of G*. Then *f* is isomorphism between G and G*, and G and G* are called isomorphic graphs. Normally, we do not distinguish between isomorphic graphs (even though their diagrams may "look different").

MATCHINGS

Maximal Maching

It is a matching to which no edge in the graph can be added.

e.g. in a complete graph of three vertices (i.e. a triangle) any single edge is a maximal matching. The edges shown by heavy lines.

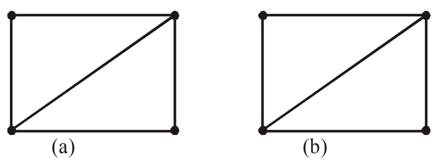


Fig. Graph and two of its maximal matchings.

Clearly, a graph may have many different maximal matchings and of different sizes. Among these, the maximal matchings with the largest number of edges are called largest maximal matchings.

In Fig. (b), a largest maximal matching is shown in heavy lines. The number of edges in a largest maximal matching called *matching number of the graph.*

COVERINGS

In a graph G a set g of edges is said to cover G if every vertex in G is incident on at least one edge in g. A set of edges that covers a graph G is called edge covering, covering subgraph, or simply a covering of G.

e.g. a graph G is trivially its own covering. Spanning tree in a connected graph (or a spanning forest in an unconnected graph) is another covering. Hamiltonian circuit (if it exists) in a graph is also a covering.

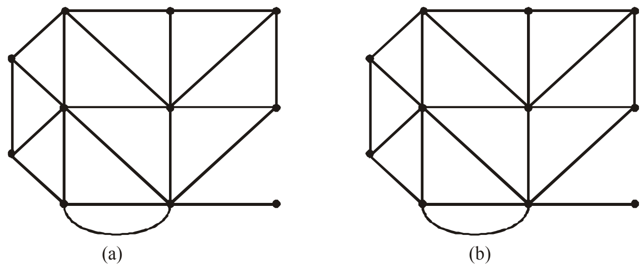


Fig. Graph and two of its minimal coverings

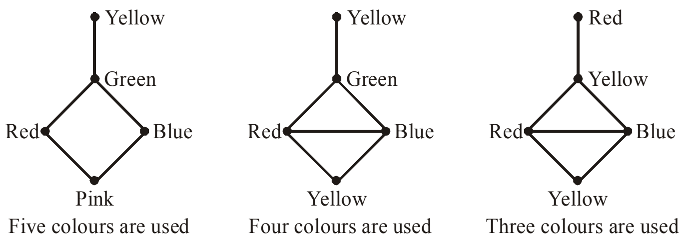
Detection of Planarity

Every subgraph of a planar graph is planar and that every graph which has non-planar graph is also non-planar. A disconnected graph is planar iff each of its components is planar. Similarly, in a separable graph, planarity of each block can be considered independently. Thus, a separable graph is planar iff each of its block is planar. Thus, we need to consider only non-separable connected graph.

Colouring of Graphs

A colouring of a graph G is an assignment of colours to its vertices such that no adjacent vertices have the same colour. A given graph can be properly coloured in many different ways. We are finding the minimum number of colours with which a given graph can be properly coloured.

e.g., There are three different proper colouring of a graph with different number of colours.



Question Exercise

Proportional and First Order Logic

1. The statement $(\neg p) \Rightarrow (\neg q)$ is logically equivalent to which of the statements below? **[2017, Set - 1, 1 Mark]**

   I. $p \Rightarrow q$ II. $q \Rightarrow p$

   III. $(\neg q) \vee p$ IV. $(\neg p) \vee q$

   (a) I only   (b) I and IV only

   (c) II only   (d) II and III only

2. Consider the first-order logic sentence $F: \forall_x (\exists_y R(x,y))$. Assuming non-empty logical domains. which of the sentences below are *implied* by F? **[2017, Set - 1, 1 Mark]**

   I. $\exists_y (\exists_y R(x, y))$ II. $\exists_y (\forall_x R(x, y))$

   III. $\forall_y (\exists_x R(x, y))$ IV. $\neg \exists_x (\forall_y \neg R(x, y))$

   (a) IV only   (b) I and IV only

   (c) II only   (d) II and III only

3. Let p, q, r denote the statements "*It is raining*". "It is *cold*", and "*It is pleasant*". respectively. Then the statement "*It is not raining and it is pleasant, and It is not pleasant only if it is raining and it is cold*" is represented by **[2017, Set - 2, 1 Mark]**

   (a) $(\neg p \wedge r) \wedge (\neg r \rightarrow (p \wedge q))$

   (b) $(\neg p \wedge r) \wedge (p \wedge q) \rightarrow \neg r)$

   (c) $(\neg p \wedge r) \vee (p \wedge q) \rightarrow \neg r)$

   (d) $(\neg p \wedge r) \vee (r \rightarrow (p \wedge q))$

4. If the ordinary generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ is $\frac{1+z}{(1-z)^3}$, then $(a_3 - a_0)$ is equal to ________.

   [2017, Set - 2, 2 Mark]

5. Let p, q, r, s represent the following propositions: **[2016, Set 1, 1 Mark]**

   p: $x \in \{8, 9, 10, 11, 12\}$

   q: x is a composite number

   r: x is a perfect square

   s: x is a prime number

   The integer $x \geq 2$ which satisfies $\neg((p \Rightarrow q) \wedge (\neg r \vee \neg s))$ is_____.

6. Let a_n be the number of n-bit strings that do NOT contain two consecutive 1s. Which one of the following is the recurrence relation for a_n? **[2016, Set 1, 1 Mark]**

   (a) $a_n = a_{n-1} + 2a_{n-2}$ (b) $a_n = a_{n-1} + a_{n-2}$

   (c) $a_n = 2a_{n-1} + a_{n-2}$ (d) $a_n = 2a_{n-1} + 2a_{n-2}$

7. The coefficient of x^{12} in $(x^3 + x^4 + x^5 + x^6 + ...)^3$ is _______.

   [2016, Set 1, 2 Mark]

8. A function $f: \mathbb{N}^+ \rightarrow \mathbb{N}^+$, defined on the set of positive integers $\mathbb{N}^+$, satisfies the following properties: **[2016, Set 1, 2 Marks]**

   $f(n) = f(n/2)$ **if** n is even

   $f(n) = f(n+5)$ **if** n is odd

   Let $R = \{i | \exists j : f(j) = i\}$ be the set of distinct values that f takes.The maximum possible size of R is_______.

9. Consider the following expressions: **[2016, Set 2, 1 Mark]**

   (i) false   (ii) Q

   (iii) true   (iv) $P \vee Q$

   (v) $\neg Q \vee P$

   The number of expressions given above that are logically implied by $\Rightarrow P \wedge (P \Rightarrow Q)$ is ________.

10. A binary relation R on $\mathbb{N} \times \mathbb{N}$ is defined as follows: $(a, b)R(c, d)$. If $a \leq c$ or $b \leq d$, consider the following propositions: **[2016, Set 2, 2 Marks]**

    P: R is reflexive

    Q: R is transitive

    Which one of the following statements is TRUE?

    (a) Both P and Q are true.

    (b) P is true and Q is false.

    (c) P is false and Q is true.

    (d) Both P and Q are false.

11. Which one of the following well-formed formulae in predicate calculus is NOT valid? **[2016, Set 2, 2 Marks]**

    (a) $(\forall x\, p(x) \Rightarrow \forall x\, q(x)) \Rightarrow (\exists x \neg p(x) \vee \forall x\, q(x))$

    (b) $(\exists x\, p(x) \vee \exists x\, q(x)) \Rightarrow \exists x\, (p(x) \vee q(x))$

    (c) $\exists x\, (p(x) \wedge q(x)) \Rightarrow (\exists x\, p(x) \wedge \exists x\, q(x))$

    (d) $\forall x\, (p(x) \vee q(x)) \Rightarrow (\forall x\, p(x) \vee \forall x\, q(x))$

12. Consider a set U of 23 different compounds in a Chemistry lab. There is a subset S of U of 9 compounds, each of which reacts with exactly 3 compounds of U. Consider the following statements: **[2016, Set 2, 2 Marks]**

    I. Each compound in U \ S reacts with an odd number of compounds.

    II. At least one compound in U \ S reacts with an odd number of compounds.

    III. Each compound in U \ S reacts with an even number of compounds.

    Which one of the above statements is **ALWAYS TRUE**?

    (a) Only I   (b) Only II

    (c) Only III   (d) None

13. The value of the expression 13^{99} (mod 17), in the range 0 to 16, is ______ . **[2016, Set 2, 2 Marks]**

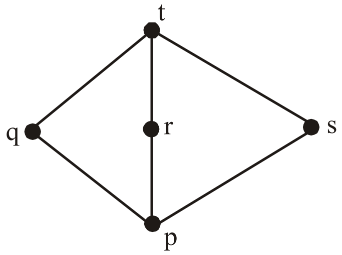
14. If $g(x) = 1 - x$ and $h(x) = \dfrac{x}{x-1}$, then $\dfrac{g(h(x))}{h(g(x))}$ is

[2015, Set 1, 1 Mark]

(a) $\dfrac{h(x)}{g(x)}$ (b) $\dfrac{-1}{x}$

(c) $\dfrac{g(x)}{h(x)}$ (d) $\dfrac{x}{(1-x)^2}$

15. Suppose $L = \{p,q,r,s,t\}$ is a lattice represented by the following Hasse diagram:



For any $x, y \in L$, not necessarily distinct, $x \vee y$ and $x \wedge y$ are join and meet of x, y respectively. Let $L^3 = \{(x,y,z): x,y,z \in L\}$ be the set of all ordered triplets of the elements of L. Let pr be the probability that an element $(x, y, z) \in L^3$ chosen equiprobably satisfies $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Then

[2015, Set 1, 2 Marks]

(a) $p_r = 0$ (b) $p_r = 1$

(c) $0 < p_r \le \frac{1}{5}$ (d) $\frac{1}{5} < p_r < 1$

16. For a set A, the power set of A is denoted by 2^A. If $A = \{5, \{6\}, \{7\}\}$, which of the following options are TRUE?

I. $\phi \in 2^A$ II. $\phi \in 2^A$

III. $\{5, \{6\}\} \in 2A$ IV. $\{5\{6\}\} \subseteq 2A$

[2015, Set 1, 1 Mark]

(a) I and III only (b) II and III only

(c) I, II and III only (d) I, II and IV only

17. Let a_n represent the number of bit strings of length n containing two consecutive 1s. What is the recurrence relation for a_n? **[2015, Set 1, 2 Marks]**

(a) $a_{n-2} + a_{n-1} + 2^{n-2}$ (b) $a_{n-2} + 2a_{n-1} + 2^{n-2}$

(c) $2a_{n-2} + a_{n-1} + 2^{n-2}$ (d) $2a_{n-2} + 2a_{n-1} + 2^{n-2}$

18. The cardinality of the power set of {0, 1, 2,…, 10} is _______.

19. If p, q, r, s are distinct integers such that: **[2015, Set 2, 1 Mark]**

f(p, q, r, s) = max(p, q, r, s)

g(p, q, r, s) = min(p, q, r, s)

h(p, q, r, s) = remainder of (p×q)/(r×s) if

(p × q) > (r × s) or remainder of (r×s)/(p×q) if (r×s) > (p×q)

Also a function fgh(p, q, r, s) = f(p, q, r, s) × g(p, q, r, s) × h(p, q, r, s)

Also the same operations are valid with two variable functions of the form f(p, q).

What is the value of fg(h(2,5,7,3), 4,6,8)?

20. Consider the following two statements. **[2015, Set 2, 1 Mark]**

S1: If a candidate is known to be corrupt, then he will not be elected

S2: If a candidate is kind, he will be elected.

Which one of the following statements follows from S1 and S2 as per sound inference rules of logic?

(a) If a person is known to be corrupt, he is kind

(b) If a person is not known to be corrupt, he is not kind

(c) If a person is kind, he is not known to be corrupt

(d) If a person is not kind, he is not known to be corrupt

21. Let R be the relation on the set of positive integers such that aRb if and only if a and b distinct and have a common divisor other than 1. Which one of the following statements about R is true? **[2015, Set 2, 1 Mark]**

(a) R is symmetric and reflexive but not transitive

(b) R is reflexive but not symmetric and not transitive

(c) R is transitive but not reflexive and not symmetric

(d) R is symmetric but not reflexive and not transitive

22. Suppose U is the power set of the set $S = \{1, 2, 3, 4, 5, 6\}$. For any $T \in U$, let $|T|$ denote the number of elements in T and T^1 denote the complement of T. For any $T, R \in U$, let $T \setminus R$ be the set of all elements in T which are not in R. Which one of the following is true?

(a) $\forall X \in U (|X| = |X'|)$

(b) $\exists X \in U \exists Y \in U (|X| = 2, |Y| = 5 \text{ and } X \cap Y = 0)$

(c) $\forall X \in U \forall Y \in U (|X| = 2, |Y| = 3 \text{ and } X / Y = 0)$

(d) $\forall X \in U \forall Y \in U (X/Y = Y'/X')$

23. The CORRECT formula for the sentence, "not all rainy days are cold" is **[2014, Set-3, 2 Marks]**

(a) $\forall d\, (\text{Rainy}(d) \wedge \sim \text{Cold}(d))$

(b) $\forall d\, (\sim \text{Rainy}(d) \rightarrow \text{Cold}(d))$

(c) $\exists d\, (\sim \text{Rainy}(d) \rightarrow \text{Cold}(d))$

(d) $\exists d\, (\text{Rainy}(d) \wedge \sim \text{Cold}(d))$

24. Which one of the following propositional logic formulas is TRUE when exactly two of *p*, *q*, and *r* are **TRUE**?

[2014, Set-1, 2 Marks]

(a) $((p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$

(b) $(\sim(p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$

(c) $((p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$

(d) $(\sim(p \leftrightarrow q) \wedge r) \wedge (p \wedge q \wedge \sim r)$

25. Consider the statement

"Not all that glitters is gold"

Predicate *glitters* (*x*) is true if *x glitters* and predicate *gold* (*x*) is true if *x* is gold. Which one of the following logical formulae represents the above statement?

[2014, Set-1, 1 Mark]

(a) $\forall x : glitters(x) \Rightarrow \neg gold(x)$

(b) $\forall x : gold(x) \Rightarrow glitters(x)$

(c) $\exists x : gold(x) \wedge \neg glitters(x)$

(d) $\exists x : glitters(x) \wedge \neg gold(x)$

26. Consider the following statements:
P: Good mobile phones are not cheap
Q: Cheap mobile phones are not good
L: P implies Q
M: Q implies P
N: P is equivalent to Q
Which one of the following about L, M, and N is **CORRECT**? **[2014, Set-3, 1 Mark]**
(a) Only L is TRUE.
(b) Only M is TRUE.
(c) Only N is TRUE.
(d) L, M and N are TRUE.

27. Which one of the following Boolean expressions is NOT a tautology? **[2014, Set-2, 2 Marks]**
(a) $((a \to b) \wedge (b \to c)) \to (a \to c)$
(b) $(a \leftrightarrow c) \to (\sim b \to (a \wedge c))$
(c) $(a \wedge b \wedge c) \to (c \vee a)$
(d) $a \to (b \to a)$

Proportional and First Order Logic

28. Let $G(x) = 1/(1-x)^2 = \sum_{i=0}^{\infty} g(i)x^i$, where $|x| < 1$. What is g(i)? **[2005, 2 Marks]**
(a) i (b) i + 1
(c) 2i (d) 2^i

29. Identify the correct translation into logical notation of the following assertion. Some boys in the class are taller than all the girls.
Note Taller (x, y) is true, if x is taller than y. **[2004, 1 Mark]**
(a) $(\exists x)$ (boy (x) $\to$ $(\forall y)$ (girl (y) $\wedge$ taller (x, y)))
(b) $(\exists x)$ (boy (x) $\wedge$ $(\forall y)$ (girl (y) $\wedge$ taller (x, y)))
(c) $(\exists x)$ (boy (x) $\to$ $(\forall y)$ (girl (y) $\to$ taller (x, y)))
(d) $(\exists x)$ (boy (x) $\wedge$ $(\forall y)$ (girl (y) $\to$ taller (x, y)))

30. The following propositional statement is
$(P \to (Q \vee R)) \to ((P \wedge Q) \to R)$ **[2004, 2 Marks]**
(a) satisfiable but not vaild
(b) valid
(c) a contradiction
(d) None of these

31. The inclusion of which of the following sets into S = {{1, 2}, {1, 2, 3}, {1, 3, 5}, {1, 2, 4}, {1, 2, 3, 4, 5}} is necessary and sufficient to make S a complete lattice under the partial order defined by set containment? **[2004, 2 Marks]**
(a) {1}
(b) {1}, {2, 3}
(c) {1}, {1, 3}
(d) {1}, {1, 3}, {1, 2, 3, 4}, {1, 2, 3, 5}

32. The following is the incomplete operation table of a 4-element group

*	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b				
c				

The last row of the table is **[2004, 2 Marks]**
(a) c a e b (b) c b a e
(c) c b e a (d) c e a b

33. How many graphs on n labelled vertices exist which have at least $(n^2 - 3n)/2$ edges? **[2004, 2 Marks]**
(a) ${}^{(n^2-n)/2}C_{(n^2-3n)/2}$ (b) $\sum_{k=0}^{(n^2-3n)/2} {}^{(n^2-n)}C_k$
(c) ${}^{(n^2-n)/2}C_n$ (d) $\sum_{k=0}^{n} {}^{\frac{(n^2-n)}{2}}C_k$

34. Which of the following is a valid first order formula? (Here α and β are first order formulae with x as their only free variable) **[2003, 2 Marks]**
(a) $(\forall x)[\alpha] \Rightarrow (\forall x)[\beta] \Rightarrow (\forall x)[\alpha \Rightarrow \beta]$
(b) $(\forall x)[\alpha] \Rightarrow [\exists x][\alpha \wedge \beta]$
(c) $(\forall x)[\alpha \vee \beta] \Rightarrow [\exists x][\alpha]\,(\forall x)[\alpha]$
(d) $(\forall x)[\alpha \Rightarrow \beta] \Rightarrow ((\forall x)[\alpha] \Rightarrow (\forall x)[\beta])$

35. Consider the following logic program P :
$A(x) \leftarrow B(x, y), C(y)$
$\leftarrow B(x, x)$
Which of the following first order sentences is equivalent to P? **[2003, 2 Marks]**
(a) $(\forall x)[\exists y][B(x, y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)[B(xx)]$
(b) $(\forall x)[\forall y][B(x, y) \wedge C(y)] \Rightarrow A(x)] \wedge \neg(\exists x)[B(xx)]$
(c) $(\forall x)[\exists y][B(x, y) \wedge C(y)] \Rightarrow A(x)] \vee \neg(\exists x)[B(xx)]$
(d) $(\forall x)[\forall y][B(x, y) \wedge C(y)] \Rightarrow A(x) \wedge \neg(\exists x)[B(xx)]$

36. Consider the following formula α and its two interpretations l_1 and l_2
$\alpha : (\forall x)[P_x \Leftrightarrow (\forall y)[Q_{xy} \Leftrightarrow \neg Q_{xy}]] \Leftrightarrow (\forall x)[\neg P_x]$
l_1 : domain : the set of natural numbers
$P_x \equiv$ x is a prime number
$Q_{xy} \equiv$ y divides x
l_2 : same as l_1 except that P_x = x is a composite number
Which of the following statements is true? **[2003, 2 Mark]**
(a) l_1 satisfies α, l_2 does not
(b) l_2 satisfies α, l_1 does not
(c) Neither l_1 nor l_2 satisfies α
(d) Both l_1 and l_2 satisfy α

37. The following resolution rule is used in logic programming
Derive clause $(P \vee Q)$ from clauses $(P \vee R)$, $(Q \vee \neg R)$
Which of the following statements related to this rule is false? **[2003, 2 Marks]**

(a) $(P \vee R) \wedge (Q \vee \neg R) \Rightarrow (P \vee Q)$ is logically valid
(b) $(P \vee Q) \Rightarrow (P \vee Q) \wedge (Q \vee \neg R)$ is logically valid
(c) $(P \vee Q)$ is satisfiable if and only if $(P \vee R) \wedge (Q \vee \neg R)$
(d) $(P \vee R) \Rightarrow$ False if and only if both P and Q are unsatisfiable

38. Let $\Sigma = \{a, b, c, d, e\}$ be an alphabet. We define an encoding scheme as follows g(a) = 3, g(b) = 5, g(c) = 7, g(d)= 9, g(e) = 11. Let P_i denotes the i th prime number ($p_i = 2$)
For a non-empty string $s = a_1....a_n$, where each $a_1 \in \Sigma$, define $f(s) = \prod_{i=1}^{n} p_i^{g(ai)}$.
For a non-empty sequence $<s_1 ...s_n>$ of strings from Σ^*.
define $h(<s_1....s_n>) = \prod_{i=1}^{n} p_i^{f(s_i)}$
Which of the following numbers is the encoding h of a non-empty sequence of strings? **[2003, 2 Marks]**
(a) $2^7\ 3^7\ 5^7$
(b) $2^8\ 3^8\ 5^8$
(c) $2^9\ 3^9\ 5^9$
(d) $2^{10}\ 5^{10}\ 7^{10}$

39. Consider the set Σ^* of all strings over the alphabet $\Sigma = \{0, 1\}$ Σ^* with the concatenation operator for strings **[2003, 1 Mark]**
(a) does not form a group
(b) forms a non-commutative group
(c) does not have a right identify element
(d) forms a group, if the empty string is removed from Σ^*

Sets & Relations,Functions, Partial orders, Latices & Groups

40. Consider the following relation on subsets of the set *S* of integers between 1 and 2014. For two distinct subsets U and V of S we say U < V if the minimum element in the symmetric difference of the two sets is in U.
Consider the following two statements:
S1: There is a subset of *S* that is larger than every other subset.
S2: There is a subset of *S* that is smaller than every other subset.
Which one of the following is CORRECT?
[2014, Set-2, 2 Marks]
(a) Both S1 and S2 are true
(b) S1 is true and S2 is false
(c) S2 is true and S1 is false
(d) Neither S1 nor S2 is true

41. Let X and Y be finite sets and $f : X \to Y$ be a function. Which one of the following statements is TRUE?
[2014, Set-3, 1 Mark]
(a) For any subsets A and B of X, $|f(A \cup B)| = |f(A)| + |f(B)|$
(b) For any subsets A and B of X, $f(A \cap B) = f(A) \cap f(B)$
(c) For any subsets A and B of X, $|f(A \cap B)| = \min\{|f(A)|, |f(B)|\}$
(d) For any subsets S and T of Y, $f^{-1}(S \cap T) = f^{-1}(S) \cap f^{-1}(T)$

42. Consider the set of all functions $f: \{0, 1, ..., 2014\} \to \{0, 1, ..., 2014\}$ such that $f(f(i)) = i$, for all $0 \le i \le 2014$. Consider the following statements:
P. For each such function it must be the case that for every i, f (i) = i.
Q. For each such function it must be the case that for some i, f (i) = i.
R. Each such function must be onto.
Which one of the following is CORRECT?
[2014, Set-3, 2 Marks]
(a) P, Q and R are true
(b) Only Q and R are true
(c) Only P and Q are true
(d) Only R is true

43. Let S denote the set of all functions f: $\{0, 1\}^4 \to \{0,1\}$. Denote by *N* the number of functions from *S* to the set {0,1}. The value of $\log_2 \log_2 N$ is ________ .
[2014, Set-1, 2 Marks]

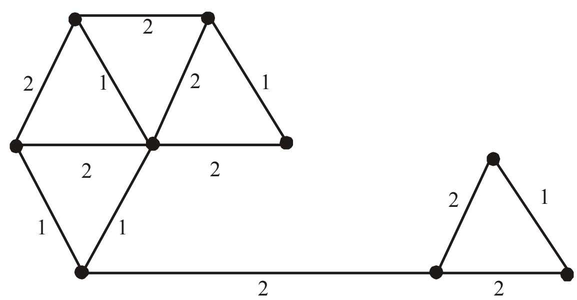
44. A pennant is a sequence of numbers, each number being 1 or 2. An n-pennant is a sequence of numbers with sum equal to n. For example, (1, 1, 2) is a 4-pennant. The set of all possible 1-pennants is {(1)}, the set of all possible 2-pennants is {(2), (1,1)}and the set of all 3-pennants is {(2,1), (1,1,1), (1,2)}. Note that the pennant (1,2) is not the same as the pennant (2,1). The number of 10-pennants is ______________. **[2014, Set-1, 2 Marks]**

45. If V_1 and V_2 are 4-dimensional subspaces of a 6-dimensional vector space V, then the smallest possible dimension of $V_1 \cap V_2$ is ______. **[2014, Set-3, 1 Mark]**

Graphs

46. The maximum number of edges in a bipartite graph on 12 vertices is ________________. **[2014, Set-2, 1 Mark]**

47. The number of distinct minimum spanning trees for the weighted graph below is _____ **[2014, Set-2, 2 Marks]**



48. A cycle on *n* vertices is isomorphic to its complement. The value of *n* is _____. **[2014, Set-2, 2 Marks]**

49. Let G be a group with 15 elements. Let L be a subgroup of G. It is known that $L \neq G$ and that the size of L is at least 4. The size of L is __________. **[2014, Set-3, 1 Mark]**

50. A binary operation $\oplus$ on a set of integers is defined as $x \oplus y = x^2 + y^2$. Which one of the following statements is true about $\oplus$? **[2013, 1 Mark]**

(a) Commutative but not associative
(b) Both commutative and associative
(c) Associative but not commutative
(d) Neither commutative nor associative

51. Which of the following statements is/are true for undirected graphs?

P: Number of odd degree vertices is even.

Q: Sum of degrees of all vertices is even.

[2013, 1 Mark]

(a) P only (b) Q only
(c) Both P and Q (d) Neither P nor Q

52. What is the correct translation of the following statement into mathematical logic? **[2012, 1 Mark]**

"Some real numbers are rational"?

(a) $\exists x$ (real) (x) v rational (x))
(b) $\forall x$ (real) (x) $\rightarrow$ rational (x))
(c) $\exists x$ (real) (x) $\wedge$ rational (x))
(d) $\exists x$ (rational (x) $\rightarrow$ real (x))

53. Consider the following logical inferences :

I_1: If it rains, then the cricket match will not be played.
The cricket match was played. **[2012, 1 Mark]**

Inference There was no rain.

I_2 : If it rains, then the cricket match will not be played.
It did not rain.

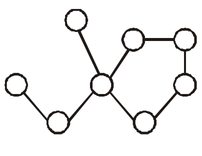
Inference The cricket match was played.

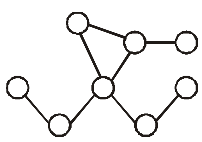
Which of the following is true?

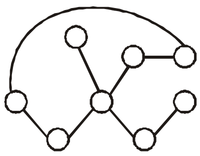
(a) Both I_1 and I_2 are correct inferences
(b) I_1 is correct but I_2 is not a correct inferences
(c) I_1 is not correct but I_2 is a correct inference
(d) Both I_1 and I_2 are not correct inferences

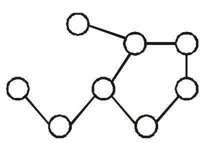
54. Which of the following graphs is isomorphic to given graph?

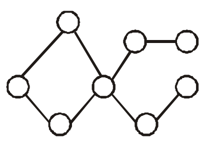
[2012, 2 marks]



(a) 

(b) 

(c) 

(d) 

55. Let G be a complete undirected graph on 6 vertices. If vertices of G are labeled, then the number of distinct cycles of length 4 in G is equal to **[2012, 2 Marks]**

(a) 15 (b) 30
(c) 90 (d) 360

56. Let G be a simple undirected planar graph on 10 vertices with 15 edges. If G is a connnected graph, then the number of bounded faces in any embedding of G on the plane is equal to **[2012, 1 Mark]**

(a) 3 (b) 4
(c) 5 (d) 6

57. K_4 and Q_3 are graphs with the following structure

Which one of the following statements is true in relation to these graphs? **[2011, 1 Mark]**

(a) K_4 is planar while Q_3 is not
(b) Both K_4 and Q_3 are planar
(c) Q_3 is planar while K_4 is not
(d) Neither K_4 nor Q_3 is planar

58. What is the possible number of reflexive relations on a set of 5 elements? **[2010, 1 Mark]**

(a) 2^{10} (b) 2^{15}
(c) 2^{20} (d) 2^{25}

59. Consider the set S = {1, ω, ω^2}, where ω and ω^2 are cube roots of unity. If * denotes the multiplication operation, the structure {S, *} forms **[2010, 1 Mark]**

(a) a group (b) a ring
(c) an integral domain (d) a field

60. Let $G = (V, E)$ be a graph. Define $\xi(G) = \sum_d i_d \times d$, where i_d is the number of vertices of degree d in G. If S and T are two different trees with $\xi(S) = \xi(T)$, then **[2010, 1 Mark]**

(a) $|S| = 2|T|$ (b) $|S| = |T| - 1$
(c) $|S| = |T|$ (d) $|S| = |T| + 1$

61. Let X, Y, Z be sets of sizes x, y and z respectively. Let W = X × Y and E be the set of all subsets of W. The number of functions from Z to E is **[2006, 1Mark]**

(a) z (b) $z \times 2^{xy}$
(c) z^2 (d) 2^{xyz}

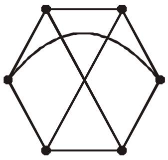
62. Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is **[2006, 1 Mark]**

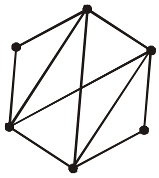
(a) 0 (b) 8
(c) 9 (d) 13

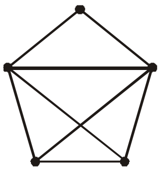
63. Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is **[2006, 1 Mark]**

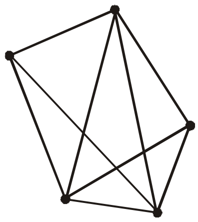
(a) 0 (b) 8
(c) 9 (d) 13

64. Which of the following graphs is not planar? **[2005, 1 Mark]**

(a) G_1 : 

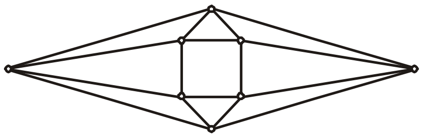
(b) G_2 : 

(c) G_3 : 

(d) 

65. Let G be a simple connected planar graph with 13 vertices and 19 edges. Then, the number of faces in the planar embedding of the graph is **[2005, 1 Mark]**
(a) 0 (b) 8
(c) 9 (d) 13

66. The minimum number of colours require to colour the following graph, such that no two adjacent vertices are assigned the same colour, is **[2004, 2 Marks]**



(a) 2 (b) 3
(c) 4 (d) 5

67. Let G be an arbitary graph with n nodes and k components. If a vetex is remvoed form G, the nubmer of components in the resultant graph must necessarily lie between **[2003, 1 Mark]**
(a) k and n (b) k – 1 and k + 1
(c) k – 1 and n – 1 (d) k + 1 and n – k

68. A graph G = (V, E) satisfies $|E| \le 3\,|V| - 6$. The min-degree of G is defined as $\min_{v \in V}$ {degree (v)}. Therefore, min-degree of G cannot be **[2003, 2 Marks]**
(a) 3 (b) 4
(c) 5 (d) 6

69. How many perfect matchings are there in a complete graph of 6 vertices? **[2003, 2 Marks]**
(a) 15 (b) 24
(c) 30 (d) 60

Combinatorics

70. In a binary tree with n nodes, every node has an odd number of descendants. Every node is considered to be its own descendent. What is the number of nodes in the tree that have exactly one child? **[2010, 2 Marks]**
(a) 0 (b) 1
(c) $\frac{n-2}{2}$ (d) $n-1$

71. Which one of the following is the most appropriate logical formula to represent the statement **[2009, 1 Mark]**
'Gold and silver ornaments are precious'?
The following notations are used
$G(x)$: x is a gold ornament
$S(x)$: x is a silver ornament
$P(x)$: x is precious
(a) $\forall x\,(P(x)) \rightarrow (G(x) \wedge S(x))$
(b) $\forall x\,(G(x)) \wedge S(x) \rightarrow P(x))$
(c) $\exists x\,((G(x) \wedge S(x)) \rightarrow P(x))$
(d) $\forall x\,(G(x) \rightarrow S(x)) \rightarrow P(x))$

72. The binary operation is defined as follows

P	Q	PQ
True	True	True
True	False	True
False	True	False
False	False	True

Which one of the following is equivalent to $P \vee Q$? **[2009, 1 Mark]**
(a) $\neg Q \vee \neg P$ (b) $P \vee \neg Q$
(c) $\neg P \vee Q$ (d) $\neg P \neg Q$

73. Which one of the following is not necessarily a property of a group? **[2009, 1 Mark]**
(a) Commutativity
(b) Associativity
(c) Existence of inverse for every element
(d) Existence of identity

74. Consider the binary relation $R = \{(x, y), (x, z), (z, x), (z, y)\}$ on the set $\{x, y, z\}$ which one of the following is true?
(a) R is symmetric but not anti-symmetric **[2009, 1Mark]**
(b) R is not symmetric but anti-symmetric
(c) R is both symmetric and anti-symmetric
(d) R is neither symmetric nor anti-symmetric

75. Which one of the following is true for any simple connected undirected graph with more than 2 vertices? **[2009, 2 Marks]**
(a) No two vertices have the same degree
(b) Atleast two vertices have the same degree
(c) Atleast three vertices have the same degree
(d) All vertices have the same degree

76. What is the chromatic number of an n-vertex simple connected graph which does not contain any odd length cycle? Assume $n \ge 2$ **[2009, 2 Marks]**
(a) 2 (b) 3
(c) $n-1$ (d) n

77. Let graph (x) be a predicate which denoted that x is a graph. Let graph (x) is connected. Which of the following first order logic sentences does not represent the statement 'Not every graph is connected'? **[2008, 1 Mark]**
(a) $\neg \forall$ x (graph (x) $\Rightarrow$ connected (x))
(b) $\exists$ x (graph (x) $\wedge \neg$ connected (x))
(c) $\neg \forall$ ($\neg$ graph (x)) $\vee$ (connected (x))
(d) $\forall$ x (graph (x)) $\Rightarrow \neg$ connected (x))

78. Let fsa and pda be two predicates such that fsa(x) means x is a finite state automation and pda(y), means y is a push down automation. Let equivalent be another predicate such that equivalent (a, b) means a and b are equivalents. Which of the following first order logic statements represents the following: **[2008, 1 Mark]**

(a) $(\forall x \text{ fsa}(x)) \Rightarrow (\exists y \text{ pda}(y) \wedge \text{equivalent}(x, y))$
(b) $\sim \forall y (\exists x \text{ fsa}(x) \Rightarrow (\exists y \text{ pda}(y) \wedge \text{equivalent}(x, y))$
(c) $\forall x \exists y (\text{fsa}(x) \wedge \text{pda}(y) \wedge \text{equivalent}(x, y))$
(d) $\forall x \exists y (\text{fsa}(x) \wedge \text{pda}(x) \wedge \text{equivalent}(x, y))$

79. P and Q two propositions. Which of the following logical expressions are equivalent? **[2008, 1 Mark]**

1. $P \vee \sim Q$
2. $\sim(\sim P \wedge Q)$
3. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$
4. $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

(a) 1 and 2 (b) 1, 2 and 3
(c) 1, 2 and 4 (d) All of 1, 2, 3 and 4

80. If P, Q, R are subsets of the universal set U, then $(P \cap Q \cap R) \cup (P^c \cap Q \cap R) \cup Q^c \cup R^c$ is **[2008, 1 Mark]**

(a) $Q^c \cup R^c$
(b) $P \cup Q^c \cup R^c$
(c) $P^c \cup Q^c \cup R^c$
(d) U

81. Which one of the first order predicate calculus statements given below, correctly expresses the following english statement? **[2007, 1 Mark]**

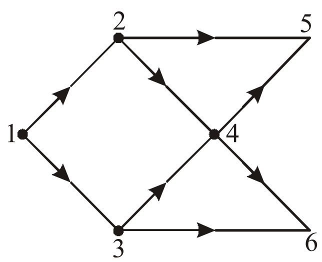
Tiger or lions attack, if they are hungry or threatened.

(a) $\forall \times [(\text{tiger}(\times) \wedge \text{lion}(\times)) \rightarrow \{(\text{hungry}(\times) \vee \text{threatened}(\times)) \rightarrow \text{attacks}(\times)\}]$
(b) $\forall \times [(\text{tiger}(\times) \vee \text{lion}(\times)) \rightarrow \{(\text{hungry}(\times) \vee \text{threatened}(\times)) \rightarrow \text{attacks}(\times)\}]$
(c) $\forall \times [(\text{tiger}(\times) \vee \text{lion}(\times)) \rightarrow \{(\text{attacks}(\times) \vee \text{hungry}(\times)) \vee \text{threatened}(\times))\}]$
(d) $\forall \times [(\text{tiger}(\times) \vee \text{lion}(\times)) \rightarrow \{(\text{hungry}(\times) \vee \text{threatened}(\times)) \rightarrow \text{attacks}(\times)\}]$

82. Let S be a set of n elements. The number of ordered pairs in the largest and the smallest equivalence relations on S are **[[2007, 1 Mark]**

(a) n and n (b) n^2 and n
(c) n and 0 (d) n and 1

83. Consider the DAG with V = {1, 2, 3, 4, 5, 6}, shown below: **[2007, 1 Mark]**



Which of the following is not a topological ordering?

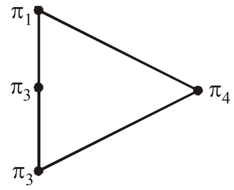
(a) 1 2 3 4 5 6 (b) 1 3 2 4 5 6
(c) 1 3 2 4 6 5 (d) 3 2 4 1 6 5

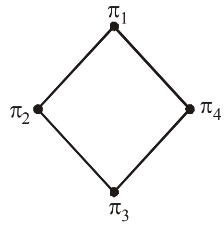
84. Consider the set $S = \{a, b, c, d\}$. Consider the following 4 partitions $\pi_1, \pi_2, \pi_3, \pi_4$ on S : $\pi_1 = \{\overline{abcd}\}$, $\pi_2 = \{\overline{ab}, \overline{cd}\}, \pi_3 = \{\overline{abc}, \overline{d}\}, \pi_4 = \{\overline{a}, \overline{b}, \overline{c}, \overline{d}\}$

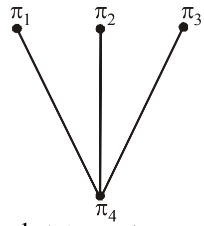
Let < be partial.

Order on the set of partitions $S' = (\pi_1, \pi_2, \pi_3, \pi_4)$ defined as follows : $\pi_i < \pi_j$ if and only if π_i refines π_j. The poset diagram for $(S', <)$ is **[2007, 1 Mark]**

(a)


(b)


(c)


(d)


85. Consider the following propositional statements **[2006, 1 Mark]**

$P_1 : ((A \wedge B) \rightarrow C) \equiv ((A \rightarrow C) \wedge (B \rightarrow C))$
$P_2 : ((A \vee B) \rightarrow C) \equiv ((A \rightarrow C) \vee (B \rightarrow C))$

Which one of the following is true?

(a) P_1 is a tautology, but not P_2
(b) P_2 is a tautology, but not P_1
(c) P_1 and P_2 are both tautologies
(d) Both P_1 and P_2 are not tautologies

86. A relation R is defined an ordered pair of integers as follows (x, y) R (u, v), if x < u and y < v. The, R is **[2006, 1 Mark]**

(a) neither a Partial Order nor an Equivalence Relation
(b) a Partial Order but not a Total Order
(c) a Total Order
(d) an equivalence Relation

87. Let $S = \{1, 2, 3, ..., m\}$, $m > 3$. Lt $X_1, ..., X_n$ be subsets of S each of size 3. Define a function f from S to the set of natural numbers as, $f(i)$ is the number of sets X_j that contain the element i. i.e., $f(i) = \{j | i \in X_j\}$, then $\sum_{i=1}^{m} f(i)$ is **[2006, 1 Mark]**

(a) $3m$ (b) $3n$
(c) $2m+1$ (d) $2n+1$

88. Let E, F and G be finite sets. Let $X = (E \cap F) - (F \cap G)$ and $Y = (E - (E \cap G) - E - F)$. Which one of the following is true? **[2006, 2 Marks]**

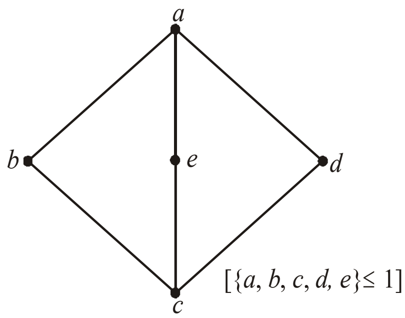
(a) $X \subset Y$ (b) $X \supset Y$
(c) $X = Y$ (d) $X - Y \neq \phi$ and $Y - X \neq \phi$

89. The set {1, 2, 4, 7, 8 ,11, 13, 14} is a group under multiplication modulo 15. The inverse of 4 and 7. are respectively **[2006, 2 Marks]**

(a) 3 and 13 (b) 2 and 11
(c) 4 and 13 (d) 8 and 14

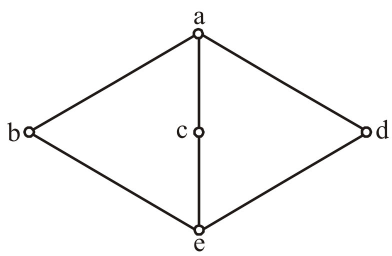
Lattice

90. The following is the Hasse diagram of the poset [{a, b, c, d, e} ≤ 1]. The poset is **[2006, 2 Marks]**



(a) not a lattice
(b) a lattice but not a distributive lattice
(c) a distributive lattice but not a Boolean algebra
(d) a Boolean algebra

91. The following is the hasse diagram of the poset [{a, b, c, d, e}, ≤]. **[2005, 1 Mark]**



The poset is
(a) not a lattice
(b) a lattice but not a distributive lattice
(c) a distributive lattice but not a Boolean algebra
(d) A Boolean algebra

Set & Relations

92. Let A, B and C be non-empty set and let X= (A – B) – C and Y = (A – C) – (B – C). Which one of the following is true?
(a) X = Y (b) $X \subset Y$ **[2006, 2 Marks]**
(c) $Y \subset X$ (d) None of these

93. The set {1, 2, 3, 5, 7, 8, 9} under multiplication modulo 10 is not a group. Given below are four possible reasons. Which one of them is false? **[2006, 1 Mark]**
(a) It is not closed
(b) 2 does not have an inverse
(c) 3 does not have an inverse
(d) 8 does not have an inverse

94. Let P, Q and R be three atomic propositional assertions. Let X denotes $(P \vee Q) \to R$ and Y denotes $(P \to R) \vee (Q \to R)$. Which one of he following is a tautology? **[2005, 1 Mark]**
(a) $X \equiv Y$ (b) $X \to Y$
(c) $Y \to 1$ (d) $\neg Y \to X$

95. What is the first order predicate calculus statement equivalent to the following? **[2005, 1 Mark]**
Every teacher is liked by some student.
(a) $\forall$ (x) [teacher (x) $\to \exists$ (y) {student (y) $\to$ likes (y, x)}]
(b) $\forall$ (x) [teacher (x) $\to \exists$ (y) {student (y) $\wedge$ likes (y, x)}]
(c) $\exists$ (y), $\forall$ (x) [teacher (x) $\to$ {student (y) $\wedge$ likes (y, x)}]
(d) $\forall$ (x) [teacher (x) $\to \exists$ (y) {student (y) $\to$ likes (y, x)}]

96. Let P, Q, R be three atomic propositional assertions. Let X denotes $(P \vee Q) \to R$ and Y denotes $(P \to R) \vee (Q \to R)$. Which one of the following is tautology? **[2005, 2 Marks]**
(a) $X \equiv Y$ (b) $X \to Y$
(c) $Y \to X$ (d) $\neg Y \to X$

97. Let f : B → C and g : A → B be two functions and let h = fog. Given that h is an onto function. Which one of the following is true? **[2005, 2 Marks]**
(a) f and g should both be onto functions
(b) f should be onto but g needs not be onto
(c) g should be onto but not be onto
(d) both f and g need not be onto

98. Let R and S be any two equivalence relations on a non-empty set A. Which one of the following statements is true? **[2005, 2 Marks]**
(a) $R \cap S$, $R \cup S$ are both equivalence relations
(b) $R \cup S$ is an equivalence relation
(c) $R \cap S$ is an equivalence relation
(d) Neither $R \cup S$ nor $R \cap S$ is an equivalence relation

99. What is the minimum number of ordered pairs of non-negative numbers that should be chosen to ensure that ther are two paris (a, b) and (c, d) in the chosen set such that a ≡ c and mod 3 and b ≡ d mod 5? **[2005, 2 Marks]**
(a) 4 (b) 6
(c) 16 (d) 24

100. The set {1, 2, 4, 7, 8, 11, 13, 14} is a group under multiplication modulo 15. The inverses of 4 and 7 are respectively **[2005, 1 Mark]**
(a) 3 and 13 (b) 2 and 11
(c) 4 and 13 (d) 8 and 14

101. Let A, B and C be non-empty sets and let X = (A – B) – C and Y = (A – C) – (B – C) **[2005, 1 Mark]**
Which one of the following is true?
(a) X = Y (b) $X \subset Y$
(c) $Y \subset X$ (d) None of these

102. Consider the binary relation :
S = {(x, y) | y = x + 1 and x, y ∈ {0, 1, 2,...}}
The reflexive transitive closure of S is **[2004, 1 Mark]**
(a) {(x, y) | y > x and x, y ∈ {0, 1, 2,...}}
(b) {(x, y) | y ≥ x and x, y ∈ {0, 1, 2,...}}
(c) {(x, y) | y < x and x, y ∈ {0, 1, 2,...}}
(d) {(x, y) | y ≥ x and x, y ∈ {0, 1, 2,...}}

103. Let (S, $\leq$) be a partial order with two minimal elemetns a and b, and a maximum element c. Let P : S $\to$ {True, False} be a predicate defined on S. Suppose that P(a) = True, P(b) = False and $P(x) \Rightarrow P(y)$ for all x, y $\in$ S satisfying $x \leq y$, where $\Rightarrow$ stands for logical implication. Which of the following statements cannot be true? **[2003, 2 Marks]**

(a) P(x) = True for all x $\in$ S such that x $\in$ such that $x \neq b$
(b) P(x) = False for all x $\in$ S such that $x \neq a$ and $x \neq c$
(c) P(x) = False for all x $\in$ S such that $b \leq x$ such that $x \neq c$
(d) P(x) = False for all x $\in$ S such that $a \leq x$ and $b \leq c$

104. Let $f : A \to B$ be an injective (one-to-one) function. Define $g : 2^A \to 2^B$ as :g(C) = {f(x) x $\in$ C), for all subsets C of A. Define $h : 2^B \to 2^A$ as : h(D) = {x | x $\in$ A, f(x) $\in$ D}, for all subsects D of B.
Which of the following stattements is always true? **[2003, 2 Marks]**

(a) $g(h(D)) \subseteq D$
(b) $g(h(D)) \supseteq D$
(c) $g(h(D)) \cap D = \phi$
(d) $g(h(D)) \cap (B - D) \neq \phi$

105. Consider the set {a, b, c} with binary operators + and $\times$ defined as follows :

+	a	b	c	$\times$	a	b	c
a	b	a	c	a	a	b	c
b	a	b	c	b	b	c	a
c	a	c	b	c	c	c	b

For example, a + c = c, c + a = a, c $\times$ b = c and b $\times$ c = a.
Given the following set of equations

$(a \times x) + (a \times y) = c$
$(b \times x) + (c \times y) = c$

The number of solutions [i.e., pair (s) (x, y) that satisfies the equations] is **[2003, 2 Marks]**

(a) 0
(b) 1
(c) 2
(d) 3

106. Consider the following reccurrence relation:
T(1) = 1

$T(n + 1) = T(n) + \lfloor \sqrt{n+1} \rfloor$ for all $n \geq 1$

The value of $T(m^2)$ for $m \geq 1$ is **[2003, 2 Marks]**

(a) $\frac{m}{6}(21m - 39) + 4$

(b) $\frac{m}{6}(4m^2 - 3m + 5)$

(c) $\frac{m}{2}(3m^{2.5} - 11m + 20) - 5$

(d) $\frac{m}{6}(5m^3 - 34m^2 + 137m - 104) + \frac{5}{6}$

Hints & Solutions

Propostional and First Order Logic

1. (d) By the rule of contrapositive.

The given statement $(\neg p) \Rightarrow (\neg q)$ is logically equivalent to $q \Rightarrow p$, and $(\neg q) \vee p$ because

$\therefore (\neg p) \Rightarrow (\neg q) \Leftrightarrow q \Rightarrow p$

$\therefore q \Rightarrow p \Leftrightarrow (\neg q) \vee p$

By taking

L.H.S. :-

$q \Rightarrow p \Leftrightarrow (q' + p)$

Now, R.H.S.:-

$(\neg q) \vee p \Leftrightarrow (q + p)$

So, $q \Rightarrow p \Leftrightarrow (\neg q) \vee p$

Hence, clearly the given statement is same as (ii) and (iii), only. So option (d) is correct.

2. (b) The given statement $F : \forall_x (\exists_y R(x, y))$

1. $\forall_x (\exists_y R(x, y)) \longrightarrow \exists_y (\exists_x R(x, y))$ is correct.

Since $\exists_y (\exists_x R(x, y)) \equiv \exists_x (\exists_y R(x, y))$

$\equiv \forall_x (\exists_y R(x, y))$

2. $\forall_x (\exists_y R(x, y)) \longrightarrow \exists_y (\forall_x R(x, y))$ is incorrect,

because $\exists_y$ (it is stronger when it is outside).

3. $\forall_x (\exists_y R(x, y)) \longrightarrow \forall_y \exists_x R(x, y)$ is incorrect,

because $R(x, y)$ may not be symmetric in x and y.

4. $\forall_x (\exists_y R(x, y)) \longrightarrow (\neg \exists_x \forall_y \neg R(x, y))$ is correct because

$\neg(\exists_x \forall_y \neg R(x, y)) \equiv (\neg \exists_x)(\neg \forall_y)(\neg(\neg R(x, y)))$

$(\neg(\exists_x)(\forall_y)\neg(\neg R(x, y))) \equiv \forall_x (\exists_y R(x, y))$

$\therefore \neg(\exists_x) = \forall_x, \neg(\forall_y) = \exists_y, \neg\neg R(x, y) = R(x, y)$

So, it will be reduced to

$\forall_x (\exists_y R(x, y)) \longrightarrow \forall_x \exists_y R(x, y)$

which is trivially correct.

So correct answer is I and IV only.

Hence option (b) is correct.

3. (a) As given statements are :-

p : "It is raining"

q : "It is cold"

r : "It is pleasant".

Then, the statement is:-

"It is not raining and it is pleasant" $\Rightarrow (\neg p \wedge r)$

And "It is not pleasant only if it is raining and it is cold"

$\Rightarrow \neg r \to (p \wedge q)$

$(\neg p \wedge r) \wedge (\neg r \to (p \wedge q))$

Hence, option (a) is correct

4. The given sequence is

$$\{a_n\}_{n=0}^{\infty} = \frac{1+z}{(1-z)^3},$$

Consider, $\dfrac{1+z}{(1-z)^3} = (1+z)(1-z)^{-3}$

$$(1-z)^{-3} = (1+z) * (1 + {}^3C_1 z + {}^4C_2 z^2 + {}^5C_3 z^3 + ...\infty)$$

a_0 is first term in the expansion of above series and a_3 is the fourth term or coefficient of z^3.

Then, we find the value of a_0 and a_3, so

a_0 (coefficient of z^0) = 1

a_3 (coefficient of z^3) $= {}^5C_3 + {}^4C_2$

$= 10 + 6 = 16$

Then, $a_3 - a_0 = 16 - 1 = 15$.

Hence 15 is correct answer.

5. $(p \Rightarrow q)$ will result {8, 9, 10, 12}

$\neg r$ will result {8,10, 11, 12}

$\neg s$ will result {8, 9, 10, 12}

$(\neg r \vee \neg s)$ will result {8, 9, 10, 11, 12}

$(p \Rightarrow q) \wedge (\neg r \vee \neg s)$ will result {8, 9, 10, 12}

$\neg((p \Rightarrow q) \wedge (\neg r \vee \neg s))$ will result 11.

6. Case I $\to$ First bit is '0'

$0\underbrace{\_\ \_\ a_n - 1\ \_\ \_\ \_}$

Case II $\to$ First bit is '1'

$0\underbrace{\_\ \_\ a_{n-2}\ \_\ \_}$ 10 $\to$ (It must be zero)

$\therefore a_n = a_{n-1} + a_{n-2}$

disha
Nurturing Ambitions

7. $(x^3+x^4+x^5+x^6+....)^3$
$= x^9(1+x+x^2+.......)^3$
$= x^9((1-x)^{-1})^3$
$= x^9(1-x)^{-3}$

$$= x^9\sum_{n=0}^{\infty}\frac{(n+1)(n+2)}{2}x^n$$

To find the coefficient of x^{12}, put $n = 3$

$$= \frac{4\times 5}{2} = 10$$

8. In the given properties, we can observe that f(1) = f(2) = f(3) = f(4) = f(6) = f(7) and f(5) = f(10) = f(15) = ...
Obviously, the range of f(x) will contain two distinct elements only.

9. Given expression is :
$= P \wedge (P \Rightarrow Q)$
$= P \wedge (P' \vee Q)$
$= (P \wedge P') \vee (P \wedge Q)$
$= F \wedge (P \wedge Q)$
$= (P \wedge Q)$

1. $(P \wedge Q) \Rightarrow F = (P \wedge Q)' \vee F = (P' \vee Q') \vee F = (P' \vee Q')$
= Contingency but not Tautology
2. $(P \wedge Q) \Rightarrow Q = (P \wedge Q)' \vee Q = (P' \vee Q') \vee Q = P' \vee (Q' \vee Q) = P' \vee T = T$
= Tautology
3. $(P \wedge Q) \Rightarrow T = (P \wedge Q)' \vee T = T =$ Tautology
4. $(P \wedge Q) \Rightarrow P \vee Q = (P \wedge Q)' \vee P \vee Q = (P' \vee Q') \vee P \vee Q = (P' \vee P) \vee (Q \vee Q')$
$= T \vee T = T =$ Tautology
5. $(P \wedge Q) \Rightarrow Q' \vee P = (P \wedge Q)' \vee Q' \vee P = (P' \vee Q') \vee Q' \vee P = (P' \vee P) \vee (Q' \vee Q')$
$= T \vee Q' = T =$ Tautology

Therefore, statements (2), (3) (4) and (5) satisfies.

10. (b) R is reflexive as every ordered pair is related to itself.
(a, b) R (a, b) since $a \le a$ or $b \le b$
It is not transitive as (2, 4) R (3, 2) and (3, 2) R(1, 3) but (2, 4) R (1, 3).

11. (d) P and Q are unary relations. Therefore one can model each of them as sets P_s and Q_s, where P(x) iff $x \in P_s$ and $Q(x)$ iff $x \in Q_s$. Let W be every object in the universe. Now we can convert each statement into sets:

(i) $(\forall xp(x) \Rightarrow \forall xq(x)) \Rightarrow (\exists x \neg p(x) \vee \forall xq(x))$

Converting the $\vee$ to $\Rightarrow$, we get

$$(\forall xp(x) \Rightarrow \forall xq(x)) \Rightarrow (\neg\exists x\neg p(x) \Rightarrow \forall xq(x))$$

$$(\forall xp(x) \Rightarrow \forall xq(x)) \Rightarrow (\forall xp(x) \Rightarrow \forall xq(x))$$

Hence, the formula is valid if, for any model where $(\forall xp(x)) \Rightarrow \forall xq(x))$ holds, that $(\forall xq(x) \Rightarrow \forall xq(x))$ also holds.

(ii) $(\exists xp(x) \vee \exists xq(x)) \Rightarrow \exists x(p(x) \vee q(x))$

If P_s or Q_s is not empty, then $P_s \cup Q_s$ is not empty.
It can be easily showed that this holds for any model.

(iii) $\exists x(p(x) \wedge q(x)) \Rightarrow (\exists xp(x) \wedge \exists xq(x))$

If $P_s \cap Q_s$ is non-empty then P_s is non-empty and Q_s is non-empty. Anything can model this.

(iv) $\forall x(p(x) \vee q(x)) \Rightarrow x\forall(p(x) \vee \forall xq(x))$

If $P_s \cup Q_s = W$, then $P_s = W$ or $Q_s = W$. This is obviously not true for any choice of P_s and Q_s. Actually the only times it will be true is:
(a) When $P_s = W$, meaning P(x) = True, or
(b) When $Q_s = W$, meaning Q(x) = True, or
(c) When $P_s \cap Q_s \neq W$, i.e., when there is a value of x where both P(x) and Q(x) do not hold.
In any other case the choice of P() and Q() will not satisfy (iv).

12. (b) The given problem can be solved by taking an undirected graph with 23 vertices and 9 of these with degree 3.
Suppose that if two compounds react with each other, then there exists an edge between the vertices.
According to Theorem of Degree, atleast one of the remaining vertices must have odd degree (since number of vertices of odd degree is always even).

13. We have 13 * 13 * 13 * ... * 13 (total 99 terms)
By Remainder theorem, ⇒ (-4) * (-4) * ... * (-4) (total 99 terms) ⇒ 16 * 16 * ... * 16 * (-4) (total 50 terms, 49 terms for 16 and one term for -4). Reapplying Remainder theorem, ⇒ (-1) * (-1) * ... * (-1) * (-4) (total 50 terms, 49 terms for -1 and one term for -4) ⇒ (-1) * (-4) ⇒ 4

14. (a) $g(x) = 1 - x$ ...(1)

$$h(x) = \frac{x}{x-1} \quad ...(2)$$

Replace x by h (x) in ...(1)
Replace x by g (x) in ...(2)
$g(h(x)) = 1 - h(x)$

$$= 1 - \frac{x}{x-1} = \frac{-1}{x-1}$$

$$h(g(x)) = \frac{g(x)}{g(x)-1} = \frac{1-x}{-x}$$

$$\frac{g(h(x))}{h(g(x))} = \frac{x}{(x-1)(1-x)}$$

$$\frac{g(h(x))}{h(g(x))} = \frac{\frac{x}{x-1}}{1-x}$$

$$\frac{g(h(x))}{h(g(x))} = \frac{h(x)}{g(x)}$$

15. (d) Number of elements in L^3 = Number of ways in which we can choose 3 elements from 5 with repetition $= 5 \times 5 \times 5 = 125$ Now, when we take x = t, then also the given condition is satisfied for any y and z. Here, y and z can be taken in $5 \times 5 = 25$ ways.
Take x = r, y = p, z = p, these, also the given condition is satisfied. So, $p_r > \frac{25}{125} > \frac{1}{5}$.
For $x = p$, $y = r$, $z = t$, the given condition is not satisfied.
So, $P_r \neq 1$.
So, (d) corret option.

16. (c) $2^A \rightarrow$ Power set of A i.e., set of all subsets of A. Since empty set is a subset of every set
$\therefore \varphi \subseteq 2^A$ and $\varphi \in 2^A$
Since $\{5, \langle 6 \rangle\} \subseteq$ and $5 \notin 2A$
$\therefore \{5, \langle 6 \rangle\} \in 2^A$ and $\{5, \langle 6 \rangle\} \subseteq 2^A$
$\therefore$ I, II and III are TRUE

17. (a)

18. Cardinality of the power set of $\{0, 1, 2, \ldots, 10\}$ is 2^{11} i.e., 2048

19. h (2, 5, 7, 3) = Remainder of $\frac{7 \times 3}{2 \times 5} \rightarrow 1$
fg (h (2, 5, 7, 3), 4, 6, 8) = fg (1, 4, 6, 8)
= f (1, 4, 6, 8) × g (1, 4, 6, 8)
= 8 × 1 = 8

20. (c) Let p: candidate known to be corrupt
q: candidate will be elected
r: candidate is kind
then $S_1 = p \rightarrow \sim q$
$= q \rightarrow \sim p$ (contrapositive rule)
and $s_2 : r \rightarrow q$
$\Rightarrow r \rightarrow \sim p$ (transitive rule)
i.e., If a person is kind, he is not known to be corrupt
$\therefore$ Option is (c).

21. (d) R is not reflexive as each element can't be related to itself.
R is symmetric
Let a = 3, b = 6 and c = 10 then 3 and 6 have a common division other than 1
6 and 10 have a common division other than 1
but 3 &10 have no common division other than 1
3R6 and 6R10 but $3 \not R\ 10$
R is not transitive.

22. (d) Counter example :
(a) Let X = {1} $\Rightarrow X' = S - X$ {2, 3, 4, 5, 6} $\Rightarrow |X| \neq |X'|$
(b) Since |S| = 6 and |X| = 5 = |Y|
$\therefore$ Atleast 4 elements common in X and Y
$\Rightarrow X \cap Y = \phi$ is false
(c) Counter example : Let X = {1, 2} and Y = {2, 3, 4} then X/Y = $\{1\} \neq \phi$
(d) $X/Y = X - Y = X \cap Y'$ and $\frac{Y'}{X'} = Y' - X' = Y' \cap (X')$
$= Y' \cap X = X \cap Y'$
$\therefore \frac{X}{Y} = \frac{Y'}{X'}, \cancel{\forall} X, Y \in U$

23. (d)

24. (b) $(\sim (p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$
We prove the truth table as follows:–
As the 2^{nd} term in each given option is common we have included this term only once in the truth table:

p	q	r	~r	p ↔ q	~(p ↔ q)	p → q	Option (A) 1^{st} term	(B) Ist	(C) Ist	(D) Ist	2^{nd} Common term	Final truth value Option (A)	(B)	(C)	(D)
1	1	0	1	1	0	1	0	0	0	0	1	1	1	1	0
1	0	1	0	0	1	0	0	1	0	1	0	0	1	0	0
0	1	1	0	0	1	1	0	1	1	1	0	0	1	1	0

Also as given exactly two of p, q and r are to be true, so we need to consider only three combinations as above.

We see that only option (B) $(\sim (p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$ is true in all three cases when exactly two of p, q and r are true.

25. (d) The Given statement is:
"Not all that glitters is Gold"
Here "Glitters" means the "Golden shiny surface" that looks like gold.
So, the statement is saying that "Not all the shiny surface is Gold." From the statement it can be deduced that there is "Some glitters that is "definitely" not gold". However, nothing can be deduced that whether "some glitters is Gold" or "No glitters is Gold."

(A) For all x : glitters $(x) \Rightarrow \neg$ gold (x)
It says that if something is glitters then it must not be Gold, which may not be true according to the above discussion.

(B) For all x: gold $(x) \Rightarrow$ glitters (x)

It says that if something is Gold then it must be Glitters, which can't be deduced from given statement.

(C) $\exists x$: gold (x) $\wedge \neg$ glitters (x)

It says that if something x is available which is not glitters and is Gold. However this also can't be deduced from given statement.

(D) $\exists x$: glitters $(x) \wedge \neg$ gold (x)

It says that some x exists which is glitters and is not Gold. We note that from the given statement we deduced this only as discussed above.

26. (d) g : mobile in good

c : mobile in cheap.

P : Good mobile phones are not cheap

$\approx g \to \neg c \approx (\neg g \vee \neg c)$

Q : cheap mobile phones are not good

$\approx c \to \neg g \approx (\neg c \vee \neg g)$

$\therefore$ Both P and Q are equivalant

$\therefore$ L, M and N are correct

27. (b) $(a \leftrightarrow c) \to (\sim b \to (a \wedge c))$

Proportional and First order logic

28. (b) $\dfrac{1}{(1-x)^2} = (1-x)^{-2}$

$= 1 + {}^2C_1x + {}^3C_2x^2 + \ldots\ldots\ |x| < 1$

$= g(0) + g(1)x + g(2)\,x^2 + \ldots\ldots$

$\therefore$ $g(0) = 1$

$g(1) = {}^2C_1 = 2$

$g(2) = {}^3C_2 = {}^3C_1 = 3$

$\ldots \quad \ldots \quad \ldots$

$g(i) = {}^{i+1}C_i = {}^{i+1}C_1 = i + 1$

29. (d) The statement some boys in the class are taller than all the girls.

So, the logical notation is

$(\exists x)$ (boy $(x) \wedge (\forall y)$ (girl $(y) \to$ taller (x, y)))

30. (a) $P \to (Q \vee R) \equiv P \to Q \vee R$

$= P' + Q + R$

$(P \wedge Q) \to R \equiv PQ \to R$

$\equiv (PQ)' + R$

$\equiv P' + Q' + R$

$\therefore$ $P \to (Q \vee R) \to (P \wedge Q) \to R$

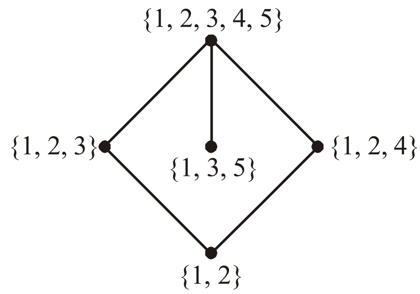
$\equiv (P' + Q + R) \to P' + Q' + R$

$\equiv (P' + Q + R)' + P' + Q' + R$

$\equiv PQ'R' + P' + Q' + R$ (By absorption rule)

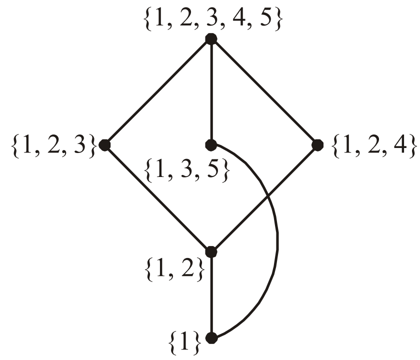
$\equiv P' + Q' + R$

31. (a) The hasse diagram of the given poset is



In a complete lattice L, every non-empty subset of L, has both LUB and GLB.

Now add {1}. The hasse diagram now becomes



Now the above hasse diagram represents a complete lattice, since every non-empty subset has both LUB and GLB.

32. (d) From given table,

$e * a = a,\ e * b = b,\ e * c = c$

$\Rightarrow$ e is the identity element,

$\therefore$ $a * e = a,\ b * e = b,\ c * e = c$

Again from given column, we have

$a * c = e$

$\therefore$ $c * a = e$

(If a is the inverse of c then c is the inverse of a)

$\therefore$ First two elements of the last row will be ce.

33. (d) Number of edges available in n labelled vertices is

$= {}^nC_2 = \dfrac{n(n-1)}{2}$

$= \dfrac{n^2 - n}{2}$ edges.

From this, we need to choose $\dfrac{n^2 - 3n}{2}$ edges or more

upto maximum $\dfrac{n^2 - n}{2}$.

Taking $\dfrac{n^2 - n}{2} = m$

and $\dfrac{n^2 - 3n}{2} = p$

$\Rightarrow m - n = p,\ m - p = n$

$\therefore$ Total number of such graphs

$= {}^mC_p + {}^mC_{p+1} + {}^mC_{p+2} + \ldots + {}^mC_{p+n}$

$= {}^mC_n + {}^mC_{n-1} + {}^mC_{n-2} + \ldots + {}^mC_0$

$= {}^mC_0 + {}^mC_1 + {}^mC_2 + \ldots + {}^mC_n$

$= \sum_{k=0}^{n}$

$= \sum_{k=0}^{n} \frac{n^2 - n}{2} C_k$

34. (d) $(\forall x)[\alpha \Rightarrow \beta] \Rightarrow ((\forall x)[\alpha] \Rightarrow \forall(x)[\beta])$ is a logical

35. (c) $p \Rightarrow q \equiv \sim p \vee q$

$(B(x, x) \rightarrow [B(x, y), C(y) \rightarrow A(x)]$

$\equiv \neg B(x, x) \vee [B(x, y) \wedge C(y) \rightarrow A(x)]$

36. (d) $Q_{yy} \equiv$ "y" divides y is always true.

$\therefore Q_{xy} \Leftrightarrow \neg Q_{yy}$ is same as $Q_{xy} \Leftrightarrow$ False.

Now α becomes

$(\forall x)[P(x) \Leftrightarrow (\forall y)(Q_{xy} \Leftrightarrow \text{False})]$

$\Rightarrow \quad (\forall x)[\neg P(x)]$

Now consider $l_1 : P(x) \equiv$ "x is a prime number".

α becomes $\forall x$ (x is a prime number if and only if $\forall y$ (y does not divide x) is always false (since x divides x always). α now becomes

$\forall x$ (x is a prime number $\Leftrightarrow$ false) $\Leftrightarrow \forall x$ (x is not a prime).

which is true.

Now consider $l_2 : P(x) \equiv$ "x is a composite number".

Now α becomes

$\forall x$ (x is a composite number if and only if $\forall y$ (y does not divide x)) $\forall x$ (x is not a composite number)

By same reasoning used above, α now becomes $\forall x$ (x is a composite number $\Leftrightarrow$ false) $\forall x$ (x is not composite) is also true.

Since, if no number divides x, x cannot be a composite number, is true.

$\therefore$ Both l_1 and l_2 satisfy α.

37. (b) Derive caluse $P \vee Q$ from clauses $P \vee R$, $Q \vee \neg R$ means that

$(P \vee R) \wedge (Q \vee \neg R) \Rightarrow P \vee Q$

$\therefore$ (a) is true

Since, $x \Rightarrow y$ does not imply that $y \Rightarrow x$

$\therefore P \vee Q \Rightarrow (P \vee R) \wedge (Q \vee \neg R)$

$\therefore$ may or may not be true. Hence, (b) is not true.

38. (b) If S = {a, b, c, d, e}

g(a) = 3

g(b) = 5

g(c) = 7

g(d) = 9

g(e) = 11

Now, consider the string aaa, where s_1 =a, s_2 = a and s_3 = a

$$f(s_1) = f(s_2) = f(s_3) = \prod_{i=1}^{1} p_i^{g(a_i)}$$

$$= p_1^{g(a_1)} = p_1^{g(a)} = 2^3 = 8$$

$$h(s_1, s_2\ s_3) = \prod_{i=1}^{3} p_i^{f(s_i)}$$

$$= 2^{f(s_1)} . 3^{f(s_2)} . 5^{f(s_3)}$$

$$= 2^8\ 3^8\ 5^8$$

39. (a) $\Sigma = \{0, 1\}$

$\Sigma^* = \{0, 1\}^*$

$= \{\in, 0, 1, 01, 10, 11, 000, \ldots\}$

So, $(\Sigma^*, \bullet)$ is an algebraic system, where $\bullet$ (concatenation) is a binary operation.

So $(\Sigma^*, \bullet)$ is a group, if and only if the following conditions are satisfied.

1. • (Concatenation) is a closed operation.
2. • is an associative operation.
3. There is an identity.
4. Every element of Σ^* has a left inverse.

Condition 1 * is a closed operation because for any $\omega_1 \in \Sigma^*$ and $\omega_2 \in \Sigma^*$, $\omega_1 . \omega_2 \in \Sigma^*$

Condition 2 For any string x, y, z $\in \Sigma^*$, x. (y . z) = (x . y) . z

So, it is associative for example, Let

x = 01, y = 11, z = 00 then

LHS = x . (y . z)

= 01. (11.00) = 01.(1100) = 011100

RHS = (x. y) . z

= (01.11) .00 = (0111).00 = 011100

Condition 3 The identity is $\in$ or empty string because for any string $\omega \in \Sigma^*$,

$\in \omega = \omega \in = \omega$

Now, since $\in$ belongs to Σ^*, identity exists.

Condition 4 There is no inverse exist for Σ^* because any string $\omega \in \Sigma$, there is no string ω^{-1} such that $\omega . \omega^{-1} = \in$.

So, Σ^* with the concatenation operator for strings doesn't form a group but it does form a monoid.

Sets & Relations,Functions, Partial orders, Latices & Groups

40. (a) Nullset is larger than any other set.

Universal set is smaller than any other set.

41. (d) f : X $\rightarrow$ Y defined as f (a) = 1, f(b) = 1, f(c) = 2

where, X = {a, b, c}

Y = {1, 2}

Let A = {a, c}, B = {b, c} be subsets of X

(A) $|f(A \cup B)| = 2$

$|f(A)| = 2 : |f(B)| = 2$

$\therefore$ (A) is false

(B) $f(A \cap B) = \{i\}$

f(A) = {1, 2) ; f{B} = {1, 2}

f(A) $\cap$ f (B) = {1, 2}

$\therefore$ (B) is false.

(C) $|f(A \cap B)| = 1$

$\therefore$ (C) is false.

(D) is true

42. (b) Consider a function as :
$f(0) = 1, f(1) = 0, f(2) = 3, f(3) = 2, ..., f(2012) = 2013,$
$f(2013) = 2012$ and $f(2014) = 2014$
Clearly $f(f(i)) = i$ for $0 \le i \le 2014$
Here $f(i) \ne i$ for every i and $f(i) = i$ for some i
Also f is onto
Hence, only Q and R are true

43. The value of $\log_2 \log_2 N$ is 16.
We know that the total no. of functions $f : A \to B$
= (Cardinality of B) (Cardinality of A) = $|B|^{|A|}$
Now for the given questions:
$\{0, 1\}^4$ represents all 4-tubles on the set $\{0, 1\}$,
e.g. (1, 0, 1, 0,), (1, 1, 1, 0) etc.
Total no. of 4-tubles on $\{0, 1\} = 2^4 = 16$
(Since each element of four tuble has 2 choice)
Total no. of functions $f : \underset{A}{\{0, 1\}^4} \to \underset{B}{\{0, 1\}} = 2^{16}$ ($|B|^{|A|}$)
Set S contains all these functions, so cardinality (S) = 2^{16}
Now, N denotes the no. of func. $g : S \to \{0, 1\}$.

$\Rightarrow$ $N = |\{0, 1\}|^{|S|} = 2^{2^{16}}$

$\Rightarrow$ $N = 2^{2^{16}}$

$\Rightarrow$ $\log_2 \log_2 \left(2^{2^{16}}\right) = 16$

44. A 10 percent will be a sequence of any no. of 1s and 2s such that sum is 10.
p.g. (1, 1, 2, 2, 1, 1, 2) is a 10 pennant.

I. When sequence contains all 1s:
no. of such pennants = 1 (1, 1, 1, 1, 1, 1, 1, 1, 1, 1)

II. When sequence contains Eight 1s and one '2' : (8 × 1, 1 × 2)

no. of such pennants = $^9C_1 = \frac{9!}{1!8!} = 9$

III. When sequence contains six 1s and Two 2s :
6 × 1, 2 × 2 (e.g. 1, 1, 1, 1, 1, 1, 2, 2,)

no. of pennants = $^{18}C_2 = \frac{18!}{2!6!} = 28$

IV. When sequence contains 4 1s and 3 2s:-
4 × 1, 3 × 2 (e.g. (1, 1, 1, 1, 2, 2, 2))

no. of pennants = $^7C_3 = \frac{7!}{3!4!} = 35$

V. When sequence contains 2 1s and 4 2s :
2 × 1, 4 × 2 (e.g. (1, 1, 2, 2, 2, 2,))

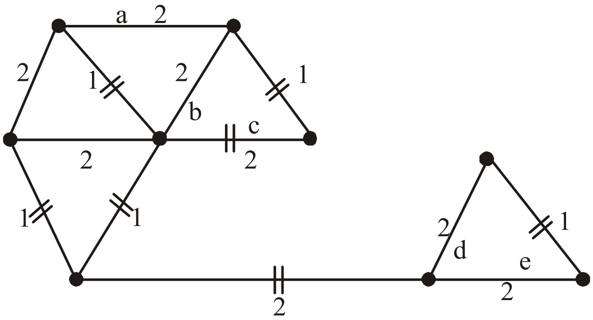
no. of pennants = $^6C_4 = \frac{6!}{4!2!} = 15$

VI. When sequence contains 5 2s and no. 1s:
no. of pennants = 1 (i.e. (2, 2, 2, 2, 2))
Total no. of 10 pennants
= 1 + 9 + 28 + 35 + 15 + 1 = 89.

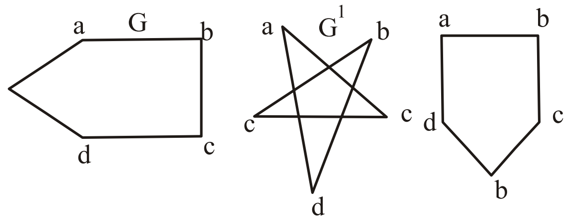
45. Let the vector space, V be {a, b, c, d, e, f}
For smallest possible dimension of $V_1 \cap V_2$,
V_1 = {a, b, c, d} and V_2 = {c, d, e, f}
$\therefore V_1 \cap V_2 = \{c, d\}$
$\therefore$ Smallest possible dimension = 2

Graphs

46. (1) 1, 11 $\Rightarrow 1 \times 11 = 11$ edges
(2) 2, 10 $\Rightarrow 2 \times 10 = 20$ edges
(3) 3, 9 $\Rightarrow 3 \times 9 = 27$ edges
(4) 4, 8 $\Rightarrow 4 \times 8 = 32$ edges
(5) 5, 7 $\Rightarrow 5 \times 7 = 35$ edges
(6) 6, 6 $\Rightarrow 6 \times 6 = 36$ edges (maximum)

47. 

Edges picked to make MST is given the double line.
In the right side of MST, we could either pick 'd' or 'e'
In the left side we could either pick 'a', 'b' or 'c' in MST.
There are two options for one edge to be picked and three option for other edge to be picked.
$\therefore$ Total possible MSTs = 2 × 3 = 6

48. 

G and G^1 are complement and isometric to each other when there are 5 vertices.

49. As per Lagrange's theorem, order of subgroup divides order of group
$\therefore$ 3, 5 and 15 can be order of subgroup.
Given, $L \ne G$ $\therefore$ it is not 15
and L is atleast 4 $\therefore$ it is not 3
$\therefore$ size of L is 5

50. (a) A binary operation $\oplus$ on a set of integers is defined as $x \oplus y = x^2 + y^2$
$x \oplus y = x^2 + y^2$
$y \oplus x = y^2 + x^2$
$= x^2 + y^2$ ($\because$ Addition is commutative)
$\therefore \oplus$ is commutative.
$(x \oplus y) \oplus z = (x^2 + y^2) \oplus z = (x^2 + y^2)^2 + z^2$
$x \oplus (y \oplus z) = x \oplus (y^2 + z^2) = x^2 + (y^2 + z^2)^2$
Clearly, $(x \oplus y) \oplus z \ne x \oplus (y \oplus z)$
$\therefore \oplus$ is not associative.
Hence, $\oplus$ is commutative but not associative.

51. (c) P : Number of odd degree vertices is even.
Q : Sum of degrees of all vertices is even
Q is true
Explanation Calculating the sum of degrees of all vertices.
Considering an edge, it is joining two vertices (not necessarily distinct), hence 2 is the sum of degrees.

Hence, for 'e' edges, the sum of degrees of all vertices is $2e$ (i.e., even)

This is valid even in the case of self loops.

P is true

Explanation Let us assume, number of odd degree vertices is odd. So, the contribution of odd degree vertices in total sum is odd.

Now, the contribution of even degree vertices is also even (whether the number is even or odd).

Including vertex of degree zero in even degree vertices.

So, total sum becomes odd, which is not possible. Hence, P is true.

$\therefore$ Both P and Q statements are ture for undirected graphs. Hence answer is (c).

52. (c) The given statement or sentence is—

'Some real number are rational.' We can also write this sentence as follows:

'There exists real number that are rational' therefore we have to use $\exists$ (Existential) quantifier consider number as random variable x. Then,

$\exists$x (real) (x)) $\Rightarrow$ This, indicates 'There exists number that are real.'

$\exists$ x (rational (x)) $\Rightarrow$ This indicates 'There exists number that are rational.'

Hence, $\exists$x (real (x) $\wedge$ rational (x) $\Rightarrow$

This indicates 'There exists real number that are rational.

53. (b) I_1 : If it rains, then then the cricket match will not be played.

The cricket match was played.

Inference : There was no rain.

Hence, I_1 is correct.

Proof

Suppose P : It rains

Q : The cricket match will not be played.

It is given $P \to Q$

We know that $P \to Q \equiv \sim Q \to \sim P$

$\therefore$ $\sim Q$: The circket match will be played.

$\sim P$: It is not rain.

$\therefore$ If the cricket match was played, then there was no rain $[P \to Q = \sim PQ \to \sim P]$

$\therefore$ I_1 is correct.

I_2 : If it rains, then the cricket match will not be played.

It did not rain.

Inference : The cricket match was played.

I_2 is not correct.

Proof

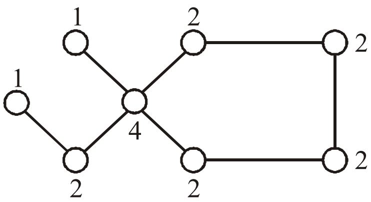
According to logic table $P \to Q$ i.e.,

P	Q	$P \to Q$
F	F	T
F	T	T
T	F	F
T	T	T

i.e., if there was not rain [i.e., P is F in logic table] then match may or may not be played because in both case $P \to Q$ is true.

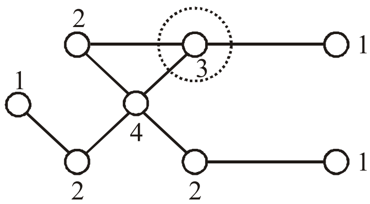
So, the correct inference is—the cricket match might be played.

54. (b) The given graph is with degree of each node.



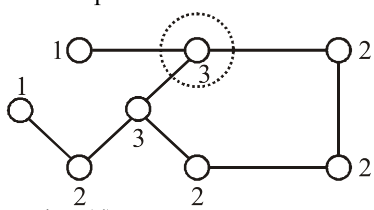
Consider option (a) :

In this graph, one vertex has degree of 3 but in the original graph, there is no vertex of degree 3. So, this graph is not isomorphic.



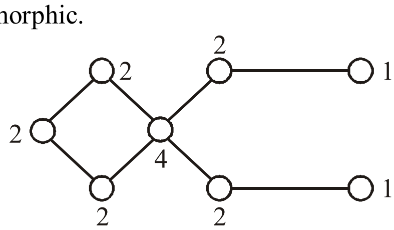
Consider option (c) :

Here, the same explanation as above given. This graph is also not isomorphic.



Consider option (d) :

In this graph, there is a 4 length but in the original graph, there is no cycle of length 4. So this graph is not isomorphic.



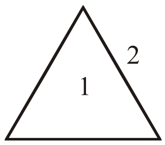
55. (c) Here, given graph is a complete graph and we can choose any 4 vertices from given 6 vertices to make cycle of length 4.

We can arrange 4 vertices in (4 – 1)! ways or 3! ways because there is a cycle.

Hence, number of cycles of length

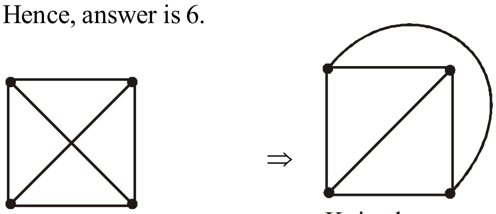
$4 = 6 \times {}^6C_4 = 6 \times \dfrac{6!}{4!\,2!} = 6 \times \dfrac{6 \times 5}{2} = 90$ cycle

56. (d) In any graph, bounded faces or bounded region and unbounded regions are present.
For example

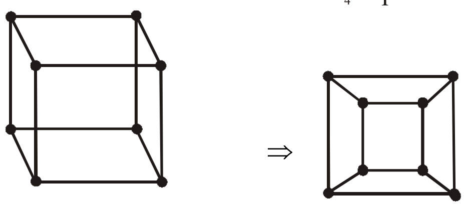


In the above graph, there are 2 regions, 1 st is bounded region or bounded face graph and 2nd is unbounded.
So, we have to calculate the regions of the graph.
In the question, G is a simple undirected graph on 10 vertices and 15 edges and G is connected garph, so there are total number of regions (r).
$r = e - v + 2$, $\quad r = 15 - 10 + 2$
$r = 7$
$\therefore$ Total number of regions in the given graph are 7 but one extra region is unbounded and 6 are bounded. Hence, answer is 6.

57. (b)



K_4 is planar



Q_3 is planar

Both K_4 and Q_3 is planar

58. (c) Let the set
$S = \{a, b, c, d, e\}$
Relation is subset of S × S
There are totally $5^2 = 25$ ordered pairs in S × S including five ordered pairs (a, a), (b, b), (c, c), (d, d) and (e, e) that must be included in order to find reflexive
$\therefore$ Total number of reflexive relation on S
$S = 2^{25-5} = 2^{20}$

59. (a) Here, we have onlyh one operation given.
Therefore, given options *viz*; ring, integral domain and field are ruled out.
As it requires two binary operations.
Alternative Method
(i) For all a, b, $\in$ S
$a \times b \in S$
$a * b \in S$
i.e.,
$\therefore$ Closure property hold son set S.
(ii) × is an associative operation.
(iii) $1 \in S$ is the identity element.
(iv) $1^{-1} = 1, \omega^{-1} = (w^2)^{-1} = \omega$
$\therefore$ Inverse axiom is satisfied on given set.
$\therefore$ (S, *) is a group

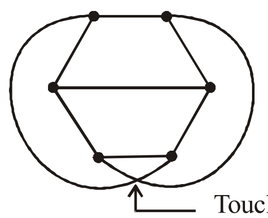
60. (c) Given, S and T are two different trees with
$\xi(S) = \xi(T)$
$\Rightarrow \quad 2|E_S| = 2|E_T|$
(By handshaking theorem $\xi(G) = 2|E_G|$)
$\Rightarrow \quad |E_S| = |E_T|$
$\Rightarrow \quad |S| - 1 = |T| - 1$
$\Rightarrow \quad |S| = |T|$
(In a tree $|E_S| = |S| - 1$ and $|E_T| = |T| - 1$, where $|S|$ and $|T|$ are numbers of vertices of trees S and T respectively)

61. (d) Let n (X) denotes the number of elements of set X.
Given, $n(X) = x$
$n(Y) = y$
$n(Z) = z$
n (E) = Number of all subsets of W
= Number of all subsets of X × Y
$= 2^{n(X).n(Y)} = 2^{xy}$
Therefore, the number of functions from Z to E
$= [n(E)]^{n(Z)}$
$= (2^{xy})^z = 2^{xyz}$

62. (b) Given, $V = 13, E = 19$
Let R be the number of regions.
Applying Euler's formula, (Here faces and regions mean one and the same).
$R = E - V + 2$
or $\quad R = 19 - 13 + 2 = 8$

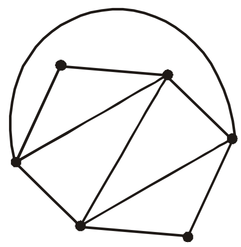
63. (b) Given, V = 13, E = 19
Let, R be the number of regions
Applying Euler's formula (Here faces and regions mean one and the same)

64. (a) From figure (a)

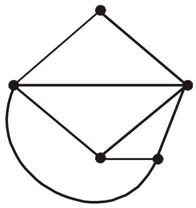


Touch each other.
So, not planar.

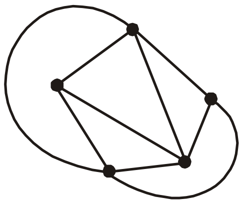
From figure (b)



From figure (c)

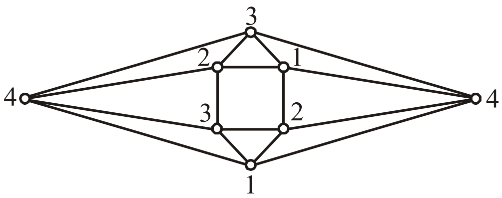


From figure (d)



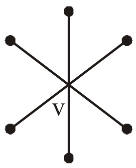
65. (b) Given, V = 13, E = 19
Let R be the number of regions.
Applying Euler's formula, (Here faces and regions mean one and the same).
$R = E - V + 2$
or $R = 19 - 13 + 2 = 8$

66. (c) An assignment of the colors 1, 2, 3 and 4 to the vertices of the graph is shown below such that the graph is shown below such that the graph is properly colored.

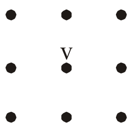


So, 4 colors are required.

67. (c) Maximum components will result after removal of a node, if graph G is a star graph as shown below



or a null graph of n vertices as shown below



In either case, if node v is removed, the number of components will be n – 1, where n is the total number of nodes in the star graph.
∴ n – 1 is the maximum number of components possible. Minimum components will result, if the node being removed is a lone vertex in which case, the nubmer of components will be K – 1.
∴ The number of components must necessarily lie between K – 1 nad n – 1.

68. (d) Given, $|E| \le 3\,|V| - 6$
∴ $2\,|E| \le 6|V| - 12$ ...(i)
Now $\sum$ degrees = 2|E|
If minimum degree = k, then $\sum$ degrees $\ge k|V|$
i.e., $2|E| \ge k\,|V|$ ...(ii)
From Eqs. (i) and (ii), we can say that
$k\,|V| \le 6\,|V| - 12$
Now substituting k = 3, 4 , 5 , 6 in order
$k = 3 \Rightarrow \quad 3\,|V| \le 6\,|V| - 12$
$\Rightarrow \quad |V| \ge 4$ which is possible
$k = 4 \Rightarrow \quad 4\,|V| \le 6\,|V| - 12$
$\Rightarrow \quad |V| \ge 6$ which is possible
$k = 5 \Rightarrow \quad 5\,|V| \le 6\,|V| - 12$
$\Rightarrow \quad |V| \ge 12$ which is possible
$k = 6 \Rightarrow \quad 6\,|V| \le 6\,|V| - 12$
which is not possible.
∴ Minimum degree cannot be 6.

69. (a) The number of perfect matchings in a complete graph of n vertices, where n is even, reduces of the problem of finding unordered partitions of the vertex set of the type p(2n; 2, 2, 2, ...n times)

$$\frac{(2n)!}{(2!)^n\,n!}$$

For n =3, 2n = 6, i.e., complete garph K_6, we have

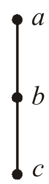
Number of perfect matchings $= \dfrac{6!}{(2!)^n\,n!}$

$$= \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2 \times 2 \times 6} = 15$$

Combinatorics

70. (a) A tree with 1 node is not possible, since it is given that every node has exactly 1 child.

Now, consider a tree with 2 nodes (0 is the root)
Now, *a* has exactly one child, number of descendants of *a* = 2 but this contradicts the given fact that every node has an odd number of descendants.
Now, consider a tree with 3 nodes. Since, every node has exactly one child, it must be of the form.



Here, a has 3 descendants, b has 2 descendants and c has one. Again we have contradiction in that b does not have odd number of descendants. Similarly, we can show that for tree with 4, 5, 6, ... nodes, it is not possible to have all nodes with odd number of descendants. So, correct answer is the trees has 0 nodes.

71. (c) The correct translation of gold and silver ornaments are precious is choice (d).

$$\forall x\,((G(x) \wedge S(x)) \rightarrow P(x))$$

which reads as "if an ornament is gold or silver, then it is precious". Since, a given ornament cannot be both gold and silver at the same time.

72. (b) The given table can be converted into Boolean function by adding minterms corresponding to true-row.

Translates $P \vee Q = PQ + PQ' + P'Q'$

Using this we can check the choices one by one to see which is equivalent to $P + Q$

Choice (a) $\neg Q \vee \neg P \equiv Q' \vee P'$

$\equiv Q'P' + Q'(P')' + (Q')'(P')'$

$\equiv Q'P' + Q'P + QP$

$\equiv Q'P' + \mathrm{P}(Q + Q')$

$\equiv Q'P' + \mathrm{P}$ ($\because$ by complement law)

$\equiv (Q' + P) \cdot (P' + \mathrm{P})$ ($\because$ by distribution law)

$\equiv Q' + P$

Choice (b) $P \vee \neg Q = P \vee Q'$

$= PQ' + P(Q')' + P'(Q')'$

$\equiv PQ' + PQ + P'Q$

$\equiv P(Q' + Q) + P'Q$

$\equiv P + P'Q$ ($\because$ by complement law)

$\equiv (P + P')\ (P + Q)$ ($\because$ by distribution law)

$= P + Q$

Choice (c) $\neg P \vee Q = P' \vee Q$

$\equiv P'Q + P'Q' + (P')'Q'$

$\equiv P'Q + P'Q' + PQ'$

$\equiv P'(Q + Q') + PQ'$

$= P' + PQ'$ ($\because$ by complement law)

$= (P' + P)(P' + Q') \equiv P' + Q'$ ($\because$ by distribution law)

Choice (d) $\neg P \neg Q \equiv P'Q'$

$\equiv P'Q' + P'(Q')' + (P')'(Q')'$

$\equiv P'Q' + P'Q + PQ$

$\equiv P'(Q' + Q) + PQ$

$\equiv P' + PQ$ ($\because$ by complement law)

$\equiv (P' + P)(P' + Q)$ ($\because$ by distribution law)

$= P' + Q$

We can clearly see only choice (b) $P \vee \neg Q$ is equivalent to $P + Q$.

73. (a) Group properties are closure, associativity, existence of identity and existence of inverse for every element. Commutativity is not required for a mathematical structure to become a group.

74. (d) Given, R = $\{x, y\}$, (x, z), (z, x), $(z, y)\}$ on set $\{x, y, z\}$

Now, since $(x, y) \in$ R and $(y, x) \notin$ R, R is not symmetric. Also since $(x, z) \in$ R and $(z, x) \in$ R, R is not anti-symmetric either.

$\therefore$ R is neither symmetric nor anti-symmetric.

75. (b) In a simple connected undirected graph (with more than two vertices) atleast two vertices must have same degree, since, if this is not true then all vertices would have different degree. A graph with all vertices would have different degree. A graph with all vertices having different degrees is not possible to construct. Notice that it is possible to construct graph satisfying choices (a), (c) and (d).

76. (c) If an n-vertex simple contex simple connected graph contains no cycles of odd length, then its chromatic number is two, since the vertices cna be alternately coloured with first colour, then the second colour, then the first colour and then the second colour and so on. Alternatively, since a simple connected graph with no cycles of odd length must be bipartite and since, the chromatic number of a bipartite graph is always 2 (in a bipartite graph each partition requires one colour (there are no edge within a partition of a bipartite graph) and there are only two partitions).

77. (b) The statement 'Not every graph is connected' is same as 'There exists some graph which is not connected' which is same as $\exists$ x {graph (x) $\wedge \neg$ connected (x)}.

78. (a) For x, the predicate fsa there exists y and the corresponding predicate pda which is equivalent to x.

$\therefore$ (A x fsa (x))

$\Rightarrow$ ($\exists$ y pda (y) ^ (equivalent (x, y))

79. (b) (i) $P \wedge \sim Q \equiv P + Q'$

(ii) $\sim(\sim P \wedge Q)\ (P' + Q)' \equiv P + Q'$

(iii) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge \sim Q)$

$= (PQ) + (PQ') + (P'Q')$

$= P(Q + Q') + P'Q'$

$= P + P'Q'$

$= (P + P')\ (P + Q')$

$= P + Q'$

(iv) $(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q)$

$= (PQ) + (PQ') + (P'Q)$

$= P(Q + Q') + P'Q$

$= P + P'Q$

$= (P + P')\ (P + Q) = P + Q$

Clearly, (i), (ii), (iii) are equivalent. Correct choice is (b).

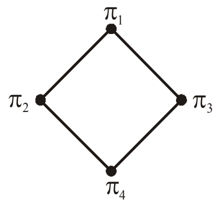
80. (d) $\because Q^c \cup R^c = (Q \cap R)^c$
and $(P \cap Q \cap R) \cup (P^c \cap Q \cap R) = Q \cap R$
Therefore, $(P \cap Q \cap R) \cup (P^c \cap Q \cap R) \cup Q^c \cup R^c$
$= (Q \cap R) \cup (Q \cap R)^c$
= U (Universal set)

81. (d) The given statement should be read as 'If an animal is a tiger or a lion, then (if the animal is hungry or threatened) it will attack.'
Therefore, correct option is (d).

82. (b) Let S = {1, 2, 3,..., n} be a set of n elements number of ordered pairs in the smallest equivalent relation on S is n, it must contain all the reflexive elements *viz.*, {(1, 1), (2, 2), (3, 3), ..., (n, n)}
and the largest equivalence relation on S has $n \times n = n^2$ ordered pairs.

83. (d) In topological sorting, the partial ordering of the DAG, must be preserved *i.e.*, if $a \leq b$ in the DAG, then in the topological order, b must come after a, not before. Consider the ordering 3 2 4 1 6 5. $1 \leq 4$ in the given DAG but 4 is coming before 1 in 3 2 4 1 6 5 order which means that 3 2 4 1 6 5 is not a topological order of the given DAG.

84. (c) A partition P_1 is called a refinement of the partitions P_2 is every set in P_1 is a subset of one of the sets in P_2.
π_4 is a refinement of π_2, π_3 and π_1.
π_2 and π_3 are refinements of π_1.
π_2 and π_3 are not comparable since, neither is a refinement of the other.
So, the poset diagram for (S′, <) would be



85. (d) $P_1 : ((A \wedge B) \to C) \equiv ((A \to C) \wedge (B \to C))$
LHS $(A \wedge B) \to C \equiv AB \to C$
$= (AB)' + C$
$= A' + B' + C$
RHS $(A \to C) \wedge (B \to C) \equiv (A' + C)(B' + C)$
$= A'B' + C$
$\Rightarrow$ RHS $\neq$ LHS
$\therefore P_1$ is not tautology.
$P_2 : ((A \wedge B) \to C \equiv ((A \to C) \vee (B \to C)$
LHS $(A \vee B) \to C \equiv A + B \to C$
$\equiv (A + B)' + C$
$= A'B' + C$
RHS $(A \to C) \vee (B \to C)$
$\equiv (A' + C) + (B' + C)$
$= A' + B' + C$
$\Rightarrow$ LHS $\neq$ RHS
$\Rightarrow P_2$ is also not tautology.

86. (a) (x, y) R (u, v) if x < u and y > v (x, x) R, (x, x)
since x $\not<$ x adn x $\not>$ x.
So. R is not reflexive.
$\therefore$ is neither a partial order, nor an equivalence relation.

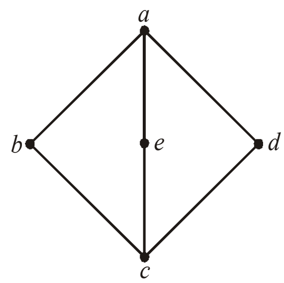
87. (b) Here, $n = {}^mC_3$
and $f(1) = f(2)... = f(m) = {}^{m-1}C_2$
$$\Rightarrow \sum_{i=1}^{m} f(i) = m^{m-1}C_2$$
$$= 3 \times {}^mC_3 = 3 \times n = 3n$$

88. (c) X $= (E \cap F) - (F \cap G)$
$= (E \cap F) \cap (F \cap G)'$
$= (E \cap F) \cap (F' \cap G')$
$= (E \cap F \cap F') \cup (E \cap F \cap G')$
$= E \cap F \cap G'$
Y $= (E - (E \cap G)) - (E - F)$
$= (E \cap (E \cap G')) - (E \cap F')$
$= (E \cap (E' \cup G') \cap (E \cap F')'$
$= (E \cap E') \cup (E \cap G') \cap (E \cup F)$
$= (E \cap G') \cap (E' \cup F)$
$= (E \cap G' \cap E') \cup (E \cap G' \cap F)$
$= E \cap F \cap G'$
$\Rightarrow$ X = Y

89. (c) The set S = {1, 2, 4, 7, 8, 11, 13, 14} is a group under multiplication modulo 15.
The identity element for this group is $e = 1$
Since, $\forall\, x \notin S$, 1. x and 15 = x
Now, let the inverse of 4 be 4^{-1}.
Now, $(4.\ 4^{-1})$ mod 15 = e = 1
Now, (4. 4) mod 15 = 1
Since, (4. 4) mod 15 = 1
$\therefore$ $4^{-1} = 4$ (This inverse is unique)
Similarly, let the inverse of 7 by 7^{-1}
$(7.\ 7^{-1})$ mod 15 = 1
Putting each element of set as 7^{-1} by trial and error, we get
(7.13) mod 15 = 91 mod 15 = 1
$7^{-1} = 13$
So, 4^{-1} and 7^{-1} are respectively 4 and 13.

Lattics

90. (b)



The poset $[\{a, b, d, e\}, \leq 1]$ is a lattice (since, every pair of element has LUB and GLB) but it is not a distributive lattice. Because distributive lattice satisfying the following conditions. For any x, y, z

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

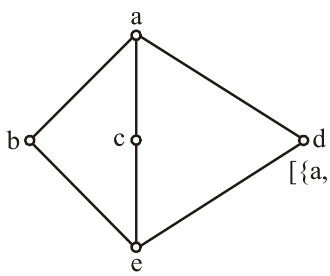
where, $\wedge$ and $\vee$ are meet and join operations but for given poset $[\{a, b, c, d, e\}, \leq 1]$

$b \wedge (c \vee d) = b \wedge a = b$

$(b \wedge c) \vee (b \wedge d) = e \vee e = e$

So, it is not distributive. (Also, element b has 2 complements c and d, which is not possible in the distributive lattice, since in a distributive lattice, complement if it exist, is always unique).

91. (b)



$[\{a, b, c, d, e\}, \leq]$

The poset $[\{a, b, c, d, e\}, \leq]$ is a lattice (since every pair of elements has LUB and GLB) but it is not a distributive lattice. Because distributive latice satisfies the following conditions. For any x, y, z

$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

where, $\wedge$ and $\vee$ are meet and join operations but for given poset $[\{a, b, c, d, e\} \leq]$

$b \wedge (c \vee d) = b \wedge a = b$

$(b \wedge c) \vee (b \wedge d) = e \vee e = e$

So, it is not distributive. (Also element b has 2 complements c and d, which is not possible in the distributive lattice, since, in a distributive lattice, complement, if it exists, is always unique).

Sets & Relations

92. (a) $X = (A - B) - C$

$= (A \cap B') \cap C'$

$= A \cap B' \cap C'$

$Y = (A - C) - (B - C)$

$= (A \cap C') - (B \cap C')$

$= (A \cap C') \cap (B \cap C')'$

$= (A \cap C') \cap (B' \cup C)$

$= (A \cap C' \cap B') \cup (A \cap C' \cap C)$

$= A \cap B' \cap C'$

$\therefore$ $X = Y$

93. (a) Let $A = \{1, 2, 3, 5, 7, 8, 9\}$

Construct the table for any $x, y \in A$ such that

$x * y = (x . y) \bmod 10$

•	1	2	3	5	7	8	9
1	1	2	3	5	7	8	9
2	2	4	6	0	4	6	8
3	3	6	9	5	1	4	7
5	5	0	5	5	5	0	5
7	7	4	1	5	9	6	3
8	8	6	4	0	6	4	2
9	9	8	7	5	3	2	1

We know that $0 \notin A$. So, it is not closed. Therefore, option (a) is true.

The identity element = 1

$\therefore$ $(2.\ 2^{-1}) \bmod 10 = 1$

From the table we see that 2^{-1} does not exist

Since, $(3.7) \bmod 10 = 1$

$\therefore$ 7 is the inverse of 3 and $7 \in A$.

(i) $E' = E - V + 2 = 19 - 13 + 2 = 8$

94. (b) $X : (P \vee Q) \to R$

$Y : (P \to R) \vee (Q \to R)$

$X : P + Q \to R \equiv (P + Q)' + R \equiv P'Q' + R$

$Y : (P' + R) + (Q' + R) \equiv P' + Q' + R$

Clearly, $X \neq Y$

Consider $X \to Y$

$\equiv (P'Q' + R) \to (P' + Q' + R)$

$\equiv (P'Q' + R)' + P' + Q' + R$

$\equiv (P'Q')' R' + P' + Q' + R$

$\equiv (P + Q) R' + P' + Q' + R$ ($\because$ by distribution law)

$\equiv PR' + QR' + P' + Q' + R$

$\equiv (PR' + R) + (QR' + Q') + P'$

$\equiv (P + R)(R' + R) + (Q + Q') \times (R' + Q') + P'$

$\equiv P + P' + R + R' + Q'$ (by complente law)

$\equiv 1 + 1 + Q' = 1$

$\therefore$ $X \to Y$ tautology.

95. (b) Every teacher is liked by some students, then the logical expression is $\forall (x)$ [teacher $(x) \to \exists (y)$ $\{$student $(y) \wedge$ likes $y(x)\}]$

where, likes (y, x) mean y likes x, such that y represents the student and x represents the teacher.equivalence and therefore, a valid first order formula.

96. (b) $X : (P \vee Q) \to R$

$Y : (P \to R) \vee (Q \to R)$

$X : P + Q \to R \equiv (P + Q)' + R = P'\,Q' + R$

$Y : (P' + R) + (Q' + R) \equiv P' + Q' + R$

Clearly, $X \neq Y$

Consider $X \to Y$

$\equiv (PQ)' + R) \to (P' + Q' + R)$

$\equiv (P'Q' + R)' + P' + Q' + R$

$\equiv (P'Q')'R' + P' + Q' + R$

$\equiv (P + Q)R' + P' + Q' + R$

$\equiv PR' + QR' + P' + Q' + R$

$\equiv (PR' + R) + (QR' + Q') + P'$

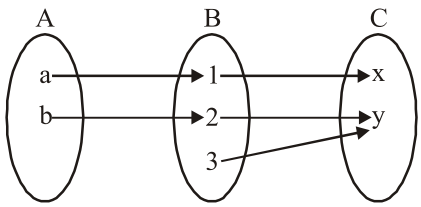
$\equiv (P + R)(R' + R) + (Q + Q') \times (R' + Q') + P'$

$\equiv P + P' + R + R' + Q'$

$= 1 + 1 + Q' \equiv 1$

$\therefore$ $X \to Y$ is a tautology.

97. (b) Clearly form the shown diagram,



g is into

f is onto

and h = g of is onto

98. (c) Given R and S be two equivalence relations on a non-empty set A. Then both are reflexive, symmatric and transitive.

$\forall\, x \in A,\ y \in A,\ z \in A$

(a) $(x, x) \in R$ and $(x, x) \in S$

$\Rightarrow$ $(x, x) \in R \cap S$

$\therefore$ $R \cap S$ is reflexive.

(a) $(x, y) \in R \cap S$

$\Rightarrow$ $(x, y) \in R$ and $(x, y) \in S$

$\Rightarrow$ $(y, x) \in R$ and $(y, x) \in S$

($\because$ R, S are symmetric)

$\Rightarrow$ $(y, x) \in R \cap S$

Thus, $R \cap S$ is symmetric

(c) (x, y) $R \cap S$ and $(y, z) \in R \cap S$

$\Rightarrow$ $(x, y) \in R$ and $(x, y) \in S$

and $(y, z) \in R$ and $(y, z) \in S$

$\Rightarrow$ $[(x, y) \in R$ and $(y, z) \in R]$

and $[(x, y) \in S$ and $(y, z) \in R]$

$\Rightarrow$ $(x, z) \in R$ and $(x, z) \in S$

(By transitivity of R and S)

$\Rightarrow$ $(x, y) \in R \cap S$

Hence, $R \cap S$ is transitive.

Thus, $R \cap S$ is an equivalence relation.

But $R \cup S$ is not necessarily an equivalence relation on A which can be explained by an example.

Let $A = \{1, 2, 3\}$

and $R = \{(1,1), (2, 2), (3, 3)\ (1, 2), (2, 1)\}$

$S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3)\ (3, 2)\}$

Then, $R \cup S = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$

Cleanly, R and S are equivalence relations on A but $R \cup S$ is not equivalence relations on A as it is not transitive because (1, 2) 6 $R \cup S$ and $(2, 3) \in R \cup S$ but $(1, 3) \notin R \cup S$.

99. (c) The number of combinations of pairs (a mod 3, b mod 5) is

$3 \times 5 = 15$

(since, a mod 3 can be 0, 1 or 2) and b mod 5 can be 0, 1, 2, 3 or 4

$\therefore$ If 16 different ordered pairs are chosen at least 2 of them must have (a mod 3, b mod 5) as same (basic pigeon hole principle).

Let such two pairs be (a, b) and (c, d) then

a mod 3 $\equiv$ c mod 3 $\Rightarrow$ a $\equiv$ c mod 3

and b mod 5 $\equiv$ d mod 5 $\Rightarrow$ b $\equiv$ d mod 5.

100. (c) The set $S = \{1, 2, 4, 7, 8, 11, 13, 14\}$ is a group under multiplication modulo 15.

The identity element for this group is e = 1 since,

$$\forall x \in S, 1 \cdot x \bmod 15 = x$$

Now, let the inverse of 4 be 4^{-1}.

Now, $(4 \cdot 4) \bmod 15 = 1$

$\therefore$ $4^{-1} = 4$ (This inverse is unique)

Similarly, let the inverse of 7 be 7^{-1}.

$$(7 \cdot 7^{-1}) \bmod 15 = 1$$

Putting each element of set as 7^{-1} by trial and error, we get

$(7 \cdot 13) \bmod 15 = 91 \bmod 15 = 1$

$\therefore$ $7^{-1} = 13$

So, 4^{-1} and 7^{-1} are respectively 4 and 13.

101. (a) $X = (A - B) - C$

$= (A \cap B') \cap C'$

$= A \cap B' \cap C'$

$Y = (A - C) - (B - C)$

$= (A - C') - (B \cap C')$

$= (A \cap C') \cap (B \cap C')'$

$= (A \cap C') \cap (B' \cup C)$

$= (A \cap C' \cap B') \cup (A \cap C' \cap C)$

$= A \cap B' \cap C'$

$\therefore$ $X = Y$

102. (b) $S = \{(x, y) \mid y = x + 1 \text{ and } x,\ y \in \{0, 1, 2, ...\}\}$

$= \{(0, 1), (1, 2), (2, 3), (3, 4), ...\}$

Now let T_1 be the reflexive closure of S.

$T = \{(0, 0), (1, 1), (2, 2), (3, 3)...\}$

$\cup \{(0, 1), (1, 2), (2, 3), (3, 4)....\}$

$= \{(0, 0), (0, 1), (1, 1), (1, 2), (2, 2), (2, 3)\ (3, 3), (3, 4)....\}$

Let T_2 be the transitive closure of S.

$(0, 1), (1, 2) \in S \Rightarrow (0, 2) \in T_2$

$(0, 2), (2, 3) \in S \Rightarrow (0, 3) \in T_2$

$(0, 3), (3, 4) \in S \Rightarrow (0, 4) \in T_2$

and so on....

Also, $(1, 2), (2, 3) \in S \Rightarrow (1, 3) \in T_2$

$(1, 3), (3, 4) \in S \Rightarrow (1, 4) \in T_2$

$(1, 4), (4, 5) \in S \Rightarrow (1, 5) \in T_2$

and so on....

$\therefore T_2 = \{(0, 1), (0, 2), (0, 3),....,(1, 2), (1, 3), (1, 4),...\}$

Now the reflexive, transition closure of S will be

$T_3 = T_1 \cup T_2$

$= \{(0, 0), (0, 1), (0, 2)...., (1, 1), (1, 2)\},$

$(1, 3)...(2, 2)\ (2, 3), (2, 4)....\}$ and so on

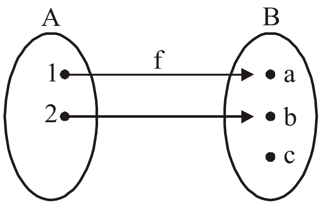
103. (d) If $a \le x$, since $p(x) \Rightarrow p(y)$ whenever $x \le y$

$\therefore p(a) \Rightarrow p(x)$

Now since p(a) = True, p(x) = cannot be false.

$\therefore$ (d) cannot be true.

104. (a) consider A, B and f as defined below (since f needs not be onto)



Consider a subset D of B such that

$D = \{a, c\}$

$h(D) = \{1\}$

$g(h(D)) = \{a\}$

$g(h(D)) \subseteq D$ is true

Notice, that if the function f were both one-to-one and onto then $g(h(D)) = D$.

105. (c) The possible solution pairs are (a, a), (a, b), (a, c), (b, a), (b, b), (c, a), (c, b) and (c, c).

Substitute them one by one in both equations and see which of them satisfies both the equations.

The given equations are

$(a \times x) + (a \times y) = c$ ...(i)

$(b \times x) + (c \times y) = c$ ....(ii)

Substitue first (x, y) = (a, a)

LHS of Eq. (i) becomes

$(a \times a) + (a \times a) = a + a = b$

Now, RHS of Eq. (i) = c.

Therefore, LHS $\neq$ RHS. This means that (a, a) is not a solution pair. Similarly try each of the remaining seven possible solution pairs. It will be found that only two pairs (b, c) and (c, b) will satisfy both Eqs. (i) and (ii) simultaneously.

106. (b) $T(1) = 1$

$T(2) = T(1) + \lfloor \sqrt{2} \rfloor$

$= 1 + 1 = 2$

$T(3) = T(2) + \lfloor \sqrt{3} \rfloor$

$= 2 + 1 = 3$

$T(4) = T(3) + \lfloor \sqrt{4} \rfloor$

$= 3 + 2 = 5$

$T(5) = T(4) + \lfloor \sqrt{5} \rfloor$

$= 5 + 2 = 7$

$T(6) = T(5) + \lfloor \sqrt{6} \rfloor$

$= 7 + 2 = 9$

$T(7) = 9 + 2 = 11$

$T(8) = 11 + 2 = 13$

$T(9) = 13 + \sqrt{9} = 16$ and so on till.

$T(16) = 16 + 6 \times 3 + 4 = 38$

$\therefore$ T(1) = 1, T(4) = 5, T(9) = 16 and T(16) = 38.

Choice (a) does not satisfy T(16).

Choice (c) does not satisfy T(4).

Choice (d) does not satisfy T(1).

$\therefore$ Answer is choice (b) which satisfies T(1) upto T(16).