



---

College Admissions and the Stability of Marriage

Author(s): D. Gale and L. S. Shapley

Source: *The American Mathematical Monthly*, Vol. 69, No. 1 (Jan., 1962), pp. 9-15

Published by: Mathematical Association of America

Stable URL: <http://www.jstor.org/stable/2312726>

Accessed: 11/02/2009 21:25

---

Your use of the JSTOR archive indicates your acceptance of JSTOR's Terms and Conditions of Use, available at <http://www.jstor.org/page/info/about/policies/terms.jsp>. JSTOR's Terms and Conditions of Use provides, in part, that unless you have obtained prior permission, you may not download an entire issue of a journal or multiple copies of articles, and you may use content in the JSTOR archive only for your personal, non-commercial use.

Please contact the publisher regarding any further use of this work. Publisher contact information may be obtained at <http://www.jstor.org/action/showPublisher?publisherCode=maa>.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

JSTOR is a not-for-profit organization founded in 1995 to build trusted digital archives for scholarship. We work with the scholarly community to preserve their work and the materials they rely upon, and to build a common research platform that promotes the discovery and use of these resources. For more information about JSTOR, please contact support@jstor.org.



*Mathematical Association of America* is collaborating with JSTOR to digitize, preserve and extend access to *The American Mathematical Monthly*.

<http://www.jstor.org>

①

# Consider a PROFILE of PREFERENCES

of MEN

A:	b	a	d	c
B:	c	d	a	b
C:	c	d	b	a
D:	d	a	b	c

of WOMEN

a:	c	B	A	D
b:	B	C	A	D
c:	A	D	C	B
d:	A	C	D	B

Is the MATCHING

$$\begin{pmatrix} Aa \\ Bc \\ Cd \\ Db \end{pmatrix}$$

STABLE ?

NO because  $(A, b)$  can disrupt it

[... So can  $(C, c)$  ...]

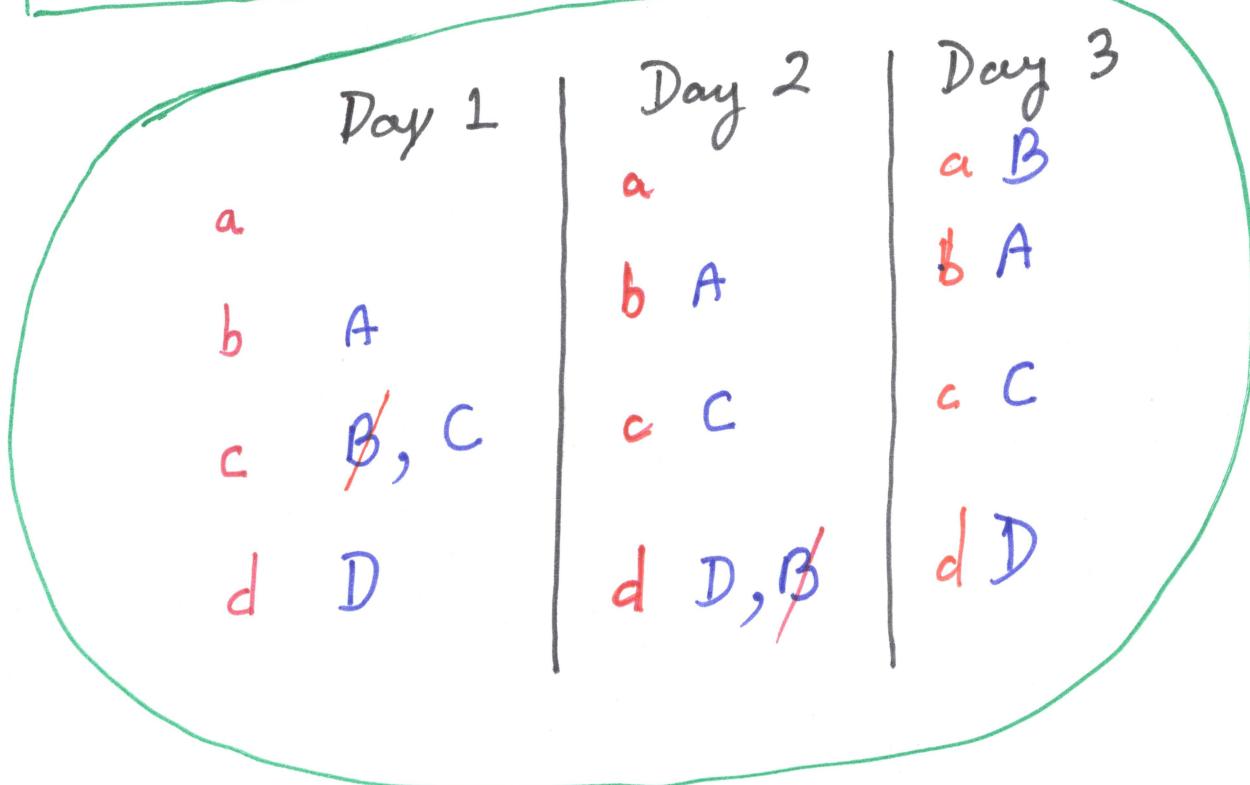
# MAN PROPOSES, WOMAN DISPOSES

(2)

Recall PROFILE

(MPP)

A	b	a	d	c		a	c	B	A	D
B	c	d	a	b		b	B	C	A	D
C	c	d	b	a		c	A	D	C	B
D	d	a	b	c		d	A	C	D	B



M P P  
 (Men Proposing Procedure)  
 for the above profile

(3)

# MPP or WPP ?

Consider the PROFILE

	a	B	C	A
a	A	b	c	
b	B	b	c	a
c	C	c	a	b

MPP  $\Rightarrow \begin{pmatrix} A \\ B \\ C \end{pmatrix}$ , i.e., each man gets his BEST choice  
 " woman " her WORST "

WPP  $\Rightarrow \begin{pmatrix} A \\ C \\ B \end{pmatrix}$ , i.e., each man gets his WORST choice  
 " woman " her BEST "

# PROFILE

(4)

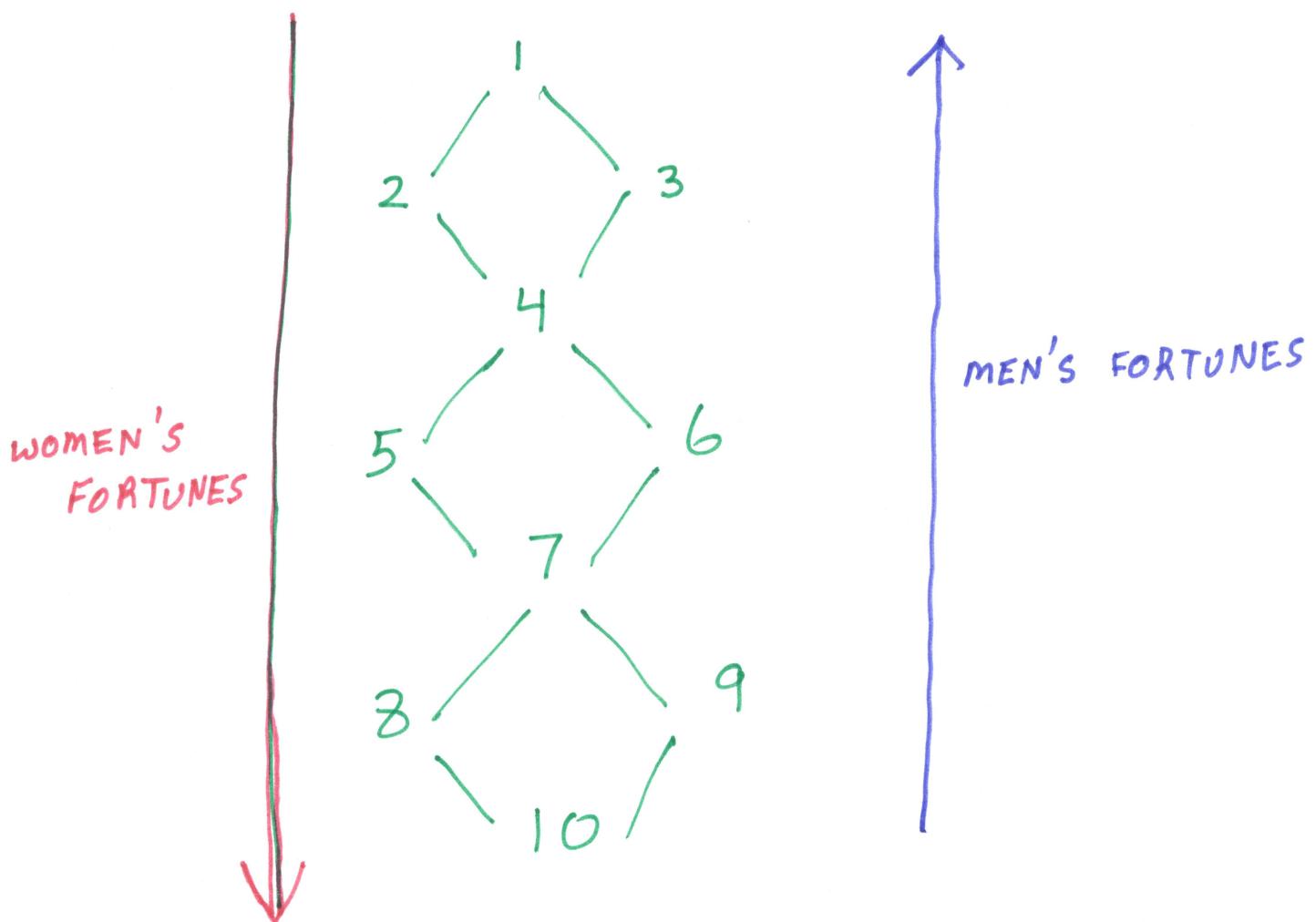
A	a	b	c	d	a	D	C	B	A
B	b	a	d	c	b	C	D	A	B
C	c	d	a	b	c	B	A	D	C
D	d	c	b	a	d	A	B	C	D

MANY STABLE MATCHINGS  
(10 out of 24)

	a	b	c	d					
matched to:	a	b	c	d					
	A	B	C	D	... (1)				
	B	A	C	D	... (2)				
etc	A	B	D	C	... (3)				
	B	A	D	C	... (4)				
	C	A	D	B	... (5)				
	B	D	A	C	... (6)				
	C	D	A	B	... (7)				
	D	C	A	B	... (8)				
	C	D	B	A	... (9)				
	D	C	B	A	... (10)				

(5)

# "LATTICE STRUCTURE"



**FIX** a profile of preferences

List **ALL** stable matchings w.r.t the profile

For any man  $m$ , let

$\text{Poss}(m) = \text{set of wives of } m \text{ in the list}$

e.g.

$$\left( \begin{array}{c} \vdots \\ \text{John-} \\ \text{Mary} \\ \vdots \end{array} \right), \quad \left( \begin{array}{c} \vdots \\ \text{John-} \\ \text{Carol} \\ \vdots \end{array} \right), \quad \left( \begin{array}{c} \vdots \\ \text{John-} \\ \text{Mary} \\ \vdots \end{array} \right), \quad \left( \begin{array}{c} \vdots \\ \text{John-} \\ \text{Eva} \\ \vdots \end{array} \right), \quad \left( \begin{array}{c} \vdots \\ \text{John-} \\ \text{Susan} \\ \vdots \end{array} \right)$$

$$\text{Poss}(\text{John}) = \{\text{Mary}, \text{Carol}, \text{Eva}, \text{Susan}\}$$

(6)

Thm When MPP is followed  
**EVERY** man  $m$  winds up with  
his **BEST** choice in  $\text{Poss}(m)$  ....  
(any every woman  $w$  with her worst choice  
in  $\text{Poss}(w)$  !! )

(7)

## PROOF (Focus throughout on MPP for the fixed profile)

Note: If John ends up with Mary for wife then

John must have proposed to, and been rejected by, ALL the women he prefers to Mary

So must show:

NONE of the women who reject John, in the course of the MPP, are possible for John

Must show this not just for John, but also for Bob, Harry, .... for ALL men

In short, show

In MPP: **NO** man  $m$  is rejected by  
 $\Leftarrow$  a woman in **Poss**( $m$ )

~~Define~~

"**BAD EVENT**"  $\equiv$  a man  $m$  is rejected  
 by a woman in **Poss**( $m$ )

To show:

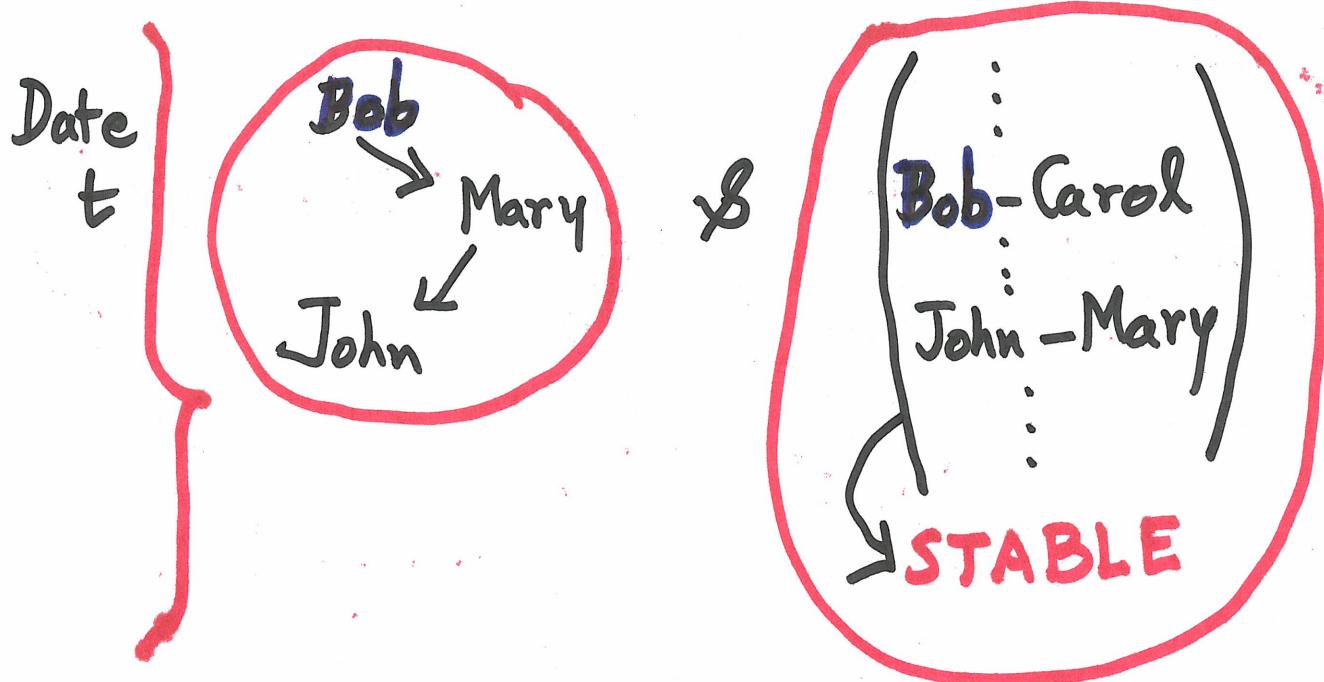
**BAD EVENTS** cannot happen in MPP

Will show  $\uparrow$  by "contradiction"  
 Suppose "bad events" **DO** happen!  
 Let  $t$  be the **EARLIEST** date  
 on which a "bad event occurs

i.e.

9

NO BAD EVENT on dates 1, ..., t-1  
BAD EVENT<sup>\*</sup> on date t



Mary : ..... Bob ..... John .....

Bob : ..... Mary .....



(10)

all rejected Bob on dates  $1, \dots, t-1$ ; so  
not in Poss(Bob)

Bob: ..... Mary

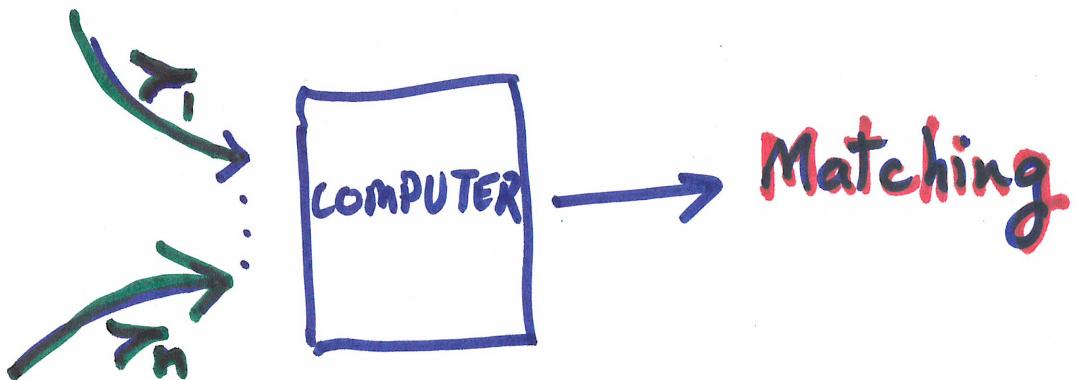
BUT Carol is in Poss(Bob)

So have

Bob: ..... Mary .....  
Carol ↑

Reinterpret

MPP = Computer Program



Incentive Compatibility  
(Dubins-Friedman)

Real world applications

(Alvin Roth)

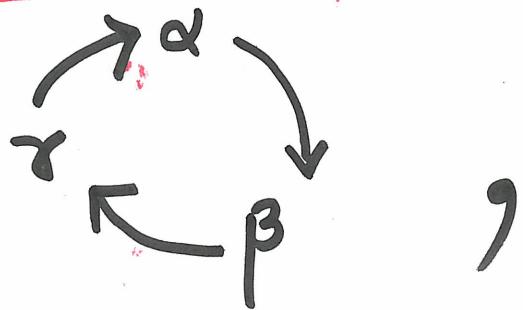
NRMP

NYC

Public schools -  
kids

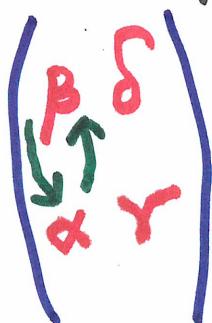
## ROOMMATES PROBLEM

(12)



\alpha, \beta, \gamma each like \delta least

Suppose  $\beta\delta$  are paired  
\* matching is



NO stable matching!

Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with.

Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures," or that they "know a lot of formulas." At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.

What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematical, and the reason that your friends and ours cannot understand mathematics is not because they have no head for figures, but because they are unable to achieve the degree of concentration required to follow a moderately involved sequence of inferences. This observation will hardly be news to those engaged in the teaching of mathematics, but it may not be so readily accepted by people outside of the profession. For them the foregoing may serve as a useful illustration.

---

### GRADUATED INTEREST RATES IN SMALL LOANS

HUGH E. STELSON, Michigan State University

Many small loan companies charge a graduated interest rate in accordance with various state laws. For example, 3% per month is charged on the first \$150 of a loan, and 2% on the portion of the loan in excess of \$150. Rates may be graduated in two, three or more brackets. A three-bracket loan might be at the rate of  $2\frac{1}{2}\%$  on that part of the loan or loan balance which is \$100 or less, at the rate of 2% on that part of a loan which is in excess of \$100 but less than \$200, and at the rate of 1% on that part of a loan which is in excess of \$200. Such a graduated rate is written:  $2\frac{1}{2}\%/2\%/1\%/\$100/\$200$ .

The main problem considered in this paper is that of finding the level monthly rent payment which will amortize a loan in a given time at a graduated rate.

# KIDNEY EXCHANGE

## (Shapley-Scarf)

13  
a

A "owns" kidney a

B . . . . . b etc.

⋮

Profile of preferences



A :	c	d	b a
B :	c	a	b d
C :	d	b	a c
D :	a	b	c d

PROFILE				
A	c	d	b	a
B	c	a	b	d
C	d	b	a	c
D	a	b	c	d

Proposed allocation

$$A = \langle A_d, B_a, C_b, D_c \rangle$$

Coalition  $\{B, C\}$  can block  $A$

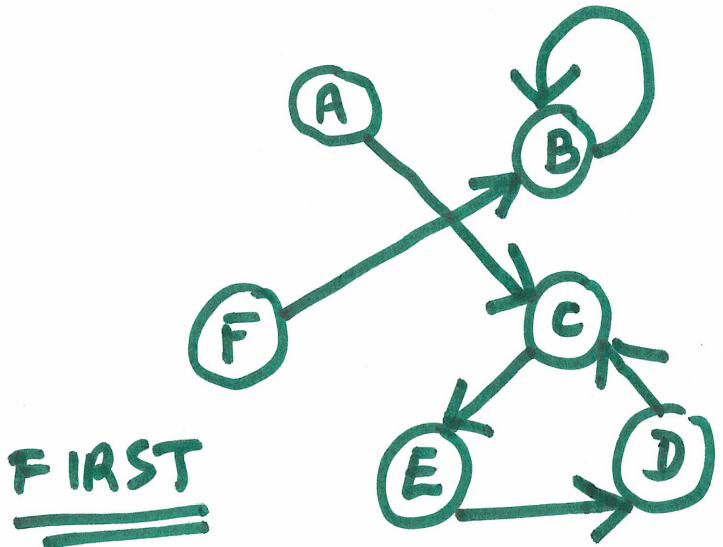
via  $\langle B_c, C_b \rangle$  since  
 $B^{\uparrow}$  and  $C^{\rightarrow}$

DEF<sup>n</sup> An allocation is STABLE  
if no coalition can block it.

THEOREM There exists one, and  
only one, stable allocation

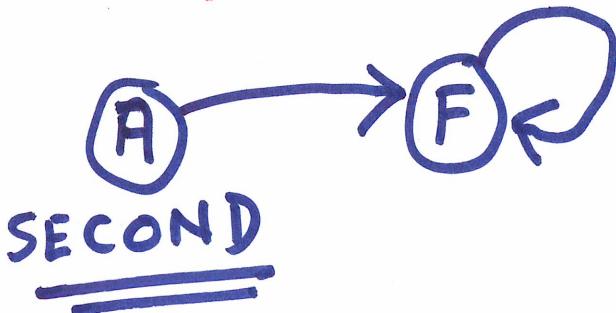
Proof (Gale)

(14)



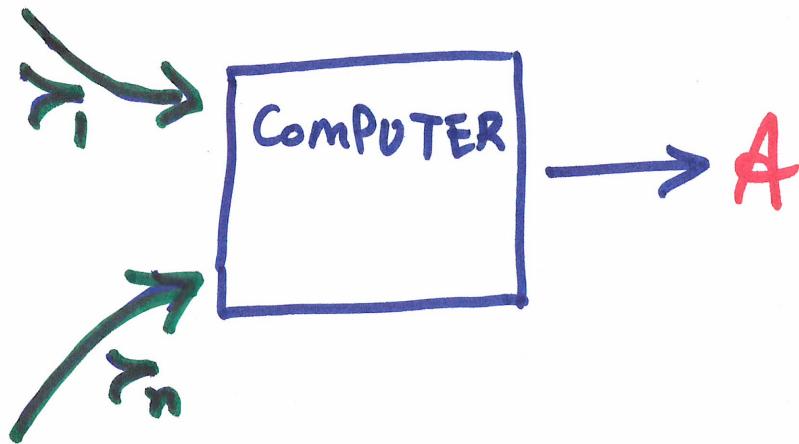
PROFILE	
A	c e f a b d
B	b a c e f d
C	e f c a d b
D	c a b e d f
E	d c b f e a
F	b d e f a c

$$A = \langle A?, Bb, Ce, Dc, Ed, F? \rangle$$



$$A = \langle Aa, Bb, Ce, Dc, Ed, Ff \rangle$$

## REINTERPRET



Also Incentive Compatible!