4. The House Swapping Game.

Let $N = \{A, B, C, \ldots\}$ be the set of <u>traders</u>. At the start of the game, A owns house <u>a</u>, B owns house <u>b</u>, C owns house <u>c</u>, etc. The traders can transfer ownership amongst themselves in any way they please, except that at the end no one is allowed to own more than one house. The traders have only ordinal preferences, i.e., each has a simple ranking of all the houses, with no ties. There are no side payments; indeed, as this is a "barter economy", questions of monetary value do not arise.

Here is a six-trader example.

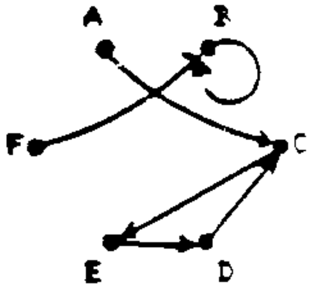
TABLE OF PREFERENCES

A:	c e f a!b d
B:	b!a c e f d
C:	e f c!a d b
D:	c a b e d!f
E:	d c b f e!a
F:	b d e f!a c

The symbol "!" marks the trader's own house; beyond that point the ordering does not matter. We see that while B likes her own house best, the others have possibilities of "trading up" to something better. For example, E and F would each move up a notch if they exchanged houses.

We shall see that there is always at least one outcome in the core of the game, i.e. a re-allocation of the houses such that no coalition of traders could have done better for <u>all</u> of its members by trading (from the beginning) only among themselves. Moreover, we shall see that while there may be several outcomes in the core, the "strict core" is unique (see the footnote on page 4.7). That is, there is only one way to redistribute the houses so that no coalition could have bettered even one member's position without making some other member worse off.

The following algorithm, due to David Gale, is based on the idea of <u>top trading cycles</u> (TTC) and produces that unique strict-core outcome. We shall use the above six-trader example to illustrate the method.



<u>Step 1</u>. Make a directed graph, as at right, with each trader represented by a vertex from which an edge points to the owner of his/her top-ranked house.

<u>Step 2</u>. Find the top trading cycle(s) by starting at any vertex and following the arrows until the path loops back on itself. In the example, starting at A, C D or E yields the cycle [CED], and starting at B or F yields the cycle [B].

T_1 = [CED]
T_2 = [B]

Preferences:

A:	f	a
F:	f	a

<u>Step 3</u>. Delete from the preference table all mention of the traders appearing in the TTCs discovered in Step 2, and return to Step 1 if any traders are left.

T_3 = [F]

Preferences:

A:	a

T_4 = [A]

<u>Step 4</u>. When every trader has been assigned to a TTC in this manner, execute all the indicated trades. In other words, award to each trader the house originally owned by his successor in the TTC. Thus, in the example C gets e, E gets d, B gets b, etc. The final allocation is

$$\mathcal{A} = \langle Aa,\ Bb,\ Ce,\ Dc,\ Ed,\ Ff \rangle.$$

> THEOREM 3. The allocation $\mathcal{a}$ obtained by the "top trading cycle" algorithm is <u>stable</u>, in the sense that no coalition of traders, trading only with each other, could have achieved any allocation in which all members would be better off.

<u>Proof</u>. Let $\mathcal{a}$ be the allocation determined by the TTC algorithm. Clearly, $\mathcal{a}$ is a feasible outcome. But suppose that some set S of traders could have "improved", i.e., could have traded only amongst themselves to arrive at an allocation $\mathcal{a}'$ in which every member has a better house than in $\mathcal{a}$. Let i_0 be the first member of S to be assigned to a TTC during the algorithm. Then i_0 is getting the best house available <u>at that stage</u>. If he is to do better in $\mathcal{a}'$, he must get the house of a trader who was assigned to a TTC at an earlier stage. But no such trader exists in S, since i_0 was the first to be assigned. So S can't "improve" after all, and it follows that $\mathcal{a}$ is in the core, as claimed.

> THEOREM 4. The allocation $\mathcal{a}$ obtained by the "top trading cycle" algorithm is <u>strongly stable</u>, in the sense that no coalition of traders, trading only with each other, can achieve an allocation in which at least one member is better off and none are worse off. Moreover, it is the only allocation of this kind.

This stronger result is proved in the appendix.

For an example of a core allocation that is <u>not</u> strongly stable, see Exercise 8 below. Curiously enough, if such a "weak" core allocation is used as the starting point of another game, then it will not be in the core of the second game.

EXERCISES

Exercise 6. Solve the following house swapping games by the method of top trading cycles.

(a)

A:	f	c	g	a	b	d	h	e
B:	e	f	g	a	c	b	h	d
C:	g	e	d	c	b	a	h	f
D:	c	h	g	e	d	a	b	f
E:	g	a	c	f	b	h	d	e
F:	c	b	a	d	g	h	f	e
G:	f	e	d	c	b	a	h	g
H:	b	c	d	e	f	g	h	a

(b)

A:	e	d	f	b	a	c
B:	d	f	c	e	a	b
C:	c	a	b	e	f	d
D:	f	e	c	d	a	b
E:	c	e	a	d	b	f
F:	e	c	f	d	b	a

Exercise 7. Show that how a player ranks the houses he considers worse than his own has no effect on the core of the game (not merely the strict core).

Exercise 8. Consider the four-person house swapping game defined by the following preferences:

A:	b	d	a	c
B:	c	a	d	b
C:	d	b	c	a
D:	a	c	b	d

(a) Show that the allocation $\mathcal{a}' = \langle Ab, Ba, Cd, Dc \rangle$ is in the core.

(b) Show that it is weakly dominated by the allocation that gives all traders their first choice.

(c) Formulate the game in which the same four traders start with the allocation $\mathcal{a}'$, and determine its core and strict core.