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NOTE

A Non-constructive Elementary Proof of the Existence of Stable Marriages

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Gale and Shapley showed in their well known paper of 1962 (*Amer. Math. Monthly* **69**, 9–14) that stable matchings always exist for the marriage market. Their proof was constructed by means of an algorithm. Except for the existence of stable matchings, all the results for the marriage market which were proved by making use of the Gale and Shapley algorithm could also be proved without the algorithm. The purpose of this note is to fill out this case. We present here a nonconstructive proof of the existence of a stable matching for the marriage market, which is quite short and simple and applies directly to both cases of preferences: strict and nonstrict.

1. INTRODUCTION

This is an old mathematical problem: How to get, if it exists, a nonconstructive proof of the existence of stable marriages?

Although the existence proof presented in Gale and Shapley (1962) is satisfactory in all aspects, there seems to be a certain mystery concerning the existence of a nonconstructive proof, whose lack constitutes a lacuna in the mathematical theory of the marriage market.

This note fills the lacuna mentioned above. The main feature of the proof presented is that it is quite short and simple. Furthermore, it does not make any restriction on the kind of preferences (strict or nonstrict). That is, it applies to both cases of preferences, without the need of separating them or using any tie-breaking rule.

The practical advantage of being nonconstructive is that it gives insights to be used in more complex models, just as happens with proofs of other results valid

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for this market which do not make use of the algorithm. See Roth and Sotomayor (1990) for a comprehensive account.

In the next section we describe the model. The existence theorem is proved in Section 3.

2. THE MARRIAGE MODEL

There are two finite and disjoint sets M and W (men and women). Each man m in M has complete and transitive preferences on the set $W \cup \{m\}$, the set of all women and himself, which may be strict or nonstrict. Hence they can be given by an ordered list of preferences. The position in which the man places himself in the list has the meaning that the only women he is willing to be matched with are those whom he weakly prefers to himself. These women are said to be *acceptable* to him. Similarly, each w in W has complete and transitive preferences on the set $M \cup \{w\}$. By convenience we will consider that every man (resp. woman) is acceptable to himself (resp. herself). We will denote by (m, w) a pair in $M \times W$.

A *matching* μ is a function from the set $M \cup W$ onto itself of order two, (that is, $\mu^2(x) = x$), such that if $\mu(x) \neq x$ then x and $\mu(x)$ are mutually acceptable to each other. We call $\mu(x)$ the mate of x.

We will say that a man or a woman prefers x to y if he or she strictly prefers x to y.

The main concept of the theory is the following:

DEFINITION 1. The pair (m, w) *blocks* the matching μ if m and w are not matched by μ but m prefers w to $\mu(m)$ and w prefers m to $\mu(w)$. The matching μ is *stable* if it is not blocked by any pair.

Stable matchings are thus "divorce proof."

DEFINITION 2. The matching μ is *simple* if, in the case a blocking pair (m, w) exists, w is single.

In other words, μ is simple if either everyone is single or the matching obtained by removing all single women is stable for the resulting market. Therefore the *set of simple matchings is nonempty*, since it contains the matching in which everyone is single. Another example of a simple matching is the one which matches a given woman to her most preferred acceptable man among those who accept her (if he exists) and leaves all the other players single. From our definition, if no woman is single, the matching is simple if and only if it is stable.

DEFINITION 3. The matching μ is *weakly Pareto optimal for the men* (among all simple matchings), if it is simple and there is no simple matching μ' such that:

(i) all men like μ' at least as well as μ, and
(ii) at least one man prefers μ' to μ.

That is, if μ' is simple and m prefers μ' to μ then there is some m' who prefers μ to μ'.

The existence of μ is guaranteed by the fact that the *set of simple matchings is nonempty and finite and the preferences are transitive*. However, much more information would be necessary if we wanted to construct such a matching. In general, it would be necessary to obtain all the simple matchings, which would require us to know the whole set of stable matchings! Another way to see the existence of μ is to use numerical values, a_{mj}. Thus m prefers j to k if and only if $a_{mj} > a_{mk'}$ and m is indifferent between j and k if and only if $a_{mj} = a_{mk'}$. It is clear that there exists a matching μ such that $\sum_{m\in M} a_{m\mu(m)} \geq \sum_{m\in M} a_{m\mu'(m)}$, for all simple μ', because every nonempty and finite set of real numbers has a maximal element. The simple matching μ obtained in this way is clearly weakly Pareto optimal for the men.

3. THE EXISTENCE THEOREM

THEOREM. *The set of stable matchings for the marriage market is nonempty*.

Proof. Let μ be some weakly Pareto optimal matching for the men, among all simple matchings. We claim that μ is stable. In fact, suppose not. Then μ is blocked by some (m, w), where w is single, by Definition 2. Choose m such that if (m', w) also blocks μ then w does not prefer m' to m. Now match w to m, leave $\mu(m)$ single, in case $\mu(m) \in W$, and keep the same mates for the other players. This new matching is clearly simple since w does not belong to any blocking pair for the new matching. Furthermore, this matching is weakly preferred by all men and strictly preferred by m to the original matching μ, which contradicts the Pareto optimality of μ. Hence μ is stable and the proof is complete. ■

Remark. Note that if the preferences are strict, there is only one weakly Pareto optimal matching for the men among all simple matchings. Furthermore, this matching is weakly preferred by every man to any other stable matching. It is called the M-optimal stable matching and it is the matching obtained when we use the Gale and Shapley algorithm. To prove this, suppose that μ is some weakly Pareto optimal matching for the men and μ' is the M-optimal stable matching. The M-optimality of μ' requires that $\mu'(m)$ is weakly preferred to $\mu(m)$ by all m. However, no man can strictly prefer μ' to μ, because this contradicts the Pareto optimality of μ. Hence $\mu = \mu'$. ■

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