Homework 3 Suggested Solutions

1. Consider a production and exchange economy with two commodities, two households, labeled 1 and 2, and one firm with production set $Y = \{(x, y)|x \leq 0, y \leq \sqrt{-x}\}$. Household 1 has endowment $e^1 = (2, 1)$, utility $u^1(x_1, y_1) = \frac{1}{2}\ln x_1 + \ln y_1$ and $\frac{1}{3}$ share of the firm. Household 2 has endowment $e^2 = (6, 1)$, utility $u^2(x_2, y_2) = \ln x_1 + \frac{1}{2}\ln y_1$, and $\frac{2}{3}$ share of the firm. Compute a competitive equilibrium.

Easy to verify that no price of commodities can be zero. If $p_x = 0$, household 1, household 2 and firm's demand for x will all be infinite, but supply of x is finite, so no CE exists. If $p_y = 0$, both household 1 and 2's demand for y are infinite, but firm will not produce any y since $p_y = 0$, thus the supply of y is finite, no CE exists. So we can normalize $p_y = 1$ and let $p_x = p$. Firstly, we calculate the supply correspondence and the profit function for the firm. Notice that the firm's production set is given by $Y = \{(y, x) \in \mathbb{R} \times \mathbb{R}_- |y \leq \sqrt{-x}\}$. Thus the firms solves

$$\pi(p) = \max_{(y,x)\in Y} \{y + px\}.$$

Note that since the technology is strictly convex the first order conditions are not only necessary but also sufficient. Furthermore we know that the firm will choose to produce at its maximum amount, ie. $y = \sqrt{-x}$. Thus we can restate the firm's problem as

$$\pi(p) = \max_{x\in\mathbb{R}_+} \{\sqrt{x} - px\}.$$

Since the marginal productivity of x is infinite as x approaches 0 from the left, we have that the First order condition will hold with equality:

$$\frac{1}{2}x^{-1/2} = p,$$

which implies that the firm input demand of x^*, supply of y^* and profit function $\pi(p)$ will be, respectively:

$$x^*(p) = \frac{1}{4p^2}, \quad y^*(p) = \frac{1}{2p}, \quad \pi(p) = \frac{1}{4p}.$$

Household shares of the profits are, for 1 and 2 respectively, $\frac{1}{12p}$ and $\frac{1}{6p}$. Thus demands are given by:

$$x^1(p) = \frac{2p + \frac{1}{12p} + 1}{3p}, \qquad y^1(p) = \frac{2}{3}\left(2p + \frac{1}{12p} + 1\right),$$
$$x^2(p) = \frac{2\left(6p + \frac{1}{6p} + 1\right)}{3p} \qquad y^2(p) = \frac{1}{3}\left(6p + \frac{1}{6p} + 1\right).$$

Using market clearing we have that p has to solve simultaneously:

$$\frac{2p + \frac{1}{12p} + 1}{3p} + \frac{2\left(6p + \frac{1}{6p} + 1\right)}{3p} + \frac{1}{4p^2} = 8,$$
$$\frac{2}{3}\left(2p + \frac{1}{12p} + 1\right) + \frac{1}{3}\left(6p + \frac{1}{6p} + 1\right) = 2 + \frac{1}{2p},$$

which implies that $p = \frac{1}{60}\left(9 + \sqrt{501}\right) \approx 0.52305$. Thus we have that the equilibrium allocations are $x^1 = 1.40549, y^1 = 1.47028, x^2 = 5.68071, y^2 = 1.48565, x^* = 0.9138, y^* = 0.9559$.

2. Consider a production and exchange economy with two commodities x and y, two households 1 and 2 and one firm with production set

$$Y = \left\{(x, y) \in \mathbb{R}^2 \;\middle|\; x \leq 0, y \leq -\frac{1}{2}x\right\}.$$

Household 1 has endowment $e^1 = (4, 1)$, utility function $u^1(x, y) = x^{\frac{1}{3}}y^{\frac{2}{3}}$ and $\frac{1}{3}$ share of the firm. Household 2 has an endowment $e^2 = (2, 0)$, utility function $u^2(x, y) = \sqrt{xy}$ and $\frac{2}{3}$ share of the firm.

(a) Compute a competitive equilibrium of this economy. Is it unique?

(b) If firm shares are changed from $(\frac{1}{3}, \frac{2}{3})$ to $(\frac{1}{2}, \frac{1}{2})$, how will the competitive equilibrium change?

(a) The same reason as question 1 to argue that $p_x, p_y \neq 0$. Normalize $p_y = 1$ and let $p_x = p$. Notice that the firm has constant returns to scale. Therefore the profit function and the supply correspondence are

$$\pi(p) = \begin{cases} \infty & \text{if } p < \frac{1}{2} \\ 0 & \text{if } p \geq \frac{1}{2} \end{cases}, \quad (x_3, y_3) = \begin{cases} \text{undefined} & \text{if } p < \frac{1}{2} \\ \lambda(-1/2, 1), \text{for } \lambda > 0, & \text{if } p = (1/2, 1) \\ (0, 0) & \text{if } p > 1/2 \end{cases}$$

If $p < 1/2$ we have unbounded returns for the firm which is not possible in equilibrium since households' wealth would be unbounded. Then $p \geq 1/2$. Solving both households' utility maximization problem we can get the following :

$$x_1 = 4/3 + 1/3p, y_1 = 8p/3 + 2/3$$

$$x_2 = 1, y_2 = p$$

The MC gives

$$x_1 + x_2 = 6 + x_3$$

$$y_1 + y_2 = 1 + y_3$$

Then we have

$$x_3 = 1/3p - 11/3, \quad y_3 = 11p/3 - 1/3$$

If $p > 1/2$ we have $x_3 = y_3 = 0$. But then $p = 1/11$, contradiction. Then $p = 1/2$ and demands are:

$$x_1 = 2, \quad y_1 = 2, \quad x_2 = 1, \quad y_2 = \frac{1}{2},$$

and from this we have that the firm input demand is $x_3 = -3$ and production $y_3 = \frac{3}{2}$.

(b) Since the profit of the firm is always zero in CE. No matter what the firm shares change to, the budget constraint for HHs will not change. So the allocations and the CE will not change.

3. Suppose that we have $n \geq 2$ individuals and $k > 2$ alternatives. Assume for simplicity that individuals' rankings of the alternatives are strict. Consider the following social welfare functions (SWF):

(a) Each individual, $i = 1, \ldots, n$ gives k points to the alternative he likes most, $k - 1$ to the alternative he likes second most, etc. The social ranking is according to the total points received from individuals to alternatives.

(b) There is an individual i so that x is socially prefered to y if and only if $y \succ_i x$.

For (a) and (b), check whether *transitivity, IIA* and *unanimity* hold. If a property holds, provide a proof, otherwise provide a counterexample.

Solution: Procedure (a) is known as the *Borda Count.* Notice that if an alternative x is in position n, then we have that each individual i assign points $\pi_i(x) = k - n + 1$. Thus the alternative x is socially preferred to y, i.e. $x \succ y$ if and only if $\sum_i \pi_i(x) > \sum_i \pi_i(y)$. Thus, the SWF it induces is *transitive*. Furthermore it satisfies unanimity since if for every i we have that $x_i \succ_i y_i$, this implies that $\pi_i(x) > \pi_i(y)$ so that $\sum_i \pi_i(x) > \sum_i \pi_i(y)$. The Borda method does not satisfy IIA, which can be shown by the following example. Consider two agents and three alternatives $\{x, y, z\}$. For the profile

$$x \succ_1 z \succ_1 y,$$

$$y \succ_2 x \succ_2 z$$

we get that $x \succ y$ since x gets 5 points and y gets 4 points. But for the profile

$$x \succ_1' y \succ_1' z,$$

$$y \succ_2' z \succ_2' x$$

we get that $y \succ x$ since y gets 5 points and x gets 4 points. Yet the relative ranking of x and y in both profiles has not changed.

Procedure (b) is called an *anti-dictatorship*. Indeed if i is the anti-dictator notice that x is socially preferred to y, i.e. $x \succ y$ if and only if $y \succ_i x$. Since $\succ_i$ is a preference relation, then it satisfies transitivity and IIA. To see the latter, fix two profiles $(\succ_j)$ and $(\succ_j')$ such that $x \succ_j y$ if and only if $x \succ_j' y$. This condition holds in particular for i, then if follows that $y \succ x$ if and only if $y \succ' x$. Notice that it can be easily seen that an anti-dictatorship does not satisfy unanimity. Suppose we have again two agents and three alternatives and suppose agent 1 is the anti-dictator. Consider the profile

$$x \succ_1 z \succ_1 y,$$

$$x \succ_2 y \succ_2 z$$

where we get that $y \succ z \succ x$, even though all members of society ranked x at the top.