SHAPLEY VALUE + CORE
(for side-payment or TU cooperative games)
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Chapter 3: The Shapley Value

In this and subsequent chapters, we turn to the theory of "cooperaive games," where the focus of interest is the way in which the players argain together over the division of the available payoff, rather than he way this payoff can be attained by the use of certain strategies.

3.1 Definition: A game in coalitional form consists of

1) a set N (the players), and

2) a function $v: 2^N \to \mathbb{R}$ such that $v(\emptyset) = 0$. ($2^N = \{S: S \subset N\}$)

A subset of N is called a coalition; $v(S)$ is called the worth of the coalition S.

3.2 Agreement: If $\{i_1, i_2, \ldots, i_j\}$ is a set of players, we will sometimes write $v(i_1 i_2 \ldots i_j)$ rather than $v(\{i_1, i_2, \ldots, i_j\})$ for the worth of $\{i_1, i_2, \ldots, i_j\}$.

3.3 Example (2-person bargaining game):

$$N = \{1,2\} \quad , \quad v(N) = 1 \quad , \quad v(1) = v(2) = 0 \quad .$$

3.4 Example (Market for a perfectly divisible good with one buyer and two sellers):

$$N = \{1,2,3\} \quad , v(N) = v(12) = v(13) = 1 \quad , \quad v(23) = v(1) = v(2) = v(3$$

3.5 Example (Pure bargaining game with n players, or unanimity game with n players):

$$v(N) = 1 \quad , \quad v(S) = 0 \qquad \text{for } S \neq N \quad .$$

3.6 Example (3-person majority game):

$$N = \{1,2,3\} \quad , v(N) = v(12) = v(13) = v(23) = 1 \quad , \quad v(1) = v(2) = v(3$$

3.7 Example (Weighted majority game):

$$N = \{1,2,3,4\} \qquad v(S) = \begin{cases} 1 & \text{if } \sum_{i\in S} w^i \geq 3 \\ 0 & \text{if } \sum_{i\in S} w^i \leq 2 \end{cases},$$

with $w^1 = 2$ and $w^i = 1$ for $i = 2,3,4$.

(w^i is the "weight" of player i.)

3.8 Definition: Let N be the set of players. An n-person weighted majority game with weights $\{w^i\}_{i\in N}$ and quota q is defined by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i\in S} w^i \geq q \\ 0 & \text{if } \sum_{i\in S} w^i < q \end{cases}.$$

3.9 Definition: v is monotonic if $S \supset T$ implies $v(S) \geq v(T)$. (Note that this does not mean that $|S| \geq |T|$ implies $v(S) \geq v(T)$ (where $|S|$ is the cardinality of S).) v is superadditive if $S \cap T = \emptyset$ implies $v(S \cup T) \geq v(S) + v(T)$.

Unless specifically stated, it will not be assumed that v is monotonic or superadditive.

3.10 Definition: A game is 0-normalized if $v(i) = 0$ for all i in N; it is 0-1 normalized if it is 0-normalized and $v(N) = 1$.

3.11 Definition: i and j, elements of N, are substitutes in v if for all S containing neither i nor j, $v(S \cup \{i\}) = v(S \cup \{j\})$.

3.12 Definition: An element i of N is called a null player if $v(S \cup \{i\}) = v(S)$ for all $S \subset N$.

3.13 Definition: E^N is the Euclidean space whose dimension is the cardinality of N, and whose coordinates are indexed by the members of N themselves.

We now introduce the solution concept studied in this chapter.

3.14 Definition: Let $N = \{1,2,\ldots,n\}$ and let G^N be the set of all games whose player set is N. A Shapley value or value on N is a function ϕ: $G^N \to E^N$ satisfying the following conditions:

1. (Symmetry condition): if i and j are substitutes in v, then $(\phi v)_i = (\phi v)_j$.
2. (Null player condition): if i is a null player, then $(\phi v)_i = 0$.
3. (Efficiency condition): $\sum_{i=1}^{n} (\phi v)_i = v(N)$.
4. (Additivity condition): $(\phi(v + w))_i = (\phi v)_i + (\phi w)_i$.

3.15 Remark: $(\phi v)_i$, the i-th coordinate of the image vector $\phi(v)$ (sometimes denoted ϕv) is interpreted as the "power" of player i in the game v, or what it is worth to i to participate in the game v (in brief, v's "value" for i).

3.16 Remark: Conditions 1, 2, and 4 are weak restrictions which are easy to accept as "reasonable," while 3 is much stronger (to require an efficient outcome in game situations is as strong an assumption as requiring it in a traditional economic problem).

3.17 <u>Theorem</u> (Shapley [1953a]): <u>There exists a unique value on</u> G^N <u>for every</u> N.

<u>Proof</u>: First we prove uniqueness. Let ϕ be a value on G^N. Define for each coalition $T \subseteq N$ with $T \neq \emptyset$, a game v_T by

$$v_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} .$$

Note that for any real α, members of $N \setminus T$ are null players in αv_T, and members of T are substitutes for each other in αv_T. Hence by the null player condition, $\phi(\alpha v_T)_i = 0$ when $i \notin T$, and by the symmetry condition $\phi(\alpha v_T)_i = \phi(\alpha v_T)_j$ when $i,j \in T$. Hence, by the efficiency condition $\sum_{i \in N} \phi(\alpha v_T)_i = (\alpha v_T)(N) = \alpha v_T(N) = \alpha$. Thus $\alpha = \sum_{i \in T} \phi(\alpha v_T)_i = |T| \phi(\alpha v_T)_i$ for any $i \in T$. Hence,

$$\phi(\alpha v_T)_i = \begin{cases} \frac{\alpha}{|T|} & \text{for } i \in T \\ 0 & \text{for } i \notin T \end{cases} .$$

Now, G^N is a Euclidean space of dimension $2^{|N|} - 1$ and there are $2^{|N|} - 1$ games v_T. We know $\phi(\alpha v_T)$ for all α and T, so by additivity we know $\phi(\sum_{i=1}^{k} \alpha_i v_{T_i})$ for all linear combinations $\sum_{i=1}^{k} \alpha_i v_{T_i}$ of the v_T's. Hence if we prove that the v_T's are linearly independent, we will have shown uniqueness. Suppose they are not; then we may write

$$v_T = \sum_{i=1}^{j} \beta_i v_{T_i} ,$$

where $|T| \leq |T_i|$ for all i and all T_i's are different from each other and from T. Then

$$1 = v_T(T) = \sum_{i=1}^{j} \beta_i v_{T_i}(T) = \sum_{i=1}^{j} \beta_i \cdot 0 = 0 ,$$

a contradiction. We therefore conclude that the v_T's are indeed linearly independent, which completes the uniqueness proof.

For the existence proof, suppose that the players in N are ordered, and suppose that according to this order, each player gets his marginal incremental worth to the coalition formed by the players preceding him. That is, the i-th player gets

$$v(1,2,3,\ldots,i-1,i) - v(1,2,3,\ldots,i-1) ,$$

where $1,\ldots,i-1$ denotes the players before i in the order under consideration. The function on N thus obtained does not always satisfy the conditions of the Shapley value. But if we take all possible orders of the players and average the corresponding marginal contributions, this average turns out to satisfy all the conditions of the Shapley value. Thus, a null player has zero incremental worth in all orders, and the symmetry of the set of all orderings ensures that the symmetry condition is satisfied. The efficiency condition may also be verified, and the additivity follows from

$$\begin{aligned} &(v+w)(1,2,\ldots,i-1,i) - (v+w)(1,2,\ldots,i-1) \\ &= [v(1,2,\ldots,i-1,i) - v(1,2,\ldots,i-1)] + [w(1,2,\ldots,i-1,i) \\ &\qquad - w(1,2,\ldots,i-1)] \end{aligned}$$

for any two games v and w and any order. This establishes the existence of a Shapley value, so that the proof of the theorem is now complete. The above argument also establishes the following.

3.18 <u>Theorem</u> (Shapley [1953a]): $(\phi v)_i = (1/|N|!)\sum_R [v(S_i \cup \{i\}) - v(S_i)]$ <u>where</u> R <u>runs over all</u> $|N|!$ <u>different orders on</u> N, <u>and</u> S_i <u>is the set of players preceding</u> i <u>in the order</u> R.

We will now compute the Shapley value for some simple games.

3.19 <u>Example</u>: 2-person bargaining game. One has

$$N = \{1,2\} \quad v(12) = 1 \ , \ v(1) = v(2) = 0 \ ,$$

so the formula gives:

$$(\phi v)_1 = (\phi v)_2 = \frac{1}{2} \ .$$

3.20 <u>Example</u>: 3-person majority game. One has

$$N = \{1,2,3\} \quad v(1) = v(2) = v(3) = 0 \ ,$$
$$v(12) = v(23) = v(31) = v(123) = 1 \ ,$$

so the formula gives:

$$(\phi v)_1 = (\phi v)_2 = (\phi v)_3 = \frac{1}{3} \ .$$

In both these examples one can also deduce the value directly from the symmetry and efficiency conditions.

3.21 <u>Example</u>: Market with two sellers and one buyer. Here

$$N = \{1,2,3\} \ , \ v(123) = v(12) = v(13) = 1 \ ,$$
$$\text{and} \quad v(S) = 0 \quad \text{for all other} \quad S \subset N \ .$$

In order to compute the Shapley value for this game, we first notice that there are $3! = 6$ orderings of the 3 players. Since this game is a <u>simple game</u> (i.e. the worth of every coalition is either 0 or 1), the following definition is useful: player i is a <u>key player</u> with respect to the coalition S if $v(S) = 0$ and $v(S \cup \{i\}) = 1$. The Shapley value for a player i is his average incremental worth, so we obtain it by computing the proportion of orderings in which player i is a key player with respect to the set of players which precedes him in the ordering. The six orderings are:

$$\{1,2,3\}, \{1,3,2\}, \{2,1,3\}, \{2,3,1\}, \{3,1,2\}, \text{ and } \{3,2,1\}.$$

Player 1 is key in $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, and $\{3,2,1\}$. So

$$(\phi v)_1 = \frac{4}{6} = \frac{2}{3} \ .$$

Since 2 and 3 are substitutes $(\phi v)_2 = (\phi v)_3$. The efficiency condition is $\sum_{i=1}^{3} (\phi v)_i = v(123) = 1$, so

$$(\phi v)_2 = (\phi v)_3 = \frac{1}{6} \ ,$$

so that

$$\phi v = (\tfrac{2}{3},\tfrac{1}{6},\tfrac{1}{6}) \ .$$

This example illustrates the fact that the Shapley value gives a measure of the power of the players in a situation free of any institutions. Thus, one might think that if the available payoff above were distributed according to the players' strengths, the outcome would be $(1/2,1/4,1/4)$, since the two sellers can form a cartel which will put them on an equal footing with the buyer. The Shapley value, however, reflects the fact that the buyer is actually in a stronger position since each of the sellers may be willing to deal with him separately.

3.22 Examples: Weighted majority games. The Shapley value gives interesting insights into some multi-party political situations. For instance, the political arena in Israel is characterized by the existence of a large party (the Labor Party) which counts for approximately 1/3 of the votes, whereas until several years ago the remaining votes were split among many relatively small parties. In spite of the fact that it controlled only 1/3 of the votes, however, the Labor Party has, since the creation of the state, always held all four major ministries (Prime Minister, Finance, Foreign, Defense).

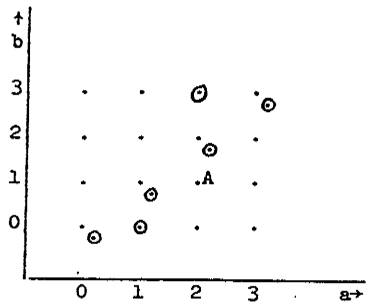
To try to gain some insight into this situation, let us compute the Shapley value for a weighted majority game (N,v) with quota $q = 1/2$ and a vector of weights $w = (1/3,2/9,2/9,2/9)$. We get $\phi v = (1/2,1/6,1/6,1/6)$. This result provides some understanding of the situation in Israel: although it has only 1/3 of the votes, the Labor Party has half the "power" within parliament.

Consider next a situation in which there are 100 parties: one large party has 1/3 of the votes, and the remaining 99 parties share the other 2/3 equally. The large party is a key player in all orderings in which there are more than 1/4 and less than 3/4 of the 99 players before him. So he is key in half of the orderings, so that again 1/3 of the votes gives the large party 1/2 of the power: $(\phi v)_1 = 1/2$.

Now consider a situation in which there are two large parties, each with 1/3 of the votes, and 3 small ones with 1/9 of the votes each; i.e.

$$w = (\tfrac{1}{3},\tfrac{1}{3},\tfrac{1}{9},\tfrac{1}{9},\tfrac{1}{9}) \ .$$

We will compute the Shapley value for the corresponding weighted majority game with $q = 1/2$. Let the two large players be denoted by x and y. For each order on the small players, one can characterize the order on all the players by a pair (a,b), where a (resp. b) is the number of small players after which x (resp. y) appears. Corresponding to a pair (α,α) there are two orders of all the players--one where x precedes y, and one where the reverse is true; corresponding to every other pair there is just one order. Hence possible orders are illustrated in the diagram below: for example, the point A corresponds to the order (p_1,y,p_2,x,p_3) (where (p_1,p_2,p_3) is the ordering of the small players); each position on the diagonal corresponds to two possible orders of all players. So for every ordering of the small players there are 20 possible orderings of all the players, and x is a key player in the six positions which

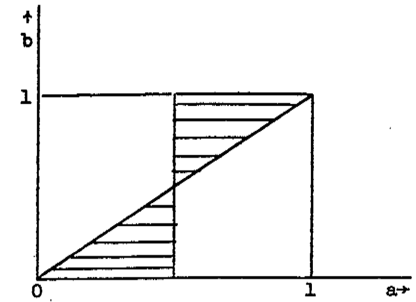


are circled. So, since the order of the small players is irrelevant at present, the value of x is $6/20 = 3/10$. By symmetry the value of y is also $3/10$, and by symmetry and efficiency the value of the game is

$$\phi v = \left(\frac{3}{10},\frac{3}{10},\frac{2}{15},\frac{2}{15},\frac{2}{15}\right) .$$

So in this case, the Shapley value imputes to each of the large players a share of the power smaller than his share of the votes.

Let us consider now a more general case in which there are two large parties (each with 1/3 of the votes) and $n - 2$ small ones of equal size. We are interested in the Shapley value of this game for n arbitrarily large. The characterization of orderings used above can be modified by letting a and b be the proportions of the small players after which x and y respectively appear. The diagram below then illustrates the situation, the shaded area corresponding to those orderings in which x is a key player. The value of each of the large parties is, then, approximately $1/4$ when n is large; for $n = 5$, it was $3/10$. If there is a large number of small parties it will, then, be better for



them <u>not</u> to get together in larger groups. The intuitive rationale is as follows. Whatever the number of small parties, each large party does not need all their votes to form a majority, but if there are few <u>small</u> parties the large ones will have no choice but to bargain over large blocks of votes. If there is a large number of small parties, the large parties can bargain for just the number of votes they need, and can consequently offer more per vote: the small parties will then actually be more powerful.

This result may account for another aspect of the political scene in Israel: the fact that the relatively small religious parties have not gotten together, but have remained independent, in the presence of the two large parties on the right and on the left.

<u>Chapter 4: The Core</u>

4.1 <u>Definition</u>: A <u>payoff vector</u> is a member of E^N (the Euclidean $|N|$-dimensional space whose coordinates are indexed by the members of N).

4.2 <u>Definition</u>: A payoff vector x is called <u>individually rational</u> (in the game (N,v)) if $x^i \geq v(i)$ for all players $i \in N$.

4.3 Definition: A payoff vector x is called group rational (or efficient) if $\sum_{i \in N} x^i = v(N)$.

4.4 Remark: If $\sum_{i \in N} x^i < v(N)$, then all players could improve their payoff by forming the coalition N; hence x is inefficient. If v is superadditive, then for any partition $\{S_1,\ldots,S_k\}$ of the players (i.e. $\bigcup_{i=1}^{k} S_i = N$ and $S_i \cap S_j = \emptyset$ for all $i \neq j$), we have $v(N) \geq \sum_{i=1}^{k} v(S_i)$; therefore there is no way for the players to obtain a total payoff greater than $v(N)$. Hence under the assumption of superadditivity, it is to be expected that payoff vectors that actually occur will be group rational. However, superadditivity will not be assumed here unless specifically stated.

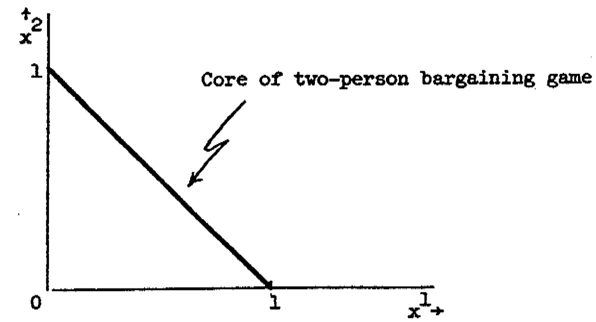
4.5 Definition: An imputation is a payoff vector that is individually and group rational.

4.6 Definition: The core of the game (N,v) is the set of all imputations x such that $v(S) \leq \sum_{i \in S} x^i$ for all $S \subset N$.

4.7 Example: Two-person bargaining game. We have $N = \{1,2\}$, $v(N) = 1$, and $v(1) = v(2) = 0$. Then (x^1,x^2) is in the core if and only if

$$x^1 \geq 0 \quad , \quad x^2 \geq 0 \quad , \quad \text{and} \quad x^1 + x^2 = 1 \quad .$$

So the core is the set of all imputations, as shown in the diagram.

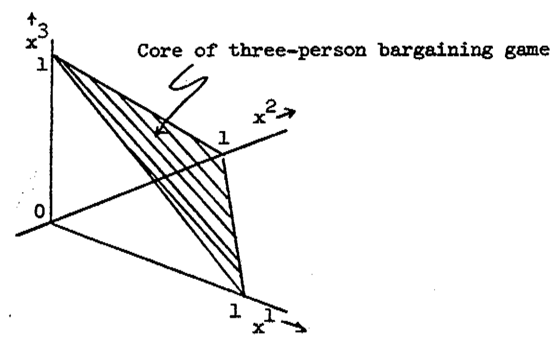


4.8 Example: Three-person bargaining game. In this game $N = \{1,2,3\}$, $v(N) = 1$ and $v(S) = 0$ for all other $S \subset N$. So (x^1,x^2,x^3) is in the core if and only if

$$x^1 + x^2 + x^3 = v(N) = 1 \quad , \quad x^i \geq v(i) = 0 \quad \text{for all} \quad i \in N \quad , \quad \text{and}$$

$$\sum_{i \in S} x^i \geq v(S) = 0 \quad \text{for all} \quad S \subset N \quad , \quad S \neq N \quad .$$

The core is therefore the set of all imputations once again; it is shown in the diagram below.



4.9 Example: Market with 2 sellers and a buyer. In this game $N = \{1,2,3\}$, $v(123) = v(12) = v(13) = 1$, and $v(S) = 0$ for all other $S \subset N$. So x is in the core if and only if

$$x^1 + x^2 + x^3 = 1 \quad , \quad x^1 + x^2 \geq 1 \quad , \quad x^1 + x^3 \geq 1 \quad ,$$

$$x^1 \geq 0 \quad , \quad x^2 \geq 0 \quad , \text{ and } \quad x^3 \geq 0 \quad .$$

Hence the core is $\{(1,0,0)\}$.

4.10 Remark: Note that the core in the example above $(\{(1,0,0)\})$ differs considerably from the Shapley value of the game considered there (which is $(2/3,1/6,1/6)$). One can interpret the zero payoff to players 2 and 3 in the core allocation as the result of cutthroat competition between them.

4.11 Example: 3-person majority game. Here $N = \{1,2,3\}$, $v(123) = v(12) = v(13) = v(23) = 1$, and $v(i) = 0$ for all $i \in N$. For x to be in the core, we need $x^1 + x^2 + x^3 = 1$, $x^i \geq 0$ for all $i \in N$, $x^1 + x^2 \geq 1$, $x^1 + x^3 \geq 1$, and $x^2 + x^3 \geq 1$. There exists no x satisfying these conditions, so the core is empty.

We now wish to study conditions on v which will ensure that the core of (N,v) be non-empty. Consider first a 0-1 normalized 3-person game. Let us suppose that the core is non-empty, i.e. there exists an imputation $x = (x^1,x^2,x^3)$ such that

$$x^1 + x^2 \geqq v(12) \quad ,$$

$$x^1 + x^3 \geqq v(13) \quad ,$$

and

$$x^2 + x^3 \geqq v(23) \quad .$$

In this case we have

$$2(x^1 + x^2 + x^3) \geqq v(12) + v(13) + v(23) \quad ,$$

or

$$v(N) = 1 \geqq \frac{[v(12) + v(13) + v(23)]}{2} \quad ,$$

and

$$v(12) \leqq 1 \quad , \quad v(13) \leqq 1 \quad , \quad v(23) \leqq 1 \quad .$$

So a necessary condition for a 0-1 normalized 3-person game to have a non-empty core is that $1 \geqq [v(12) + v(13) + v(23)]/2$ and $v(ij) \leqq 1$ for all $\{i,j\} \subset N$.

Exercise 4: Prove that the condition $1 \leqq [v(12) + v(13) + v(23)]/2$ and $v(ij) \leqq 1$ for all $\{i,j\} \subset N$ is also a sufficient condition for the 0-1 normalized 3-person game (N,v) to have a non-empty core.

Let us now consider the conditions under which a general game (N,v) has a non-empty core.

4.12 Definition: Let $S \subset N$. The characteristic vector of S is the element x_S of E^N defined by

$$x_S^i = \begin{cases} 1 & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} .$$

4.13 Definition: A family $\mathcal{S}$ of coalitions is called balanced if there exists a sequence of non-negative numbers $\{\delta_S\}_{S\in\mathcal{S}}$ such that

$$\sum_{S\in\mathcal{S}} \delta_S x_S = x_N .$$

$\{\delta_S\}_{S\in\mathcal{S}}$ are called balancing weights for $\mathcal{S}$.

A natural interpretation of this definition is the following. Each player is endowed with one unit of time that he allocates among the coalitions S in $\mathcal{S}$; δ_S is the fraction of his time that each member of S allocates to the coalition S; the condition $\sum_{S\in\mathcal{S}} \delta_S x_S = x_N$ is a feasibility condition (for every individual the sum of the amounts of his time he spends with each coalition must equal exactly the amount of time he is endowed with).

4.14 Theorem (Bondareva [1962], [1963], and Shapley [1967]): A necessary and sufficient condition for the core of (N,v) to be non-empty is that for all balanced families $\mathcal{S}$ and corresponding balancing weights $\{\delta_S\}_{S\in\mathcal{S}}$, we have $\sum_{S\in\mathcal{S}} \delta_S v(S) \le v(N)$.

Proof: We will assume that v is 0-1 normalized; the extension to the general case is left to the reader.

1. The condition is necessary.

Let x be in the core. Then $\sum_{i\in N} x^i = v(N)$ and $\sum_{i\in S} x^i \ge v(S)$ for all $S \subset N$. Let $\mathcal{S}$ be a balanced family with weights $\{\delta_S\}$. Then

$$\delta_S \sum_{i\in S} x^i \ge \delta_S v(S) ,$$

so

$$\sum_{S\in\mathcal{S}} \sum_{i\in S} \delta_S x^i \ge \sum_{S\in\mathcal{S}} \delta_S v(S) .$$

Since we are dealing with a finite sum we can reverse the double summation sign:

$$\sum_{S\in\mathcal{S}} \sum_{i\in S} \delta_S x^i = \sum_{i\in N} \sum_{\substack{S\in\mathcal{S} \\ S\ni i}} \delta_S x^i = \sum_{i\in N} x^i \sum_{\substack{S\in\mathcal{S} \\ S\ni i}} \delta_S = \sum_{i\in N} x^i = v(N) .$$

Hence

$$v(N) \ge \sum_{S\in\mathcal{S}} \delta_S v(S) .$$

This establishes necessity.

2. The condition is sufficient.

Assume $v(N) \ge \sum_{S\in\mathcal{S}} \delta_S v(S)$ for all balanced families $\mathcal{S}$ and corresponding weights. Define a 2-person 0-sum game as follows. Player I chooses a player i in the game (N,v). Player II chooses a coalition S in the game (N,v), such that $v(S) > 0$. The payoff to Player I is:

$$h(i,S) = \begin{cases} \frac{1}{v(S)} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases} .$$

<u>Assertion</u>: In order to prove that the condition is sufficient, it is enough to prove that the minimax value of this 2-person game is greater than or equal to 1.

<u>Proof</u>: If the minimax value is greater than or equal to 1, there is a mixed strategy x of Player I that yields at least 1, no matter which pure strategy S is chosen by Player II. That is,

$$1 \leq \sum_{i \in N} x^i h(i,S) = \frac{1}{v(S)} \sum_{i \in S} x^i$$

for all $S \subset N$ with $v(S) > 0$. Hence

$$v(S) \leq \sum_{i \in S} x^i$$

for all $S \subset N$ such that $v(S) > 0$. When $v(S) = 0$, the inequality holds since $x^i \geq 0$ for all i. Hence

$$v(S) \leq \sum_{i \in S} x^i$$

for all $S \subset N$. Together with the condition $\sum_{i \in N} x^i = 1$, this means

that x is in the core of the game (N,v), so that that core is non-empty. This establishes the assertion.

So we must now prove that the 2-person 0-sum game has a minimax value greater than or equal to 1. Suppose contrariwise that the minimax value is less than 1; let it be $0 < \xi < 1$ (notice that $\xi > 0$, since if Player I chooses a strictly positive probability for every player, he will be guaranteed a positive payoff). There is then a mixed strategy for Player II that guarantees that the payoff to I will at most be ξ. Let this mixed strategy assign probability $\theta_S > 0$ to each coalition in a family $\mathcal{S}$ with $v(S) > 0$ for all $S \in \mathcal{S}$. For each i, we have

$$\xi \geq \sum_{S \in \mathcal{S}} \theta_S h(i,S) = \sum_{\substack{S \in \mathcal{S} \\ S \ni i}} \frac{\theta_S}{v(S)} ,$$

so

$$1 \geq \sum_{\substack{S \in \mathcal{S} \\ S \ni i}} \frac{\theta_S}{\xi v(S)} .$$

Let us define $\delta_S = \theta_S / \xi v(S)$ for all $S \in \mathcal{S}$. Then we have

$$1 \geq \sum_{\substack{S \in \mathcal{S} \\ S \ni i}} \delta_S .$$

In order to construct a balanced family of coalitions, define

$$\delta_i = 1 - \sum_{\substack{S \in \mathcal{S} \\ S \ni i}} \delta_S \quad .$$

Consider the collection $\mathcal{T}$ consisting of $\mathcal{S}$ and all singletons $\{i\}$. Then for all i,

$$\sum_{\substack{S \in \mathcal{T} \\ S \ni i}} \delta_S = \sum_{\substack{S \in \mathcal{S} \\ S \ni i}} \delta_S + \delta_i = 1 \quad ,$$

so $\mathcal{T}$ is a balanced family with balancing weights $\{\delta_S\}$. Hence by assumption

$$\sum_{S \in \mathcal{T}} \delta_S v(S) \leq v(N) \quad ,$$

so that, since $v(\{i\}) = 0$ for all i,

$$\sum_{S \in \mathcal{S}} \delta_S v(S) \leq v(N) \quad .$$

So

$$\sum_{S \in \mathcal{S}} \frac{\theta_S}{\xi} \leq v(N) = 1 \quad ,$$

or

$$\sum_{S \in \mathcal{S}} \theta_S \leq \xi < 1$$

(we have supposed $\xi < 1$). But this result contradicts the fact that $\{\theta_S\}_{S \in \mathcal{S}}$ is a strategy for Player II: we need $\sum_{S \in \mathcal{S}} \theta_S = 1$. Hence

the minimax value is greater than or equal to 1, which, using the above assertion, establishes sufficiency.

The following leads up to an exercise in the use of the Bondareva-Shapley theorem.

4.15 <u>Definition</u>: S is a <u>winning</u> coalition in a simple game if $v(S) = 1$; a <u>veto player</u> in such a game is a player who is a member of every winning coalition.

<u>Exercise 5</u>: Prove that a 0-1 normalized weighted majority game has a non-empty core if and only if there is at least one veto player.

<u>Exercise 6</u>: Find the core of a 0-1 normalized weighted majority with $p \geq 1$ veto players.

We may sum up some basic features of the Shapley value and the core as follows:

The Shapley value of a game is a single payoff vector. It is always group rational; in superadditive games it is individually rational, but this is not necessarily so in general.

The core is a set of payoff vectors. It is a subset of the set of imputations. It may be empty, and even when it is not the Shapley value may not be a member of it.

Intuitively the Shapley value represents a "reasonable compromise", whereas the core represents a set of payoff vectors which are in a certain sense "stable". There is no general relationship between the two, though for certain classes of games (not considered in these Lectures) a close relationship can be established.

Chapter 5: Market Games

Let us now consider an economic application of the concepts we have developed. The situation we will describe is that of a "market game". In a market game, there is one consumption good, ℓ production goods and n players. Each player i has a <u>production function</u> $u^i(x_1,x_2,\dots,x_\ell)$, defined for all $x_j \geq 0$ and with values in $\mathbb{R}$. The quantity $u^i(x_1,x_2,\dots,x_\ell)$ represents the amount of the single consumption good that i can produce from inputs $x_1,x_2,\dots,x_\ell$. Each player i also has an <u>endowment</u> $(a_1^i,a_2^i,\dots,a_\ell^i)$ of production goods. Each coalition produces as much of the consumption good as possible so that

(1) $$v(S) = \max \{ \sum_{i\in S} u^i(x^i): \quad \sum_{i\in S} x^i = \sum_{i\in S} a^i \text{ and } x^i \geq 0 \text{ for all } i\}$$

where $x^i = (x_1^i,x_2^i,\dots,x_\ell^i)$ and a^i is similarly defined.

5.1 <u>Remark</u>: If the u^i's are continuous, then the above maximum is attained.

5.2 <u>Definition</u>: A function u is called <u>concave</u> if its domain is convex and for x and y in the domain of u and all α in $[0,1]$,

$$u(\alpha x + (1-\alpha)y) \geq \alpha u(x) + (1-\alpha)u(y) \quad .$$

5.3 <u>Definition</u>: Assume that the u^i's are concave and continuous. Then the game (N,v) defined by (1) is called a <u>market game</u>.

5.4 <u>Proposition</u> (Shapley and Shubik [1969]): <u>Every market game has a non-empty core</u>.

<u>Proof</u>: We will use the Bondareva-Shapley theorem. Let $\mathcal{S}$ be a balanced collection of coalitions with corresponding weights $\{\delta_S\}_{S\in\mathcal{S}}$. We must prove that

$$\sum_{S\in\mathcal{S}} \delta_S v(S) \leq v(N) \quad .$$

Let $v(S) = \sum_{i\in S} u^i(x_S^i)$ where x_S^i is the point of the set $\{y^i \in E_+^\ell: \sum_{i\in S} y^i = \sum_{i\in S} a^i\}$ at which the function $\sum_{i\in S} u^i(y^i)$ attains its maximum. Define

$$x^i = \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S x_S^i \quad .$$

(One can think of player i spending a fraction δ_S of his time in coalition S; x^i is then his total input vector.) We can then prove that $(x^i)_{i\in N}$ is a feasible allocation for N:

$$\sum_{i\in N} x^i = \sum_{i\in N} \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S x_S^i = \sum_{S\in\mathcal{S}} \sum_{i\in S} \delta_S x_S^i = \sum_{S\in\mathcal{S}} \delta_S \sum_{i\in S} x_S^i$$

$$= \sum_{S\in\mathcal{S}} \delta_S \sum_{i\in S} a^i = \sum_{S\in\mathcal{S}} \sum_{i\in S} \delta_S a^i = \sum_{i\in N} \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S a^i = \sum_{i\in N} a^i \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S = \sum_{i\in}$$

(since $\sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S = 1$ for all i). Hence

$$\sum_{i\in N} x^i = \sum_{i\in N} a^i \quad .$$

Moreover, the x^i's are non-negative since they are averages of non-negative numbers with positive weights. So $(x^i)_{i\in N}$ is a feasible allocation for N. Hence by the definition of $v(N)$

$$v(N) \geq \sum_{i\in N} u^i(x^i) \quad .$$

Since u^i is a concave function

$$u^i(x^i) \geq \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S u^i(x^i_S) \quad .$$

Hence

$$v(N) \geq \sum_{i\in N} \sum_{\substack{S\in\mathcal{S}\\ S\ni i}} \delta_S u^i(x^i_S)$$

$$= \sum_{S\in\mathcal{S}} \delta_S \sum_{i\in S} u^i(x^i_S) = \sum_{S\in\mathcal{S}} \delta_S v(S) \quad .$$

So

$$\sum_{S\in\mathcal{S}} \delta_S v(S) \leq v(N) \quad .$$

So by the Bondareva-Shapley theorem the core is non-empty, which establishes the proposition.

The converse of Proposition 5.4 is false--not every game with a non-empty core is a market game. For example, the four-person game defined by $v(1234) = 2$, $v(S) = 1$ if $|S| = 2$ or 3, and $v(S) = 0$ if $|S| = 0$ or 1, is not a market game, although $(1/2,1/2,1/2,1/2)$ is in the core.

5.5 Definition: Let (N,v) be a game, and $T \subset N$. The subgame (T,v_T) defined by T is the game whose player set is T and whose worth function is defined by $v_T(S) = v(S)$ for all $S \subset T$.

Obviously every subgame of a market game is itself a market game, and so from Proposition 5.4 we obtain

5.6 Corollary: Every subgame of a market game has a non-empty core.

The 4-person game defined above has a subgame (defined by $T = \{1,2,3\}$, say) with an empty core. This raises the question whether every game, all of whose subgames have non-empty cores, is a market game. This is indeed the case; we have

5.7 Theorem (Shapley and Shubik [1969]): A necessary and sufficient condition for a game (N,v) to be a market game is that it and all of its subgames have non-empty cores.

Proof: We have already proved that the condition is necessary. To prove that the condition is sufficient, we consider a game (N,v) such that it and all of its subgames have non-empty cores. We will construct a market game such that its value function is precisely v.

Define a market by $\ell = n$; good i is the labor time of player i. The endowment of player i is defined by the i-th unit vector of the Euclidean space E^n: $a^i = (\underbrace{0,0,0,\ldots,0,1}_{i},0,\ldots,0)$ (i.e. each player is endowed with one unit of his own labor time). The players have the same production functions, defined by

$$u^i(x) = u(x) = \max_{\{\alpha_T\}} \{ \sum_{T\subset N} \alpha_T v(T) : \alpha_T \geq 0 \quad \text{and} \quad \sum_{T\subset N} \alpha_T \chi_T = x\} \quad .$$

Let

$$w(S) = \max \{ \sum_{i\in S} u(x^i) : \sum_{i\in S} x^i = \chi_S\} \quad .$$

(N,w) is, then, a market game; we will show that $w(S) = v(S)$ for all $S \subset N$. By the definition of $w(S)$,

$$\begin{aligned} w(S) &\leq u(\chi_S) \\ &= \max_{\{\alpha_T\}} \{ \sum_{T\subset N} \alpha_T v(T) : \alpha_T \geq 0 \quad \text{and} \quad \sum_{T\subset N} \alpha_T \chi_T = \chi_S\} \\ &\geq v(S) \end{aligned}$$

(taking $\alpha_S = 1$, and $\alpha_T = 0$ for all $T \neq S$). So we have proved that $w(S) \geq v(S)$. In order to prove the reverse inequality, we are going to use the hypothesis that every subgame has a non-empty core. We want to prove that $w(S) \leq v(S)$. We will first prove that $w(S) \leq u(\chi_S)$, and then that $u(\chi_S) \leq v(S)$.

Let the maximum in the definition of w by attained at x_S, so that

$$w(S) = \sum_{i\in S} u(x^i_S) \quad .$$

We will show that $\sum_{i\in S} u(x^i_S) \leq u(\sum_{i\in S} x^i_S)$ for all $S \subset N$, i.e. that u is superadditive.

<u>Assertion</u>: u is homogeneous of degree 1, i.e. for all $\alpha \geq 0$, $u(\alpha x) = \alpha u(x)$.

<u>Proof</u>:

$$\begin{aligned} u(\alpha x) &= \max_{\{\alpha_T\}} \{ \sum_{T\subset N} \alpha_T v(T) : \alpha_T \geq 0 \quad \text{and} \quad \sum_{T\subset N} \alpha_T \chi_T = \alpha x\} \\ &= \max_{\{\alpha_T\}} \{\alpha \sum_{T\subset N} \frac{\alpha_T}{\alpha} v(T) : \alpha_T \geq 0 \quad \text{and} \quad \sum_{T\subset N} \frac{\alpha_T}{\alpha} \chi_T = x\} \\ &= \alpha \max_{\{\beta_T\}} \{ \sum_{T\subset N} \beta_T v(T) : \beta_T \geq 0 \quad \text{and} \quad \sum_{T\subset N} \beta_T \chi_S = x\} \\ &= \alpha u(x) \end{aligned}$$

(with $\beta_T = \alpha_T/\alpha$).

<u>Assertion</u>: u is a concave function, i.e. for all $1 \geq \alpha \geq 0$, $u[\alpha x + (1-\alpha)y] \geq \alpha u(x) + (1-\alpha)u(y)$.

<u>Proof</u>: Let

$$u(x) = \sum \alpha_T v(T) \qquad \text{with} \quad \sum \alpha_T \chi_T = x$$

and

$$u(y) = \sum\beta_T v(T) \qquad \text{with} \quad \sum\beta_T \chi_T = y \quad .$$

By definition

$$u[\alpha x + (1 - \alpha)y] \geq \sum[\alpha\alpha_T + (1 - \alpha)\beta_T]v(T) \quad ,$$

since

$$\sum[\alpha\alpha_T + (1 - \alpha)\beta_T]\chi_T = \alpha\sum\alpha_T\chi_T + (1 - \alpha)\sum\beta_T\chi_T$$
$$= \alpha x + (1 - \alpha)y \quad .$$

Hence

$$u[\alpha x + (1 - \alpha)y] \geq \alpha\sum\alpha_T v(T) + (1 - \alpha)\sum\beta_T v(T)$$
$$= \alpha u(x) + (1 - \alpha)u(y) \quad .$$

Exercise 7: Prove that the following is true for every concave function $f\colon\ E^n \to \mathbb{R}$ and for all $m \geq 1$:

$$\forall\ \alpha \in E^m_+ \quad , \quad \sum_{i=1}^{m} \alpha_i = 1 \Rightarrow f(\sum_{i=1}^{m} \alpha_i x_i) \geq \sum_{i=1}^{m} \alpha_i f(x_i) \quad .$$

We deduce the superadditivity of u from the two assertions above:

$$\sum_{i\in S} u(x^i_S) = n \sum_{i\in S} \frac{1}{n} u(x^i_S) \leq n \cdot u(\sum_{i\in S} \frac{1}{n} x^i_S) = n \cdot \frac{1}{n} u(\sum_{i\in S} x^i_S) \quad .$$

Hence

$$\sum_{i\in S} u(x^i_S) \leq u(\sum_{i\in S} x^i_S) \quad ,$$

or

$$w(S) \leq u(\chi_S) \quad ,$$

by the definitions of $w(S)$ and $u(\chi_S)$. Let us now prove that $u(\chi_S) \leq v(S)$. We have

$$u(\chi_S) = \max_{\{\alpha_T\}} \{ \sum_{T\subset N} \alpha_T v(T) : \alpha_T \geq 0 \quad , \quad \sum_{T\subset N} \alpha_T \chi_T = \chi_S \} \quad .$$

Let the maximum be $\sum_{T\subset N} \hat{\alpha}_T v(T)$, and consider the subgame corresponding to S. Since $\sum\hat{\alpha}_T\chi_T = \chi_S$, all the members of T of every feasible collection are subsets of S. Therefore if we consider each T as a coalition for the subgame,

$$\sum\hat{\alpha}_T\chi_T = 1 \quad .$$

Thus the collection of T's is balanced in the subgame, with balancing weights $\{\hat{\alpha}_T\}$. So by the Bondareva-Shapley theorem applied to the subgame

$$\sum\hat{\alpha}_T v(T) \leq v(S) \quad ,$$

or

$$u(\chi_S) \leq v(S) \quad .$$

Above it was established that $w(S) \leqq u(\chi_S)$, so we have $w(S) \leqq v(S)$. This, together with the conclusion above that $w(S) \geqq v(S)$ yields

$$v(S) = w(S) \quad \text{for all} \quad S \subset N \ .$$

The initial game (N,v) is then a market game (since (N,w) is) and this completes the proof of the theorem.