Basics on Lattice Polytopes and Ehrhart Theory

Day # 1

Exercise 1.
i) Prove that an empty triangle is a unimodular triangle.
ii) Give several proofs why the normalised volume of a lattice polytope is an integer.
iii)* Let $S(p,q) = \text{conv}(\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}p\\q\\1\end{pmatrix})$ where $0 \leq p \leq q$ with $\gcd(p,q) = 1$. For which p, q and p', q' are the empty tetrahedra $S(p,q)$ and $S(p',q')$ unimodularly equivalent?

Exercise 2.
i) Prove hierarchy: SFP implies reflexive implies canonical.
ii) A canonical centrally-symmetric lattice polytope has at most 3^d lattice points where d is the dimension of the polytope.
iii)** Show that every 3-dimensional smooth polytope has a facet with at most 4 vertices. (Recall that a lattice polytope is smooth if its normal fan has unimodular cones.)

Exercise 3.
i) Convince yourself that the leading coefficient of the Ehrhart polynomial is the (Euclidean) volume.
ii) Verify Ehrhart reciprocity for the standard simplex.
iii) What is the linear coefficient of the h^*-polynomial?

Exercise 4.
i) Let P be a reflexive polytope and $x, y \in P$ two lattice points. Show that if $x + y \neq 0$ and x, y are not contained in a common facet then there is a lattice point $0 \neq z \in P$ such that there are two facets F, F' with $x, z \in F$ and $z, y \in F'$.
ii) Show that the diameter of the edge graph of a simplicial reflexive polytope is bounded by 3.
iii) Let $P \subset \mathbb{R}^d$ be a reflexive polytope, F a facet and $v \in P$ a non-zero lattice point that has lattice distance 1 from F. Show that v is contained in a facet F' such that $\dim(F \cap F') = d - 2$.