Homework 4 Suggested Solutions

1. Consider a situation in which there are 3 agents. Assume that the set of alternatives, X, is the interval $[0,1]$, and that each individual's preference is *single-peaked*, that is, for each i there is an alternative a_i^* such that if $a_i^* \geq b > c$ or $c > b \geq a_i^*$, then $b \succ_i c$. Consider the following voting procedure to chose a social alternative. Each agent writes in a sealed bid his vote for one alternative. The votes are counted and the median is calculated, which is then chosen as the social alternative. Show that the voting procedure described above is strategy proof, i.e., for each agent it is a dominant strategy to vote for her preferred alternative.

Solution: Fix i and let a^* be the preferred alternative of individual i. Fix $(b,c) \in [0,1]^2$ as the alternatives voted by the other two individuals, without loss of generality, assume that $b \leq c$. We have to consider three cases:

- $a^* \in (b,c)$: In this case, ax^* is the median and thus the alternative chosen. Since $a^* \succ_i y, \forall y \in [0,1]$ then it dominates any other vote for another alternative.
- $a^* \in [0,b]$: In this case, i cannot manipulate the voting in any way, since b will be the outcome, which by single-peakedness, is better for i than any $a \in [b,c]$.
- $a^* \in [c,1]$: Analogous to the second case.

2. Consider an agency relationship in which the principal contracts the agent, whose effort determines the result. Assume that the uncertainty present is represented by three states of nature. The agent can choose between two effort levels. The results are shown in the following table.

	states of Nature		
	ϵ_1	ϵ_2	ϵ_3
$e=6$	60,000	60,000	30,000
$e=4$	30,000	60,000	30,000

The principal and the agent both believe that the probability of each state is one third. The objective functions of the principal and the agent are, respectively:

$$\mathrm{B}(x,w) = x - w$$
$$\mathrm{U}(w,e) = \sqrt{w} - e^2,$$

where $x = x(e, \epsilon)$ is the monetary result of the relationship and $w = w(x)$ is the monetary pay-off that the agent receives. Assume that the agent will only accept the contract if he obtains an expected utility of at least 114.

(a) What can be deduced from the participants' objective functions?

(b) What would be the effort and the wage in a situation of symmetric information?

(c) What happens in a situation of asymmetric information? What pay-off scheme allows an effort of $e = 4$ to be obtained? What pay-off scheme allows the effort level of $e = 6$ to be obtained? Which effort level that the principal prefer?

Solution:

(a) The principal is risk neutral and the agent is risk averse.

(b) General information symmetric case with risk neutral principal and risk averse agent:

Let p_i be the probability that the result is x_i, $w(x_i)$ the wage paid to the agent when the results obtained is x_i and $v(e)$ the disutility from effort.

$$\max_{[e, \{w(x_i)\}_{i=1,\ldots,n}]} \quad \sum_{i=1}^{n} p_i(e) \mathrm{B}(x_i - w(x_i))$$
$$s.t. \quad \sum_{i=1}^{n} p_i(e) u(w(x_i)) - v(e) \geq \underline{\mathrm{U}}$$

$$\frac{\partial \mathrm{L}}{\partial w(x_i)} = -p_i(e)\mathrm{B}'(x_i - w(x_i)) + \lambda p_i u'(w(x_i)) = 0$$

$$\Rightarrow \lambda = \frac{\mathrm{B}'(x_i - w(x_i))}{u'(w(x_i))}, \quad \forall i \in \{1, 2, \ldots, n\}.$$

Since λ is constant, we have that

$$\frac{\mathrm{B}'(x_2 - w_2)}{\mathrm{B}'(x_1 - w_1)} = \frac{u'(w_2)}{u'(w_1)}.$$

Thus, since $\mathrm{B}'(\cdot)$ is constant (risk neutral), the efficiency condition requires that $u'(w(x_i) =$ constant for all i. Since the agent is risk averse, it must be that $w(x_1) = w(x_2) = \cdots = w(x_n)$.

In our exercise, then:

The optimal contracts are derived from (i) the principal accepts all the risk, and (ii) the participation constraint binds. If $e = 6$, then w is such that $w^{1/2} - 6^2 = 114$, which is $w = 22,500$. In this case, $\mathrm{B} = 50,000 - 22,500 = 27,500$. If $e = 4$, then, $w = 16,900$ and $\mathrm{B} = 23,100$. The information symmetric solution is: $e^* = 6$, $w^* = 22,500$. If the principal was not risk-neutral, then both participants would share the risk inherent in the relationship.

(c) The optimal contract if $e = 4$ is the same as before: $w = 16,900$, since given a constant wage the agent will always choose the lowest effort level! That is, if the principal wants the agent to exert effort level 4, she knows that with a fix wage she will choose the least costly effort. Thus, as we know from section (b), the fix wage when aiming for $e = 4$ is $w = 16,900$.

In order to achieve $e = 6$, the principal must offer a contract that is contingent on the result. She will pay $w(60)$ if the result is 60,000 and $w(30)$ if the result is 30,000. The contract must simultaneously satisfy the participation and the incentive constraints:

$$\frac{2}{3}[w(60)]^{1/2} + \frac{1}{3}[w(30)]^{1/2} - 36 \geq 114 \qquad \text{[PC]}$$

$$\frac{2}{3}[w(60)]^{1/2} + \frac{1}{3}[w(30)]^{1/2} - 36 \geq \frac{2}{3}[w(30)]^{1/2} + \frac{1}{3}[w(60)]^{1/2} - 16. \qquad \text{[ICC]}$$

$$\Rightarrow [w(60)]^{1/2} - w(30)^{1/2}] \geq 60 \qquad \text{[ICC]}$$

Both restriction will bind in the solution to the principle's problem of "spending the least possible amount."

- Participation constraint binding: If it were not, the principal can reduce wages a little bit. This would increase the principal's profits while the agent would still want to participate in the relationship. Thus, the principal will keep on reducing until the PC binds.
- Incentive compatibility constraint: If it were not, the principal would be "giving to much incentives." The principal would reduce the difference between the wages which would benefit the profits. The principal will keep decreasing it until it binds, since that is the minimum difference between the wages that incentivizes the agent to choose effort level 6.

We have two equations in two unknowns that lead to the solution $w(60) = 28,900$ and $w(30) = 12,100$. The principal's expected utility is $U_p = 26,700$. Under asymmetric information the principal also chooses $e = 6$, since $26,700 > 23,100$, but with an efficiency loss measured by the reduction in the expected profits of the principal.

3. There's a fixed, finite supply of cars, and infinitely many buyers in the market. The quality distribution of cars as well as the valuation of different quality cars by buyers and sellers are given in the following table:

	q_1	q_2	q_3
Buyer	1100	1800	2500
Seller	1000	1500	2000
Fraction	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) Suppose sellers observe the quality of the car, but buyers do not. Compute the market equilibrium prices.

(b) Suppose sellers have an option to credibly disclose the quality of the car. A seller chooses: He either sends a signal fully and truthfully disclosing the quality of the car, or does not disclose it at all. The cost of the signal to the sellers is 400. What will the equilibrium be in this case?

Solution:

(a) Firstly, note the following:

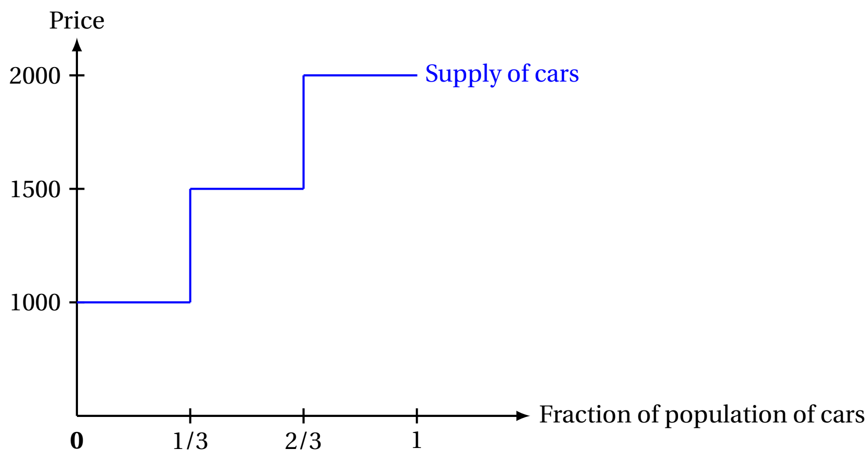

Figure 1 Cars Market

(i) For the three qualities to be sold in the market, the price needs to be $p \geq 2000$. Then the expected value of the buyer is given by

$$\mathrm{E}\left[\mathrm{V_B}|p \geq 2000\right] = \frac{1}{3}1100 + \frac{1}{3}1800 + \frac{1}{3}2500 = 1800 < p.$$

Thus no equilibrium in which the three qualities are found in the market is possible.

(ii) If $p \in [1500, 2000)$ then q_1 and q_2 can be found in the market.

$$\mathrm{E}\left[\mathrm{V_B}|p \in [1500, 2000)\right] = \frac{1/3}{2/3}1100 + \frac{1/3}{2/3}1800 = 1450 < p.$$

Thus no equilibrium in which the two qualities are found in the market is possible.

(iii) If $p \in [1000, 1500)$ then only the lowest quality is sold in the market and the value to the buyer is 1100.

Thus the market equilibrium price is 1100 and only the low quality cars are sold in the market.

(b) Seller can send a credible signal at cost of 400. Let us start from the equilibrium found in section (a).

(i) Suppose sellers consider to send a signal to truthfully disclose that a car is of quality q_2. Once the buyer observes the signal, he will know the car has quality q_2 with probability 1 and his value will be of 1800. However, the value of the car to the seller is 1500 plus the 400 of the costly signal, which is greater than the value to the buyer. The price would have to be 1900 and the demand of q_2 cars is zero.

(ii) Suppose sellers consider sending a signal to truthfully disclose that a car is of quality q_3. Once the buyer observes the signal, he will know the quality with probability 1 and his value will be 2500.

The value of the car to the seller is of 2000 plus 400 of the costly signal. Thus in this new equilibrium with a signal, qualities q_1 and q_3 are sold in the market at prices of 1100 and 2500, respectively, and sellers disclose high quality cars.