Homework 2 Suggested Solutions

1. Edgeworth Box. Let $e^1 = (1, 2)$ and $e^2 = (2, 3)$. Find (compute and picture) the Pareto set, core and competitive equilibrium for each of the following cases:
(a) $u^1(x, y) = x + y$, $u^2(x, y) = 2x + 3y$
(b) $u^1(x, y) = x^{1/2}y^{1/2}$, $u^2(x, y) = x^{2/3}y^{1/3}$
(c) $u^1(x, y) = \min\{x, y\}$, $u^2(x, y) = \min\{2x, 3y\}$
(d) $u^1(x, y) = x + y$, $u^2(x, y) = \min\{2x, 3y\}$
(e) $u^1(x, y) = x^{1/2}y^{1/2}$, $u^2(x, y) = 2x + 3y$

Here I only record the results, for the graphs, see your recitation notes.

(a) $\mathcal{PS} = \{(x_1, y_1), (x_2, y_2) | y_1 = 0, y_2 = 5, x_1 + x_2 = 3\} \cup \{(x_1, y_1), (x_2, y_2) | x_1 = 3, x_2 = 0, y_1 + y_2 = 5\}$, $\mathcal{C} = \{(x_1, y_1), (x_2, y_2) | x_1 = 3, x_2 = 0, y_2 \geq 13/3, y_1 + y_2 = 5\}$, $CE = \langle p = (1, 1), x_1 = 3, y_1 = 0, x_2 = 0, y_2 = 5\rangle$.

(b) $\mathcal{PS} = \{(x_1, y_1), (x_2, y_2) | y_1 = \frac{10x_1}{3+x_1}, x_1 + x_2 = 3, y_1 + y_2 = 5\}$, $\mathcal{C} = \{(x_1, y_1), (x_2, y_2) | x_1 \in [0.88, 0.9]\} \cap \mathcal{PS}$, $CE = \langle p = (18/7, 1), x_1 = 8/9, x_2 = 19/9, y_1 = 16/7, y_2 = 19/7\rangle$.

(c) $\mathcal{PS} = \{(x_1, y_1), (x_2, y_2) | y_1 \geq x_1, y_2 \geq 2/3x_2, x_1 + x_2 = 3, y_1 + y_2 = 5\}$, $\mathcal{C} = \{(x_1, y_1), (x_2, y_2) | x_1 = 1, x_2 = 2, y_1 \geq 1, y_2 \geq 4/3, y_1 + y_2 = 5\}$. The CE is not unique, $CE = \langle p \in \{p' \in \mathbb{R}^2 | p_1 > 0, p_2 = 0\}, ((x_1, y_1), (x_2, y_2)) \in \mathcal{C}\rangle$.

(d) $\mathcal{PS} = \{(x_1, y_1), (x_2, y_2) | y_2 = 2/3x_2, x_1 + x_2 = 3, y_1 + y_2 = 5\}$, $\mathcal{C} = \{(x_1, y_1), (x_2, y_2) | y_2 = 2/3x_2, x_2 \geq 2, x_1 + x_2 = 3, y_1 + y_2 = 5\}$, $CE = \langle p = (1, 1), x_1 = 0, x_2 = 3, y_1 = 3, y_2 = 2\rangle$.

(e) $\mathcal{PS} = \{(x_1, y_1), (x_2, y_2) | y_1 = 2/3x_1, x_1 + x_2 = 3, y_1 + y_2 = 5\} \cup \{(x_1, y_1), (x_2, y_2) | x_1 = 3, x_2 = 0, y_1 \geq 2, y_1 + y_2 = 5\}$, $\mathcal{C} = \{(x_1, y_1), (x_2, y_2) | y_1 = 2/3x_1, \sqrt{3} \leq x_1 \leq 2, x_1 + x_2 = 3, y_1 + y_2 = 5\}$, $CE = \langle p = (1, 2/3), x_1 = 2, x_2 = 1, y_1 = 4/3, y_2 = 11/3\rangle$.

2. Let $u^h(x_1, ..., x_L) = x_1^{\alpha_1} x_2^{\alpha_2} ... x_L^{\alpha_L}$. Assume $\alpha_i \geq 0$ for all i and $\sum_{i=1}^{L} \alpha_i = 1$. Show that if $u^h(z)$ maximizes utility on $B^h(p) = \{y \in R_+^L | p \cdot y \leq p \cdot e^h\}$, then $p_l z_l = \alpha_l p \cdot e^h$ for all $l = 1, ..., L$.

Notice that u^h is concave and is strictly monotone, hence we have an interior optimum. Suppose that z maximizes utility on $B^h(p)$. Let $u^* = \prod_{i=1}^n z_i^{\alpha_i}$. Then the first order conditions imply:

$$\alpha_l u^* = \lambda^* p_l z_l, \quad \forall l, \tag{1}$$

where λ is the optimal value of the Lagrange multiplier of the budget constraint. Summing over l yields:

$$u^* = \lambda p \cdot e^h,$$

since $\sum_{i=1}^L \alpha_i = 1$. Given this, (1) becomes

$$\alpha_l \lambda p \cdot e^h = \lambda p_l z_l$$

which simplifies to $p_l z_l = \alpha_l p \cdot e^h$ for all $l \in L$.

3. Consider an exchange economy in which there are four agents and three goods. Agents' utility functions are $u^1(x,y,z) = x^{1/2}y^{1/4}z^{1/4}$, $u^2(x,y,z) = x^{1/3}y^{1/3}z^{1/3}$, $u^3(x,y,z) = x^{2/3}y^{1/4}z^{1/12}$ and $u^4(x,y,z) = x^{1/4}y^{1/4}z^{1/2}$ respectively. Their endowments are $e^1 = (1,2,0)$, $e^2 = (0,2,3)$, $e^3 = (1,1,1)$ and $e^4 = (1,0,0)$ respectively. Compute the competitve equilibrium for this economy.

First, verify that in the CE the prices of all commodities can not be zero. If one of them is 0, say, $p_x = 0$, then all agents will have infinite demand for x because of the weak monotonicity property of Cobb-Douglas utility functions. Total demand for x then is also infinite. But supply of x is finite then no CE exists. Then $p_x, p_y, p_z \neq 0$.

Then, we can normalize that $p_x = 1, p_y = p_1, p_z = p_2$. Given price, consider each agent's utility maximization problem, for example, for agent 1:

$$\max \quad x_1^{1/2} x_2^{1/4} x_3^{1/4}$$

$$s.t. \quad x_1 + p_1 y_1 + p_2 z_1 \leq 1 + 2p_1$$

The Focs give

$$\frac{y_1}{x_1} = \frac{1}{2p_1} \quad \frac{z_1}{x_1} = \frac{1}{2p_2}$$

Plug into the BC we can get

$$x_1 = p_1 + 1/2 \quad y_1 = 1/2 + 1/4p_1 \quad z_1 = p_1/2p_2 + 1/4p_2$$

Similarly, solving other agents' utility maximization problem gives

$$x_2 = 2p_1/3 + p_2 \quad y_2 = 2/3 + p_2/p_1 \quad z_2 = 2p_1/3p_2 + 1$$

$$x_3 = 2/3 + 2/3p_1 + 2/3p_2 \quad y_3 = 1/4p_1 + 1/4 + p_2/4p_1 \quad z_3 = 1/12p_1 + p_1/12p_2 + 1/12$$

$$x_4 = 1/4 \quad y_4 = 1/4p_1 \quad z_4 = 1/2p_2$$

Lastly, the market clearing conditions give

$$x_1 + x_2 + x_3 + x_4 = 3$$

$$y_1 + y_2 + y_3 + y_4 = 5$$

$$z_1 + z_2 + z_3 + z_4 = 4$$

We just need to plug into two equations here to calcaute p_1, p_2 and allocations. The CE is:

$$p_x = 1 \quad p_y = 93/256 \quad p_z = 113/256$$

$$x_1 = 221/256 \quad y_1 = 221/186 \quad z_1 = 221/226$$

$$x_2 = 175/256 \quad y_2 = 175/93 \quad z_2 = 175/113$$

$$x_3 = 308/256 \quad y_3 = 231/186 \quad z_3 = 77/226$$

$$x_4 = 1/4 \quad y_4 = 64/93 \quad z_4 = 128/113$$

4. Consider an exchange economy in which there is a commodity l such that

(a) Household 1 owns (i.e. is endowed with) only commodity l and likes only commodity l.

(b) Households $2, .., H$ each own commodity l but none of them like commodity l.

Show that a CE doesn't exist.

We will analyze two cases. Consider first $p_l = 0$. Then the demand of good l by household 1 is unbounded since it can increase its utility by consuming more of the *free* good l. Thus, there is unbounded excess demand of good l which contradicts market clearing.

Consider now the case of $p_l > 0$. Notice that $p \cdot x^1 = p_l e_l^1$, which implies that household 1 demands $x_l^1 = e_l^1$ and $x_j^1 = 0, j \neq l$. On the other hand, households $h = 2, \ldots, L$ demand $x_l^h = 0$, so we have that

$$\sum_{h \in H} x_l^h = x_l^1 = e_l^1 < \sum_{h \in H} e_l^h,$$

which contradicts market clearing.

5. Consider an exchange economy in which there are two goods and two agents. Agent 1 has utility function $u^1(x, y) = -\sqrt{(x - 1/4)^2 + (y - 1/4)^2}$ and agent 2 has utility function $u^2(x, y) = \log(x) + \log(y)$. Endowments are $e^i = (1/2, 1/2)$ for $i = 1, 2$. Show that this economy has a competitive equilibrium which is not Pareto Optimal. Does this example contradict the First Welfare Theorem?

For the sake of completeness, I record the demand correspondences for the households for arbitrary endowments and prices. For household 1 we have:

$$\begin{pmatrix} x^1(p,e^1) \\ y^1(p,e^1) \end{pmatrix} = \begin{cases} \left(\frac{4p_1(p\cdot e^1)+p_2^2-p_1p_2}{4(p_1^2+p_2^2)}, \frac{4p_2(p\cdot e^1)+p_1^2-p_1p_2}{4(p_1^2+p_2^2)}\right)^T & \text{if } p_1(e_1^1-1/4)+p_2(e_2^1-1/4)<0, \\ (1/4,1/4)^T & \text{if } p_1(e_1^1-1/4)+p_2(e_2^1-1/4)\geq 0. \end{cases} \quad (2)$$

For household 2 we have that

$$\begin{pmatrix} x^2(p,e^2) \\ y^2(p,e^2) \end{pmatrix} = \begin{pmatrix} \frac{p\cdot e^2}{2p_1} \\ \frac{p\cdot e^2}{2p_2} \end{pmatrix}. \quad (3)$$

For this economy, any price vector p such that $1/2 \leq \frac{p_1}{p_2} \leq 2$ and allocations $(x^1,y^1)=(1/4,1/4)$ and $(x^2,y^2)=(1/4+p_1/4p_2, p_2/4p_1+1/4)$ can constitute a *competitive equilibrium with free disposal*. The free disposal property, as discussed in class, is the ability of any household to dispose at free cost any extra commodity that they do not like. Note that at the proposed allocations, households are indeed maximizing utility in their respective budget sets, which satisfies property (i) of competitive equilibrium. On the other hand note that we have

$$(x^1,y^1)+(x^2,y^2)=(1/2+p_1/4p_2, p_2/4p_1+1/2)\leq(1,1)=e^1+e^2 \quad for \quad any \quad 1/2\leq\frac{p_1}{p_2}\leq 2,$$

Thus the allocation is feasible and some units of commodity 1 and some units of commodity 2 are discarded. However, notice that this equilibrium is not Pareto Optimal since household 2 could be made better off by consuming $(3/4,3/4)$ without affecting the utility of household 1.

This example does not contradict the First Welfare Theorem because the preferences of household 1 violate monotonicity. It is obvious that household 1 has a bliss point which is the reason that the inequality in the feasibility condition is strict.

6. Consider traders $h\in H=\{1,2,..,H\}$ with endowments $e^h\in R^K_{++}$ and monotonic, strictly concave utility function u^h. For any price vector $p\in R^K_{++}$, there will be a unique consumption bundle $y^h(p)$ in the budget set $B^h(p)=\{x\in R^K_+ : p\cdot x\leq p\cdot e^h\}$ which maximizes u^h. (Note: the uniqueness follows from strict concavity). Define the aggregate excess demand function $z: R^K_{++}\longrightarrow R^K$ by $z(p)=\sum_{h\in H}[y^h(p)-e^h]$. The function $z(.)$ has the gross substitute (GS) property if whenever p' and p are such that, for some l, $p'_l>p_l$ and $p'_k=p_k$ for $k\neq l$, we have $z_k(p')>z_k(p)$ for $k\neq l$.

(a) Using the fact that the aggregate excess demand functions are homogenneous of degree zero, prove that $z_l(p')<z_l(p)$.

(b) Prove that if the aggregate excess demand function $z(.)$ satisfies the gross subsitute property, then there is at most one normalized price vector p such that $z(p)=0$. (Note: normalized means $\sum_{l=1}^K p_l=1$.)

(a) Let p' and p be related as in the question. Define $\bar{p} = \alpha p$ where $\alpha = \frac{p'_\ell}{p_\ell}$. Notice that $\bar{p}_\ell = p'_\ell$ and $\bar{p}_k > p'_k$ for $k \neq \ell$. Then homogeneity of degree zero of z implies that

$$0 = z_\ell(\bar{p}) - z_\ell(p) = z_\ell(\bar{p}) - z_\ell(p') + z_\ell(p') - z_\ell(p).$$

Gross substitution (applied sequentially by changing one at a time each $p'_k, k \neq \ell$ to $\bar{p}_k$) implies that $z_\ell(\bar{p}) - z_\ell(p') > 0$. Thus, we must have that $z_\ell(p') - z_\ell(p) < 0$ which is what we wanted to show.

(b) Since we are considering normalized prices, it is sufficient for us to show that $z(p) = z(p')$ cannot occur whenever p and p' are not colinear. By homogeneity of degree zero, we can assume, without loss of generality, that $p' \geq p$ and $p_\ell = p'_\ell$ for some ℓ, and the inequality is strict for at least one component. Now, consider a process in which we alter p' to get p in $L-1$ steps: we lower (or keep unaltered) the price of every commodity $k \neq \ell$ one at a time. By gross substitution (using a), the excess demand for good ℓ cannot decrease in any step, and since $p' \neq p$ it will actually increase (again, by a) in at least one step, i.e. $z_\ell(p) > z_\ell(p')$.