SaHHN(e) Combinatorial Coworkspace

Adding Lattice Points in Polytopes

Day # 2

Exercise 1. Consider $P := \text{conv}\left(\begin{smallmatrix} 0&1&0&0&1&0&1&1 \\ 0&0&1&0&0&1&1&1 \\ 0&0&0&1&1&1&7&8 \end{smallmatrix}\right) \subset \mathbb{R}^3$.

i) Show that P is not IDP.

ii) Show that for every vertex v of P, the lattice points in $P - v$ generate the semigroup cone$(P - v)$.

Exercise 2. i) Show that $2 \cdot S(p, q)$ has a unimodular cover.

ii) Show that $2 \cdot S(1, q)$ has a regular unimodular triangulation.

iii)* Show that $2 \cdot S(2, 5)$ does not have a unimodular triangulation.

Exercise 3. Describe the Gröbner basis for the toric ideal of an order polytope corresponding to the canonical triangulation.

Exercise 4. Suppose $P \subset \mathbb{R}^d$ and $Q \subset \mathbb{R}^e$ have regular unimodular triangulations.

i) Show that $P \times Q$ also has a regular unimodular triangulation.

ii) Can you say something about Gröbner basis degrees?

Exercise 5. The unique empty 4-simplex S of width 4 is the convex hull of the standard unit vectors in $\mathbb{R}^4$ together with $(6, 14, 17, 65)$.

i)** Can you give a $k \in \mathbb{N}$ so that kS has a unimodular triangulation?

ii)*** What is the smallest such k?

iii)* Does $2S$ have a unimodular triangulation?

Exercise 6. i) Let $P \subset \mathbb{R}^2$ be a lattice polygon with all edges of length ≥ 2. Show that the corresponding toric ideal is quadratically generated.

ii) Show that it even has a quadratic Gröbner basis.

iii)*** Let $P \subset \mathbb{R}^3$ be a lattice polytope which is covered by kth dilates of unimodular simplices for some $k \in \mathbb{N}$. Show that the corresponding toric ideal is quadratically generated.

iv)*** Is it enough to assume that all edges have length $\geq k$?