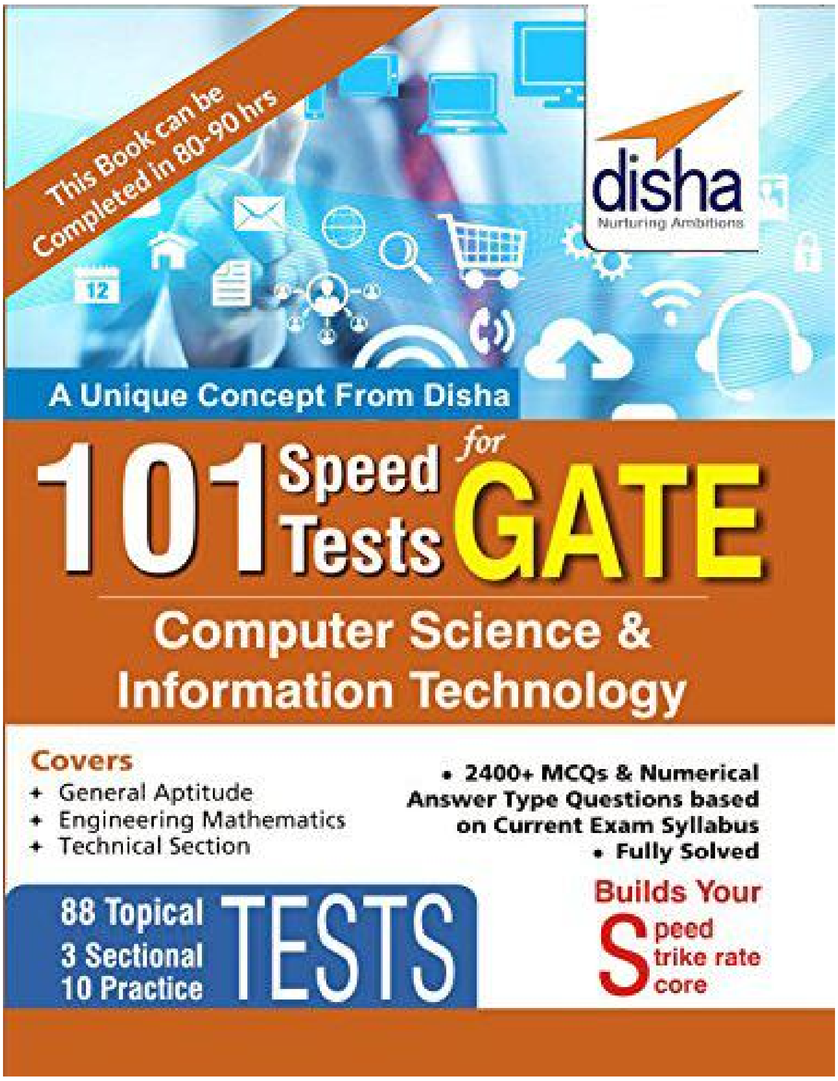
Minimum Spanning Trees

This Chapter is taken from our Book:



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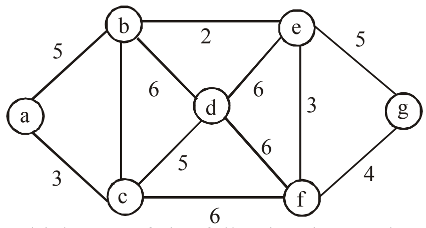
Minimum Spanning Trees

Max. Marks : 21 **No. of Qs. 21** **Time : 30 min.** **Date : ..../......../...............**

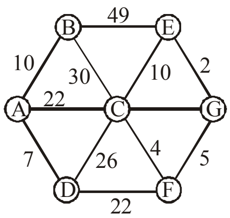
1. Let G be a weighted graph with edge weights greater than one and G be the graph constructed by squaring the weights of edges in G. Let T and T' be the minimum spanning trees of G and G' respectively, with total weights t and t'. Which of the following statements is true?
   (a) T' = T with total weight $t' = t^2$
   (b) T' = T with total weight $t' = t^2$
   (c) T' ¹ T but total weight $t' = t^2$
   (d) None of the above
2. An undirected graph G has n nodes. Its adjacency matrix is given by an n × n square matrix whose (i) diagonal elements are 0's and (ii) non-diagonal elements are 1's. Which one of the following is true?
   (a) Graph G has no Minimum Spanning Tree (MST)
   (b) Graph G has a unique MST of cost n – 1
   (c) Graph G has multiple distinct MSTs, each of cost n – 1
   (d) Graph G has multiple spanning trees of different costs
3. Consider a weighted complete graph G on the vertex set $\{v_1, v_2, ... v_n\}$ such that the weight of the edge (v_i, v_j) is $2|i - j|$. The weight of a minimum spanning tree of G is
   (a) n – 1 (b) 2n – 2 (c) $\binom{n}{2}$ (d) n^2
4. Let w be the minimum weight among all edge weights in an undirected connected graph. Let e be a specific edge of weight w. Which of the following is false?
   (a) There is a minimum spanning tree containing e
   (b) If e is not in a minimum spanning tree T, then in the cycle formed by adding e to T, all edges have the same weight.
   (c) Every minimum spanning tree has an edge of weight w
   (d) e is present in every minimum spanning tree.
5. G is a graph on n vertices and 2n – 2 edges. The edges of G can be partitioned into two-edge-disjoint spanning trees. Which of the following is not true for G?
   (a) For every subset of k vertices, the induced subgraph has at most 2k – 2 edges
   (b) The minimum cut in G has at least two edges
   (c) There are two edge-disjoint paths between every pair of vertices
   (d) There are two vertex-disjoint paths between every pair of vertices.
6. Consider the following graph:



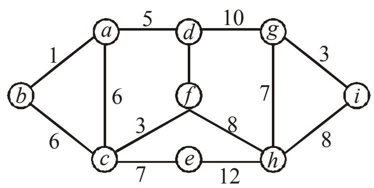
   Which one of the following is not the sequence of edges added to the minimum spanning using Kruskal's algorithm?
   (a) (b, e) (e, f) (a, c) (b, c) (f, g) (c, d)
   (b) (b, e) (e, f) (a, c) (f, g) (b, c) (c, d)
   (c) (b, e) (a, c) (e, f) (b, c) (f, g) (c, d)
   (d) (b, e) (e, f) (b, c) (a, c) (f, g) (c, d)
7. Consider a complete undirected graph with vertex set {0, 1, 2, 3, 4}. Entry W_{ij} in the matrix W below is the weight of the edge {i, j}

$$W = \begin{pmatrix} 0 & 1 & 8 & 1 & 4 \\ 1 & 0 & 12 & 4 & 9 \\ 8 & 12 & 0 & 7 & 3 \\ 1 & 4 & 7 & 0 & 2 \\ 4 & 9 & 3 & 2 & 0 \end{pmatrix}$$

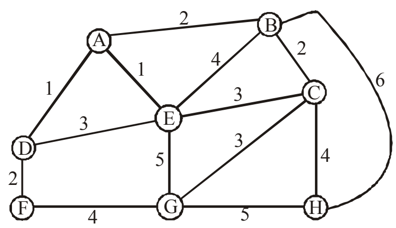
   What is the minimum possible weight of a spanning tree T in this graph such that vertex 0 is a leaf node in the tree T?
   (a) 7 (b) 8
   (c) 9 (d) 10
8. Consider this undirected graph and using Prim's algorithm. Construct a minimum spanning tree starting with node A. Then find the sequence of MST is :-



   (a) (A, C), (A, B), (A, D), (A, D), (F, G), (C, F), (E, G)
   (b) (A, C), (A, D), (A, B), (A, D), (E, G), (F, G), (C, G)
   (c) (E, G), (C, F), (F, G), (A, D), (A, D), (A, B), (A, C)
   (d) None of these
9. For the undirected, weighted graph given below, which of the following sequences of edges represents a correct execution of Prim's algorithm to construct a minimum spanning tree?

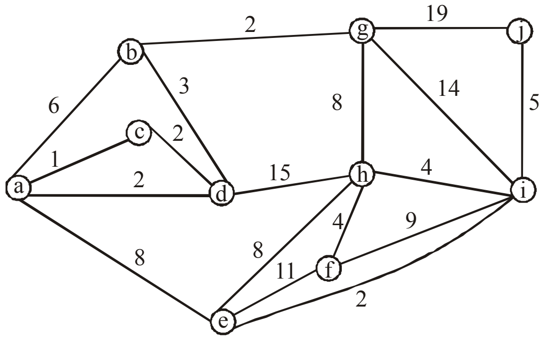


   (a) (a, b), (d, f), (f, c), (g, i), (d, a), (g, n), (c, e), (f, h)
   (b) (c, e), (c, f), (f, d), (d, a), (a, b), (g, h), (h, f), (g, i)
   (c) (d, f), (f, c), (d, a), (a, b), (c, e), (f, n), (g, h), (g, i)
   (d) (h, g), (g, i), (h, f), (f, c), (f, d), (d, a), (a, b), (c, e)
10. The following graph has a minimum spanning tree that is a



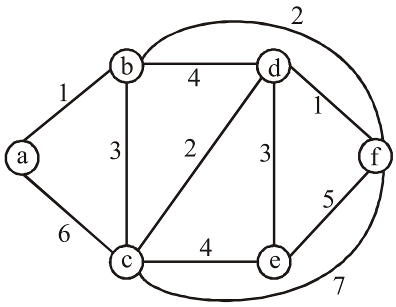
   (a) strictly binary tree but not complete
   (b) complete binary tree
   (c) almost complete tree
   (d) none of the above

11. What is the weight of a minimum spanning tree of the following graph?



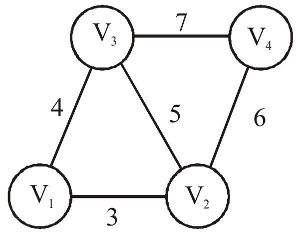
(a) 29 (b) 31
(c) 38 (d) 41

12. Consider the following graph?



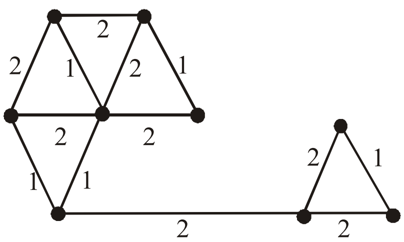
Which one of the following cannot be the sequence of edges added, in that order, to a minimum spanning tree using Kruskal's algorithm?
(a) (a – b), (d – f), (b – f), (d – c), (d – e)
(b) (a – b), (d – f), (b – c), (b – f), (d – e)
(c) (a – f), (a – b), (d – c), (b – f), (d – e)
(d) (d – f), (a – b), (b – f), (d – e), (d – c)

13. An undirected graph G(V, E) contains n (n > 2) nodes named v1 , v2 ,….vn. Two nodes vi , vj are connected if and only if 0 < |i - j| <= 2. Each edge (vi, vj) is assigned a weight i + j. A sample graph with n = 4 is shown below. What will be the cost of the minimum spanning tree (MST) of such a graph with n nodes?



(a) 1/12(11n^2 - 5n) (b) n^2 - n + 1
(c) 6n - 11 (d) 2n + 1

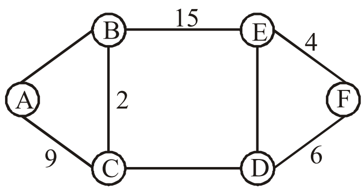
14. The number of distinct minimum spanning trees for the weighted graph below is ____.



(a) 4 (b) 5
(c) 6 (d) 7

15. Let s and t be two vertices in a undirected graph G + (V, E) having distinct positive edge weights. Let [X, Y] be a partition of V such that s ∈ X and t ∈ Y. Consider the edge e having the minimum weight amongst all those edges that have one vertex in X and one vertex in Y The edge e must definitely belong to:
(a) The minimum weighted spanning tree of G
(b) The weighted shortest path from s to t
(c) Each path from s to t
(d) The weighted longest path from s to t

16. The graph shown below 8 edges with distinct integer edge weights. The minimum spanning tree (MST) is of weight 36 and contains the edges: {(A, C), (B, C), (B, E), (E, F), (D, F)}. The edge weights of only those edges which are in the MST are given in the figure shown below. The minimum possible sum of weights of all 8 edges of this graph is ______________.



(a) 66 (b) 68
(c) 69 (d) 70

17. Let G be connected undirected graph of 100 vertices and 300 edges. The weight of a minimum spanning tree of G is 500. When the weight of each edge of G is increased by five, the weight of a minimum spanning tree becomes ________.
(a) 1000 (b) 995
(c) 2000 (d) 1995

18. Let G be a weighted connected undirected graph with distinct positive edge weights. If every edge weight is increased by the same value, then which of the following statements is/are TRUE?
P : Minimum spanning tree of G does not change
Q : Shortest path between any pair of vertices does not change
(a) P only (b) Q only
(c) Neither P nor Q (d) Both P and Q

19. Let G be a complete undirected graph on 4 vertices, having 6 edges with weights being 1, 2, 3, 4, 5, and 6. The maximum possible weight that a minimum weight spanning tree of G can have is. …
(a) 6 (b) 7
(c) 8 (d) 9

20. G = (V, E) is an undirected simple graph in which each edge has a distinct weight, and e is a particular edge of G. Which of the following statements about the minimum spanning trees (MSTs) of G is/are TRUE
I. If e is the lightest edge of some cycle in G, then every MST of G includes e
II. If e is the heaviest edge of some cycle in G, then every MST of G excludes e
(a) I only (b) II only
(c) both I and II (d) neither I nor II

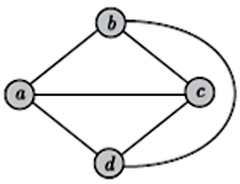
21. What is the largest integer m such that every simple connected graph with n vertices and n edges contains at least m different spanning trees?
(a) 1 (b) 2
(c) 3 (d) n

Hints & Solutions

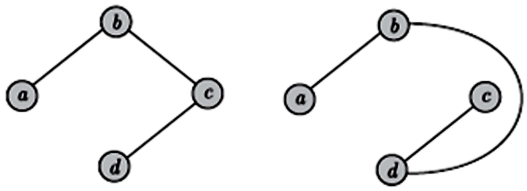
1. **(d)** Squaring the weights of the edges in a weighted graph will not change the minimum spanning tree. Assume the opposite to obtain a contradiction. If the minimum spanning tree changes then at least one edge from the old graph G in the old minimum spanning tree T must be replaced by a new edge in tree T' from the graph G' with squared edge weights. The new edge from G' must have a lower weight than the edge from G. This implies that there exists some weights C_1 and C_2 such that $C_1 < C_2$ and $C_{12} >= C_{22}$. This is a contradiction. Sums of squares of two or more numbers is always smaller than square of sum.
Example: $2^2 + 2^2 < 4^2$

2. **(c)** **Given adjacency matrix of order 4 is 4*4**

0	0	1	1	1
1	1	0	1	1
2	1	1	0	1
3	1	1	1	0

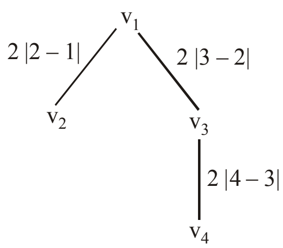


There can be many min spaning but all of $n - 1$ cost



So on.

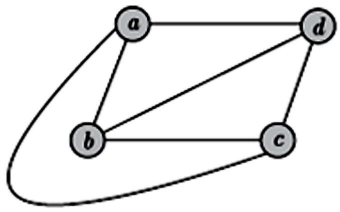
3. **(b)** The minimum spanning tree is formed when
$V_i \leftarrow \{1, 2, ...\}$
Lets construct a tree on the basis of
(v_i, v_j) is $2|i - j|$
Considering the maximum value (n) to be 4.



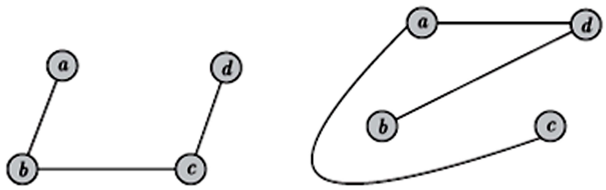
As V_i is connected to V_{i+1},
The minimum weight of each edge = 2
Therefore, the weight of tree with n – 1 edges
= 2(n – 1) = 2n – 2.

4. **(d)** Minimum weight w
Edge e
Weight w
Now, w is the minimum weight among the edges.
There is a possibility that two edges may contain the same weight w. This would then be added to minimum spanning tree.
Now, when the edge, e is added to the minimum tree we get a circuit.
Therefore, to avoid the circuit, e cannot be included in the minimum spanning tree.

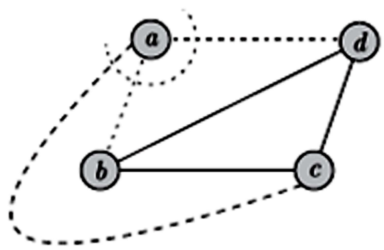
5. **(d)** Consider this graph with $n = 4$



Two spanning trees



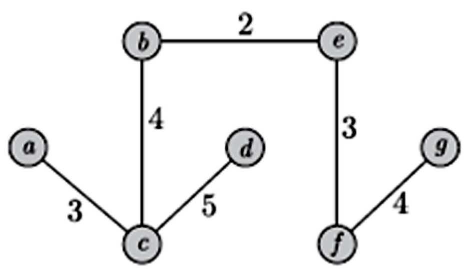
Statement (B), (C) & (D) are correct.



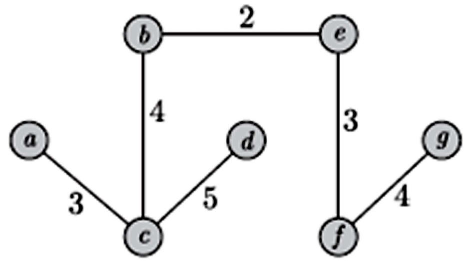
min cent has 3 edges.
Two edge & vertex disjoint paths are present can be seen in two spanning trees but option (A) is false for $K = 2$ $\quad 2K - 2 =$ i.e 2 edges
should be there but it is not true.

6. **(d)** Kruskal's algorithm, arranging edges in ascending order.
{2, 3, 3, 4, 4, 5, 5, 6, 6, 6, 6}

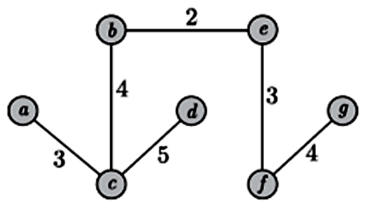
(a)



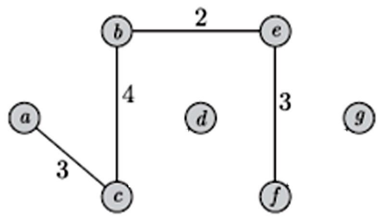
(b)



(c)

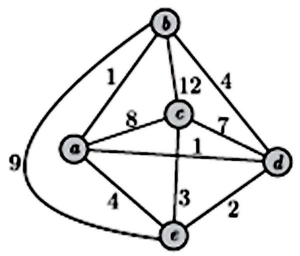


(d)



(A, C) can't be taken after (B, C).
Hence (D) is correct option.

7. **(a)** let {0,1,2,3,4} be {*a*,*b*,*c*,*d*,*e*}



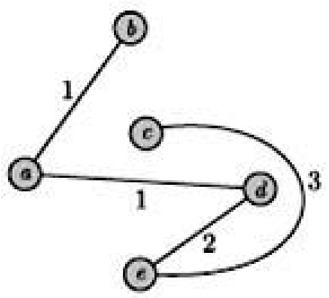
Drawing spanning true using Prim's Algorithms start with *a*

(A) &(*a*,*b*),(*a*,*c*),(*a*,*d*),(*a*,*e*) (*a*,*b*) = 1

(B) &(*a*,*c*), (*a*,*d*), (*a*,*e*), (*b*,*c*), (*b*,*d*), (*b*,*e*) (*a*,*d*) = 1

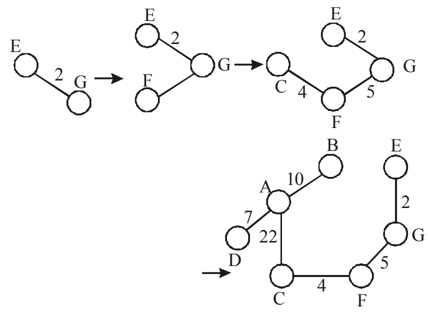
(D) &(*a*,*c*), (*a*,*e*), (*b*,*c*), (*b*,*d*), (*b*,*e*), (*c*,*d*) (*d*,*e*) = 2

(E) &(*a*,*c*), (*a*,*e*), (*b*,*c*), (*b*,*d*), (*b*,*e*), (*c*,*d*), (*c*,*e*) (*c*,*e*) = 3



Weight = 1 + 1 + 2 + 3 = 7

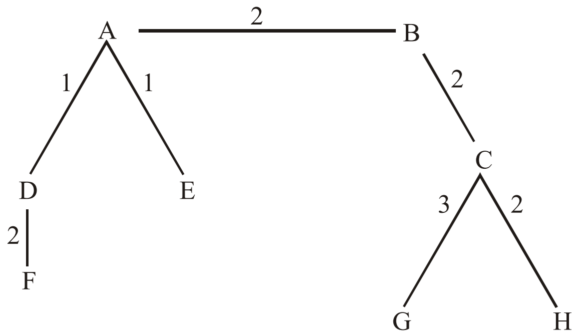
8. **(c)** In Prim's alogorithm, the edge with smallest weight added first, so sequence of addition of edge in spanning tree so that cycle does not exist will be



Hence, sequence = (E, G), (C, F), (F, G), (A, D), (A, D), (A, B), (A, C)

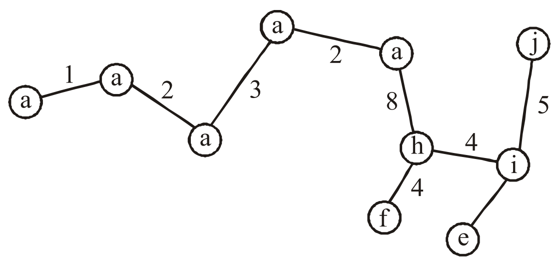
9. **(c)**

10. **(c)** Spanning tree



It is almost complete tree.

11. **(a)** Spanning tree is minimum, when all weight assigned to its edges between two vertices where every vertex is connected.



Height of minimum spanning tree
= 1 + 2 + 3 + 2 + 8 + 4 + 4 + 2 + 5 = 31

12. **(d)** Ordering the weights, the sequence obtained is
1, 1, 2, 2, 3, 3, 4, 4, 5, 6, 7
Now, in order to obtain minimum spanning tree using Kruskal's algorithm, we have to add edges with weights in increasing order such that weight of the spanning tree is minimum.
In option (d), weights of the edges are 1, 1, 2, 3, 2 which contradicts Kruskal's algorithm.

13. **(b)** Minimum spanning tree for 2 nodes would be
(v1) _ (v2)
Total weight 3
Minimum spanning tree for 3 nodes would be
(v1) _ (v2)
|
(v3)
Total weight= 3 + 4 = 7
Minimum spanning tree for 4 nodes would be
(v1) _ (v2) _ (v4)
|
(v3)

Total weight= 3 + 4 + 6 = 13
Minimum spanning tree for 5 nodes would be
(v1) _ (v2) _ (v4)
|
(v3)
|
(v5)
Total weight= 3 + 4 + 6 + 8 = 21
Minimum spanning tree for 6 nodes would be
(v1) _ (v2) _ (v4) _ (v6)
|
(v3)
|
(v5)

Total weight= 3 + 4 + 6 + 8 + 10 = 31

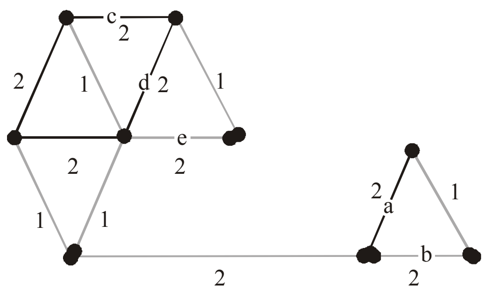
We can observe from above examples that when we add kth node, the weight of spanning tree increases by 2k-2. Let T(n) be the weight of minimum spanning tree. T(n) can be written as

T(n) = T(n-1) + (2n-2) for n > 2

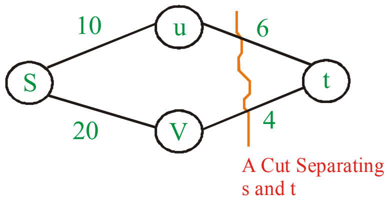
T(1) = 0, T(2) = 0 and T(2) = 3

The recurrence can be written as sum of series (2n - 2) + (2n-4) + (2n-6) + (2n-8) + ... 3 and solution of this recurrence is $n^2 – n + 1$.

14. **(c)** Highlighted (in green) are the edges picked to make a MST. In the right side of MST, we could either pick edge 'a' or 'b'. In the left side, we could either pick 'c' or 'd' or 'e' in MST. There are 2 options for one edge to be picked and 3 options for another edge to be picked. Therefore, total 2*3 possible MSTs.



15. **(b)** The minimum weight edge on any s-t cut is always part of MST. This is called Cut Property. This is the idea used in Prim's algorithm. The minimum weight cut edge is always a minimum spanning tree edge. See below example, edge 4 (lightest in highlighted red cut from s to t) is not part of



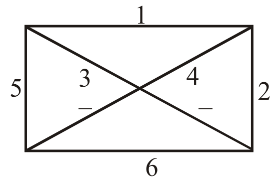
shortest path.

16. **(c)** In every cycle, the weight of an edge that is not part of MST must by greater than or equal to weights of other edges which are part of MST. Since all edge weights are distinct, the weight must be greater. So the minimum possible weight of ED is 7, minimum possible weight of CD is 16 and minimum possible weight of AB is 10. Therefore minimum possible sum of weights is 69.

17. **(b)** Since there are 100 vertices, there must be 99 edges in Minimum Spanning Tree (MST). When weight of every edge is increased by 5, the increment in weight of MST is = 99 * 5 = 495 So new weight of MST is 500 + 495 which is 995.

18. **(a)** The shortest path may change. The reason is, there may be different number of edges in different paths from s to t. For example, let shortest path be of weight 15 and has 5 edges. Let there be another path with 2 edges and total weight 25. The weight of the shortest path is increased by 5*10 and becomes 15 + 50. Weight of the other path is increased by 2*10 and becomes 25 + 20. So the shortest path changes to the other path with weight as 45. The Minimum Spanning Tree doesn't change. Remember the Kruskal's algorithm where we sort the edges first. IF we increase all weights, then order of edges won't change.

19. **(c)** One graph that has maximum possible weight of spanning tree



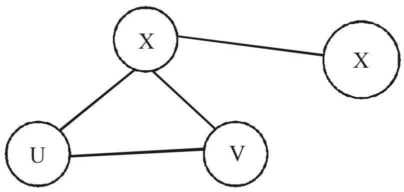
20. **(b)** **I. is NOT true.** Let G=(V, E) be a rectangular graph where V = {a, b, c, d} and E = {ab, bc, cd, da, ac}. Let the edges have weights: ab = 1, bc = 2, cd = 4, da = 5, ac = 3. Then, clearly, ac is the lightest edge of the cycle cd,ac, however, the MST abcd with cost 7 (= ab + bc + cd) does not include it. Let the edges have weights: ab = 6, bc - 7, cd = 4, da = 5, ac = 3. Then, again, ac is the lightest edge of the cycle cd,ac, and, the MST ba,cd with cost 13 (= ba + ac + cd) includes it. So, the MSTs of G may or may not include the lightest edge

II. is true Let the heavies edge be e. Suppose the minimum spanning tree which contains e. If we add one more edge to the spanning tree we will create a cycle. Suppose we add edge e' to the spanning tree which generated cycle C. We can reduce the cost of the minimum spanning tree if we choose an edge other than e from C for removal which implies that e must not be in minimum spanning tree.

21. **(c)** A graph is connected if all nodes can be traversed from each node. For a graph with n nodes, there will be n-1 minimum number of edges.

Given that there are n edges, that means a cycle is there in the graph.

The simplex graph with these conditions may be:



Now we can make a different spanning tree by removing one edge from the cycle, one at a time.

Minimum cycle length can be 3, So, there must be atleast 3 spanning trees in any such Graph.