Int. Journal of Game Theory, Vol. 4, Issue 3, page 131–139, Physica-Verlag, Vienna.

On the Uniqueness of the Shapley Value[1])

By *P. Dubey*, Ithaca[2])

Abstract: *L.S. Shapley* [1953] showed that there is a unique value defined on the class D of all superadditive cooperative games in characteristic function form (over a finite player set N) which satisfies certain intuitively plausible axioms. Moreover, he raised the question whether an axiomatic foundation could be obtained for a value (not necessarily the *Shapley* value) in the context of the subclass C (respectively C', C'') of simple (respectively simple monotonic, simple superadditive) games *alone*. This paper shows that it is possible to do this.

Theorem I gives a new simple proof of *Shapley*'s theorem for the class G of *all* games (not necessarily superadditive) over N. The proof contains a procedure for showing that the axioms also uniquely specify the *Shapley* value when they are restricted to certain subclasses of G, e.g., C. In addition it provides insight into *Shapley*'s theorem for D itself.

Restricted to C' or C'', *Shapley*'s axioms do *not* specify a unique value. However it is shown in theorem II that, with a reasonable variant of one of his axioms, a unique value is obtained and, fortunately, it is just the *Shapley* value again.

Notation: For a set S we denote by $|S|$ the number of elements that S contains and frequently write it as s; similarly t abbreviates $|T|$ for a set T, etc. 2^S denotes the class of all subsets of the set S. $\emptyset$ stands for the empty set. R, as usual, represents the real line and Z^+ the set of positive integers. For a vector v in R^n, v_i is the i^{th} component of v. The symbol i is used both as a number and as the name of a player in N, but its meaning will be clear from the context.

1. Introduction

An *n-person cooperative game in characteristic function form* is a pair (N, v) where $N = (1, 2, \ldots, n)$ is a set of n players, and v is a function

$$v: 2^N \to R$$

with the property $v(\emptyset) = 0$. Intuitively $v(S)$ represents the "worth" ("value", "power") of the coalition S of players, i.e., the least payoff that S can quarantee itself no matter what the other players (that are not in S) do. Given a game v it is desirable to have a measure of the apriori "value" of each player in v.

Denote the class of all games on N by G.

Let ϕ be a function

[1]) This research was supported in part by the Office of Naval Research under contract N00014–67–A–0077–0014, task NR 047–094 and by the National Science Foundation under grant GP–32314 X in the Department of Operations Research in Cornell University.

[2]) *P. Dubey*, Cornell University, Ithaca, N.Y. 14850.

$$\phi: G \to R^n$$

which we interpret as follows: $\phi_i(v)$ is the *value* of the i^{th} player in the game v.

Shapley [1953] proposes three axioms which the function ϕ ought to satisfy. In order to state them it is necessary to first define a few concepts. All games in the definitions below are assumed to be in G.

1. S is called a *carrier* for v if

$$v(T) = v(T \cap S) \text{ for all } T \subset N.$$

2. If π: $N \to N$ is a permutation of N, then the game πv is defined by

$$\pi_v(T) = v(\pi(T)) \text{ for all } T \subset N.$$

3. Given any two games v_1 and v_2, the game $v_1 + v_2$ is defined by

$$(v_1 + v_2)(T) = v_1(T) + v_2(T) \text{ for all } T \subset N.$$

Shapley's axioms are:

$S1$. If S is any carrier for v, then $\sum_{i \in S} \phi_i(v) = v(S)$.

$S2$. For any permutation π and $i \in N$,

$$\phi_{\pi(i)}(\pi v) = \phi_i(v)$$

$S3$. If v_1 and v_2 are any games, then

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2).$$

Shapley [1953] proved the following

Theorem I. There is a unique function ϕ, defined on G, which satisfies the axioms $S1$, $S2$, $S3$.

Proof. For each coalition S define the game $v_{S,c}$ by

$$v_{S,c}(T) = \begin{cases} 0 & \text{if } S \not\subset T \\ c & \text{if } S \subset T. \end{cases}$$

Then it is clear that S and its supersets are all carriers for $v_{S,c}$. Therefore, by $S1$,

$$\sum_{i \in S} \phi_i(v_{S,c}) \qquad = c, \text{ and}$$

$$\sum_{i \in S \cup \{j\}} \phi_i (v_{S,c}) \quad = c \text{ whenever } j \notin S$$

This implies that $\phi_j (v_{S,c}) = 0$ whenever $j \notin S$. Also if π is a permutation of N which interchanges i and j (for any $i \in S$ and $j \in S$) and leaves the other players fixed, then it is clear that $\pi v_{S,c} = v_{S,c}$ and thus, by $S2$,

$$\phi_i (v_{S,c}) = \phi_j (v_{S,c}) \text{ for any } i \in S \text{ and } j \in S.$$

Therefore $\phi (v_{S,c})$ is unique, if ϕ exists, and is given by

$$\phi_i (v_{S,c}) = \begin{cases} c/\mid S \mid & \text{if } i \in S \\ 0 & \text{if } i \notin S \end{cases}$$

Now the games $\{v_{S,c} \mid \emptyset \neq S \subset N, c \in R\}$ form an additive basis for the vector space G, and a proof of the theorem could be obtained by showing this [*Shapley*, 1953]. However, for our purposes, it is useful to consider the games $\{v'_{S,c} \mid \emptyset \neq S \subset N, c \in R\}$ defined by

$$v'_{S,c} (T) = \begin{cases} c & \text{if } T = S \\ 0 & \text{if } T \neq S. \end{cases}$$

Any game v can be written as a finite sum of games of the type $v'_{S,c}$. Hence the uniqueness of ϕ follows, using $S3$, if we can show that each $\phi (v'_{S,c})$ is unique.

Assume that $\phi (v'_{S,c})$ is unique for $\mid S \mid = k + 1, \ldots, n$. (This is obviously true for $\mid S \mid = n$ because $v'_{N,c} = v_{N,c}$.) We will then show that $\phi (v'_{S,c})$ is unique for $\mid S \mid = k$.

Let $S_1, \ldots, S_l$ be all of the proper supersets of S. Note that $\mid S_i \mid > k$ for $i = 1, \ldots, l$, thus $\phi (v'_{S_i,c})$ is unique by the inductive assumption.
But

$$v_{S,c} = v'_{S,c} + v'_{S_1,c} + \ldots + v'_{S_l c}$$

Therefore, by $S3$,

$$\phi (v_{S,c}) = \phi (v'_{S,c}) + \phi (v'_{S_1,c}) + \ldots + \phi (v'_{S_l c}) \tag{1}$$

Since all the terms except $\phi (v'_{S,c})$ are unique, so is $\phi (v'_{S,c})$. This concludes the proof that ϕ, if it exists, is unique.

The proof of uniqueness has implicit in it, as was to be expected, a recipe for constructing ϕ. Suppose

$$\phi_i (v'_{S,c}) = \frac{(s-1)!\,(n-s)!}{n!} \cdot c \quad \text{if } i \in S$$

$$- \frac{s}{n-s} \quad \frac{(s-1)!\,(n-s)!}{n!} \cdot c \quad \text{if } i \notin S$$

for $s = |S| = k + 1, \ldots, n$. This is obviously true for $|S| = n$ since $v'_{N,c} = v_{N,c}$. It follows, using (1), that

$$\phi_i(v'_{S,c}) = \begin{cases} \dfrac{(s-1)!\,(n-s)!}{n!} \cdot c & \text{if } i \in S \\ -\dfrac{s}{n-s} \cdot \dfrac{(s-1)!\,(n-s)!}{n!}\, c & \text{if } i \notin S \end{cases}$$

for $|S| = k$.

It is now straightforward to obtain $\phi(v)$ for any v.

Since $v = \sum_{\emptyset \neq S \subset N} v'_{S,v(S)}$

$$\phi(v) = \sum_{\emptyset \neq S \subset N} \phi(v'_{S,v(S)}) \text{ by } S3.$$

The right-hand-side, when simplified, gives

$$\phi_i(v) = \sum_{\{i \in T \subset N\}} \frac{(t-1)!\,(n-t)!}{n!}\; v(T) - v(T - \{i\}),$$

Shapley's familiar formula. It is easy to verify that ϕ, defined as above, satisfies the axioms $S1, S2, S3$. This completes the proof of theorem I.

2. Uniqueness for Subclasses

One can restrict $S1, S2, S3$ to subclasses K of G. $S2$ is then required to hold only if $\pi v \in K$ whenever $v \in K$, and $S3$ only if $v_1 + v_2 \in K$ whenever $v_1 \in K$ and $v_2 \in K$. The question arises whether the axioms, so restricted, specify a unique ϕ on K. That they specify at least one ϕ is clear by considering the restriction of the *Shapley* value on G to K. By following the procedure given in the above proof we can establish the uniqueness of ϕ for certain K. It needs to be emphasized that *in each case* the proof of the uniqueness of ϕ on K is given by a recursive construction of ϕ on K which parallels the construction in the proof of theorem I. (The case $K = D$ requires a somewhat special treatment which is outlined below). Therefore it is *not* necessary to turn to ϕ on G and restrict its domain to K in order to prove the existence of ϕ on K.

The case $K = D$. D is the subclass of G which consists of all *superadditive games* in G, i.e., games v in G for which $v(S \cup T) \geqslant v(S) + v(T)$ whenever $S \cap T = \emptyset$. Though *Shapley*'s proof in [*Shapley*, 1953] is also a proof of theorem I, it is essentially concerned with D, which is perhaps why games of the type $v'_{S,c}$ are not considered in it. (Recall that $v'_{S,c}$ is not in D if $S \neq N$). However $\{v'_{S,c} \mid \emptyset \neq S \subset N, c \in R\}$ does help one to construct *Shapley*'s proof also. We first show that $\{v_{S,c} \mid \emptyset \neq S \subset N, c \in R\}$ forms a basis for G. Suppose that $v'_{S,1}$ is in the linear span of $\{v_{T,1} \mid T$ is a superset of $S\}$ when $|S| = k + 1, \ldots, n$. This is trivially true for $|S| = n$ because, as we have remarked before, $v_{N,c} = v'_{N,c}$. Let $|S^*| = k$. Since

$$v'_{S^*,1} = v_{S^*,1} - v'_{S_1^*,1} - \ldots - v'_{S_j^*,1} \tag{2}$$

where $S_1^*, \ldots, S_j^*$ are all the proper supersets of S^*, and since by the inductive assumption each $v'_{S_i^*,1}$ is in the linear span of $\{v_{T,1} \mid T$ is a superset of $S_i^*\}$, it follows that $v'_{S^*,1}$ is in the linear span of $\{v_{T,1} \mid T$ is a superset of $S^*\}$. From the fact that $\{v'_{S,1} \mid \emptyset \neq S \subset N\}$ spans G, we now see that $\{v_{S,1} \mid \emptyset \neq S \subset N\}$ also spans G. It is in fact a basis for G because it has the same number of elements as $\{v'_{S,1} \mid \emptyset \neq S \subset N\}$ which is well known to be a basis.

Express a v in D *uniquely* as: $v = \Sigma c_S v_{S,1}$. Some of the c_S on the right hand side may be negative so that the equation may contain games that are not in D. This would prevent an application of $S3$ which is restricted to D. To overcome this, transpose terms with negative c_S coefficients to the left. Then it is easy to see that the new equation will only contain games that are in D. An application of $S3$ now proves the uniqueness of ϕ on D. To find c_S explicitly, first express each $v'_{S,1}$ in terms of the basis $\{v_{S,1} \mid \emptyset \neq S \subset N\}$ using (2) and induction, and then substitute into $v = \sum_{\emptyset \neq T \subset N} v'_{T,v(T)}$. It can be shown in this way that $c_S = \sum_{T \subset S} (-1)^{s-t} v(T)$, which of course enables us to write out an explicit formula for $\phi(v)$ as is done by *Shapley* [1953]. This is not simple, however, and it is easier to show the existence of ϕ on D by restricting the previously obtained ϕ on G to D.

Others cases, $K \neq D$. In the following examples (which are by no means exhaustive) the proof of the uniqueness of ϕ on the given K is completely parallel to the proof of theorem I, and involves a similar recursive construction of ϕ.

A. The subclass of all *simple games*, i.e., all games v for which $v(S) = 0$ or 1, for any $S \subset N$.

B. The subclass of all games v for which $v(S) = 0$ whenever $|S| \leqslant k$; as well as the subclass of all simple games with this restriction.

C. The subclass of all games v in which certain players $i_1, \ldots, i_k$ are distinguished and $v(S) = 0$ if $\{i_1, \ldots, i_k\} \not\subset S$; as well as the subclass of all simple games with this restriction.

Remarks. (I). The convex cone generated by the simple games with veto players (i.e., players i such that $v(S) = 0$ if $i \notin S$, for all $S \subset N$) is the subclass L of all games with non-empty cores [*Spinneto*, 1971]. Therefore case C shows that the axioms uniquely specify the *Shapley* value on L. In fact this is true for convex cones generated by the class of games in any one of A, B, or C or their unions.

(II). For any $P \subset G$, $|P| < \infty$, we can determine in a finite number of steps whether or not the axioms uniquely specify the *Shapley* value on P; and if they do not, we can construct different ϕ's on P which satisfy the axioms. Indeed, this corresponds to checking whether a certain system of linear equations has a unique solution or not. The size of this system can be cut down using a procedure which mimics the proof of theorem I. (We omit the details.)

3. Monotonic Simple Games

Let C' be the subclass of all *monotonic simple games* in G, i.e., simple games v for which $v(S) = 1$ implies that $v(T) = 1$ whenever $S \subset T$. And let C'' be the subclass of all *superadditive simple* games in G.

The axioms $S1, S2, S3$ do *not* uniquely specify the *Shapley* value on C' or C'' if $|N| > 2$. First note that the games in C' or C'' for which the value is determined by $S1$ and $S2$ alone are precisely of the type $v_{S,1}$. Pick a game v in C'' (and thus also in C' since $C'' \subset C'$) which is not of the type $v_{S,1}$. An example of one is:

$$v(N - \{i\}) = v(N - \{j\}) = v(N) = 1, \text{ and}$$

$$v(S) = 0 \text{ for all other } S \subset N$$

where i and j are any two distinct players in N, and where we assume that $|N| > 2$.

Set $\phi_i(v) = \phi_j(v) = p$, where p is an arbitrary real number, and set

$$\phi_k(v) = \frac{1 - 2p}{|N - \{i,j\}|} \text{ for } k \neq i, j.$$

Then it is obvious that $\phi(v)$ satisfies $S1$ and $S2$. It also satisfies $S3$ vacuously. For suppose $v + v' = v''$ for a $v' \in C'$ and a $v'' \in C'$. Then $v(N) + v'(N) = v''(N)$. But $v(N) = 1$, therefore $v''(N) = 1$, which implies that $v'(N) = 0$. Thus $v' = 0$ since v' is monotonic. Also, if $v - v' = v''$ for a $v' \in C'$ and a $v'' \in C'$, then two cases arise: (a) $v''(N) = 1$, therefore $v'(N) = 0$, and so $v' = 0$. (b) $v''(N) = 0$ which implies that $v'' = 0$, and hence $v' = v''$. There is no question, therefore, of $S3$ being violated for any choice of p, and so ϕ is not uniquely specified on C' or C'' by $S1, S2, S3$.

However, if we replace $S3$ by a variant of it, $S3'$ (which will be stated below), then a unique ϕ is specified on C' or C'' and it is just the *Shapley* value.

In what follows we will write out only the case for C'', because the case for C' is obtained by replacing C'' by C' throughout.

First we make a few definitions. For $v \in C''$ and $v' \in C''$ let $v \vee v'$ denote the game given by

$$(v \vee v')(S) = \begin{cases} 1 & \text{if either } v(S) = 1 \text{ or } v'(S) = 1 \\ 0 & \text{if } v(S) = 0 \text{ and } v'(S) = 0. \end{cases}$$

Note that $v \vee v'$ may not always be in C'' for a v in C'' and a v' in C''. (However $v \wedge v'$ is in C' whenever v is in C' and v' is in C'.). Let $v \wedge v'$ denote the game given by

$$(v \wedge v')(S) = \begin{cases} 1 & \text{if } v(S) = 1 \text{ and } v'(S) = 1 \\ 0 & \text{if } v(S) = 0 \text{ or } v'(S) = 0. \end{cases}$$

Let us make a simple check to see that $v \wedge v' \in C''$ whenever $v \in C''$ and $v' \in C''$. If $v \wedge v' \notin C''$, then there are coalitions S and T, $S \cap T = \emptyset$, such that $(v \wedge v')(S \cup T) < (v \wedge v')(S) + (v \wedge v')(T)$. But by the definition of $v \wedge v'$ this means that either $\nu(S \cup T) < \nu(S) + \nu(T)$ or $\nu'(S \cup T) < \nu'(S) + \nu'(T)$, which is a contradiction. (A similar argument shows that C' is closed under $\wedge$).

We are now in a position to state $S3'$:

$S3'$. If $\nu \vee \nu' \in C''$ whenever $\nu \in C''$ and $\nu' \in C''$ then

$$\phi(\nu \vee \nu') + \phi(\nu \wedge \nu') = \phi(\nu) + \phi(\nu').$$

(In stating $S3'$ for C' we may drop the "if" because $\nu \quad \nu' \in C'$ always.)

Theorem II. There is a unique function ϕ, defined on C'', which satisfies the axioms $S1, S2, S3'$. Moreover, this ϕ is just the *Shapley* value.

Proof. Every ν in C'' has a finite number of minimal winning coalitions $S_1, \ldots, S_k$, i.e. coalitions S_i such that $\nu(T) = 1$ if $S_i \subset T$ for some i and $\nu(T) = 0$ if $S_i \not\subset T$ for all i. Clearly

$$\nu = \nu_{S_1,1} \vee \nu_{S_2,1} \vee \ldots \vee \nu_{S_k,1}$$

where the right hand side is defined associatively. Let $n^1(\nu) = \min\{p \in Z^+ \mid$ there exists a minimal winning coalition T of ν such that $|T| = p\}$ and let $n^2(\nu)$ = the number of winning coalitions T of ν such that $|T| = n^1(\nu)$.

The proof of the uniqueness of ϕ will be by induction on $n^1(\nu)$ and $n^2(\nu)$.

For $n^1(\nu) = n$, $\nu = \nu_{N,1}$, in which case $\phi(\nu)$ is obviously unique.

Suppose $\phi(\nu)$ has been shown to be unique for all ν such that $n^1(\nu) = k+1$, $k+2, \ldots, n$. Then $\phi(\nu)$ is unique when $n^1(\nu) = k$ *and* $n^2(\nu) = 1$.

Let S be the unique minimal winning coalition with k players. If S is the only minimal winning coalition of ν, then $\nu = \nu_{S,1}$ and $\phi(\nu)$ is unique. Otherwise let $S_1, \ldots, S_m$ denote all of the minimal winning coalitions of ν apart from S.

Note: $|S_i| > k$ for $1 \leqslant i \leqslant m$ since $n^2(\nu) = 1$. Now

$$(\nu_{S_1,1} \vee \nu_{S_2,1} \vee \ldots \vee \nu_{S_m,1}) \vee \nu_{S,1} = \nu$$

say, $\nu' \vee \nu_{S,1} = \nu$

It follows that $n^1(\nu') > k$. Therefore $\phi(\nu')$ is unique by the inductive assumption. Further, $n^1(\nu_{S,1} \wedge \nu') > k$. This is obvious from the definition of $\wedge$. Therefore $\phi(\nu \vee \nu')$ is also unique by the inductive assumption. Invoke axiom $S3'$. Then

$$\phi(\nu) = \phi(\nu' \vee \nu_{S,1}) = \phi(\nu') + \phi(\nu_{S,1}) - \phi(\nu_{S,1} \wedge \nu')$$

Since all the three vectors on the right hand side are unique, so is $\phi(\nu)$.

Next, suppose $\phi(\nu)$ has been shown to be unique for all ν such that either

$$n^1(\nu) = k+1, \ldots, n \tag{3}$$

or $$n^1(\nu) = k \text{ and } n^2(\nu) = 1, \ldots, j \tag{4}$$

Then $\phi(v)$ is unique when $n^1(v) = k$ *and* $n^2(v) = j + 1$.

Indeed, let $S_1, \ldots, S_{j+1}$ be the minimal winning coalitions of v with k players each. And let $T_1, \ldots, T_m$ be all the other minimal winning coalitions of v. By the conditions on $n^1(v)$ and $n^2(v)$ it is clear that $|T_i| > k$ for $1 \leqslant i \leqslant m$. Now

$$(v_{T_1,1} \vee \ldots \vee v_{T_m,1} \vee v_{S_1,1} \vee \ldots \vee v_{S_j,1}) \vee v_{S_{j+1},1} = v.$$

$$\text{say, } v'' \vee v_{S_{j+1},1} = v$$

clearly v'' satisfies (4) and $v'' \wedge v_{S_{j+1},1}$ satisfies (3). Therefore $\phi(v'')$ and $\phi(v'' \wedge v_{S_{j+1},1})$ are both unique by the inductive assumption.
By $S3'$,

$$\phi(v) = \phi(v'' \vee v_{S_{j+1},1}) = \phi(v'') + \phi(v_{S_{j+1},1}) - \phi(v'' \wedge v_{S_{j+1},1})$$

which proves the uniqueness of $\phi(v)$.

Putting together the two results we get that $\phi(v)$ is unique for any feasible numbers $n^1(v)$ and $n^2(v)$, i.e., for all $v \in C''$.

It is clear that the *Shapley* value ϕ on G satisfies $S1, S2, S3'$ when it is restricted to C''. Indeed $v + v' = (v \vee v') + (v \wedge v')$ where we regard the $+$ as taking place in the vector space G. Hence by $S3$ $\phi(v) + \phi(v') = \phi(v \vee v') + \phi(v \wedge v')$. Thus the *Shapley* value is *the* unique ϕ on C''which satisfies $S1, S2, S3'$.

However, we need not depend on the ϕ already defined on G to establish the existence of ϕ on C''. It is quite clear that implicit in the proof of uniqueness is a recursive construction of ϕ. (We omit this because it is straightforward.)

Remarks: (III) Theorem II holds when we replace C' (respectively C'') by certain subclasses of C' (respectively C''). The proofs are similar and involve stopping the induction at appropriate stages, and considering games that take on values in $\{c, 0\}$ instead of $\{1, 0\}$. We give just two examples: Subclasses of C' (or C'') for which (1) $v(S) = 0$ if $|S| \leqslant k$, (2) $v(S) = 0$ if $\{i_1, \ldots, i_k\} \not\subset S$.

(IV) By changing $S1$, but retaining $S2$ and $S3'$, we can obtain an axiomatic foundation for the *Banzhaf* value in its unnormalized form [*Lucas*, 1973] when it is restricted to C' or C''. The proof of this is similar to the proof of theorem II, and will appear in a forthcoming paper.

Acknowledgments

It is a pleasure to thank *W.F. Lucas* for suggesting these problems, and both him and *L.J. Billera* for helpful discussions.

References

Lucas, W.F.: Measuring Power in Weighted Voting Systems, Cornell University, Ithaca, N.Y. 14850, October 1973, (Draft).

Shapley, L.S.: A Value for n-person Games, Annals of Mathematics Study No. **28**, Princeton University Press, Princeton, N.J., 1953, pp. 307–317.

Spinneto, R.D.: Solution Concepts of n-person Cooperative Games as Points in the Game Space, Technical Report No. **138**, Department of Operations Research, College of Engineering, Cornell University, Ithaca, N.Y. 14850, 1971.

Received October, 1974