Lecture Notes of Prof. Pradeep Dubey's lecture at Paris I. on Kidney Matching Problem

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I. Background of the problem

Assumption: Each patient has a donor who is willing to donate a kidney to him/her. So each patient "owns" a kidney.

For each patient, now rank all the kidneys according to how suitable they are for him on medical grounds.
To save life with higher probability, trading kidneys is highly desirable if it can make everyone better off.

There is a paper by Shapley & Scarf (1974) on this, where they use "houses" instead of "kidneys".

II. Model & Theorems

Examples.

suppose A owns house a,
B owns house b, etc.

profile of preferences

A : c d b a

B : c a b d

C : d b a c

D : a b c d

X : not blocking

v : blocking

There are $4! = 24$ possible allocations of houses among A, B, C, D.
We shall define "stable" and "strongly stable" allocations.

Suppose an allocation is proposed but not carried out.
Then a "coalition" can form and wonder if its members can be better off than the proposal by simply trading among themselves.

Two notions of "blocking":

1. on "Earth", people are selfish. So an allocation is carried out by a coalition S only when everyone in S is better off.

2. on "Mars", people are more generous. An allocation is ok if nobody is worse off and somebody is better off.

Suppose the allocation $\mathcal{A}$ = < Ad, Ba, Cb, Dc > is proposed and not executed, there are 2^4 possible coalitions (in or out, 2 choices for each person).

③

① Let the coalition be {B, D} → <Bb, Dd>
→ <Bd, Db>.

Does {B, D} block $\mathcal{A}$? Earth × Mars ×

(B: a > d, a > b. B is worse off in both trades. hence {B, D} does not block A on either Mars or earth.)

② Let the coalition be {B, C} → <Bb, Cc>
→ < Bc, Cb >.

Can {B, C} block $\mathcal{A}$? Earth × Mars ✓

(B: c > a > b, c: b > c. So, compared to $\mathcal{A}$, B is better off in the trade < Bc, Cb > and C is the same. Not both are better off, so there's no blocking on Earth. But {B, C} blocks $\mathcal{A}$ via < Bc, Cb > on Mars.

③ Let the coalition be {c} → < Cc >

Can {c} block $\mathcal{A}$? Earth × Mars ×

(C: b > c. C itself can form a coalition, but C is better off in the proposal. So there's no blocking by {c} either on Earth or on Mars.)

④ Let the coalition be {A, C, D} → <Ac, Cd, Da>

blocking? Earth ✓ Mars ✓

(A: c > d, C: d > b, D: a > c, so everyone is better off. Thus {A, C, D} blocks $\mathcal{A}$ via <Ac, Cd, Da > on both Earth and Mars.)

⑤ Is $\langle Ac, Cb, Da \rangle$ an allocation? No.

({A, C, D} do not have access to B's house because B is not a member of their coalition.)

Remarks.

* Is it possible a coalition can block the proposal on Earth but not on Mars?

No. Any blocking on Earth is automatically a blocking on Mars.

A blocking occurs on Earth when everyone in the coalition is better off, on Mars there are only more possibilities of being better off. So a blocking on Earth is a blocking on Mars.

Now introduce the definitions on "stable".

Definition.

1. An allocation is called "stable" if no coalition can block it on Earth.

2. An allocation is called "strongly stable" if no coalition can block it on Mars.

3. strict core = set of strongly stable allocations

4. core = set of stable allocations

Remarks. * stable on Earth is not automatically stable on Mars.

* Clearly, strict cone $\subset$ cone

Now consider the following two allocations:

$\tilde{\mathcal{A}} = \langle Ab, Bc, Cd, Da \rangle$
$\mathcal{A}^* = \langle Ac, Bb, Cd, Da \rangle$

Claim 1: $\tilde{\mathcal{A}}$ is stable (wrt the given profile)

Pf. B, C, D are getting their top choices in $\tilde{\mathcal{A}}$.
So the only coalition on Earth is $\{A\} \longrightarrow \langle Aa \rangle$
$(A: b > a)$
which cannot block $\tilde{\mathcal{A}}$ on Earth. $\square$

Claim 2: $\tilde{\mathcal{A}}$ is not strongly stable.

Pf. Consider the coalition $\{A, C, D\} \longrightarrow \langle Ac, Cd, Da \rangle$.
(A: $c > b$. A is better off in $\langle Ac, Cd, Da \rangle$ compared to $\tilde{\mathcal{A}}$, C and D stay the same)
So $\{A, C, D\}$ blocks $\tilde{\mathcal{A}}$ via $\langle Ac, Cd, Da \rangle$ on Mars.
Hence $\tilde{\mathcal{A}}$ is not strongly stable. $\square$

Remarks.

* $\tilde{\mathcal{A}} \in$ cone, $\tilde{\mathcal{A}} \notin$ strict cone.

* We can see why A^* is strongly stable later.

* $A^* \in$ strict cone $\subset$ cone.