Lecture Note 1: Consumer's Behavior

Ting Liu

Econ 500 – Microeconomic Theory I

State University of New York– Stony Brook

An individual's consumption decsion is affected two aspects: her prefrence and her budget. Hence

$$\begin{array}{lcl} \text{preference} & & \\ & \rightarrow & \text{Choice} \\ \text{budget} & & \end{array}$$

1 Preference

There are two approaches to study individual decision making. The traditional approach makes assumptions on individul preferences and draws implications on individual's choices under different circumstances. For example,we assume that an individual prefers apple to pear and prefers pear to peaches. Suppose that these fruits are sold at the same price and the indivudal can only affort one. Then we can predict from her prefernce that she will buy apples. The second approach starts from dividual's choices and we can infer her preferences from her choices. Take the fruit buying example. Suppose we know nothing about the indivdual's preferences but only observe that she buys apples. Then, we can infer that she prefer apple to the other two fruits.

We will focus on the preference based theory.

1.1 Notation

- When considering a consumer we assume that consumption set $X \subseteq R_{+}^{L}$, where L is the number of goods that can be potentially consumed. A typical element $x \in X$ is therefore a vector $x = (x_1, x_2, ..., x_L)$, wheredenotes the (non-negative) quantity consumed of good k.

- Our decision maker has preferences $\succsim$ defined over a set X. The preferences are defined by a **binary** relationship $\succsim$, where $x \succsim y$, with $x \in X$ and $y \in X$ means 'x is at least as preferred as y'.

- As usual, a negation means that the relationship does not hold, that is $x \not\succsim y$ means 'it is not true that $x \succsim y$'.

- We will use the symbol $\succ$ to denote the relation of strict preference. The strict preference is defined by : $x \succ y$ iff $x \succsim y$ and $y \not\succsim x$.

- We will use the symbol $\sim$ to denote the relation of indiference. The definition of indiference is: $x \sim y$ iff $x \succsim y$ and $y \succsim x$

1.2 Properties of preference

Economics do not make judgement on individual's value system. When we say someone is "irrational" we don't mean that the person enjoys doing crazy things like diving into Niragara fall or marrying for 100 times. What we mean is that this person's preference is inconsistent. We need individual's preference to be consistent in order to make predicitons on their choices.

We will say that a decision maker is *rational* if the preferences $\succsim$ satisfy the two following properties:

Completeness For each pair (x, y) we have either $x \succsim y$ or $y \succsim x$ (or both).

Transitivity If $x \succsim y$ and $y \succsim z$ then $x \succsim z$.

Completeness simply says that whenever presented with a choice, the decision maker can decide what she likes best (or at least that she is indifferent between the two alternatives). Transitivity is a consistency condition. If it were not satisfied then, when having to pick up an object in the set (x, y, z), the decision maker would be unable to choose.

Suppose that $x \succsim y$ and $y \succsim z$ and $z \succ x$, so that transitivity is violated. We can now see that no consistent choice is possible. Clearly, x cannot be a choice since it is strictly preferred by z. If y is chosen then x must be also chosen, since $x \succsim y$, but we have just established that this is impossible. Finally, if z is chosen then y must also be chosen (since $y \succsim z$), but then x should also be chosen. While it is perfectly conceivable to imagine situations in which preferences do not satisfy either completeness or transitivity, the economic study of consumer's behavior usually maintains the assumption of rational preferences.

We often assume that the preferences of our consumers satisfy the following axioms:

Monotonicity The preference realtion $\succsim$ on X is *monotone* if $x, y \in X$ and $y \gg x$ implies $y \succ x$. $\succsim$ is *strongly monotone* if $y \geq x$ & $y \neq x$ implies $y \succ x$.

Monotonicity says "the more the better". This is a natural assumption when the goods are desirable. This assumption implies a downward sloping thin indifference curve. Monotonicity is violated when goods are undesirable. For example, pollution, junk food, etc. This assumption implies a downward sloping indifference curve. Monotone says that a consumer may not be better off when there is an increase in the consumption of certain but not all goods. By contrast, strongly monotone says that the consumer is better off as long as there is an increase the consumption of one good while there is no reduction in the consumption of all other goods.

Convexity The prference relation $\succeq$ on X is *convex* if for every $x \in X$, the upper contour set $\{y \in X : y \succeq x\}$ is convex; that is if $y_1 \succsim x$ and $y_2 \succsim x$ then $\alpha y_1 + (1-\alpha) y_2 \succsim x$ for each $\alpha \in (0,1)$. Convexity is *strict* if $\alpha y_1 + (1-\alpha) y_2 \succ x$ for $y_1 \neq y_2$..

Figure 11 & 12

Convexity implies that the indifference curve is convex. Illustrate in figure that a concave indifference curve violates *convexity.* Convexity implies *diminishing marginal rates of substitution.* Given this assumption, consumers are less willing to substitute one good for an additional unit of another good. One interpretation is that economic agents appreciate *diversification.* For example, if x is indifferent to y, then $\frac{1}{2}x + \frac{1}{2}y$ cannot be worse than either x or y. A context in which the assumption is realistic is that of decisions under uncertainty, that we will consider later.

Monotonicity and Convexity certainly do not fit each and every individual's preference but can capture the majority of individuals' preferences.

Thought Experiments :

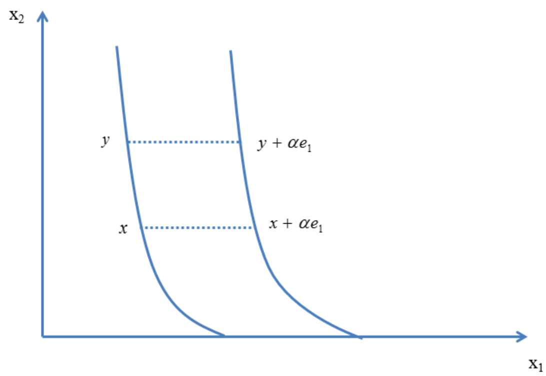
i) cirular indifference curve

ii) Concave preference:

Quasilinear preference Quasilinear Preference Relation: $\succsim$ on $X = (-\infty, \infty) \times R_+^{L-1}$ is quasilinear with respect to commodity 1 (the numeraire) if

(i) $x \sim y$ implies $(x + \alpha e_1) \sim (y + \alpha e_1)$ for $e_1 = (1, 0, 0, ..., 0)$ and any $\alpha \in R$.

(ii) Good one is desirable: $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.



Exercises:

1. Proposition 1.B.1: *If* $\succsim$ *is rational, then:*

(i) $\succ$ *is both irreflective (i.e.,* $\neg\, x \succ x$*) and transitive.*

(ii) $\sim$ *is reflective (i.e.,* $x \sim x$*), transitive and symmetric.*

(iii) if $x \succ y \succsim z$*, then* $x \succ z$*.*

Proof:

(i) Suppose $x \succ x$, then $x \succsim x$ and $x \not\precsim x$. Contradiction.

Suppose $x \succ y$, $y \succ z$ and $z \succsim x$. Then $x \succsim y$ and $y \succsim z$ imply $x \succsim z$. Then $z \succsim x$ & $y \succsim z$ imply $y \succsim x$. This contradicts $x \succ y$.

(ii) Completeness implies $x \succsim x \Longrightarrow x \sim x \Longrightarrow$ reflective

Suppose $x \sim y$, $y \sim z$. Then (a) $x \succsim y, y \succsim z$ & (b) $y \succsim x$, $z \succsim y$. By transitivity, (a) and (b) respectively imply $x \succsim z$, $z \succsim x \Longrightarrow x \sim z$.

Suppose $x \sim y$, then $x \succsim y$ and $y \succsim x \Longrightarrow y \sim x$.

(iii) Suppose $x \succ y \succsim z$ and $z \succsim x$. Then $y \succsim z$ & $z \succsim x$ imply $y \succsim x$. This contradicts $x \succ y$. Q.E.D.

2) 3.B.1

1.3 Preference and Utility Functions

- We will use utility functions to represent preferences. We say that the function $u : X \rightarrow R$ represents preferences $\succsim$ if and only if

$$u(x) \geq u(y) \qquad \Longleftrightarrow \qquad x \succsim y.$$

  In other words, the function $u(x)$ assigns a higher value to bundles which are most preferred. The properties of the utility function depend on the properties of the preference relation $\succsim$.

Proposition 1 *A preference relation $\succeq$ can be represented by a utility function only if it is rational.*

Proof Show that $\exists\ u(x)$ for $\succeq \Rightarrow \succeq$ is rational.

Completeness: For any $x, y \in X$, $u()$ assigns two real numbers $u(x)$ and $u(y)$. Either $u(x) \geq u(y)$ or $u(x) \leq u(y)$. Hence, either $x \succsim y$ or $y \succsim x$.

Transitivity: Suppose $x \succeq y$ and $y \succeq z$. By the definition of $u(.)$, $u(x) \geq u(y)$ and $u(y) \geq u(z)$. This implies $u(x) \geq u(z)$ and hence $x \succsim z$.

- Rationality is the necessary condition for the existence of a utility function. Unfortunately, not every rational preference $\succsim$ can be represented by a utility function.

In order to ensure the existence of a utility function, we need to assume that $\succsim$ is **continuous**. In fact, a rational and continuous relation $\succsim$ can be representated by a utility function.

Sequence A sequence in R^L assigns to every positive integer $m = 1, 2, ...$a vector $x^m \in R^L$. We denote the sequence by $\{x^m\}_{m=1}^{m=\infty}$ or $\{x^m\}$.

For example, $\{\frac{1}{m}\} \to 0$.

Continuity Consider **any** two converging sequences $\{x^n\}$ and $\{y^n\}$ such that $x^n \succsim y^n$ for each n. Let $x = \lim_{n\to+\infty} x^n$ and $y = \lim_{n\to+\infty} y^n$. Then $x \succsim y$.

- An equivalent statement of continuity is to say that for all x, the upper contour set $\{y \in X : y \succeq x\}$ and the lower contour set $\{y \in X : y \preceq x\}$ are both *closed.* Continuity says that if bundle x is strictly preferred to bundle y in the sequences, then bundles which are sufficiently close to x are still strictly preferred to y. The assumption is needed to ensure the existence of a utility function. We give an example of preferences that are not continous and cannot be represented by a utility function.

Sketch of the proof (3.B.3. Won't cover the proof in class)

.Suppose there exist two sequences $\{x^n\} \to x$, $\{y^n\} \to y$ and $x^n \succeq y^n$ but $y \succ x$.

Since the set $\{z|y \succ z\}$ is open. There exists a positive integer N_1 such that $y \succ x^n$ for $n > N_1$.

Now, consider two cases about the sequence $\{y^n\}$.

Case 1. $\{y^n\} \succeq y$, $\forall n$. Then, by transitivity $y^n \succ x^n$ for $n > N_1$. This contradicts $x^n \succeq y^n$, $\forall n$.

Case 2 There exists a subsequence $\{y^{k(n)}\}$ such that $y \succ \{y^{k(n)}\}$, $\forall n$.

Since the set $\{z|z \succ x\}$ is open, there exists a positive integer N_2 such that $y^n \succ x$ for $n > N_2$.

There exists a positive integer m, such that $k(m) > N_2$.

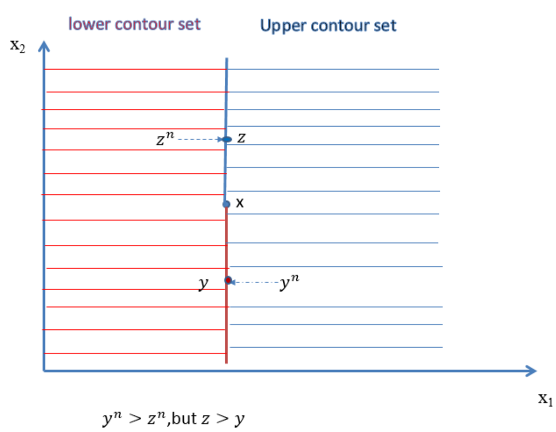
$y \succ \{y^{k(n)}\}$, $\forall n \rightarrow y \succ y^{k(m)}$

$\{z|z \succ y^{k(m)}\}$ is open$\rightarrow \exists$ a positive integer N_3 such that $y^n \succ y^{k(m)}$ for $n > N_3$

$\rightarrow x^n \succ y^{k(m)}$ for $n > N_3$. Closed upper contour set $\rightarrow x \succeq y^{k(m)}$.

Since $k(m) > N_2$, $y^{k(m)} \succ x$. A contradiction. Q.E.D.

Example: *The lexicographic preference.* Assume $X = R^2_+$. Define $x \gtrsim y$ if either "$x_1 > y_1$" or "$x_1 = y_1$ and $x_2 \geq y_2$". This preference gives good 1 the highest priority. When a consumer has lexicographic preference, her indifferent set is singleton. That is, no two distinctive bundles are indifferent. Let $x^n = (1/n, 0)$ and $y^n = (0, 1)$. We have $x^n \succeq y^n$ for each n. However, $\lim_{n\rightarrow+\infty} x^n = (0, 0) \preceq (0, 1) = \lim_{n\rightarrow+\infty} y^n$. In other words, there is a jump in consumer's preferences when $n \rightarrow +\infty$.



Lemma 1 *There does not exist a utility function which represents the lexicographic preference*

Proof: Suppose the lexicographic preference can be represented by a utility funciton $U(.)$. For any real number x_1, we can find a **rational** number $r(x_1)$ such that

$$U(x_1, 2) > r(x_1) > U(x_1, 1).$$

(This step establishes that $r(x_1)$ is a function from $R \to Q$)

In addition if $x_1 > x_1'$, it must be true that

$$r(x_1) > r(x_1').$$

This is because

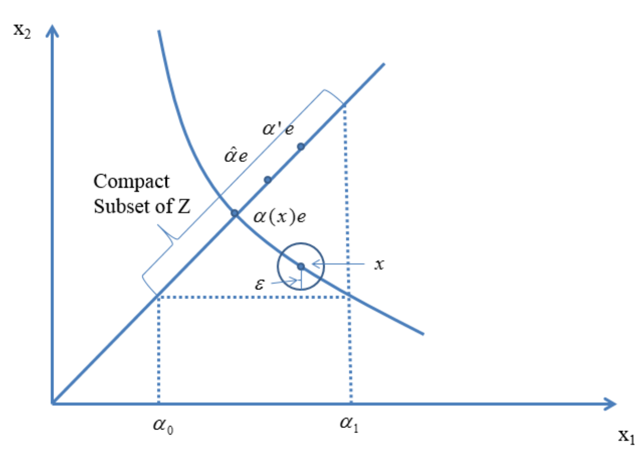
$$r(x_1) > U(x_1, 1) > U(x_1', 2) > r(x_1').$$

(this step establishes that $r(x_1)$ is monotonically increasing. Hence, it is a one-to-one mapping)

Therefore, $r(.)$ privides a one-to-one relationship from the set of real numbers (uncountable) to the set of rational numbers (countable). This is a mathematical impossibility.Q.E.D.

Theorem 1 *If preferences $\succsim$ satisfy completeness, transitivity and continuity then they can be represented by a continuous utility function. If the preferences also satisfy (strict) convexity, then they can be represented by a (strictly) quasi-concave utility function.*

The figure in MWG which illustrates the idea of the proof:



Proof: For case of proof, we focus on monotone preference relation and $X = R^L_+$.

Def $Z :=\{x \in R^L_+ : x_l = x_k \text{ for all } k, l = 1, ..., L\}$

Def $e := (1, ..., 1)$ with L elements.

$\alpha e \in Z$ for $\alpha \geq 0$.

Step 1: For all $x \in X$, $\exists$ unique $\alpha \geq 0$ such that $\alpha e \sim x$. (this step constructs a function $\alpha(x)$)

a) First, we show there exists α, such that $\alpha e \sim x$.

Define $A^- = \{\alpha \geq 0 : x \succsim \alpha e\}$ and $A^+ = \{\alpha \geq 0 : \alpha e \succsim x\}$. Since $\succsim$ is continuous, the upper contour set and lower contour set of x are closed. Hence, A^+ and A^- are nonempty (by monotonicity, $x \succsim 0$. Hence, $0 \in A^-$. Since the upper countour set is closed, there exists $\alpha \in A^+$.) and closed.

Now, we show that $R_+ \subset \{A^- \cup A^+\}$. Suppose $\exists \widetilde{\alpha} \in R_+$ and $\widetilde{\alpha} \notin \{A^- \cup A^+\}$. Then, $\widetilde{\alpha} \notin A^-$ and $\widetilde{\alpha} \notin A^+ \Rightarrow \widetilde{\alpha}e \succ x$ and $x \succ \widetilde{\alpha}e$. By completeness, $\widetilde{\alpha}e \succ x \Rightarrow \widetilde{\alpha}e \succsim x$ and $x \not\succsim \widetilde{\alpha}e$ which contradicts $x \succ \widetilde{\alpha}e$.

$R_+ \subset \{A^- \cup A^+\}$ and R_+ is connected$\Rightarrow$ $A^- \cap A^+ \neq \varnothing$. Hence, there exists a $\alpha \in A^- \cap A^+ \Leftrightarrow$ there exists α, such that $\alpha e \succsim x$ and $x \succsim \alpha e \Leftrightarrow \alpha e \sim x$.

b) Next, we show there exists a **unique** α, such that $\alpha e \sim x$. Suppose there exsit $\alpha_1 \neq \alpha_2$ and $\alpha_1 e \sim x \sim \alpha_2 e$.

WLOG, let $\alpha_1 > \alpha_2$. By monotonicity, $\alpha_1 e \succ \alpha_2 e$. This contradicts $\alpha_1 e \sim \alpha_2 e$ by completeness.

Set $u(x) = \alpha(x)$

Step 2: We show $\alpha(x)$ represents $\succsim$. i.e., $\alpha(x) \geq \alpha(y)$ iff $x \succsim y$.

Note that $\alpha(x)e \sim x$ (Step 1)

If $\alpha(x) \geq \alpha(y)$, then by monotonicity $\alpha(x)e \succsim \alpha(y)e$

By transitivity $x \succsim y$.

If $x \succsim y$, then $\xrightarrow{\text{transitivity}}$ $\alpha(x)e \succsim \alpha(y)e$ $\xrightarrow{\text{monotonicity}}$ $\alpha(x) \geq \alpha(y)$.

Step 3: (do not show this proof in class) $\alpha(x)$ is continuous, i.e., take any sequence $\{x^n\}_{n=1}^{\infty}$ with $x = \lim_{n \longrightarrow \infty} x^n$, we have $\lim_{n \longrightarrow \infty} \alpha(x^n) = \alpha(x)$.

Key steps: Suppose $\lim_{n \longrightarrow \infty} \alpha(x^n) = \alpha' > \alpha(x)$. ($\alpha(x^n)$ subsequence: every bounded sequence has a convergent subsequence.) [Proof similar for $\alpha(x) > \alpha'$]

Let $\widehat{\alpha} = \frac{1}{2}(\alpha' + \alpha(x))$, so $\alpha(x) < \widehat{\alpha} < \alpha'$, $x \sim \alpha(x)e \prec \widehat{\alpha}e \prec \alpha' e$

Recall that $x^n \sim \alpha(x^n)e$ by construction.

Since $\lim_{n \longrightarrow \infty} \alpha(x^n) = \alpha', \exists N$ $s.t.$ $\forall n \geq N$, $x^n \sim \overset{\rightarrow \alpha'}{\alpha(x^n)}e \succ \widehat{\alpha}e \overset{\text{Monotonicity}}{\succ} \alpha(x)e \sim x$

On the other hand, continuity of preference implies $\lim_{n \longrightarrow \infty} x^n = x \sim \alpha(x)e \overset{\text{monotonicity}}{\prec} \widehat{\alpha}e$

Therefore, $\exists \widetilde{N}$ $s.t.$ $\forall n \geq \widetilde{N}$, $x^n \prec \widehat{\alpha}e \Longrightarrow \forall n \geq \max\{N, \widetilde{N}\}$, $\widehat{\alpha}e \succ x^n \sim \alpha(x^n)e \succ \widehat{\alpha}e$ $\Longrightarrow$ Contradiction.

Definition: Homothetic preference A monotone preference relation $\succsim$ on X is homothetic if the following is true: if $x \sim y$, then $\alpha x \sim \alpha y$ for any $\alpha \geq 0$.

Definition: quasi-concavity The function $f : X \rightarrow R$, defined on the convex set $X \subset R^L$, is quasiconcave if and only if

$$f(\alpha x + (1-\alpha)x') \geq \text{Min}\{f(x), f(x')\}$$

for all $x, x' \in X$ and $\alpha \in [0, 1]$.

Quasi-concavity is a weaker notion than concavity. A concave function is quasiconcave but not vice versa. For example, a monoton function is quasiconcave but is not necessarily concave.

Lemma 2 *convexity of $\succsim$ implies quasi-concavity of $U(x)$*

Proof: Consider two arbitrary consumption bundle x_1, x_2. Because $\succsim$ is rational, we can rank the two consumption bundles. Without loss of generality, suppose $x_1 \succsim x_2$. This implies $U(x_1) \geq U(x_2)$. Convexity of $\succsim$ implies that for any $\alpha \in [0,1]$, the following is true

$$\alpha x_1 + (1-\alpha)x_2 \succsim x_2,$$

which implies

$$U(\alpha x_1 + (1-\alpha)x_2) \geq U(x_2) = \min\{U(x_1), U(x_2)\}.$$

Remarks:

1. A preference relation $\succsim$ can be representated by a utility function only if it is rational. Hence, Rationality is the necessary condition for the existence of a utility function and Rationality+Continuity is the sufficient condition.

2. Utility is ordinal. Positive monotone transformations of the utility function necessarily represent the same preferences. That is, suppose that $\phi : R \rightarrow R$ is a strictly increasing function. Then, if the utility function u represents preferences $\succsim$, then the function $\widehat{u}(x) = \phi(u(x))$ also represents exactly the same preferences.

3. convexity of $\succsim$ implies quasiconcavity but not concavity of $U(.)$.

4. While quasi-concavity and strict quasi-concavity are preserved by monotone transformations, concavity is not. Thus, assuming concavity of the utility function does not make sense in this context. For example, consider a concave function $f(x) = x$. Make the monotone transformation $g(f(x)) = x^3$. The function $g(x)$ is convex for $x \geq 0$.

assumption on preference		indifference curve
rational+continuity	$\rightarrow$	existience
montonicity	$\rightarrow$	downward sloping, thin
convexity	$\rightarrow$	convex

assumption on preference		property of utility function
rational+continuity	$\rightarrow$	Existence
montonicity	$\rightarrow$	increasing
convexity	$\rightarrow$	quasiconcave

Exercises 3.C.5. ii)

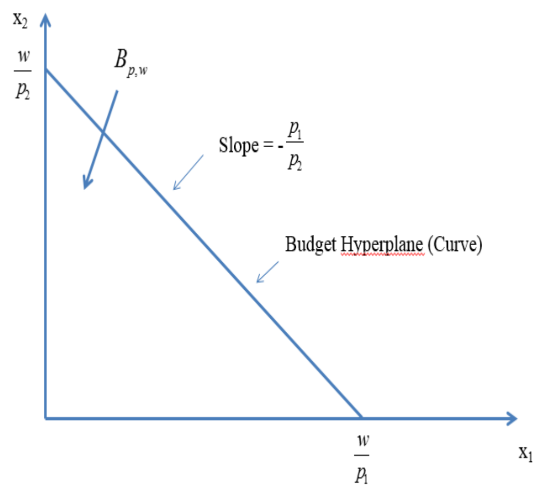
2 Budget Constraints

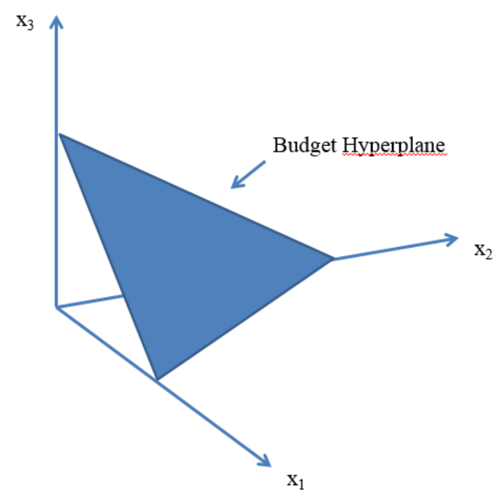
The consumer is not free to choose any vector x she likes. In general, in order to buy one unit of good i the consumer will have to pay a price p_i. Thus, if x_i is the amount consumed of good i then the consumer will have to pay an amount $p_i x_i$. The consumer is endowed with a quantity w of money, and this poses a limit on the total amount that can be spent buying the goods.

- In particular, consider a bundle $x = (x_1, x_2, \ldots, x_L)$. If the prices of the goods are given by the vector $p = (p_1, p_2, \ldots, p_L)$ then the total cost of bundle x is given by

$$p.x = \begin{bmatrix} p_1 & . & . & . & p_L \end{bmatrix} \begin{bmatrix} x_1 \\ . \\ . \\ . \\ x_L \end{bmatrix} = \sum_{i=1}^{L} p_i x_i.$$

A consumer with wealth w can obtain the bundle x only if $p.x \leq w$, that is only if the cost of the bundle is inferior to her wealth. Therefore, the only bundles that the consumer can consider are those belonging to the set: $B(p, w) = \{x \in X \,|px \leq w\}$





- This is the simplest representation of the budget set, corresponding to the case of *linear pricing*, that is each unit has the same price. When pricing is non-linear the budget set takes different forms.

Example 1: Two-part tariffs. Some goods, such as many utilities, require the payment of a fixed fee independent of the amount consumed, plus a price for each unit consumed. Let F be the fixed fee, and p_i the unit price of the good. Let us call $T(x_i)$ the total cost of consuming x_i. Then

$$T(x_i) = \begin{cases} 0 & \text{if} \quad x_i = 0 \\ F + p_i x_i & \text{if} \quad x_i > 0 \end{cases}$$

Suppose now that consumers only consume bread (x_1), which is linearly priced at p_1, and water, which is supplied through a two-part tariff scheme with fee F and unit price p_2. Then their budget set is:

$$B((p_1, p_2, F), w) = \left\{(x_1, x_2) \in R^2_+ \mid p_1 x_1 + T(x_2) \leq w \right\}.$$

In other words, the bundles (x_1, x_2) that the consumer can buy are either those such that $x_2 = 0$ and $p_1 x_1 \leq w$, or those such that $x_2 > 0$ and $p_1 x_1 + p_2 x_2 + F \leq w$.

Example 2: quantity discount

3 Consumer Problem

The consumer problem can now be written as follows:

$$\max_{x \in B(p,w)} \quad u(x)$$

where $B(p,w)$ is the budget set.

- Consider the case of linear pricing and $X = R_+^L$. Then we can write the problem as:

  $$\max_{x_1,\ldots,x_L} \quad u\left(x_1, \ldots, x_L\right)$$

  subject to:

  $$p.x \leq w,$$

  $$x_i \geq 0, \quad i = 1, ..., L.$$

- This is a standard constrained maximization problem. If u is continuous then a solution always exists, since the feasible set is compact (this follows from the Weierstrass theorem).

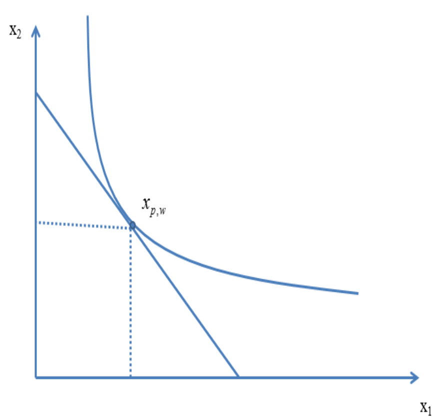
(the Weierstrass Theorem) Let $D \subset R^n$ be compact, and let $f : D \to R$ be a continuous function on D. Then, f attains a maximum and a minimum on D. That is, there exist points $z_1, z_2 \in D$ such that

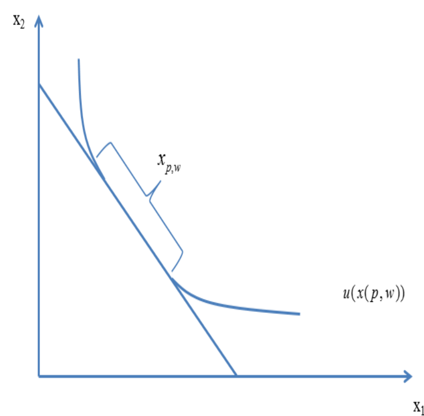
$$f(z_1) \geq f(x) \geq f(z_2), \quad x \in D.$$

Theorem 2 *Let $D \subset R^n$ be compact, and let $f : D \to R$ be a continuous function on D. Then, f attains a maximum and a minimum on D. That is, there exist points z_1, $z_2 \in D$ such that*

$$f(z_1) \geq f(x) \geq f(z_2),\ \forall x \in D.$$

- However, there may be multiple solutions, i.e. various bundles of goods may be seen by the consumer as optimal. The set $x^*(p, w)$ of bundles solving the consumers' problem is called the *demand correspondence*. When $x^*(p, w)$ has a single element for each (p, w) then we call it *demand function*.





Example. Suppose that the utility function is $u(x_1, x_2) = x_1 + x_2$. In this case it can be checked that the demand correspondence takes the following form:

- If $p_1 > p_2$ then $x_1 = 0$ and $x_2 = \frac{w}{p_2}$.
- If $p_1 = p_2$ then any pair $(x_1, x_2) \in R^2_+$ such that $x_1 + x_2 = \frac{w}{p_1}$ is optimal.
- If $p_1 < p_2$ then $x_1 = \frac{w}{p_1}$ and $x_2 = 0$.

3.1 Properties of Individual Demands

We now investigate the general properties of the demand correspondence.

- **Definition 2.E.1:** $x(p, w)$ is homogeneous of degree zero (H.D.$\emptyset$) if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

  Proposition 2.E.1: *If the Walrasian demand function $x(p, w)$ is homogeneous of degree zero, then for* ***all*** *p & w,* $\sum_{k=1}^{L} \frac{\partial x_l(p,w)}{\partial p_k} p_k + \frac{\partial x_l(p,w)}{\partial w} w = 0$, *for* $i = 1, ..., L$

$$or \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & . \; . \; . & \frac{\partial x_1(p,w)}{\partial p_L} \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial x_L(p,w)}{\partial p_1} & . \; . \; . & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} \begin{bmatrix} p_1 \\ . \\ . \\ . \\ p_L \end{bmatrix} + \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ . \\ . \\ . \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} w = 0$$

or $D_p x(p, w)p + D_w x(p, w)w = 0.$

Proof: Homogenous degree zero$\Leftrightarrow x(\alpha p, \alpha w) = x(p, w)$
$\Rightarrow \frac{\partial x_i(\alpha p, \alpha w)}{\partial \alpha} = 0 \Rightarrow \sum_{k=1}^{L} \frac{\partial x_i(\alpha p, \alpha w)}{\partial p_k} p_k + \frac{\partial x_i(\alpha p, \alpha w)}{\partial w} w = 0$, *for* $i = 1, ..., L$
$\Rightarrow D_p x(\alpha p, \alpha w)p + D_w x(\alpha p, \alpha w)w = 0$, setting $\alpha = 1$ implies the result. Q.E.D

- **Walras' Law: Definition 2.E.1:** $x(p, w)$ satisfies Walras' law if for every $p \gg 0$ and $w > 0$, we have $p \cdot x(p, w) = w$.

- **Walras' Law has two implications:**

  Proposition 2 *(2.E.2)* If the Walrasian demand function $x(p, w)$ satisfies the Walras' Law, then for all p and w: $\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p, w) = 0$, for $k = 1, 2, ..., L.$

Proof:

$$
\begin{aligned}
\frac{\partial}{\partial p_k}(p \cdot x(p,w)) &= 0 \quad \left(\text{i.e., } \frac{\partial}{\partial p_k}(\sum_{l=1}^{L} p_l x_l(p,w)) = 0\right) \\
p \cdot \frac{\partial x(p,w)}{\partial p_k} + x_k(p,w) &= 0 \quad \left(\text{i.e., } \sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial p_k} + x_k(p,w) = 0\right)
\end{aligned}
$$

Q.E.D.

Alternative representation of Proposition 2.E.2:

$$
[p_1, p_2, ..., p_L] \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdot \; \cdot \; \cdot & \frac{\partial x_1(p,w)}{\partial p_L} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdot \; \cdot \; \cdot & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} + \begin{bmatrix} x_1(p,w) & \cdot \; \cdot \; \cdot & x_L(p,w) \end{bmatrix} = \begin{bmatrix} 0 & \cdot \; \cdot \; \cdot & 0 \end{bmatrix},
$$

or

$$
p \cdot D_p x(p,w) + x(p,w) = 0.
$$

- Proposition 2.E.2 states that when price changes, total expenditure remain unchanged. This is because the wealth does not change and the consumer always use up the wealth when Walras' law holds.

- If we differentiate the budget curve w.r.t w, we immediately have $\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$ or $p \cdot D_w x(p,w) = 1$. And the following proposition follows.

  Proposition 2.E.3: *If the Walrasian demand function $x(p,w)$ satisfies Walras' Law, then for ALL p and w :* $\sum_{l=1}^{L} p_l \frac{\partial x_l(p,w)}{\partial w} = 1$.

Alternative representation of Proposition 2.E.3:

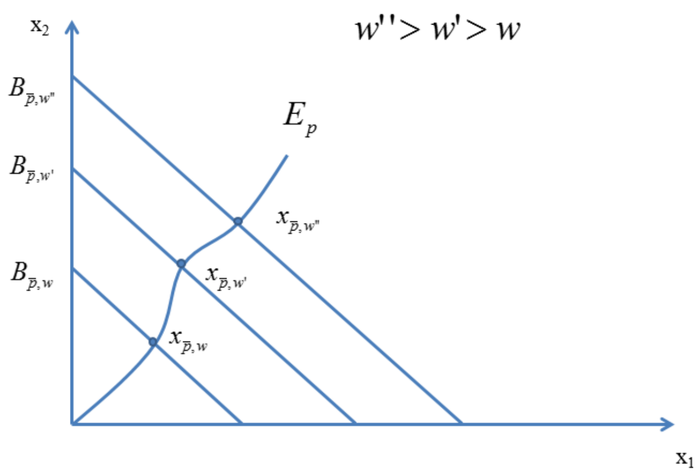
$$\begin{aligned} [p_1, ..., p_L] \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ . \\ . \\ . \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} &= 1, \text{ or} \\ p \cdot D_w x(p, w) &= 1. \end{aligned}$$

This propositon states that when wealth increases by \$1, total expenditure must also increase by \$1.

3.2 Comparative Statics (with respect to p and w)

Wealth Effect

- Engel function: $x(\bar{p}, w)$ for fixed $\bar{p}.E_{\bar{p}} = \{x(\bar{p}, w) : w > 0\}$: the wealth expansion path



Wealth Effect: $\frac{\partial x_l(p,w)}{\partial w} \geq 0$ normal; $\frac{\partial x_l(p,w)}{\partial w} < 0$ inferior

Price Effect

- $\frac{\partial x_l(p,w)}{\partial p_k} > 0$ substitutes; $\frac{\partial x_l(p,w)}{\partial p_k} = 0$ independent; $\frac{\partial x_l(p,w)}{\partial p_k} < 0$ complements. $\frac{\partial x_l(p,w)}{\partial p_l} > 0$ Giffen Good.

- Notation $D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & . \quad . \quad . & \frac{\partial x_1(p,w)}{\partial p_L} \\ . & . & . \\ . & . & . \\ . & . & . \\ \frac{\partial x_L(p,w)}{\partial p_1} & . \quad . \quad . & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}$, $D_w x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ . \\ . \\ . \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix}$

We first state some general properties which can be obtained with very few assumptions on preferences.

Theorem 3 *If the rational preferences $\succeq$ satisfy continuity and monotonicity, then the demand correspondence $x^*(p,w)$satisfies the following properties: i) If $x \in x^*(p,w)$ then $p.x = w$ (Walras law), ii) If $x \in x^*(p,w)$ and $\alpha > 0$, then $x \in x^*(\alpha p, \alpha w)$(homogeneous degree zero), and iii) If $\succeq$ is convex, so that $u(x)$ is quasiconcave, then $x^*(p,w)$ is a convex set. Moreover, if $\succeq$ is strictly convex, so that $u(x)$ is strictly quasiconcave, then $x^*(p,w)$ is a singleton.*

Proof:

i) To prove the first point, suppose by contradiction that $x \in x^*(p,w)$ and $px < w$. Then it is possible to find $\varepsilon > 0$ such that the bundle $x' = (x_1 + \varepsilon, x_2 + \varepsilon, \ldots, x_L + \varepsilon)$ satisfies $px' \leq w$. But monotonicity implies that $x' \succ x$, contradicting that x is the preferred bundle among the feasible ones.

ii) To prove the second point observe that $B(p,w) = B(\alpha p, \alpha w)$ when prices are linear (this is obtained dividing by α both sides of the budget constraint in $B(\alpha p, \alpha w)$). Therefore the consumer's problem is the same under (p,w) and under $(\alpha p, \alpha w)$; the solution must therefore be identical.

iii) To prove the third point, Suppose that there are two vectors x and y which are considered optimal by a consumer facing prices p and having budget w. Denote $u^* = u(x) = u(y)$. Notice that both $px \leq w$ and $py \leq w$ must hold. Now consider the new vector

$$x^\alpha = \alpha x + (1 - \alpha)\, y$$

for some $\alpha \in (0, 1)$. We have $x^\alpha \in B\,(p, w)$, because

$$px^\alpha = \alpha px + (1 - \alpha)\, py \leq \alpha w + (1 - \alpha)\, w = w.$$

By quasiconcavity,

$$u(x^\alpha) \geq \min\{u(x), u(y)\} = u^*.$$

Because u^* is the optimal utility, $x^\alpha \in x^*\,(p, w)$.

Lastly, consider strictly convex $\succeq$ and hence strictly quasiconcave utility function. By the argument above we have $x^\alpha \in B\,(p, w)$. Strictly quasiconcavity implies

$$u(x^\alpha) > \min\{u(x), u(y)\} = u^*.$$

This contradicts the assumption that x and y are optimal bundles.

- The first property states that the consumer always spends all the budget, and it is a straightforward consequence of the fact that more is preferred to less (i.e. monotonicity of preferences).

- The second implies that rational consumers have no ‘monetary illusion’. If all prices and the wealth double (or, more generally, are multiplied by the same positive number), they understand that the situation has not changed, and therefore make the same choices. One important implication of this result is that we can choose an arbitrary unit of account when defining prices and income without changing the problem. In other words, suppose that the consumer faces prices and income $(p, w) = (p_1, p_2, \ldots, p_L, w)$. We may decide to redefine units so that the price of the first good is 1. This is equivalent to dividing all prices and income by p_1, so that the consumer now faces:

$$(p', w') = \left(1, \frac{p_2}{p_1}, \ldots, \frac{p_L}{p_1}, \frac{w}{p_1}\right).$$

The second point in Theorem (**??**) makes sure that the consumer's problem is the same when facing (p, w) and when facing (p', w'). When $x^*(p, w)$ is a function, the result implies $x^*(p, w) = x^*(\alpha p, \alpha w)$ for each $\alpha > 0$, that is *the demand function is homogeneous of degree zero.*

- In order to characterize further the solution of the consumer problem we now analyze the case of a differentiable utility function, that is we make the following assumption: The function $u(x)$ is quasi-concave and differentiable. When this assumption is satisfied we can solve the consumer problem applying the techniques for constrained optimization from calculus.

- Set up the Lagrange function

$$\underset{x \in R^L_+, \lambda \geq 0}{L(x, \lambda)} = u(x) + \lambda(w - p \cdot x),$$

where $\lambda \geq 0$ is the *Lagrange multiplier.*

- Then, the solution of the UMP satisfies

$$\frac{\partial u(x^*)}{\partial x_i} \leq \lambda p_i, \text{ with equality if } x_i^* > 0 \tag{1}$$

$$x_i^*\left[\frac{\partial u(x^*)}{\partial x_i} - \lambda p_i\right] = 0 \tag{2}$$

$$\lambda\left[\sum_{i=1}^{L} p_i x_i^* - w\right] = 0 \tag{3}$$

for all $i = 1, ..., L$

Let $\nabla u(x^*) = (\frac{\partial u(x^*)}{\partial x_1} ... \frac{\partial u(x^*)}{\partial x_l})$. The FOC in matrix notation is

$$\begin{aligned} \nabla u(x^*) &\leq \lambda p, \\ x^*.[\nabla u(x^*) - \lambda p] &= 0 \\ \lambda[px^* - w] &= 0 \end{aligned}$$

- Notice that the assumption of quasi-concavity ensures that first-order conditions are also sufficient.???

- $\lambda \geq 0$ measures the marginal utility of \$1 of wealth **at the optimum**. To see this explicitly, assume $x^* \gg 0$.

$$\begin{aligned} \frac{\partial u(x^*(p,w))}{\partial w} &= \sum_{i=1}^{L} \frac{\partial u(x^*)}{\partial x_i} \frac{\partial x_i}{\partial w} \\ &= \sum_{i=1}^{L} \lambda p_i \frac{\partial x_i}{\partial w} \end{aligned}$$

The second equality follows from the F.O.C

$$\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i$$

Recall that by Walras' Law (Prop 2.E.3)

$$\sum_{i=1}^{L} p_i \frac{\partial x_i}{\partial w} = 1.$$

Hence

$$\frac{\partial u(x^*(p,w))}{\partial w} = \lambda$$

- Suppose $x_i^* = 0$ (corner solution) for some i. Then, it is possible to have $\frac{\partial u(x^*)}{\partial x_i} < \lambda p_i$ which suggests that utility is increased if x_i^* is decreased. But then, constraint (2) requires that $x_i^* = 0$. In other words, the nonnegative constraint on x_i is binding.

- If $px^* - w < 0$, the budget constraint is slack at the optimal consumption bundle. Condition (3) requires $\lambda = 0$. In this case, relaxing the budget doesn't increase utility

- We focus on the easiest case: assume that $x^* \gg 0$ (this occurs, for example, when $\frac{\partial u}{\partial x_i}$ becomes arbitrarily large when x_i goes to zero) and $\lambda > 0$ (this occurs when the consumer spends entirely the budget; monotonicity of preferences is a sufficient condition). The problem can be written as a system of $L + 1$ equations (the L conditions $\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i$ and the budget constraint $p.x^* = w$) with $L+1$ unknowns (the Lagrange multiplier λ and the L quantities $x_1^*, \ldots, x_L^*$). The solution to this problem in general depends on the vector of prices $p = (p_1, \ldots p_L)$ and on the wealth w. Therefore we write the solution as:

$$x^* (p, w) = (x_1^* (p, w), \ldots, x_L^* (p, w))$$

- Another way to understand the interior solution $\nabla u (x^*) = \lambda p$ is that the marginal utility at the optimal consumption bundle is proportional to price. Suppose we change the consumption bunle slightly along the budget hyperplane and call the new consumption bundle x'. Then, $p(x^* - x') = 0$ because both consumption bundles cost w. So, the price vector is orthogoal to $\Delta x = (x^* - x')$ (illustrate this using the two goods example). Since x^* is the optimal consumption bundle, a small purterbation to it on the budget hyperplane should not change the utility (again, illustrate this using the two goods indifference curve). Hence, $\nabla u (x^*) \Delta x = 0$. This sugges that $\nabla u (x^*)$ is proportional to p and the proportion is captured by λ.

- $\nabla u (x^*) = \lambda p$ is the "last dollar rule"

Example: The Cobb-Douglas function. Suppose that there are two goods, x_1 and x_2. Consumer's preferences are represented by the utility function:

$$u(x_1, x_2) = A\, x_1^a x_2^b$$

where $A > 0, a\blacksquare b \in (0, 1)$.

The consumer's problem can therefore be written as:

$$\max \quad A\, x_1^a x_2^b$$

subject to: $p_1 x_1 + p_2 x_2 = w$. The Lagrangian is:

$$L = A\, x_1^a x_2^b - \lambda \left[p_1 x_1 + p_2 x_2 - w \right]$$

(we are ignoring the positivity constraints, since setting $x_i = 0$ for some i is clearly sub-optimal). The system of the first order conditions is:

$$\begin{aligned} A a x_1^{a-1} x_2^b &= \lambda p_1 \\ A b x_1^a x_2^{b-1} &= \lambda p_2 \\ p_1 x_1 + p_2 x_2 &= w \end{aligned}$$

Dividing side by side the first and the second equations we have:

$$\begin{aligned} \frac{a x_2}{b x_1} &= \frac{p_1}{p_2} \\ p_1 x_1 + p_2 x_2 &= w \end{aligned}$$

The solution gives us the following demand functions:

$$x_1^* (p_1, p_2, w) = \frac{a}{(a+b)\, p_1} w \qquad x_2^* (p_1, p_2, w) = \frac{b}{(a+b)\, p_2} w \tag{4}$$

Notice that we can simplify the analysis. Since monotone positive transformations of the utility function leave the result unchanged, we can do the following:

- divide the utility function by the constant A;
- take the power $\frac{1}{a+b}$ of the resulting expression.

We obtain the utility function:

$$\overline{u}(x_1, x_2) = x_1^{\frac{a}{a+b}} x_2^{\frac{b}{a+b}} = x_1^{\alpha} x_2^{1-\alpha}$$

where $\alpha = \frac{a}{a+b}$ is a number between zero and one. This utility function represents the same preferences as the original one, and therefore yields exactly the same demand functions. Furthermore, we already observed that we can exclude without loss of generality from the analysis the bundles in which $x_1 = 0$ or $x_2 = 0$, since such bundles are obviously sub-optimal. Then another admissible monotonic transformation is obtained taking the natural logarithm; we end up with the utility function

$$\overline{\overline{u}}(x_1, x_2) = \ln\left(x_1^{\alpha} x_2^{1-\alpha}\right) = \alpha \ln x_1 + (1-\alpha) \ln x_2.$$

Again, the utility functions u, $\overline{u}$ and $\overline{\overline{u}}$ represent the same preferences, and we can choose any of them to solve the consumer's problem. The demand functions will always be the same, namely the ones given in (4).

Example: Quasi-linear function. Suppose that the utility function is *quasi-linear*, that is,

$$u(x) = x_1 + f(x_2, ..., x_L).$$

Then it is convenient to use good 1 as the numeraire and rewrite the budget constraint as

$$x_1 + \sum_{i=2}^{L} \widehat{p}_i \, x_i = \widehat{w},$$

where $\widehat{p}_i \equiv p_i/p_1$ and $\widehat{w} \equiv w/p_1$. The budget constraint is binding since $u(x)$ is increasing in x_1. Assume that x_1 can take any real value, and in particular that it can be negative.

Substituting into the utility function simplifies the consumer problem to:

$$\max_{x_2,\dots,x_L} \quad f(x_2,\dots,x_L) + \widehat{w} - \sum_{i=2}^{n} \widehat{p}_i \, x_i$$

The FOCs are:

$$\begin{aligned} \underbrace{\frac{\partial f(x_2,...,x_n)}{\partial x_i}} &= \widehat{p}_i, \quad i = 2, n \\ \begin{array}{c}\text{marginal}\\ \text{utility}\end{array} &= \quad \text{price} \end{aligned}$$

Thus the demands for goods $2, ...L$, do not depend on income, but only on prices: an increase in income goes entirely into consumption of the first good.

3.3 The Marginal Rate of Substitution

Definition 1 *The* marginal rate of substitution (MRS) *of good i for good j at consumption bundle x^* is the amount of good j which should be given to compensate her for a one-unit reduction in her consumtion of good i.*

$MRS_{ij}(x^*)$ is the slope of the indifference curve at consumption bundle x^*. To understand better the concept, suppose that there are only two goods, x_1 and x_2, and fix a consumption bundle $x^* = (x_1^*, x_2^*)$. Total differentiate the utility function at x^*, we have

$$\frac{\partial u}{\partial x_1}(x_1^*, x_2^*)\, dx_1 + \frac{\partial u}{\partial x_x}(x_1^*, x_2^*)\, dx_2 = 0$$

Then the marginal rate of substitution between 1 and 2 is given by:

$$MRS_{1,2}(x^*) = \frac{dx_2}{dx_1} = -\frac{\frac{\partial u}{\partial x_1}(x_1^*, x_2^*)}{\frac{\partial u}{\partial x_2}(x_1^*, x_2^*)}$$

More in general, given a vector $x^* = (x_1^*, \dots, x_L^*)$, we have:

$$MRS_{i,j}(x^*) = -\frac{\frac{\partial u}{\partial x_i}(x^*)}{\frac{\partial u}{\partial x_j}(x^*)}.$$

The marginal rate of substitution of good i for good j represents, in a sense, the price that the consumer is willing to pay in terms of good j for one unit of good i, and this is given by the ratio between the marginal increase in utility given by good i and the marginal increase in utility given by good j.

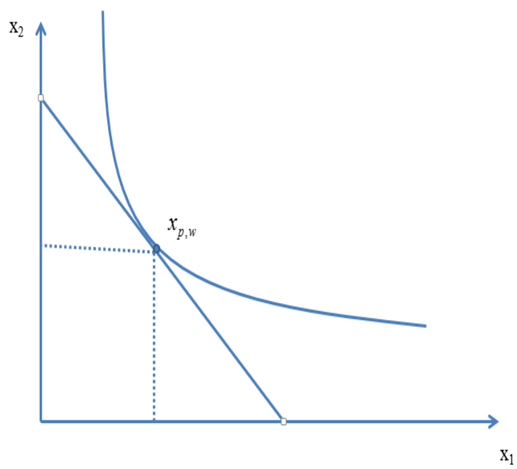
- **Remark.** The concept of MRS relates to preferences, not to a particular utility function used to represent preferences. This must imply that for *any* utility function that represents the same preferences, the MRS must be the same. In particular, let $\phi : R \to R$ be a strictly increasing and differentiable function, and let $\phi' > 0$ be its derivative. Then we know that $\phi(u(x))$ is a utility function that represents the same preferences as $u(x)$. The marginal rate of substitution is given by:
$$MRS_{i,j}(x^*) = \frac{\phi'(u(x^*))\frac{\partial u}{\partial x_i}(x^*)}{\phi'(u(x^*))\frac{\partial u}{\partial x_j}(x^*)} = \frac{\frac{\partial u}{\partial x_i}(x^*)}{\frac{\partial u}{\partial x_j}(x^*)},$$
so it is the same as before.

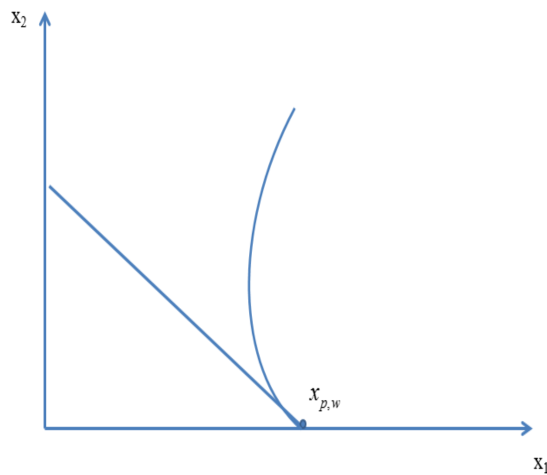
- Let us now go back to the consumer maximization problem. We have seen that, when the solution involves strictly positive consumption of all goods, the first order conditions take the form:
$$\frac{\partial u(x^*)}{\partial x_i} = \lambda p_i.$$
This in turn implies, taking ratios side by side:
$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{\frac{\partial u(x^*)}{\partial x_j}} = -MRS_{i,j}(x^*) = \frac{p_i}{p_j}, \qquad \text{for all } i, j = 1, ...L. \tag{5}$$

Another way to understand the optimal consumption bundle is to write (**??**) as

$$\frac{\frac{\partial u(x^*)}{\partial x_i}}{p_i} = \frac{\frac{\partial u(x^*)}{\partial x_j}}{p_j}, \qquad \text{for all } i, j = 1, ...L.$$

This is often called "the last dollar rule".





4 The Indirect Utility Function

We have seen that, given a utility function $u(x)$, for each price vector p and level of wealth w we can compute an optimal choice $x^*(p, w)$. This in turn implies that we can compute the utility achieved by the consumer as a function of (p, w). This function is called the *indirect utility function*, and it is defined as:

$$v(p, w) \equiv u(x^*(p, w)).$$

The indirect utility function is often a useful object to work with.

Example: Cobb-Douglas. We have seen that in this case the demand functions are:

$$x_1(p, w) = \frac{a}{(a+b)\, p_1} w \qquad\qquad x_2(p, w) = \frac{b}{(a+b)\, p_2} w$$

Substituting into the utility function we have the following indirect utility function:

$$v(p_1, p_2, w) = A \frac{a^a b^b}{(a+b)^{a+b}} \cdot \frac{w^{a+b}}{p_1^a p_2^b}.$$

What are the properties of the indirect utility function $v(p,w)$? We expect consumers to be worse off when prices go up, and better off when wealth is increased. This implies that the function $v(p_1,\ldots,p_L,w)$ should be decreasing in each p_i and increasing in w. Another intuitive property is that the indirect utility should remain the same when the consumer's choices do not change. This in particular implies that, since $x^*(p,w) = x^*(\alpha p,\alpha w)$ (the demand does not change when all prices and wealth are multiplied by the same positive constant), we should also have $v(p,w) = v(\alpha p,\alpha w)$. This discussion leads to the following theorem, which we state without a formal proof.

Suppose that $u(\cdot)$ is a continuous utility function representing a monotone preference relation $\succsim$ defined on the consumption set $X = R^L_+$. The indirect utility function $v(p,w)$ is: (i)Homogeneous of degree zero (ii) **Strictly** increasing in w and nonincreasing in p_l for any l, (iii) Quansiconvex; that is, the set $\{(p,w) : v(p,w) \leq \overline{v}\}$ is convex for any $\overline{v}$. and (iv)Continuous in $p \gg 0$ and w.

Proposition 3 *Suppose that $u(.)$ is a continuous utility function representing a monotone preference relation $\succeq$ defined on the consumption set $X = R^L_+$. The indirect utility function $v(p,w)$ is: (i)Homogeneous of degree zero (ii) (strictly) increasing in w and nonincreasing in p_l, $\forall l$. (iii) Quansiconvex; that is, the set $\{(p,w) : v(p,w) \leq \overline{v}\}$ is convex for any $\overline{v}$ and (iv)Continuous in p and w.*

Proof:

(i) $\Rightarrow B(p,w) = B(\alpha p,\alpha w) \Rightarrow x(p,w) = x(\alpha p,\alpha w) \Rightarrow u(x(p,w)) = u(x(\alpha p,\alpha w)) \Longleftrightarrow v(p,w) = v(\alpha p,\alpha w)$.

Alternatively, we say that $B_{p,w}$ is unaffected. $\max u(x)$ s.t. $p \cdot x \leq w$ is equivalent to $\max u(x)$ s.t. $\alpha p \cdot x \leq \alpha w$.

(ii) $p \cdot x(p,w) = w$ by Walras' Law.

$v(p,w)$ non-increasing in p : Suppose $p' \geq p$, then $B(p',w) \subseteq B(p,w)$. Therefore, $v(p,w) \geq v(p',w)$.

$v(p,w)$ strictly increasing in w : Suppose $w' > w$ and $v(p,w') \leq v(p,w)$. Let the corresponding demand be $x(p,w')$ and $x(p,w)$. By Walras' Law $px(p,w) = w < w'$. So, $x(p,w) \in B(p,w')$. Since the set $\{x \in X | px < w'\}$ is an open set, $\exists$ a small $\varepsilon > 0$ such that $\widetilde{x} = (x_1 + \varepsilon, ...x_L + \varepsilon)$ satisfies $p\widetilde{x} < w'$. Hence, $\widetilde{x} \in B(p,w')$. By monotonicity of $\succsim$, $u(\widetilde{x}(p,w')) > u(x(p,w)) \geq u(x(p,w'))$. This contradicts the assumption that $x(p,w')$ is the demand given p and w'.

(iii) Consider (p,w) and (p',w') such that $v(p,w) \leq \overline{v}$ and $v(p',w') \leq \overline{v}$. The corresponding budgetssets are denoted by $B(p,w)$ and $B(p',w')$.Consider a convex combination of (p,w) and (p',w'):

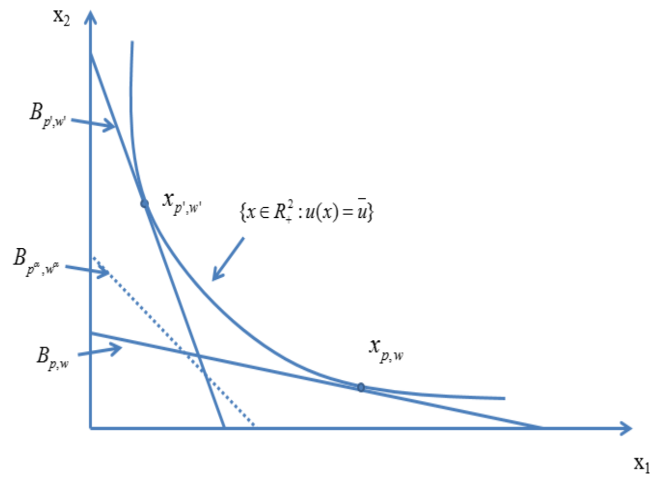
$$(p^\alpha, w^\alpha) = (\alpha p + (1-\alpha)p', \alpha w + (1-\alpha)w').$$

Given the (p^α, w^α), the new budget set is

$$\begin{aligned} B(p^\alpha, w^\alpha) &= \{x \in R^L_+ : [\alpha p + (1-\alpha)p']x \leq \alpha w + (1-\alpha)w'\} \\ &= \{x \in R^L_+ : \alpha p.x + (1-\alpha)p'.x \leq \alpha w + (1-\alpha)w'\}. \end{aligned}$$

This implies for any $x \in B(p^\alpha, w^\alpha)$, either $p \cdot x \leq w$ or $p' \cdot x \leq w'$ or both. Hence,

$x \in B_{p,w} \cup B_{p',w'} \Longrightarrow u(x) \leq \overline{v}$.



(iv) $v(p,w)$ is continuous follows from the continuity of $U(.)$ and $x(p,w)$.

Relationship between demand and indirect utility function.

Roy's identity Suppose $u(x)$ is continuous, monotone and strictly quasi-concave. Suppose also $v(p,w)$ is differentiable at $(\overline{p},\overline{w}) > 0$. Then

$$x_i(\overline{p},\overline{w}) = -\frac{\partial v(\overline{p},\overline{w})/\partial p_i}{\partial v(\overline{p},\overline{w})/\partial w},$$

for all $i = 1, ..., L$.

Proof: Assume $x(p,w)$ is differentiable and $x(p,w) \gg 0$. Differentiate $v(\overline{p},\overline{w})$ with respect to p_i at $(\overline{p},\overline{w})$, we have

$$\begin{aligned}\frac{\partial v(\overline{p},\overline{w})}{\partial p_i} &= \frac{\partial u\left(x_1(\overline{p},\overline{w}), x_2(\overline{p},\overline{w}), ..., x_L(\overline{p},\overline{w})\right)}{\partial p_i} \\ &= \sum_{j=1}^{L} \frac{\partial u(x)}{\partial x_j}\frac{\partial x_j}{\partial p_i}.\end{aligned}$$

By the F.O.C. of the utility maximization program, we have

$$\frac{\partial u(x)}{\partial x_j} = \lambda p_j,$$

so

$$\frac{\partial v(\overline{p},\overline{w})}{\partial p_i} = \sum_{j=1}^{L} \lambda p_j \frac{\partial x_j}{\partial p_i}.$$

By Walras' law (Prop 2.E.2),

$$\sum_{j=1}^{L} p_j \frac{\partial x_j(\overline{p},\overline{w})}{\partial p_i} = -x_i(\overline{p},\overline{w}).$$

Hence,

$$\frac{\partial v(\overline{p},\overline{w})}{\partial p_i} = -\lambda x_i(\overline{p},\overline{w}).$$

Finally, we show $\lambda = \dfrac{\partial v(\overline{p},\overline{w})}{\partial w}$.

$$\begin{aligned}
\frac{\partial v(\overline{p},\overline{w})}{\partial w} &= \frac{\partial u\left(x_1(\overline{p},\overline{w}), x_2(\overline{p},\overline{w}), ..., x_L(\overline{p},\overline{w})\right)}{\partial w} \\
&= \sum_{j=1}^{L} \frac{\partial u(x)}{\partial x_j} \frac{\partial x_j(\overline{p},\overline{w})}{\partial w}. \\
&= \sum_{j=1}^{L} \lambda p_j \frac{\partial x_j(\overline{p},\overline{w})}{\partial w}
\end{aligned}$$

The last equality follows from the F.O.C. By the By Walras' Law (Prop 2.E.3), we have

$$\sum_{j=1}^{L} p_j \frac{\partial x_j(\overline{p},\overline{w})}{\partial w} = 1.$$

Hence, $\dfrac{\partial v(\overline{p},\overline{w})}{\partial w} = \lambda$ and accordingly

$$\begin{aligned}
\frac{\partial v(\overline{p},\overline{w})}{\partial p_i} &= -x_i(\overline{p},\overline{w}) \frac{\partial v(\overline{p},\overline{w})}{\partial w} \\
x_i(\overline{p},\overline{w}) &= -\frac{\dfrac{\partial v(\overline{p},\overline{w})}{\partial p_i}}{\dfrac{\partial v(\overline{p},\overline{w})}{\partial w}}.
\end{aligned}$$

Q.E.D.

5 Expenditure minimization problem

- The inverse of $v(p,w)$ with respect to w, for fixed p, which is denoted by the symbol $e(p,u)$ is called *the expenditure function*. The expenditure function is obtained answering the following question: When the price vector is p, what is the minimum amount of wealth that should be given to the consumer for her to be able to achieve a utility of u? In other words, let $x^h(p,u) \in X$ be the vector of goods solving:

$$\min_{x \in X} \qquad p \cdot x$$

subject to:

$$\begin{aligned} u(x) &\geq u \\ x &\geq 0. \end{aligned}$$

It is clear that when $u(x)$ is monotonic, we can write the constraint as equality. Then in this case the first order conditions of the problem are obtained writing the Lagrangian:

$$\sum_{i=1}^{L} p_i x_i - \lambda \left(u \left(x_1, \ldots, x_L \right) - u \right)$$

and differentiating to obtain:

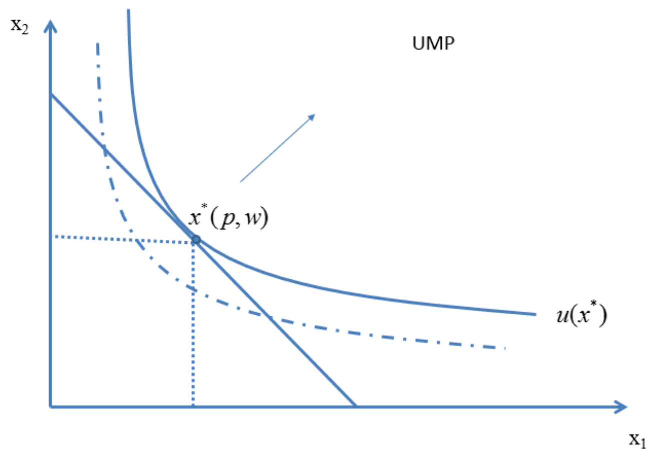
$$p_i = \lambda \frac{\partial u}{\partial x_i} \qquad\qquad i = 1, \ldots, L \tag{6}$$

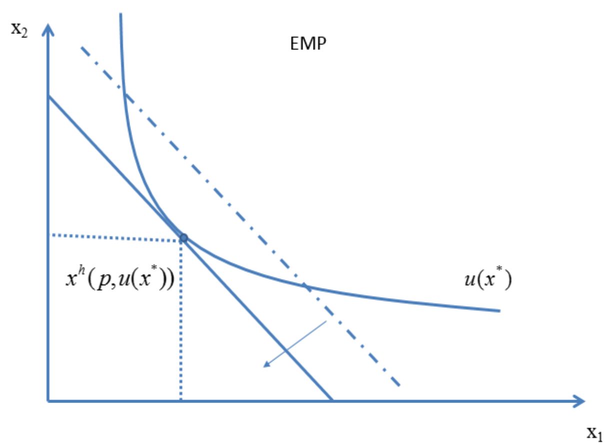
$$u \left(x_1, \ldots, x_L \right) = u$$

The solution of this program is the *compensated (Hicksian) demand function* $x^h(p,u)$, and the expenditure function is given by:

$$e(p,u) = p \cdot x^h(p,u) = \sum_{i=1}^{L} p_i x_i^h(p,u)$$

Proposition 4 *(3.E.1): Suppose $u(.)$ is a continuous utility function representing a monotone preference relation $\succeq$ defined on the consumption set $X = R_+^L$ and that the price vector is $p \gg 0$. We have (i) If $x^*(p, w)$ is optimal in the UMP when wealth is $w > 0$, then x^* is optimal in the EMP when the required utility is $u(x^*)$. Moreover, the minimized expenditure in the EMP is w.(ii) If $x^*(p, u)$ is optimal in the EMP when the required utility level is $u > u(0)$, then x^* is optimal in the UMP when wealth is $p \cdot x^*$. Moreover, the maximized utility in the UMP is u. (*No excess utility)*





Proof (skipped):

(i) Suppose $x^* = \underset{x \in B(p,w)}{\arg\max} u(x)$ is not optimal in the EMP given utility $u(x^*)$. Then, $\exists x'$ such that $p \cdot x' < p \cdot x^* \leq w$ and $u(x') \geq u(x^*)$. Monotonicity implies $\exists x''$ very close to x', $x'' = (x_1 + \varepsilon, ..., x_L + \varepsilon)$, $\varepsilon > 0$ and $\varepsilon \to 0$, such that $u(x'') > u(x') \geq u(x^*)$ and $p \cdot x'' \leq p \cdot x^* \leq w$. This contradicts the assumption that x^* is optimal in the UMP. $\Longrightarrow x^*$ must solve the EMP when the required utility is $u(x^*)$. The minimized expenditure is $p \cdot x^*$. By Walras' Law in the UMP, $p \cdot x^* = w$.

(ii) Suppose x^* is the solution of EMP given u but is not optimal in the UMP given $w = p.x^*$. Then, $\exists x'$ such that $u(x') > u(x^*) \geq u$ and $p \cdot x' \leq p \cdot x^*$. Therefore, $\exists x'' < x'$, $x'' = (x_1 - \varepsilon), ..., x_L - \varepsilon)$, $\varepsilon > 0$ and $\varepsilon \to 0$, such that $u(x'') > u$ and $p \cdot x'' < p \cdot x' \leq p \cdot x^*$. This contradicts that x^* solves the EMP given u. So, x^* must solve the UMP. The maximized utility is $u(x^*)$.

By the proposition

$$\begin{aligned} e(p, v(p, w)) &= w \\ v(p, e(p, u)) &= u. \end{aligned}$$

The Expenditure Function

Let $x^h(p, u)$ be the/a solution to the EMP. Then $p \cdot x^h(p, u)$ is the minimized expenditure. Let this be called the Expenditure Function and denoted by $e(p, u)$.(Unlike $\widetilde{x}(p, u)$, this is single-valued.)

Proposition 5 *(3.E.2) Suppose that $u(.)$ is a continuous utility representing a monotone preference relation $\succeq$ defined on the consumption set $X = R_+^L$. The expenditure function $e(p, u)$ is (i) Homogeneous of degree one in p.(ii) (Strictly) increasing in u and nondecreasing in p_l $\forall\, l$.(iii) Concave in p, i.e., $\alpha\ e(p, u) + (1-\alpha)e(p', u) \leq e(ap + (1-\alpha)p', u)$, (iv) Continuous in $p \gg 0$ and u.*

Proof:

(i) The constraint set $u(x) \geq u$ is unaffected by the change in p. The solution to $\min\limits_{x\geq 0} \alpha p \cdot x$ and $\min\limits_{x\geq 0} p \cdot x$ s.t. $u(x) \geq u$ are identical. $e(\alpha p, u) = \alpha p \cdot x^h = \alpha e(p, u)$.

(ii) Show $e(p, u)$ is strictly increasing in u. Suppose $e(p, u)$ is **NOT** strictly increasing in u. Consider change from u' to u'', $u'' > u'$. Suppose. $p \cdot x^h(p, u'') = e(p, u'') \leq e(p, u') = p \cdot x^h(p, u')$. Since $u(.)$ is contnuous and $u(x^h(p, u'')) \geq u'' > u'$, $\exists\ \alpha \in (0, 1)$ sufficiently close to 1 such that $u(\alpha x^h(p, u'')) > u'$and $p \cdot \alpha x^h(p, u'') < p \cdot x^h(p, u')$. This contradicts that $x^h(p, u')$ minimizes expenditure subject to the constraint $u \geq u'$.

Show $e(p, u)$ is nondecreasing in p_l. Let $e_l = (0, ..., 0, 1, 0, ..., 0)$ (1 in the l^{th} element). Suppose price changes from p' to $p'' = p' + \alpha e_l$, and that $p' \cdot x^h(p', u) = e(p', u) > e(p'', u) = p'' \cdot x^h(p'', u)$.Since $p' \leq p''$,

$$p' \cdot x^h(p'', u) \leq p'' x^h(p'', u) = e(p'', u) < e(p', u) = p' \cdot x^h(p', u).$$

Since $u(x^h(p'', u)) \geq u$, $x^h(p'', u)$ satisifes the minimum utility constraint and yields a smaller expenditure than $x^h(p', u)$ given p'. This contradicts that $x^h(p', u)$ minimizes expenditure subject to constraint $u(x) \geq u$.

(iii) $p'' = ap + (1 - \alpha)p'$ for $\alpha \in [0, 1]$.

$$\begin{aligned}
e(p'', u) &= p'' \cdot x^h(p'', u) \\
&= \alpha p \cdot x^h(p'', u) + (1 - \alpha)p' \cdot x^h(p'', u) \\
&\geq \alpha p \cdot x^h(p, u) + (1 - \alpha)p' \cdot x^h(p', u) \\
&= \alpha e(p, u) + (1 - \alpha)e(p', u)
\end{aligned}$$

.The inequality follows from the definition of $e(p, u)$.

(iv) (Skipped)Boundedness of $e(p^n, u)$: Let $e(p, u) = p \cdot \widetilde{x}(p, u)$. $e(p^n, u) \leq p^n \cdot \widetilde{x}(p, u)$ $\because u(\widetilde{x}(p, u)) \geq u$.

Suppose $\lim_{n \longrightarrow \infty} p_n = p$ for $\{p_n\}_{n=1}^{\infty}$ but $\lim_{n \longrightarrow \infty} e(p_n, u) > e(p, u)$ (replace n by $m(n)$ for some converging subsequence).

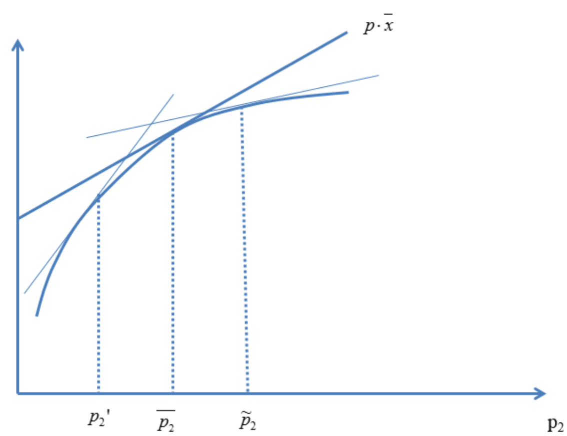
$\exists N > 0$ s.t. $\forall n \geq N, e(p_n, u) = p_n \cdot \widetilde{x}(p_n, u) > e(p, u) = p \cdot \widetilde{x}(p, u)$

$\because p_n \longrightarrow p, \exists \widetilde{N} > 0$ s.t. $\forall n \geq \widetilde{N}$, $p_n \cdot \widetilde{x}(p_n, u) > p_n \cdot \widetilde{x}(p, u)$. Since $u(\widetilde{x}(p, u)) \geq u$, this contradicts the assumption that $\widetilde{x}(p, u)$ minimizes $p_n \cdot x$.

The case of $\lim_{n \longrightarrow \infty} e(p_n, u) < e(p, u)$ is similarly proved.

$\exists N > 0$ s.t. $\forall n \geq N$, $p_n \cdot \widetilde{x}(p_n, u) < p \cdot \widetilde{x}(p, u)$, contradicting the assumption that $\widetilde{x}(p, u)$ minimizes $p \cdot x$.

- Intuition of Concavity of $e(p, u)$.$p_2 \uparrow$, if x stays at $\overline{x}$, e grows with p_2 linearly. However, the consumer can lower x_2 and raise x_{-2} to more cost effectively achieve u.



Example: Cobb-Douglas. Write the problem as:

$$\min \quad p_1 x_1 + p_2 x_2$$

subject to:

$$x_1^{\alpha} x_2^{1-\alpha} = u.$$

Write the Lagrangian as:

$$L = p_1 x_1 + p_2 x_2 - \lambda \left(x_1^{\alpha} x_2^{1-\alpha} - u \right)$$

The FOCs are:

$$\begin{aligned} p_1 &= \lambda \alpha x_1^{\alpha-1} x_2^{1-\alpha} \\ p_2 &= \lambda (1-\alpha) x_1^{\alpha} x_2^{-\alpha} \end{aligned}$$

so that dividing side by side:

$$\frac{\alpha x_2}{(1-\alpha) x_1} = \frac{p_1}{p_2}$$

which, together with the condition $x_1^{\alpha} x_2^{1-\alpha} = u$ yields the solution:

$$\begin{aligned} x_1^h(p,u) &= \frac{u}{\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{\alpha-1}} = \left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{p_2}{p_1}\right)^{1-\alpha} u \\ x_2^h(p,u) &= \frac{u}{\left(\frac{\alpha p_2}{(1-\alpha)p_1}\right)^{\alpha}} = \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{p_2}{p_1}\right)^{-\alpha} u \end{aligned}$$

which gives us the compensated demand function. The expenditure function is given by:

$$\begin{aligned} e(p_1, p_2, u) &= p_1 \left[\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} \left(\frac{p_2}{p_1}\right)^{1-\alpha} u \right] + p_2 \left[\left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \left(\frac{p_2}{p_1}\right)^{-\alpha} u \right] \\ &= \left(\left(\frac{\alpha}{1-\alpha}\right)^{1-\alpha} + \left(\frac{\alpha}{1-\alpha}\right)^{-\alpha} \right) p_2^{1-\alpha} p_1^{\alpha} u. \end{aligned}$$

Property of Hicksian demand

Proposition 6 *(3.E.3): Suppose that* $u(.)$ *is a continuous utility representing a monotone preference relation* $\succeq$ *defined on the consumption set* $X = R_+^L$*. Then for any* $p \gg 0$*, the Hicksian demand correspondence* $x^h(p,u)$ *possesses the following properties:(i) Homogeneity of degree zero in p:* $x^h(p,u) = x^h(\alpha p, u)$ *for all* p, u *and* $\alpha > 0$*. (ii) No excess* *utility: For any $x \in x^h(p,u)$, $u(x) = u$.(iii) Convexity/uniqueness: If $\succeq$ is convex, then $x^h(p,u)$ is a convex set; and if $\succeq$ is strictly convex, then there is a unique element in $x^h(p,u)$.*

Proof:

(i) $\underset{x\geq 0}{argmin}\ \alpha p \cdot x$ s.t. $u(x) \geq u \ \equiv \underset{x\geq 0}{argmin}\ p \cdot x$ s.t. $u(x) \geq u$.

(ii) Suppose $u(x') > u$ for some $x' \in x^h(p,u)$, by continuity of $u(\cdot), \exists \theta \in (0,1)$ such that $u(\theta x') > u$. However, $p \cdot \theta x' < p \cdot x'$. This contradicts that $x' \in x^h(p,u)$.

(iii) Suppose $x, x' \in x^h(p,u)$, then $p \cdot x = p \cdot x' \equiv e(p,u)$. By (ii), $u(x) = u(x') = u$.

Let $x'' = \alpha x + (1-\alpha)x', \alpha \in (0,1)$. $p \cdot x'' = \alpha p \cdot x + (1-\alpha) p \cdot x' = e^*$

Convexity of $\succsim$ implies quasiconcavity of $u(.)$. Hence, $u(x'') \geq \min\{u(x), u(x')\} = u$.

Suppose $x \neq x'$ & $x, x' \in x^h(p,u)$.Strict convexity implies strictly quasic concave $u(.) \Rightarrow u(x'') > u$.Applying the logic in (ii), $\exists \theta \in (0,1)$ s.t. $u(\theta x'') > u$ but $p \cdot \theta x'' < e^*$ $\Longrightarrow$ Contradiction.

Hicksian demand and compensated law of demand

A fairly intuitive property we expect from a demand function is that the amount of quantity consumed of good i should decrease when the price of that good goes up (this is sometimes called 'the law of demand'). This is always true for the compensated demand function $x^h(p,u)$, **but it is not necessarily true for the demand function** $x^*(p,w)$. We have the following result.

Suppose that the utility function $u(x)$ is continuous and monotonic and strictly quasi-concave ($x^h(p,u)$ is function). Then, for each pair p, p':

$$(p' - p)\left(x^h(p',u) - x^h(p,u)\right) \leq 0.$$

Proof. Both $x^h(p', u)$ and $x^h(p, u)$ guarantee a utility of u to the consumer. Furthermore, $x^h(p', u)$ is the least costly bundle achieving u when the price vector is p'. This implies $p'x^h(p', u) \leq p'x^h(p, u)$. A similar reasoning yields $px^h(p, u) \leq px^h(p', u)$. Adding side by side the two inequalities

$$\begin{aligned} p'x^h(p', u) + px^h(p, u) &\leq p'x^h(p, u) + px^h(p', u) \\ (p' - p)x^h(p', u) &\leq (p' - p)x^h(p, u) \\ (p' - p)\left(x^h(p', u) - x^h(p, u)\right) &\leq 0 \end{aligned}$$

.

Hicksian demand & the expenditure function

Recall $e(p, u) = p \cdot x^h(p, u)$. Now we show $x^h(p, u) = \nabla_p e(p, u)$

3.G.1: Suppose that $u(\cdot)$ is continuous, representing monotone and **strictly** convex preference relation $\succsim$ defined on $X = R_+^L$. For all p and u,

$$x^h(p, u) = \nabla_p e(p, u)$$

or

$$x_i^h(p, u) = \frac{\partial e(p, u)}{\partial p_i}, \quad \forall i$$

Proof: By direct computation we have:

$$\frac{\partial e(p, u)}{\partial p_i} = x_i^h(p, u) + \sum_{j=1}^{L} p_j \frac{\partial x_j^h}{\partial p_i}.$$

Our task is to show that the second term on the right hand side is equal to zero. Using (6) we observe that:

$$\sum_{j=1}^{L} p_j \frac{\partial x_j^h}{\partial p_i} = \lambda \sum_{j=1}^{L} \frac{\partial u}{\partial x_j} \frac{\partial x_j^h}{\partial p_i}$$

Now observe that the relation

$$u\left(x^h(p,u)\right) = u$$

must hold for every p. Differentiating both sides of the equality with respect to p_i we have:

$$\sum_{j=1}^{L} \frac{\partial u}{\partial x_j} \frac{\partial x_j^h}{\partial p_i} = 0.$$

The Hicksian and Walrasian Demand Functions

We now discuss the relation between the compensated demand $x^h(p,u)$ and the demand $x(p,w)$. For simplicity, we assume that both x^h and x are single-valued and differentiable. The following relation must hold:

$$x^h(p,u) = x(p, e(p,u)). \tag{7}$$

$$x^h(p, v(p,w)) = x(p,w)$$

This says the following. Suppose that the consumer is given exactly the amount of money necessary to achieve a level of utility u when prices are p; this, by definition, is $e(p,u)$. Then her demand is $x(p, e(p,u))$. Also, $x(p, e(p,u))$ allows the consumer to achieve a utility level of u. But then it must be equal to $x^h(p,u)$, since this is the least-costly vector which allows the consumer to achieve a utility of u.

This also explains why $x^h(p,u)$ is called 'compensated demand'. The function $x^h(p,u)$ describes how the consumption vector changes when prices change *and* the utility achieved by the consumer is kept constant. As we will see more in detail later, a change in the price p_i of good i has two effects: A *substitution effect* (the cost of i changes with respect to other goods) and a *wealth effect* (the consumer becomes richer when p_i decreases and poorer when p_i increases). **The function $x^h(p,u)$ can be seen**

as the demand of the consumer when the wealth effect is eliminated; that is, when p changes the wealth of the consumer is correspondingly changed to make sure that the level of utility u is still achieved. **In other words, wealth is varied according to the function** $e(p,u)$. Another way of putting it is to say that the consumer is *compensated* for changes in purchasing power due to changes in price.

Insert Figure: decomposing change in Walras' demand into a subsitution effect and an income effect.

3.G.3 (The Slutsky Equation) Suppose that $u(\cdot)$ is continuous & representing monotone & strictly convex $\succsim$ on $X = R^L_+$. Then for all (p,w), and $u = v(p,w)$, we have for all l, k,

$$\frac{\partial x^h_l(p,u)}{\partial p_k} = \frac{\partial x_l(p,w)}{\partial p_k} + \frac{\partial x_l(p,w)}{\partial w} \cdot x_k(p,w)$$

In matrix notation

$$D_P x^h(p,u) = D_P x(p,u) + D_w x(p,w) x(p,w)^T,$$

which is

$$\begin{bmatrix} \frac{\partial x_1^h(p,u)}{\partial p_1} & \cdot & \cdot & \frac{\partial x_1^h(p,u)}{\partial p_L} \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ \frac{\partial x_L^h(p,u)}{\partial p_1} & \cdot & \cdot & \frac{\partial x_L^h(p,u)}{\partial p_L} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdot & \cdot & \frac{\partial x_1(p,w)}{\partial p_L} \\ \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & \cdot \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdot & \cdot & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix} + \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial x_L(p,w)}{\partial w} \end{bmatrix} \begin{bmatrix} x_1(p,w) & \cdot & \cdot & x_L(p,w) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} + \frac{\partial x_1(p,w)}{\partial w} x_1(p,w) & \cdot & \cdot & \frac{\partial x_1(p,w)}{\partial p_L} + \frac{\partial x_1(p,w)}{\partial w} x_L(p,w) \\ \cdot & \cdot & & \cdot \\ \cdot & & & \cdot \\ \cdot & & \cdot & \cdot \\ \frac{\partial x_L(p,w)}{\partial p_1} + \frac{\partial x_L(p,w)}{\partial w} x_1(p,w) & \cdot & \cdot & \frac{\partial x_L(p,w)}{\partial p_L} + \frac{\partial x_L(p,w)}{\partial w} x_L(p,w) \end{bmatrix}}_{\text{Slutsky substitution matrix}}$$

(Calculating $\frac{\partial x_l^h(p,u)}{\partial p_k}$ based on observables.)

Proof: Consider $\overline{p}, \overline{w}$ and $\overline{u}$ such that $\overline{w} = e(p, \overline{u})$ or equivalently $\overline{u} = \arg\max_{x \in B(\overline{p}, \overline{w})} u(x)$

$$\begin{aligned} \frac{\partial x_l^h(\overline{p}, \overline{u})}{\partial p_k} &= \frac{\partial x_l(\overline{p}, e(p, \overline{u}))}{\partial p_k} \\ &= \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial p_k} + \frac{\partial x_l(\overline{p}, e(\overline{p}, \overline{u}))}{\partial w} \frac{\partial e(\overline{p}, \overline{u})}{\partial p_k} \\ &= \frac{\partial x_l(\overline{p}, \overline{w})}{\partial p_k} + \frac{\partial x_l \overline{p}, \overline{w})}{\partial w} x_k^h(\overline{p}, \overline{u}) \\ &= \frac{\partial x_l(\overline{p}, \overline{w})}{\partial p_k} + \frac{\partial x_l(\overline{p}, \overline{w})}{\partial w} \cdot x_k(\overline{p}, \overline{w}) \end{aligned}$$

The last equality follows from

$$x_l^h(\overline{p}, \overline{u}) = x_l(\overline{p}, e(\overline{p}, \overline{u})) = x_l(\overline{p}, \overline{w})$$

Rearranging we obtain

$$\begin{array}{ccccc} \frac{\partial x_l(\overline{p},\overline{w})}{\partial p_k} & = & \frac{\partial x_l^h(\overline{p},\overline{u})}{\partial p_k} & - & \frac{\partial x_l(\overline{p},\overline{w})}{\partial w} \cdot x_k(\overline{p},\overline{w}) \\ & & & & \\ \text{total} & & \text{substitution} & & \text{income} \\ & = & & - & \\ \text{change} & & \text{effect} & & \text{effect} \end{array}$$

The equation says that the change in the demand of good l when p_k changes can be decomposed in two parts. The first part is called 'substitution effect', because it captures the effect of the change in relative prices ignoring the impact of the price change on the wealth of the consumer. This is captured by the second part, the 'income effect'.

Consider now the response of the demand for good i to a change in its own price p_i, that is $\frac{\partial x_i}{\partial p_i}$. The propsition tells us that $\frac{\partial x_i^h}{\partial p_i} \leq 0$. However, the term $\frac{\partial x_i}{\partial w}$ can take any sign. A good is called *normal* if $\frac{\partial x_i}{\partial w} \geq 0$, that is the consumption of the good increases with wealth. It is called *inferior* if $\frac{\partial x_i}{\partial w} < 0$, that is the consumption of the good decreases with wealth; this may happen because as the consumers become richer they move to goods which are deemed superior.

If the good is inferior and the income effect is strong enough then the sign of $\frac{\partial x_i}{\partial p_i}$ could be positive; an increase (decrease) in price increases (decreases) the demand for the good (these are called *Giffen goods*). This anomaly is due to the income effect. In fact, for normal goods it is always the case that $\frac{\partial x_i}{\partial p_i} \leq 0$.

Insert a figure of Giffen good

Example: Quasi-linear function. We have seen that when the consumer has quasi-linear preferences then the demands for goods $2, ...L$, do not depend on income, but only on prices, so that we can write $x_i^*(p)$ for $i = 2, ...L$. The demand for good 1 is given by the budget constraint:

$$x_1^*(p, w) = w - \sum_{i=2}^{L} p_i x_i^*(p).$$

Since the demand for goods $2, ...L$ does not depend on income, the income effect is zero. Therefore, all goods $2, ...L$ have a downward sloping demand. Good x_1 is normal, since $\frac{\partial x_1^*}{\partial w} = 1 > 0$, so that it also has a downward sloping demand.

Summary of the relationship between Walrasian demand, Hicksian Demand, indirect utility function and expenditure function.

Figure 3.G.3 in MWG