Hollow polytopes and width

Day # 2

Exercise 1. Using Exercise 1 from sheet 1 ("every empty lattice triangle is unimodular") derive
Pick's Theorem: a lattice polygon with i lattice points in the interior and b in the boundary, has (normalized) area $2i + b - 2$.

Exercise 2. **Hollow 2-polytopes.** Show that every hollow lattice polygon either has width one or is isomorphic to $P \cong 2\Delta_2 :=$ conv$\{(0,0),(2,0),(0,2)\}$. **Clue**: this involves a little case study. One way to simplify it is to prove that either P has at most four lattice points (easy case, since then Area$(P) \leq 2$ by Pick's Thm) or P has three collinear points, and derive consequences of the latter.

Exercise 3. Show that no empty tetrahedron projects to $2\Delta_2$. Deduce that all except (perhaps) finitely many empty tetrahedra have width one. **Clue**: Use that "in each dimension d there are only finitely many hollow lattice d-polytopes that do not project to a hollow lattice $(d-1)$-polytope" (Nill-Ziegler)

Exercise 4. For each

$$v = (v_1, \ldots, v_d) \in \mathbb{Z}^d \cap \left\{ \sum_i x_i \geq 2 \right\}$$

consider the simplex $\Delta_v :=$ conv$(e_1, \ldots, e_d, v)$.

i) What condition on a lattice d-simplex Δ is necessary and sufficient for Δ to be isomorphic to some Δ_v?

ii) Show that Vol$(\Delta_v) = \sum_i v_i - 1$.

iii) Let $w \in \mathbb{N}$ and abbreviate $V = \sum_i v_i - 1$. Show that width$(\Delta_v) \leq w$ is equivalent to any of the following conditions:

a) There is a $b \in \mathbb{Z}$ and $a_0, \ldots, a_d \in \{b, \ldots, b+w\}$, with not all a_is equal, such that $\sum_{i=1}^d a_i v_i = a_0 \pmod V$.

b) For every $b \in \mathbb{Z}$, there are $a_0, \ldots, a_d \in \{b, \ldots, b+w\}$, with not all a_is equal, such that $\sum_{i=1}^d a_i v_i = a_0 \pmod V$.

iv) Use (iii) to show that $\Delta_{6,14,17,65}$, of volume 102, has width four. (Note: this simplex is empty, and is the only empty 4-simplex of width> 3).

Exercise 5. Let $P \subset \mathbb{R}^p$ and $Q \subset \mathbb{R}^q$ be (perhaps non-lattice) polytopes containing the origin (perhaps in the boundary). Their **direct sum** is defined as

$$\begin{aligned} P \oplus Q :=& \{(\lambda x, (1-\lambda)y \in \mathbb{R}^p \times \mathbb{R}^q : x \in P, y \in Q, \lambda \in [0,1]\} \\ =& \text{conv}(P \times \{0\} \cup \{0\} \times Q) \subset \mathbb{R}^{p+q}. \end{aligned}$$

Show that:

i)* width$(P \oplus Q) = \min\{$width(P), width$(Q)\}$.

ii)* If P and Q are hollow and $\frac{1}{k_1} + \frac{1}{k_2} \geq 1$ then $k_1 P \oplus k_2 Q$ is hollow.

iii) Conclude that if P and Q are hollow and have widths w_p and w_q then there is a hollow $(p+q)$-polytope of width $w_p + w_q$.