Microeconomics II(ECO 501)

Questions on the comprehensive exam will be chosen from the list below(with possible minor variations)

CALCULATORS ARE ALLOWED

Matching

1. Consider the Gale-Shapley marriage problem with the following profile of preferences:

$$
\begin{array}{ccccc|cccccc}
 & & M & & & & & W & & & \\
A: & a & b & c & d & a: & B & C & A & E & \\
B: & d & b & c & a & b: & C & A & B & D & E \\
C: & d & c & a & & c: & E & D & B & C & \\
D: & a & d & b & & d: & A & D & E & B & C \\
E: & a & b & d & & & & & & &
\end{array}
$$

Compute the woman-optimal stable matching.

2. Consider the following profile of preferences:

$$
\begin{array}{ccccc|cccccc}
A: & a & b & c & d & a: & B & C & A & D & E \\
B: & d & b & c & a & b: & C & A & B & D & E \\
C: & d & c & a & b & c: & E & D & A & B & C \\
D: & a & d & c & b & d: & A & D & E & B & C \\
E: & a & b & d & & & & & & &
\end{array}
$$

(a) Show that the men-proposing procedure (MPP) leads to

$$\mu_M = \langle Aa, Bb, Cc, Dd, E \rangle,$$

as the man-optimal stable matching, and that the woman-proposing procedure leads to

$$\mu_W = \langle Ad, Ba, Cb, Dc, E \rangle$$

as the woman-optimal stable matching.

(b) Let a misrepresent her preferences by stating

$$a: \quad B \quad C \quad D \quad E \quad A$$

while all others state preferences according to the profile given above. Compute the new matching according to the MPP and show that a obtains a husband she prefers.

3. Find the stable assignments that are "boy-best" and "girl-best" with the following profile of preferences:

$$\begin{array}{lcccc|lcccc} & & B & & & & & G & & \\ W: & a & b & c & d & a: & Y & Z & W & X \\ X: & a & b & c & d & b: & Z & W & X & Y \\ Y: & b & c & a & d & c: & W & X & Y & Z \\ Z: & c & a & b & d & d: & Z & Y & W & X \end{array}$$

4. Find the stable assignments that are "boy-best" and "girl-best" with the following profile of preferences:

$$\begin{array}{lcccc|lccccc} & & B & & & & & & G & & \\ U: & a & b & c & d & a: & V & Y & U & Z & X \\ V: & b & d & a & c & b: & W & U & Y & & \\ W: & d & c & b & & c: & V & Z & Y & W & \\ X: & c & b & & & d: & Z & Y & X & U & \\ Y: & a & b & c & & & & & & & \\ Z: & d & c & b & a & & & & & & \end{array}$$

5. Find the optimal (i.e. "applicant-best") stable assignment of applicants to colleges, based on the following preference orderings:

$$\begin{array}{llccccccccccc} (4) & X: & i & h & f & e & c & d & g & k & a & b & j \\ (3) & Y: & g & i & c & e & k & f & j & a & h & b & d \\ (2) & Z: & a & k & i & g & e & d & h & b & j & c & f \end{array}$$

$a:$	X	Y	Z		$g:$	Y	Z	X
$b:$	X	Y	Z		$h:$	Z	X	Y
$c:$	X	Z	Y		$i:$	Z	X	Y
$d:$	Y	X	Z		$j:$	Z	Y	X
$e:$	Y	X	Z		$k:$	Z	Y	X
$f:$	Y	Z	X					

6. Find the optimal (i.e. "applicant-best") stable assignment of applicants to colleges, based on the following preference orderings:

$$\begin{array}{llcccccccccccccc} (6) & X: & n & m & l & k & j & i & h & g & f & e & d & c & b & a \\ (6) & Y: & a & b & c & d & e & f & g & h & i & j & k & l & m & n \end{array}$$

$a:$	X	Y		$h:$	X	Y
$b:$	X	Y		$i:$	X	Y
$c:$	X	Y		$j:$	X	Y
$d:$	X	Y		$k:$	Y	X
$e:$	X	Y		$l:$	Y	X
$f:$	X	Y		$m:$	Y	X
$g:$	X	Y		$n:$	Y	X

7. Fix a profile of preferences and let μ and $\tilde{\mu}$ be two stable matchings for the profile. Let each man point to the better of his two wives in μ and $\tilde{\mu}$. Show

(a) no two men point to the same woman (so we get a matching).

(b) the matching obtained in (a) is stable. Notation: This matching is denoted $\mu \bigvee_{\mathrm{M}} \tilde{\mu}$; read "$\bigvee_{\mathrm{M}}$" as "better for men".

(c) Each woman obtains the *worse* of her two husbands in $\mu, \tilde{\mu}$. Notation: so the matching is also denoted $\mu \bigwedge_{\mathrm{W}} \tilde{\mu}$; read "$\bigwedge_{\mathrm{W}}$" as "worse for women".[1,2]

8. Prove that if the result of the men-proposing procedure yields the same result as the women-proposing procedure, then this resulting matching is the unique stable matching.

[1] This question is Conway's theorem on the lattice structure of stable matchings.

[2] (c) says: $\mu \bigwedge_{\mathrm{W}} \tilde{\mu} = \mu \bigvee_{\mathrm{M}} \tilde{\mu}$ showing an inherent conflict of interests between men and woman.

9. In Julius' list of preferences, Agrippina appears first, Messalina appears second, and Cleopatra appears third. Suppose there is a stable matching under which Julius is matched to Agrippina, and that there is a stable matching under which Julius is matched to Cleopatra. Is there necessarily a stable matching under which Julius is matched to Messalina? Either prove this statement or provide a counterexample.

10. Consider the Gale-Shapley marriage problem with n men and n women. Prove that if in stage t of the men-proposing procedure, a particular man is dismissed for the $(n-1)$-th time, then the algorithm terminates at stage $(t+1)$.

11. Consider the Gale-Shapley marriage problem with n men and n women. Prove that the men-proposing procedure terminates after at most $(n-1)^2+1$ stages.

12. Consider the Gale-Shapley marriage problem with n men and $k < n$ women. Assume that each man prefers to be matched than staying single. Show that in every stable matching, the same set of $n-k$ men are single. (Hint: You may use, without proof, the theorem proved in Gale-Shapley the that MPP gives a men-optimal matching.)

Housing Market

13. Consider the five-person house swapping game for the following preferences (where A owns a, B owns b, etc):

$$\begin{array}{llllll} A: & c & a & b & d & e \\ B: & d & b & c & e & a \\ C: & e & c & a & b & d \\ D: & a & d & b & c & e \\ E: & b & a & e & d & c \end{array}$$

(a) Show that the allocation $\mu =< Ac, Bb, Ce, Dd, Ea >$ is in the core but not in the strict core.

(b) Formulate the game in which the same five traders start with the allocation μ, and determine the core and strict core.

14. Solve the following house swapping games by the method of top trading cycles.

$$\begin{array}{lllllllll} A: & f & c & g & a & b & d & h & e \\ B: & e & f & g & a & c & b & h & d \\ C: & g & e & d & c & b & a & h & f \\ D: & c & h & g & e & d & a & b & f \\ E: & g & a & c & f & b & h & d & e \\ F: & c & b & a & d & g & h & f & e \\ G: & f & e & d & c & b & a & h & g \\ H: & b & c & d & e & f & g & h & a \end{array}$$

15. Show that how a player ranks the houses he considers worse than his own has no effect on the core of the game (not merely the strict core).

16. Consider the five-person house swapping game for the following preferences (where A owns a, B owns b, etc):

$$\begin{array}{lllll} A: & b & d & a & c \\ B: & c & a & d & b \\ C: & d & b & c & a \\ D: & a & c & b & d \end{array}$$

(a) Show that the allocation $\mu =< Ab, Ba, Cd, Dc >$ is in the core but not in the strict core.

(b) Show that it is weakly dominated by the allocation that gives all traders their first choice.

(c) Formulate the game in which the same five traders start with the allocation μ, and determine the core and strict core.

17. Consider the housing market with the profile:

$$\begin{array}{ll} A: & b \sim c \succ a \\ B: & a \succ c \succ b \\ C: & a \succ b \succ c \end{array}$$

Here A ranks b and c equally in the first place, and a in the second place.

Define strict-core as in class and show that it is empty. (This example shows the need for strict preferences in our analysis.

18. Show that the *top-trading cycle* (TTC) allocation is unique.

19. Show that the TTC allocation is strategy-proof.

Competitive Equilibrium (CE)

(Exercises 20 to 27 are all about the Edgeworth Box, with different types of preferences. So it is worthwhile working them all out.)

20. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = x+y$ and $u^2(x, y) = 2x+3y$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

21. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = x^{1/2}y^{1/2}$ and $u^2(x, y) = x^{2/3}y^{1/3}$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

22. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = \min\{x, y\}$ and $u^2(x, y) = \min\{2x, 3y\}$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

23. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = x + y$ and $u^2(x, y) = \min\{2x, 3y\}$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

24. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = x^{1/2}y^{1/2}$ and $u^2(x, y) = 2x + 3y$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

25. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 2)$ and $e^2 = (2, 3)$, and household preferences are $u^1(x, y) = x^{1/2}y^{1/2}$ and $u^2(x, y) = \min\{2x, 3y\}$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

26. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (1, 1)$ and $e^2 = (1, 2)$, and household preferences are $u^1(x, y) = x^{1/3}y^{2/3}$ and $u^2(x, y) = 3x + 2y$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

27. Consider the exchange economy in which there are two agents and two goods. Endowments are $e^1 = (2, 1)$ and $e^2 = (1, 3)$, and household preferences are $u^1(x, y) = 2x+y$ and $u^2(x, y) = x+2y$. Find (compute and picture) the Pareto set, core and competitive equilibria of this exchange economy.

28. Let $u^h(x_1, \ldots, x_L) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_L^{\alpha_L}$. Assume $\alpha_i \geq 0$ for all i and $\sum_{i=1}^{L} \alpha_i = 1$. Show that if $u^h(z)$ maximizes utility on $B^h(p) = \{y \in \mathbb{R}_+^L | py \leq pe^h\}$, then $p_l z_l = \alpha_l pe^h$.

29. Consider an exchange economy in which there are four agents and three goods. Agents' utility functions are $u^1(x, y, z) = x^{1/2} y^{1/4} z^{1/4}$, $u^2(x, y, z) = x^{1/3} y^{1/3} z^{1/3}$, $u^3(x, y, z) = x^{2/3} y^{1/4} z^{1/12}$ and $u^4(x, y, z) = x^{1/4} y^{1/4} z^{1/2}$ respectively. Their endowments are $e^1 = (1, 2, 0)$, $e^2 = (0, 2, 3)$, $e^3 = (1, 1, 1)$ and $e^4 = (1, 0, 0)$ respectively. Compute a competitve equilibrium for this economy. Are there other competitive equilibria?

30. Consider the following economy. There are three goods (x, y, z), two consumers (Alice and Bob), and two firms. Good z is used as an input in each firm and it provides no direct utility of consumption. Firm 1, which is owned entirely by Alice, has a technology that allows good z to be made into good x, according to the simple linear technology $x \leq -3z$.(In other words, Alice's firm produces output of freely disposable good x that is equal to three times the input of good z.) Frim 2, owned entirely by Bob, uses good z to make good y, and its technology is described by $y \leq -4z$. Each consumer initially owns 5 units of good z, i.e. $e^A = e^B = (0, 0, 5)$. Alice's utility function is $u^A(x, y, z) = 0.4 \ln(x) + 0.6 \ln(y)$, while Bob's utility function is $u^B(x, y, z) = 0.5 \ln(x) + 0.5 \ln(y)$. Compute a competitive equilibrium of this economy. Are there other competitive equilibria?

31. Consider an exchange economy in which there is a commodity l such that

(a) household 1 owns (i.e. is endowed with) only commodity l and likes only commodity l.

(b) households $2, \ldots, H$ each own commodity l but none of them like commodity l.

Show that a CE doesn't exist.

32. Consider an exchange economy in which there are two goods and two agents. Agent 1 has utility function $u^1(x, y) = -\sqrt{(x - 1/4)^2 + (y - 1/4)^2}$ and agent 2 has utility function $u^2(x, y) = \log(x) + \log(y)$. Endowments are $e^i = (1/2, 1/2)$ for $i = 1, 2$. Show that this economy has a competitive equilibrium which is not Pareto Optimal. Does this example contradict the First Welfare Theorem?

33. Assume that $u^i : X \to \mathbb{R}$ is C^2, strongly monotonic increasing[3] and strictly quasi-concave for all $i \in H$. Furthermore, assume that $e^i \in \mathbb{R}_{++}^L$ for all $i \in H$.

[3] In this problem we are assuming differentiability so strongly monotonic increasing means $\nabla u^i \gg 0$.

(a) Show that *Pareto efficient allocations* are solutions of the following problem:

$$\begin{aligned} &\max_{x \in X^H} u^i(x^i) \\ \text{s.t.} \quad &\sum_{i \in H} x^i \leq \sum_{i \in H} e^i \\ &u^j(x^j) \geq \bar{u}^j \text{ for all } j \neq i \end{aligned} \tag{PE}$$

for some given vector $(\bar{u}^j)_{j \neq i} \in \mathbb{R}^{H-1}$. Note that by varying the values of $\bar{u}^j$ for $j \neq i$ we obtain the *set of Pareto efficient allocations.*

(b) Characterize the set of Pareto efficient allocations under the previous assumptions by characterizing the solutions to problem (PE).

34. Consider traders $h \in H = \{1, 2, .., H\}$ with endowments $e^h \in R^K_{++}$ and monotonic, strictly concave utility function u^h. For any price vector $p \in R^K_{++}$, there will be a unique consumption bundle $y^h(p)$ in the budget set $B^h(p) = \{x \in R^K_+ : p \cdot x \leq p \cdot e^h\}$ which maximizes u^h. (Note: the uniqueness follows from strict concavity). Define the aggregate excess demand function $z : R^K_{++} \longrightarrow R^K$ by $z(p) = \sum_{h \in H}[y^h(p) - e^h]$. The function $z(.)$ has the gross substitute (GS) property if whenever p' and p are such that, for some l, $p'_l > p_l$ and $p'_k = p_k$ for $k \neq l$, we have $z_k(p') > z_k(p)$ for $k \neq l$.

(a) Using the fact that the aggregate excess demand functions are homogeneous of degree zero, prove that $z_l(p') < z_l(p)$.

(b) Prove that if the aggregate excess demand function $z(.)$ satisfies the gross subsitute property, then there is at most one normalized price vector p such that $z(p) = 0$. (Note: normalized means $\sum_{l=1}^K p_l = 1$.)

35. Consider a pure exchange economy $E = (e^h, u^h)_{h \in H}$. Define Core E and Competitive Equilibrium of E. Prove that a Competitive Equilibrium allocation is in Core E.

36. Consider a production and exchange economy with two commodities, two households, labeled 1 and 2, and one firm with production set $Y = \{(x, y) | x \leq 0, y \leq \sqrt{-x}\}$. Household 1 has endowment $e^1 = (2, 1)$, utility $u^1(x_1, y_1) = \frac{1}{2} \ln x_1 + \ln y_1$ and $\frac{1}{3}$ share of the firm. Household 2 has endowment $e^2 = (6, 1)$, utility $u^2(x_2, y_2) = \ln x_2 + \frac{1}{2} \ln y_2$, and $\frac{2}{3}$ share of the firm. Compute a competitive equilibrium.

37. Consider a production and exchange economy with two commodities x and y, two households 1 and 2 and one firm with production set

$$Y = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x \leq 0, y \leq -\frac{1}{2}x \right\}.$$

Household 1 has endowment $e^1 = (4, 1)$, utility function $u^1(x, y) = x^{\frac{1}{3}} y^{\frac{2}{3}}$ and $\frac{1}{3}$ share of the firm. Household 2 has an endowment $e^2 = (2, 0)$, utility function $u^2(x, y) = \sqrt{xy}$ and $\frac{2}{3}$ share of the firm.

(a) Compute a competitive equilibrium of this economy. Show that it is the unique CE.

(b) If firm shares are changed from $(\frac{1}{3}, \frac{2}{3})$ to $(\frac{1}{2}, \frac{1}{2})$, how will the competitive equilibrium change?

38. Let $E = \{(e_h, u_h)\}_{h \in \{1,2\}}$ be an exchange economy where the endowments are $e_1 = (1,0)$ and $e_2 = (0,1)$. Compute the set of Pareto optimal allocations, core allocations and competitive equilibrium allocations for the following utilities:

(a) $u_1(x_1, y_1) = y_1$ and $u_2(x_2, y_2) = x_2$

(b) $u_1(x_1, y_1) = x_1 y_1$ and $u_2(x_2, y_2) = 2x_2 + y_2$

39. Let $\varepsilon = \{(e^h, u^h, \theta^h)_{h \in H}, (Y_j)_{j \in J}\}$ be an Arrow-Debreu production and exchange economy where $e^h \in R^L_+$, $u^h : R^L_+ \longrightarrow R$ and $\theta^h \in R^J_+$ give the endowment, utility and shares of agent $h \in H = 1, ..., H$, and Y^j is the production set of firm $j \in J = 1, ..., J$.

(a) Define a competitive equilibrium and prove that it is Pareto-optimal (using the minimal hypotheses on the economy).

(b) Calculate the CE when $H = 1, 2$ and there is just one firm. Let $\theta^1 = \frac{1}{4}$, $\theta^2 = \frac{3}{4}$, $e^1 = (2, 0, 0)$, $e^2 = (0, 1, 0)$, $u^h(x, y, z) = \log x + \log y + \log z$ for $h = 1, 2$ and the production of the firm is $z = x^{\frac{1}{3}} y^{\frac{1}{3}}$.

40. Consider the following economy: $e^1 = (0, 1)$, $e^2 = (1, 0)$ and households preferences are represented by $u^1(x, y) = \sqrt{x} + y$ and $u^2(x, y) = x$.

(a) In an Edgeworth Box, show the Pareto Set, and Core. Is the allocation (e^1, e^2) Pareto efficient?

(b) Show that (e^1, e^2) is not a competitive equilibrium.

(c) Does this result contradicts the Second Welfare Theorem?

41. Suppose there are two households which have the same utility functions:

$$u(x,y) = \begin{cases} \sqrt{x} + \frac{1}{2}\sqrt{y} & \text{if } x \leq y, \\ \frac{1}{2}\sqrt{x} + \sqrt{y} & \text{if } x > y. \end{cases}$$

Notice that the preferences display a kink.

(a) Suppose that the initial endowments are $e^i = (4, 4), \forall i = 1, 2$. Find the Pareto Set, core and CE allocations (just the allocations).

(b) Calculate the prices that support the competitive equilibrium in part (a) for each allocation.

(c) Does the First Welfare Theorem hold in this economy? What about the Second Welfare Theorem?

42. Consider a pure exchange economy: $I = \{1, 2\}$, $L = \{1, 2\}$, $e^1 = (1, 2)$, $u^1(x_1, y_1) = \min\{2x_1, y_1\}$, $e^2 = (2, 2)$, $u^2(x_2, y_2) = x_2 + 2y_2$.

(a) Compute Pareto set, core and competitive equilibrium.

(b) Suppose we also add a firm with a production set $Y = \{(x, y)|x \leq 0, y \leq \min\{-x, 3\}\}$. The agents' ownership shares of the firm are $\theta^1 = \theta^2 = \frac{1}{2}$. How do CE change?

Asymmetric Information

43. Consider a principal who hires a salesman to approach a client. The reservation utility level of the salesman is 3. A high level of effort gives no sale with probability 0.2, a $100 sale with probability 0.4 and a $400 sale with probability 0.4. For a low level of effort, the three probabilities are 0.4, 0.4, 0.2 respectively. The principal is risk neutral. The agent is risk averse, and if the wage is w and the effor is a, he gets utility $u(w, a) = \sqrt{w} - a$. (High effort requires $a = 3$ and low effort requires $a = 0$.) Contracts can be made contingent on the size of the sale only (effort is unobservable). Answer the following questions: (Display the optimization problems, but *do not* solve them numerically.)

(a) What is the optimal way to induce the salesman to put in a low level of effort?

(b) What is the optimal way to induce the salesman to put in a high level of effort?

(c) What is the optimal contract (from the principal's perspective) to offer the salesman?

44. A risk-neutral principal hires an agent to perform a task. The agent can perform at high level of effort (with disutility 4) or at low level of effort (with disutility 1). The agent has reservation utility of 2 and utility function $U(w, a) = \sqrt{w} - a$, where w is the wage and a is the disutility of effort. The productivity is given by probabilities in the table below.

Efforts/Output	0	100
high	0.2	0.8
low	0.8	0.2

(a) Suppose the principal can observe the level of effort. What is the best contract for the principal?

(b) Suppose the principal cannot observe the level of effort but only the output level. What is the best contract for the principal?

45. Consider a risk-neutral principal who wishes to hire a risk-averse agent. The agent gets utility $m^{\frac{3}{5}}$ for a wage of m dollars. The agent can exert three effort levels. Both his disutility and his productivity (probabilities of achieving sales of 100,200, 400 dollars) from effort are given in the table below:

Disutility of effort/ Sales	100	200	400
5	0.8	0.1	0.1
7	0.5	0.3	0.2
9	0.2	0.2	0.6

The agent has a reservation value of 9. The principal wants to announce a wage schedule based on observable output in order to maximize his expected profit. Denote by w_1, w_2, w_3 the wages announced for sales 100, 200, 400. Write down a minimization problem with linear constraints whose solution will be an optimal wage schedule. (Do not solve the minimization problem.)

46. There is a fixed, finite supply of cars, and infinitely many buyers in the market. The quality distribution of cars as well as the valuation of different quality cars by buyers and sellers are given in the following table:

	q_1	q_2	q_3
Buyer	1100	1800	2500
Seller	1000	1500	2000
Fraction	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$

(a) Suppose sellers observe the quality of the car, but buyers do not. Compute the market equilibrium prices.

(b) Suppose sellers have an option to credibly disclose the quality of the car. A seller chooses: He either sends a signal fully and truthfully disclosing the quality of the car, or does not disclose it at all. The cost of the signal to the sellers is 400. What will the equilibrium be in this case?

47. There is a finite number of sellers of cars, and infinitely many buyers. The distribution and valuation of the qualities (of cars) is given by:

	q_1	q_2	q_3	q_4
Buyer	1100	1800	2500	3000
Seller	1000	1500	2000	2500
Fraction(of cars)	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(a) Suppose sellers observe the quality of the car, but buyers do not. Compute the market equilibrium.

(b) Replace the first "$\frac{1}{3}$" in the table by x and adjust the remaining fractions so that they are in the same proportion as before, while leaving the buyers' and sellers' valuations unchanged. What is the smallest value of x for which only cars of quality q_1 and q_2 will sell on the market?

48. There are 3 types of used cars: bad (B), medium (M), good (G) in the proportion 4:3:2 among the finitely many sellers. There is a perfectly competitive sector of (infinitely many) buyers. The valuations of the cars are as follows:

Quality	Seller	Buyer
B	100	150
M	200	250
G	300	350

As usual sellers know the quality of the car and buyers don't.

(a) Find all the equilibria.

(b) Suppose the proportions are $4 : 3 : x$. What's the minimum value of x so that all the cars will be sold.

(c) Suppose the proportions are $4 : y : 2$. What's the minimum value of y so that B and M will be sold, but not G.

49. Suppose there are equal portions of low-ability ($t = 1$) and high-ability ($t = 2$) workers. The productivity of a worker of type $t = 1, 2$ is given by $\eta_t(e) = 2te$, where e is the education level. The utility of wage w and education e to a student of type t is $u_t(w, e) = 4\sqrt{w} - \frac{2e}{t}$. Find the Rothschild-Stiglitz equilibrium.

50. Suppose the fraction of low-ability ($t = 1$) workers is $\frac{1}{4}$ and fraction of high-ability ($t = 2$) workers is $\frac{3}{4}$. The productivity of a type 1 worker is $2e$ and the productivity of a type 2 worker is $\frac{9}{4}e$, where e is the education level. The utility of wage w and education e to a worker of type 1 is $u_1(w, e) = 4\sqrt{w} - 2e$. The utility of wage w and education e to a worker of type 2 is $u_2(w, e) = 4\sqrt{w} - 1.8e$. Find the Rothschild-Stiglitz equilibrium.

51. Suppose the productivity of a type 1 worker is $2e$ and the productivity of a type 2 worker is $\frac{9}{4}e$, where e is the education level. The utility of wage w and education e to a worker of type 1 is $u_1(w,e) = 4\sqrt{w} - 2e$. The utility of wage w and education e to a worker of type 2 is $u_2(w,e) = 4\sqrt{w} - 1.8e$.

(a) If the fraction of low-ability $(t = 1)$ workers is $\frac{1}{2}$, find the Rothschild-Stiglitz equilibrium.

(b) What's the minimum fraction of low-ability workers such that the Rothschild-Stiglitz equilibrium exists?

52. Suppose there is an equal proportion of low-ability worker and high-ability workers. The productivity of a worker of low-ability and high-ability worker is e and $2e$ respectively, where e is the education level. The utility functions of wage w and education e of low-ability and high-ability workers are respectively $u_1(w,e) = 4w - \frac{e^2}{2}$, $u_2(w,e) = 4w - \frac{2e^2}{7}$. Does a Rothschild-Stiglitz equilibrium exist? If so, find it.

53. Suppose there are equal proportions of low-ability and high-ability workers. Their productivities are e and $2e$ respectively, where e is the education level. The utility functions of wage w and education e of low-ability and high-ability workers are respectively $u_1(w,e) = 16\sqrt{w} - 4e$, $u_2(w,e) = 16\sqrt{w} - 3e$.

(a) Find the Rothschild-Stiglitz equilibrium.

(b) Suppose the proportion of low-ability to high-ability is changed from $1 : 1$ to $1 : \alpha$. For which values of α will a Rothschild-Stiglitz equilibrium exist?

54. Suppose there is an equal proportion of low-ability worker and high-ability workers. The productivity of a worker of low-ability and high-ability worker is e and $2e$ respectively, where e is the education level. The utility functions of wage w and education e of low-ability and high-ability workers are respectively $u_1(w,e) = 4w - \frac{e^2}{2}$, $u_2(w,e) = 4w - \frac{e^2}{3}$. Find the Rothschild-Stiglitz equilibrium.

Social Choice

55. Suppose that we have $n \geq 2$ individuals and $k > 2$ alternatives. Assume for simplicity that individuals' rankings of the alternatives are strict. Consider the following social welfare functions (SWF): Each individual, $i = 1, \ldots, n$ gives k points to the alternative he likes most, $k - 1$ to the alternative he likes second most, etc. The social ranking is according to the total points received from individuals to alternatives. For this social welfare function, check whether *transitivity, IIA* and *unanimity* hold. If a property holds, provide a proof, otherwise provide a counterexample.

56. Suppose that we have $n \geq 2$ individuals and $k > 2$ alternatives. Assume for simplicity that individuals' rankings of the alternatives are strict. Consider the following social welfare functions (SWF): There is an individual i so that x is socially prefered to y if and only if $y \succ_i x$. For this social welfare function, check whether *transitivity, IIA* and *unanimity* hold. If a property holds, provide a proof, otherwise provide a counterexample.

57. Consider a situation in which there are 3 agents. Assume that the set of alternatives, X, is the interval $[0, 1]$, and that each individual's preference is *single-peaked*, that is, for each i there is an alternative a_i^* such that if $a_i^* \geq b > c$ or $c > b \geq a_i^*$, then $b \succ_i c$. Consider the following voting procedure to chose a social alternative. Each agent writes in a sealed bid his vote for one alternative. The votes are counted and the median is calculated, which is then chosen as the social alternative. Show that the voting procedure described above is strategyproof, i.e., for each agent it is a dominant strategy to vote for her preferred alternative.

58. Suppose the set of alternatives is $A = \{x, y, z, t\}$. Consider the social choice function specified as follows: Any pair of alternatives except for the pair x, y, are ranked by simple majority voting (a tie in the vote means indifference). As for x, y the rule for comparing them is as follows: If an even number of individuals prefer x to y, then society will prefer x to y; if an odd number of individuals prefer x to y, then society will prefer y to x. Consider the following examples of preference profiles:

Profile 1				Profile 2				Profile 3		
1	2	3	4	1	2	3	4	1	2	3
x	z	y	t	x	z	t	y	x	z	t
y	t	x	x	y	x	y	x	y	x	x
t	x	t	y	z	y	x	t	z	y	y
z	y	z	z	t	t	z	z	t	t	z

(a) For each of the profiles, find the social preference.

(b) Using only the preference profiles provided above, determine if any of the axioms of transitivity, unanimity and independence of irrelevant alternatives is being violated and explain why if it is indeed the case.

Pivot Mechanism

59. Suppose a bridge costs 13 to build[4] and that the valuation of the bridge by the four individuals is given by

individual	1	2	3	4
valuation	5	4	1	2

[4] 1 unit of money here represents \$10,000

(a) Apply the pivot mechanism to decide whether or not the bridge is built, and the taxes (negative or positive) levied on the individuals; and the surplus (if any) collected by the government.

(b) Now suppose the bridge cost is 11. Answer the questions in part (a) again.

60. Show that if $f : \Theta \to X$ is truthfully implementable in dominant strategies when the set of possible types is Θ_i, for $i = 1, \ldots, I$, then when each agent i's set of possible types is $\hat{\Theta}_i \subset \Theta_i$ (for $i = 1, \ldots, I$) the social choice function $\hat{f} : \hat{\Theta}_i \to X$ satisfying $\hat{f}(\theta) = f(\theta)$ for all $\theta \in \hat{\Theta}$ is truthfully implementable in dominant strategies.

61. Suppose a bridge costs 24 to build and that the valuation of the bridge by the six individuals is given by

individual i	1	2	3	4	5	6
valuation u_i	2	5	6	4	1	6
valuation u_i'	4	-7	12	8	-5	6

Apply the Groves-Clark mechanism to both valuation profiles u and u' in order to decide whether or not the bridge is built, and the taxes (negative or positive) levied on the individuals, and the surplus (if any) collected by the government.

Shapley Value and Core

62. A glove-market game has three players: players 1 and 2 each have a left-hand glove and player 3 has a right-hand glove. The worth of a coalition is the amount that it will get for the gloves in its possession. Every pair of gloves (left and right) can be sold in the market for 50. A single glove cannot be sold in the market. Describe the game in coalition function form. Compute the core and the Shapley Value of the game. Is the Shapley Value in the core?

63. Suppose that if k workers are employed by a landlord, they produce $f(k)$ units of food (for $1 \leq k \leq 10$). Without the landlord, there can be no production. Consider the eleven-person game consisting of the landlord and the ten identical workers. Compute the Shapley Value of this game.

64. There are two landlords and 6 workers. If k workers till the land, the output is worth k^2 dollars.

Case A: Both landlords need to be present in a coalition for the land to be tilled.

Case B: The presence of any one landlord suffices for the land to be tilled.

(a) Compute the Shapley value of the eight players in both cases A and B.

(b) In both cases, write a general formula for the Shapley value when the number of workers is n.

(Hint: It may be simpler to do part (b) before part (a).)

65. The Security Council has 5 permanent members and 10 non-permanent members. For a coalition to win, it must contain all 5 permanent members and at least 4 non-permanent members. View this situation as a simple (voting) game and compute the Shapley Value of the 15 members.

66. Let v be a simple game on players set N and let T be the set of veto players in v. Prove that

$$\text{Core } v = \{x \in R_+^N : \sum_{i \in T} x_i = 1\}$$

(Note: this implies that if $T = \emptyset$, then Core $v = \emptyset$)

67. Consider the three-person cooperative game v given by

$$v(1) = v(2) = v(3) = 0$$

$$v(12) = v(23) = x, \quad v(13) = 1, \quad v(123) = 4$$

(a) For which values of x is Core v non-empty?

(b) Take $x = 2$. Compute all the vertices of Core v and the Shapley Value of v. Is the Shapley Value in Core v?

68. Consider the symmetric game on ten players where, (denoting $|S|$ = number of players in coalition S), let

$$v(S) = \begin{cases} f(|S|) & \text{for } |S| \leq 9 \\ x & \text{for } |S| = 10 \end{cases}$$

Case I: $f(|S|) = \sqrt{|S|}$

Case II: $f(|S|) = |S|^2$

In both cases what is the minimum value of x for which the core of the game v is non-empty.

69. (a) Consider the game where player 1 and 2 have right gloves and player 3 has a left glove. Any coalition gets one dollar for every pair of left-right glove it has. Express the game as a characteristic function v and compute the Core of v and the Shapley Value of v.

(b) Can you say what Core v and Shapley Value of v are when players 1,2,3 each have right golves and 4,5 each have left golves; and, as before, every pair is worth one dollar?

(c) Can you do the case where $k + 1$ players have right gloves and k players have left gloves ($2k + 1$ players in all)?

70. Calculate the Shapley Value in the following simple majority games:

a. [10; 7, 5, 4, 3]

b. [12; 4, 4, 9, 5]

c. [17; 7, 8, 9, 9]

d. [7; 4, 2, 2, 2, 2]

e. [5; 3, 3, 1, 1, 1]

f. [9; 4, 4, 2, 2, 2, 2]

g. [6; 3, 1, 1, 1, 1, 1, 1, 1]

71. Which of Shapley's axioms for the value (i.e, Efficiency,Symmetry, Dummy, and Additivity) are violated by the functions defined in (a) and (b) below. Give reasons for your answer.

(a) For any game v and any $i \in N$, let $\varphi_i(v)$ be the average marginal contribution of player i over all the $(|N|-1)!$ orderings of N in which player 1 is first.

(b) For any game v let $\varphi_i(v) = \frac{v(N)}{|N|}$.