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AN ELEMENTARY CORE EQUIVALENCE THEOREM

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We give an elementary statement and proof of a core equivalence theorem.

1. INTRODUCTION

IN 1881, F. Y. EDGEWORTH [**8**] defined the contract curve, a solution concept in economies which coincides in the cases he considered with what we now call the core. He showed that, under very special hypotheses, any element in the core of an economy with sufficiently many traders is close to a competitive equilibrium. He conjectured that this behavior holds in a wide class of economies.

In recent years, extensions of Edgeworth's result have been a major focus of the literature in mathematical economics. Notable contributions have been made by Arrow–Hahn [**2**], Aumann [**3**], Bewley [**4**], Brown–Robinson [**5**], Debreu–Scarf [**6**], Dierker [**7**], Hildenbrand [**9**], Keiding [**10**], Vind [**14**], and others. Some of these papers have involved only elementary techniques; others have involved techniques such as advanced measure theory, nonstandard analysis, or differential topology.

In most of the above papers, Edgeworth's Conjecture has been formulated in terms of sequences of economies. The theorems state that, for suitable sequences, some measure of the degree of non-competitiveness tends to 0. Hildenbrand obtained very general theorems of this type with a very lengthy argument using sophisticated techniques in measure theory. In an unpublished paper [**10**], Keiding outlined a short and elementary proof of an improved version of Hildenbrand's limit theorem for economies without convex preferences [**9**, Theorem 3, p. 202].

A few of the above papers have been formulated in terms of a single fixed economy, rather than a sequence. Our paper lies within this group. Vind [**14**] obtained a bound on the number of agents violating a certain weak competitiveness condition; his theorem was later used in the derivation of some of the sequential theorems. Arrow–Hahn [**2**] obtained a bound on the sum of the measure of non-competitiveness of the agents, assuming bounded non-convexity of sets depending on the core allocation and the initial endowments. Dierker [**7**], in a paper concerned with finding a coalition blocking a given allocation, obtained a bound on the sum of the measure of non-competitiveness without the restrictive assumptions of Arrow–Hahn.

In this paper, we shall state and prove a very simple yet very general version of Edgeworth's Conjecture. The only tools used are the Shapley–Folkman

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Theorem and a separating hyperplane argument. Theorem 1 gives an explicit bound on the degree of non-competitiveness of a fixed finite economy.

We feel that the fixed economy formulation is superior to the sequential formulation because it provides an explicit computable bound on the degree of non-competitiveness of a fixed economy. In order to indicate the connection between the approaches, however, we shall show how our result immediately implies an improved version (Theorem 2) of Hildenbrand's Theorem 3 [**9**, p. 202].

Apart from minor differences in the transitivity assumption and the constant, Dierker's result is bound (ii) of our Theorem 1, and our bound (i) follows easily. We believe that Dierker's paper drew less attention than it deserved. This was perhaps due to the length and technical demands of his argument, and also to the fact that he was not explicit regarding the relationship of his results to those in Hildenbrand's book. With the simple treatment provided here, the efficiency of an approach to Edgeworth's Conjecture via the bounds (i) and (ii) becomes clear.

Throughout the paper, we make no convexity assumptions on preferences. An improved version of Hildenbrand's limit theorem for convex economies [**9**, Theorem 1, p. 179] is included in the author's dissertation [**1**], and will be presented in a separate article.

The proof was suggested by the proof of Hildenbrand [**9**, Theorem 1, p. 133] and the special properties of nonstandard economies. However, a simplification process resulted in the elimination of all measure-theoretic and nonstandard arguments from the final proof. The work of Rashid [**12**] relating nonstandard and measure-theoretic concepts in economies, and the treatment of very general preferences in Khan-Rashid [**11**] proved very helpful in carrying out this simplification.

In hindsight, the final version of our proof is very similar to the proof given by Arrow and Hahn [**2**], with one crucial change sufficing to eliminate their extra assumption. In this sense, our result confirms their intuition that core equivalence can be viewed naturally as a property of a fixed finite economy.

2. STATEMENT AND PROOF

In this section we state and prove the principal theorem. We begin with some notation. Suppose $x, y \in R^k$ and $A, B \subset R^k$. Then x^i denotes the ith component of x and

$$\|x\|_\infty = \max_{1 \leq i \leq k} |x^i|, \qquad \|x\|_1 = \sum_{i=1}^{k} |x^i|;$$

$u = (1, \ldots, 1) \in R^k$; $x \leq y$ if $x^i \leq y^i (1 \leq i \leq k)$; $x \ll y$ if $x^i < y^i (1 \leq i \leq k)$; $A + B = \{x + y \colon x \in A,\ y \in B\}$; con A is the convex hull of A; and $R^k_+ = \{x \in R^k : x \geq 0\}$.

Let $\mathcal{P}$ denote the set of preferences (i.e. binary relations $\succ$ on R^k_+) satisfying the following conditions: (i) weak monotonicity: $x \gg y \Rightarrow x \succ y$, and (ii) free disposal: $x \gg y,\ y \succ z \Rightarrow x \succ z$. Note that in place of the transitivity commonly assumed, we assume the weaker condition of free disposal. Note further that no continuity is assumed.

DEFINITION: An exchange economy is a map $\varepsilon : A \to \mathcal{P} \times R^k_+$, where A is a finite set. For $a \in A$, let $>_a$ be the projection of $\varepsilon(a)$ onto $\mathcal{P}$, and $e(a)$ the projection of $\varepsilon(a)$ onto R^k_+. $>_a$ is interpreted as the preference of trader a, and $e(a)$ his initial endowment. An allocation is a map $f: A \to R^k_+$ such that $\Sigma_{a\in A} f(a) = \Sigma_{a \in A} e(a)$. A coalition is a non-empty subset of A. An allocation, f, is blocked by a coalition S if there exists $g: S \to R^k_+$ with $\Sigma_{a \in S} g(a) = \Sigma_{a \in S} e(a)$ such that $g(a) >_a f(a)$ for all $a \in S$. The core of ε, $C(\varepsilon)$, is the set of all allocations which are not blocked by any coalition. A price p is an element of R^k_+ such that $\|p\|_1 = 1$. Let $\mathcal{S}$ be the set of all prices.

THEOREM 1: *Let* $\varepsilon : A \to \mathcal{P} \times R^k_+$ *be a finite exchange economy, with* $|A| = n$. *Let* $M = \sup \{\|e(a_1) + \ldots + e(a_k)\|_\infty . a_1, \ldots, a_k \in A\}$. *If* $f \in C(\varepsilon)$, *there exists* $p \in \mathcal{S}$ *such that* (i) $(1/n)\Sigma_{a\in A}|p \cdot (f(a) - e(a))| \leqslant 2M/n$; (ii) $(1/n)\Sigma_{a \in A}|\inf\{p \cdot (x - e(a)): X >_a f(a)\}| \leqslant 2M/n$.

REMARK: Condition (i) says that the average budget deviation—the average amount by which the cost of the core allocation differs from the proceeds of sale of the initial endowment—is small. Condition (ii) asserts that, if one shrinks the budget set $\{x: p \cdot x \leqslant p \cdot e(a)\}$ slightly on average, no element of the reduced budget set is preferred to the core allocation. This captures the idea that nothing in the original budget set is preferred to the core allocation by very much. In some applications, other normalizations of price are more convenient. Let $\|x\|_q = (\Sigma_{i=1}^k |x^i|^q)^{1/q}$ $(1 \leqslant q < \infty)$. If we normalize prices by $\|p\|_q = 1$ $(1 \leqslant q \leqslant \infty)$, the theorem remains valid with $M = k^{1/r} \sup \{\|e(a_1) + \ldots + e(a_k)\|_\infty : a_1, \ldots, a_k \in A\}$, where $(1/r) + (1/q) = 1$.

PROOF OF THEOREM 1: For $a \in A$, let $\phi(a) = \{x - e(a): x >_a f(a)\} \cup \{0\}$. Define $\Phi = (1/n) \Sigma_{a \in A} \phi(a)$.

Suppose there exists $G \in \Phi$, $G \ll 0$. Then there exists $g: A \to R^k$ with $g(a) \in \phi(a)$ for all a and such that $G = (1/n) \Sigma_{a\in A} g(a)$. Let $B = \{a \in A: g(a) \neq 0\}$ and $h(a) = g(a) + e(a) - (n/|B|)G$ for $a \in B$. Then $h(a) \gg g(a) + e(a)$ and $g(a) + e(a) >_a f(a)$. Since $>_a \in \mathcal{P}$, $h(a) >_a f(a)$ for all $a \in B$.

$$\sum_{a\in B} h(a) = \sum_{a \in B} (g(a) + e(a) - (n/|B|)G) = \sum_{a\in B} g(a) + \sum_{a \in B} e(a) - nG$$

$$= nG + \sum_{a\in B} e(a) - nG = \sum_{a \in B} e(a).$$

Hence B blocks f, so $f \notin C(\varepsilon)$, contradiction. Hence $G \ll p \Rightarrow G \notin \Phi$.

Let $z = (M/n)u$. Suppose $x \in (\text{con } \Phi) \cap \{w \in R^k: w \ll -z\}$. By the Shapley–Folkman Theorem (Starr [**13**, p. 35]), we can write x in the form $x = (1/n)\Sigma_{a\in A} g(a)$, where $g(a) \in \text{con } \phi(a)$ for all $a \in A$ and $g(a) \in \phi(a)$ for all but m

values of a, with $m \leq k$. Let those values be $\{a_1, \ldots, a_m\}$. Let $g'(a)$ equal 0 if a is one of the a_i's and let $g'(a) = g(a)$ otherwise. Since $\phi(a_i) \geq -e(a_i)$, con $\phi(a_i) \geq -e(a_i)$. Let $y = (1/n)\Sigma_{a \in A}\, g'(a) \in \Phi$. Then

$$y = x - (1/n) \sum_{i=1}^{m} g(a_i) \leq x + (1/n) \sum_{i=1}^{m} e(a_i) \leq x + z \ll 0.$$

Since $y \in \Phi$, this is a contradiction.

Hence $(\text{con}\, \Phi) \cap \{w \in R^k : w \ll -z\}$ is empty. Since these sets are convex, Minkowski's Theorem implies that there exists $p \in \mathcal{S}$ such that p separates Φ from $\{w \in R^k : w \ll -z\}$. Therefore $\inf p \cdot \Phi \geq \sup\{p \cdot w : w \ll -z\} = -p \cdot z = -(M/n)$. Since $0 \in \phi(a)$ for all a, $0 \geq (1/n)\Sigma_{a \in A} \inf p \cdot \phi(a) = \inf p \cdot \Phi \geq -(M/n)$.

Since $f(a) - e(a) + u/m \in \phi(a)$ for any natural number m, $p \cdot (f(a) - e(a)) \geq \inf p \cdot \phi(a)$. If $S = \{a \in A : p \cdot (f(a) - e(a)) < 0\}$,

$$\frac{1}{n} \sum_{a \in S} p \cdot (f(a) - e(a)) \geq \frac{1}{n} \sum_{a \in S} \inf p \cdot \phi(a) \geq -\frac{M}{n}.$$

$$\frac{1}{n} \sum_{a \in A} p \cdot (f(a) - e(a)) = \frac{1}{n} p \cdot \Big(\sum_{a \in A} f(a) - \sum_{a \in A} e(a) \Big) = p \cdot 0 = 0.$$

Therefore,

$$\frac{1}{n} \sum_{a \in A} |p \cdot (f(a) - e(a))| = \frac{2}{n} \sum_{a \in S} |p \cdot (f(a) - e(a))| \geq \frac{2M}{n}.$$

$$\begin{aligned} &\frac{1}{n} \sum_{a \in A} |\inf\{p \cdot (x - e(a)) : x \succ_a f(a)\}| \\ &\quad \leq -\frac{1}{n} \sum_{a \in A} \inf p \cdot \phi(a) + \frac{1}{n} \sum_{a \notin S} p \cdot (f(a) - e(a)) \\ &\quad \leq \frac{M}{n} + \frac{M}{n} = \frac{2M}{n}. \end{aligned}$$

This completes the proof.

The following theorem applies to more general sequences than Hildenbrand [**9**, Theorem 3, p. 202]. The principal differences are (i) the replacement of the uniform integrability condition $|E_n|/|A_n| \to 0$ implies $\|\Sigma_{E_n} e_n(a)\|/|A_n| \to 0$ by the weaker condition $M_n/|A_n| \to 0$, and (ii) the removal of several assumptions on individual preferences, as well as the tightness assumption on the collection of preferences implicit in the definition of weak convergence. The conclusion is basically stronger, since it gives convergence in L^1 instead of the weaker convergence in measure obtained by Hildenbrand. However, Hildenbrand does obtain the additional information that one can take all $p_n = p$, for some equilibrium price p of a limit economy; since our hypotheses are not sufficient to guarantee the existence of a limit economy, we cannot make such a statement. Further comments on the relationship of these results to Hildenbrand's are given in Chapter V of the author's dissertation [**1**].

THEOREM 2: *Let* $\varepsilon_n: A_n \to \mathcal{P} \times R_+^k$ *be a sequence of exchange economies such that* $M_n/|A_n| \to 0$. *If* $f_n \in C(\varepsilon_n)$, *there exists prices* $p_n \in \mathcal{S}$ *such that*

(i) $$\frac{1}{|A_n|} \sum_{a \in A_n} |p_n \cdot (f_n(a) - e_n(a))| \to 0$$

and

(ii) $$\frac{1}{|A_n|} \sum_{a \in A_n} |inf\{p_n \cdot (x - e_n(a)) \colon x \succ_a f_n(a)\}| \to 0.$$

PROOF: Apply Theorem 1.

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