Final

Friday, January 22, 2021 9:33 AM

1. $\begin{cases} x'' - 3x' - 10x = e^{2t} + e^{-5t} \\ x(0) = x'(0) = 0 \end{cases}$

$r^2 - 3r - 10 = 0$

$r_1 = 5, \ r_2 = -2$

$x_c = c_1 e^{5t} + c_2 e^{-2t}$

let $x_p = A e^{-5t} + B e^{2t}$

$x_p' = -5A e^{-5t} + 2B e^{2t}$

$x_p'' = 25A e^{-5t} + 4B e^{2t}$

$\begin{cases} 25A + 15A - 10A = 1 \\ 4B - 6B - 10B = 1 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{30} \\ B = -\frac{1}{12} \end{cases}$

$\begin{cases} c_1 + c_2 + A + B = 0 \\ 5c_1 - 2c_2 - 5A + 2B = 0 \end{cases} \Rightarrow \begin{cases} c_1 = \frac{13}{210} \\ c_2 = -\frac{1}{84} \end{cases}$

$x = x_c + x_p = \frac{13}{210} e^{5t} - \frac{1}{84} e^{-2t} + \frac{1}{30} e^{-5t} - \frac{1}{12} e^{2t}$

2. $\begin{cases} X'(t) = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix} X(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \\ X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{cases}$

$A = \begin{pmatrix} 2 & 4 \\ 1 & -1 \end{pmatrix}$

$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 4 \\ 1 & -1-\lambda \end{vmatrix} = (\lambda+2)(\lambda-3) = 0$

$$\lambda_1 = -2, \quad \lambda_2 = 3$$

$$V_1 = (-1, 1)^T, \quad V_2 = (4, 1)^T$$

$$X_c = C_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + C_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t}$$

Let $x_p = B_1 t e^{-2t} + B_2 e^{-2t}$

$$x_p' = B_1 e^{-2t} - 2B_1 t e^{-2t} - 2B_2 e^{-2t}$$

$$\because \quad x_p' = A x_p + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$$

$$\begin{cases} -2B_1 = AB_1 \\ B_1 - 2B_2 = AB_2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\begin{cases} B_1 = \frac{1}{5} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \\ B_2 = -\frac{1}{5} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

又 $\because \quad x\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0,$

$$\therefore \quad \begin{cases} -C_1 + 4C_2 - \frac{1}{5} = 0 \\ C_1 + C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = -\frac{1}{25} \\ C_2 = \frac{1}{25} \end{cases}$$

P.S.

$$X = X_c + X_p = -\frac{1}{25} \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} + \frac{1}{25} \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{3t}$$

$$x = x_c + x_p = \cdots \frac{\cdots}{25}\begin{pmatrix} \cdots \\ 1 \end{pmatrix} - \cdots \frac{\cdots}{25}\begin{pmatrix} \cdots \\ 1 \end{pmatrix}$$

$$+\frac{1}{5}\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{-2t} - \frac{1}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$$

3. $\begin{cases} x' = 2x - y \\ y' = x + 4y \end{cases}$

Method 1: $x = y' - 4y$

$$x' = y'' - 4y'$$

$$y'' - 4y' = 2(y' - 4y) - y$$

$$\Rightarrow y'' - 6y' + 9y = 0$$

$$r_{1,2} = 3$$

$$y = (c_1 + c_2 t)e^{3t}$$

$$x = y' - 4y$$

$$= c_2 e^{3t} + 3y - 4y$$

$$= (c_2 - c_1 - c_2 t)e^{3t}$$

Method 2:

$$\begin{cases} (D-2)x + y = 0 \\ (D-4)y - x = 0 \end{cases}$$

$$(D-2)(D-4)\,y + y = 0$$

$$(D^2 - 6D + 9)\,y = 0$$

$$r_{1,2} = 3$$

$$y = (c_1 + c_2 t)\,e^{3t}$$

$$x = y' - 4y$$

$$= (c_2 - c_1 - c_2 t)\,e^{3t}$$

4. $y'' + 4y = \sin 2x$

VOP: $r_{1,2} = \pm 2i$

$$y_c = c_1 \cos 2x + c_2 \sin 2x$$

Let $y_1(x) = \cos(2x)$, $y_2(x) = \sin(2x)$, $f(x) = \sin(2x)$

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$y_p = \int^x \frac{y_2(x)\,y_1(s) - y_1(x)\,y_2(s)}{W(x)} f(s)\,ds$$

$$= \frac{\sin 2x}{2} \int^x \cos 2s \cdot \sin 2s\,ds - \frac{\cos 2x}{2} \int^x \sin^2 2s\,ds$$

$$= \frac{1}{4}\sin 2x \int^x \sin 4s\,ds - \frac{\cos 2x}{4} \int 1 + \cos 4s\,ds$$

$$= -\frac{1}{\ }\sin 2x \cos 4x \qquad \frac{x\cos 2x}{\ } - \frac{1}{\ }\cos 2x \sin 4x$$

$$= -\frac{1}{16}\sin 2x \cos 4x - \frac{x\cos 2x}{4} - \frac{1}{16}\cos 2x \sin 4x$$

$$= -\frac{1}{16}\sin 6x - \frac{x\cos 2x}{4}$$

$$\therefore\ y = y_c + y_p$$

$$= C_1 \cos 2x + C_2 \sin 2x - \frac{1}{16}\sin 6x - \frac{x\cos 2x}{4}$$

5. $r^3 - 3r^2 + 7r - 5 = 0$

$$(r-1)(r^2 - 2r + 5) = 0$$

$$r_1 = 1, \quad r_{2,3} = 1 \pm 2i$$

$$y = c_1 e^x + e^x(C_2 \cos 2x + C_3 \sin 2x)$$

$$\begin{cases} c_1 + c_2 = 1 \\ c_1 + c_2 + 2c_3 = 0 \\ c_1 + c_2 + 2c_3 - 4c_2 + 2c_3 = 0 \end{cases}$$

$$\begin{cases} c_1 = \frac{5}{4} \\ c_2 = -\frac{1}{4} \\ c_3 = -\frac{1}{2} \end{cases}$$

$$\therefore\ y = \frac{5}{4}e^x + e^x\left(-\frac{1}{4}\cos 2x - \frac{1}{2}\sin 2x\right)$$

6. Let $x = e^t$

$$\frac{d^2y}{dt^2} - 16y = e^{4t} + e^{-4t}$$

$$r = \pm 4$$

$$y_c = c_1 e^{4t} + c_2 e^{-4t}$$

Let $y_p = A t e^{4t} + B t e^{-4t}$

$$y''_p = 16 A t e^{4t} + 16 B t e^{-4t} + 8 A e^{4t} - 8B e^{-4t}$$

$$\begin{cases} 8A = 1 \\ -8B = 1 \end{cases} \Rightarrow \begin{array}{l} A = \frac{1}{8} \\ B = -\frac{1}{8} \end{array}$$

$\therefore y = y_c + y_p$

$$= c_1 x^4 + c_2 x^{-4} + \frac{1}{8} \ln x \cdot x^4 - \frac{1}{8} \ln x \cdot x^{-4}$$

7. $(r+1)(r^2+1) = 0$

$r_1 = -1, \quad r_{2,3} = \pm i$

$$y_c = c_1 e^{-x} + c_2 \cos x + c_3 \sin x$$

Let $y_p = A + Be^x + Cxe^{-x} + De^{2x} + Ee^{-2x}$

Let $y_p = A + Be^{x} + Cxe^{-x} + De + Ee$

$$y_p' = Be^{x} + Ce^{-x} - Cxe^{-x} + 2De^{2x} - 2Ee^{-2x}$$

$$y_p'' = Be^{x} - 2Ce^{-x} + Cxe^{-x} + 4De^{2x} + 4Ee^{-2x}$$

$$y_p''' = Be^{x} + 3Ce^{-x} - Cxe^{-x} + 8De^{2x} - 8Ee^{-2x}$$

$$\begin{cases} A = 1 \\ 4B = 1 \\ 3C = 1 \\ 15D = 1 \\ -5E = 1 \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = \frac{1}{4} \\ C = \frac{1}{3} \\ D = \frac{1}{15} \\ E = -\frac{1}{5} \end{cases}$$

$$\therefore y = y_c + y_p$$

$$= C_1 e^{-x} + C_2 \cos x + C_3 \sin x + 1 + \frac{1}{4}e^{x} + \frac{1}{3}xe^{-x} + \frac{1}{4}e^{2x} - \frac{1}{5}e^{-2x}$$