2. The Marriage Game.

In the marriage game the players are of two types, say boys and girls. In the absence of a strategic description of the game, the basic "rule" is simply that any boy/girl pair can marry if they wish. Polygamy, however, is frowned upon. Thus, the possible outcomes, called matchings, consist of the different sets of marriages that the players might enter into. How well a player likes a particular matching, we assume, depends only on how well that player likes his/her spouse. In other words, a player's ranking of the outcomes is independent of the other marriages that may or may not occur. Finally, we rule out all side payments -- or other means of compromise like tossing a coin.

The essential data for a marriage game is therefore nothing more than a double list of preference orderings. Here is an example:

Table of Preferences

Boys						Girls					
	A:	b	a	d	c		a:	C	B	A	D
	B:	c	d	a	b		b:	B	C	A	D
	C:	c	d	b	a		c:	A	D	C	B
	D:	d	a	b	c		d:	A	C	D	B

There are four boys (capital letters) and four girls (small letters). In this case, there are the same number of boys and girls, and we assume that not getting married at all is last on everyone's list. It is plain that not everyone in this game can get his or her first choice; thus, "A" most prefers "b", who most prefers "B", who most prefers "c",...

... etc. The question is what will happen (or what ought to happen?) if eight rational individuals, endowed with these preferences, are turned loose to bargain freely for a mate. What sort of matching will they arrive at, and why and how?

The "core" principle gives an answer of sorts, by suggesting that a matching will not be stable if some coalition is not satisfied — e.g. if some boy and girl, not matched to each other, prefer each other to their partners in the matching. Of course, the core principle refers to all coalitions, not just boy/girl pairs. But if there is some other coalition that can improve on a given matching, then it necessarily includes a boy/girl pair who could have made an improvement by forming their own coalition.*

For example, if the matching

$$\langle Aa, Bc, Cd, Db \rangle$$

were proposed in the light of the above preferences, the four players $\{A, b, C, c\}$ might correctly observe that they would all be happier if they formed a coalition and swapped partners:

$$\langle Ab, Ba, Cc, Dd \rangle.$$

But in this case, the instability of the given matching

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* "Improve" is understood to mean that <u>every</u> member of the coalition is better off. A more restrictive core concept could be based on <u>weak</u> improvements (i.e. no one worse off and at least one better off.) In the case of the marriage game this "strict core" is identical to the regular core, but this is not always the case (see Sec. 4).

would be equally apparent from the fact that either of the boy/girl pairs (C, c) or (A, b) could have made an improvement on their own. So in hunting for the core, it suffices to check just the boy/girl pairs and not worry about other coalitions.

To sum up, a proposed matching $\mathfrak{M}$ belongs to the core of the game if and only if it is _stable_, in the sense that no boy and girl, not matched in $\mathfrak{M}$, prefer each other to their "$\mathfrak{M}$-mates".

The basic technique for finding stable matchings is the _deferred acceptance_ algorithm first described by Gale and Shapley.* It is a highly intuitive procedure, easy to implement and motivate. Each boy proposes to his favorite girl. Any girl receiving more than one proposal immediately rejects all but the one she likes best. But the unrejected boys are not immediately accepted; instead they are put "on hold" while the boys who were rejected propose to their second choices. Again, any girl with more than one suitor rejects all but the best, ..., and the cycle repeats until either there are no boys rejected or the rejected boys have exhausted their lists.**

It is important to distinguish between the game and

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* _American Mathematical Monthly_ _69_ (1962), 9-15.

** As must happen if there are more boys than girls, and as may happen if some boys or girls have short lists, i.e., consider some potential mates as worse than no mate at all.

the algorithm.* The game is obviously symmetric between the sexes in its definition, but the algorithm is not. Thus if the girls do the proposing, a different set of marriages may well result. Both matchings will be stable, however, as we shall show presently. But first some examples.

The tables on page 4.6 (repeated below) provides our first example. If the boys take the initiative, the course of true love may be traced in the chart on the right:

Boys:

A:	b	a	d	c
B:	c	d	a	b
C:	c	d	b	a
D:	d	a	b	c

Girls:

a:	C	B	A	D
b:	B	C	A	D
c:	A	D	C	B
d:	A	C	D	B

	1st day	2nd day	3rd day
a:	–	–	B
b:	A	A	A
c:	~~B~~C	C	C
d:	D	D~~B~~	D
Rej:	B	B	

Day One: A proposes to b and is told to wait. Also, B and C both propose to c, who rejects B and puts C "on hold". Also D proposes to d and is told to wait, while nobody proposes to a. Day Two: B proposes to d and is again rejected. Day Three: B proposes to a and is not rejected. So the "mating season" is at an end, the girls all accept their current suitors, and the matching ⟨Ab, Ba, Cc, Dd⟩ results.

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* It is noteworthy that, although both game and solution concept are non-strategic, the solution is most effectively presented as the end result of a stylized courtship ritual. A basic assumption of cooperative game theory is that pre-play negotiations are open and unstructured, particularly in the matter of forming coalitions. In the present case, however, a detailed protocol of bids and offers emerges anyway — not as an assumption of the model or as a prediction of the theory, but merely as a kind of analog device that imparts a bit of life to an otherwise dry computation.

If the girls propose, the result is quite different:

	1st day	2nd day
A:	/cd	d
B:	b	b
C:	a	a
D:	–	c
Rej:	c	

In fact, the resulting matching ⟨Ad, Bb, Ca, Dc⟩ has <u>no</u> pairs in common with the previous one. We invite the reader to verify (1) that both are stable, and (2) that the first one is better for all of the boys and none of the girls.*

The next example has an imbalance of the sexes:

<u>Boys</u>

U:	a	b	c	d	e	f	g
V:	b	d	a	f	c	e	g
W:	d	c	b	a	g	f	e
X:	c	a	b	e	f	d	g
Y:	a	g	f	b	c	d	e
Z:	c	b	e	d	g	a	f

<u>Girls</u>

a:	V	Y	U	Z	X	W
b:	W	U	Y	V	Z	X
c:	V	X	Y	W	U	Z
d:	Z	Y	X	U	W	V
e:	X	Z	W	V	Y	U
f:	Y	X	V	W	Z	U
g:	Z	U	Y	W	V	X

If the boys propose ...

a:	~~U~~Y	Y	Y	~~Y~~V	V
b:	V	~~Y~~U~~Z~~	U	U	U
c:	X~~Z~~	X	X	X	X
d:	W	W	W~~Y~~	W	W
e:	–	–	Z	Z	Z
f:	–	–	–	–	–
g:	–	–	–	–	Y
Rej:	UZ	VZ	V	Y	

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* Theorems 1 and 2 below. There are two other matchings in the core of this example, namely ⟨Ab, Ba, Cd, Dc⟩ and ⟨Ad, Ba, Cb, Dc⟩. Observe that each person ranks each of them <u>between</u> the "boys propose" and "girls-propose" solutions. (N.B.: This is a contrived example. It is unusual to find such a large core with so few players.)

and the solution is ⟨Ub, Va, Wd, Xc, Yg, Ze⟩, with <u>f</u> the extra girl. If the girls propose the "mating season" is considerably longer ...

U:	-	g	g	g	g	~~g~~d	d	d	d	d	d	d~~f~~	d
V:	a~~c~~	a	a	a	a	a	a	a	a~~f~~	a	a	a	a
W:	b	b	b	b	b	b	b	b	b	b~~f~~	b	b	b
X:	e	~~e~~c	c	c	c~~d~~	c	c	c~~f~~	c	c	c	c	c
Y:	f	f	f	f~~d~~	f	f	~~f~~g	g	g	g	g	g	g
Z:	d~~g~~	d	~~d~~e	e	e	e	e	e	e	e	e~~f~~	e	e
Rej:	cg	e	d	d	d	g	f	f	f	f	f	f	f

but the solution is almost the same: ⟨Ud, Va, Wb, Xc, Yg, Ze⟩, with <u>f</u> left out again. Note that the last seven "days" are devoted to <u>f</u> running out her string, without success. (But examples are easily constructed having much longer "mating seasons".)

We now state and prove two theorems.

> THEOREM 1. The matchings obtained by the deferred acceptance algorithm are <u>stable</u>, in the sense that no persons not matched by the algorithm would prefer to have been matched.

<u>Proof</u>. Without loss of generality we consider only the boys-propose version. Suppose that John prefers Jane to the wife he gets from the algorithm. That means he proposed to Jane during the procedure but was rejected in favor of a boy higher on her list. Eventually, Jane either marries that boy or marries another boy whom she likes even better. So there's no instability, since Jane does not prefer John to her husband.

Call a boy [girl] <u>feasible</u> for a girl [boy] if there is a stable matching in which they are paired.

> THEOREM 2. The matchings obtained by the deferred acceptance are <u>uniformly optimal</u> for the proposers, in the sense that each gets the best feasible mate.

<u>Proof</u>. Without loss of generality, we may consider only the girls-propose version. The theorem will be proved if we can show that in the course of the algorithm no girl is ever rejected by a boy who is feasible for her.

If a rejection by a feasible boy does occur during the course of the girls-propose algorithm, then there has to be a first time. Suppose that no such rejection has yet taken place when, say, Jack rejects Mary in favor of Kate. We shall show that Jack is not feasible for Mary.

Indeed, if Jack is feasible for Mary then there is a stable matching $\mathfrak{M}$ which matches Jack with Mary and matches Kate with someone else who is feasible for her, say, Larry:

$$\mathfrak{M} = \langle Jm, Lk, \ldots \ldots \rangle.$$

But the only boys whom Kate prefers to Jack are those who have <u>already</u> rejected her. By assumption, they are not feasible for her, hence Larry, who does not reject Kate, is below Jack on Kate's list. So Jack and Kate would gladly "dump" Mary and Larry, respectively, contradicting the assumed stability of $\mathfrak{M}$. Hence Jack is <u>not</u> feasible for Mary, and the theorem is proved.

EXERCISES

Exercise 1. Find the stable assignments that are "boy-best" and "girl-best" in each of the following marriage games:

(a) Table of preferences

W:	a	b	c	d
X:	a	b	c	d
Y:	b	c	a	d
Z:	c	a	b	d

a:	Y	Z	W	X
b:	Z	W	X	Y
c:	W	X	Y	Z
d:	Z	Y	W	X

(b) Table of preferences

W:	a	b	c	d
X:	a	b	c	d
Y:	a	b	c	d
Z:	a	b	c	d

a:	Z	X	Y	W
b:	W	Y	X	Z
c:	Z	W	Y	X
d:	X	Y	Z	W

(c) Table of preferences

U:	a	b	c	d
V:	b	d	a	c
W:	d	c	b	a
X:	c	b	a	d
Y:	a	b	c	d
Z:	d	c	b	a

a:	V	Y	U	Z	X	W
b:	W	U	Y	V	Z	X
c:	V	Z	Y	W	U	X
d:	Z	Y	X	U	W	V

(d) Table of preferences

A:	v	w	x	y	z
B:	y	z	v	x	w
C:	v	x	w	z	y
D:	y	x	z	w	v

v:	B	A	C	D
w:	D	B	C	A
x:	C	A	B	D
y:	C	B	A	D
z:	C	A	D	B

Exercise 2. Show that if there are several stable matchings, the same player or set of players gets left out each time.

Exercise 3. Show that the example on pages 4.10 and 4.11 has just two matchings in its core.

3. The College Admissions Game.

In essence, the "college admissions" game is the marriage game with polygamy. A player of one type (a college) can "marry" more than one player of the other type (an applicant), up to some stated capacity. Preferences are based on simple orderings of the players of the opposite type; this involves the assumption that applicants rank only the colleges, not their prospective classmates, and that the colleges rank only individuals, not sets of individuals.*

Here is an example. The small letters represent applicants, the capitals colleges, the numerals capacities.

The colleges rank the applicants ...

(4)	W:	a	l	q	i	f	c	s	b	v	k	t	d	j	r	p	n	m	g	e	u	o	h
(3)	X:	j	n	d	c	l	p	e	g	m	a	h	i	o	r	b	q	f	u	k	s	t	v
(5)	Y:	g	l	f	t	k	s	u	q	v	o	c	i	d	p	a	r	m	j	h	n	b	e
(6)	Z:	p	k	l	q	m	h	a	n	o	g	u	b	e	t	r	j	v	f	i	d	s	c

The applicants rank the colleges ...

a:	X	Y	Z	W	l:	X	Z	Y	W
b:	X	Z	W	Y	m:	X	W	Z	Y
c:	X	W	Y	Z	n:	W	X	Z	Y
d:	Z	X	W	Y	o:	W	Y	Z	X
e:	Z	Y	X	W	p:	Z	W	X	Y
f:	Z	W	Y	X	q:	Z	Y	W	X
g:	Z	X	Y	W	r:	W	Y	X	Z
h:	Y	Z	X	W	s:	W	Z	Y	X
i:	Y	W	Z	X	t:	W	X	Y	Z
j:	Y	X	W	Z	u:	W	Y	Z	X
k:	X	Y	W	Z	v:	W	Z	X	Y

----

* In effect, the colleges are ranking sets of applicants "lexicographically" — i.e., the top person in a set determines its position in the order. Thus, a college with room for three applicants is considered to prefer the two it ranks 1 and 22 to the three it ranks 2, 3 and 4.

The solution criterion is easily stated. An assignment of applicants to colleges is deemed <u>unstable</u> if it sends an applicant to one college when there is another college that is preferred by the applicant, and either has room for him or could make room by rejecting someone else it likes less. For example, if a goes to W and y, g, c, m, and j go to Y, the situation is unstable because both a and Y would prefer that Y reject j and accept a instead. The core of the game is defined to be the set of all assignments that stay within the capacity limits and are not unstable.

We note that if all capacities are 1, then the college admissions game reduces to the marriage game. It is therefore not surprising that a similar "deferred acceptance" algorithm can be used to find an outcome in the core. On the debatable philosophy that colleges exist for the benefit of students and not vice versa,* we let the applicants do the "proposing" and so obtain the "applicant-best" point in the core.

The solution of the example proceeds as shown on the following page. The commas separate off the holdovers from the preceding "day", re-ordered for convenience according to the college's preference list.

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* The opposite philosophy might be more realistic for the case of the "athlete recruiting game"!

	1st day	2nd day	3rd	4th	5th	
(4) W:	~~n~~prst~~p~~v	svtr,	svtr,~~m~~	sv~~tr~~,cf	fcs~~v~~,b	
(3) X:	~~ap~~c~~k~~lm	cl~~m~~,n	n~~dl~~,jd	jnd,	jnd,~~t~~	...
(5) Y:	hij	i~~jk~~,ouak	kuoia,	kuoia,	kuoia,~~ef~~	
(6) Z:	defgpq	pqgef~~d~~,b	pqgbe~~f~~,h	pqhg~~be~~,lm	plqmhg,	

	6th	7th	8th	9th	10th	
(4) W:	fcsb,	fcsb,~~e~~	fcsb,	fcs~~b~~,i	ifcs,	
(3) X:	jnd,~~ef~~	jnd,~~y~~	jnd,~~a~~	jnd,	jnd,	...
(5) Y:	kuoi~~a~~,t	tkuoi,	tkuo~~i~~,v	tkuv~~o~~,g	gtkuv,~~b~~	
(6) Z:	plqmhg,~~y~~	plqmh~~g~~,a~~f~~	plqmha,	plqmha,	plqmha,~~o~~	
Out:			(er)	(er)	(er)	

	11th	12th
(4) W:	ifcs,	ifcs.
(3) X:	jnd,~~o~~	jnd.
(5) Y:	gtkuv,	gtkuv.
(6) Z:	plqmha,	plqmha.
Out:	(erb)	(erbo)

The applicant-best solution is therefore

$$\langle W\!: ifcs,\ X\!: jdn,\ Y\!: gtkuv,\ Z\!: plqmha \rangle,$$

with e, r, b and o losing out.*

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* As with the marriage problem, it can be shown that in any stable assignment the same four applicants would have been rejected.

EXERCISES

Exercise 4. Find the optimal (i.e. "applicant-best") stable assignments of applicants to colleges, based on the following preference orderings:

(a)

(4) X: i h f e c d g k a b j
(2) Y: g i c e k f j a h b d
(3) Z: a k i g e d h b j c f

a: X Y Z		g: Y Z X	
b: X Y Z		h: Z X Y	
c: X Z Y		i: Z X Y	
d: Y X Z		j: Z Y X	
e: Y X Z		k: Z Y X	
f: Y Z X			

(b)

(6) X: n m l k j i h g f e d c b a
(6) Y: a b c d e f g h i j k l m n

a: X Y	h: X Y
b: X Y	i: X Y
c: X Y	j: X Y
d: X Y	k: Y X
e: X Y	l: Y X
f: X Y	m: Y X
g: X Y	n: Y X

Exercise 5. Devise a "college best" algorithm and use it to show that the core of the example of this section consists of a single assignment.