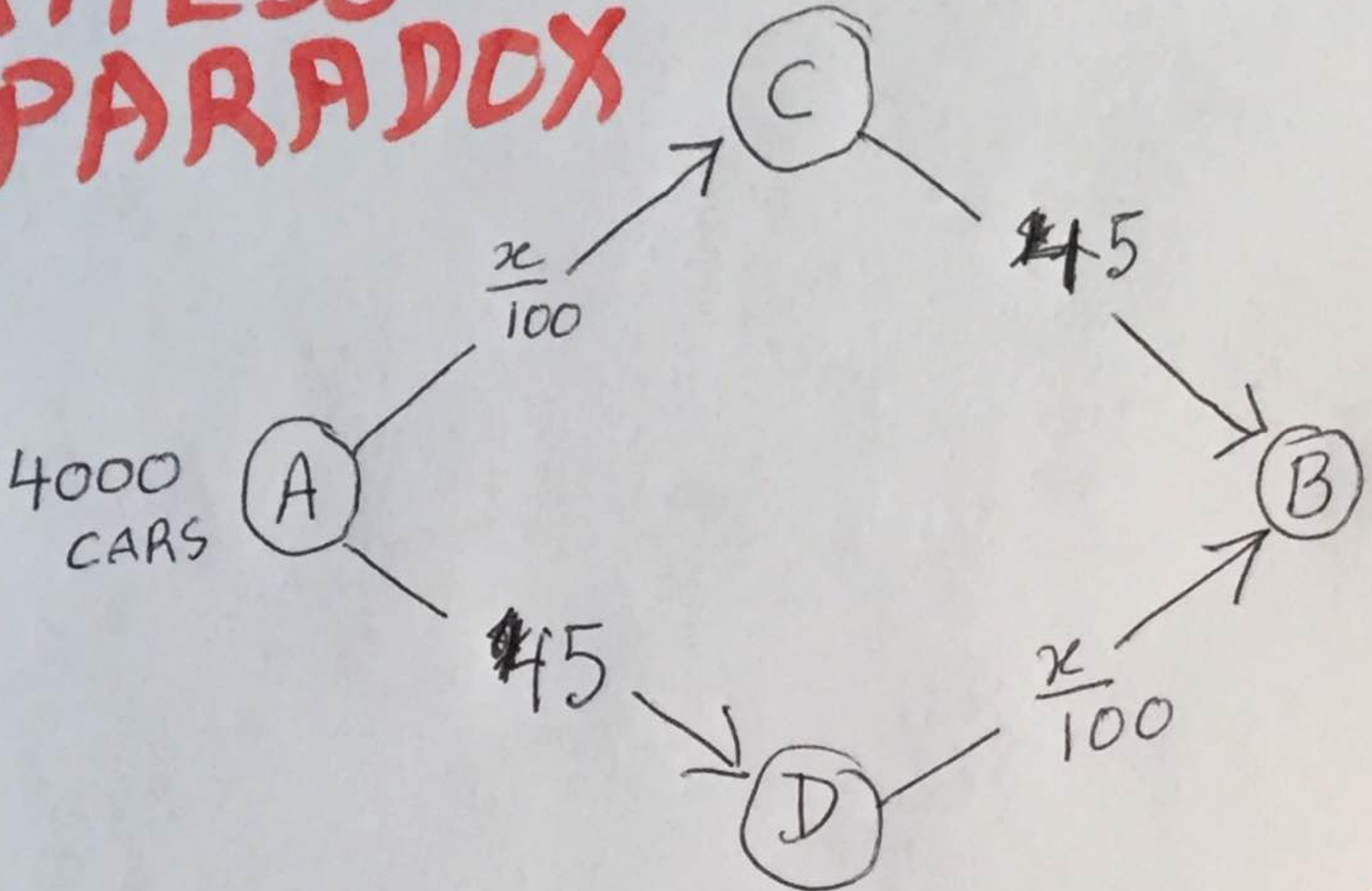
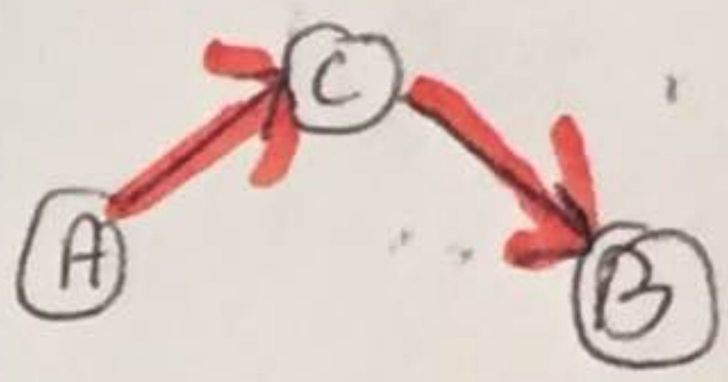
TRAFFIC IN NETWORKS

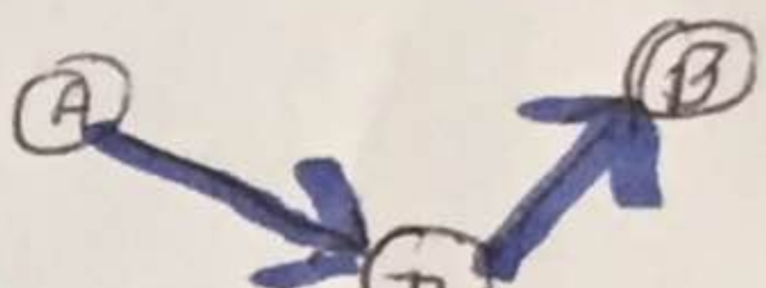
BRAESS' PARADOX



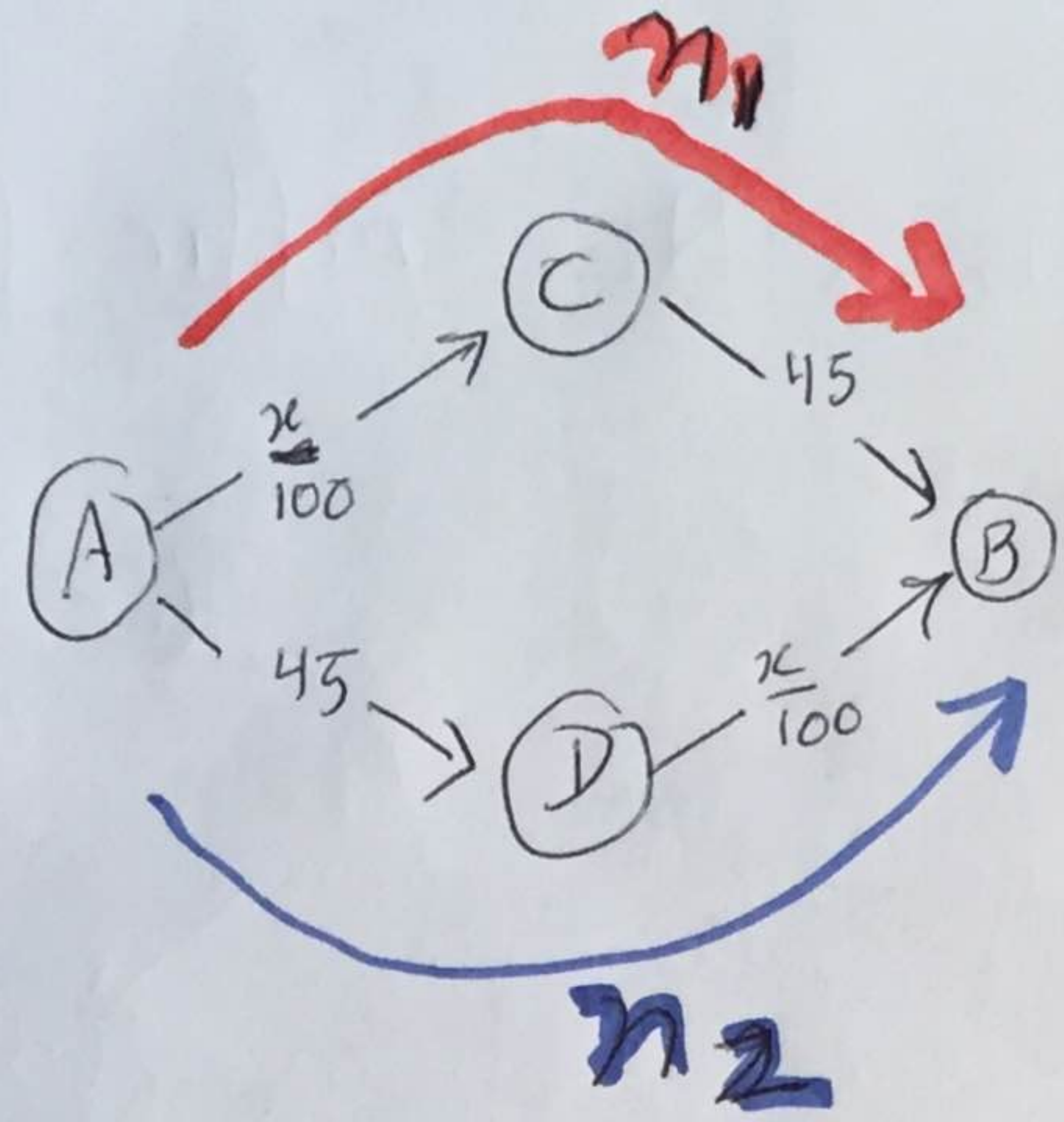
- Each car has 2 STRATEGIES:



or

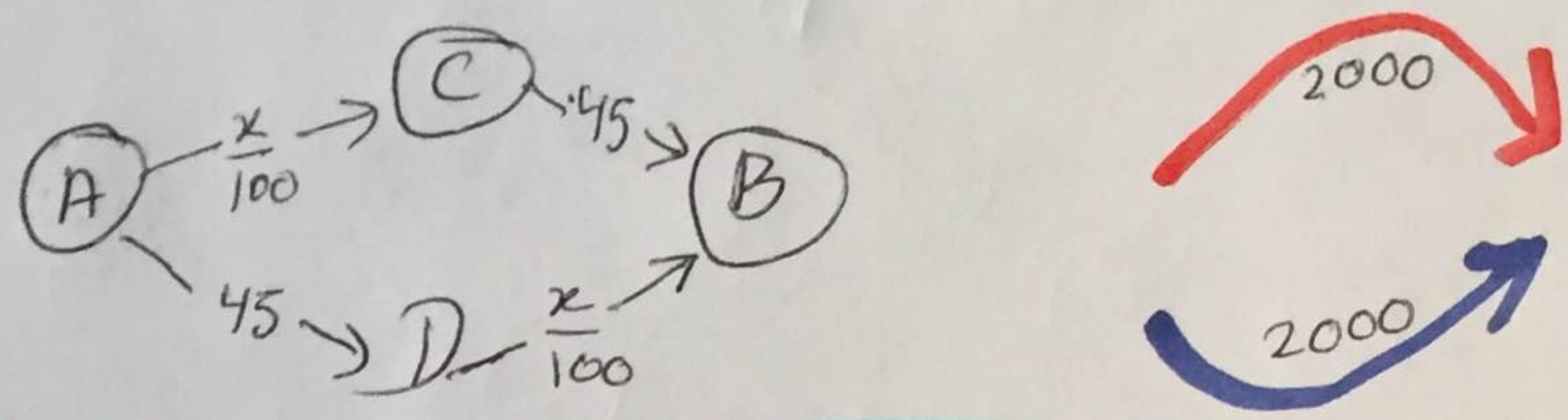


$\frac{x}{100} \equiv$ If x cars are on $A \longrightarrow C$ then each car takes $\frac{x}{100}$ minutes to traverse $A \longrightarrow C$



- Suppose $n_1 > n_2$
- Consider any car which took 
  and let it switch to 
  - Before the switch, it took
    $$\frac{n_1}{100} + 45 \text{ min}$$
    Afterwards it took $\frac{n_2+1}{100} + 45$ min
  - Since $n_1 + n_2 = 4000$ and $n_1 > n_2$ → (need $n_1 + n_2 = \text{EVEN}$)
    we see that $n_1 > n_2 + 1$
    So the car benefits by the switch.

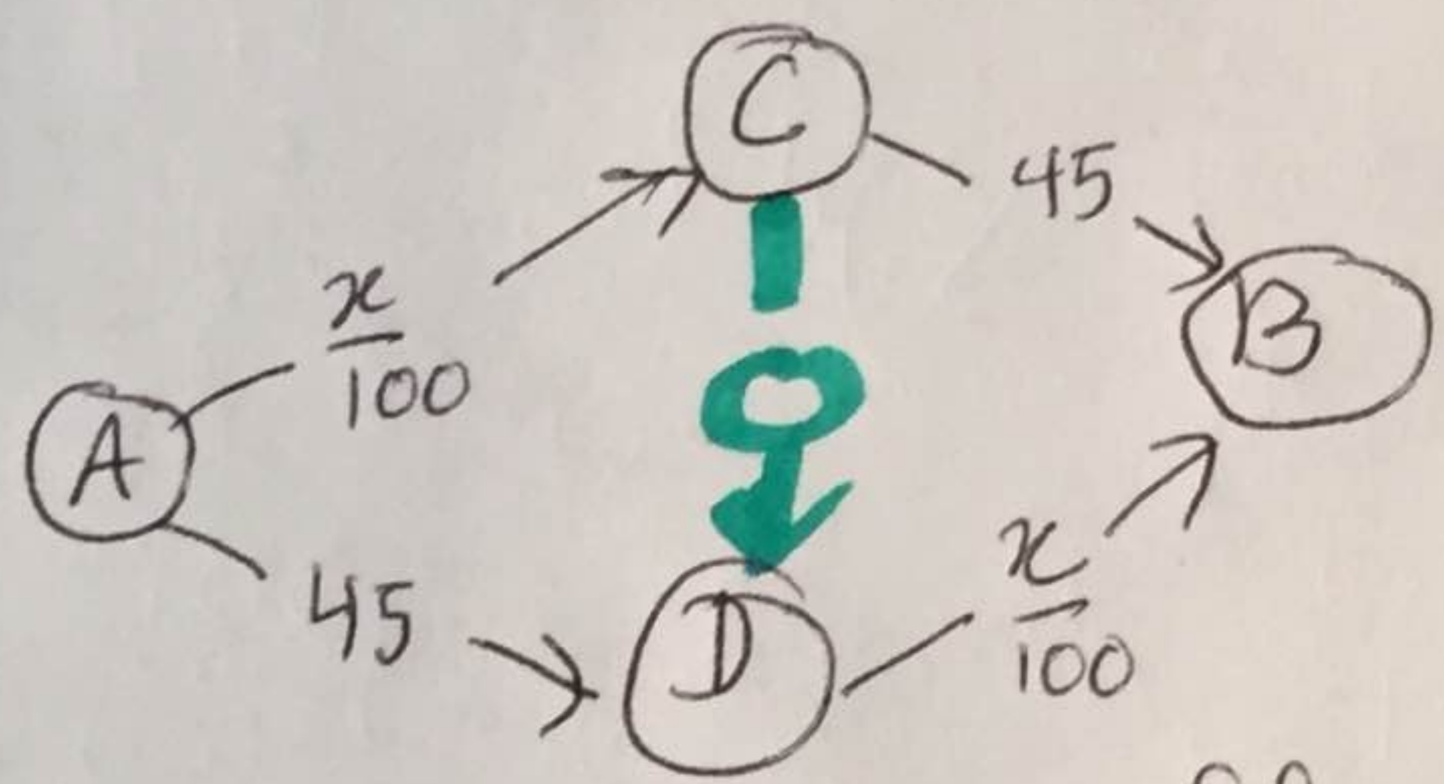
We conclude that the only NE is ③



And each car takes $\frac{2000}{100} + 45$

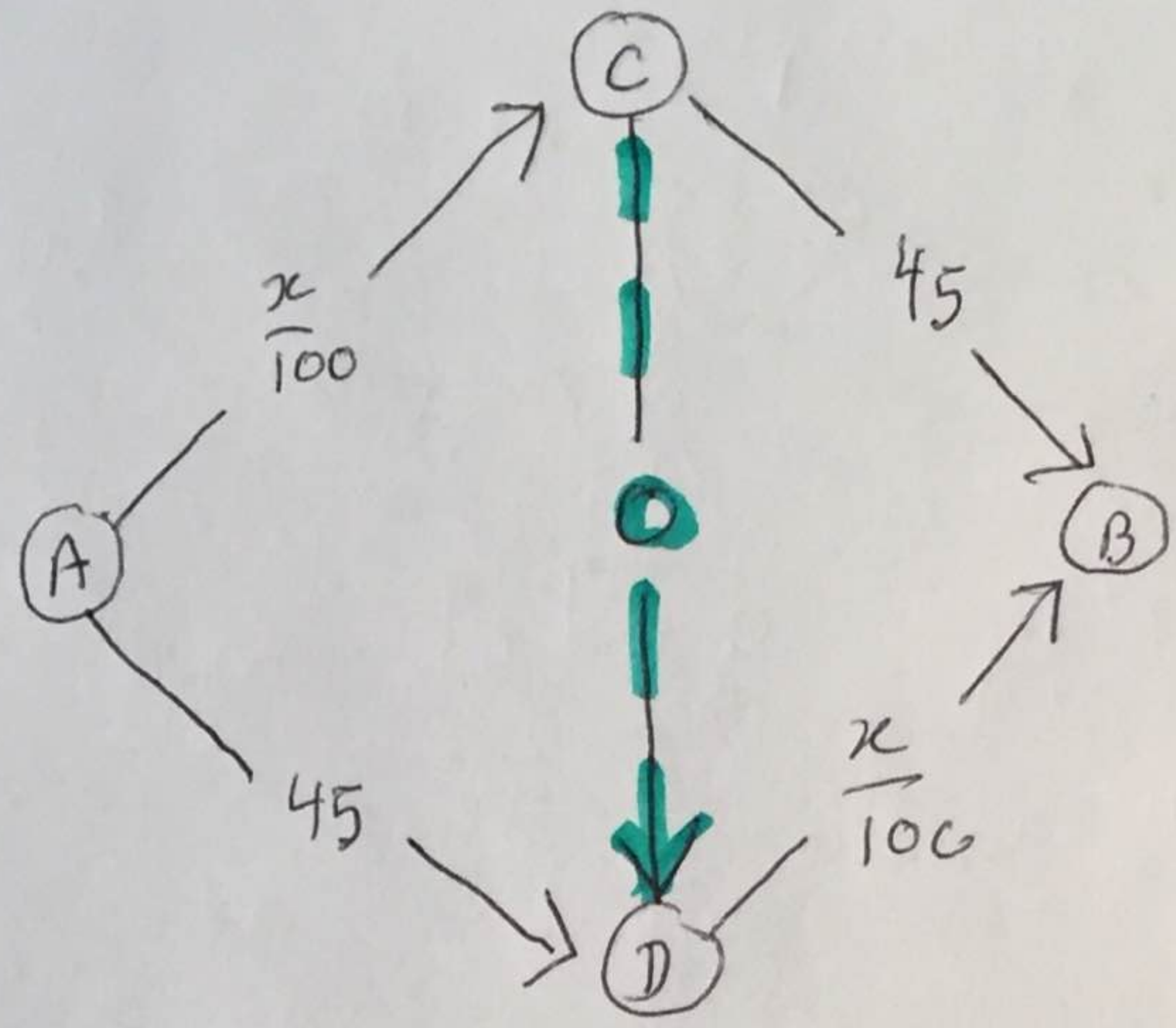
$= 65$ min.

ADD A "SUPER-HIGHWAY" CD

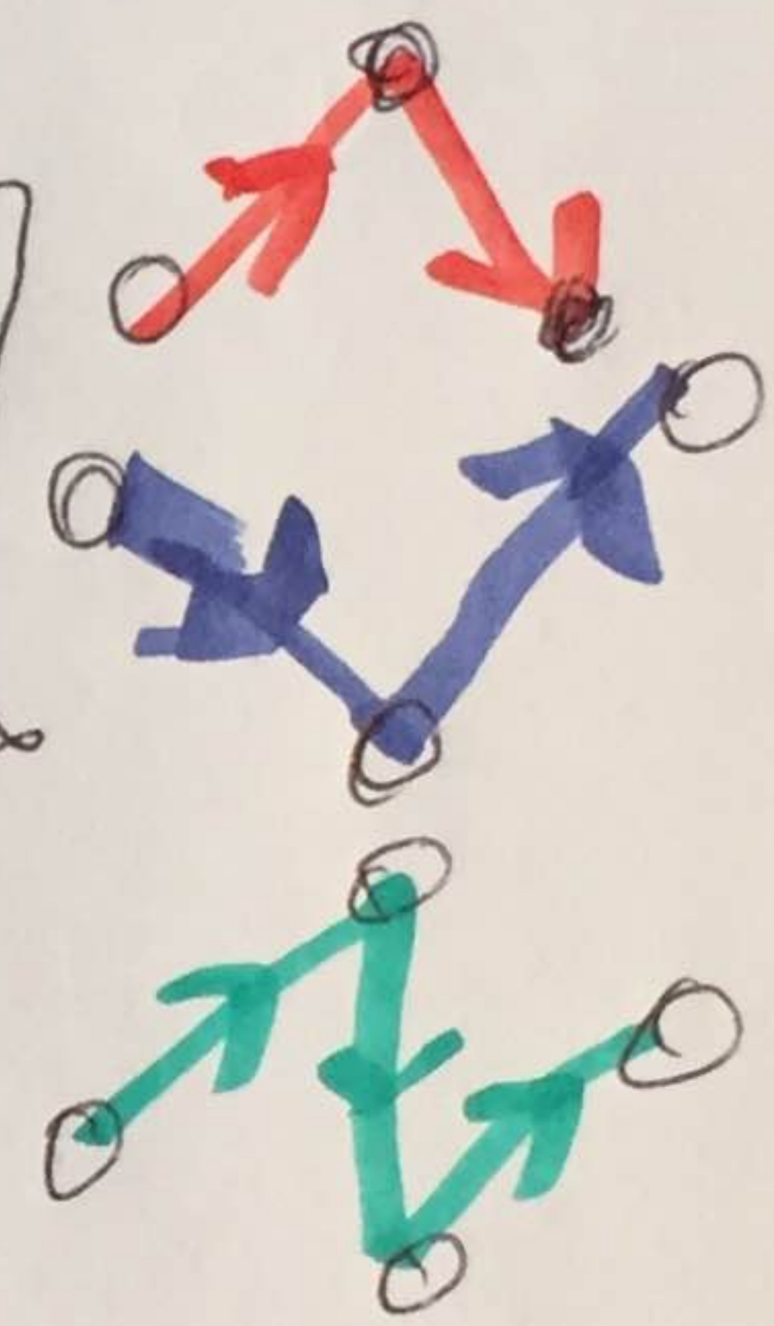


How will traffic flow now?

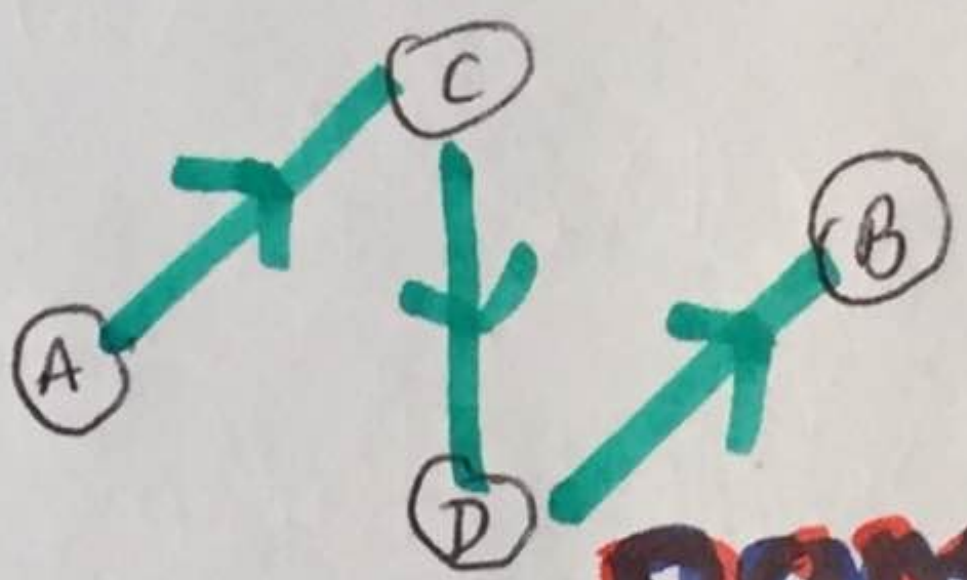
(NE = behavior under "selfish routing")



Now each car has 3 strategies



CLAIM

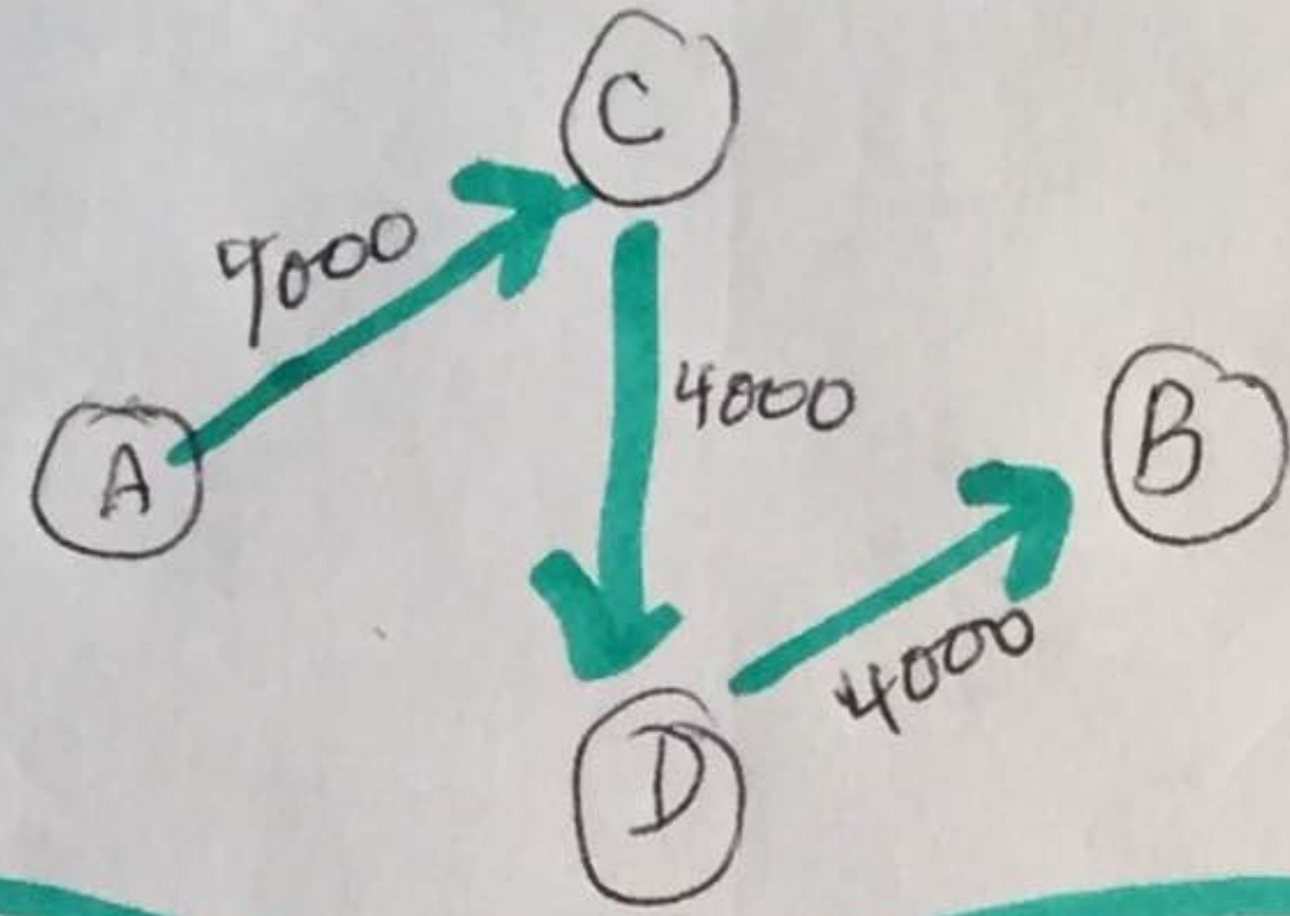


IS a DOMINANT strategy for each player

PROOF Quite obvious

(Say in words)

THEREFORE the UNIQUE NE is

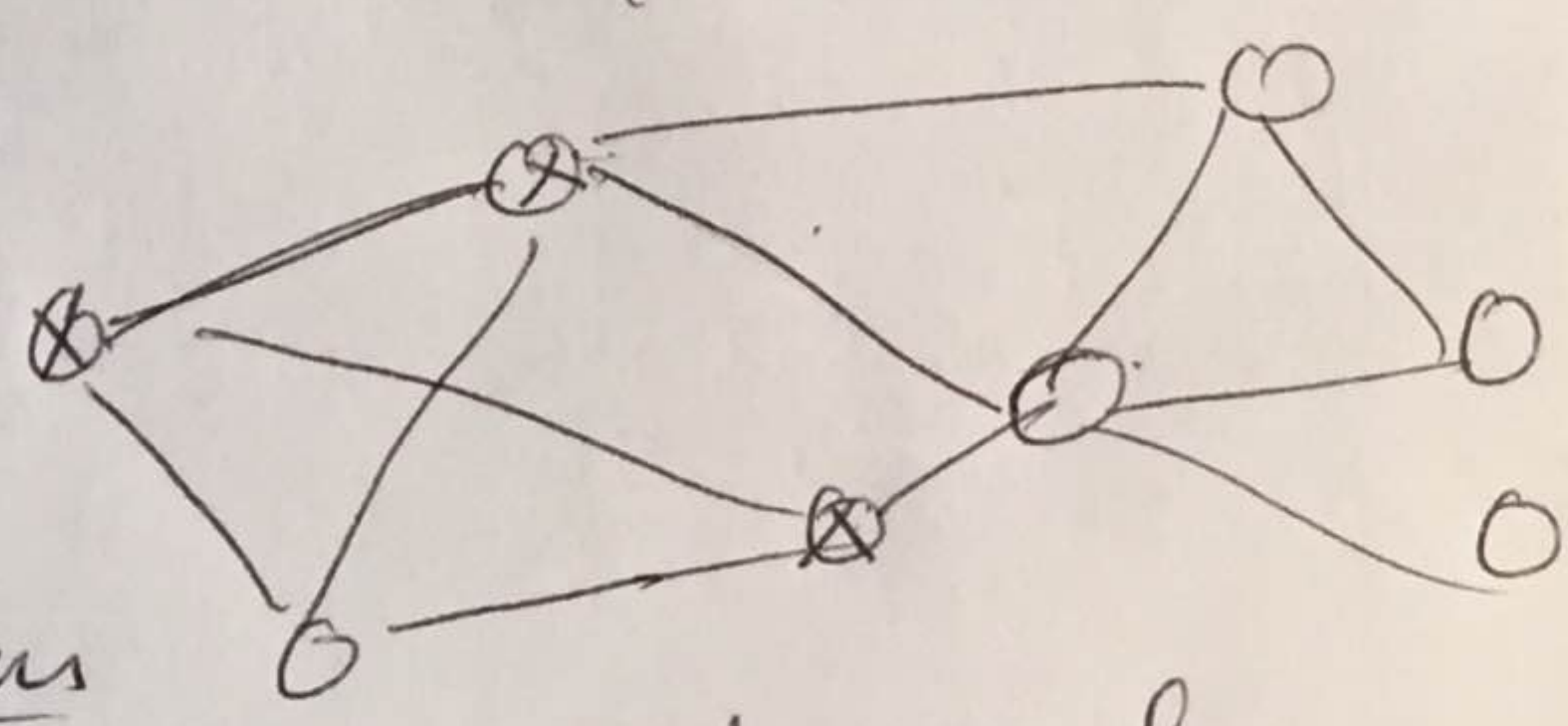


AND NOW EACH CAR TAKES

$$\frac{4000}{100} + 0 + \frac{4000}{100} = 80 \text{ minutes!}$$

Examples (Wikipedia)

- In Seoul $\xrightarrow[\text{restoration}]{}\!\!\!\!\!\!\!/$ and traffic improved
- Stuttgart $\Rightarrow$ better
- NYC $\not\to$ $\Rightarrow$ 42 St (Earth Day)
- Power Grid



- electrons in nanoscopic network etc.

Braess' Paradox is not an anamoly

Steinberg & Zangwill (1983) show that in a general network, when a new route is added, Braess' paradox occurs with probility $\frac{1}{2}$

(Network is "random")

See also "~~Steinberg~~ Valiant & Roughgarden" (2006)

A GENERAL MODEL OF TRAFFIC FLOW

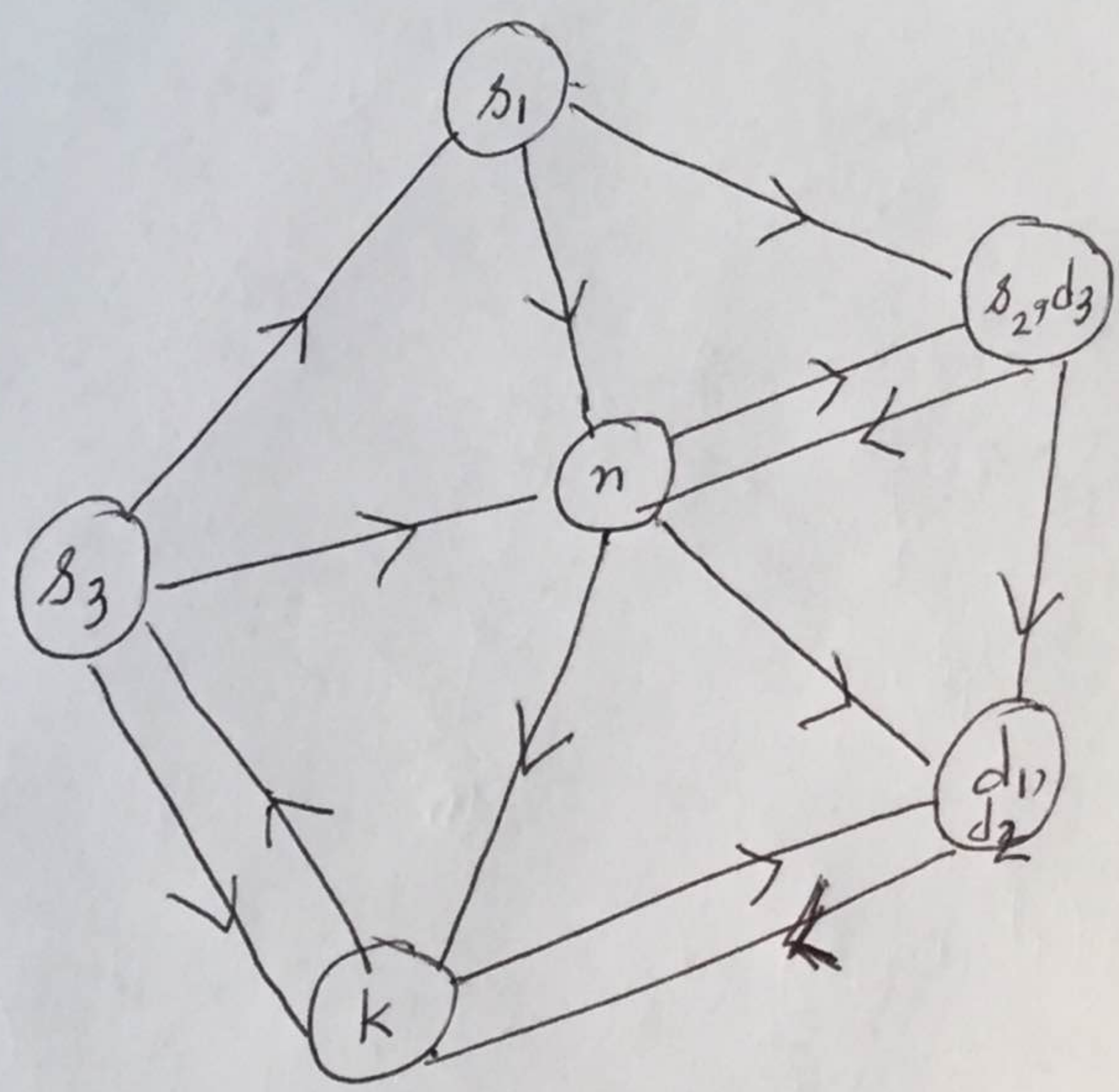
Let G be a directed graph on a finite set of vertices V.
E will denote the set of edges of G
There is a set of players N.
Each player $n \in N$ wants to go from vertex $s(n)$ to another vertex $d(n)$.

(Pure) Strategy set of n is

$$\Sigma^n = \text{set of all directed paths from } s(n) \text{ to } d(n)$$

Note Σ^n is a finite set (non-empty by assumption)

EXAMPLE WITH $N = \{1, 2, 3\}$



$$\Sigma^1 = \{ (s_1, s_2, d_1), (s_1, n, s_2, d_1), (s_1, n, k, d_1), (s_1, n, d_1) \}$$

$$\Sigma^3 = \{ (s_3, n, d_3), (s_3, s_1, d_3), (s_3, s_1, n, d_3), \ldots\ldots \}$$

Define, for any edge $e \in E$ and integer x,

$T_e(x)$ = time taken by each user to traverse e, when there are x users of e

Assume

$$T_e(1) \leq T_e(2) \leq T_e(3) < \cdots$$

($T_e(0) = 0$ by convention)

Then, for any choice of strategies $(\sigma^1, \ldots, \sigma^N)$, where $\sigma^n \in \Sigma^n$, the payoff of n is: $-$(time taken by n to travel on the path σ^n)

This defines the noncooperative "traffic game"

CLAIM There exists a Nash Equilibrium (in pure strategies)
(Rosenthal)

Proof Define the "ENERGY" of edge e when there are x users of e by

$$\boxed{E_e(x) = T_e(1) + T_e(2) + \cdots + T_e(x)}$$

Let τ = any flow of traffic (after every player picks a route), & denote

$$\boxed{\tau_e = \# \text{ of users of edge } e \text{ in traffic flow } \tau}$$

Define

$$E(\tau) = \sum_{e \in E} E_e(\tau_e)$$

Consider any flow τ

Let one player n change his route (keeping other players fixed).

Denote the new flow by $\tilde{\tau}$ (obtained by his switch)

Then we submit

$$\boxed{\text{time saved by } n = E(\tau) - E(\tilde{\tau})} \quad (*)$$

To prove (*), suppose n gives up edges $e_1, \dots, e_k$ and uses new edges $\tilde{e}_1, \dots, \tilde{e}_n$ (when he changes his route).

~~Then he [illegible] "saves" time~~

As n changes his route energy falls on edges $e_1, \ldots, e_k$ in the amounts $T_{e_1}(\tau_{e_1}), \ldots, T_{e_k}(\tau_{e_k})$

and

rises on edges $\tilde{e}_1, \ldots, \tilde{e}_n$ in the amounts $T_{\tilde{e}_1}(\tau_{e_1}+1), \ldots, T_{\tilde{e}_n}(\tau_{\tilde{e}_n}+1)$

So

$$E(\tilde{\tau}) - E(\tau) =$$

$$- \left[T_{e_1}(\tau_{e_1}) + \cdots + T_{e_k}(\tau_{e_k})\right]$$

$$+ \left[T_{\tilde{e}_1}(\tau_{e_1}+1) + \cdots + T_{\tilde{e}_n}(\tau_{\tilde{e}_n}+1)\right]$$

= Time of travel for n in $\tilde{\tau}$

− Time of travel for n in τ

$\Longrightarrow$ Time saved by n = $E(\tau) - E(\tilde{\tau})$

1st method (CONSTRUCTIVE)

To show existence of NE start with any flow $\tau_1 = (\sigma_1, \dots, \sigma_N) \in \Sigma^1 \times \dots \times \Sigma^N$

Suppose there exist players who could profit by unilaterally deviating to a different route.

Select any one of them arbitrarily and let him deviate*, resulting in a new flow τ_2. [$\tau_2 = (\sigma_1, \dots, \sigma_{n-1}, \tilde{\sigma}_n, \sigma_{n+1}, \dots, \sigma_N)$ if n deviates]

Continue these unilateral deviations (sequentially, one player at a time) resulting in flows $\tau_1, \tau_2, \tau_3, \dots$

By (*) $E(\tau_1) > E(\tau_2) > E(\tau_3) > \dots$

But energy cannot go down forever as there are finitely many flows possible, and, corresponding to them, finitely many energy levels possible.

When we cannot descend further we have reached an NE

(* DEVIATE TO ANY STRATEGY THAT IS BETTER FOR HIM, NOT NECESSARILY SOME BEST REPLY)

(15.1)

2^{nd} method (EXISTENTIAL)

Let τ^* be a flow with minimum energy. (Such a τ^* clearly exists since there are finitely many flows)

(Recall any flow $\tau \equiv (\sigma_n)_{n\in N} \in \left(\bigtimes_{n \in N} \Sigma^n\right)$

$\bigtimes_{n\in N} \Sigma^n$ is a finite set)

Now, if a player could make a profitable unilateral deviation, then — denoting $\tilde{\tau}$ the flow that results from τ after he deviates — we must have by *

$$E(\tilde{\tau}) < E(\tau^*)$$

But this contradicts that τ^* minimized energy

So τ^* is an NE

- There could be MANY NE

Sequential unilateral deviations lead to ~~se~~ an NE, but which?

That depends on where we start and who we pick in order to make a deviation (~~as we~~ i.e on the sequence selected).

- The NE τ^* which minimizes energy has a special property as we shall see.

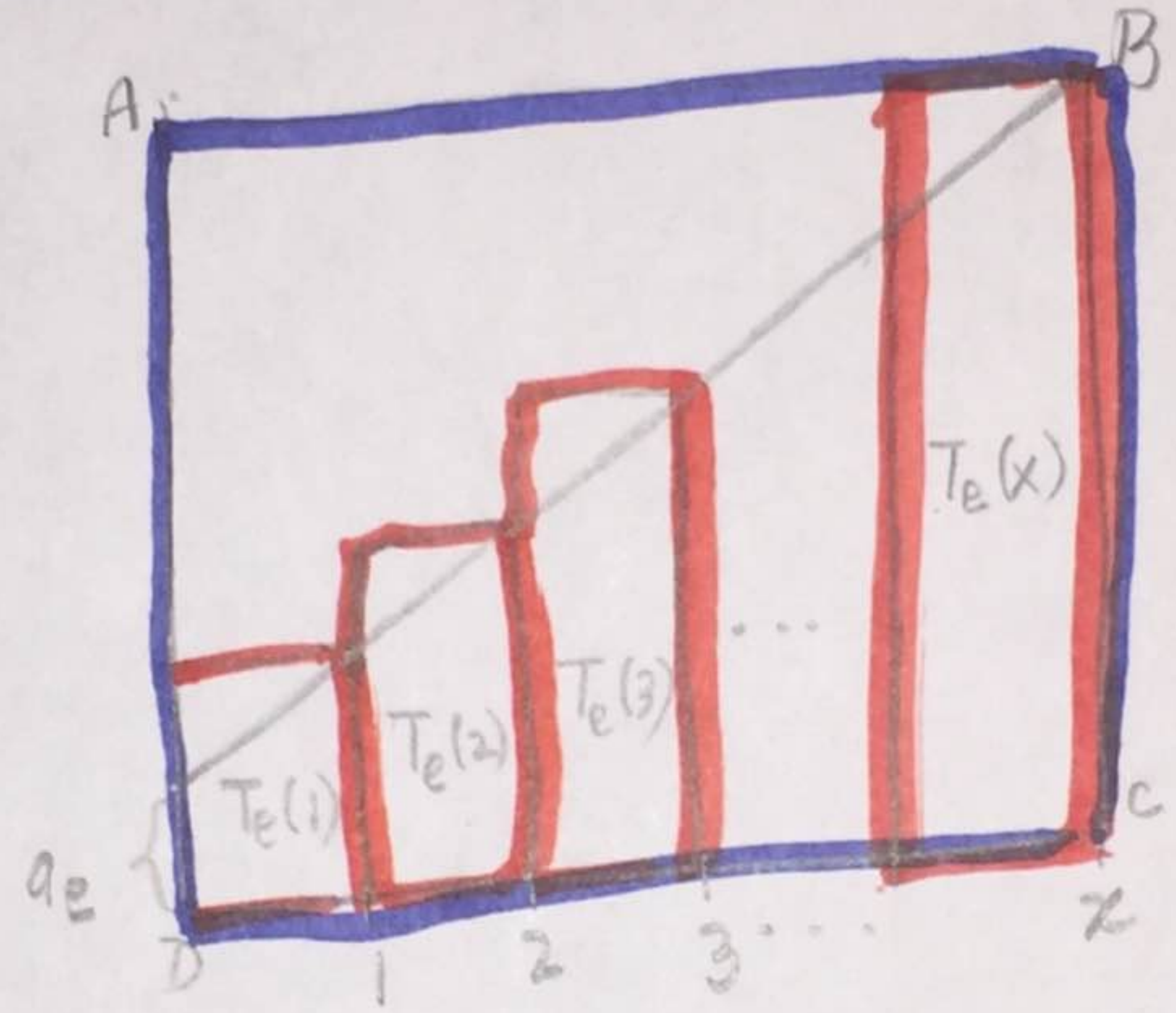
BOUND ON THE INEFFICIENCY OF τ^* WITH LINEAR COSTS

Lemma 1 Assume $T_e(x) = a_e + b_e x$ where $a_e \geq 0, b_e \geq 0$

Then

$$\boxed{\frac{1}{2} x T_e(x) \leq E_e(x) \leq x T_e(x)} \quad **$$

Proof



/ $\equiv$ slope b_e

$E_e(x) = T_e(1) + T_e(2) + T_e(3) + \cdots + T_e(x) =$ + + + ... +

$x\, T_e(x) =$ area of big

(Algebraically: $T_e(1) + T_e(2) + \cdots + T_e(x)$

$$= \underbrace{a_e + b_e}_{T_e(1)} + \underbrace{a_e + 2b_e}_{T_e(2)} + \cdots + \underbrace{a_e + x b_e}_{T_e(x)}$$

$$= x a_e + b_e (1 + 2 + \cdots + x)$$

$$= x\left[a_e + b_e\left(\frac{x+1}{2}\right)\right] \geq \frac{1}{2} x (a_e + b_e x)$$

$$= \frac{1}{2} x T_e(x)$$

Summing $**$ over all $e \in E$

Time = Total Time

$$\boxed{\tfrac{1}{2}\,\text{Time}\,\tau \le E(\tau) \le \text{Time}(\tau)}$$

for <u>any flow τ</u>

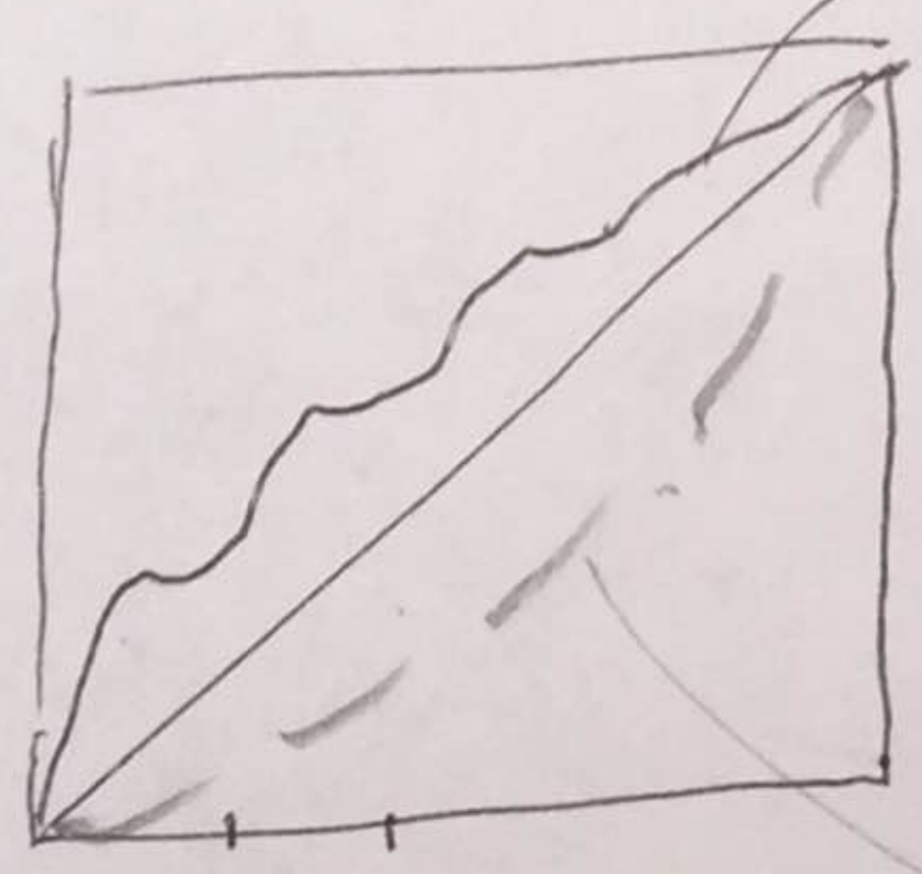
Then $\frac{1}{2}\,\text{Time}\,\tau^* \le E(\tau^*) \le E(\tau) \le \text{Time}(\tau)$ for any flow τ.

$$\Longrightarrow \quad \text{Time}\,\tau^* \le 2\,\text{Time}(\tau) \quad \text{for any flow } \tau$$

$$\Longrightarrow \text{Time}\,\tau^* \le 2 \min_{\tau}(\text{Time}\,\tau)$$

(Remark: Proof also works if

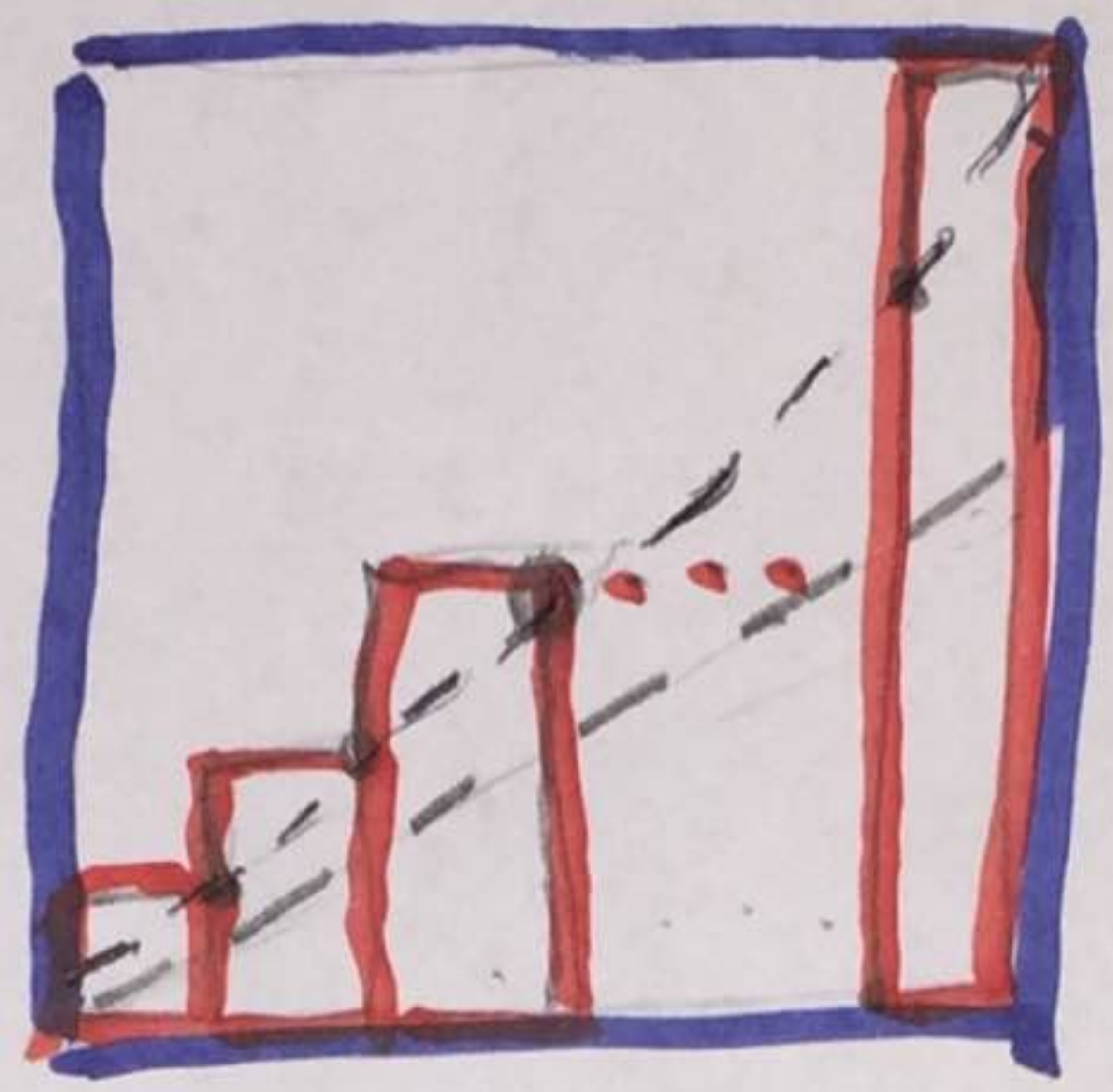
Time

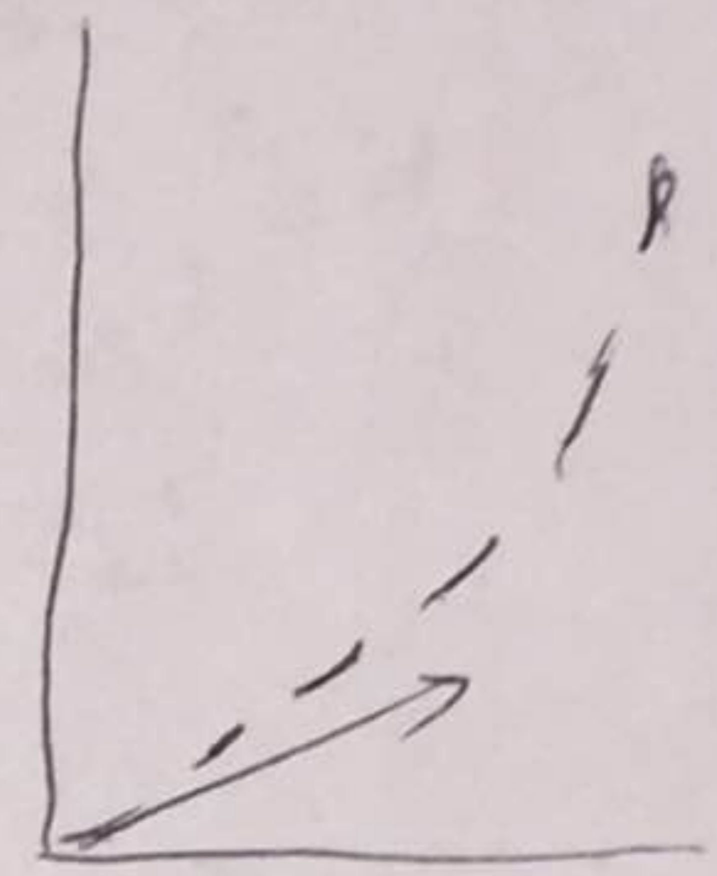


but this is not realistic,

realistic is convex)

REMARK





If we had $\square + \cdots + \square \geq \frac{1}{3} \square$

Then the above argument shows that

$$\text{Time } \tau^* \leq 3 \min_{\tau} (\text{Time } \tau)$$

etc.

"Potential Games" (Monderer & Shapley)

GAME

$$\begin{cases} 1, \ldots, n & \text{(players)} \\ \Sigma^1, \ldots, \Sigma^n & \text{(finite strategy sets)} \\ \Pi^k(\sigma^1, \ldots, \sigma^n) & \text{payoff } f^n \text{ of player } k \end{cases}$$

(here $\sigma_1 \in \Sigma^1, \ldots, \sigma^n \in \Sigma^n$)

The GAME has an (ordinal) potential if there exists a function

$$\Sigma^1 \times \cdots \times \Sigma^n \xrightarrow{P^*} \mathbb{R}$$

such that

$$P^*(\sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_n) - P^*(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) > 0$$

$$\Longleftrightarrow \quad \Pi^k(\sigma_1, \ldots, \sigma_{k-1}, \tilde{\sigma}_k, \sigma_{k+1}, \ldots, \sigma_n) - \Pi^k(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k, \sigma_{k+1}, \ldots, \sigma_n) > 0$$

Note P^* ← no player here

P^* is independent of players

(Same theorems obviously hold)

EXCURSION → (The paper of John Nash on "equilibrium points" $\underset{\text{new}}{=}$ Nash Equilibrium (NE))

Brouwer's Fixed Point Theorem

$S \xrightarrow{f} S$

Assume

- S is compact & convex
- f is continuous

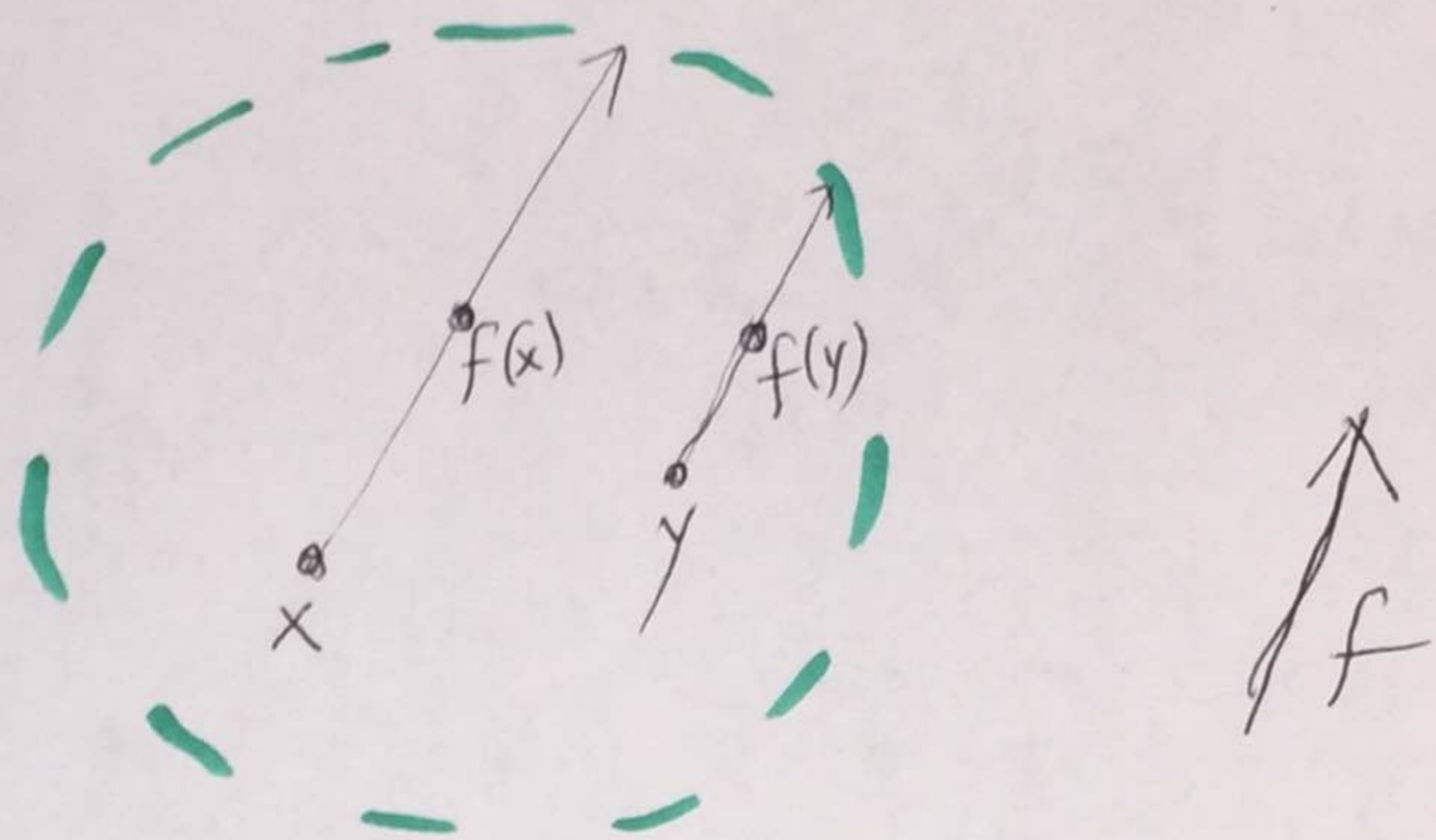
Then there exists $x \in S$ such that $f(x) = x$

REMARK

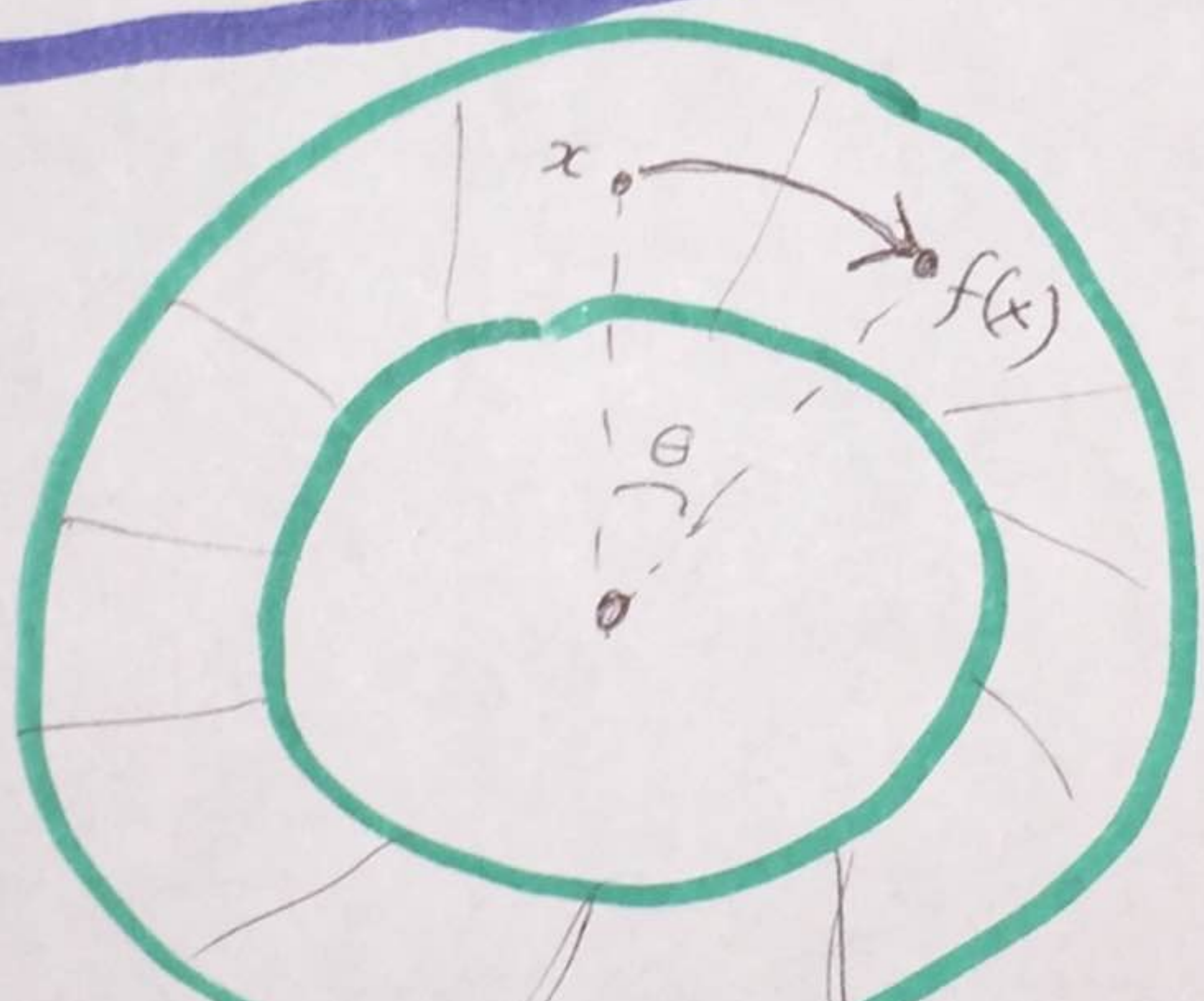
Convexity is an overkill

Suffices that S be "contractible"

[every loop in S can be continuously deformed into a point, without leaving S]



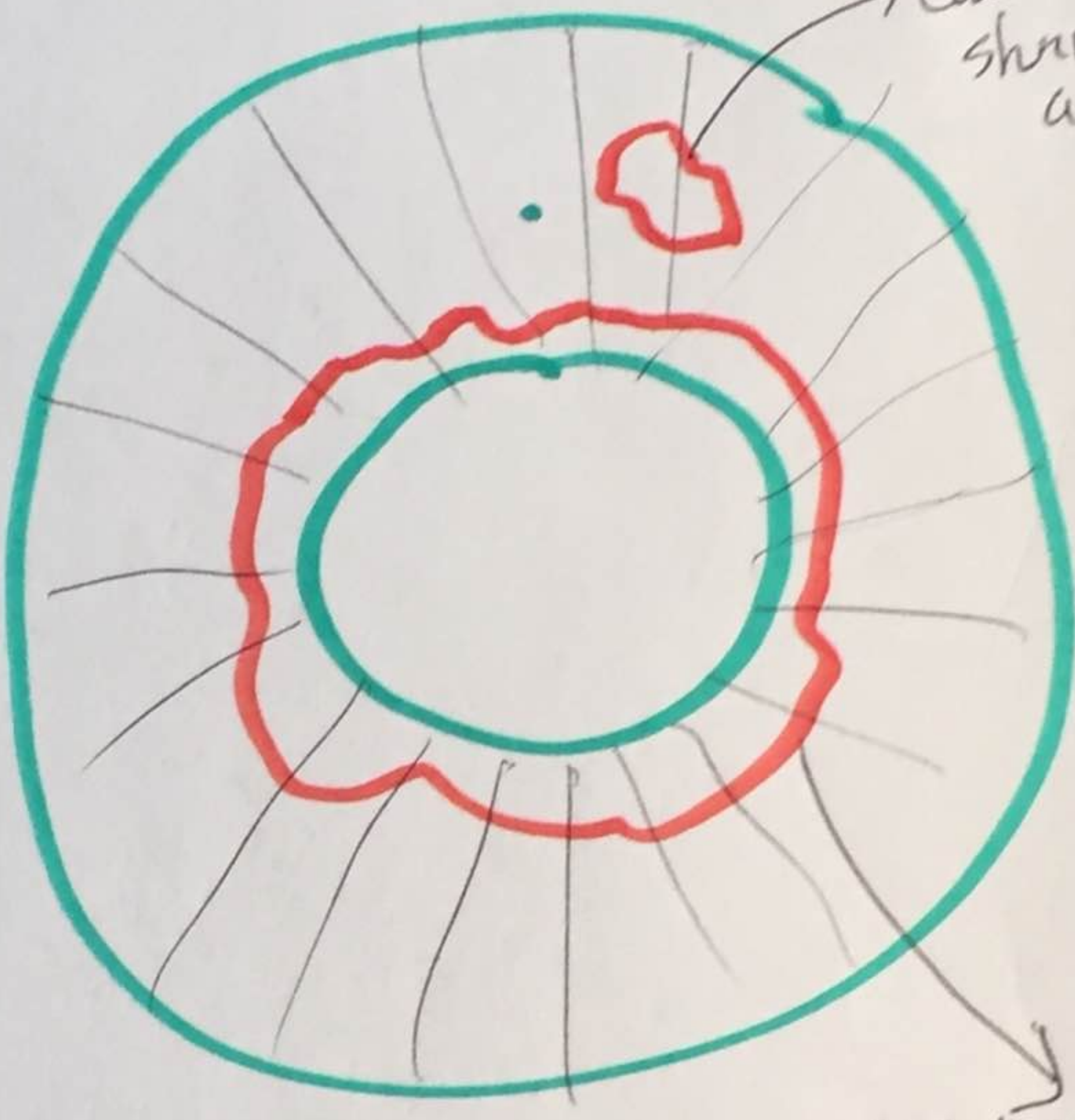
S is convex but not compact
f has no fixed point

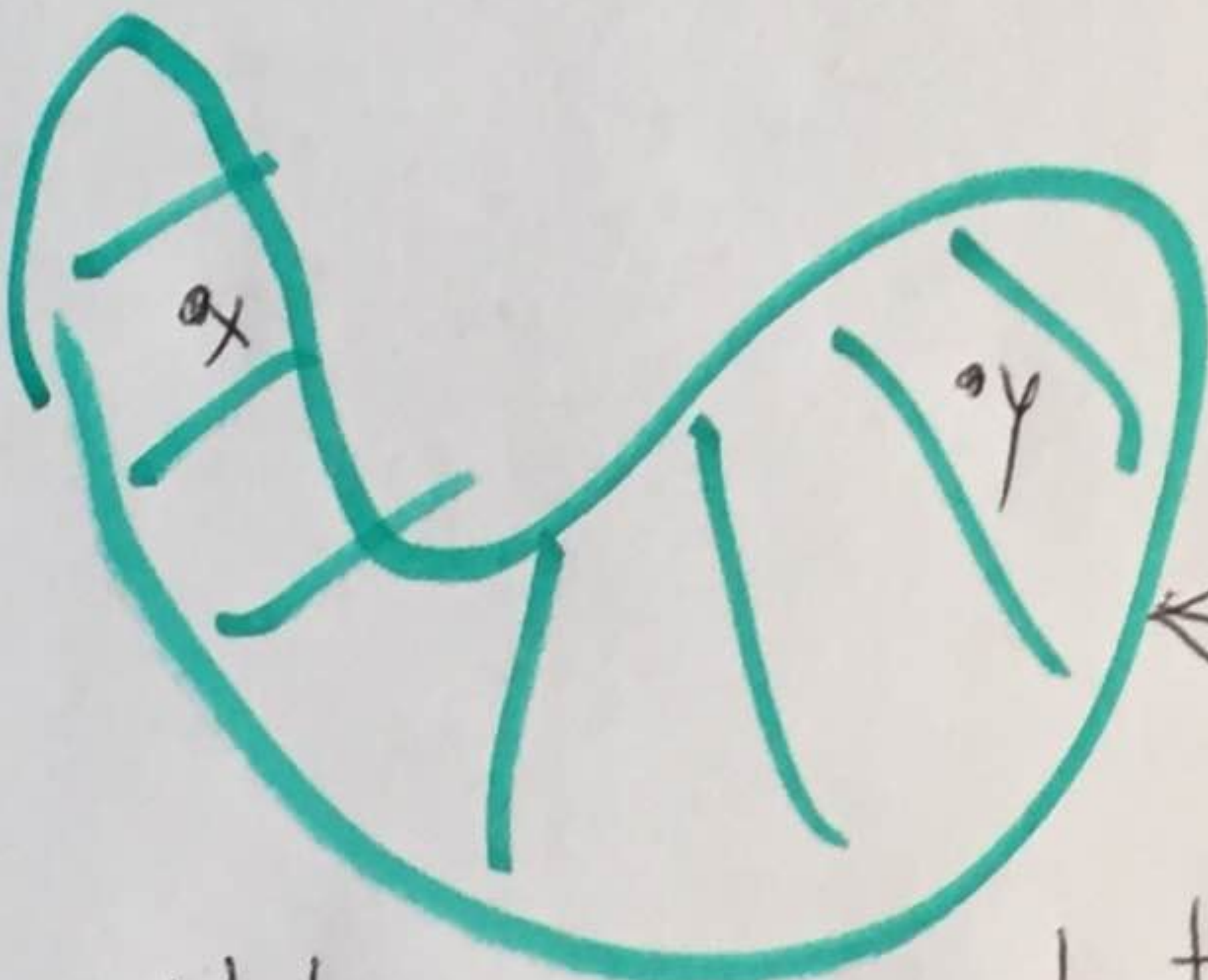


f is a rotation by angle θ

S is compact but not convex
f has no fixed point

Actually





Not convex, but contractible

1 $\qquad\qquad\qquad$ n

Σ^1 $\qquad\qquad\qquad$ Σ^n

compact & convex $\qquad\qquad$ compact & convex

For $1 \le k \le n$, assume

$$\Pi^k(\overbrace{\sigma^1, \ldots, \sigma^{k-1}, \sigma^k, \sigma^{k+1}, \ldots, \sigma^n}^{\sigma})$$

is continuous in all its variables and concave in σ^k (i.e. as we vary $\sigma^k \in \Sigma^k$ holding strategies of all others fixed).

- Define best reply $\beta^k(\sigma) \subset \Sigma^k$ of k

$$\beta^k(\sigma) = \text{set of all } \tilde{\sigma}^k \text{ which maximize } \Pi^k \text{ holding others strategies fixed and varying only } \sigma^k \in \Sigma^k$$

Then
- $\beta^k(\sigma)$ is a nonempty, closed set
- $\beta^k(\sigma)$ is also convex

IF Π^k is not just concave in σ^k but STRICTLY concave, then $\beta^k(\sigma)$ is a singleton.

Assume (provisionally) strict concavity of Π^k in σ^k (for all $1 \le k \le n$)

Then the map

$$\underbrace{(\sigma_1, \dots, \sigma^n)}_{\sigma} \xrightarrow{\beta} \left(\beta^1(\sigma), \dots, \beta^n(\sigma)\right)$$

where $\Sigma = \Sigma^1 \times \dots \times \Sigma^n$

~~from~~ $\Sigma \xrightarrow{\beta} \Sigma$ by Brouwer has a fixed pt $\tilde{\sigma}$

i.e. $\tilde{\sigma}^k = \beta^k(\tilde{\sigma})$ for $k = 1, \dots, n$

which is NE

(NE = set of fixed points)

Now suppose strict concavity fails and we have only concavity

Let $\Sigma^k \xrightarrow{g^k} \mathbb{R}$ be a strictly concave function

Consider the ε-perturbed game in which the payoff $\Pi^k(\sigma)$ is replaced by

$$\Pi_\varepsilon^k(\sigma) = \Pi^k(\sigma^1, \dots, \sigma^{k-1}, \sigma^k, \sigma^{k+1}, \dots, \sigma^n) + \varepsilon g^k(\sigma^k)$$

strictly concave in σ^k

Now ε-perturbed game, has a NE $\tilde{\sigma}_\varepsilon \in \Sigma^1 \times \dots \times \Sigma^n$

Let $\tilde{\sigma} = \lim_{\varepsilon \to 0} \tilde{\sigma}_\varepsilon$ (some cluster point)

Easily checked that $\tilde{\sigma}$ is a NE of the real unperturbed game

John Nash (in the paper I posted)

3 paragraphs in ENGLISH!

did not do this.

He looked at the point-to-set map

$$\underbrace{(\sigma^1, \dots, \sigma^n)}_{\sigma} \xrightarrow{\beta} \underset{\Sigma^1}{\underset{\cap}{\beta^1(\sigma)}} \times \dots \times \underset{\Sigma^n}{\underset{\cap}{\beta^n(\sigma)}}$$

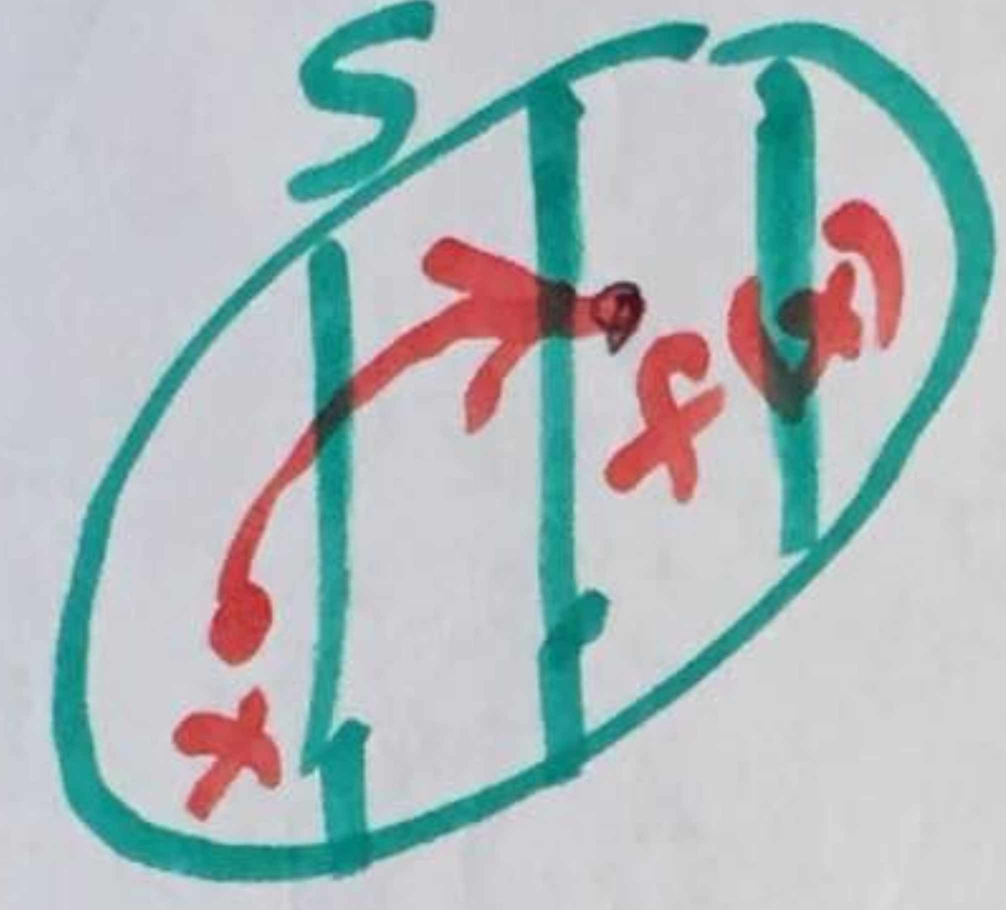
And invoked Kakutani's thm to show that there exists

$(\tilde{\sigma}^1, \dots, \tilde{\sigma}^n)$ such that

$$\tilde{\sigma}^k \in \beta^k(\tilde{\sigma}) \quad \forall\ 1 \le k \le n.$$

EXCURSION → (The paper of John Nash on "equilibrium points" $\xrightarrow[\text{now}]{}$ Nash Equilibrium (NE)

Brouwer's Fixed Point Theorem



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