Lecture Notes of Microeconomics: Equilibrium

Lecturer: Pradeep Dubey

Recorded and Edited by Yan Liu, Not Revised

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Contents

Chapter 1

Pure Exchange Economy

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In this lecture notes, $\mathbb{R}^K_+ = \{x \in \mathbb{R}^K | x_k \geq 0, k = 1, \ldots, K\}$; for $x \in \mathbb{R}^K$, $x \gg 0 \Leftrightarrow x_k > 0$; $x \geq 0 \Leftrightarrow x_k \geq 0$, $\forall k$; $x \gneq 0 \Leftrightarrow x \geq 0$, and $\exists\, k$, *st* $x_k > 0$.

Let $H = \{1, \ldots, H\}$ be a set of households, and $L = \{1, \ldots, L\}$ be a set of commodities, in which labor but not food may be the most important one (naturally $\mathbb{R}^L_+$ represents the commodity space). For each household $h \in H$, the most important two characteristics are preferences and initial endowments. For the former one, we use a utility function $u^h : \mathbb{R}^L_+ \to \mathbb{R}$ to represent the preference for h, and until now no properties are impose to the preferences (i.e., no monotonicity or continuity); for the latter one, use $e^h = (e^h_1, \ldots, e^h_L) \in \mathbb{R}^L_+$ to denote the initial endowment of h.

Put things together, we define a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$.

1.1 Competitive Equilibrium in $\mathscr{E}$

Given a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$, we could discuss the concept of competitive equilibrium of $\mathscr{E}$.

Before given the former definition of competitive equilibrium, we should clarify some issues related to the setup of the model for competitive equilibrium. First, we assume there are markets for each commodity $\ell \in L$. Second, all the commodities are private good and perfect divisible. Third, there is no tax or transaction cost in this model. Forth, households in this model just care about the "prices" in the markets. They don't know how many people are in the markets or what is the total amount for a certain kind of commodity, and in fact they don't need to know these issues. By using the word "price" here, we don't mean to involve money into this model, but on the contrary, there is no money here. We talk about prices, but we don't use this term (in the sense of monetary price). All we concern are exchange rates (relative prices).

Given prices $p = (p_1, \ldots, p_L) \in \mathbb{R}^L \backslash \{0\}$[1], we can always scale the prices to get $\sum_{\ell \in L} p_\ell = 1$. Double all prices in this model doesn't matter anything, as you will see the budget set doesn't change subject to this kind of scaling. However, in real life, this can not be true since all the prices go up but your income remains the same.

We put two more assumptions about initial endowments and preferences in this economy.

1. $e^h \in \mathbb{R}^L_+ \setminus \{0\}$, reads every household has something, and $\sum_{h \in H} e^h \gg 0$, means every named commodity does present in the aggregate.

2. For all $h \in H$ and x, $y \in \mathbb{R}^L_+$, $x \gg y \Rightarrow u^h(x) > u^h(y)$, and $x \geq y \Rightarrow u^h(x) \geq u^h(y)$. This is also called weak monotonicity of utility (preference).

Now we give the formal definition for competitive equilibrium of $\mathscr{E}$.

Definition 1.1.1. *A set of prices and allocations $\langle p, x^1, \ldots, x^H \rangle$ is a competitive equilibrium of pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$ if*

- *Households maximize utility on budget sets.*
  $\forall h \in H$, $x^h \in \text{argmax}_{y \in B^h(p)} u^h(y)$ [2], *where* $B^h(p) \equiv \{y \in \mathbb{R}^L_+ : p \cdot y \leq p \cdot e^h\}$ *is the budget set for* h.

- *Market clear.*
  $\sum_{h \in H}(x^h - e^h) = 0$.

We add three remarks on this definition.

1. From now on, we postulate that price vector $p \in \mathbb{R}^L_+ \setminus \{0\}$ in our following lectures. However, in certain circumstance we could device negative price in order to get an equilibrium. A formal treatment will be left until we discuss the existence of competitive equilibrium.

[1] Generally speaking, if all the commodities are goods (which in fact we have not yet demonstrate the exact meaning of this term), then we could always consider $p \in \mathbb{R}^L_+ \setminus \{0\}$. However, if some commodities are bads, then maybe we need to extent the domain for price vector to get reasonable answers, at least if we want to treat these bads in the same way as goods

[2] For a function $f : Z \subset \mathbb{R}^n \to \mathbb{R}$, define $\text{argmax}\, f(\cdot) = \{y \in Z | f(y) = \max_{t \in Z} f(t)\}$. If the maximum is not achieved, then $\text{argmax}\, f(\cdot)$ is empty.

2. In the definition of budget set, $y \in B^h(p)$ can be interpreted as affordable (final) consumption, and $p \cdot y \leq p \cdot e^h$ can be viewed as first selling everything then buying what you want. Another way of looking at the budget constraint is change the inequality into $\sum_{\ell \in L} p_\ell(y_\ell - e^h_\ell) = p \cdot (y - e^h) \leq 0$, then $y_\ell - e^h_\ell > 0 \Leftrightarrow h$ is buying and $y_\ell - e^h_\ell < 0 \Leftrightarrow h$ is selling. In a sequential way, h can buy things with a total "value" less than what he/she has initially.

3. If we assume the freedom of agents (households) on choosing maximal bundles by themselves, then the equilibrium fails to achieved generally. In fact, we economists choose the maximal bundles for the agents. So we write down $\langle p, x^1, \ldots, x^H\rangle$ *s.t.* ....

1.2 The Core of $\mathscr{E}$

For any subset $S \subset H$ (the number of elements in this set is also denoted as S), we say that $(x^h)_S \in \mathbb{R}^{LS}_+$ [3] is an S-allocation, if $\sum_{h\in S} x^h \leq \sum_{h \in S} e^h$. And H-allocation is sometimes just called an allocation.

Definition 1.2.1. *An H-allocation $(x^h)_{h\in H}$ in $\mathscr{E} \equiv (e^h,\, u^h)_{h\in H}$ is called a core allocation of $\mathscr{E}$, if $\nexists$ an S-allocation $(y^h)_{h\in S}$ $(\forall S \subset H)$, such that*

$$\begin{aligned} &u^h(y^h) \geq u^h(x^h) \quad \forall h \in S, \\ &u^h(y^h) > u^h(x^h) \quad \textit{for at least one } h \in S. \end{aligned}$$

Remark: when $S = H$, the definition above meets with the definition of *Pareto Optimal*, and thus a core allocation is automatically Pareto optimal. Somehow, the converse is not true. Consider an allocation of giving all the goods to one individual and everyone else gets nothing. This allocation is Pareto optimal, however, it is not a core allocation.

The following theorem is left as a quasi-homework which will be discussed in next class. Note, the same assumptions about the preferences and endowments of $\mathscr{E}$ will be hold on as in section 2.

Theorem 1.2.1. *Let $\langle p, x^1, \ldots, x^H\rangle$ be a competitive equilibrium of $\mathscr{E} \equiv (e^h,\, u^h)_{h\in H}$, then $\langle x^1, \ldots, x^H\rangle$ is a core allocation of $\mathscr{E}$.*

Definition 1.2.2. *$\langle x^1, \ldots, x^H\rangle$ is a competitive allocation of $\mathscr{E}$, if $\exists$ p s.t. $\langle p, x^1, \ldots, x^H\rangle$ is a competitive equilibrium of $\mathscr{E}$.*

With the definition, we give the following result as corollary of above theorem.

[3] Here $(x^h)_S$ can be viewed as a vector in $\mathbb{R}^{LS}$.

Corollary 1.2.2. *Every competitive allocation is Pareto optimal.*

► Feb.3, 2010

We will prove the theorem 1.2.1 today. Let's just make the assumption of weak monotonicity of utility to see whether it's enough for the proof.

Proof. Let $\langle p, x^1, \ldots, x^H \rangle$ be a C.E. of $\mathscr{E}$. Suppose conversely the competitive allocation is not a core allocation, then $\exists$ an S-allocation $(y^h)_{h\in S}$ s.t. $u^h(y^h) \geq u^h(x^h)$ for any $h \in S$ with some h the strict inequality holds.

First, we claim that $p \cdot y^h \geq p \cdot e^h$, $\forall h \in S$. If not, there exists a $h \in S$ s.t. $p \cdot y^h < p \cdot e^h$. Then we can find $\epsilon > 0$ small enough, which satisfies $y^h \ll y^h_\epsilon \triangleq y^h + (\epsilon, \ldots, \epsilon)$ and $p \cdot y^h_\epsilon < p \cdot e^h$, i.e. $y^h_\epsilon \in B^h(p)$. Since $y^h_\epsilon \gg y^h$ and by the assumption of weak monotonicity, we have $u^h(y^h_\epsilon) > u^h(y^h)$. By the definition of S-allocation, we have $u^h(y^h) \geq u^h(x^h)$. Thus $u^h(y^h_\epsilon) > u^h(x^h)$, but this contradicts with the fact that x^h maximizes h's utility in $B^h(p)$ since y^h_ϵ is also in the budget set. Therefore, the claim must be true.

Second, we claim that $\exists\ h$ s.t $p \cdot y^h > p \cdot e^h$. By definition, there must be at least one $h \in S$ that $u^h(y^h) > u^h(x^h)$. Fix this h, if $p \cdot y^h \leq p \cdot e^h$, then $y^h \in B^h(p)$. But this leads to a contradiction to the fact that x^h is a utility maximizer in $B^h(p)$. This proves the claim.

Combine the two claims, we must have $\sum_{h\in S} p \cdot y^h > \sum_{h \in S} p \cdot e^h$. However, since $(y^h)_{h\in S}$ is an S-allocation, we have $\sum_{h\in S} y^h \leq \sum_{h\in S} e^h$, and multiply each side by p[4], we get $\sum_{h\in S} p \cdot y^h \leq \sum_{h\in S} p \cdot e^h$. This contradiction proves our initial assumption is not true and thus complete the proof. ■

Remark: as shown above, we only need following property of preference to be hold instead of the whole properties of weak monotonicity (if we postulate price vector is non-negative),

$$x \gg y \Rightarrow u^h(x) > u^h(y).$$

Now, let us think about a five sentences proof for a little modified version of this theorem, which known as the *Debreu's proof.*[5]

For this, we impose strong monotonicity (i.e. $x \gneq y \Rightarrow u^h(x) > u^h(y)$) and continuity[6] to the preferences, and notice a fact under these assumptions that if an S-allocation violates the conditions for a competitive allocation to be an core allocation, then the agent who has

[4] Recall that we postulate that $p \geq 0$

[5] Somebody said Debreu demonstrated this result and gave an five sentences proof in his Nobel Prize speech. However, there isn't such an argument in neither Debreu's speech nor his prize lecture.

[6] Continuity is in fact necessary, which professor didn't mention in the class.

a strictly higher utility in S-allocation than in competitive allocation can move something he has to other persons in the S-allocation and make all people in S have a strictly higher utility.[7] And from the proof of theorem 1.2.1 we know this immediately means that everyone has a higher total value of commodities in this S-allocation than in competitive allocation.

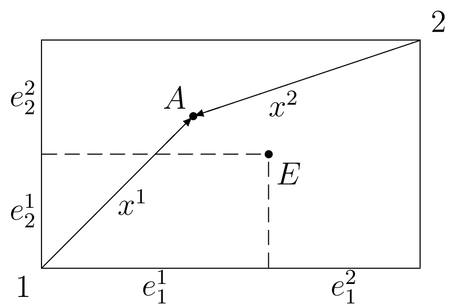
With this fact, the *Debreu's proof* can be showed as following: imagine a blue C.E. and a red S in which everyone is strictly better than in blue C.E.. This will be impossible, since if so, the total value of this S-allocation will exceed the aggregate value of S in the C.E..

The term of "core" origins from von Neumann and Morgenstern (1947). They defined "stable set", something in the center geometrically, like the core of an apple. However the idea of core allocation dates back to Edgeworth. In a 2 people economy, rationality of no worse than endowments they initially have and Pareto optimal is equivalent to core allocation.

1.3 Edgeworth Box

Consider a the simplest pure exchange economy with two households and two commodities, saying, let $H = \{1, 2\}$ and $L = \{1, 2\}$, together with the utility $u^1(x_1^1, x_2^1) \quad u^2(x_1^2, x_2^2)$ and initial endowments $e^1 = (e_1^1, e_2^1)$, $e^2 = (e_1^2, e_2^2)$. Let $T = [0, e_1^1 + e_1^2] \times [0, e_2^1 + e_2^2]$ be the total endowment set for this economy.

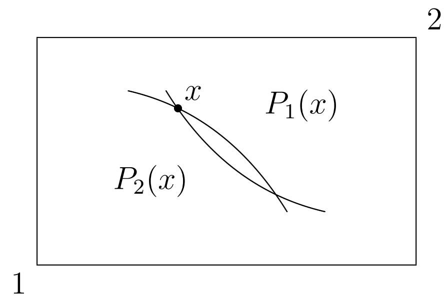
We can use a rectangular box to express this economy, as the following figure, in particular, this rectangular represents T.



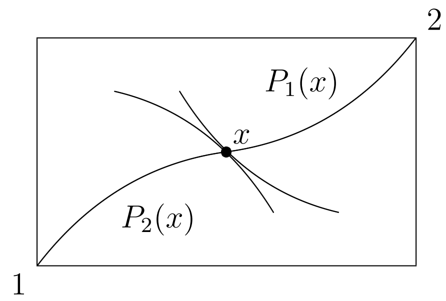
In this figure, E represents the initial endowment of two people, and A represents a possible allocation for them within in the total endowment they have. We can also draw the indifference curves for both of them in this same box, with which we could show the relationship among Pareto optimal, *contract curve*, competitive allocation and core allocation.

[7] If for some $h \in S$, $u^h(y^h) > u^h(x^h)$, then use the same argument as in the proof of theorem 1.2.1, we have $p \cdot y^h > p \cdot e^h$. And this implies h could reduce some commodities he has by a sufficiently small amount and give them fairly to everybody else in S, while his utility is still higher than $u^h(x^h)$ (by continuity). Now, everyone else in S will achieve a higher utility because the strong monotonicity of preference.

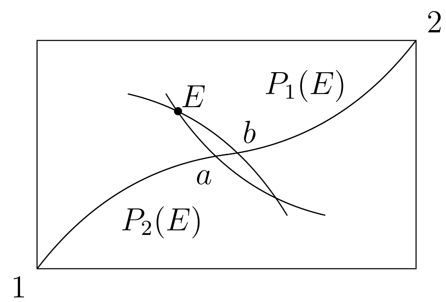
Let us consider the simplest case, namely, the utility functions of H are strictly quasi-concave, strictly increasing and continuously differentiable. Let $P_i(x) = \{y \in T | u^i(y) \geq u^i(x)\}$ be the set that $i \in H$ will be at least as happy as having x, and now, we can observe from the following figure that if the intersection of $P_1(x)$ and $P_2(x)$ has a non-empty interior, then the allocation x must not be a Pareto optimal.



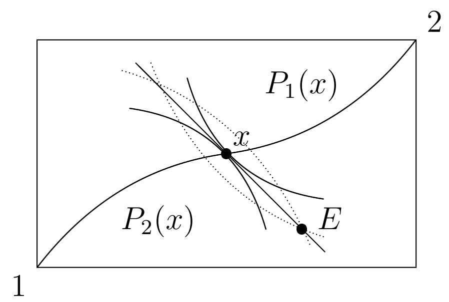
Thus, Pareto optimal can be achieved only in the case that two indifference curves intersect at a single point, i.e. they tangent each other at this point. Now, imagine you plot all these points in the box and then you can get a curve, where all the points are Pareto optimal. This curve is called *contract curve*, as shown in the following figure.



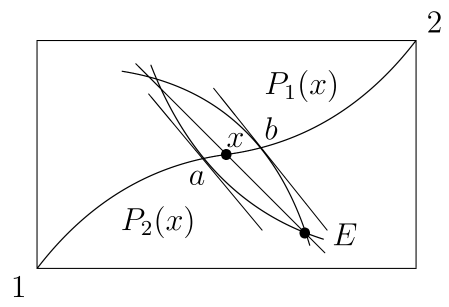
Given initial endowments E, we can discuss core allocation in this economy. Notice in this case the possible S-allocations consist of only sets, two individuals separately and H as a whole, so once an allocation is Pareto optimal and each individual is at least as happy as with his/her own endowment, then it must be a core allocation. It can be easily shown in a graph that a certain part of contract curve consists the set of all core allocation with this E. In the following figure, core allocation set is the contract curve between a and b.



Now, consider the competitive allocation in this economy. Given E, a C.E. $\langle p, x\rangle$ can be viewed geometrically as a separating plain (in our 2 people 2 commodities economy, it's a separating line) which go through the endowments point E, and the allocation point x is the tangent point both by $P_i(x)$ and the separating plain, while the price vector is the normal vector of the separating plain. From this point of view, it is also obvious that competitive allocation x is pareto optimal (at least in this 2 by 2 economy, since x is the intersection point by separating plain and contract curve), thus x is also a core allocation. All above description can be shown in the following Edgeworth box.



The discussion above provides an argument for *the first welfare theorem* (every C.E. is Pareto optimal). Moreover, using a similar argument in Edgeworth box, we can show the existence of C.E. in this 2 by 2 economy. Given an endowments point E, we can draw two indifference curves for 1 and 2. Let a and b be two intersection points with contract curve. Notice the preferences are concave (utility functions are quasi-concave), the tangent line of 1's indifference curve at a is on the left side of E, and for 2 it's on the right side. By the continuous differentiability assumption of utility functions, the separating line between a and b will change continuously, and thus one of them must go through E.



Chapter 2

Nash Equilibrium

▶ Feb.8, 2010

2.1 N.E. in A Game

Let $N \equiv \{1, \dots, N\}$ be a set of players, and for each $n \in N$, $S^n \equiv$ strategy set of player n. Put $S \equiv S^1 \times \cdots \times S^N$, and define $\pi^n : S \to \mathbb{R}$ as payoff function for n. Together, we call $\Gamma = (S^n, \pi^n)_{n \in N}$ to be a game.

Given $s \equiv (s^1, \dots, s^N) \in S$ and $t \in S^n$, denote $(s|_n t) \equiv (s^1, \dots, s^{n-1}, t, s^{n+1}, \dots, s^N) \in S$ to be *unilateral deviation by player n to t*.

Definition 2.1.1. *$\beta^n(s) = \mathrm{argmax}_{t \in S^n} \pi^n(s|_n t)$ is called the best reply correspondence of n.*

Definition 2.1.2. *$s = (s^1, \dots, s^N)$ is a Nash equilibrium (N.E.) of $\Gamma = (S^n, \pi^n)_{n \in N}$ if $s^n \in \beta^n(s)$ $\forall n \in N$.*

Definition 2.1.3. *Let best reply correspondence $\beta : S \rightrightarrows S$ of Γ be defined in the following way, $\forall s \in S$*

$$\begin{array}{ccccccc} \beta(s) \equiv & \beta^1(s) & \times & \cdots & \times & \beta^N(s) & \\ & \cap & & \cdots & & \cap & \\ & S^1 & \times & \cdots & \times & S^N & \equiv S. \end{array}$$

Where " $\rightrightarrows$ " represents correspondence.

With this notation, s is a *N.E.* iff $s \in \beta(s)$.

In order to prove the existence of *N.E.*, we need following mathematical preliminary.

We have talked about correspondences several times, now let's give a formal definition.[1]

[1] This is not the most general definition for correspondence, however, it's enough for our discussion in this course within the restriction of Euclidean space.

Definition 2.1.4. *$\varphi : X \subset \mathbb{R}^m \rightrightarrows Y \subset \mathbb{R}^n$ is called a correspondence from X to $\mathbb{R}^n$ if $\forall x \in X$, $\varphi(x)$ is a non-empty subset of $\mathbb{R}^n$.*

Since a function is a point-to-point relationship, apparently correspondence is a generalization of function.

We want correspondences to have some kind of continuity. It turns out that following definitions are most useful for economic application.[2] There are several steps for the definition of continuous correspondence.

Definition 2.1.5 (Graph)**.** *Let $\varphi : X \rightrightarrows Y$ be a correspondence, $G_\varphi = \{(x, y) \in X \times Y | y \in \varphi(x)\}$ is called the graph of φ.*

Definition 2.1.6 (u.s.c)**.** *φ is upper semi-continuous [3] (u.s.c) in X if G_φ is closed. This is also equivalent to either one of the following statements:*

- *φ is u.s.c at $x \in X$, if $(x_n, y_n) \to (x, y) \in X \times Y$ and $(x_n, y_n) \in G_\varphi$, then $(x, y) \in G_\varphi$; φ is u.s.c in X, if it's u.s.c $\forall\, x \in X$.*
- *φ is u.s.c at $x \in X$, if $x_n \to x$, $y_n \to y \in Y$ and $y_n \in \varphi(x_n)$, then $y \in \varphi(x)$; φ is u.s.c in X, if it's u.s.c $\forall\, x \in X$.*

We will also use the term *upper hemi-continuous* (u.h.c) with the same meaning as u.s.c.

Definition 2.1.7 (l.s.c)**.** *φ is lower semi-continuous (l.s.c) at $x \in X$ if $\forall$ sequence $x_n \to x$ and $\forall\, y \in \varphi(x)$, $\exists\, y_n \in \varphi(x_n)$ $\forall n$ s.t. $y_n \to y$; if φ is l.s.c at every point of X, the it's l.s.c in X.*

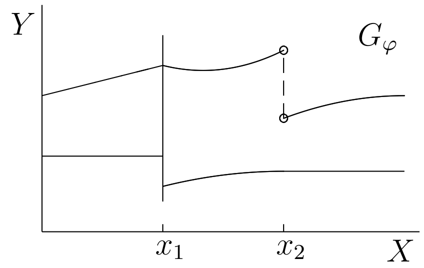
We will also use the term *lower hemi-continuous* (l.h.c) with the same meaning as l.s.c.

Definition 2.1.8. *φ is continuous if it is both u.s.c and l.s.c.*

Here is an illustration for above definitions. φ is u.s.c at x_1 but not l.s.c, and on the contrary, l.s.c at x_2 but not u.s.c.

[2] The following definitions for continuity are also quite straight forward from a stand point of viewing correspondence as function from X to the power set of $\mathbb{R}^n$, and under certain conditions, a continuous correspondence defined as following is indeed a continuous function from X to some subset of the power set of $\mathbb{R}^n$.

[3] In a general context, the definition here is for *closed correspondence*. However, since we mainly consider correspondences of which the imagine space Y is compact, closed correspondences are equivalent to u.s.c correspondence (in the general definition for u.s.c).



The following propositions show some basic properties and applications of correspondence. In particular, theorem 2.1.1 characterizes the fundamental property of optimization with constraints using the language of correspondence. It is also called *Berge's maximal theorem.*

Theorem 2.1.1 (Berge)**.** *Let $f : Y \to \mathbb{R}$ be a continuous function, and $\varphi : X \rightrightarrows Y$ be continuous and compact valued at $x \in X$. Then $\eta : X \rightrightarrows Y$ is a u.s.c and compact valued correspondence at x where $\eta(x) = \operatorname{argmax}_{y\in\varphi(x)} f(y)$.*

Proof. (*This proof is supplemented by Yan Liu.*)

First of all, since $\varphi(x)$ is compact and $f(\cdot)$ is continuous on $\varphi(x)$, there exists $y \in \varphi(x)$ that maximizes $f(\cdot)$ by Weierstrass theorem. Thus $\beta(x)$ is non-empty.

Let $(x_n, y_n) \to (x, y) \in (X \times Y)$ where $y_n \in \beta(x_n)\ \forall\, n$, we want to show that $y \in \beta(x)$. First, by u.h.c of φ and $y_n \in \beta(x_n) \subset \varphi(x_n)$, $y \in \varphi(x)$. Second, by l.h.c of *varphi*, $\forall\, z \in \varphi(x)$, there exists a sequence of $\{z_n\}$ that goes to z and $z_n \in \varphi(x_n)$. Since $y_n \in \beta(x_n)$ maximizes $f(\cdot)$ in $\varphi(x_n)$, it follows $f(y_n) \geq f(z_n)\ \forall\, n$. Then we have $f(y) = \lim_n f(y_n) \leq \lim_n f(z_n) = f(z)$ from the continuity of $f(\cdot)$. Moreover, this implies $y \in \beta(x)$.

In order to prove $\beta(x)$ is compact, we only need to show that $\beta(x)$ is closed since it's a subset of compact set $\varphi(x)$. Let $\{y_m\} \in \beta(x)$ and $\lim_m y_m = y$. By definition, $f(y_m) = \max_{t\in\varphi(x)} f(t) \equiv M\ \forall\, m$. Obviously, $f(y) = M$, hence $y \in \beta(x)$. Therefore $\beta(x)$ is compact. ■

A more general form of theorem 2.1.1 which can often be seen in the literature is stated as a corollary.[4]

Corollary 2.1.2. *Let $f : X \times Y \to \mathbb{R}$ be a continuous function, and $\varphi : X \rightrightarrows Y$ be continuous and compact valued. Then*

- *$\eta : X \rightrightarrows Y$ is a u.s.c correspondence where $\eta(x) = \operatorname{argmax}_{y\in\varphi(x)} f(x, y)$;*

[4] In fact, this is the primitive form of *Berge's maximal theorem.* However the essential idea and the method of the proof in both theorem are exactly the same. Just for convenience of referring, we just label the first one as Berge's theorem.

- *the maximal function* $g: X \to \mathbb{R}$ *is continuous, where* $g(x) = \max_{y \in \varphi(x)} f(x, y)$.

Lemma 2.1.3. *Finite product of u.s.c (l.s.c) correspondences is also a u.s.c (l.s.c) correspondence. Product of correspondences is defined as following.* $\varphi_k : X \rightrightarrows Y_k$, $k = 1, \dots, K$, *then* $\varphi : X \rightrightarrows \prod_{k=1}^{K} Y_k$, *where* $\varphi(x) = \prod_{k=1}^{K} \varphi_k(x)$.

Now we state two useful fixed point theorems. The first one is Brouwer's fixed point theorem.

Theorem (Brouwer)**.** *Let* D *be a compact convex* [5] *set in* $\mathbb{R}^k$, *and let* $f : D \to D$ *be a continuous function. Then* $\exists\, x \in D$, *s.t.* $f(x) = x$.

From this theorem we can easily prove the following Kakutani's fixed point theorem, which plays a key role in establishing the existence of equilibria.

Theorem 2.1.4 (Kakutani)**.** *Let* $\varphi : X \rightrightarrows X$ *be u.s.c and convex valued, where* X *is a compact convex set in Euclidean space, then* $\exists\, x \in X$ *s.t.* $x \in \varphi(x)$.

With these preparation, we now state and prove the existence of Nash equilibrium.

Theorem 2.1.5 (Nash Equilibrium)**.** *Let* $\Gamma = (S^n, \pi^n)_{n \in N}$ *be a game satisfying*

- S^n *is a compact convex set in* $\mathbb{R}^{k(n)}$ [6] *for each* $n \in N$.
- $\pi^n : S^n \to \mathbb{R}$ *is continuous for each* $n \in N$.
- $\pi^n(s|_n t)$ *is quasi-concave in* $t \in S^n$, $\forall\, s \in S$ *and for each* $n \in N$.

Then, there exists a Nash equilibrium.

Proof. By theorem 2.1.1 $\beta^n(s)$ is u.s.c in S. And by lemma 2.1.3, $\beta(s) = \beta^1(s) \times \cdots \times \beta^N(s)$ is also u.s.c. From quasi-concavity of $\pi^n(S|_n t)$, β is convex-valued. Therefore, by theorem 2.1.4, $\exists\, s^*$ s.t. $s^* \in \beta(s^*)$. ■

► Feb.15, 2010

[5] Mathematically, this condition is not necessary. A mathematically good condition should be there's no "hole" in the space, or more accurately, it should be a space which is homeomorphical to a sphere. However, convexity is somehow a common property in economic models, and particularly, preferences of players in a game can be easily and reasonably formulated in a way of satisfying convexity, with which mathematical tools can be conveniently applied. So we keep this convex condition, which is also an analogous case for Kakutani's theorem.

[6] Strategy space of different players may have different dimension.

2.2 N.E. in A Generalized Game

In order to prove the existence of competitive equilibrium, we introduce *generalized game* (pseudo game) in this section.

Let $N \equiv \{1, \dots, N\}$ be a set of players. For each $n \in N$, $S^n \equiv$ (all possible) strategy set, and assume S^n to be a compact convex subset of Euclidean space. $S \equiv S^1 \times \cdots \times S^N$, and define $\pi^n : S \to \mathbb{R}$ as payoff function for n. Let $\varphi_n : S \rightrightarrows S^n$ be a correspondence where $\varphi_n(S) \subset S^n$[7] for each $n \in N$. Define $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ as a generalized game.

Let $\beta^n : S \rightrightarrows S^n$ be *best reply correspondence* of n for $\widetilde{\Gamma}$, where

$$\beta^n(s) = \operatorname*{argmax}_{t \in \varphi^n(s)} \pi^n(s|_n t).$$

And let $\beta = \beta^1 \times \cdots \times S^N$. Define $s \in S$ to be a *Nash equilibrium* for $\widetilde{\Gamma}$ if $s \in \beta(s)$.

The following theorem plays a fundamental role in the approach employed by Debreu and Arrow to prove the existence of competitive equilibrium.

Theorem 2.2.1 (N.E. of Generalized Game)**.** *Let* $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ *be a generalized game.If* $\forall\, n$*,* π^n *is continuous in* s *and quasi-concave in* s^n*, and* φ_n *is continuous, compact and convex valued, then there exists a N.E of* $\widetilde{\Gamma}$*.*

Proof. The proof of this theorem is exactly the same as theorem 2.1.5. ■

[7] If $\varphi_n(S) = S^n$, then such a generalized game is a real game.

Chapter 3

Existence of C.E. in Pure Exchange Economy

3.1 With Positive Endowment Assumption

In this section we prove the existence of competitive equilibrium in a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h\in H}$. The method used here, which involves generalized game, is due to Arrow and Debreu (1954). And at this moment, we only consider somehow the standard conditions among which strictly positive endowments are assumed, and latter on, we'll see how this condition can be weakened.

Theorem 3.1.1. *Let $\mathscr{E} = (e^h, u^h)_{h\in H}$ be an economy where $e^h \in \mathbb{R}^L_+$ and $u^h : \mathbb{R}^L_+ \to \mathbb{R}\ \forall\, h$. If*

- *$e^h \gg 0$, $\forall\, h$;*
- *u^h is continuous, quasi-concave and weakly monotonic*[1], *$\forall\, h$;*
- *for each $\ell \in L$, there is some h who likes ℓ, which means $u^h(x) > u^h(y)$, if $x \geq y$ and $x_\ell > y_\ell$.*

Then, there exists a competitive equilibrium of $\mathscr{E}$.

Proof. Let $\triangle$ = price simplex $\equiv \{p \in \mathbb{R}^L_+ | \sum_{\ell\in L} p_\ell = 1\}$. Fix $M > \max_{\ell\in L} \sum_{h\in H} e^h_\ell$, and let $\square = \{x \in \mathbb{R}^L_+ | x_j \leq M\}$. Define strategy set $S = \triangle \times \underbrace{\square \times \cdots \times \square}_{H}$, which is obviously a compact and convex set in Euclidean space.

[1] See the definition of weak monotonicity on page 6.

View H households to be H players with payoff functions

$$\pi^h(p, x^1, \ldots, x^h, \ldots, x^H) = u^h(x^h), \quad x^h \in \Box.$$

And imagine there is one more player pr called *price player* with following payoff function

$$\pi^{pr}(p, x^1, \ldots, x^H) = p \cdot (\sum x^h - \sum e^h), \quad p \in \triangle.$$

For each $h \in H$, define a correspondence $\varphi^h : S \rightrightarrows \Box$ as following,

$$\varphi^h(p, x^1, \ldots, x^H) = B^h(p) \cap \Box \equiv \bar{B}^h(p).$$

We can prove such a correspondence is continuous[2] (See lemma 3.1.2 for the proof).

Moreover, define $\varphi^{pr} : S \rightrightarrows \triangle$ for pr as $\varphi^{pr}(p, x^1, \ldots, x^H) = \triangle$, which is a constant valued correspondence, thus it is obviously continuous.

Now let $N = \{1, \ldots, H, pr\}$, $S^h = \Box\ \forall h$ and $S^{pr} = \triangle$. Then $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ is a generalized game satisfying the conditions of theorem 2.2.1, thus a N.E. $s = \langle p, x^1, \ldots, x^H \rangle \in S$ exists. We will verify this s is also a competitive equilibrium for $\mathscr{E}$.

Step 1: $\sum x^h \leq \sum e^h$.

$\forall h$, since $x^h \in \bar{B}^h(p) \subset B^h(p)$, we have $p \cdot x^h \leq p \cdot e^h$. So, adding over h, then

$$p \cdot (\sum x^h - \sum e^h) \leq 0. \tag{3.1}$$

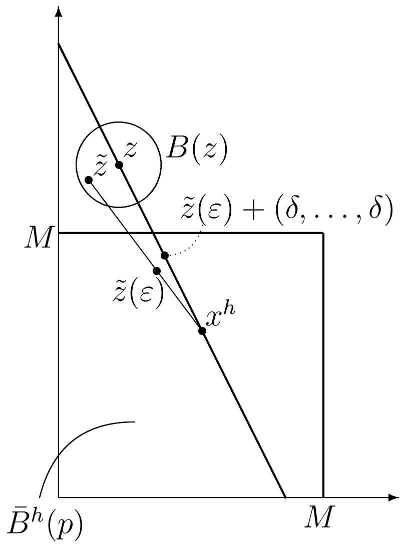
But if $\sum x^h_\ell - \sum e^h_\ell > 0$ for some ℓ. By choosing

$$p = 1_\ell \equiv (0, \ldots, 0, \underset{\ell^{\text{th}}}{\underline{1}}, 0, \ldots, 0) \in \triangle,$$

the price player can get positive payoff, contradicting (3.1) that he gets non-positive payoff in the N.E.

Step 2: *$\forall h$, x^h maximizes u^h on $B^h(p)$, not just on $\bar{B}^h(p)$.*

[2] In order to prove φ^h is l.h.c, the first condition in the theorem is necessary. For u.h.c, that is not necessary. By the way, it should be noticed that Prof. Dubey made a mistake in stating theorem 2.1.1, that the correspondence φ should also be u.h.c besides being l.h.c.



The main idea of the proof in this section is showed in above picture.

First, by step 1, we have

$$x^h_\ell < M \quad \forall\, \ell \in L. \tag{3.2}$$

Second, suppose $\exists\, z \in B^h(p) \setminus \Box$ s.t. $u^h(z) > u^h(x^h)$. By continuity of $u^h(\cdot)$, there exists a small open ball $B(z)$ s.t. $\forall\, y \in B(z)\ u^h(y) > u^h(x^h)$. Choose one point $\tilde{z} \in B(z)$ with $p \cdot \tilde{z} < p \cdot e^h$. By (3.2), x^h will never reach the bound of $\Box$, thus $\exists\, \varepsilon > 0$ small enough satisfying $\tilde{z}(\varepsilon) \equiv (1-\varepsilon)x^h + \varepsilon\tilde{z} \in \bar{B}^h(p)$, and $\tilde{z}(\varepsilon)$ doesn't reach the bound of $\Box$ also. Therefor $\exists\, \delta > 0$ small enough s.t. $z^* \equiv \tilde{z}(\varepsilon) + (\delta,\ldots,\delta) \in \bar{B}^h(p)$, and then by weak monotonicity assumption $u^h(z^*) > u^h(\tilde{z}(\varepsilon))$. Since quasi-concavity of $u^h(\cdot)$ implies $u^h(\tilde{z}(\varepsilon)) \geq \min\{u^h(\tilde{z}), u^h(x^h)\} \geq u^h(x^h)$, we have $u^h(z^*) > u^h(x^h)$, contradicting that x^h maximizes $u^h(\cdot)$ in $\bar{B}^h(p)$.

Step 3: $p \gg 0$.

Suppose $p_\ell = 0$ for some $\ell \in L$. By the third assumption, there is a h who likes ℓ. But then $x^h + 1_\ell \in B^h(p)$, and by weak monotonicity there is $u^h(x^h + 1_\ell) > u^h(x^h)$, contradicting that x^h maximizes $u^h(\cdot)$ on $B^h(p)$.

Step 4: $p \cdot x^h = p \cdot e^h$, $\forall\, h \in H$.

Suppose $p \cdot x^h < p \cdot e^h$ (notice that $x^h \in B^h(p)$, i.e. $p \cdot x^h \leq p \cdot e^h$). Since step 3 and the first assumption imply $p \cdot e^h > 0$, then $x^h + (\delta,\ldots,\delta) \in B^h(p)$ for small $\delta > 0$. But $u^h(x^h + (\delta,\ldots,\delta)) > u^h(x^h)$ by weak monotonicity. Contradicting again.

Step 5: *Market clear* $\sum_h x^h_\ell = \sum_h e^h_\ell$, $\forall\, \ell \in L$.

By step 1 we have $\sum_h x^h_\ell \leq \sum_h e^h_\ell$, $\forall\, \ell \in L$. So the only violation is $\sum_h x^h_\ell < \sum_h e^h_\ell$ for some ℓ. But since $p \gg 0$, this implies $p \cdot (\sum_h x^h - \sum_h e^h) = \sum_\ell p_\ell \cdot \sum_h (x^h_\ell - e^h_\ell) < 0$, which is contradicting with step 4.

■

(*Following lemma and the proof is supplemented by Yan Liu*)

Lemma 3.1.2. *Assume $e^h \gg 0$, then the correspondence $p \rightrightarrows B^h(p)$ is continuous on price simplex $\triangle$, where $B^h(p) = \{x \in \mathbb{R}^L_+ | p \cdot x \leq p \cdot e^h\}$.*

Proof. First we show such a correspondence is u.h.c. Let $(p_n, x_n) \to (p^*, x) \in \triangle \times \mathbb{R}^L_+$ with $x_n \in B^h(p_n)$, then by definition we have $p_n \cdot x_n \leq p_n \cdot e^h$. Since inner product is a continuous function, there is $p^* \cdot x \leq p^* \cdot e^h$, i.e. $x \in B^h(p^*)$.

Second we show such a correspondence is also l.h.c. Let $\{p_n\} \in \triangle$ and $p_n \to p^* \in \triangle$, and fix an (arbitrary) $x \in B^h(p^*)$, we want to find a sequence $\{x_n\} \in B^h(p_n)$ with $x_n \to x$.[3]

Case 1. Suppose $p^* \cdot x = 0$. Since $p \cdot x$ is a continuous function of p, we have $p_n \cdot x \to 0$. Observe $p \cdot e^h$ is continuous on a compact set $\triangle$ and $e^h \gg 0$, we have $m \equiv \min_{p \in \triangle} p \cdot e^h > 0$. Therefore, for n large enough we have $p_n \cdot x < m \leq p_n \cdot e^h$, i.e. $x \in B^h(p_n)$. So, let $x_n = x$ for large n and set the beginning elements of the sequence to be 0 will meet with our need.

Case 2. Suppose $p^* \cdot x > 0$. Let $t(p) = \frac{p \cdot e^h}{p \cdot x}$ be a scalar function of $p \in \triangle$. Obviously, $t^* \equiv t(p^*) \geq 1$ and $t(p)$ is continuous at p^*, hence $t_n \equiv t(p_n) \to t(p^*)$. Define $x_n = \frac{t_n}{t^*} x$. Observe $p_n \cdot x_n = p_n \cdot \frac{t_n}{t^*} x \leq t(p_n) p_n \cdot x = p_n \cdot e^h$, we have $x_n \in B^h(p_n)$. Moreover, $\lim_n x_n = \frac{1}{t^*} x \lim_n t_n = x$. This completes the proof. ■

▶ Feb.17, 2010

3.2 C.E. without Positive Endowment Assumption

Keep on considering economy $\mathscr{E} = (e^h, u^h)_{h \in H}$. The positive endowment assumption $e^h \gg 0$ in previous theorem 3.1.1 is in fact very strong and not realistic at all (obviously, not everyone has everything). In this section, we weaken this assumption, and prove the existence of C.E. in $\mathscr{E}$ under the new assumption. This result can be viewed as a generalization of previous theorem. In proceeding the prove, we first introduce following necessary concepts.

Definition 3.2.1. *We say household $h \in H$ want's commodity $\ell \in L$, if $u^h(x + \delta 1_\ell) > u^h(x)$, $\forall\, x \in \mathbb{R}^L_+$ and $\delta > 0$. And say h has ℓ if $e^h_\ell > 0$.*

Definition 3.2.2. *For ℓ, $k \in L$, say directed arc (ℓ, k) exists if $\exists\, h \in H$ who has ℓ and likes k. Say commodity i is connected to commodity j, if $\exists$ a directed path from i to j.*[4]

[3] It will be really helpful to draw a picture to understand the main routine of the proof here.

[4] It means there exist several commodities $\ell_1, \ldots, \ell_n$, and the path consists of the corresponding directed arcs $(i, \ell_1), (\ell_1, \ell_2), \ldots, (\ell_n, j)$.

The commodity set L can be viewed as a directed graph, if we define the directed arc by above *having-wanting* property. In this sense, we would call L as *having-wanting graph.*

Definition 3.2.3 (Connected Graph)**.** *Graph L is connected, if $\forall$ pair $(i, j) \in L^2$, i is connected to j.*

Theorem 3.2.1. *Let $\mathscr{E} = (e^h, u^h)_{h \in H}$ be an economy where $e^h \in \mathbb{R}^L_+$ and $u^h : \mathbb{R}^L_+ \to \mathbb{R}\ \forall\, h$. If*

- *u^h is continuous, concave and weakly monotonic, $\forall\, h$;*
- *the* having-wanting graph *of $\mathscr{E}$ is connected.*

Then there exists a competitive equilibrium of $\mathscr{E}$.

We first give several remarks.

1. It should be quite clear that if a (quasi-concave) utility function can be expressed as a monotonic (strictly increasing) transformation of a concave function, then the above theorem will also be true for such a utility function assumption. However, since not every quasi-concave function can be expressed as a monotonic transformation of a concave function, the method used in this prove will not be applicable for such a case. However, using the similar idea to be expressed here and with more advanced tools, the result will also be true under quasi-concavity condition.

2. The first and third assumptions in theorem 3.1.1 together satisfy the condition for connected graph here. More directly, those two assumptions read everyone has every commodity and every commodity is likes by someone. So, now it should be quite clear that this theorem is a generalization of theorem 3.1.1. However, actually we use theorem 3.1.1 to prove this theorem.

3. The intuition for connected graph condition is quite straight forward. Let (i, j) be an arbitrary pair of commodities in L, there exist a path from i to j, i.e. there are $\{\ell_1, \ldots, \ell_m\} \in L$ s.t. $i \longrightarrow \ell_1 \longrightarrow \cdots \longrightarrow \ell_m \longrightarrow j$; this further implies there are $m+1$ household $\{h^1, \ldots, h^{m+1}\}$ that h^1 has i and likes ℓ_1, …, h^{m+1} has ℓ_m and likes j. At the same time there's another path from j to i, thus you can imagine an buying-selling process from i to j, e.g h^1 sells i to who likes it and buys ℓ_1, …, h^{m+1} sells ℓ_m to h^m and buys j …. By such exchanging procedure, all individuals will be satisfied and no commodity will be discarded.

4. Arrow and Debreu (1954) provided a different approach to relax the positive endowment assumption. However, their weakened condition involved the assumption that each

household all likes at least one same commodity, which is stronger than our assumption. Yet they considered a production economy, that condition turned out to be necessary for such a set up.

(The following proof and lemmas is completed and supplemented by Yan Liu.)

Proof. For any $\varepsilon > 0$, define $\mathscr{E}(\varepsilon) = (e^h(\varepsilon), u^h)_{h\in H}$, where $e^h(\varepsilon) = e^h + \varepsilon(1,\ldots,1)_{1\times L}$. By theorem 3.1.1, there exists a C.E. $\langle p(\varepsilon), (x^h(\varepsilon))_{h\in H}\rangle$ for each ε. Observe that $\forall\,\varepsilon$, $p(\varepsilon) \in \triangle$ which is a compact set, and $(x^h(\varepsilon))_{h\in H} \in \underbrace{\square\times\cdots\times\square}_{H} \equiv \square^H$[5] which is also a compact set, thus we can find a sequence ε_n with $\varepsilon_n \to 0$, s.t.

$$\langle p(\varepsilon_n), (x^h(\varepsilon_n))_{h\in H}\rangle \to \langle p, (x^h)_{h\in H}\rangle$$

We're going to prove $\langle p, (x^h)_{h\in H}\rangle$ is the C.E of $\mathscr{E}$.

Step 1: $p \gg 0$.

Suppose there is an $\ell \in L$ s.t. $p_\ell = 0$. Since $p \in \triangle$, there exists $k \in L$ s.t. $p_k > 0$. By the connected graph assumption, there is a path from k to ℓ denoted as

$$k \to \ell_1 \to \cdots \to \ell_m \to \ell.$$

Starting from k, search the first position on the path that a zero price occurs. Let ℓ_{j+1} to be this first position[6] that $p_{\ell j+1} = 0$. By the procedure, we have $p_{\ell_j} > 0$, hence $p_{\ell_{j+1}}(\varepsilon_n)/p_{\ell_j}(\varepsilon_n) \to p_{\ell_{j+1}}/p_{\ell_j} = 0$. Note ℓ_j and ℓ_{j+1} are two commodities connected by a directed arc, therefore $\exists\, h^* \in H$ who has ℓ_j and likes ℓ_{j+1}.

We want to show, for large n, $x^{h^*}(\varepsilon_n)$ does not maximize $u^{h^*}(\cdot)$ on $B^{h^*}(p(\varepsilon_n))$.

Take box $\square$ as in the first paragraph of the proof. For each $z \in \square$, let $H_z : \mathbb{R}^L_+ \to \mathbb{R}$ be the family of supporting hyperplanes[7] to the graph of $u^{h^*}(\cdot)$ at the point $(z, u^{h^*}(z))$. Now, for h^*, define a modified utility function $\tilde{u}^{h^*}(x) = \inf_{z\in\square} H_z(x)$. Note that so defined $\tilde{u}^{h^*}(\cdot)$ coincides with $u^{h^*}(\cdot)$ in $\square$, i.e. $\tilde{u}^{h^*}(x) = u^{h^*}(x)\ \forall\, x \in \square$.

Consider the family of modified economies $\tilde{\mathscr{E}}(\varepsilon) = (e^h(\varepsilon), u^h)_{h\in H\setminus\{h^*\}} \cup (e^{h^*}, \tilde{u}^{h^*})$. Since all C.E allocations of $\tilde{\mathscr{E}}(\varepsilon)$ must also located in $\square^H$ as $\mathscr{E}(\varepsilon)$[8], and $\tilde{u}^{h^*}(x) = u^{h^*}(x)$ in $\square$, thus

[5] In fact, for each ε, the scale of the $\square_\varepsilon$ is associated with $e^h(\varepsilon)$. However, since $e^h(\varepsilon) \to e^h$, we can choose a $\square$ large enough, s.t. $\square_\varepsilon \subset \square\ \forall\,\varepsilon$.

[6] It is possible that $\ell_{j+1} = \ell$, i.e. $p_{\ell_1}, \ldots, p_{\ell_m} > 0$.

[7] Note we only assume the utility function to be continuous and concave, therefore there may be more than one supporting hyperplane at a given point. By the way, the supporting hyperplane could also be viewed as a function.

[8] $\square$ does not depend on utility function, but only on endowment.

for each ε, $\tilde{\mathscr{E}}(\varepsilon)$ has the same C.E as $\mathscr{E}(\varepsilon)$. Therefore $x^{h^*}(\varepsilon_n)$ must also maximize $\tilde{u}^{h^*}(\cdot)$ on $B^{h^*}(p(\varepsilon_n))$.

Since we know $e^{h^*}_{\ell_j} > 0$ and $p_{\ell_j} > 0$, we can choose a point $y^{h^*} \in \mathbb{R}^L_+$ s.t. $y^{h^*} \in B^{h^*}(p(\varepsilon_n))$ $\forall\, n$[9]. Let $d = \sup_n \tilde{u}^{h^*}(x^{h*}(\varepsilon_n)) - \tilde{u}^{h^*}(y^{h^*})$, since $x^{h*}(\varepsilon_n) \to x^{h^*}$ and $\tilde{u}^{h^*}(\cdot)$ is continuous, we know $d < \infty$. Define $\hat{y}^{h^*} = (y^{h^*}_1, \dots, y^{h^*}_{\ell_{j+1}-1}, \hat{y}^{h^*}_{\ell_{j+1}}, y^{h^*}_{\ell_{j+1}+1}, \dots, y^{h^*}_L) \in \mathbb{R}^L_+$. Observe the budget set of h can be written in the following form

$$\sum_{\ell \neq \ell_{j+1}} \frac{p_\ell}{p_{\ell_j}} z_{\ell_j} + \frac{p_{\ell_{j+1}}}{p_{\ell_j}} z_{\ell_{j+1}} \leq \sum_{\ell \neq \ell_j} \frac{p_\ell}{p_{\ell_j}} e^{h^*}_\ell + e^{h^*}_{\ell_j},$$

and since $p_{\ell_{j+1}}(\varepsilon_n)/p_{\ell_j}(\varepsilon_n) \to 0$, $e^{h^*} > 0$, hence we could choose $\hat{y}^{h^*}_{\ell_{j+1}}(\varepsilon_n) \to \infty$ with $y^{h^*}_{\ell_{j+1}}(\varepsilon_n) \in B^{h^*}(p(\varepsilon_n))\ \forall\, n$.

However, since $u^{h^*}(\cdot)$ is strictly increasing in its ℓ_{j+1}th argument, by lemma 3.2.4 (followed with this proof), $\exists\, \kappa > 0$ s.t. $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) \geq \tilde{u}^{h^*}(y^{h^*}) + \kappa(\hat{y}^{h^*}_{\ell_{j+1}}(\varepsilon_n) - y^{h^*}_{\ell_{j+1}})$. Obviously, for large n, $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) - \tilde{u}^{h^*}(y^{h^*}) > d$, and this leads to $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) > \tilde{u}^{h^*}(x^{h^*}(\varepsilon_n))$, which means $x^{h^*}(\varepsilon_n)$ does not maximize $\tilde{u}^{h^*}(\cdot)$ in $B^{h^*}(p(\varepsilon_n))$.

Step 2: x^h maximizes $u^h(\cdot)$ in $B^h(p)$, $\forall\, h \in H$.

The basic idea to prove $x^h \in \operatorname{argmax}_{y \in B^h(p)} u^h(y)$ is using Berge's maximal theorem (theorem 2.1.1). For this purpose, we introduce some notations.

$\forall\, h \in H$, define $\varphi^h : \triangle \times \square \rightrightarrows \square$ where

$$\varphi^h(q, w^h) = B^h(q, w^h) \cap \square \equiv \{y^h \in \mathbb{R}^L_+ | q \cdot y^h \leq q \cdot w^h\} \cap \square.$$

Here we use q and w to represent price and endowment, and use y for a possible allocation point. Let $\beta^h(q, w^h) = \operatorname{argmax}_{z \in \varphi^h(q,w^h)} u^h(z)$ be the best reply correspondence from $\triangle \times \square$ to $\square$. Since $u^h(\cdot)$ is continuous and concave, thus if we can prove φ^h is continuous (compactness is obvious) at (p, e^h), then by theorem 2.1.1, β^h is u.h.c at (p, e^h), and therefore, since $x^h(\varepsilon_n) \in \beta^h(p(\varepsilon_n), e^h(\varepsilon_n))$ and $\big((p(\varepsilon_n), e^h(\varepsilon_n)), x^h(\varepsilon_n)\big) \to \big((p, e^h), x^h\big)$, we have $x^h \in \beta^h(p, e^h)$.

Since $\varphi^h = B^h \cap \square$ and $\square$ could be viewed as a constant valued correspondence from $\triangle \times \square$ to $\square$, so we only need to show B^h is a continuous correspondence at (p, e^h). We use similar techniques as in lemma 3.1.2 to prove the continuity of B^h at (p, e^h).

First we show B^h is u.h.c at (p, e^h). Let $\big((p_n, e^h_n), y^h_n\big) \to \big((p, e^h), y^h\big)$ with $y^h_n \in B^h(p_n, e^h_n)$. Since $p_n \cdot y^h_n \leq p_n \cdot e^h_n\ \forall\, n$, and inner product is continuous function, there is $p \cdot y^h \leq p \cdot e^h$, i.e. $y^h \in B^h(p, e^h)$.

Second we show B^h is l.h.c at (p, e^h). Let $(p_n, e^h_n) \to (p, e^h)$ and $\forall\, y^h \in B^h(p, e^h)$, we want to find a sequence $\{y^h_n\}$ s.t. $y^h_n \in B^h(p_n, e^h_n)$ and $y^h_n \to y^h$.

[9] E.g., we can choose $y^{h^*} = (0, \dots, e^{h^*}_{\ell_j}, \dots, 0)$.

Case 1. Suppose $p \cdot y^h = 0$. By step 1, $p \gg 0$, thus $y^h = (0, \dots, 0)$. Therefore let $y_n^h = (0, \dots, 0)\ \forall\, n$ will be enough.

Case 2. Suppose $p \cdot y^h > 0$. Define $t(q, w^h) = \frac{q \cdot w^h}{q \cdot x^h}$, and $t(q, w^h)$ is continuous at (p, e^h). Obviously, $t^* \equiv t(p, e^h) \geq 1$. Let $t_n = t(p_n, e_n^h)$ and $y_n^h = \frac{t_n}{t^*} y^h$, then $t_n \to t^*$ and $y_n^h \to y^h$. Observe $p_n \cdot y_n^h = \frac{t_n}{t^*} p_n \cdot y^h \leq t_n p_n \cdot y^H = p_n \cdot e_n^h$, i.e. $y_n^h \in B^h(p_n, e_n^h)$. So $\{y_n^h\}$ will do the job.

Step 3: Market clear $\sum_h x^h = \sum_h e^h$

Since $(x^h(\varepsilon_n))_{h \in H}$ is competitive allocation in $\mathscr{E}(\varepsilon_n)\ \forall, n$, there is $\sum_h x^h(\varepsilon_n) = \sum_h e^h(\varepsilon_n)$. Take the limit on both sides, we get $\sum_h x^h = \sum_h e^h$. ■

In order to prove $\tilde{u}^h$ satisfies the desired increasing condition (lemma 3.2.4), we first prove a basic property for one variable concave function, and then give the proof of lemma 3.2.4.

Lemma 3.2.2. *Let $f(x)$ be a concave function on $\mathbb{R}$, then $\forall\, a < b < c$,*

$$\frac{f(b) - f(a)}{b - a} \geq \frac{f(c) - f(a)}{c - a} \geq \frac{f(c) - f(b)}{c - b}.$$

Proof. Let $t = \frac{c-b}{c-a}$, we have $0 < t < 1$. By concavity of $f(x)$, there is $f(b) = f(ta + (1-t)c) \geq tf(a) + (1-t)f(b)$. Thus we have following inequality,

$$\begin{aligned} \frac{f(b) - f(a)}{b - a} &\geq \frac{[tf(a) + (1-t)f(c)] - f(a)}{b - a} \\ &= \frac{f(c) - f(a)}{c - a} \\ &= \frac{f(c) - [tf(a) + (1-t)f(c)]}{c - b} \\ &\geq \frac{f(c) - f(b)}{c - b}. \end{aligned}$$

■

Corollary 3.2.3. *Let $f(x)$ be a concave function on $\mathbb{R}$, then its left and right derivatives exist respectively, and $\forall\, x$*

$$f'_-(x) \equiv \lim_{\epsilon \to 0+} \frac{f(x) - f(x - \epsilon)}{\epsilon} \geq \lim_{\epsilon \to 0+} \frac{f(x + \epsilon) - f(x)}{\epsilon} \equiv f'_+(x).$$

Moreover, $\forall\, x < y$, there is

$$f'_+(x) \geq f'_-(x) \geq f'_+(y) \geq f'_-(y).$$

Proof. $\forall\, \epsilon_n \searrow 0$ [10], observe $\frac{f(x) - f(x - \epsilon_n)}{\epsilon_n}$ is an decreasing sequence with a lower bound by above lemma, thus the limit exists. Similarly, the limit of $\frac{f(x+\epsilon_n) - f(x)}{\epsilon_n}$ exists. Moreover, observe for any n, $\frac{f(x) - f(x - \epsilon_n)}{\epsilon_n} \geq \frac{f(x+\epsilon_n) - f(x)}{\epsilon_n}$, thus $\lim_{\epsilon \to 0+} \frac{f(x) - f(x-\epsilon)}{\epsilon} \geq \lim_{\epsilon \to 0+} \frac{f(x+\epsilon) - f(x)}{\epsilon}$.

[10] " $\searrow$ " means converging from above.

For the second part, observe $\exists\, z$ s.t. $x < z < y$, thus for large n,

$$\frac{f(x+\epsilon_n)-f(x)}{\epsilon_n} \geq \frac{f(z)-f(x)}{z-x} \geq \frac{f(y)-f(z)}{y-z} \geq \frac{f(y)-f(y-\epsilon_n)}{\epsilon_n}.$$

Taking the limit, we get the desired result. ■

Lemma 3.2.4. *Let function* $u : \mathbb{R}^L_+ \to \mathbb{R}$ *be continuous, concave and strictly increasing in its first argument, i.e.* $u(x+\delta(1,0,\ldots,0)) > u(x)$ $\forall x \in \mathbb{R}^L_+$ *and* $\delta > 0$. $\square = \{x \in \mathbb{R}^L_+ | 0 \leq x_\ell \leq M, \ell = 1,\ldots,L\}$ *where* $M > 0$ *is a given number. For each* $z \in \square$, *let* $H_z = \{H^i_z : \mathbb{R}^L_+ \to \mathbb{R}, i \in I_z\}$ *be the set of supporting hyperplane to the graph of* $u(\cdot)$ *at* $(z, u(z))$ *where* I_z *is the index set* [11], *and define* $\tilde{u}(x) = \inf_{z\in\square}\{H^i_z(x), i \in I_z\}$ $\forall\, x \in \mathbb{R}^L_+$.

Then, $\exists\, \kappa > 0$ *s.t.* $\tilde{u}(y_1, x_2, \ldots, x_L) \geq \tilde{u}(x_1, x_2, \ldots, x_L) + \kappa(y_1 - x_1)$, $\forall\, x = (x_1,\ldots,x_L) \in \mathbb{R}^L_+$ *and* $\forall\, y_1 > x_1$.

Proof. Let $\hat{x} = (x_2, \ldots, x_L)$ be the last $L-1$ coordinates of x, and $\forall\, z \in \square$, define $h(z) = \inf_{i\in I_z}\{\partial_1 H^i_z\}$[12].

Step 1: $\exists\, \kappa > 0$ s.t. $\partial_1 H^i_z > \kappa$, $\forall\, z \in \square$ and $\forall\, i \in I_z$.

First, $\forall\ \hat{x} \in \widehat{\square} = \{x \in \mathbb{R}^{L-1}_+ | 0 \leq x_\ell \leq M, \ell = 2,\ldots,L\}$, let $y = (y_1, \hat{x})$, then $f_{\hat{x}}(y_1) = u(y) \equiv u(y_1, \hat{x})$ is concave and strictly increasing in y_1, $\forall\, y_1 \in [0,\infty)$. Hence for this $\hat{x}$, $\kappa(\hat{x}) \equiv \frac{f_{\hat{x}}(M+\delta) - f_{\hat{x}}(M)}{\delta} > 0$ where $\delta > 0$ is a given number. By above corollary, we know $f'_{\hat{x}-}(y_1) \geq f'_{\hat{x}-}(M) \geq \kappa(\hat{x}) > 0$, $\forall\, y_1 \in [0, M]$. Then, $\forall\, y = (y_1, \hat{x})$ with $y_1 \in [0, M]$, observe $h(y) = f'_{\hat{x}-}(y_1)$, thus $\forall\, i \in I_y$, $\partial_1 H^i_y \geq h(y) \geq \kappa(\hat{x})$.

Second, let $\kappa = \inf_{\hat{x}\in\widehat{\square}} \kappa(\hat{x})$. Obviously, $\kappa \geq 0$. Suppose $\kappa = 0$, i.e. there is a sequence $\{\hat{x}_n\}$ s.t. $\kappa(\hat{x}_n) \to 0$. Since $\{\hat{x}_n\} \in \widehat{\square}$ which is a compact set, hence there is a subsequence $\{\hat{x}_{n_k}\}$ s.t. $\hat{x}_{n_k} \to \hat{x}^* \in \widehat{\square}$. Observe $\kappa(\hat{x}) = \delta^{-1}[u(M+\delta, \hat{x}) - u(M, \hat{x})]$ is a continuous function of $\hat{x}$, therefore $\kappa(\hat{x}^*) = \lim_k \kappa(\hat{x}_{n_k}) = 0$. However, since $\hat{x}^* \in \widehat{\square}$, there is $\kappa(\hat{x}^*) > 0$. This contradiction implies $\kappa > 0$.

Third, $\forall\, z \in \square$ and $\forall\, i \in I_z$, since $h(z) \geq \kappa(\hat{z})$, there is $\partial_1 H^i_z \geq h(z) \geq \kappa > 0$.

Step 2: $\tilde{u}(x)$ is concave in $\mathbb{R}^L_+$.

Observe $\tilde{u}(x) = \inf_{z\in\square}\{H^i_z(x), i \in I_z\} = \inf_{z\in\square}\inf_{i\in I_z}\{H^i_z(x)\}$, and in the same hyperplane $H^i_z(\cdot)$ there is $H^i_z(tx + (1-t)y) = tH^i_z(x) + (1-t)H^i_z(y)$, $\forall\, x, y \in \mathbb{R}^L_+$ and $\forall\, t \in [0,1]$.

[11] Since we only assume $u(\cdot)$ to be continuous, at each point z, there may be more than one supporting hyperplane, thus we use $i \in I$ to label all these hyperplane. Note, this index set I may depend on z.

[12] "∂_1" means the partial derivative with respect to the first argument. Of course, H^i_z is a linear function, thus $\partial_1 H^i_z$ is a constant.

Therefore,

$$\begin{aligned}
\tilde{u}(tx+(1-t)y) &= \inf_{z\in\Box}\inf_{i\in I_z}\{H_z^i(tx+(1-t)y)\} \\
&= \inf_{z\in\Box}\inf_{i\in I_z}\{tH_z^i(x)+(1-t)H_z^i(y)\} \\
&\geq \inf_{z\in\Box}\left\{t\inf_{i\in I_z}\{H_z^i(x)\}+(1-t)\inf_{i\in I_z}\{H_z^i(y)\}\right\} \\
&\geq t\inf_{z\in\Box}\inf_{i\in I_z}\{H_z^i(x)\}+(1-t)\inf_{z\in\Box}\inf_{i\in I_z}\{H_z^i(y)\} \\
&= t\tilde{u}(x)+(1-t)\tilde{u}(y).
\end{aligned}$$

Thus $\tilde{u}(x)$ is concave.

Step 3

$\forall\, x=(x_1,\hat{x})\in\mathbb{R}_+^L$ and $\forall\, y>x_1$, $g(y)\equiv\tilde{u}(y,\hat{x})$ is concave in y $\forall\, y>x_1$, since $\tilde{u}(x)$ is concave and $g(y)$ is a restriction of $\tilde{u}(x)$ on a straight line. Let $y_1>x_1$ be fixed, by lemma 3.2.2 and its corollary, there is

$$\frac{g(y_1)-g(x_1)}{y_1-x_1}\geq g'_-(y_1).$$

By definition, $g(y_1)=\tilde{u}(y_1,\hat{x})=\inf_{z\in\Box}\inf_{i\in I_z}\{H_z^i(y_1,\hat{x})$, thus there is some $z\in\Box$ and $i\in I_z$ s.t. $g'_-(y_1)=\partial_1 H_z^i$. From step 1 we know $\partial_1 H_z^i>\kappa$, $\forall\, z\in\Box$ and $\forall\, i\in I_z$. Therefore,

$$\frac{g(y_1)-g(x_1)}{y_1-x_1}\geq g'_-(y_1)\geq\kappa.$$

Rearrange last expression, we have

$$\tilde{u}(y_1,x_2,\ldots,x_L)\geq\tilde{u}(x_1,x_2,\ldots,x_L)+\kappa(y_1-x_1).$$

■

The proof of above lemma can be significantly simplified if we assume $u^h(\cdot)$ is continuously differentiable. Under this assumption, at each point $z\in\Box$, the only supporting hyperplane is also tangent plane, and $\partial_1 H_z=\partial_1 u^h(z)>0$. Observe $\partial_1 u^h(z)$ is a continuous function in $\Box$, thus the minimum of $\partial_1 u^h(z)$ can be achieved, then there is $\kappa\equiv\min_{z\in\Box}\partial_1 u^h(z)>0$.

Final remarks about the proof:

The main routine of this proof is straight forward and without much trick.

First, since the endowment may have zero components, we can not use a generalized game framework to prove the existence of N.E equilibrium (the difficulty for this step is the budget set correspondence may not be l.h.c. within a context of theorem 3.1.1).

However, by small perturbations of the primitive economy, we can get a sequence of C.Es for the disturbed economies, and then taking a limit of these sequence C.Es, we get a limiting price vector and a set of limiting allocations. Therefore, all we need to do is to prove the limiting price and limiting allocations consist of a C.E for the primitive economy.

It's trivial to that limiting allocation also satisfies the market clear condition, but it is not easy to prove the allocations are optimal under the limiting price. The difficulty again arises from the lack of l.h.c for the budget set correspondence (within the context of this proof, not the same as the previous one) as the consequence of the endowment assumption.

At this time, the most important observation is if we know the limiting price vector is strictly positive, then we can assure the continuity of the correspondence. Thus, in fact, all we need to do is to prove limiting price does posses this property. Fortunately, by assuming connectedness of having-wanting graph for this economy, we can show (quite uneasily) this to be true. The technical difficulty for this part is to modified the primitive utility function a little bit by taking the infimum of its supporting hyperplane on a compact set, and it turns out that this specific modified utility function has a particular increasing property (no less than a linear increasing) along a particular direction.

After overcome all these technical difficulties, we reach the final intuitive conclusion.

Two examples:

The first example is an illustration of the necessity of connected graph of an economy for the existence of C.E. Consider a 2×2 economy with $e^1 = (1,1)$, $e^2 = (2,0)$, and $u^1(x,y) = y$, $u^2(x,y) = x$.

- If $p = (p_1, p_2) \gg 0$, then $\operatorname{argmax}_{B^1(p)} u^1$ is $(0, \frac{p_1}{p_2} + 1)$ and $\operatorname{argmax}_{B^2(p)} u^2$ is $(2,0)$, thus market is not clear.
- If $p = (1,0)$, then $\operatorname{argmax}_{B^1(p)} u^1$ is (s,t) with $t \to \infty$ and $s \in [0,1]$, while $\operatorname{argmax}_{B^2(p)} u^2$ is $(2,r)$ with $r \geq 0$, thus market is not clear too.
- If $p = (0,1)$, then $\operatorname{argmax}_{B^1(p)} u^1$ is $(t,1)$ with $t \geq 0$, and $\operatorname{argmax}_{B^2(p)} u^2$ is $(r,0)$ with $r \to \infty$, thus market is not clear once more.

Therefore, no C.E. exists. It is clear that the second commodity is not connected with the first commodity, since the only household 1 who has the second commodity doesn't like it.

The second example shows how the result of this theorem is stronger than theorem 3.1.1. Consider a 2×2 economy with $e^1 = (1,0)$, $e^2 = (0,2)$, and $u^1(x,y) = y$, $u^2(x,y) = x$. Since initial endowment is not strictly positive, theorem 3.1.1 can not ensure the existence of a C.E. of this economy. However, since household 1 has commodity 1 and likes commodity 2, while household 2 has commodity 2 and likes commodity 1, the graph of this economy is connected, thus by theorem 3.2.1 there is a C.E. It can be easily verified that $\langle p, x^1, x^2 \rangle = \langle (\frac{2}{3}, \frac{1}{3}), (0,2), (1,0) \rangle$ is a C.E.

Chapter 4

Production in An Economy

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In this section, we consider competitive equilibria in an economy with production. Instead of using production function to describe production, we use a more general concept *production set* to characterize production. Intuitively, we assume all production activities are carried out by firms. For each firm, we use a production set, or more properly, a set of all feasible production plans, to characterize the technological properties associated with the production activities of this firm. A production plan (allocation) of a firm, like a consumption plan (allocation) of a consumer, can be viewed as a point in the commodity space (a vector space), with each coordinate representing the quantity of a certain kind of commodity. Conventionally, if the quantity is positive, then the corresponding commodity is output; if negative, then input; if the quantity is zero, then this commodity is irrelevant to the production activity of this firm.

What we concern is the production plan only but not the specific production process, because production activities here have been abstracted as an input-output relation. A firm receives (buys) inputs and then releases (sells) outputs, and the concrete production process has been put into a black box. In this point of view, any form of management is absent from this setting-up. No human being manager is needed. It could be a machine, who merely executes the production plan automatically.

With the concept of production set, we will discuss the competitive equilibrium in an economy with production. More over, the basic setting of the property right in this economy will be private ownership, i.e. all firms are completely owned by households.

Let the household set $H = \{1, \dots, H\}$ and the firm set $J = \{1, \dots, J\}$, both of which are finite set. $\theta_j^h \equiv$ is the share fraction of firm j owned by household h, of course $\theta_h^j \geq 0$ and $\sum_h \theta_h^j = 1$, $\forall\, j \in J$. Use Y_j to denote the production set of firm j, and $Y_j \subset \mathbb{R}^L$ where L is the number of commodities in the economy, of course Y^j is nonempty. Put things together, we have following expression for a economy with production $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$.

It's sure that not an arbitrary subset of $\mathbb{R}^L$ can be viewed as a realistic production set. We would like to have production sets posses particular properties according to our empirical observation, thus following assumptions are made.

(1). $Y^j \cap \mathbb{R}^L_+ = \{0\}$, for all $j \in J$.
This assumption means no firm can produce anything from nothing, however they could do nothing.

(2). Y^j is convex and closed.
Since at current stage, all commodities are perfect divisible, hence it seems reasonable to assume the production set to be closed. However, convexity is a really strong assumption, since it implies (combined with the first assumption) no firm could be increasing return to scale.

(3). Y^j is comprehensive, i.e. $Y^j + \mathbb{R}^L_- \subset Y^j$ [1], where $\mathbb{R}^L_- \equiv \{x \in \mathbb{R}^L : x_i \leq 0, i = 1, \dots, L\}$.
This assumption is interpreted as free disposal. This assumption is very strong and unrealistic somehow. In reality, it is always very costly to dispose things, especially for certain by-products.

(4). $Y \cap \mathbb{R}^L_+ \equiv Y^1 + \cdots + Y^J \cap \mathbb{R}^L_+ = \{0\}$.
Y is the *aggregate* production set for this economy. This assumption is independent from the first assumption. It means the firms together can not do arbitrage, which is not implied by the first assumption.

(5). $Y \cap (-Y) = \{0\}$. [2]
Intuitively, if y is a possible aggregate production plan, $-y$ is a production plan which reverses the whole production process. This assumption excludes the possibility for the economy to reverse its aggregate production, which is quite reasonable since in reality all productions take time.

Before proceeding the discussion of the competitive equilibria in this economy, we first show an interesting, and at the same time, crucial property implied by above assumptions.

[1] Let A, $B \subset \mathbb{R}^L$, $A + B$ is defined as $\{a + b : a \in A, b \in B\}$, and $-A$ is defined as $\{-a : a \in A\}$.

[2] This highly reasonable assumption is not mentioned in the class, however, in fact, it plays a key role in the existence of C.E. for an economy with production. In particular, this assumption ensures the compactness of *attainable production set* (Debreu 1959a: 5.3, 5.4), which is analogous to the strategy space defined for firms in our proof later.

Lemma 4.0.5. *Let $Y^1, \dots, Y^J$ be production sets satisfying assumption* (2) *to* (5)*, and e be a nonzero vector of $\mathbb{R}^L_+$. If $Y \cap (-Y) = \{0\}$ where $Y = Y^1 + \cdots + Y^J$ is the aggregate production set, then $(Y + e) \cap (-Y - e)$ is non-empty and bounded.*

Proof. First we show $(Y + e) \cap (-Y - e)$ is non-empty. By (3), $Y + \mathbb{R}^L_- \subset Y$, hence by (4) $-e = 0 + (-e) \in Y$. So we have $0 = (-e) + e \in Y + e$. Same argument shows $0 = e + (-e) \in (-Y - e)$. Therefore $(Y + e) \cap (-Y - e)$ is non-empty.

Second we show it is bounded. Suppose conversely, there is a sequence $\{y^k\} \subset (Y + e) \cap (-Y - e)$ with $||y^k|| \equiv \max\{|y^k_1|, \dots, |y^k_n|\} \to \infty$. Obviously, $y^k - e \in Y$ and $y^k + e \in (-Y)$. Without loss of generality, assume $||y^k|| > 1$, so $0 < 1/||y^k|| < 1$. By assumption (2) and (5), Y is also convex and closed[3], thus $y^k/||y^k|| - e/||y^k|| \in Y$ and $y^k/||y^k|| + e/||y^k|| \in (-Y)$ are uniformly bounded, hence there are subsequence for each sequence that converge to the same point y with $||y|| = 1$. Clearly, $y \in Y$ and $y \in (-Y)$, therefore by assumption (5) there is $y = 0$, contradicting. ■

We'll always assume the objective for any firm is to maximize its profit. With our notation, the profit of a certain production plan y^j of firm j is $p \cdot y^j$, where p is price vector.

[3] Closedness is non-trivial, see Debreu (1959a: 3.3, (1)).

Chapter 5

C.E. in An Economy with Production

With the conceptual preparation in previous section, now we're able to define the competitive equilibrium for this economy.

Definition 5.0.4. *$\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a competitive equilibrium (C.E.) of an economy with production $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$, if*

- *Firms maximize profit.*
  $y^j \in \operatorname{argmax}_{z\in Y^j} p\cdot z$, $\forall\, j\in J$.

- *Households maximize utility.*
  $x^h \in \operatorname{argmax}_{z\in B^h(p,(y^j)_{j\in J})} u^h(z)$, $\forall\, h\in H$, *where*

$$B^h(p,(y^j)_{j\in J}) = \left\{ x\in \mathbb{R}^L_+ : p\cdot x \le p\cdot e^h + \sum_{j\in J} \theta^h_j\, p\cdot y^j \right\}.$$

- *Market clear.*
  $\sum_h (x^h - e^h) - \sum_j y^j = 0$.

Theorem 5.0.6. *Let $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ be an economy with production, if*

- $e^h \gg 0$, *for all h;*

- $u^h(\cdot)$ *is continuous, quasi-concave and weakly monotonic for all h;*

- $\forall\, \ell \in L$, $\exists\, h$ *who likes it;*

- *production sets satisfy above assumptions* (1) *to* (5)*;*

- $\sum_h \theta^h_j = 1$, *for all j.*

Then there exists a competitive equilibrium of $\mathscr{E}$.

The main idea and method of the proof for this theorem is exactly the same as theorem 3.1.1. We will write the proof here in a more concise way, and omit some details. The original statement of this theorem is in Arrow and Debreu (1954) as theorem 1, and essence of the proof here coincides with the original one, regardless slightly different assumptions with respect to preference and production[1]. It is highly recommended to read their original paper, and once understand the argument in the following proof, no difficulty is there for the original proof.

Proof. Let $e = \sum_h e^h$ and $Y = \sum_j Y^j$. By lemma 4.0.5 $(Y + e) \cap (-Y - e)$ is non-empty and bounded, thus fix $M > \max\{||z|| : z \in (Y+e) \cap (-Y-e)\}$[2] where $||z|| \equiv \max\{|z_1|, \dots, |z_L|\}$. Let $\square = \{x \in \mathbb{R}^L | x_j \in [-M, M]\}$ and $\triangle$ be the price simplex. Define $S = \triangle \times \square^{H+J}$, and a strategy in S looks like $s = \big(p, (x^h)_{h\in H}, (y^j)_{j\in J}\big)$. Each household h is a player with a payoff function $\pi^h(s) = u^h(x^h)$ and each firm is also a player with a payoff function $\pi^j(s) = p \cdot y^j$. Define the price player with a payoff function $\pi^{pr}(s) = p \cdot \Big(\sum_h (x^h - e^h) - \sum_j y^j\Big)$. Further, define a modified budget set[3] for each household as

$$\tilde{B}^h(p, (y^j)_{j\in J}) = \left\{ x^h \in \mathbb{R}^L_+ | p \cdot x^h \le p \cdot e^h + \max\left\{0, \sum_j \theta^h_j p \cdot y^j \right\} \right\}.$$

Now, for all $s \in S$ define following correspondence,

$$\begin{aligned} \varphi^h(s) &= \tilde{B}^h(p, (y^j)_{j\in J}) \cap \square, \quad \forall\, h; \\ \varphi^j(s) &= Y^j \cap \square, \quad \forall\, j; \\ \varphi^{pr}(s) &= \triangle. \end{aligned}$$

Following this theorem, lemma 5.0.7 will prove φ^h is continuous, and the continuity of φ^j and φ^{pr} is triviality. By now, we have a well-defined generalized game satisfying theorem 2.2.1,

[1] Arrow and Debreu (1954) imposed somehow weaker assumptions on both preferences and production sets, i.e. some commodities may not be liked by any household and no free disposal is assumed. For the former one, since no guarantee on positive equilibrium price anymore, a different market clear condition is adopted and a slightly adjusted argument is involved to have the strategy space compact; for the latter one, it's quite easy to show that free disposal is of no necessity for the existence of competitive equilibrium for production economy.

[2] If choosing M according to the method mentioned in the class, i.e. define $M > \max\{z : z \in (y+e) \cap \mathbb{R}^L_+\}$, one can not prove the equilibrium production plan to be a interior point of $\square$, thus can not prove this production plan maximizes profit on the whole production set.

[3] Introducing such a budget set is purely a technical trick to overcome the difficulty with continuity if we use the primitive form of the budget set. This difficulty arises from the fact that, without proper restriction, $e^h + \sum_j \theta^h_j y^j$ may not be necessarily strictly positive, and as summarized at the end of theorem 3.2.1, definitely this will lead to the lack of l.h.c for the budget correspondence. As we'll see, this modification has no effect in the equilibrium.

thus an N.E. $\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ exists. We shall verify this is a C.E. of $\mathscr{E}$. Notice, now p, x^h and y^j all denote the particular equilibrium values.

Step 1: $p \cdot y^j \geq 0$, for all j.

Since $0 \in Y^j \cap \Box$, zero profit is always possible, thus $p \cdot y^j \geq 0$. This also shows that $\tilde{B}^h(p, (y^j)_{j\in J}) = B^h(p, (y^j)_{j\in J})$, since $\sum_j \theta_j^h p \cdot y^j$ is non-negative.

Step 2: $\sum_h (x^h - e^h) - \sum_j y^j \leq 0$.

Observe $x^h \in B^h(p, (y^j)_{j\in J})$, hence $p \cdot x^h \leq p \cdot e^h + \sum_{j\in J} \theta_j^h \, p \cdot y^j$, and sum across h together with the last condition gives us

$$p \cdot \left(\sum_h (x^h - e^h) - \sum_j y^j \right) \leq 0.$$

Thus the optimizing by price player implies the desired result.

Step 3: x^h maximizes $u^h(\cdot)$ on $B^h(p, (y^j)_{j\in J})$.

First observe that $x^h \leq \sum_h x^h \leq e + \sum_j y^j \subset Y + e$, and $x^h \in \mathbb{R}_+^L \subset (-Y - e)$, thus x^h is an interior point of $\Box$.

Suppose there is $z \in B^h(p, (y^j)_{j\in J}) \setminus \Box$ with $u^h(z) > u^h(x^h)$, then by continuity of utility there is a small ball $B(z)$ centering at z within which each point has a higher utility than x^h. Thus we could choose a $\hat{z}$ to be an interior point of $B^h(p, (y^j)_{j\in J})$ with $u^h(\hat{z}) > u^h(x^h)$. Let $\epsilon > 0$ small enough s.t. $\hat{z}(\epsilon) = (1 - \epsilon)x^h + \epsilon \hat{z}$ is also an interior point of $B^h(p, (y^j)_{j\in J}) \cap \Box$, hence $u^h(\hat{z}(\epsilon)) \geq u^h(x^h)$. So we can further choose a small $\delta > 0$ s.t. $y \equiv \hat{z}(\epsilon) + (\delta, \ldots, \delta) \in B^h(p, (y^j)_{j\in J}) \cap \Box$ and $u^h(y) > u^h(x^h)$. Contradiction.

Step 4: $p \gg 0$.

Suppose $p_\ell = 0$, then by the third condition there is an h who likes ℓ, thus $u^h(x^h + 1_\ell) > u^h(x^h)$. Observe $x^h + 1_\ell \in B^h(p, (y^j)_{j\in J})$, which leads to a contradiction with step 3.

Step 5: Market clear $\sum_h (x^h - e^h) - \sum_j y^j = 0$.

First observe that by step 2 and 4, there is $p \cdot x^h = p \cdot e^h + \sum_j \theta_j^h p \cdot y^j$ for all h. Otherwise, for sufficient small $\delta > 0$, $x^h + (\delta, \ldots, \delta) \in B^h(p, (y^j)_{j\in J})$ which yields a higher utility. Summing over h follows $p \cdot \left(\sum_h (x^h - e^h) - \sum_j y^j \right) = 0$. Rewrite this expression as

$$\sum_\ell p_\ell \left(\sum_h (x_\ell^h - e_\ell^h) - \sum_j y_\ell^j \right) = 0,$$

and observe the terms in the bracket are non-positive by step 1, so they must be 0 by step 4.

Step 6: y^j is interior point of $\Box$ for all j.

By step 5, $\sum_h x^h = \sum_h e^h + \sum_j y^j \in \mathbb{R}_+^L$. For any $j^* \in J$, of course $y^{j^*} \in Y^{j^*} \subset Y \subset Y + e$. On the other hand, $y^{j^*} = -(\sum_{j\neq j^*} y^j - \sum_h x^h) - e$. Notice $\sum_{j\neq j^*} y^j \in Y$ and $-\sum_h x^h \in \mathbb{R}_-^L$, hence the difference belongs to Y, thus $y^{j^*} \in (-Y - e)$. Observe $(Y + e) \cap (-Y - e)$ is contained in the interior of $\square$, therefore y^{j^*} is an interior point of $\square$.

Step 7: y^j maximizes profit in Y^j for all j.

Suppose conversely there is a production plan $z \in Y^j \setminus \square$ with a strictly higher profit, then by convexity of Y^j for any $\epsilon \in (0, 1)$, $\hat{z}(\epsilon) \equiv (1 - \epsilon)y^j + \epsilon z \in Y^j$. Further, by step 6, $\hat{z}(\epsilon) \in Y^j \cap \square$ whenever ϵ is small enough. However, this implies $p \cdot \hat{z}(\epsilon) = (1-\epsilon)p \cdot y^j + \epsilon p \cdot z > p \cdot y^j$, which is contradicting with y^j maximizing profit in $Y^j \cap \square$. ■

Lemma 5.0.7. *Let $\varphi^h(s)$ be the correspondence defined in the previous proof, then φ^h is continuous in S.*

Proof. Since $\varphi^h(s) = \tilde{B}^h(p, (y^j)_{j \in J}) \cap \square$, it suffices to show $\tilde{B}^h(p, (y^j)_{j \in J})$ viewed as a correspondence from S to $\mathbb{R}^L_+$ is continuous. For the u.h.c part, the proof is quite straight forward, provided that inner product is a continuous function. For the l.h.c part, given the particular form of this modified budget set, consider the following two cases could be of convenience.

Case 1. Suppose $x \in \tilde{B}^h(p, (y^j)_{j \in J})$, $p_n \to p$ and $y_n = (y^1_n, \ldots, y^J_n) \to y = (y^1, \ldots, y^J)$ with $\sum_j \theta^h_j p \cdot y^j < 0$. Then there is $p \cdot x \leq p \cdot e^h$, and use the same argument as in lemma 3.1.2 will clearly prove the lower hemi-continuous provided that $e^h \gg 0$.

Case 2. Suppose now $\sum_j \theta^h_j p \cdot y^j \geq 0$, then there is $p \cdot x \leq p \cdot e^h + \sum_j \theta^h_j p \cdot y^j$. Once again use the same method of lemma 3.1.2, i.e. defining $t(p, y) = (p \cdot e^h + \sum_j \theta^h_j p \cdot y^j)/(p \cdot x)$ where $p \cdot x > 0$ and let $x_n = x$ where $p \cdot x = 0$ will complete the proof. ■

Chapter 6

Welfare Properties of C.E.

6.1 Pareto Optimal in productive economy

From previous experience in the discussion about pure exchange economy we know competitive equilibria obtaining somehow “nice” welfare properties. Notice that in an economy with production, the resource constraint for each household is totally different from that in a pure exchange economy, hence we can not talk about the optimality of an allocation directly. Therefore, we need to set up new characterization for the resource constraint in an economy with production. Note, we do not assume $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ satisfies any conditions listed in theorem 5.0.6, including the assumptions about the production set.

Definition 6.1.1. *$\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is an attainable allocation in $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$, if $y^j \in Y^j$ for all j and $0 \leq \sum_h x^h \leq \sum_h e^h + \sum_j y^j$.*

Definition 6.1.2. *Let $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ and $\langle (w^h)_{h\in H}, (z^j)_{j\in J}\rangle$ be two attainable allocations in $\mathscr{E}$. We say the former one is Pareto superior to the latter one, if $u^h(x^h) \geq u^h(w^h)$ for all h and $u^h(x^h) > u^h(w^h)$ for at least one h.*

Definition 6.1.3. *An allocation $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is Pareto optimal in $\mathscr{E}$, if there is no attainable allocation which is Pareto superior to this one.*

Definition 6.1.4. *$\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a competitive allocation in $\mathscr{E}$, if there exists a price vector $p \in \mathbb{R}^L_+$ s.t. $\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a C.E. in $\mathscr{E}$.*

With these conceptual preparation, we're going to state the following first welfare theorem for $\mathscr{E}$.

Theorem 6.1.1. *Let $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ be a production economy, and assume utility to be weakly monotonic. Then, all competitive allocations in $\mathscr{E}$ are Pareto optimal.*

Proof. Let $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ be a competitive allocation, and p be the price vector associated with the allocation.

Suppose conversely there is an attainable allocation $\langle (w^h)_{h\in H}, (z^j)_{j\in J}\rangle$ in $\mathscr{E}$ which is Pareto superior to $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$.

First, since there is one h with $u^h(x^h) < u^h(w^h)$, then $p \cdot w^h > p \cdot e^h + \sum_j \theta_j^h p \cdot y^j$ for this p, otherwise $w^h \in B^h(p, (y^j)_{j\in J})$, which is contradicting the optimality of x^h in h's budget set. Further observe for each j, z^j yields a profit at most as high as $p \cdot y^j$, thus $\sum_j \theta_j^h p \cdot y^j \geq \sum_j \theta_j^h p \cdot z^j$ provided θ_j^h is non-negative. Summing up, we have $p \cdot w^h > p \cdot e^h + \sum_j \theta_j^h p \cdot z^j$.

Second, since for all $h \in H$ there is $u^h(x^h) \leq u^h(w^h)$, there is $p \cdot w^h \geq p \cdot e^h + \sum_j \theta_j^h p \cdot y^j$, otherwise by weak monotonicity there exists a $\tilde{w}^h$ which yields a higher utility than x^h. Moreover, since $p \cdot y^j \geq p \cdot z^j$ for all j, hence $p \cdot w^h \geq p \cdot e^h + \sum_j \theta_j^h p \cdot z^j$ for all h.

Combining above facts, we get $\sum_h p \cdot w^h > \sum_h p \cdot e^h + \sum_j p \cdot z^j$. However, $\sum_h w^h \leq \sum_h e^h + \sum_j z^j$ provided $\langle (w^h)_{h\in H}, (z^j)_{j\in J}\rangle$ is attainable in $\mathscr{E}$, which leads to a contradiction and therefore complete the proof. ■

6.2 Individualized Economy

Since we put no restriction on the separability on the technology, we could split a firm j into H parts and let each household h own one hundred percent share of a separated part of the primitive big production set as $\theta_j^h Y^j$. Under this point of view, a new economy is defined with $J \times H$ firms owned by H households, and we call it individualized economy $\widetilde{\mathscr{E}}$. Observe budget sets in this economy are exactly the same as in $\mathscr{E}$. And whenever y^j maximizes the profit of firm j, $\theta_j^h y^j$ will maximize the profit of firm (h, j) with $\theta_j^h Y^j$; on the other hand, once y_j^h maximizes firm (h, j) for some h who owns this firm, then y_j^h/θ_j^h will also maximize the profit of firm j in primitive economy $\mathscr{E}$. Hence we have following corollary of theorem 5.0.6.

Corollary 6.2.1. *Let $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ be an economy satisfying all conditions of theorem 5.0.6, and $\widetilde{\mathscr{E}}$ be the individualized economy of $\mathscr{E}$, then the competitive equilibria of this two economies coincide.*

▶ Feb.24, 2010

It will be more interesting to consider core allocation in this individualized economy than the trivial competitive equilibria. Since by now, all firms are individual owned, there is no conceptual difficulty of defining the core for this economy. For simplicity of notations, we denote the individualized economy $\widetilde{\mathscr{E}} = \big(e^h, u^h, (Y_j^h)_{j\in J}\big)_{h\in H}$. Note, as in previous discussion about Pareto optimal, here we do not assume $\mathscr{E}$ satisfies any conditions listed in theorem 5.0.6, including the assumptions about the production set.

Definition 6.2.1. *For a subset S of H, an allocation $\langle (x^h)_{h\in S}, (y_j^h)_{(h,j)\in S\times J}\rangle$ is called an S-allocation of $\widetilde{\mathscr{E}}$, if $\sum_{h\in S} x^h \le \sum_{h\in S} e^h + \sum_{h\in S}\sum_{j\in J} y_j^h$.*

Definition 6.2.2. *An allocation $\langle (x^h)_{h\in H}, (y_j^h)_{(h,j)\in H\times J}\rangle$ is a core allocation of $\widetilde{\mathscr{E}}$, if $\nexists$ an S-allocation $\langle (w^h)_{h\in S}, (z_j^h)_{(h,j)\in S\times J}\rangle$ s.t.*

$$\begin{aligned} &u^h(w^h) \ge u^h(x^h), \quad \forall\, h \in S;\\ &u^h(w^h) > u^h(x^h), \quad \textit{at least for one } h \in S. \end{aligned}$$

Theorem 6.2.2. *Let $\widetilde{\mathscr{E}} = \left(e^h, u^h, (Y_j^h)_{j\in J}\right)_{h\in H}$ be an individualized economy with utility satisfying weak monotonicity. Then each competitive allocation $\langle (x^h)_{h\in H}, (y_j^h)_{(h,j)\in H\times J}\rangle$ of $\widetilde{\mathscr{E}}$ is also a core allocation.*

Proof. The proof of is almost the same as for theorem 6.1.1, and the only difference is that we sum over h in S instead of in H and get $\sum_{h\in S} p\cdot w^h > \sum_{h\in S} p\cdot e^h + \sum_{h\in S}\sum_j \theta_j^h p\cdot z^j$, which is contradicting with the definition property of S-allocation. ■

Chapter 7

Miscellaneous about Competitive Equilibria

7.1 Competitive Allocation as Limiting Core

The existence theorem about competitive equilibria in an economy doesn't tell us how these equilibria can be achieved by means of market activity, or in a much weaker sense, how competitive equilibria could be viewed as results from interactive activities of market participants, saying household and firms. From another stand point, as we showed in previous section, under very weak condition, each competitive allocation is in the core of the economy. And since core allocations could be viewed as the somehow reasonable results from interactive activities of market participants, i.e. no coalition is needed for getting a higher utility thus everyone will stay in the market and trade with each other, we could regard competitive equilibria as reasonable consequences in a (private) competitive market economy. However, one problem is there may exist much more core allocations in a economy than competitive allocations, and this fact prevent us from convincing the competitive equilibria being the unique reasonable consequences of market activity. Why should the household prefer a competitive equilibrium to a core allocation?

There were some reasonable arguments advocating competitive equilibria as proper and necessary result of competitive market as early as in 19th century. In his prominent book, Edgeworth (1881) illustrated that the core of an economy would shrink to competitive equilibria as the number of consumer tending to infinity by using his famous box, of course, there are only two types of consumer with the number of each type going to infinity. However this result depended essentially on geometrical interpretation of the economy with only two types of consumers, and it is quite not clear about whether or not the desired result will also hold when there are more than two types of consumers.

This problem is solved astonishingly in an elegant paper by Debreu and Scarf (1963), in which all proof is short and easy to understand. In this article, Debreu and Scarf considered a pure exchange economy $\mathscr{E}(r)$ consisting of m types of consumers and within each type there are r identical households, i.e. with identical preference and endowment. By imposing strictly convex preference[1] for each household, they showed in each core allocation for $\mathscr{E}(r)$, each household in the same type will get the same consumption bundle. Therefore, one only need to consider m consumption bundles when changing r. They also pointed out two facts that the set of core allocations of $\mathscr{E}(r+1)$ will be contained in that of $\mathscr{E}(r)$ and the competitive allocation of the economy with one household in each type will also be a competitive allocation of $\mathscr{E}(r)$[2], thus $\cap_{r=1}^{\infty}\{\text{core allocations in } \mathscr{E}(r)\}$ is not an empty set, which means one can consider an allocation[3] that belongs to the core for all $\mathscr{E}(r)$. With this preparation, Debreu and Scarf laid out following theorem.

Theorem. *If $(x_1, \ldots, x_m)$ is in the core for all r, then it is a competitive allocation.*

Therefore, roughly, we can say competitive equilibria become the unique reasonable consequence of the competitive market activities when there are infinitely many consumers in the market, at least if we restrict our criterion of "reasonable" to core allocations.

Debreu and Scarf (1963) also give two extensions of above theorem, that one is a relaxation on preference being convex but not strictly convex, and the other one is a similar result in a set up of production economy.

No surprising, one may wonder what will happen if no identical households are assumed, or equivalently what's going on if the number of types changes. For this kind of questions, the answer is similar to above theorem, that the core will shrink to competitive equilibria when the total number of household in the economy tends to infinity, provided that the economy becomes more "competitive", which can be formalized as

$$\frac{\sup_h ||e^h||}{||\sum_h e^h||} \to 0, \text{ when } H \to \infty,$$

i.e. the ratio of endowment of any household comparing to the total endowment falls to zero as the number of households goes to infinity.

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[1] Preference $\succsim$ is strictly convex in a consumption space X, if for any x, $y \in X$ with $x \succsim y$, there is $\alpha x + (1-\alpha)y \succ$ y for all $\alpha \in (0,1)$.

[2] I.e. households of type t in $\mathscr{E}(r)$ will get the same C.E. consumption bundle as the only household of type t in $\mathscr{E}(1)$.

[3] No matter what r is, we can always treat an allocation in $\mathscr{E}(r)$ as consisting of m consumption bundles.

7.2 Uniqueness of Competitive Equilibrium

One problem with competitive equilibria, as Nash equilibria, is that there may exist more than one equilibrium. For example, consider an 2×2 Edgeworth box with $e^1 = (2, 0)$, $e^2 = (0, 2)$, and $u^1(x, y) = \min(x, y)$, $u^1(x, y) = \min(x, y)$. Then it's obvious that every point on the line from $(0, 0)$ to $(2, 2)$ is a competitive allocation.

The most famous condition addressing this problem is called *gross substitution*, which is a sufficient condition to ensure the uniqueness of competitive equilibrium.

Consider a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h\in H}$ where demand function $x^h(p) \equiv x^h(p, e^h) = \text{argmax}_{z\in B^h(p,e^h)}\, u^h(z)$ is well defined, e.g. imposing strict convexity to the preference, and let $z^h(p) = x^h(p) - e^h$ denote the excess demand function, where both $x^h(p)$ and $z^h(p)$ are vector valued function from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. Further define aggregate excess demand function as $z(p) = \sum_h z^h(p)$. Notice, $z(p) = z(\alpha \cdot p)$ for all $p \in \mathbb{R}^L_+$ and all scalar $\alpha > 0$.

Definition 7.2.1. *$z(p)$ satisfies gross substitute property, if for all $\ell \in L$ and all p, $p' \in \mathbb{R}^L_+ \setminus \{0\}$ s.t. $p'_\ell > p_\ell$ and $p'_k = p_k\ \forall\, k \neq \ell$, there is $z_k(p') > z_k(p)$, $\forall\, k \neq \ell$.*

Gross substitution characterizes a property of the excess demand function, that once you increase the price of one commodity and keep all other prices unchanged then the excess demand for all commodities except this one goes up.

As we know, a price vector p and a set of allocations $(x^h)_{h\in H}$ consist of a competitive equilibrium in $\mathscr{E}$, if and only if

$$z(p) = \sum_h x^h(p) - \sum_h e^h = \sum_h x^h - \sum_h e^h = 0.$$

With this observation, we could easily prove following uniqueness theorem about competitive equilibrium.

Theorem 7.2.1. *Let $\mathscr{E} = (e^h, u^h)_{h\in H}$ be a pure exchange economy where demand function is well defined and all possible competitive equilibria consist of strictly positive price vectors [4]. If the aggregate excess demand function $z(p)$ of $\mathscr{E}$ satisfies gross substitute property, then there is unique competitive equilibrium[5] in $\mathscr{E}$.*

Proof. It suffices to show if p^1 and p^2 are such that $z(p^1) = 0 = z(p^2)$, then they are collinear, i.e. $p^1 = \alpha p^2$ with $\alpha > 0$. Suppose conversely, then, since p^1, $p^2 \gg 0$, we can assume $p^2 \gneqq p^1$ with $p^2_\ell = p^1_\ell$ and for some component strict inequality holds. Consider a procedure in which

[4] This is a somehow technical condition which is not emphasized in MWG pp. 613.

[5] Of course, if $\langle p, (x^h)_{h\in H}\rangle$ is a C.E., then for any positive scalar α, $\langle \alpha p, (x^h)_{h\in H}\rangle$ is also a C.E., and we shall refer these two as the same competitive equilibrium in $\mathscr{E}$.

you increase the price for one commodity k other than l from p_k^1 to p_k^2. Observe $z_\ell(\cdot)$ will not decrease in this procedure, and moreover, it will increase by a strict positive amount at least in one step since $p_k^2 > p_k^1$ for some k. Hence we have $z_\ell(p^1) \gneqq z_\ell(p^2)$, which contradicts with $z(p^1) = 0 = z(p^2)$. ■

An example of gross substitution is the demand function derived from constant elasticity of substitution (CES) utility function. Let $u(x_1, \ldots, x_L) = (\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho)^{1/\rho}$ be a CES utility function where $\alpha_1, \ldots, \alpha_L > 0$ and $\rho \in (-\infty, \infty)$. Suppose initial endowment to be $e = (e_1, \ldots, e_L) \gg 0$, and the corresponding Mashallian demand function is $x(p, e) = (x_1(p, e), \ldots, x_L(p, e))'$. It can be showed that $\partial x_k / \partial p_\ell > 0$, for $k \neq \ell$.

In general, no extension of gross substitution property is made for a production economy, since it seems not reasonable to assume that whenever the price of one commodity goes up, a firm would increase its demand (as input) or reduce its supply (as output) of other commodities.

7.3 Second Welfare Theorem

In fact, second welfare theorem, sometimes also called the second fundamental theorem for welfare economics, said nothing which makes sense, and it's an empty theorem. This theorem asserts that every Pareto optimal point in an economy with convex preferences and convex production sets can be achieved as a competitive allocation by appropriate lump-sum transfers of wealth to each agent.

However, from the existence theorem for competitive equilibrium proved in previous sections, it's not too difficult to show that for each Pareto optimal point, the social planner can first redistribute the initial endowments $e = \sum_h e^h$ to each household and each firm according to the Pareto optimal allocation $\langle (x^h)_{h\in H}, (y^j)_{j\in J}$, at the same time from the Pareto optimal allocation the social planner could find a proper price vector p such that each household h maximizes its utility within $B^h(p, (y^j)_{j\in J})$ and each firm j maximizes its profit in Y^j, then the social planner could decompose all the firms in a way like constructing an individualized economy and redistribute $\theta_j^h y^j$ to household h. After all this done, the desired Pareto optimal is a natural consequence of these redistributed endowments, or if you like, lump-sum transfers of wealth, since the social planner just puts the economy into a competitive equilibrium.

More intuitively, we can consider a pure exchange economy with differentiable utility functions, and suppose we have a Pareto optimal allocation $(x^h)_{h\in H}$ which is a interior point of $\mathbb{R}_+^{LH}$. We're going to show $\nabla u^1(x^1) = \cdots = \nabla u^H(x^H)$, where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_L}\right)$ denotes the gradient of $f(\cdot)$. Suppose not, say, $\nabla u^i(x^i) \neq \nabla u^j(x^j)$, then we could find a vector Δe

s.t. $e^i + \Delta e,\ e^j - \Delta e \in \mathbb{R}^L_+$, and $\nabla u^i(x^i) \cdot \Delta e,\ \nabla u^j(x^j) \cdot (-\triangle e) > 0$[6]. But this leads to a contradiction since $e^i + \Delta e,\ e^j - \Delta e$ is also attainable within initial resource constraint. Now, let the price vector $p = \nabla u^1(x^1)$, and redistribute the endowment as $(x^h)_{h \in H}$, we get a competitive equilibrium, but in fact there is no trading here since the economy has already been put in a competitive equilibrium.

[6] You may recall a problem in the final exam by Prof. Muench last semester which showed how you could find such a vector.

Chapter 8

Uncertainty

▶ Mar.1, 2010

8.1 Time, Uncertainty and Information Structure

This section is supplemented completely by Yan Liu. A general form of information structure will be introduced, which I wish to be of some help in understanding the setup of models in various fields in economics whenever time and uncertainty play a fundamental role. Nonetheless, all our discussion in the following sections will based on a simplest version of information structure, and no difficulty should be there if you decide to skip this section.

Consider an economy whose activities extend from 0 to ∞. Denote time period $t = 0, \ldots, \infty$. Basically, the uncertainty arises because the economy may stay in an arbitrary *state* $s_t \in S_t$, where S_t, *state space at* t, is the set of all possible states at t, e.g. the economy may be in a drought or not in a particular year t, thus $S_t = \{\text{drought, normal}\}$. We'll always assume there is no uncertainty in the initial period $t = 0$ [1], i.e. S_0 has only one element; on the contrary, there are uncertainty from $t = 1, \ldots, \infty$, i.e. S_t has more than one element for all t from 1 to infinity. In general, no restriction on S_t to be finite set is imposed.

Let $S = \prod_{\tau=0}^{\infty} S_\tau \equiv S_0 \times \cdots \times S_t \times \cdots$ be the set of all possible states in the economy, and let $S^t = \prod_{\tau=0}^{t} S_\tau$ be the set of all possible states up until time t, for all $t = 1, 2, \ldots$. It is clear that $S^t \subset S^{t+1}\ \forall\, t$. Further, we call $s^t = (s_0, s_1, \ldots, s_t) \in S^t$ a *path* from 0 to t, and $s = (s_0, s_1, \ldots) \in S$ a *path* of this economy.

Of course, our objective is to distinguish different commodities at time t. It is obvious that an apple and an orange at time t should be different commodities, no matter what state

[1] We can also consider the case in which uncertainty appears from this initial period, however it seems few models are set up in this way.

it is in the economy. However, in an uncertain circumstance, not only physical property or merely time is of great importance to a complete characterization of a commodity, but the state in which the commodity is at time t, moreover, perhaps all the underlying historic states where it has stayed before time t are critical for a fully understanding of its current relevance to the economy. For example, consider an economy extends for three periods, and $S_t = \{\text{drought, normal}\}$ for $t = 1, 2$. Suppose you have one bottle of water at $t = 2$, and also assume it is in a drought at $t = 2$. In such a case, it's quite possible that your valuation of the water will differ given different states in which the economy was in the previous period, i.e. you may valuate the water much higher if it was also a drought at $t = 1$. In this sense, we would like to distinguish this particular bottle of water with a label of $\langle$drought, drought$\rangle$, and say this bottle of water is a bottle of water contingent to an *event* $\langle$drought, drought$\rangle$[2].

In above example, we make a comprehensive distinction of commodities in a uncertain world by using *event contingent* notion[3]. However, the particular event $\langle$drought, drought$\rangle$ in this example is merely a point of state space $S^2 = S_1 \times S_2$[4], and it could be more convenient if we define an event as a subset of S^2. We could elaborate this idea by following example. Suppose we use precipitation measured by millimeter to indicate a particular state where the economy stays, hence the state spaces for this economy now become $S_1 = S_2 = [0, \infty)$. Moreover, let 50mm be the criterion of drought, so whenever $s_t \leq 50$ the economy is in a drought. With this notation, the previous event $\langle$drought, drought$\rangle$ has a new form $\{(s_1, s_2) | 0 \leq s_1, s_2 \leq 50\}$ which is a subset of $S^2 = S_1 \times S_2$.

With this understanding, we want define a event e^t to be a subset of S^t. However, not an arbitrary subset could be called as an event. Consider a set defined as $e = \{(s_1, s_2) | 0 \leq s_1 \leq 50,\ 0 \leq s_2 \leq 40\}$ in the previous example. e is an subset of S^2, but it is not an event, since we have never defined what is $\{0 \leq s_2 \leq 40\}$.

Therefore, first define *spot event* e_t as a non-empty subset of S_t, and *spot event set* E_t as a collection of all spot events at t of which the union contains S_t[5]. Notice, two spot events, as subsets of state space, may have non-empty intersection.

All spot event sets, with underlying state spaces, are given *a priori* as a component of the economy, as the agent set, relevant substance and the action set (like production plans in a production economy) in this economy, and implicitly, time structure in this economy is also fully specified at the same time.

And then, define a *event* e^t as a sequence of spot events taking the form of $\langle e_0, \ldots, e_t \rangle$,

[2] Event contingent commodity is a more general concept compared with *state contingent commodity* that will be introduced in the following section, in which a two period setup is laid out.

[3] We use this term following Debreu (1959a: Ch.7).

[4] Notice S_0 is s singleton set, hence it doesn't matter to omit S_0 from S^2.

[5] This condition is a natural requirement as we don't want to see the economy stays in some state that no event occurs.

which is equivalent to the recursive form $\langle e^{t-1}, e_t \rangle$; naturally, the corresponding *event set* is defined as a set of all events at t,i.e. $E^t = E_0 \times \cdots \times E_t$. Obviously, $E^t \subset E^{t+1}$, and a path $s^t \in e^t$ iff $s_0 \in e_0, \ldots, s_t \in e_t$.

It is clear that once all event sets are specified in the economy, the spot event sets are also specified.

Uncertainty unfolds according to the time. At the beginning of each period, the economy will switch into a new state. This state may not be observable, but all events containing this state will be observed. With this information, all future events with this history are also determined.

Hence we give following definition of information structure.

Definition 8.1.1. *A collection of event spaces $(E^t)_{t\geq 0}$ with underlying state spaces $(S_t)_{t\geq 0}$, both of which are given as components of an economy $\mathscr{E}$, are called the information structure of $\mathscr{E}$, and each E^t is called the information set by time t.*

By now, we have not employed any probabilistic concepts to characterize the uncertainty in the economy, and the information structure are described in the term of events. Yet, for a more analytic framework, we would like to have the information structure compatible with a probability model, where more powerful tools can be used.

To achieve this objective, the only modification is to extend spot event set E_t to be a σ-field, of which the elements are subsets of S_t[6]. Let E_t denote this σ-field as well, and call it *spot event field.* Then, define *event field by time t*, E^t, as $E_0 \times \cdots \times E_t$, which is also a σ-field. Finally, define *event field* $E = \cup_{t\geq 0} E^t \equiv \prod_{t=0}^{\infty} E_t$, so a probability $P : E \to [0, 1]$ can be defined. Now, the probabilistic version of our information structure becomes a probability space (S, E, P). In most models, the σ-field E^t is also called *information set* by time t.

8.2 Arrow-Debreu Economy

We're going discuss uncertainty in an economy.

Let $L = \{1, \ldots, 5\}$ be a set of 5 different commodities, $S = \{1, \ldots, 4\}$ be 4 possible states in which our economy could be, and $T = \{1, 2, 3\}$ be 3 different time periods during which activities take place.

[6] Given state space Ω, a σ-field $\mathscr{F}$ is defined as a collection of subsets of Ω which satisfies: i. $\Omega \in \mathscr{F}$; ii. if $A \in \mathscr{F}$, then the complement, $A^c \in \mathscr{F}$; iii. if $A_n \in \mathscr{F}$, $n = 1, \ldots, \infty$, then $\cup_n A_n \in \mathscr{F}$. Given state space S_t and spot event set E_t, one can find a *minimal* σ-field $\mathscr{F}_t$ that contains E_t, and we call it the σ-field generated by E_t, which is also denoted $\sigma(E_t)$.

We can consider the commodities in a funny way, saying *state contingent* commodities, i.e. a commodity ℓ which is available at time t in state s, denoted as ℓts, is a different commodity from $\ell t's'$ where $t's'$ is a different time-state combination[7]. Since time could be viewed as a special kind of state, it's convenient to omit time index, and use "state contingent" only to denote different commodities. In addition, we assume there are 60 markets here, i.e. *complete market*, each of which is opened for trading a distinct state contingent commodity. Thus oranges in a sunny day could be traded with apples in a rainy day, or even with oranges from the same tree but in a rainy day, at least in the sense of promises of delivery in a specific state.

But actually, it is a very bad model to assume complete markets, since we need too many markets to support all the trading between two different state contingent commodities, especially when the number of possible states in the economy is huge. In real world, it is always the case that there are too many states, yet without enough markets.

Even though, it is of great value for us to consider a complete market setup for uncertainty in an economy, because with the notion of state contingent commodities, we could treat uncertainty exactly in the same way as an deterministic complete market model, in which existence of competitive equilibria is already established, and various concepts of welfare optimality have been investigated.

Formally, consider a two period economy consists of a set of households $H = \{1, \dots, H\}$ and a set of firms $J = \{1, \dots, J\}$, where a private ownership is assumed, saying $\left(\theta_j^h\right)_{(h,j)\in H\times J}$ is given with $\sum_h \theta_j^h = 1$. There is no uncertainty in first period $t = 0$, and the economy may stay in one state $s \in S$ in the second period $t = 1$, where $S = \{1, \dots, S\}$ is a finite set of possible states[8] in $t = 1$. At $t = 0$, no commodity is available, and at $t = 1$, a set of different kinds of commodities $L = \{1, \dots, L\}$ is available, i.e. delivery will be made and all commodities will be consumed once a specific state reveals. Therefore, consider contingent commodity space $\mathbb{R}_+^{LS}$, where each point $x = \{x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}\}$ is a collection of the quantity of every kind of commodities in every state. An endowment vector $e^h \in \mathbb{R}_+^{LS}$ is specified for each household h *a priori*, and h knows this information *ex ante*, but only $e_s^h = \{e_{1s}^h, \dots, e_{Ls}^h\}$ will be given to h if state s reveals.

For each state contingent commodity, there is a market opened at the beginning of $t = 0$, hence a (relative) price system $p = (p_{\ell s})_{(\ell,s)\in L\times S} \in \mathbb{R}_+^{LS} \setminus \{0\}$ is observable by all households and firms. Moreover, denote $p_s = (p_{1s}, \dots, p_{Ls}) \in \mathbb{R}_+^{L} \setminus \{0\}$, and $p = (p_1, \dots, p_S)$.

Each firm will choose a production plan $y^j \in Y^j \subset \mathbb{R}_+^{LS}$, where production set Y^j depicts the technological restriction for possible production allocations in every state, to maximize

[7] One index could be the same, i.e. even if one of $t = t'$ and $s = s'$ is true, ℓts is still different from $\ell t's'$.
[8] Using the notions in the previous section, each state here by itself is a event.

the *ex ante* profit[9], $p \cdot y^j$ within this price system p. Denote $y_s^j = (y_{1s}^j, \ldots, y_{Ls}^j) \in \mathbb{R}^L$, and $y^j = (y_1^j, \ldots, y_S^j)$. One could imagine this firm j writes down this production plan as a promise, and after the state at $t = 1$ reveals, it will deliver what ever written as positive entries in the promise y_s^j and receive whatever written as negative entries.

Each household h will choose a consumption plan $x^h \in \mathbb{R}_+^{LS}$ according to the observed price system, the endowment known *a priori* and the profit share from firms, by maximizing *ex ante* utility [10], given by utility function $u^h(\cdot)$ defined over $\mathbb{R}_+^{LS}$. More specifically, the *ex ante* resource constraint of h is given by a budget set as follows,

$$B^h(p, (y^j)_{j\in J}) = \left\{ x \in \mathbb{R}_+^{LS} : p \cdot x \le p \cdot e^h + \sum_{j\in J} \theta_j^h \, p \cdot y^j \right\}.$$

And h is going to maximize its utility on this budget set and make the decision about its consumption plan at $t = 1$ according to this price system. One can imagine h writes down its *net demand* $x^h - e^h - \sum_j \theta_j^h y^j$ as a promise, and acts in a similar way as firm j.

In general, we call such a setup a Arrow-Debreu economy. Evidently, we can treat this economy in the same way as a deterministic economy. We could define Arrow-Debreu equi-

[9] One might wonder why a firm should maximize a so defined profit. In fact, rewrite $p \cdot y^j$ as $\sum_s p_s \cdot y_s^j$, where y_s^j is production plan for state s. We see once j maximizes $p \cdot y^j$, the *ex ante* profit $p_s \cdot y_s^j$ in state s will also be maximized. However, since p_s represents relative prices, $p_s \cdot y_s^j$ is the *ex post* profit in state s under p_s as well, and hence is also maximized.

[10] One may wonder as in the case for firms why household h would maximize such a *ex ante* utility. Since its *ex post* utility should come from its consumption in a particular state s, hence it seem more reasonable for h to maximize its utility separately in each state. However, for such a reasoning, one conceptual problem arisen here is what utility function h should use for such a purpose?

Since $u^h(\cdot)$ is defined on $\mathbb{R}_+^{LS}$, it seems that the utility level in state s may depend on its consumption plan for other state as well, if we use $u^h(\ldots, x_s^h, \ldots)$ directly to measure its utility.

However, consider a case in which h has two states, one is in good luck and h might win \$500 or more in lottery, while the other one is in bad luck and h could only win less \$20. If possible, let $u(g, b)$ denote h's utility function with g being the money won in good luck and b in bad luck, and also assume $u(g, b)$ is differentiable. Should h's utility level $u(g, 15)$ increase as he is in bad luck already and wins only \$15 ex post while if he's in good luck and wins more and more money? This question seems ridiculous since h can only be in one possible state, either good or bad luck, but can not be in both. Thus it seems reasonable to assume $\partial u(g, \bar{b})/\partial g$=0 for any given $\bar{b}$, saying there's no cross effect between different states.

And in this sense, one should be able to write $u^h(x_1^h, \ldots, x_S^h) = \sum_s u_s^h(x_s^h)$, where $u_s^h(\cdot)$ is a state specific utility function. Once $u^h(\cdot)$ has this form, h could maximize $u(x_1^h, \ldots, x_S^h)$ with respect to x_s only, and no worried about effects from consumption plans in other states is needed.

Debreu (1959b) proves that once utility function has such a *state independent* property, then it could indeed be written in additive form. It is also worth to mention that expected utility by definition posses such a property.

Back to our original problem, whenever state independent property is assumed, then maximizing $u^h(\cdot)$ on the *ex ante* budget set is equivalent to maximize $u^h(\cdot)$ w.r.t x_s^h only on a degenerate budget set $\{x \in \mathbb{R}_+^L : p_s \cdot x_s \le p_s \cdot e_s^h + \sum_{j\in J} \theta_j^h \, p_s \cdot y_s^j\}$ for every state s. Since p_s represents relative price in s, the last assertion implies x_s^h also maximizes *ex post* utility in this particular state.

librium, which is merely competitive equilibrium in this setup, as we did before, and we could establish existence of competitive equilibria using the same theorems. Now, we just laid out the definition for equilibrium as follows.

Definition 8.2.1. $\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ *is a competitive equilibrium (C.E.) of an Arrow-Debreu economy, if*

- *Firms maximize profit.*
  $y^j \in \text{argmax}_{z\in Y^j}\, p\cdot z,\ \forall\, j\in J.$
- *Households maximize utility.*
  $x^h \in \text{argmax}_{z\in B^h(p,(y^j)_{j\in J})}\, u^h(z),\ \forall\, h\in H.$
- *Market clear.*
  $\sum_h (x^h - e^h) - \sum_j y^j = 0.$

8.3 Assets, Rational Expectation and Radner Equilibrium

As mentioned above, the disadvantage of a setup like Arrow-Debreu economy is that too many markets need to be assumed existing *ex ante* to fulfill the trading demand for the economy, and this is highly unrealistic in real economy. Fortunately, there is another approach to formulate uncertainty in an abstract economy.

For simplicity, we consider an exchange economy. The basic setup takes the same form as Arrow-Debreu economy, except there is no production sector and no firm shares for household.

However, instead trading promises in at $t = 0$ in $L \times S$ markets, households trade K assets $A^1, \ldots, A^K$ in K markets, where $A^k = (A^k_1, \ldots, A^k_S) \in R^S_+$ and whenever h has one unit A^k then h will receive a return of A^k_s dollars "money" at $t = 1$ if s reveals. Of course, here "money" should be represented by some real commodity otherwise it makes no sense for households to purchase any physical commodities with this "money", so we set $1 \in L$ for each state $s \in S$ to be the *numeraire*, and hence there are S numeraires. *Short position* is allowed, and also assume all assets are perfect divisible, so the portfolio $\theta^h = (\theta^h_1, \ldots, \theta^h_K)$ [11] held by household h is a vector in $\mathbb{R}^K$, hence $\sum_k \theta^h_k A^k_s$ is the total "money" h receives in s with this portfolio θ^h. Further, let $\pi = (\pi_1, \ldots, \pi_K)$ denote the prices for each asset observed in the K markets, and assume no initial assets for every household, therefore $\pi \cdot \theta^h \le 0$, reading as all portfolio should be *self financing*.

[11] Whenever $\theta^h_k < 0$, it is called a short position and h need to pay $\theta^h_k A^k_s$ dollars to the buyer if s reveals at $t = 1$.

Instead of evaluating any assets, households only evaluate an *ex ante* determined consumption plan as in an Arrow-Debreu economy. Therefore, no direct way exists to tell us how a household h would choose his portfolio θ^h. However, we will see how to overcome this conceptual difficulty after we investigate the procedure in which households choose their consumption plan.

Assume each household h has its own expectation at $t = 0$ *ex ante* about the *spot prices* $p_s = (p_{1s}, \dots, p_{Ls}) \in \mathbb{R}^L_+ \setminus \{0\}$ prevailing at $t = 1$ for every state s, and as before commodity 1 is set to be the numeraire. Assume h already has a portfolio θ^h, so the total "money" resource is $p_s \cdot e^h_s + p_{1s} \sum_k \theta^h_k A^k_s$ in state s. According to these resource constraints, h chooses a consumption plan $x^h = (x^h_1, \dots, x^h_S)$ s.t. $p_s \cdot x^h_s \leq p_s \cdot e^h_s + p_{1s} \sum_k \theta^h_k A^k_s$ for all s. Now observe that households may get a higher utility by adjusting their portfolios, so after forming their expectation about the spot prices, they maximize utility by choosing portfolios and consumption plans at the same time.

Summing up, the budget set for household h will take following form,

$$B^h(p, \pi) = \left\{ (\theta^h, x^h) \in \mathbb{R}^K \times \mathbb{R}^{LS}_+ : \pi \cdot \theta^h \leq 0 \text{ and } p_s \cdot (x^h_s - e^h_s) \leq p_{1s} \sum_k \theta^h_k A^k_s, \, \forall \, s \in S \right\},$$

note the profile of spot prices $p = (p_1, \dots, p_S)$ is of h's own expectation.

Every household makes the portfolio decision and chooses consumption plan for the next period, then after a particular state s is revealed at the beginning of second period, L markets open for the state contingent commodities $1s$ to Ls, in which all delivery are accomplished. But a essential problem follows: if the prices for these L commodities don't coincide with households' expectations, how could market clear be satisfied?

The breakthrough in this setup turns out to be a fundamental notion that explains, or in fact defines how peoples' expectations will be realized in the second period. This notion is *rational expectation*. By assuming all households have rational expectation, it means not only all of them have same expectations about spot prices in the future, but also these expected prices will indeed clear the markets whenever they open in a particular state. This latter characterization for rational expectation is also called *self fulfill*. Hence, once we impose rational expectation assumption to the household, an equilibrium concept analogous to competitive equilibrium could be laid out without any difficulty. Formally, such a equilibrium concept is called Radner equilibrium.

Definition 8.3.1. *$\langle \pi, p, (\theta^h, x^h)_{h \in H} \rangle$ constitutes a Radner equilibrium in a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$ under uncertainty $S = (1, \dots, S)$ with assets $A^1, \dots, A^K \in \mathbb{R}^S_+$, if*

- *Household maximizing utility,*
  for all $h \in H$, $u^h(x^h) \geq u^h(\tilde{x}^h)$, $\forall \, (\tilde{\theta}^h, \tilde{x}^h) \in B^h(p, \pi)$.

- *Asset market clear,*
  $\sum_h \theta^h = 0.$

- *Commodity market clear,*
  $\sum_h x^h = \sum_h e^h.$

8.4 Complete Market and Asset Structure

It turns out that asset structure, i.e. complete or incomplete, is a fundamental factor for the implications of Radner equilibrium. To formalize our discussion, let $A = (A^1, \dots, A^K)$ be the return matrix of these K assets where A^k is treated as a column vector.

Definition 8.4.1. *Let S be the number of states in an economy. Then, the asset structure of $A^1, \dots, A^K$ is said to be complete, if* $\operatorname{rank}(A) = S$*; and incomplete, if* $\operatorname{rank}(A) < S$*.*

Using a different notation, the asset structure is complete iff $\operatorname{span}(A^1, \dots, A^K) = \mathbb{R}^S$. Note also, the maximal number of rank A is S, since A is an $S \times K$ matrix. Also note, with this notation, the commodity budget constraint can be written as

$$\begin{bmatrix} p_1 \cdot (x_1^h - e_1^h) \\ \vdots \\ p_S \cdot (x_S^h - e_S^h) \end{bmatrix} \leq \begin{bmatrix} p_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{1S} \end{bmatrix} A\theta^h,$$

where θ^h is arranged as a column vector.

Now we consider a special kind of assets. Let $A^1, \dots, A^S$ be S unit vectors[12] in $\mathbb{R}^S$, and it is often called Arrow-Debreu security for each one of such kind of assets. Following theorem asserts that once assuming such an asset structure (which is obviously complete), then Radner equilibrium coincides with Arrow-Debreu equilibrium. And in this sense, complete asset structure and complete market are the two sides of a same coin.

Theorem 8.4.1. *Assume $A^1, \dots, A^S$ to be S assets in a pure exchange economy $\mathscr{E}$ under uncertainty with weakly monotonic utility, where each A^s is a unit vector in $\mathbb{R}^S$. Then, Arrow-Debreu equilibrium $\langle p, (x^h)_{h\in H}\rangle$ with $p \gg 0$ is the same as Radner equilibrium $\langle \pi, p, (\theta^h, x^h)_{h\in H}\rangle$ with $\pi, p \gg 0$, in the sense that their equilibrium allocations are the same.*

Proof. We divide this proof into two parts.

(i).We show how to construct π and $(\theta^h)_{h\in H}$ such that $\langle \pi, p, (\theta^h, x^h)_{h\in H}\rangle$ constitutes a Radner equilibrium where p and $(x^h)_{h\in H}$ come from an Arrow-Debreu equilibrium.

[12] I.e. A^s is a vector of which the s'th component is 1 and all other components are 0.

Let Λ be an diagonal matrix with p_{1s} as its (s, s) element, then Λ is invertible since $p_{1s} > 0$. Define $\pi = \mathbf{1}\Lambda = (p_{11}, \ldots, p_{1S})$, where $\mathbf{1} = (1, \ldots, 1)$. Let $w_h = (p_1 \cdot (x_1^h - e_1^h), \ldots, p_S \cdot (x_S^h - e_S^h))$ be a column vector. Since utility is weakly monotonic, households will exhaust their resources in an Arrow-Debreu equilibrium, so $\mathbf{1}w^h = p \cdot (x^h - e^h) = 0$. And by market clear in Arrow-Debreu equilibrium, $\sum_h w_h = 0$. Define $\theta^h = \Lambda^{-1}w^h$, hence commodity budget feasibility are satisfied by definition. And we have $\sum_h \theta^h = \Lambda^{-1}\sum_h w^h = 0$, thus assets market clear is satisfied. Further, $\pi \cdot \theta^h = \mathbf{1}\Lambda\Lambda^{-1}w^h = \mathbf{1}w^h = 0$, thus portfolio θ^h meets with the budget feasibility.

We assert (θ^h, x^h) maximizes $u^h(\cdot)$ on the budget set for Radner equilibrium. Let $(\tilde{\theta}^h, \tilde{x}^h) \in B^h(p, \pi)$ be a point in the budget set for Radner equilbrium, then $\mathbf{1}\Lambda \cdot \tilde{\theta}^h = \pi \cdot \tilde{\theta}^h \leq 0$. Observe that $\tilde{w}^h \leq \Lambda\tilde{\theta}^h$, where $\tilde{w}^h = (p_1 \cdot (\tilde{x}_1^h - e_1^h), \ldots, p_S \cdot (\tilde{x}_S^h - e_S^h))$, hence $p \cdot (\tilde{x}^h - e^h) = \mathbf{1}\tilde{w}^h \leq \mathbf{1}\Lambda\tilde{\theta}^h \leq 0$. Hence $\tilde{x}^h$ belongs to the budget set for the Arrow-Debreu equilibrium, and so there is $u^h(x^h) \geq u^h(\tilde{x}^h)$, which justifies our assertion.

(ii).We show how to construct a price vector $\hat{p}$ such that $\langle \hat{p}, x^h \rangle$ constitutes an Arrow-Debreu equilibrium where $\langle \pi, p, (\theta^h, x^h)_{h \in H} \rangle$ is a Radner equilibrium.

Without loss of generality, we assume $p_{1s} = 0$, hence Λ is an identity matrix. Define $\hat{p} = (\pi_1 p_1, \ldots, \pi_S p_S)$ where p_s is viewed as row vector. Using the same notation as in (i), we have $\pi \cdot \theta^h \leq 0$ and $w^h \leq \Lambda\theta^h = \theta^h$. Hence $\hat{p} \cdot (x^h - e^h) = \sum_s \pi_s p_s \cdot (x_s^h - e_s^h) = \pi \cdot w^h \leq \pi \cdot \theta^h \leq 0$, so x^h meets with the budget feasibility for an Arrow-Debreu equilibrium with a price vector $\hat{p}$. Since market clear is the same between these two equilibrium, only maximality of x^h remains to be shown.

Let $\tilde{x}^h$ be a point in the budget set for Arrow-Debreu equilibrium, so $\pi \cdot \tilde{w}^h = \sum_s \pi_s p_s \cdot (x_s^h - e_s^h) = \hat{p} \cdot (x^h - e^h) \leq 0$. Define $\tilde{\theta}^h = \tilde{w}^h$, so we have $\pi \cdot \tilde{\theta}^h = \pi \cdot \tilde{w}^h \leq 0$. Therefore $(\tilde{\theta}^h, \tilde{x}^h)$ is point in the budget set for the Radner equilibrium, hence $u^h(x^h) \geq u^h(\tilde{x}^h)$. ∎

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