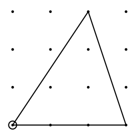
Lattice Points in Polytopes (Ehrhart Theory)

Day # 1

Exercise 1. Investigate h^*-vectors in 2 dimensions.

i) Let $h^*(t) = at^2+bt+1$ be the h^*-polynomial of a lattice polygon. Convince yourself that a is the number of interior lattice points and $b+3$ is the number of (all) lattice points in the polygon.

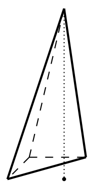
ii) Let T be a lattice triangle in Hermite normal form:



$T = \text{conv}(\binom{0}{0}, \binom{a}{0}, \binom{c}{d})$ with $0 \le c < d$. Determine the h^*-vector of T in terms of the given parameters.

iii)* Verify Scott's inequality, i.e., $b \le 3a+3$ (where a, b are as in part i).

Exercise 2. i) What is the h^*-vector of the Reeve tetrahedron, i.e.,



$$R = \text{conv}\left(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\1\\a\end{pmatrix}\right)?$$

ii) Let $P \subseteq \mathbb{R}^d$ be a full-dimensional lattice polytope with h^*-polynomial of degree s. Show that $d+1-s$ is the smallest non-negative integer such that $(d+1-s)P$ contains an interior lattice polytope. This number is called the *codegree* of P

iii) What is the smallest multiple of the Reeve simplex which contains an interior lattice point?

Exercise 3. i) Let $a, b \in \mathbb{N}$ with

$$b \le \begin{cases} 7 & \text{if } a = 1 \\ 3a+3 & \text{if } a \ge 2 \end{cases}$$

Show that $at^2 + bt + 1$ can be realised as the h^*-polynomial of a lattice polytope. (Hint: Consider polytopes up to dimension 3.)

ii) Let us call a linear inequality non-trivial if the corresponding hyperplane intersects the positive orthant in more points than the origin. Show that there cannot be non-trivial *linear* universal inequalities, i.e., every linear inequality is not satisfied by the h^*-polynomial of some lattice polytope.

Exercise 4.*** Recall the definition of a hypersimplex:

$$\Delta_{d,k} := \{\mathbf{x} \in [0,1]^d \colon \sum_{i=1}^{d} x_i = k\} \subseteq \mathbb{R}^d$$

Show/counterexample that the h^*-vector of $\Delta_{d,k}$ is log-concave/unimodal.