Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2017

Senior Section (Round 1)

Tuesday, 30 May 2017 **0930 – 1200 hrs**

Instructions to contestants

1. *Answer ALL 35 questions.*
2. *Enter your answers on the answer sheet provided.*
3. *For the multiple choice questions, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer.*
4. *For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer.*
5. *No steps are needed to justify your answers.*
6. *Each question carries 1 mark.*
7. *No calculators are allowed.*

PLEASE DO NOT TURN OVER UNTIL YOU ARE TOLD TO DO SO.

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Multiple Choice Questions

1. Find all the values of k for which the expression $x^2 + k(k-x) + 3k$ is positive for all x.

   (A) $0 < k < 4$ (B) $-4 < k < 0$ (C) $k < 0,\ k > 4$ (D) $k > 0,\ k < -4$

   (E) None of the above

2. Simplify
$$\log_3 12 + \log_9 16 - \log_{27} 8.$$

   (A) $\log_3 4$ (B) $\log_3 8$ (C) $\log_3 16$ (D) $\log_3 24$ (E) $\log_3 32$

3. Solve for x in the following equation
$$7^{x-1} + \frac{1}{7} - \frac{6}{7^x} = 0.$$

   (A) $\log_6 7$ (B) $\log_6 14$ (C) $\log_7 6$ (D) $\log_7 12$ (E) $\log_7 14$

4. When a polynomial is divided by $(x-2)$ and $x^2 - 3x + 2$, the remainders are 11 and $Ax + 5$ respectively, where A is some integer. Find the remainder when it is divided by $(x-1)$.

   (A) 7 (B) 8 (C) 9 (D) 10 (E) 11

5. Simplify
$$\frac{\sqrt{5}^3 - \sqrt{2}^3}{\sqrt{5} - \sqrt{2}} - \frac{\sqrt{5}^3 + \sqrt{2}^3}{\sqrt{5} + \sqrt{2}}.$$

   (A) $\sqrt{10}$ (B) $2\sqrt{10}$ (C) $3\sqrt{10}$ (D) $\sqrt{5} - \sqrt{2}$ (E) $\sqrt{5} + \sqrt{2}$

6. Which of the following is the largest?

   (A) 3^{100} (B) 90^{10} (C) 82^{25} (D) $8^{49} + 7^{49}$ (E) 2^{150}

7. Suppose $270° < a < 360°$. Which of the following is equal to
$$\sqrt{\cos^3 a} - \sqrt{\sin^2 a \cos a}\ ?$$

   (A) $(\sin a + \cos a)\sqrt{\cos a}$ (B) $(\sin a - \cos a)\sqrt{\cos a}$ (C) $(\cos a - \sin a)\sqrt{\cos a}$

   (D) $(-\sin a - \cos a)\sqrt{\cos a}$ (E) $\sqrt{\cos a}$

8. Find the range of values of x that satisfy both of the following inequalities:

$$(x+3)^2 \leq (3x-1)^2, \quad 3x \geq x^2.$$

(A) $0 \leq x \leq 1$ (B) $1 \leq x \leq 2$ (C) $2 \leq x \leq 3$ (D) $3 \leq x \leq 4$
(E) None of the above

9: If $f(x) = -x + 6\sqrt{x+16} - 5$, where $-16 \leq x \leq 0$, find the range of $f(x)$.

(A) $-29 \leq f(x) \leq 20$ (B) $-29 \leq f(x) \leq 19$ (C) $11 \leq f(x) \leq 19$

(D) $19 \leq f(x) \leq 20$ (E) $11 \leq f(x) \leq 20$

10. A circle is given by the equation $x^2 - 4x + y^2 + 8y = 5$. Find the equation of the tangent line to the circle at the point $(-2, -1)$.

(A) $3y - 4x - 5 = 0$ (B) $-7y + x - 5 = 0$ (C) $3y + 2x + 7 = 0$

(D) $3y - 2x - 1 = 0$ (E) $4y - 3x - 2 = 0$

Short Questions

11. Suppose $x^2 + 3x + 15$ is a factor of $2x^4 + mx^2 + 30n$ where m, n are integers. Find the value of mn.

12. How many different real numbers x, where $0° \leq x \leq 360°$, satisfy the equation

$$3\sin x - 4\sin^3 x = \frac{1}{2}\ ?$$

13. Suppose $\dfrac{1-\sin 2A}{1+\cos 2A} = \tan A - 1 \neq 0$. Find $\tan^2 A$.

14. Given that $\sin\left(\dfrac{\pi}{4} - x\right) = \dfrac{3}{5}$, $0 < x < \dfrac{\pi}{4}$, find $\dfrac{15\cos 2x}{\cos\left(\frac{\pi}{4} + x\right)}$.

15. Find the coefficient of x^3 in the expansion of $(1 + 2x + 3x^2 + 4x^3)^8$.

16. Find the maximum value of $\dfrac{1 + x - 99x^2}{x(x^2+1)}$, where $\dfrac{1}{100} \leq x \leq \dfrac{1}{10}$.

17. Four points O, P, Q, R have coordinates $(0,0)$, $(0,30)$, (x,y) and $(20,0)$ respectively, where the point Q lies on the curve $y = 50 - x^2$ in the first quadrant. Let Π be the polygon $OPQR$ with four sides OP, PQ, QR and RO. What is the maximum possible area of Π if this area must be an integer?

18. There are 5 girls and 5 boys in a junior class, and 4 girls and 9 boys in a senior class. A committee of 7 members is to be formed by selecting students from these two classes. Find the number of ways this can be done if the committee must have exactly 4 seniors and exactly 5 boys.

19. Suppose x is a real number. Find the largest possible integer that can be attained by the expression
$$\frac{7770 - |x-10|}{|x-5| + |x-15|}.$$

20. Evaluate
$$\left(10 - \frac{4\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{\pi}{7}\right) - \cos\left(\frac{2\pi}{7}\right) - 1}{2\cos\left(\frac{\pi}{7}\right)}\right)^2.$$

21. What is the last digit of 2017^{2017}?

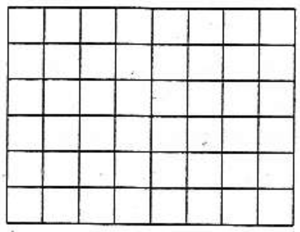
22. In how many ways can 4 integers, $a_1 < a_2 < a_3 < a_4$, be chosen from the integers 1, 2, 3, ..., 26 such that $5 \le a_i - a_{i-1} \le 7$ for all $i = 2, 3, 4$?

23. Let α and β be the roots of the quadratic equation $x^2 - 7x + 11 = 0$. Determine the value of $\alpha^4 + \beta^4$.

24. Let $\{a_1, a_2, a_3, \ldots\}$ be a sequence of real numbers. Let $\{b_1, b_2, b_3, \ldots\}$ be a sequence of real numbers such that $b_n = a_{n+1} - a_n$ for all $n \ge 1$. Determine the maximum value of a_n within the sequence $\{a_1, a_2, a_3, \ldots\}$ if $b_{m+1} - b_m = -2$ for all $m \ge 1$, $a_5 = 615$ and $a_{10} = 1045$.

25. There are 12 blue socks, 14 red socks, 16 green socks, 18 yellow socks and 20 orange socks in a drawer. Socks of the same colour are indistinguishable. A person randomly picks a certain number of socks from the drawer. Find the minimum number of socks that should be taken to ensure that he will have at least two pairs of colour X, at least two pairs of colour Y and at least two pairs of colour Z, for some three distinct colours X, Y, Z.

26. Determine the number of paths to move from the top-left cell to the bottom-right cell in the 8×6 cell grid using a sequence of downwards moves and rightwards moves such that there are an even number of direction changes.

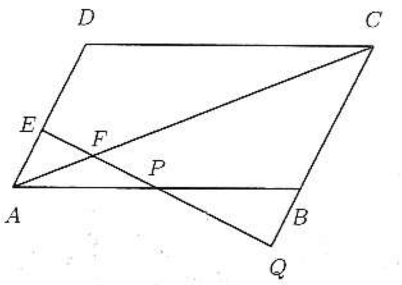


27. Consider the following equation where m and n are positive integers:

$$3^m + 3^n - 8m - 4n! = 680.$$

Determine the sum of all possible values of m.

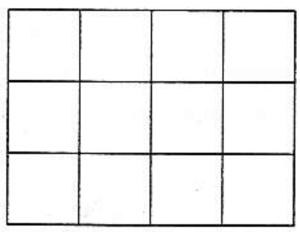
28. In the figure below, $ABCD$ is a parallelogram, where B lies on the straight line CQ. The line EQ intersects the lines AC and AB at the point F and P respectively, where P is the midpoint of the line AB. Let $|AE|$, $|AD|$, $|AC|$ and $|AF|$ denote the length of the line segment AE, AD, AC and AF respectively. If $|AE| = 3$ cm, $|AD| = 9$ cm, find $\dfrac{6 \times |AC|}{|AF|}$.



29. Find the smallest positive integer n such that

$$5(3^2 + 2^2)(3^4 + 2^4)(3^8 + 2^8) \cdots (3^{2^n} + 2^{2^n}) > 9^{256}.$$

30. The following diagram shows paths (edges in the grid) connecting 5×4 lattice points. Each path is exactly 1 meter long. Determine the shortest distance (in meters) a person needs to travel so that he will walk through each path at least once and returns to the starting position.



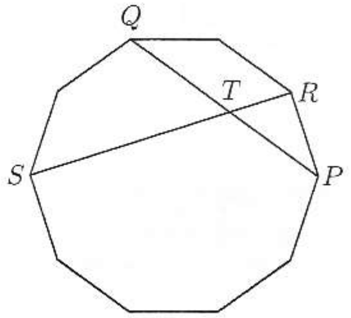
31. Let a, b, x and y be real variables such that

$$ax + by = 40, \quad ax^2 + by^2 = 110, \quad ax^3 + by^3 = 310, \quad ax^4 + by^4 = 890.$$

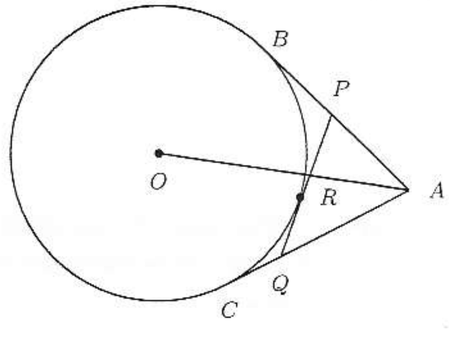
Determine the value of $ax^5 + by^5$.

32. Determine the largest prime number that cannot be expressed as the sum of three composite odd integers.

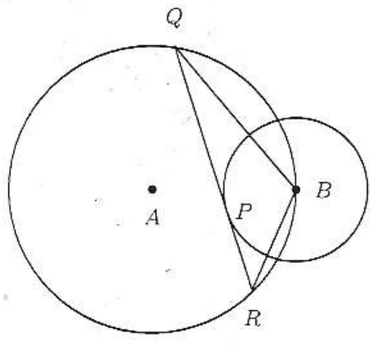
33. Two lines PQ and RS connect the vertices of a regular decagon (10-sided polygon) and intersect at the point T. Suppose $\angle PTS = x^\circ$. Find x.



34. In the figure below, the lines AB, AC and PQ are tangent to the circle (with centre O) at the points B, C and R respectively. If the circle has radius 9 cm and the length of the line segment OA is 15 cm, find the perimeter of $\triangle APQ$ in cm.



35. In the figure below, the circle centred at A has radius 20 cm and the circle centred at B has radius 5 cm. The line segment QR is tangent to the smaller circle at the point P. Let $|QB|$ and $|BR|$ denote the length of the line segment QB and BR respectively. Find the value of $|QB| \times |BR|$.



Multiple Choice Questions

1. The roots of the quadratic equation $x^2 - 3x + 1 = 0$ are α and β. Find a quadratic equation whose roots are $\alpha^2 + \alpha$ and $\beta^2 + \beta$.

   (A) $x^2 + 10x - 5 = 0$ (B) $x^2 - 10x + 5 = 0$ (C) $x^2 - 5x - 10 = 0$

   (D) $x^2 + 5x - 10 = 0$ (E) $x^2 - 5x - 5 = 0$

2. Simplify
$$\sqrt{2+\sqrt{3}} + \sqrt{2-\sqrt{3}}.$$

   (A) $\sqrt{5}$ (B) $\sqrt{6}$ (C) $\sqrt{7}$ (D) $\sqrt{8}$ (E) $\sqrt{10}$

3. Which of the following is true?

   (A) $10^{30} > 40^{15} > 30^{20}$ (B) $30^{20} > 10^{30} > 40^{15}$ (C) $10^{30} > 30^{20} > 40^{15}$

   (D) $30^{20} > 40^{15} > 10^{30}$ (E) $40^{15} > 30^{20} > 10^{30}$

4. Which of the following is the largest?

   (A) $\log_5 7 - \log_5 6$ (B) $\log_6 8 - \log_6 7$ (C) $\log_7 9 - \log_7 8$

   (D) $\log_8 10 - \log_8 9$ (E) $\log_9 11 - \log_9 10$

5. In the expansion of $\left(x^2 + \frac{1}{x}\right)^9$, find the coefficient of x^{15}.

   (A) 9 (B) 36 (C) 84 (D) 126 (E) None of the above

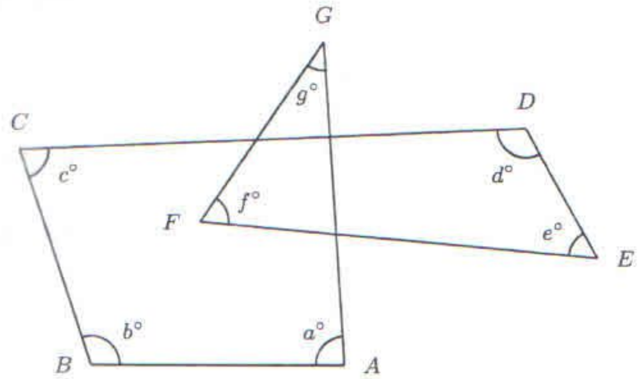
Short Questions

6. The line $y = 5x - 10$ meets the curve $x^2 - 4x + y^2 - 32 = 0$ at two points P and Q. Find the length of the line segment PQ in meters, assuming that both the x and y axis are measured in meters.

7. Suppose n is a positive integer, and x is measured in radians. If the equation $2\pi \sin nx = 2\pi - x$, where $0 \le x < 2\pi$, has exactly 2018 different solutions, find n.

8. The polynomial $p(x) = x^3 + Ax^2 + Bx - 3$ has a factor of $x^2 + 7x + 1$. Find the remainder when the polynomial $p(x)$ is divided by $x - 4$.

9. In the figure below, the angles indicated at the vertex A, B, C, D, E, F, G are given by a°, b°, c°, d°, e°, f°, g° respectively. Find $a + b + c + d + e + f + g$.



10. Find x, where $0^\circ \leq x^\circ \leq 90^\circ$, such that $\cos x^\circ = \cos 49^\circ + \cos 71^\circ$.

11. Let x and y be real numbers. Find the maximum value of $2x^2 - 3xy - 2y^2$ subject to the condition that

$$25x^2 - 20xy + 40y^2 = 36.$$

12. Suppose $\sin 2x = \frac{7}{9}$. Find $108(\sin^6 x + \cos^6 x)$.

13. Find the sum of all the positive integers x satisfying

$$(4\log_2(\log_{16} x))\,(3\log_{16}(\log_2 x) - 1) = 1.$$

14. Consider the function $f(x) = ax^2 - c$, where a and c are some constants. Suppose that $-3 \leq f(1) \leq -2$ and $1 \leq f(2) \leq 6$. Find the largest possible value of $f(4)$.

15. Suppose a is the smallest number satisfying the inequality

$$\left|\frac{|x+9|}{10} - \frac{|x-9|}{10}\right| \leq x - 1.$$

If $a = \frac{m}{n}$, where m and n are positive integers having no common factors larger than 1, find the value of $m + n$.

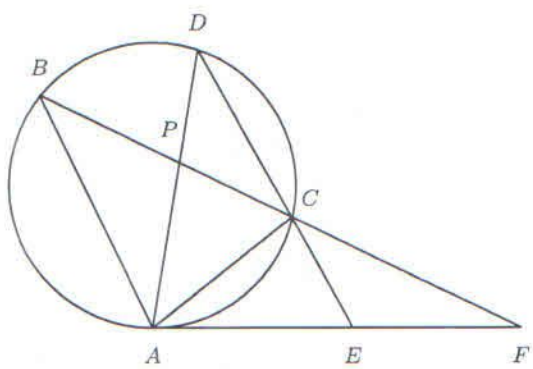
16. Let $P(x)$ be a polynomial of degree 4 such that $P(n) = \dfrac{120}{n}$ for $n = 1, 2, 3, 4, 5$. Determine the value of $P(6)$.

17. Let
$$L = \sum_{k=7}^{16} (1 + \tan(15k^\circ + 15^\circ)\tan 15k^\circ)\,.$$
Find the largest integer which is smaller than or equal to L.

18. Find the smallest three-digit positive integer whose square ends in the digits 129.

19. In the figure below, the points A, B, C and D lie on the circle such that the lines AD and BC intersect at the point P. The line AF is tangent to the circle. The point E lies on the line AF such that the lines DE and PF intersect at C.

If the line AP bisects the angle $\angle BAC$, and $\angle CEA = (22 + y)^\circ$ where $y^\circ = \angle BAP$, find the angle $\angle APC$ (in $^\circ$).



20. Eleven identical boxes are arranged in a row. In how many ways can eight identical balls be put into the boxes if each box can hold at most one ball and no three empty boxes can appear consecutively next to each other?

21. Find the minimum positive integer N such that among any N distinct positive integers, there always exist two distinct positive integers such that either their sum or their difference is a multiple of 2018.

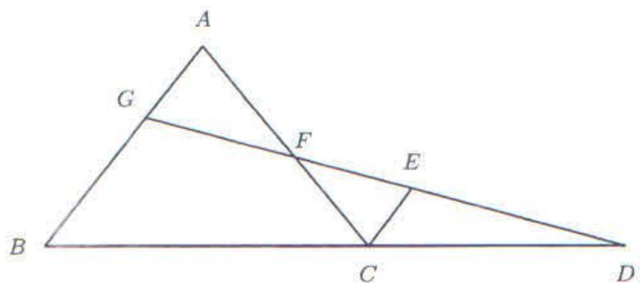
22. Suppose $f(x)$ is defined for all positive numbers x, and $2f(x - x^{-1}) + f(x^{-1} - x) = 3(x + x^{-1})^2$. Find $f(99)$.

23. Suppose x, y and z are positive real numbers satisfying the following system of equations:

$$\frac{\sqrt{xyz}}{x+y} = 3,$$
$$\frac{\sqrt{xyz}}{y+z} = \frac{5}{2},$$
$$\frac{\sqrt{xyz}}{z+x} = \frac{15}{7}.$$

If $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{N}{900}$, find N.

24. In the figure below, the lines AB and CE are parallel to each other. The point F is the midpoint of the line AC, and $AB = 3AG$. If the area of the triangle $\triangle ABC$ is 666 cm^2, find the area of the triangle $\triangle CDE$ (in cm^2).



25. For any real number x, let $\lfloor x \rfloor$ denote the largest integer smaller than or equal to x. For example, $\lfloor 3 \rfloor = 3$, $\lfloor 2.8 \rfloor = 2$, $\lfloor -2.3 \rfloor = -3$. Suppose that R is a real number such that

$$\left\lfloor R - \frac{1}{200} \right\rfloor + \left\lfloor R - \frac{2}{200} \right\rfloor + \left\lfloor R - \frac{3}{200} \right\rfloor + \cdots \left\lfloor R - \frac{99}{200} \right\rfloor = 2018.$$

Find $\lfloor 20R \rfloor$.

Singapore Mathematical Society

Singapore Mathematical Olympiad (SMO) 2019

Senior Section (Round 1)

Tuesday, 4 June 2019 — 0930-1200 hrs

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Multiple Choice Questions

1. The roots of the quadratic equation $x^2 - 7mx + 5n = 0$ are m and n, where $m \neq 0$ and $n \neq 0$. Find a quadratic equation whose roots are $\frac{m}{n}$ and $\frac{n}{m}$.

   (A) $6x^2 - 37x + 1 = 0$ (B) $6x^2 - 50x - 7 = 0$ (C) $6x^2 - 50x + 7 = 0$

   (D) $6x^2 - 37x + 6 = 0$ (E) $x^2 - 37x + 1 = 0$

2. Simplify
$$\left(\sqrt{10} - \sqrt{2}\right)^{\frac{1}{3}} \left(\sqrt{10} + \sqrt{2}\right)^{\frac{7}{3}}.$$

   (A) $24 + 4\sqrt{5}$ (B) $24 + 6\sqrt{5}$ (C) $24 + 8\sqrt{5}$ (D) $24 + 10\sqrt{5}$ (E) $24 + 12\sqrt{5}$

3. Let $a = 4^{3000}$, $b = 6^{2500}$ and $c = 7^{2000}$. Which of the following statement is true?

   (A) $a < b < c$ (B) $a < c < b$ (C) $b < a < c$

   (D) $c < a < b$ (E) $c < b < a$

4. If $\log_{21} 3 = x$, express $\log_7 9$ in terms of x.

   (A) $\dfrac{2x}{2-x}$ (B) $\dfrac{2x}{1-x}$ (C) $\dfrac{2x}{x-2}$ (D) $\dfrac{2x}{x-1}$ (E) $\dfrac{x}{1-x}$

5. Suppose that $\sin x = \dfrac{12}{13}$ and $\cos y = -\dfrac{4}{5}$, where $0° \leq x \leq 90°$ and $90° \leq y \leq 180°$. Find the value of $\cos(x+y)$.

   (A) $-\dfrac{56}{65}$ (B) $\dfrac{56}{65}$ (C) $-\dfrac{16}{65}$ (D) $\dfrac{16}{65}$ (E) None of the above

Short Questions

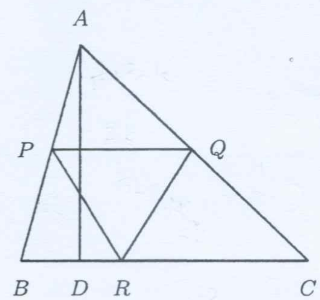
6. Find the largest positive integer n such that $n + 8$ is a factor of $n^3 + 13n^2 + 40n + 40$.

7. Suppose $\tan x = 5$. Find the value of $\dfrac{6 + \sin 2x}{1 + \cos 2x}$.

8. Suppose x and y are real numbers such that
$$\begin{aligned} |x - y| + 3x - y &= 70, \text{ and} \\ |y - x| + 3y + x &= 50. \end{aligned}$$
   Find the maximum possible value of $x + 2y$.

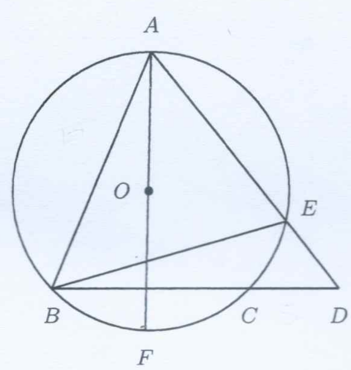
9. The coordinates of the vertices of a triangle $\triangle ABC$ are $A(6,0)$, $B(0,8)$ and $C(x,y)$ such that $x^2 - 16x + y^2 - 12y + 91 = 0$. Find the largest possible value of the area of the triangle $\triangle ABC$.

10. In the figure below, AD is perpendicular to the BC, PQ is parallel to BC, and the triangle $\triangle PQR$ is an equilateral triangle whose area (in meter^2) is equal to the length of AD (in meter). Find the smallest possible value of the length of BC.



11. Find the value of $448\left(\dfrac{\sin 12^\circ \sin 39^\circ \sin 51^\circ}{\sin 24^\circ}\right)$.

12. In the figure below, the chord AF passes through the origin O of the circle, and is perpendicular to the chord BC. It is given that $AB = 17$ cm, $CD = 5$ cm. Suppose $\dfrac{BE}{ED} = \dfrac{m}{n}$, where m and n are positive integers which are relatively prime. What is the value of $m+n$?



13. Let $P(x)$ be the polynomial that results from the expansion of the following expression:

$$(2x^3 + 3x^2 + x)^5 \left(\frac{x}{6} + \frac{1}{2}\right)^5.$$

Find the sum of the coefficients of x^{2k+1}, where $k = 0, 1, 2, 3, \ldots, 9$.

14. Find the value of the following expression:

$$\frac{2\left(1^2+2^2+3^2+\ldots+49^2+50^2\right)+(1\times 2)+(2\times 3)+(3\times 4)+\ldots+(48\times 49)+(49\times 50)}{100}.$$

15. Let M be the maximum possible value of $\dfrac{15x^2-x^3-39x-54}{x+1}$, where x is a positive integer. Find the value of $9M$.

16. Find the maximum possible value of $x+y+z$ where x, y, z are integers satisfying the following system of equations:

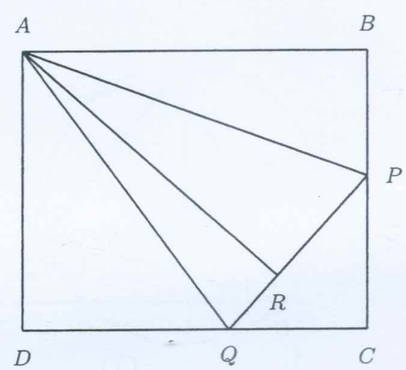
$$\begin{aligned} x^2z+y^2z+8xy &= 200, \\ 2x^2+2y^2+xyz &= 50. \end{aligned}$$

17. Find the remainder when 10^{43} is divided by 126.

18. Suppose $(\log_2 x)^2+4(\log_2(\log_3 y))^2 = 4(\log_2 x)(\log_2(\log_3 y))$. If $x=49$ and y is a positive integer, find y.

19. The figure below shows a rectangle $ABCD$ with $AB=16$ cm and $BC=15$ cm. Let P be a point on the side BC such that $BP=7$ cm, and let Q be a point on the side CD such that $CQ=6$ cm.

Find the length of AR (in cm), where R is the foot of the perpendicular from A to PQ.



20. A sequence $x_0, x_1, x_2, x_3, \ldots$ of integers satisfies the following conditions: $x_0=1$, and for any positive integer $n\geq 1$, $|x_n-1|=|x_{n-1}+2|$. Find the maximum possible value of $2019-(x_1+x_2+\cdots+x_{2018})$.

21. Consider a square $ABCD$ on the xy-plane where the coordinates of its vertices are given by $A(13,0)$, $B(23,13)$, $C(10,23)$ and $D(0,10)$. A lattice point is a point with integer coordinates. Find the number of lattice points in the interior of the square.

22. Eleven distinct chemicals $C_1, C_2, \ldots, C_{11}$ are to be stored in three different warehouses. Each warehouse stores at least one chemical. A pair $\{C_i, C_j\}$ of chemicals, where $i \neq j$, is either compatible or incompatible. Any two incompatible chemicals cannot be stored in the same warehouse. However, a pair of compatible chemicals may or may not be stored in the same warehouse. Find the maximum possible number of pairs of incompatible chemicals that can be found among the stored chemicals.

23. Let k be a positive integer and let the function f be defined as follows:

$$f(x) = \frac{\pi^x}{\pi^x + \pi^{2k-x}}.$$

Suppose the function $g(k)$ is defined as follows:

$$g(k) = f(0) + f\left(\frac{k}{2019}\right) + f\left(\frac{2k}{2019}\right) + f\left(\frac{3k}{2019}\right) + \ldots + f\left(\frac{4037k}{2019}\right) + f(2k).$$

Find the greatest positive integer n such that $g(k) \geq n$ for all $k \geq 1$.

24. Some students sat for a test. The first group of students scored an average of 91 marks and were given Grade A. The second group of students scored an average of 80 marks and were given Grade B. The last group of students scored an average of 70 marks and were given Grade C. The numbers of students in all three groups are prime numbers and the total score of all the students is 1785. Determine the total number of students.

25. Suppose a and b are positive integers satisfying

$$a^2 - 2b^2 = 1.$$

If $500 < a + b < 1000$, find $a + b$.

Multiple Choice Questions

1. Let b be a positive integer. If the minimum possible value of the quadratic function $5x^2 + bx + 506$ is 6, find the value of b.

   (A) 90 (B) 100 (C) 110 (D) 120 (E) 130

2. Which of the following is equal to

$$\sqrt{5+\sqrt{3}}+\sqrt{5-\sqrt{3}}\ ?$$

   (A) $\sqrt{10-\sqrt{22}}$ (B) $\sqrt{10+\sqrt{22}}$ (C) $\sqrt{10-2\sqrt{22}}$
   (D) $\sqrt{10+2\sqrt{22}}$ (E) None of the above

3. Simplify

$$\log_8 5 \cdot (\log_5 3 + \log_{25} 9 + \log_{125} 27).$$

   (A) $\log_2 3$ (B) $\log_3 2$ (C) $\log_2 9$ (D) $\log_3 16$ (E) $\log_2 27$

4. Let $a = 50^{\frac{1}{505}}$, $b = 10^{\frac{1}{303}}$ and $c = 6^{\frac{1}{202}}$. Which of the following is true?

   (A) $a < b < c$ (B) $a < c < b$ (C) $b < a < c$ (D) $b < c < a$ (E) $c < b < a$

5. Let $p = \log_{10}(\sin x)$, $q = (\sin x)^{10}$, $r = 10^{\sin x}$, where $0 < x < \frac{\pi}{2}$. Which of the following is true?

   (A) $p < q < r$ (B) $p < r < q$ (C) $q < r < p$ (D) $q < p < r$ (E) $r < p < q$

Short Questions

6. Find the minimum possible value of $|x-10| - |x-20| + |x-30|$, where x is any real number.

7. Parallelogram $ABCD$ has sides $AB = 39$ cm and $BC = 25$ cm. Find the length of diagonal AC (in cm) if diagonal $BD = 34$ cm.

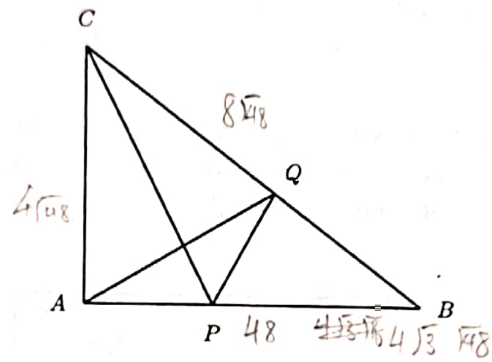
8. Suppose $\sin(45° - x) = -\frac{1}{3}$, where $45° < x < 90°$. Find $(6\sin x - \sqrt{2})^2$.

9. If $8\cos x - 8\sin x = 3$, find the value of $55\tan x + \dfrac{55}{\tan x}$.

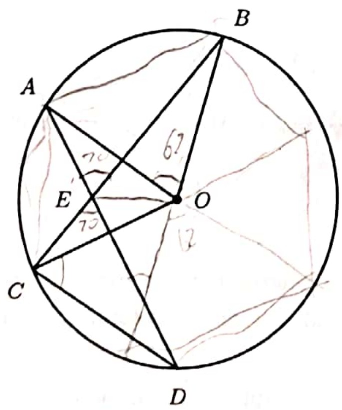
10. Find the number of ordered pairs (x, y), where x and y are integers, such that

$$x^2 + y^2 - 20x - 14y + 140 < 0.$$

11. The figure below shows a right-angled triangle ABC such that $\angle BAC = 90°$, $\angle ABC = 30°$ and $AB = 48$ cm. Let P be a point on side AB such that CP is the angle bisector of $\angle ACB$ and Q be a point on side BC such that line AQ is perpendicular to line CP. Determine the length of PQ.



12. In the figure below, the point O is the center of the circle, AD and BC intersect at E, and $\angle AEB = 70°$, $\angle AOB = 62°$. Find the angle $\angle OCD$ (in degree °).



13. Find the value of $\dfrac{4\cos 43°}{\sin 73°} - \dfrac{12\sin 43°}{\sqrt{3}\sin 253°}$.

$7x^2+5y^2=35$

14. If $\dfrac{x^2}{5} + \dfrac{y^2}{7} = 1$, find the largest possible value of $(x+y)^2$.

15. Find the coefficient of x^6 in the expansion of $(1 + x + 2x^2)^7$.

16. Suppose $(3x-y)^2 + \sqrt{x + 38 + 14\sqrt{x-11}} + |z+x-y| = 7$. Find the value of $|x+y+z|$.

17. Suppose there are real numbers x, y, z satisfying the following equations:

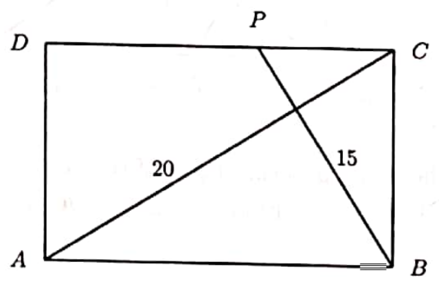
$$x + y + z = 60, \quad xy - z^2 = 900.$$

Find the maximum possible value of $|z|$.

18. Find the sum $\displaystyle\sum_{k=1}^{16} \log_2 \left(\sqrt{\sin^2 \frac{k\pi}{8} + 1} - \sin \frac{k\pi}{8} \right)$

19. Let a, b be positive real numbers, where $a > b$. Suppose there exists a real number x such that $(\log_2 ax)(\log_2 bx) + 25 = 0$. Find the minimum possible value of $\frac{a}{b}$.

20. The figure below shows a rectangle $ABCD$ such that the diagonal $AC = 20$ cm. Let P be a point on side CD such that BP is perpendicular to diagonal AC. Find the area of rectangle $ABCD$ (in cm^2) if $BP = 15$ cm.



21. Find the smallest positive integer that is greater than the following expression:

$$\left(\sqrt{7} + \sqrt{5}\right)^4.$$

22. Find the number of non-congruent right-angled triangles such that the length of all their sides are integers and that the hypotenuse has a length of 65 cm.

23. There are 6 couples, each comprising a husband and a wife. Find the number of ways to divide the 6 couples into 3 teams such that each team has exactly 4 members, and that the husband and the wife from the same couple are in different teams.

24. The **digit sum** of a number, say 987, is the sum of its digits, $9 + 8 + 7 = 24$. Let A be the digit sum of 2020^{2021}, and let B be the digit sum of A. Find the digit sum of B.

25. $40 = 2 \times 2 \times 2 \times 5$ is a positive divisor of 1440 that is a product of 4 prime numbers. $48 = 2 \times 2 \times 2 \times 2 \times 3$ is a positive divisor of 1440 that is a product of 5 prime numbers. Find the sum of all the positive divisors of 1440 that are products of an odd number of prime numbers.

2021 SMO Selected Sums (Senior 1st Round), Mon, 6 Sep 2021, 0930-1200 hrs

Instruction to contestants

1.Answer ALL 25 questions. (Usually select and answer well ±10 questions should garner at least an Honourable Mention, need not follow the order as easy sums might be slotted in the middle or towards the end of 25 sums)

2. For the MCQs, enter your answer on the answer sheet by shading the bubble containing the letter (A, B, C, D or E) corresponding to the correct answer. (Usually make use of the options to work backward for the answer)

3. For the other short questions, write your answer on the answer sheet and shade the appropriate bubble below your answer. (Usually make use of the friendly integers, like -2,-1,0,1,2 to guess and check)

4. No steps are needed to justify your answers. (Usually make use of the problem solving heuristics approach)

5. Each question carries 1 mark. (Usually target and focus on working out ±10 questions that are easy, might be in the middle or towards the end of 25 sums)

6. No calculators are allowed. (Usually make use of algebraic manipulation and the corresponding rules to compute)

7. Throughout this paper, let $\lfloor x \rfloor$ *denote the greatest integer less than or equal to x. For example,* $\lfloor 2.1 \rfloor$ *=2,* $\lfloor 3.9 \rfloor$ *=3.* (most of the times the workout of the sum could be rather easy, just need to understand the notation)

8. Throughout this paper, $[A_1A_2 \ldots A_r]$ *denote the area of the polygon* $A_1A_2 \ldots A_r$*. For example,* $[ABC]$ *=area of* Δ_{ABC}

9. Throughout this paper, let $\overline{a_{n-1}a_{n-2} \ldots a_0}$ *denote an n-digit number with the digits* a_i *in the corresponding position, i.e.* $\overline{a_{n-1}a_{n-2} \ldots a_0} = a_{n-1}10^{n-1} + a_{n-2}10^{n-2} + \cdots + a_0 10^0$. *For example,* $\overline{123} = 1(100) + 2(10) + 3(1)$.

1 Let p be a real number such that the equation $x^2 - 10x = p$ has no real solution. Which of the following is true?

(A) 0 < p < 25 (B) p = 25 (C) p > 25 (D) p < −25 (E) −25 < p < 0

Hints: Completing the Square OR Discriminant < 0

2 Which of the following is the largest?

(A) tan 50° + sin 50° (B) tan 50° + cos 50° (C) sin 50° + cos 50°

(D) tan 50° + sin²50° (E) sin²50° + cos²50°

Hints: Trigo Ratio or Trigo Curve

3 Find the value of $2021^{(\log_{2021} 2020)(\log_{2020} 2019)(\log_{2019} 2018)}$

(A) 2018 (B) 2019 (C) 2020 (D) 2021 (E) None of the above

Hints: Logarithmic rules and change base

4 Suppose $\sin\theta = \frac{n-3}{n+5}$ and $\cos\theta = \frac{4-2n}{n+5}$ for some integer n.

Find the maximum value of $160\tan^2\theta$.

(A) 80 (B) 90 (C) 100 (D) 120 (E) None of the above

Hints: Unit Circle Trigo Identities or Pythagoras Theorem

5 Select all the inequalities which hold for all real values of x and y.

(i) $x \le x^2 + y^2$

(ii) $xy \le x^2 + y^2$

(iii) $x - y \le x^2 + y^2$

(iv) $y + xy \le x^2 + y^2$

(v) $x + y - 1 \le x^2 + y^2$

(A) (i) (B) (i) & (iii) (C) (iii) & (iv) (D) (ii) (E) (ii) & (v)

Hints: Completing the Square and Inequalities

6 Let x be the integer such that $x = 5^{\sqrt{2+4\log_x 5}}$. Determine the value of x.

Hints: Taking Log_5, squaring for solving cubic equation

7 If $\cos A - \cos B = \frac{1}{2}$ and $\sin A - \sin B = -\frac{1}{4}$, find the value of $100\sin(A+B)$.

Hints: Advanced Trigo Identities, e.g. double angles, addition formula, etc.

8 Find the constant in the expansion of $(\sqrt[3]{x} + \frac{1}{\sqrt{x}})^6(\sqrt{x} + \frac{1}{x})^{10}$.

Hints: Binomial Theorem

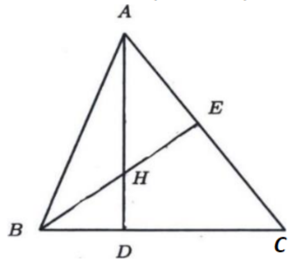
9 A quadratic polynomial $P(x) = ax^2 + bx + c$, where $a \neq 0$ has the following properties: $P(n) = \frac{1}{n^2}$ for all n = -1, 2,3.

Determine the smallest possible value of k, where $k \neq 2,\ 3$, such that $P(k) = \frac{1}{k^2}$.

Hints: Polynomial and Simultaneous Equations

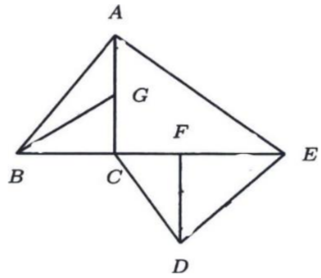
10 The figure below shows a triangle ABC such that AD and BE are altitudes to the sides BC and CA respectively. The lines AD and BE intersect at H. Determine the area (in cm^2) of the triangle ABC if AH = 50 cm, DH = 18 cm and BH = EH.

Hints: Similar Triangles and Pythagoras Theorem



11 In the figure below, ∠GCB = ∠ACE = ∠DFE = 90^0, and ∠GBC = ∠EAC = ∠EDF = θ^0. Also, GB = 6 cm, AE = 10 cm and DE = 8 cm. Let S denote the sum of the areas of the triangles ABC and CDE. Find the maximum possible value of S (in cm^2).

Hints: Trigo Ratio and R-formula



12 Find the sum of all the solutions to the equation $\sqrt[3]{x-110} - \sqrt[3]{x-381} = 1$

Hints: Apply cubic identity a^3-b^3 = (a-b)(a^2+ab+b^2)

13 If $f(x) = (2x + 4 + \frac{x-2}{x+3})^2$, where $-2 \leq x \leq 2$ find the maximum value of $f(x)$.

Hints: Partial Fraction/ Long Division for composite increasing function

14 Given that $D = \sqrt{\sqrt{x^2 + (y-1)^2} + \sqrt{(x-1)^2 + y^2}}$

for real values of x and y, find the minimum value of D^8.

Hints: Pythagoras Theorem for length/distance and co-linearity

15 Find the minimum value of $\frac{8}{\sin 2\theta} + 12\tan\theta$, where $0 < \theta < \frac{\pi}{2}$.

Hints: Trigo Identities and Arithmetic-Mean- Geometric Mean Inequality

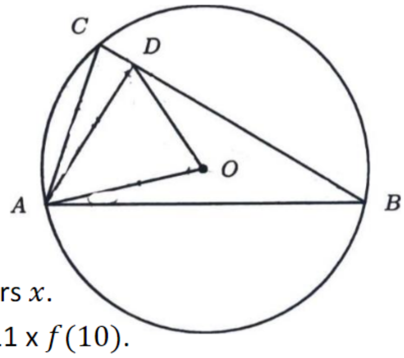
16 Determine the largest angle θ (in degree), where $0^0 < \theta < 360^0$, such that

$\sin(\theta + 18^o) + \sin(\theta + 162^o) + \sin(\theta + 234^o) + \sin(\theta + 306^o) = 1 + \cos(\theta + 60^o) + \cos(\theta + 300^o)$

Hints: *Advance Trigo*

17 Let O be the circumcentre of the triangle ABC and that $\angle ABC = 30^0$. Let D be a point on the side BC such that the length of AD is the same as the radius of the circle. Determine the value of $\angle ADO$ (in degree) if $\angle OAB = 10^0$

Hints: Circle Properties ∡ at centre



18 A function f satisfies $f(x)f(x+1) = x^2 + 3x$ for all real numbers x.

If $f(1) + f(2) = \frac{25}{6}$ and $0 < f(1) < 2$, determine the value of 11 x $f(10)$.

Hints: Telescoping Method of Differences

19 Find the value of

$\frac{1}{sin^2 0.5^o} - tan^2 0.5^o + \frac{1}{sin^2 1.5^o} - tan^2 1.5^o + \frac{1}{sin^2 2.5^o} - tan^2 2.5^o + \ldots + \frac{1}{sin^2 179.5^o} - tan^2 179.5^o$.

Hints: *Advance Trigo*

20 Let $a_1,\ a_2,\ a_3$ be three distinct integers where $1000 > a_1 > a_2 > a_3 > 0$. Suppose there exist real numbers x, y, z such that

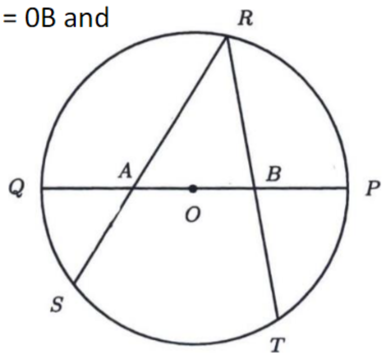
$$(a_1 - a_2)y + (a_1 - a_3)z = a_1 + a_2 + a_3$$
$$(a_1 - a_2)x + (a_2 - a_3)z = a_1 + a_2 + a_3$$
$$(a_1 - a_3)x + (a_2 - a_3)y = a_1 + a_2 + a_3$$

Find the largest possible value of $x + y + z$.

Hints: Simultaneous Equations and Elimination

21 The figure below shows a circle centred at O with radius 555 cm. If OA = 0B and $\frac{RA}{AS} + \frac{RB}{BT} = \frac{13}{6}$, find OA (in cm)

Hints: Intersecting Chord Theorem and Cosine Rule

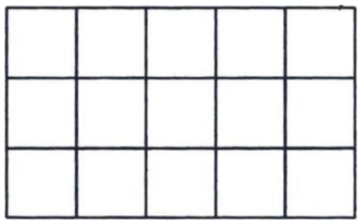


22 Find the number of real solutions (x, y) of the system of equations

$$x^3 + y^3 + y^2 = 0$$
$$x^2 + x^2y + xy^2 = 0$$

Hints: Simultaneous Equations and Elimination

23 The following 3 x 5 rectangle consists of 15 1 x 1 squares. Determine the number of ways in which 9 out of the 15 squares are to be coloured in black such that every row and every column has an odd number of black squares.



Hints: Permutation and Combinations

24 Let n be a positive integer such that

$$\frac{2021n}{2021 + n}$$

is also a positive integer. Determine the smallest possible value of n.

Hints: Number Properties with long division for partial fraction and prime factorization

25 Determine the number of 5-digit numbers with the following properties:

(i) All the digits are non-zero;
(ii) The digits can be repeated;
(iii) The difference between consecutive digits is exactly 1.

Hints: Consider Cases and Basic Counting techniques of Multiplication Rule/Permutation