Homework 6
Suggested Solutions

1. Suppose the fraction of low-ability ($t = 1$) workers is 1/4 and fraction of high-ability ($t = 2$) workers is 3/4. The productivity of a type 1 worker is $2e$ and the productivity of a type 2 worker is $\frac{9}{4}e$, where e is the education level. The utility of wage w and education e to a worker of type 1 is $u_1(w,e) = 4\sqrt{w} - 2e$. The utility of wage w and education e to a worker of type 2 is $u_2(w,e) = 4\sqrt{w} - 1.8e$. Find the R-S equilibrium.

Solution:

Low ability workers choose the optimal contract along their productivity line ($w = 2e_{\text{L}}$)

$$\max_{e_{\text{L}}} \quad 4\sqrt{2e_{\text{L}}} - 2e_{\text{L}}$$

$$\text{FOC:} \quad 4(2e_{\text{L}})^{-\frac{1}{2}} - 2 = 0 \Rightarrow e^*_{\text{L}} = 2 \Rightarrow w^*_{\text{L}} = 4.$$

Let us now find the minimum education level for the high ability required for a separating equilibrium to exist, i.e., the intersection between the indifference curve of the low ability workers that maximizes their utility with the productivity line of the high ability workers.

$$u_{\text{L}}(w^*_{\text{L}}, e^*_{\text{L}}) = 4 \quad \& \quad w = \frac{9}{4}e \quad \Rightarrow \quad 4\sqrt{\frac{9}{4}e} - 2e = 4 \quad \Rightarrow \quad \underline{\mathrm{e}} = \begin{cases} 1 \ < \ e^*_{\text{L}} & ✘ \\ 4 \ ✓ \end{cases}$$

The high ability workers choose the optimal contract along their productivity line

$$\max_{e_{\text{H}}} \quad 4\sqrt{\frac{9}{4}e_{\text{H}}} - 1.8e_{\text{H}}$$

$$\text{FOC:} \quad 3(e_{\text{H}})^{-\frac{1}{2}} - 1.8 = 0 \Rightarrow e^*_{\text{H}} = \frac{25}{9} < \underline{\mathrm{e}} \Rightarrow \underline{\mathrm{w}} = 9.$$

Thus, we need to check if the indifference curve of the high ability workers that goes through $\underline{\mathrm{e}}$ crosses the average productivity line $w = \frac{35}{16}e$.

$$u_2(\underline{\mathrm{w}}, \underline{\mathrm{e}}) = 4\sqrt{w} - 1.8e = \frac{24}{5} \Rightarrow e = \begin{cases} 2.1289 \\ 3.34017 \end{cases}$$

which implies that no Rothschild-Stiglitz equilibrium exists.

2. The Security Council has 5 permanent members and 10 non-permanent members. For a coalition to win, it must contain all 5 permanent members and at least 4 non-permanent members. View this situation as a simple (voting) game and compute the Shapley Value of the 15 members.

Solution:

Consider first a non-permanent member

$$\underbrace{\bullet \qquad \bullet \qquad \bullet}_{\substack{\text{5 permanent} \\ \text{3 non-permanent}}} \quad \text{n-p} \quad \underbrace{\bullet \qquad \bullet \qquad \bullet}_{\text{6 permanent}}$$

so that the Shapley value of any one non-permanent member is

$$\varphi(n-p) = \frac{\binom{9}{3}8!6!}{15!} = \frac{4}{2145}.$$

Then, by efficiency and symmetry,

$$\sum \varphi(p) = 1 - \sum \varphi(n-p) = \frac{421}{429},$$

so that the Shapley value of any one permanent member is

$$\varphi(p) = \frac{\sum \varphi(p)}{5} = \frac{421}{2145}$$

3. Consider a situation involving a landlord and 10 workers that till the landlord's land. If k workers till the land, the outputs is worth k^2 dollars.

(a) Compute the Shapley value of the 11 players involved in this game.

Consider now 2 landlords and 6 workers.

(b) Compute the Shapley value of the 8 players in the case where both landlords need to be present in a coalition for the land to be tilled.

(c) Compute the Shapley value of the 8 players in the case where the presence of any one landlord suffices for the land to be tilled.

Solution:

(a)

$$v(\mathrm{S}) = \begin{cases} k^2 & \text{if the landlord is in coalition S and } |\mathrm{S}| = k+1. \\ 0 & \text{if the landlord is not in coalition S.} \end{cases}$$

$$\underbrace{\bullet \qquad \bullet \qquad \bullet}_{k \text{ workers}} \quad \mathrm{L} \quad \underbrace{\bullet \qquad \bullet \qquad \bullet}_{10-k \text{ workers}}$$

So that the Shapley value of the landlord is

$$\varphi(\mathrm{L}) = \sum_{k=0}^{10} \frac{\binom{10}{k} k!(10-k)!}{11!} \cdot k^2 = \frac{1}{11} \cdot [0^2 + 1^2 + \cdots + 10^2] = 35.$$

Then, by efficiency, the sum of the Shapley values of the workers is equal to

$$\sum \varphi(w) = 10^2 - \varphi(\mathrm{L}) = 65,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{65}{10} = 6.5.$$

(b)

$$v(\mathrm{S}) = \begin{cases} k^2 & \text{if the landlord is in coalition S and } |\mathrm{S}| = k+2. \\ 0 & \text{if the landlord is not in coalition S.} \end{cases}$$

$$\underbrace{\bullet \qquad \bullet \qquad \bullet}_{k \text{ workers and 1 landlord}} \quad \mathrm{L} \quad \underbrace{\bullet \qquad \bullet \qquad \bullet}_{6-k \text{ workers}}$$

So that the Shapley value of any one landlord is

$$\varphi(\mathrm{L}) = \sum_{k=0}^{6} \frac{\binom{6}{k}(k+1)!(6-k)!}{8!} \cdot k^2 = \frac{\sum_{k=0}^{6}(k+1)k^2}{8 \cdot 7} = \frac{19}{2}.$$

Then, by efficiency, the sum of the Shapley values of the workers is equal to

$$\sum \varphi(w) = 6^2 - \varphi(\mathrm{L}) \times 2 = 17,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{17}{6}.$$

(c)

$$\underbrace{\bullet \qquad \bullet \qquad \bullet}_{k \text{ workers}} \qquad \mathrm{L} \qquad \underbrace{\bullet \qquad \bullet \qquad \bullet}_{6-k \text{ workers and the other landlord}}$$

So that the Shapley value of any one landlord is

$$\varphi(\mathrm{L}) = \sum_{k=0}^{6} \frac{\binom{6}{k}k!(6-k+1)!}{8!} \cdot k^2 = \frac{\sum_{k=0}^{6} \frac{(7-k)!}{(6-k)!} \cdot k^2}{8 \cdot 7} = \frac{7}{2}.$$

Then, by efficiency, the sum of the Shapley values of the workers is equal to

$$\sum \varphi(w) = 6^2 - \varphi(\mathrm{L}) \times 2 = 29,$$

and by symmetry, the Shapley value of any one worker is

$$\varphi(w) = \frac{29}{6}.$$