NEED FOR RANDOMIZATION

		Colin	
		1	2
Rowena	1	-2, 2	3 -3
	2	3 -3	-4 4

← WHO IS ADVATAGED?

- If Rowena plays 1, gets payoff -2
  If R plays 2, gets -4

- If R plays $\left(\frac{1}{2}, \frac{1}{2}\right)$, gets:
  - $\frac{1}{2}(-2) + \frac{1}{2}(3) = \frac{1}{2}$ (if C plays 1)
  - $\frac{1}{2}(3) + \frac{1}{2}(-4) = -\frac{1}{2}$ (if C plays 2)

What if C plays $(q, 1-q)$?
Then R gets $q\left(\frac{1}{2}\right) + (1-q)\left(-\frac{1}{2}\right) \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

		1	2
$\frac{3}{5}$	$\rightarrow$ 1	-2 2	3 -3
$\frac{2}{5}$	$\rightarrow$ 2	3 -3	-4 4

$$= \begin{array}{c} \frac{3}{5} \\ \frac{2}{5} \end{array} \begin{bmatrix} -2 & 3 \\ 3 & -4 \end{bmatrix}$$

R plays $\left(\frac{3}{5}, \frac{2}{5}\right) \Rightarrow$ R gets

$$\frac{3}{5}(-2) + \frac{2}{5}(3) = 0 \quad (\text{if C plays } 1)$$

$$\frac{3}{5}(3) + \frac{2}{5}(-4) = \frac{1}{5} \quad (\text{if C plays } 2)$$

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if C plays $(q, 1-q)$ then

R gets $q \cdot 0 + (1-q)\frac{1}{5} \in \left[0, \frac{1}{5}\right]$

$\frac{7}{12}$	-2	3
$\frac{5}{12}$	3	-4

If R plays $\left(\frac{7}{12}, \frac{5}{12}\right)$ then

R gets

- $\frac{7}{12}(-2) + \frac{5}{12}(3) = \frac{1}{12}$ (if C plays 1)
- $\frac{7}{12}(3) + \frac{5}{12}(4) = \frac{1}{12}$ (if C plays 2
- $q\left(\frac{1}{12}\right) + (1-q)\left(\frac{1}{12}\right) = \frac{1}{12}$ (if C plays $(q, 1-$

i.e. R GUARATEES HERSELF $\frac{1}{12}$

COULD R do better ?

$$\begin{array}{cc} \frac{7}{12} & \frac{5}{12} \\ \begin{array}{|c|c|} \hline -2 & 3 \\ \hline 3 & -4 \\ \hline \end{array} \end{array}$$

Let C play $\left(\frac{7}{12}, \frac{5}{12}\right)$

$\Longrightarrow$

- C gets $\frac{7}{12}\cdot(+2) + \frac{5}{12}(-3) = -\frac{1}{12}$ (if R plays 1)
- C gets $\frac{7}{12}(-3) + \frac{5}{12}(4) = -\frac{1}{12}$ (if R plays 2)

$\Longrightarrow$

C GUARANTEES losing no more than $\frac{1}{12}$ (no matter which mixed strategy $(p, 1-p)$ is picked by R)

R CANNOT do better.

5 = 6

$$
\begin{array}{c|ccc}
 & 1 & j & k \\
\hline
1 & & & \\
\vdots & & & \\
i & & a_{ij}, b_{ij} & \\
 & & & \\
n & & &
\end{array}
$$

R picks $p = (p_1, \dots, p_n)$

C picks $q = (q_1, \dots, q_k)$

$\Longrightarrow$ R's payoff is $\Pi^R(p,q) = \sum_{ij} p_i q_j a_{ij}$

C's payoff is $\Pi^C(p,q) = \sum_{ij} p_i q_j b_{ij}$

(Here $\sum_{ij} = \sum_{i=1}^{n} \sum_{j=1}^{k}$)

TWO WAYS TO DECOMPOSE PAYOFFS

(1) $\Pi^R(p,q) = \sum_{ij} p_i q_j a_{ij}$

$= \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{k} q_j a_{ij} \right)$

$= \sum_{i=1}^{n} p_i$ (what i gets by playing ~~MY~~ HER pure strategy i)

(2) $\Pi^R(p,q) = \sum_{j=1}^{k} q_j \left(\sum_{i=1}^{n} p_i a_{ij} \right)$

$= \sum_{j=1}^{k} q_j$ (what i gets when C plays HIS pure strategy j)

DEF^n $(\tilde{p}, \tilde{q})$ is a NASH EQUILIBRIUM

if $\Pi^R(\tilde{p}, \tilde{q}) = \max_p \Pi^R(p, \tilde{q})$

and $\Pi^C(\tilde{p}, \tilde{q}) = \max_q \Pi^C(\tilde{p}, q)$

(In gen^l with n players, same def^n).

THM (Nash) An eq^m always exists.
(in mixed strategies).

$\downarrow$

$$\begin{bmatrix} 2,1 & 2,0 \\ 3,5 & 1,6 \end{bmatrix}$$

Traffic Game

	L	R
L	0,0	-1,-1
R	-1,-1	0,0

Three NE

$(L),(L) = (1,0),(1,0)$ with payoff $(0,0)$

$(R),(R) = (0,1),(0,1)$ with payoff $(0,0)$

$(\frac{1}{2},\frac{1}{2}),(\frac{1}{2},\frac{1}{2})$ with payoff $(-\frac{1}{2},-\frac{1}{2})$

Prisoners' Dilemma or Disarmament Game

	A	N
A	-10,-10	-1,-15
N	-15,-1	-4,-4

Diner's Dilemma in words.

(Note: multiplicity of NE with different payoffs

inefficiency of NE)

Multiplicity is serious

a, b	0, 0	0, 0	0, 0
0, 0	c, d	0, 0	0, 0
0, 0	0, 0	e, f	0, 0
0, 0	0, 0	0, 0	g, h

Let a, b, c, d, e, f, g, h be arbitrary

So is inefficiency

2-PERSON 0-SUM GAMES

~~SAFE STRATEGIES~~

$$A = \begin{array}{c|c|c|c|} & 1 & j & k \\ \hline 1 & & & \\ \hline i & & a_{ij} & \\ \hline n & & & \\ \hline \end{array}$$

Only player 1's payoffs written here

Recall

$$\Pi^1(p,q) = \sum_{ij} p_i q_j a_{ij}$$

$$\Pi^2(p,q) = \sum_{ij} p_i q_j (-a_{ij})$$

$$= -\Pi^1(p,q)$$

(i.e $\Pi^1(p,q) + \Pi^2(p,q) = 0$ for all p,q

THE CONCEPT OF SAFE STRATEGIES

Σ^1 = set of all mixed strategies of player 1

$$= \{(p_1, \ldots, p_n) = p : \sum_{i=1}^{n} p_i = 1,\ p_i \geq 0\}$$

$$\Sigma^2 = \{q = (q_1, \ldots, q_k) : \sum_{j=1}^{k} q_j = 1,\ q_j \geq 0\}.$$

DEFINE SAFE ST OF 1.

$$p \longrightarrow \min_{q \in \Sigma^2} \Pi^1(p, q) = F(p)$$

Σ^1_* = set of those p which maximize $F(p)$ on Σ^1

So $\Sigma^1_* = \{p \in \Sigma^1 : p$ achieves $\max\limits_{p \in \Sigma^1} \min\limits_{p \in \Sigma^2} \Pi^1(p, q)\}$

$\uparrow$

SAFE ST OF 1
(or 1's MaxMin st)

SIMILARLY

$$q \longrightarrow \min_{p\in\Sigma^1} \Pi^2(p,q)$$

$$= \min_{p\in\Sigma^1} -\Pi^1(p,q)$$

$$= -\max_{p\in\Sigma^1} \Pi^1(p,q) = G(q)$$

Σ^2_* = set of those q in Σ^2 which maximize $G(q)$

= set of those q in Σ^2 which minimize $-G(q)$

$$\Sigma^2_* \; \cancel{= \{q \in \Sigma^2 : q \text{ achieves } \min_{q\in\Sigma^2} \max_{p\in\Sigma^1} \Pi^1(p,q}$$

$$= \{q \in \Sigma^2 : q \text{ achieves } \min_{q\in\Sigma^2} \max_{p\in\Sigma^1} \Pi^1(p,q)\}$$

MINMAX THM (von-Neumann)

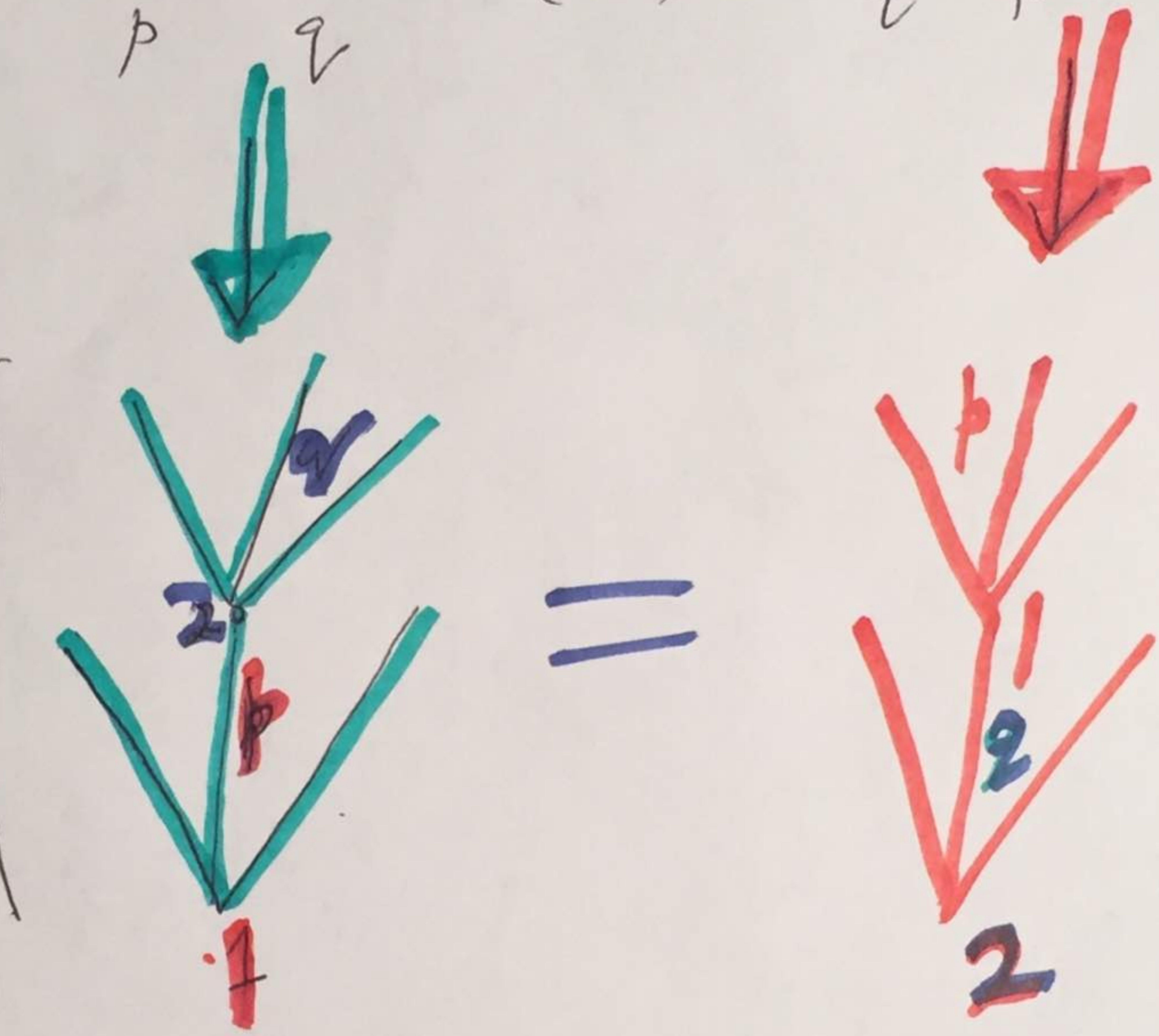
In a 2-person 0-sum game

set of NE $= \Sigma^1_* \times \Sigma^2_*$

and, at every NE, payoff of player 1 is

$$\max_p \min_q \Pi^1(p,q) = \min_q \max_p \Pi^1(p,q)$$

1 moves first, then 2 moves



2 moves first, then 1 moves

SAFE ST
Opponent's st UNKNOWN
PLAY ~~safe~~ cautious & safe

NE
Opponent's st KNOWN
PLAY agressive best reply

SAME!

COMPUTATIONS

$$\begin{tikzcd} 3 & 4 \arrow[l] \\ 2 \arrow[u] \arrow[r] & 1 \arrow[u] \end{tikzcd}$$

$(1,0),(1,0) = \text{NE}$

$$\begin{tikzcd} & q & 1-q \\ p & 3 \arrow[r] & 1 \arrow[d] \\ 1-p & 2 \arrow[u] & 4 \arrow[l] \end{tikzcd}$$

Looking at 1's payoffs

$$3q + 1(1-q) = 2q + 4(1-q)$$

$$\Rightarrow \quad q = \frac{3}{4}$$

Looking at 2's payoffs

$$3p + 2(1-p) = 1p + 4(1-p)$$

$$\Rightarrow p = \frac{1}{2}$$

$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{4}, \frac{1}{4}\right)$ is NE

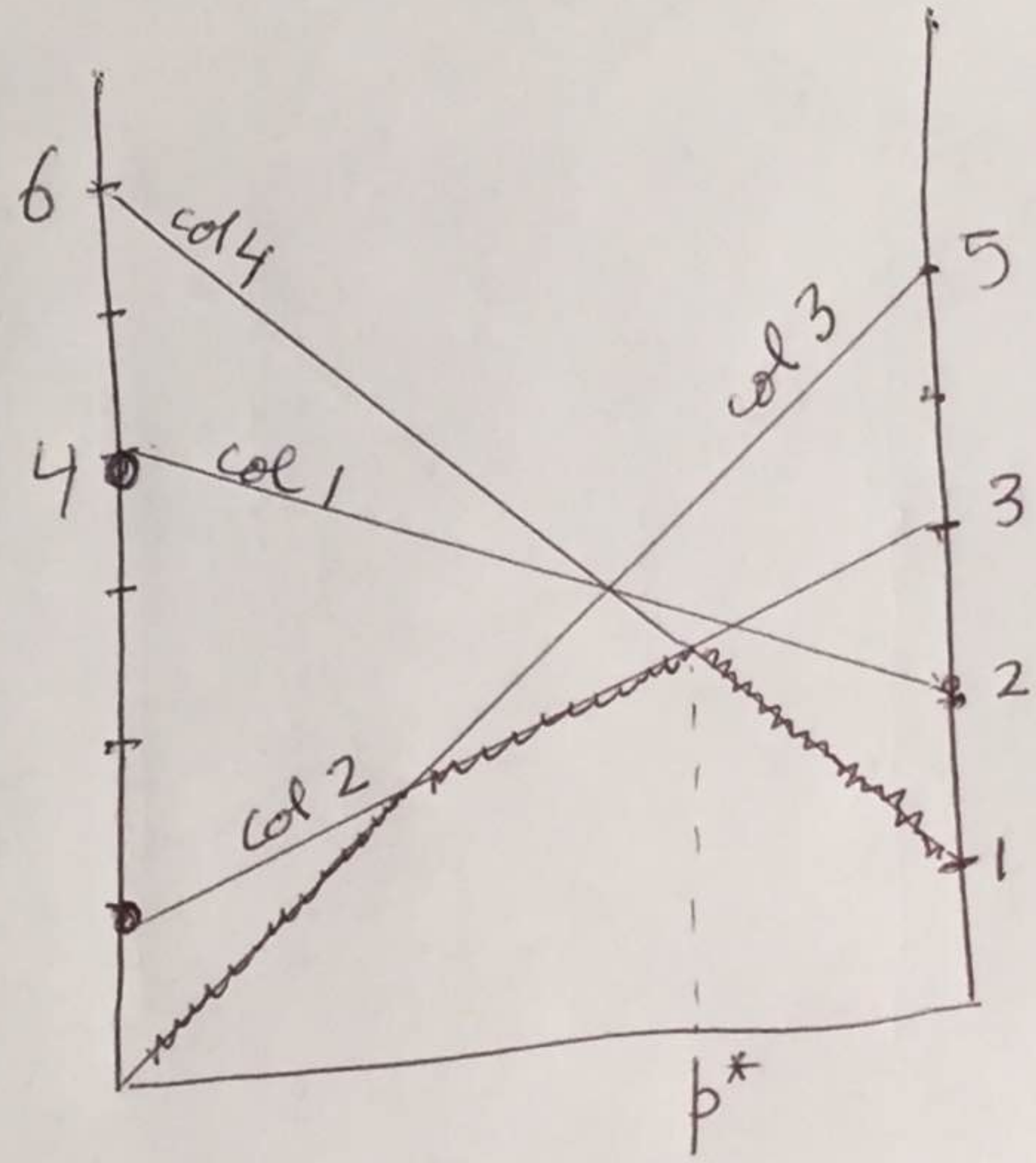
with payoffs $\left(\frac{10}{4}, -\frac{10}{4}\right)$

value of game $\frac{10}{4}$

2 x k or n x 2 games

p	2	3	5	1
$1-p$	4	1	0	6

$$\longrightarrow \begin{array}{c|cccc} & 0 & q & 0 & 1-q \\ \hline p & - & 3 & - & 1 \\ 1-p & - & 1 & - & 6 \end{array}$$



MaxMin method

p^* is intersection of column 2 & column 4 lines

$$3p + 1(1-p) = 1p + 6(1-p)$$

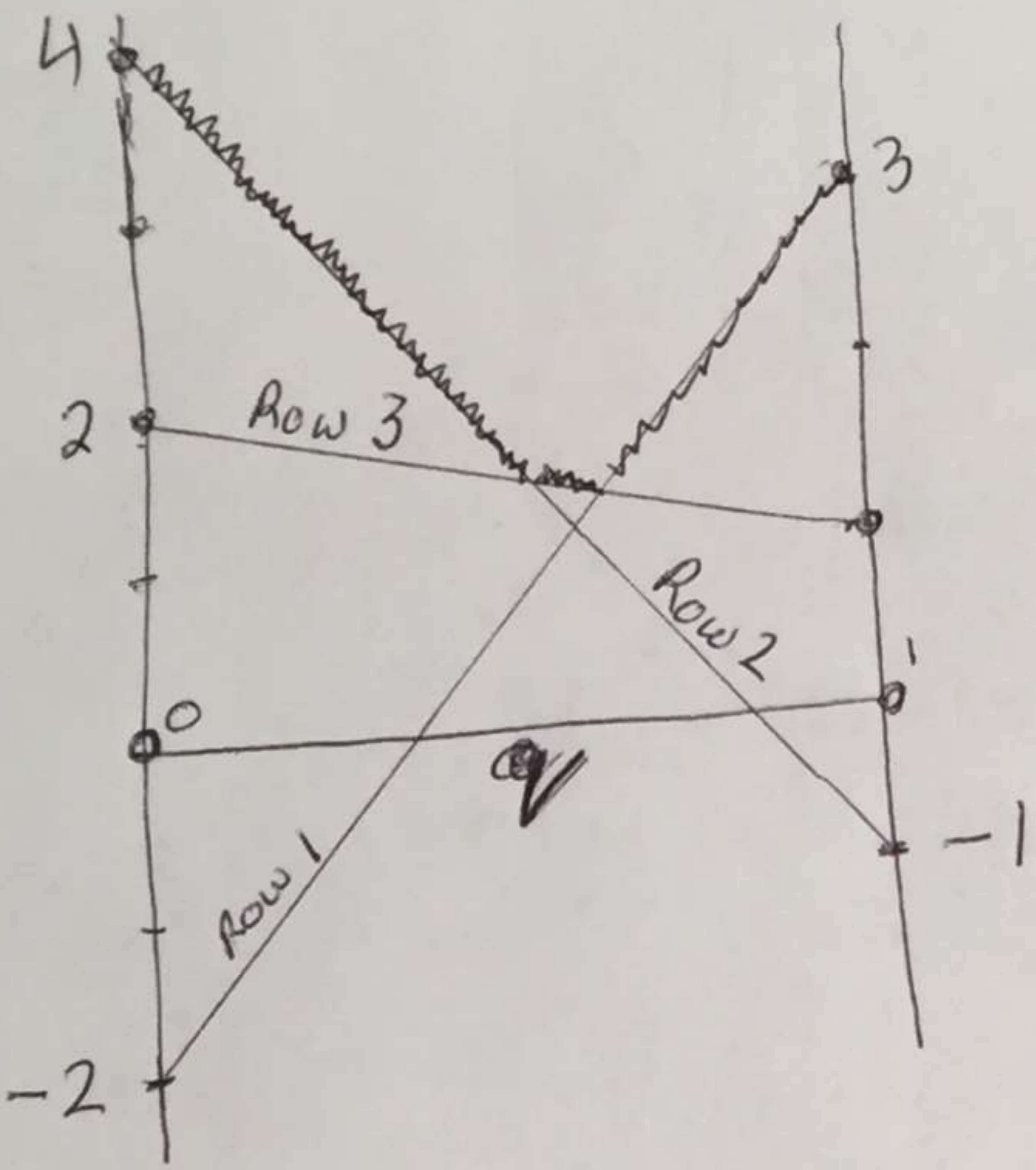
$$\Rightarrow \quad p = \frac{5}{7}$$

&

similarly find q

<u>value</u> $= \dfrac{17}{7}$

$$\begin{array}{cc} q & 1-q \\ \hline 3 & -2 \\ -1 & 4 \\ 1 & 2 \end{array}$$



q^* is the intersection of Row 1 & Row 3

$$\begin{array}{c|cc} & q & 1-q \\ \hline p & 3 & -2 \\ 0 & - & - \\ 1-p & 1 & 2 \end{array}$$

$$p^* = \left(\tfrac{1}{6}, 0, \tfrac{5}{6}\right)$$

$$q^* = \left(\tfrac{2}{3}, \tfrac{1}{3}\right)$$

$$v = \tfrac{4}{3}$$

$$3q - 2(1-q) = q + 2(1-q)$$

$$3p + 1(1-p) = -2p + 2(1-p)$$

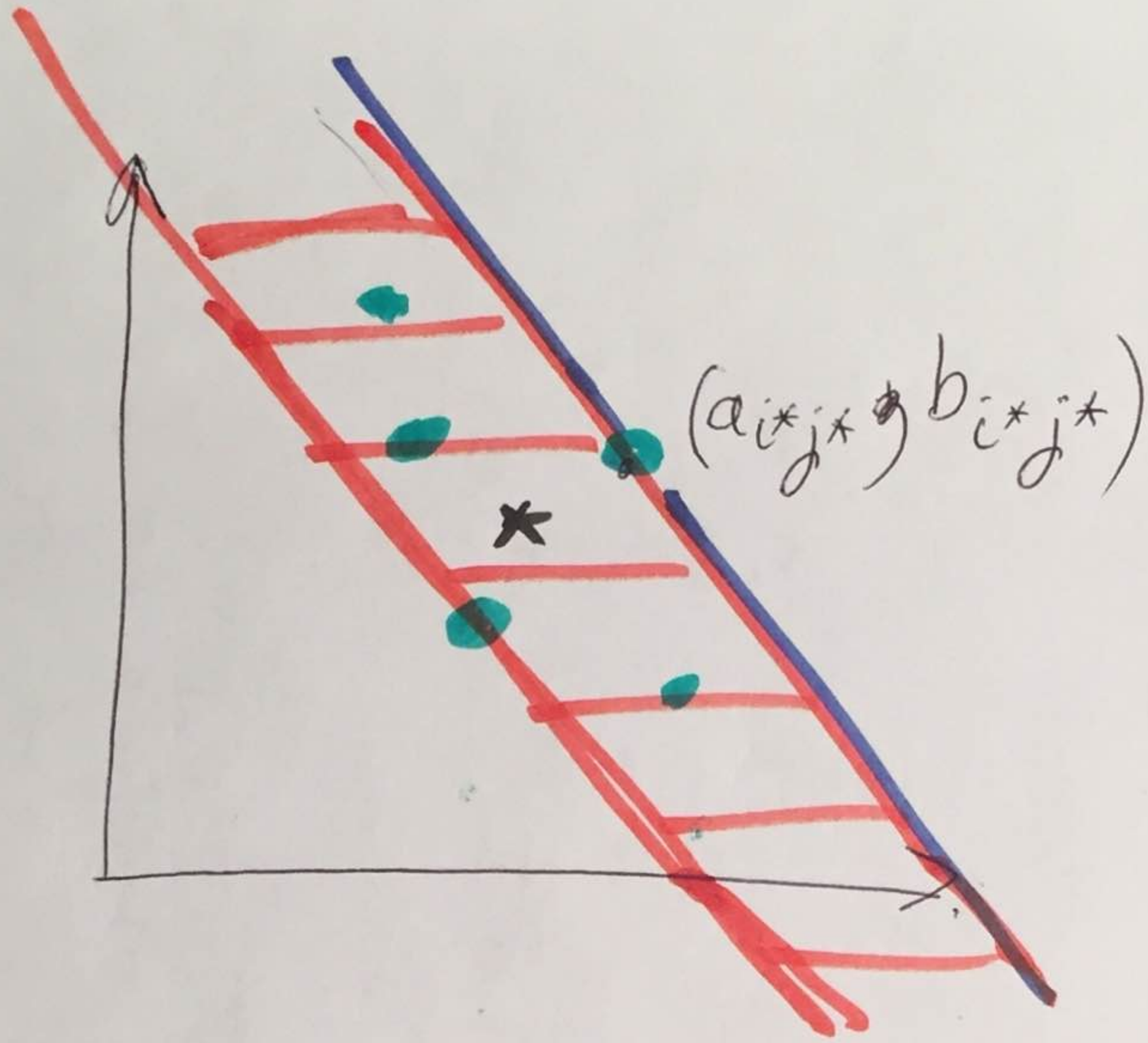
REDUCING THE MATRIX BY ~~SUCC~~ ITERATED DOMINATION

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 1 & 0 \\ 4 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ - & - & - \\ 4 & 1 & 2 \end{bmatrix}$$

$$\downarrow$$

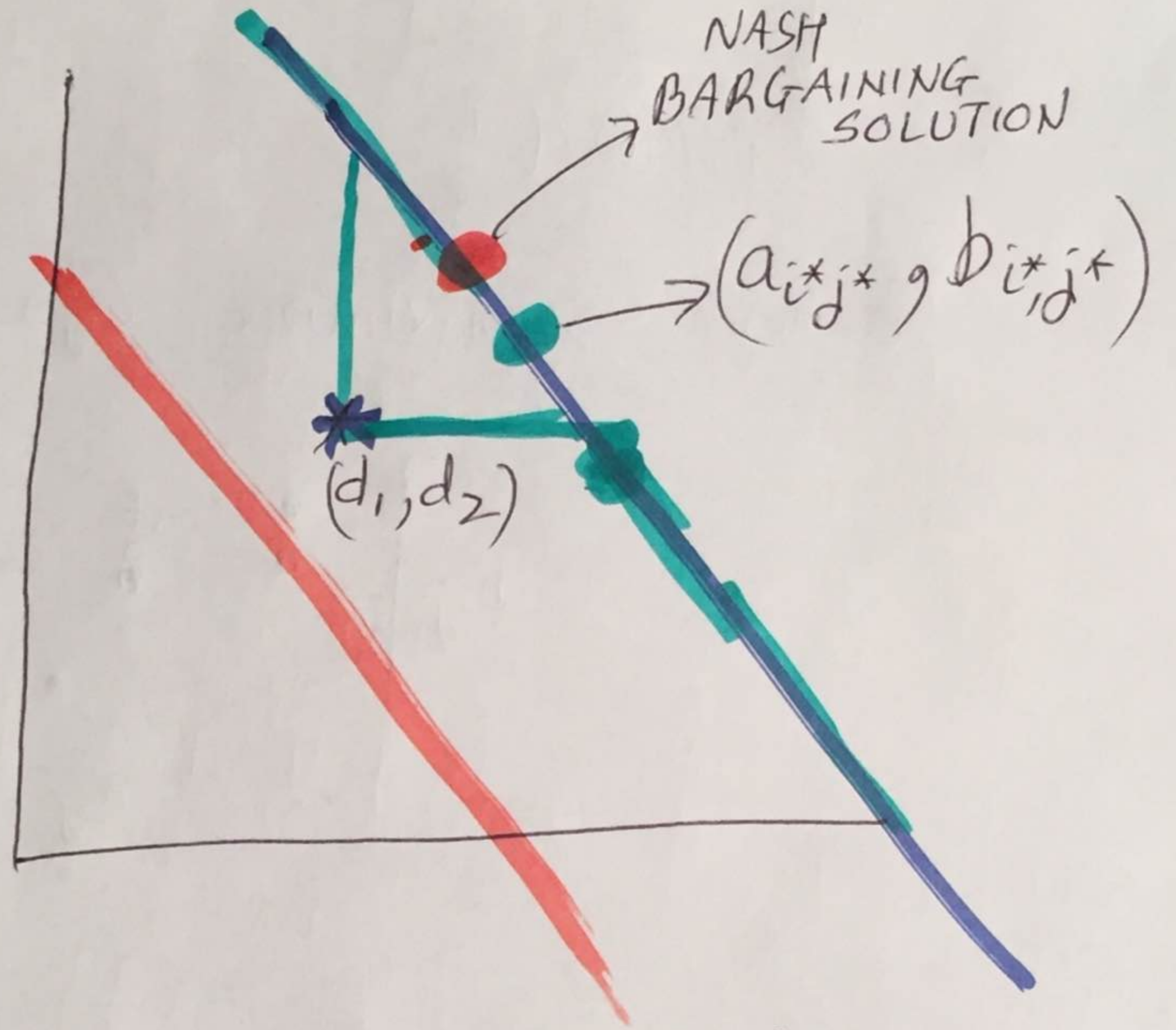
$$\begin{array}{c c} & \begin{array}{ccc} q & 1-q & 0 \end{array} \\ \begin{array}{c} p \\ 0 \\ 0 \\ 1-p \end{array} & \begin{bmatrix} 1 & 2 & - \\ - & - & - \\ - & - & - \\ 4 & 1 & - \end{bmatrix} \end{array} \leftarrow \begin{bmatrix} 1 & 2 & - \\ 2 & 0 & - \\ - & - & - \\ 4 & 1 & - \end{bmatrix}$$

$$\begin{array}{c c} & \begin{array}{cc} q & 1-q \end{array} \\ \begin{array}{c} p \\ 1-p \end{array} & \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix} \end{array} \Rightarrow \quad p =$$



$$\sigma = \max_{i,j} a_{ij} + b_{ij} = a_{i^*j^*} + b_{i^*j^*}$$

If we knew the DISAGREEMENT POINT $* = (d_1, d_2)$ then we would look at the NASH BARGAINING PROBLEM INDUCED by $*$



$\bullet$ = SPLIT THE SURPLUS

$$= \left(d_1 + \frac{\sigma - (d_1 + d_2)}{2}, \; d_2 + \frac{\sigma - (d_1 + d_2)}{2} \right)$$

(RECALL : $\sigma = a_{i^*j^*} + b_{i^*j^*}$)

Then ∗ $d_1(p,q)$, $d_2(p,q)$

where

$$d_1(p,q) = \sum_{ij} p_i q_j a_{ij}$$

$$d_2(p,q) = \sum_{ij} p_i q_j b_{ij}$$

So we get the strategic bargaining payoffs

$$R_1(p,q) = \frac{\sigma - [d_1(p,q) + d_2(p,q)]}{2} + d_1(p,q)$$

$$= \frac{\sigma}{2} + \frac{d_1(p,q) - d_2(p,q)}{2}$$

$$R_2(p,q) = \frac{\sigma}{2} + \frac{d_2(p,q) - d_1(p,q)}{2}$$

NOTE $R_1(p,q) + R_2(p,q) = \sigma$ for all p, q

So (R_1, R_2) gives a constant-sum game.

The game with payoffs R_1, R_2 is "strategically equivalent" to the game with payoffs

$$d_1 - d_2 \,,\; d_2 - d_1$$

So ~~be~~ we can find NE of the latter (it will also be an NE of R_1, R_2)

The latter is a zero-sum matrix game $A - B$.

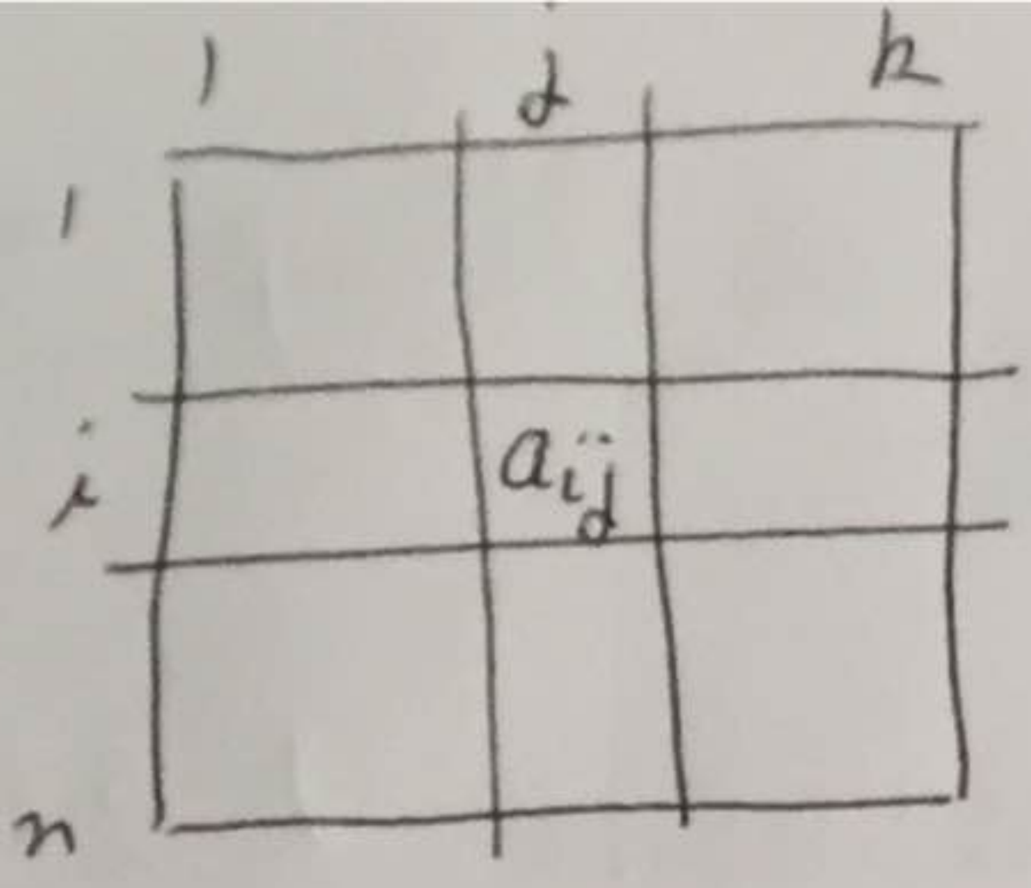
Let (p^*, q^*) be an NE of $A-B$ and let v be its value.

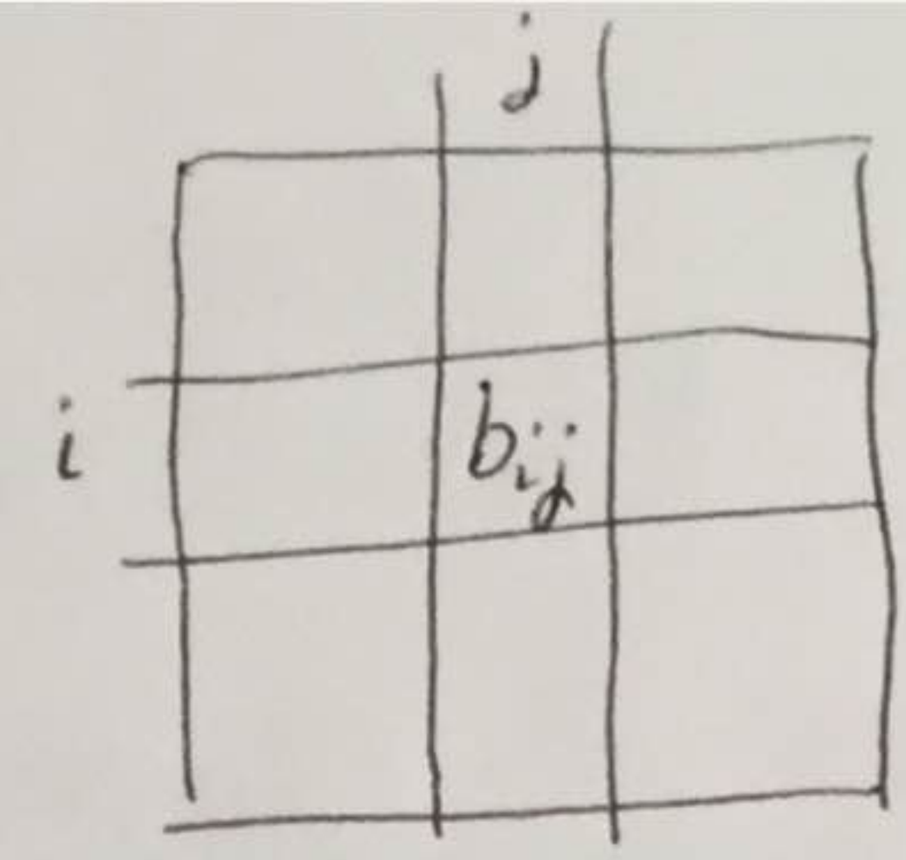
~~Then~~

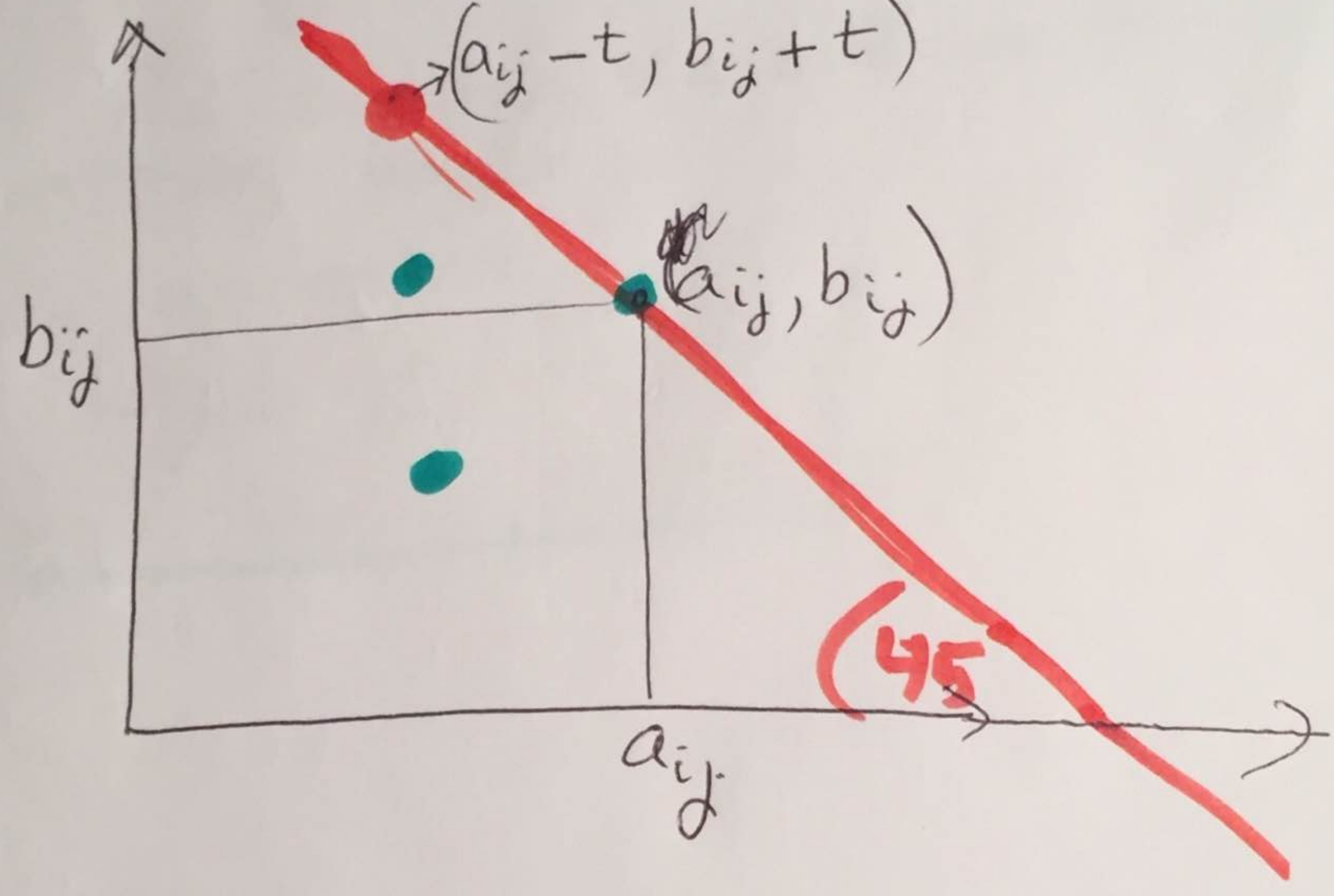
Then (p^*, q^*) is also NE of the <u>real</u> game R_1, R_2 with payoffs

$$R_1(p^*, q^*) = \frac{\sigma}{2} + \frac{v}{2} = \frac{\sigma + v}{2}$$

$$R_2(p^*, q^*) = \frac{\sigma}{2} - \frac{v}{2} = \frac{\sigma - v}{2}$$


1
j
k
1
i
n
a_{ij}


j
i
b_{ij}


$(a_{ij}-t,\ b_{ij}+t)$
$(a_{ij},\ b_{ij})$
b_{ij}
45
a_{ij}

* will be determined STRATEGICALLY.

Let 1 announce a "threat strategy"

$$p = (p_1, \dots, p_n)$$

which she declares she will use if bargaining fails

Let 2 announce

$$q = (q_1, \dots, q_k)$$

as his threat strategy