Consider a PROFILE of PREFERENCES

of MEN					of WOMEN				
A:	b	a	d	c	a:	C	B	A	D
B:	c	d	a	b	b:	B	C	A	D
C:	c	d	b	a	c:	A	D	C	B
D:	d	a	b	c	d:	A	C	D	B

Is the MATCHING $\begin{pmatrix} Aa \\ Bc \\ Cd \\ Db \end{pmatrix}$ STABLE ?

NO because (A, b) can disrupt it

[... So can (C, c) ....]

MAN PROPOSES, WOMAN DISPOSES

(MPP)

Recall **PROFILE**

A	b	a	d	c	a	C	B	A	D
B	c	d	a	b	b	B	C	A	D
C	c	d	b	a	c	A	D	C	B
D	d	a	b	c	d	A	C	D	B

Day 1		Day 2		Day 3	
a		a		a	B
b	A	b	A	b	A
c	~~B~~, C	c	C	c	C
d	D	d	D, ~~B~~	d	D

MPP (Men Proposing Procedure) for the above profile

MPP or WPP ?

Consider the PROFILE

$$
\begin{array}{cccc|cccc}
A & a & b & c & a & B & C & A \\
B & b & c & a & b & C & A & B \\
C & c & a & b & c & A & B & C
\end{array}
$$

MPP $\Rightarrow \begin{pmatrix} A & a \\ B & b \\ C & c \end{pmatrix}$, i.e., each <u>man</u> gets his <u>BEST</u> choice
" <u>woman</u> " her <u>WORST</u> "

WPP $\Rightarrow \begin{pmatrix} A & c \\ B & a \\ C & b \end{pmatrix}$, i.e., each <u>man</u> gets his WORST choice
" <u>woman</u> " her BEST "

PROFILE

A	a	b	c	d	a	D	C	B	A
B	b	a	d	c	b	C	D	A	B
C	c	d	a	b	c	B	A	D	C
D	d	c	b	a	d	A	B	C	D

MANY STABLE MATCHINGS

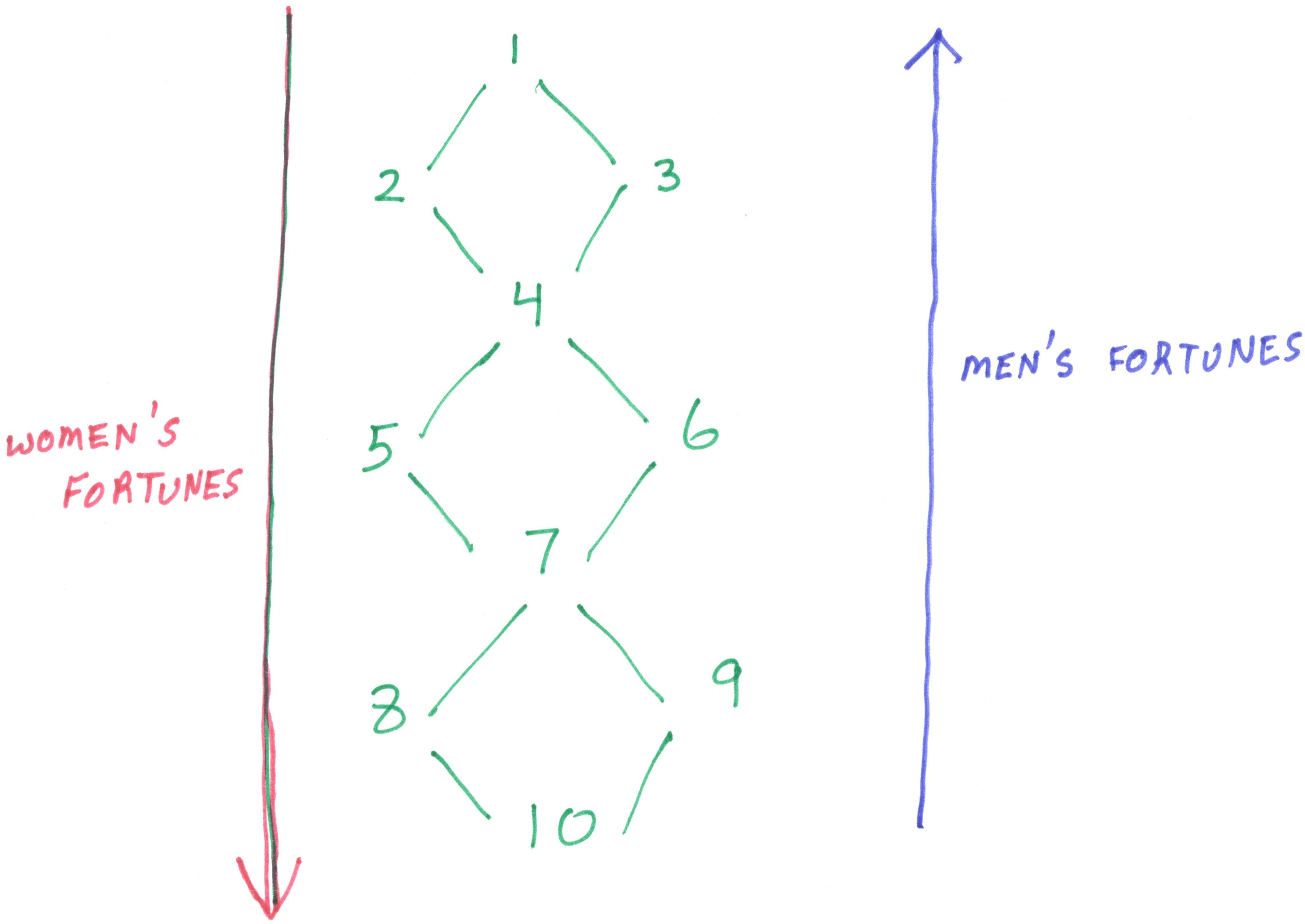
(10 out of 24)

matched to:

a	b	c	d		
A	B	C	D	... (1)	← MPD
B	A	C	D	... (2)	
A	B	D	C	... (3)	
B	A	D	C	... (4)	
C	A	D	B	... (5)	
B	D	A	C	... (6)	
C	D	A	B	... (7)	
D	C	A	B	... (8)	
C	D	B	A	... (9)	
D	C	B	A	... (10)	← WPD

etc

"LATTICE STRUCTURE"



FIX a profile of preferences

List **ALL** stable matchings w.r.t the profile

For any man m, let

Poss (m) = set of wives of m in the list

e.g.

$$\begin{pmatrix} \vdots \\ \text{John} - \text{Mary} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \text{John} - \text{Carol} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \text{John} - \text{Mary} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \text{John} - \text{Eva} \\ \vdots \end{pmatrix}, \begin{pmatrix} \vdots \\ \text{John} - \text{Susan} \\ \vdots \end{pmatrix}$$

Poss (John) = {Mary, Carol, Eva, Susan}

<u>Thm</u> When MPP is followed **EVERY** man m winds up with his **BEST** choice in $Poss(m)$ .... (any every woman w with her worst choice in $Poss(w)$!!)

PROOF (Fous throughout on MPP for the fixed profile)

Note: If John ends up with Mary for wife then

John must have proposed to, and been rejected by, **ALL** the women he prefers to Mary

So must show:

NONE of the women who reject John, in the course of the MPP, are possible for John

Must show this not just for John, but also for Bob, Harry, ...
for **ALL** men

In short, show

$\boxed{\text{In MPP: NO man } m \text{ is rejected by a woman in } Poss(m)}$

Define

$\boxed{\text{"BAD EVENT"} \equiv \text{a man } m \text{ is rejected by a woman in } Poss(m)}$

To show:

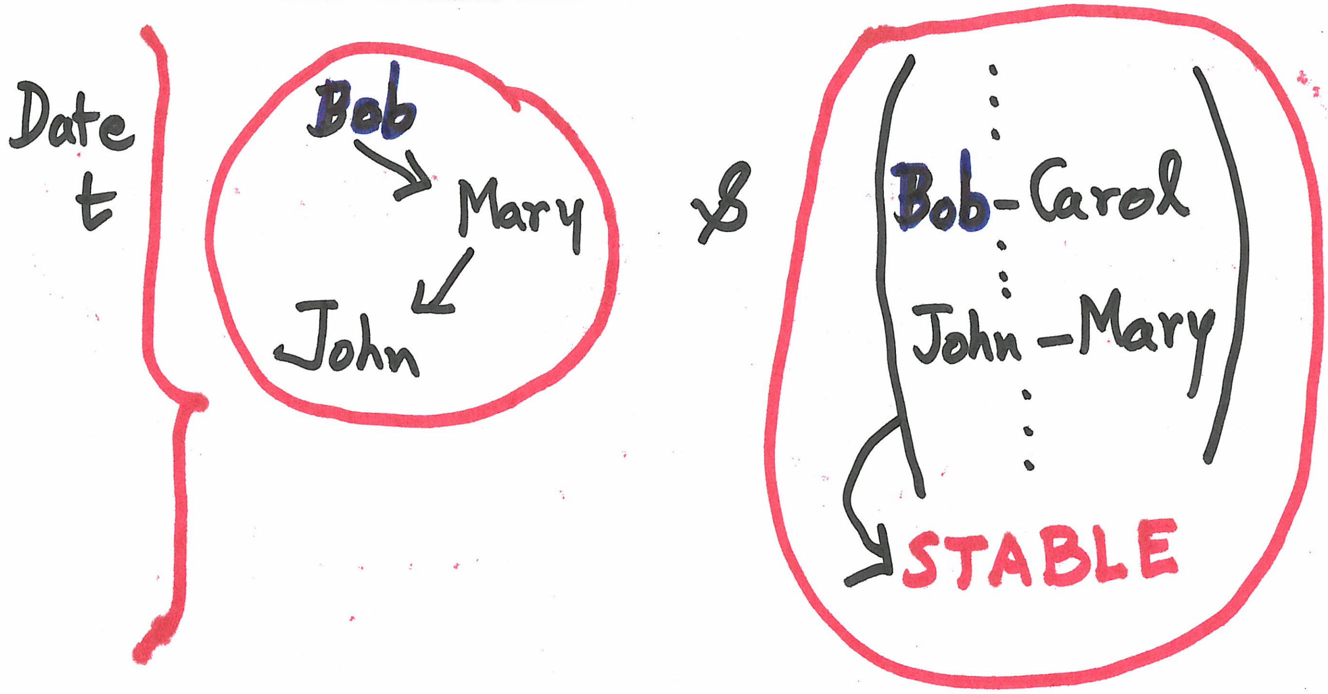
$\boxed{\text{BAD EVENTS cannot happen in MPP}}$

Will show by "contradiction"

Suppose "bad events" DO happen!

Let t be the EARLIEST date on which a "bad event" occurs i.e.

NO BAD EVENT on dates $1, \ldots, t-1$
&
BAD EVENT on date t



Mary : ..... Bob ..... John .....

Bob : .... Mary ......

? Carol ?

Bob : $\overbrace{\cdots\cdots\cdots\cdots}^{\text{all rejected Bob on dates } 1,\ldots,t-1}$ Mary ; so not in Poss(Bob)

? Carol

BUT Carol is in Poss(Bob)

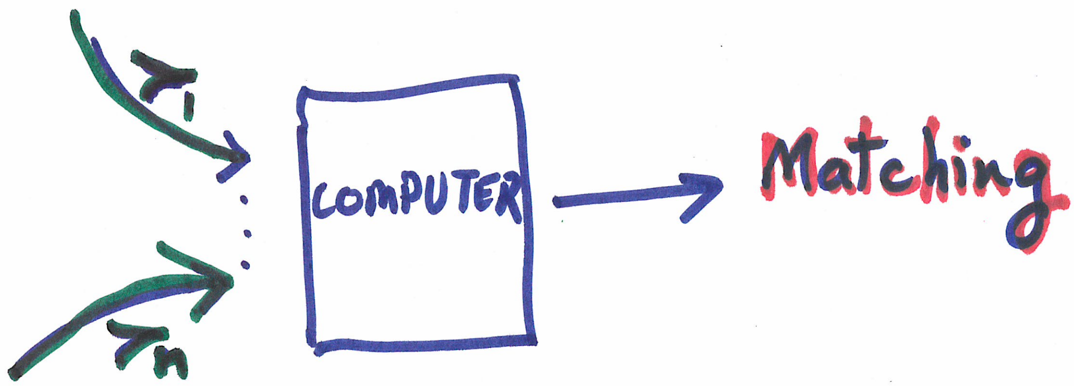
So have

Bob: . . . . . . Mary . . . . .

Carol

Reinterpret

$$\boxed{\text{MPP} = \text{Computer Program}}$$

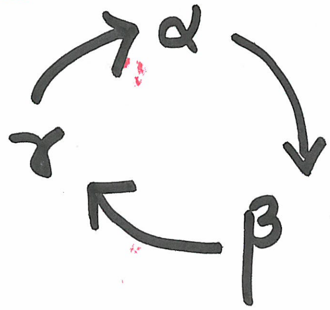


Incentive Compatibility
(Dubins-Friedman)

<u>Real world applications</u>
(Alvin Roth)

- NRMP
- NYC Public schools — kids

ROOMMATES PROBLEM



, α, β, γ each like δ least

Suppose β δ are paired

& matching is $\begin{pmatrix} \beta & \delta \\ \alpha & \gamma \end{pmatrix}$

NO stable matching!

Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with.

Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures," or that they "know a lot of formulas." At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.

What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematical, and the reason that your friends and ours cannot understand mathematics is not because they have no head for figures, but because they are unable to achieve the degree of concentration required to follow a moderately involved sequence of inferences. This observation will hardly be news to those engaged in the teaching of mathematics, but it may not be so readily accepted by people outside of the profession. For them the foregoing may serve as a useful illustration.

GRADUATED INTEREST RATES IN SMALL LOANS

HUGH E. STELSON, Michigan State University

Many small loan companies charge a graduated interest rate in accordance with various state laws. For example, 3% per month is charged on the first \$150 of a loan, and 2% on the portion of the loan in excess of \$150. Rates may be graduated in two, three or more brackets. A three-bracket loan might be at the rate of $2\frac{1}{2}\%$ on that part of the loan or loan balance which is \$100 or less, at the rate of 2% on that part of a loan which is in excess of \$100 but less than \$200, and at the rate of 1% on that part of a loan which is in excess of \$200. Such a graduated rate is written: $2\frac{1}{2}\%/2\%/1\%/\$100/\$200$.

The main problem considered in this paper is that of finding the level monthly rent payment which will amortize a loan in a given time at a graduated rate.

KIDNEY EXCHANGE

(Shapley - Scarf)

A "<u>owns</u>" kidney a

B . . . . . b etc.

.
.
.
.

Profile of preferences

A :	c	d	b	a
B :	c	a	b	d
C :	d	b	a	c
D :	a	b	c	d

PROFILE

A	c	d	b	a
B	c	a	b	d
C	d	b	a	c
D	a	b	c	d

Proposed allocation

$\mathcal{A} = \langle Ad, Ba, Cb, Dc \rangle$

Coalition $\{B, C\}$ can block $\mathcal{A}$

via $\langle Bc, Cb \rangle$ since

$B\uparrow$ and $\vec{C}$

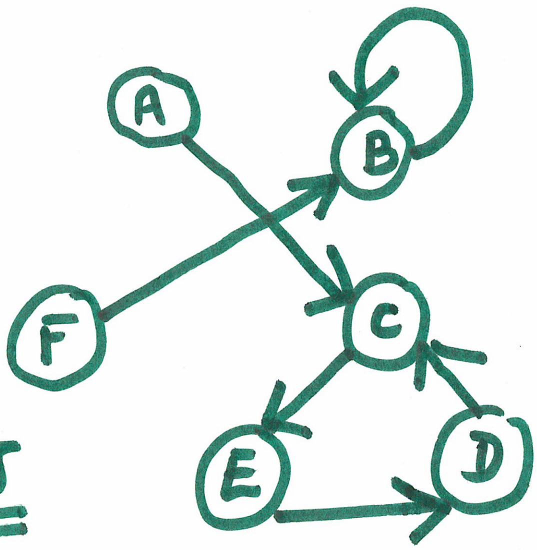
DEF^n An allocation is STABLE if no coalition can block it.

THEOREM There exists one, and only one, stable allocation

Proof (Gale)

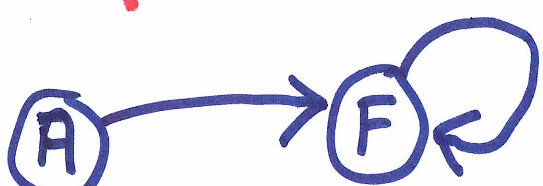
PROFILE

A	c	e	f	a	b	d
B	b	a	c	e	f	d
C	e	f	c	a	d	b
D	c	a	b	e	d	f
E	d	c	b	f	e	a
F	b	d	e	f	a	c

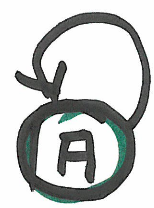


FIRST

$$\mathcal{A} = \langle A?, Bb, Ce, Dc, Ed, F? \rangle$$



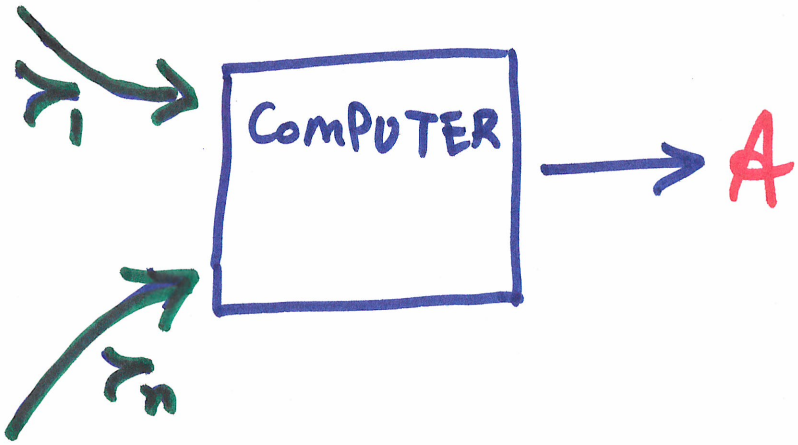
SECOND



THIRD

$$\mathcal{A} = \langle Aa, Bb, Ce, Dc, Ed, Ff \rangle$$

REINTERPRET



Also Incentive Compatible!