

Homework 6

Suggested Solutions

1. Suppose the fraction of low-ability ($t = 1$) workers is $1/4$ and fraction of high-ability ($t = 2$) workers is $3/4$. The productivity of a type 1 worker is $2e$ and the productivity of a type 2 worker is $\frac{9}{4}e$, where e is the education level. The utility of wage w and education e to a worker of type 1 is $u_1(w, e) = 4\sqrt{w} - 2e$. The utility of wage w and education e to a worker of type 2 is $u_2(w, e) = 4\sqrt{w} - 1.8e$. Find the R-S equilibrium.

Solution:

Low ability workers choose the optimal contract along their productivity line ($w = 2e_L$)

$$\max_{e_L} 4\sqrt{2e_L} - 2e_L$$

$$\text{FOC: } 4(2e_L)^{-\frac{1}{2}} - 2 = 0 \Rightarrow e_L^* = 2 \Rightarrow w_L^* = 4.$$

Let us now find the minimum education level for the high ability required for a separating equilibrium to exist, i.e., the intersection between the indifference curve of the low ability workers that maximizes their utility with the productivity line of the high ability workers.

$$u_L(w_L^*, e_L^*) = 4 \quad \& \quad w = \frac{9}{4}e \Rightarrow 4\sqrt{\frac{9}{4}e} - 2e = 4 \Rightarrow \underline{e} = \begin{cases} 1 < e_L^* \\ 4 \end{cases} \begin{matrix} \times \\ \checkmark \end{matrix}$$

The high ability workers choose the optimal contract along their productivity line

$$\max_{e_H} 4\sqrt{\frac{9}{4}e_H} - 1.8e_H$$

$$\text{FOC: } 3(e_H)^{-\frac{1}{2}} - 1.8 = 0 \Rightarrow e_H^* = \frac{25}{9} < \underline{e} \Rightarrow \underline{w} = 9.$$

Thus, we need to check if the indifference curve of the high ability workers that goes through \underline{e} crosses the average productivity line $w = \frac{35}{16}e$.

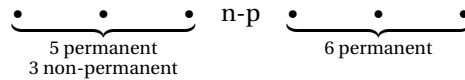
$$u_2(\underline{w}, \underline{e}) = 4\sqrt{\underline{w}} - 1.8\underline{e} = \frac{24}{5} \Rightarrow e = \begin{cases} 2.1289 \\ 3.34017 \end{cases}$$

which implies that no Rothschild-Stiglitz equilibrium exists.

2. The Security Council has 5 permanent members and 10 non-permanent members. For a coalition to win, it must contain all 5 permanent members and at least 4 non-permanent members. View this situation as a simple (voting) game and compute the Shapley Value of the 15 members.

Solution:

Consider first a non-permanent member



so that the Shapley value of any one non-permanent member is

$$\varphi(n-p) = \frac{\binom{9}{3} 8! 6!}{15!} = \frac{4}{2145}.$$

Then, by efficiency and symmetry,

$$\sum \varphi(p) = 1 - \sum \varphi(n-p) = \frac{421}{429},$$

so that the Shapley value of any one permanent member is

$$\varphi(p) = \frac{\sum \varphi(p)}{5} = \frac{421}{2145}$$

3. Consider a situation involving a landlord and 10 workers that till the landlord's land. If k workers till the land, the outputs is worth k^2 dollars.

(a) Compute the Shapley value of the 11 players involved in this game.

Consider now 2 landlords and 6 workers.

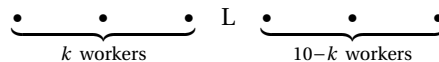
(b) Compute the Shapley value of the 8 players in the case where both landlords need to be present in a coalition for the land to be tilled.

(c) Compute the Shapley value of the 8 players in the case where the presence of any one landlord suffices for the land to be tilled.

Solution:

(a)

$$\nu(S) = \begin{cases} k^2 & \text{if the landlord is in coalition } S \text{ and } |S| = k+1. \\ 0 & \text{if the landlord is not in coalition } S. \end{cases}$$



So that the Shapley value of the landlord is

$$\varphi(L) = \sum_{k=0}^{10} \frac{\binom{10}{k} k! (10-k)!}{11!} \cdot k^2 = \frac{1}{11} \cdot [0^2 + 1^2 + \dots + 10^2] = 35.$$

Then, by efficiency, the sum of the Shapley values of the workers is equal to

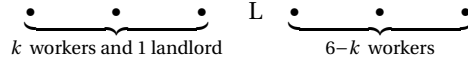
$$\sum \varphi(w) = 10^2 - \varphi(L) = 65,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{65}{10} = 6.5.$$

(b)

$$v(S) = \begin{cases} k^2 & \text{if the landlord is in coalition } S \text{ and } |S| = k + 2. \\ 0 & \text{if the landlord is not in coalition } S. \end{cases}$$



So that the Shapley value of any one landlord is

$$\varphi(L) = \sum_{k=0}^6 \frac{\binom{6}{k} (k+1)! (6-k)!}{8!} \cdot k^2 = \frac{\sum_{k=0}^6 (k+1) k^2}{8 \cdot 7} = \frac{19}{2}.$$

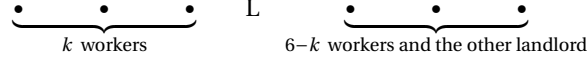
Then, by efficiency, the sum of the Shapley values of the workers is equal to

$$\sum \varphi(w) = 6^2 - \varphi(L) \times 2 = 17,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{17}{6}.$$

(c)



So that the Shapley value of any one landlord is

$$\varphi(L) = \sum_{k=0}^6 \frac{\binom{6}{k} k! (6-k+1)!}{8!} \cdot k^2 = \frac{\sum_{k=0}^6 \frac{(7-k)!}{(6-k)!} \cdot k^2}{8 \cdot 7} = \frac{7}{2}.$$

Then, by efficiency, the sum of the Shapley values of the workers is equal to

$$\sum \varphi(w) = 6^2 - \varphi(L) \times 2 = 29,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{29}{6}.$$