AMS 341 (Spring, 2010) Exam 1 - Solution notes Estie Arkin

Mean 72.16, median 71, high 100, low 36.

1. (9 points) Consider the following LP:

$$\begin{aligned} \min \quad & z = -2x_1 + x_3 \\ \text{s.t.} \quad & -x_1 + 5x_2 + x_3 && \geq 1 \\ & x_1 - 2x_2 + x_3 && = 10 \\ & x_1, x_3 && \geq 0 \\ & x_2 && \text{unrestricted} \end{aligned}$$

Rewrite the LP in standard form.

$$\begin{aligned} \min \quad & z = -2x_1 + x_3 \\ \text{s.t.} \quad & -x_1 + 5x_2' - 5x_2'' + x_3 - e_1 && = 1 \\ & x_1 - 2x_2' + 2x_2'' + x_3 && = 10 \\ & x_1, x_3, x_2', x_2'' && \geq 0 \end{aligned}$$

2. (15 points) Consider the feasible region given by the following constraints: (It may be helpful to sketch it and/or put it into standard form.)

$$x_1 + x_2 \leq 6 \qquad (1)$$
$$x_1 \leq 2 \qquad (2)$$
$$x_2 \leq 4 \qquad (3)$$
$$x_1 \geq 0 \qquad (4)$$
$$x_2 \geq 0 \qquad (5)$$

In standard form, there are 3 constraints (and the non negativity constraints) and 5 variables, so there should be 3 basic variables and 2 non basic variables.

$$x_1 + x_2 + s_1 = 6 \qquad (6)$$
$$x_1 + s_2 = 2 \qquad (7)$$
$$x_2 + s_3 = 4 \qquad (8)$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0 \qquad (9)$$

(a). Is the point $x_1 = 0$, $x_2 = 4$ a feasible point? Is it a basic solution? Yes, it is feasible (constraints are satisfied) and basic, x_2, s_1, s_2 are basic, x_1, s_3 non basic.

(b). Is the point $x_1 = 2$, $x_2 = 2$ a feasible point? Is it a basic solution? Yes, it is feasible (constraints are satisfied) but not basic, since x_1, x_2, s_1, s_3 are all positive, so we would need to have 4 basic variables, but there are only 3 constraints.

(c). The point $x_1 = 2$, $x_2 = 4$ is a basic feasible solution. Is it a degenerate basic feasible solution? Yes it is feasible and is a degenerate BFS. x_1, x_2 are basic, and all slack variables are equal to zero, one of them must be basic, and therefore it is degenerate. (Basic variable equals to zero.)

3. (15 points) You are given the tableau for a max problem. Give conditions on the unkowns a_1, a_2, a_3, b, c that make the following true. Your conditions should be as general as possible (don't just give an example, such as $a_1 = 3$.)

z	x_1	x_2	x_3	x_4	x_5	RHS
1	c	2	0	0	0	10
0	-1	a_1	1	0	0	4
0	a_2	-4	0	1	0	1
0	a_3	3	0	0	1	b

(a). The current BFS is optimal. To be feasible, $b \geq 0$, for optimality, $c \geq 0$. a_1, a_2, a_3 can be anything.

(b). The current BFS is optimal and there are multiple optimal solutions. To be feasible, $b \geq 0$, for multiple optimal solutions, $c = 0$. a_1, a_2, a_3 can be anything.

(c). The LP is unbounded. To be feasible, $b \geq 0$, for unbounded objective, $c < 0$. a_1 can be anything, $a_2, a_3 \leq 0$.

4. (8 points) Consider the following LP:

$$
\begin{aligned}
\min \quad & z = 3x_1 + x_2 & \\
\text{s.t.} \quad & x_1 - x_2 + x_3 & = 1 \\
& x_2 + x_4 & = 2 \\
& x_1 + x_2 - x_5 & = 6 \\
& x_1, x_2, x_3, x_4, x_5 & \geq 0
\end{aligned}
$$

We wish to solve this problem using the big M method. Set up the *first* tableau (in canonical form!) we should use.

The problem is in standard form, and x_3, x_4 can be basic variables for the first and second constraints, so we add an artificial variable to the third constraint and get:

z	x_1	x_2	x_3	x_4	x_5	a_3	RHS
1	-3	-1	0	0	0	-M	0
0	1	-1	1	0	0	0	1
0	0	1	0	1	0	0	2
0	1	1	0	0	-1	1	6

We then "clean-up" the objective row so that basic variable a_3 has objective coefficients zero, so the tableau is in canonical form:

z	x_1	x_2	x_3	x_4	x_5	a_3	RHS
1	-3+M	-1+M	0	0	-M	0	6M
0	1	-1	1	0	0	0	1
0	0	1	0	1	0	0	2
0	1	1	0	0	-1	1	6

5. (27 points) A bank is open Monday-Friday from 9am to 5pm. From past experiance, the bank knows that it needs (at least) the following number of tellers:

Time period	9-10	10-11	11-noon	noon-1	1-2	2-3	3-4	4-5
Tellers required	4	3	4	6	5	6	8	8

The bank hires two types of tellers. Full time tellers work 9-5 every day, except for 1 hour off for lunch. (The bank determines when a full time teller takes lunch hour, but it must be either noon-1 or 1-2.) Full time employees are paid \$8 per hour (this includes payment for the lunch hour). The bank can also hire part time tellers. Each part time teller must work exactly 3 consecutive hours each day, and gets paid \$5 per hour. To maintain quality of service, at most 5 part time tellers can be hired. Formulate an LP to minimize the cost of the bank to meet teller requirements. (Your formulation does NOT have to be put into standard form.)

(a). Define the variables you are using in the formulation. Let x_{12}, x_1 be the number of full time tellers that take their lunch starting at 12 or 1 respectively. Let y_i be the number of part time tellers that start their 3 hour shift at $i = 9, 10, 11, 12, 1, 2$. Note that full time tellers get paid 64\$ (8\$ for each of 8 hours) and part time tellers get paid 15\$. Also, since part time tellers work exactly 3 consecutive hours, I did not use variables y_3, y_4.

(b),(c). The objective function and the constraints are:

$$
\begin{aligned}
\min \quad z = 64(x_{12} + x_1) + 15(y_9 + y_{10} + y_{12} + y_1 + y_2) \\
x_{12} + x_1 + y_9 &\geq 4 \\
x_{12} + x_1 + y_9 + y_{10} &\geq 3 \\
x_{12} + x_1 + y_9 + y_{10} + y_{11} &\geq 4 \\
x_1 + y_{10} + y_{11} + y_{12} &\geq 6 \\
x_{12} + y_{11} + y_{12} + y_1 &\geq 5 \\
x_{12} + x_1 + y_{12} + y_1 + y_2 &\geq 6 \\
x_{12} + x_1 + y_1 + y_2 &\geq 8 \\
x_{12} + x_1 + y_2 &\geq 8 \\
y_9 + y_{10} + y_{11} + y_{12} + y_1 + y_2 &\leq 5 \\
x_{12}, x_1, y_9, y_{10}, y_{11}, y_{12}, y_1, y_2 &\geq 0
\end{aligned}
$$

Common mistakes: Forgetting lunch for the full time workers. Defining a variable x_i how many full (or part) time workers are working at hour i. Full time workers must work all hours except lunch! And part time workers must work exactly 3 consecutive hours.

6. (16 points) A company produces and sells wooden soldiers and wooden trains. Each soldier requires 3 board feet of lumber and 2 hours of labor. Each train requires 5 board feet of lumber and 4 hours of labor. A total of 145 board feet of lumber and 90 hours of labor are availble. Upto 50 soldiers and 50 trains can be sold. Trains sell for \$55, and soldiers for \$32. In addition to producing trains and soldiers itself, the company can buy (from an outside supplier) extra soldiers at \$27 each and extra trains at \$50 each. Let SM, TM be the number of soldiers and trains made by the company, and SB, TB the number of soldiers and trains bought from the supplier. Use the Lindo output below to answer each of the following parts.

$$
\begin{array}{llll}
\max & 32SM+55TM+5SB+5TB & & \\
\text{s.t. } 2) & 3SM+5TM & \leq 145 \\
3) & 2SM+4TM & \leq 90 \\
4) & SM+SB & \leq 50 \\
5) & TM+TB & \leq 50
\end{array}
$$

objective function value 1715.00000

variable	value	reduced cost
SM	45.000000	.000000
TM	.000000	4.000000
SB	5.000000	.000000
TB	50.000000	.000000

row	slack or surplus	dual prices
2)	10.000000	0.000000
3)	.000000	13.500000
4)	.000000	5.000000
5)	.000000	5.000000

Range in which basis remains unchanged :

OBJ coefficient ranges

variable	current coef	allowable increase	allowable decrease
SM	32.000000	infinity	2.00000
TM	55.000000	4.00000	infinity
SB	5.00000	2.00000	5.00000
TB	5.00000	infinity	4.00000

righthand side ranges

row	current RHS	allowable increase	allowable decrease
2	145.00000	infinity	10.000000
3	90.00000	6.66667	90.00000
4	50.00000	infinity	5.00000
5	50.00000	infinity	50.00000

(a). If the company can purchase trains for \$48, what would be the new optimal profit? The profit on TB increases from 5 to 7, which is within the allowable range (infinity). The new $z = 1715 + 2 \cdot 50 = 1815$.

Common mistake: Confusing the purchase price of trains vs the selling price (the coefficient of TM). Or, calling this a decrease. It is an increase in the coefficient of TB from 5 to 7.

(b). What is the most that the company should be willing to pay to for another board foot of lumber? The dual price of lumber, which is zero.

(c). If only 40 trains could be sold, what would be the new optimal solution (the z)? Decreasing the RHS in row 5 (4th constraint) from 50 to 40, is within range (the allowable decrease is 50). The new $z = 1715 + (-10)5 = 1665$.

(d). If only 40 trains could be sold, and 91 hours of labor are available, what would be the new optimal solution (the z)? Using the 100% rule, we first check the range: $\frac{50-40}{50} + \frac{91-90}{6.6667} < 1$. The new $z = 1715 + (-10)5 + 1(13.5) = 1678.5$

7. (10 points) The tableau below is for Phase I of the Two Phase Method. a_1 and a_2 are the

artificial variables or constraints 1,2, e_1 e_2 are the excess variables of constraints 1 and 2, and s_3 is the slack variable of the third constraint.

w	x_1	x_2	e_1	e_2	s_3	a_1	a_2	RHS
1	0	0	-1/2	-1	-1/2	-1/2	0	1/2
0	1	0	-3/4	0	-1/4	1	0	9/4
0	0	0	-1/2	-1	-1/2	1/2	1	1/2
0	0	1	1/2	0	1/2	-1/2	0	2

(a). At this tableau, we have:

$w = 1/2$

$x_1 = 9/4$

$x_2 = 2$

$e_1 = 0$

$e_2 = 0$

$s_3 = 0$

$a_1 = 0$

$a_2 = 1/2$

(b). The basic variables for this tableau are: w, x_1, x_2, a_2.

(c). The tableau shows an optimal solution to the Phase I LP. Is the original LP feasible? Explain briefly. Since $w = a_2 = 1/2 > 0$ the original LP is not feasible.