Homework 5

Due on Monday 13th before midnight - Recitation Tuesday 4th at 9:00am

1. Show that the second-price auction is an application of the VCG mechanism.

Solution:
Second-price auction with n players:

- Let X be the set of possible outcomes. There are $n+1$ outcomes, i.e., sell the item to one of the n bidders or not sell the item.
- Let b_i be agent i's bid.
- Agent i wins the auction if $b_i > \max_{j\neq 1} b_j$ and pays $\max_{j\neq 1} b_j$

If in outcome $x \in \mathrm{X}$ agent i wins the auction, her valuation is $v_i(x) = b_i$ and if in outcome $x' \in \mathrm{X}$ agent i does not win the auction, her valuation is $v_i(x) = 0$

In the VCG mechanism, the outcome is obtained by maximizing the sum of valuations, i.e.,

$$x^* = \arg\max_{x\in \mathrm{X}} \sum_{i=1}^{n} v_i(x),$$

that is, the outcome is that in which the highest bidder wins the auction.

Moreover, if agent i wins, she pays

$$t_i = \underbrace{\sum_{j\neq i} v_j(x^*_{-i})}_{\substack{\text{total value of}\\ \text{others if } i \text{ did}\\ \text{not participate}}} - \underbrace{\sum_{j\neq i} v_j(x^*)}_{\substack{\text{total value of}\\ \text{others if } i \text{ wins}}} = \max_{j\neq i} b_j - 0 = \underbrace{\max_{j\neq i} b_j}_{\text{second highest bid}},$$

and if agent i does not win, she pays

$$t_i = \sum_{j\neq i} v_j(x^*_{-i}) - \sum_{j\neq i} v_j(x^*) = 0$$

Thus, the second-price auction is an application of the VCG mechanism.

2. Suppose a bridge costs 13 to build[1] and that the valuation of the bridge by the four individuals is given by

Individual	1	2	3	4
Valuation	5	4	1	2

(a) Apply the pivot mechanism to decide whether or not the bridge is built, and the taxes (negative or positive) levied on the individuals; and the surplus (if any) collected by the government.
(b) Now suppose the bridge cost is 11. Answer the questions in part (a) again.

Solution:
(a) If the bridge costs 13, an equal split would mean that each individual would pay 13/4.

Individual	1	2	3	4
Net valuation	$5-\frac{13}{4}=\frac{7}{4}$	$4-\frac{13}{4}=\frac{3}{4}$	$1-\frac{13}{4}=-\frac{9}{4}$	$2-\frac{13}{4}=-\frac{5}{4}$
Valuations$\backslash\{i\}$	$\frac{3-9-5}{4}=-\frac{11}{4}$	$\frac{7-9-5}{4}=-\frac{7}{4}$	$\frac{7+3-5}{4}=\frac{5}{4}$	$\frac{7+3-9}{4}=\frac{1}{4}$
Pivot	NO	NO	YES	YES
Taxes	0	0	$\frac{5}{4}$	$\frac{1}{4}$

The bridge will not be built since the sum of net valuation is -1.

(b) If the bridge costs 11, an equal split would mean that each individual would pay 11/4.

Individual	1	2	3	4
Net valuation	$\frac{9}{4}$	$\frac{5}{4}$	$-\frac{7}{4}$	$-\frac{3}{4}$
Valuations$\backslash\{i\}$	$-\frac{5}{4}$	$-\frac{1}{4}$	$\frac{11}{4}$	$\frac{7}{4}$
Pivot	YES	YES	NO	NO
Taxes	$\frac{5}{4}$	$\frac{1}{4}$	0	0

The bridge is built since the sum of net valuation is 1 and the surplus of the government is 6/4.

[1] 1 unit of money here represents $10,000

3. Suppose there are equal portions of low-ability ($t = 1$) and high-ability ($t = 2$) workers. The productivity of a worker of type $t = 1,2$ is given by $\eta_t(e) = 2te$, where e is the education level. The utility of wage w and education e to a student of type t is $u_t(w,e) = 4\sqrt{w} - \frac{2e}{t}$. Find the Rothschild-Stiglitz equilibrium.

Solution:
Low ability workers choose the optimal contract along their productivity line ($w = 2e_{\mathrm{L}}$)

$$\max_{e_{\mathrm{L}}} \quad 4\sqrt{2e_{\mathrm{L}}} - 2e_{\mathrm{L}}$$

so the first order condition is

$$4(2e_{\mathrm{L}})^{-\frac{1}{2}} - 2 = 0 \Rightarrow e^*_{\mathrm{L}} = 2 \Rightarrow w^*_{\mathrm{L}} = 4.$$

Let us now find the minimum education level for the high ability required for a separating equilibrium to exist, i.e., the intersection between the indifference curve of the low ability workers that maximize their utility with the productivity line of the high ability workers.

$$u_{\mathrm{L}}(w^*_{\mathrm{L}}, e^*_{\mathrm{L}}) = 4 \quad \& \quad w = 4e \quad \Rightarrow \quad 4\sqrt{e} - e - 2 = 0 \quad \Rightarrow \quad \underline{e} = \begin{cases} 0.343 \; < \; e^*_{\mathrm{L}} & ✘ \\ 11.65 \quad ✓ \end{cases}$$

The high ability workers choose the optimal contract along their productivity line

$$\max_{e_{\mathrm{H}}} \quad 4\sqrt{4e_{\mathrm{H}}} - e_{\mathrm{H}}$$

so the first order condition is

$$8(4e_{\mathrm{L}})^{-\frac{1}{2}} - 1 = 0 \Rightarrow e^*_{\mathrm{H}} = 16 > \underline{e} \Rightarrow w^*_{\mathrm{H}} = 64.$$

Thus, we have find a Rothschild-Stiglitz equilibrium where low ability workers choose the contract ($e^*_{\mathrm{L}} = 2, w^*_{\mathrm{L}} = 4$) and the high ability workers choose the contract ($e^*_{\mathrm{H}} = 16, w^*_{\mathrm{H}} = 64$).

Draw the grap!!