Homework 7
Suggested Solutions

1. Consider a game involving a glove market. Players 1 and 2 each have a left-hand glove and player 3 has a right-hand glove. The worth of a coalition is the amount that it will get for the gloves in its possession. Every pair of gloves (left and right) can be sold in the market for \$50. A single glove cannot be sold in the market.

(a) Describe the game in coalition function form.

(b) Compute the Core and the Shapley Value of the game.

(c) Is the Shapley Value in the core?

Solution:

(a) The coalitional game with transferable utility that describes this game is a pair $(N; v)$ such that

(i) N={ 1,2,3 }.

(ii) $v : 2^N \to \mathbb{R}$ is the coalition function that associates every coaltion $S \in 2^N$ with a real number $v(S)$.

$$\begin{aligned} &v(\emptyset) = 0, \quad v(1) = v(2) = v(3) = 0 \\ &v(1,2) = 0, \quad v(1,3) = v(2,3) = 50 \\ &v(1,2,3) = 50 \end{aligned}$$

(b) The set of imputations is given by the triangle whose vertices are $(50,0,0)$, $(0,50,0)$ and $(0,0,50)$. An imputation $x = (x_1, x_2, x_3)$ is in the Core of this game if and only if

$$\begin{aligned} x_1 + x_2 + x_3 &= 50 \\ x_1 + x_3 &\geq 50 \\ x_2 + x_3 &\geq 50. \end{aligned}$$

Thus, the Core is simply $(x_1 = 0, x_2 = 0, x_3 = 50)$.

$$C(N; v) = \{x \in \mathbb{R}^3 | x = (0,0,50)\}$$

To compute the Shapley Value note that players 1 and 2 only contribute to a coalition when they join player 3. Moreover, player 3 contributes whenever he joins player 1, player 2 or a coalition of these two.

$$\begin{aligned} \varphi_1 = \varphi_2 &= \frac{1!(3-1-1)!}{3!}50 = \frac{25}{3} \\ \varphi_3 &= \frac{1!(3-1-1)!}{3!}50 + \frac{1!(3-1-1)!}{3!}50 + \frac{1!(3-2-1)!}{3!}50 = \frac{100}{3} \end{aligned}$$

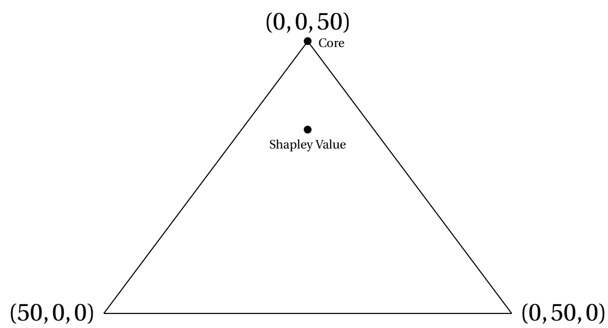

Figure 1 Core and Shapley value of the glove game.

(c) Clearly, the Shapley value is not in the Core.

$$\varphi = (\frac{25}{3}, \frac{25}{3}, \frac{100}{3}) \notin C(N; v)$$

2. Let v be a simple game on players set N and let T be the set of veto players in v. Show that the Core of a simple game is not empty if and only if there is (at least one) veto player in N.

Solution:

(i) Consider $x \in C(N; v)$.

- Since $x(N) = \sum_{i \in N} x_i = v(N) = 1$, there must exist a player i such that $x_i > 0$.
- Consider the coalition $N\backslash\{i\}$. Since it does not "block" x, it must be that $x(N\backslash\{i\}) \geq v(N\backslash\{i\})$.
- However, we know that $x(N\backslash\{i\}) < 1$. This implies $v(N\backslash\{i\}) < 1$.
- By simple game, it must be that $v(N\backslash\{i\})$=0.
- This implies that i is a veto player since $x = 1 \geq v(i)$.

(ii) Let player i be a veto player.

- We define x such that $x_j = v(N) = 1$ if $j = i$ and $x_j = 0$ if $j \neq i$ and show no coalition "blocks" x.
- For every coalition S
  - If $i \in S$ then $x(S) = 1$ since it includes i. Hence $x(S) = 1 \geq v(S)$ and S does not block x.
  - If $i \notin S$ then $x(S) = 0$. Hence $x(S) = 0 = v(S)$ and S does not block x.

3. Let (N, v) be the transferable utility game where $\mathrm{N} = \{1,2,3\}$, $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$ and $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\}) = v(\{1,2,3\}) = 1$. Show that the Core of this game is empty, i.e., $\mathrm{C}(v) = \emptyset$.

Solution:

An imputation $x = (x_1, x_2, x_3)$ is in the Core of this game if and only if

$$x_1 + x_2 \geq 1$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_3 \geq 1.$$

Hence, $2x_1 + 2x_2 + 2x_3 \geq 3$ or $x_1 + x_2 + x_3 \geq \frac{3}{2} > 1 = v(\mathrm{N})$. Thus $\mathrm{C}(\mathrm{N}; v) = \emptyset$.

4. Calculate the Shapley Value in the following simple majority games:

(a) [17;7,8,9,9]

(b) [10;7,5,4,3]

Solution: Total number of orderings: $4! = 24$.

(a) When is player 1 pivotal? Never! Null player.

When is player 2 pivotal?

- Whenever player 2 is the second to arrive, the one in front is player 3 or player 4 (2 cases) and the two behind are player 1 and whoever is not in front. (2 cases)
  - 4 cases
- When player 2 is the third to arrive and the two in front are player 1 and either player 3 or 4 ($2 \cdot 2$ cases) and the one behind is either player 3 or 4.
  - 4 cases

When is player 3 (player 4) pivotal?

- In front player 2 or player 4 (2 cases). Behind player 1 and the other of player 2 and 4 (2 cases) .
  - 4 cases
- In front player 1 and either player 2 or 4 ($2 \cdot 2$ cases) and the one behind is the other of player 2 and 4.
  - 4 cases

Shapley value of the game: $\varphi = \{0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. (Note that it is enough to compute the Shapley value of player 2 to obtain all the others.)

(b) When is player 1 pivotal?

- The one in front: either player 2, 3 or 4 (3 cases). The two behind: other two (2 cases).
  - 6 cases
- The two in front: either player 2, 3 or 4 (3 cases). The one behind: the remaining one.
  - 6 cases

When is player 2 pivotal?

- The one in front is player 1 and the two behind are players 3 and 4 (2 cases).
- The two in front are players 3 and 4 and the one behind is player 2 (2 cases).

When is player 3 pivotal? Same as 2.
When is player 4 pivotal? Same as 2.
Shapley value of the game: $\varphi = \{\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}\}$.