IV. TTC & Theorems

Note that an allocation is a permutation.

Permutation can be taken as a product of transpositions and can be broken into disjoint cycles.

So any trade can be broken into disjoint cycles.

Now we want to consider allocations in terms of disjoint cycles. Let's introduce

TTC (Top trading cycle procedure / allocation)

The idea is: first represent preferences in terms of directed graphs and nodes, then look at cycles and remove cycles. Then look at the ranking that remains and draw the graphs again.

Since the graphs are finite, eventually we can complete an allocation by obtaining cycles from each step (level).

An example is as follows:

⑦

Profile of preferences

A: c e f a b d

B: b a c e f d

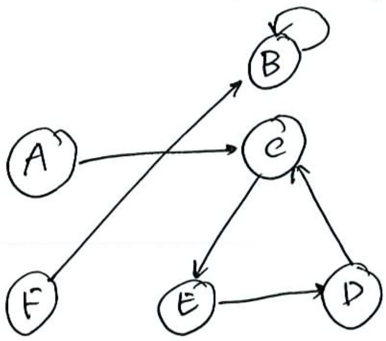
C: e f c a d b

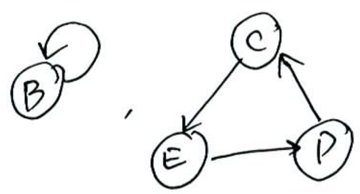
D: c a b e d f

E: d c b f e a

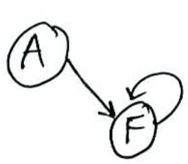
F: b d e f a c

<A?, B?, C?, D?, E?, F?>

level 1: 

At level 1,  are cycles.

So we have <A?, Bb, Ce, Dc, Ed, F?>

level 2: 

At level 2, look at ranking that remains.  is the only cycle.

So we have <A?, Bb, Ce, Dc, Ed, Ff>.

level 3: 

At level 3,  is the only and the last cycle.

Finally, we have an allocation from this procedure
$\langle Aa, Bb, Ce, Dc, Ed, Ff \rangle$.
This procedure is called TTC procedure. The allocation obtained from this procedure is called TTC allocation.

Theorem: TTC allocation is the only strongly stable allocation.

Claim 1: If an allocation $\mathcal{A}$ is strongly stable, then $\mathcal{A}$ is TTC.

Denote cycles from TTC procedure by

level 1: $C_1^1, C_2^1, \dots C_{k(1)}^1$

level 2: $C_1^2, C_2^2, \dots C_{k(2)}^2$

$\vdots$

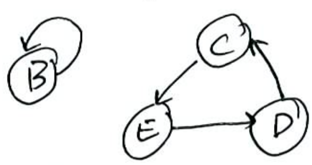
level ℓ: $C_1^\ell, C_2^\ell, \dots C_{k(\ell)}^\ell$

$k(1)$: # of cycles at level 1.

$\vdots$

$k(\ell)$: # of cycles at level ℓ.

(An example is from the previous example:

level 1: 

level 2: 

level 3: 

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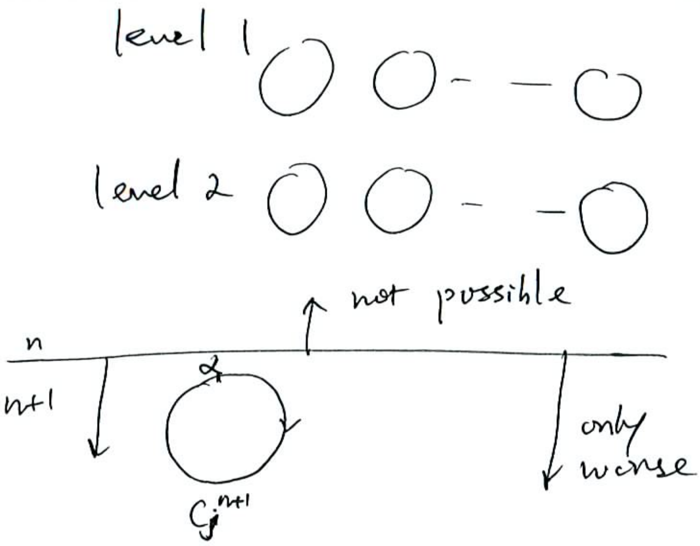
Note: the cycles of different levels are completely determined by the profile of preferences.

pf of claim 1 :

Assume $\mathcal{A}$ is strongly stable, then no coalition can block it on Mars.

We claim :



1. All cycles of level 1 are in $\mathcal{A}$.

This is true because cycles of level 1 are formed by people's top preferences.

If a cycle of level 1 is not in $\mathcal{A}$, this cycle gives a trade (or the coalition formed by the players on that cycle) which blocks $\mathcal{A}$ on Mars.

2. Suppose all cycles of level 1, 2, ... , n are in $\mathcal{A}$. Then cycles of level n+1 are also in $\mathcal{A}$.

Suppose C_j^{n+1} is not in A. Consider the coalition S of all the players in C_j^{n+1} and let them trade among themselves according to C_j^{n+1}. All the cycles of level 1, ... , n are in $\mathcal{A}$ by the inductive assumption. So in $\mathcal{A}$ everyone in S gets some house in X ≡ houses on level $\ell = n+1, n+2, \dots$

In C_j^{n+1} everyone in S gets his top choice in X.

Since C_j^{n+1} is not in A, some $\alpha \in S$ gets a house in $\mathcal{A}$ that is different from what α gets in C_j^{n+1}.

(10) So in fact α does not get in $\mathcal{A}$ his top choice from X. This shows he is better off in G_j^{n+1} compared to $\mathcal{A}$. But no other player in S can be worse off in G_j^{n+1} compared to A, because they get some house from X in $\mathcal{A}$, and they get their best house from X in G_j^{n+1}. This proves that S blocks $\mathcal{A}$ on Mars. Hence a contradiction. G_j^{n+1} must be in $\mathcal{A}$.

So $\mathcal{A}$ contains all TCC cycles, this implies $\mathcal{A}$ is a TCC allocation. □

Claim 2: $\mathcal{A}$ is a TCC allocation $\Rightarrow$ $\mathcal{A}$ is strongly stable.

pf of claim 2:

Suppose $\mathcal{A}$ is not strongly stable, then there exists a coalition S which blocks $\mathcal{A}$ on Mars via A_S.

A_S is a set of disjoint cycles called $T_1, T_2, \ldots T_\ell$.

We claim:

1. <u>If $T_i \cap G_j' \neq \emptyset$, then $T_i = G_j'$.</u>

By assumption, some α is on both T_i and G_j'. In $\mathcal{A}$, G_j' gives a trade: $\alpha \to \alpha_1 \to \alpha_2 - \cdots \to \alpha_n \to \alpha$.

Since α in S (T_i) is not worse off than in $\mathcal{A}$ (G_j'), α gets his top choice α_1 in T_i too, i.e. in S,

T_i: $\alpha \to \alpha_1 \to \cdots \to \alpha$.

(11) If we look at α_1 and continue this process, we find $T_i = G_j^1$.

2. Suppose if $T_i \cap G_j^k \neq 0$, then $T_i = G_j^k$ for $k = 1, 2, \ldots, n$. Then if $T_i \cap G_j^{n+1} \neq 0$, we still have $T_i = G_j^{n+1}$.

The idea is similar:

By assumption, some α is on both T_i and G_j^{n+1}.

In $\mathcal{A}$, G_j^{n+1} gives a trade: $\alpha \to \alpha_1 \to \alpha_2 - - - \to \alpha_m \to \alpha$.

In $\mathcal{A}$, since α is in G_j^{n+1}, α gets his best choice α_1 in $X \equiv$ houses on levels $n+1, \ldots$.

Note G_j^{n+1} and cycles of level $\ell = 1, 2, \ldots, n$ are disjoint. Because $T_i \cap G_j^{n+1} \neq 0$, we have $T_i \cap G_j^k = \phi$ for $k = 1, 2, \ldots n$, $\forall j$. Otherwise, if $T_i \cap C_\ell^k \neq \phi$ for some $k < n+1$, then by assumption $T_i = C_\ell^k$, which is a contradiction with the facts $T_i \cap G_j^{n+1} \neq \phi$ and $C_\ell^k \cap G_j^{n+1} = \phi$.

Therefore in S, α can only choose from X in T_i. But α in S is not worse off than in $\mathcal{A}$, so in S α gets his top choice α_1 in X as well, i.e.

in T_i: $\alpha \to \alpha_1 - - - \to \alpha$.

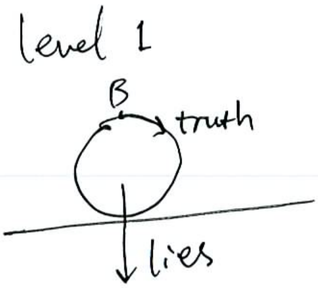
If we look at α_1 and continue this process, we find $T_i = G_j^{n+1}$.

This shows A_S is a union of TTC cycles, hence S will not block A. A is strongly stable. □

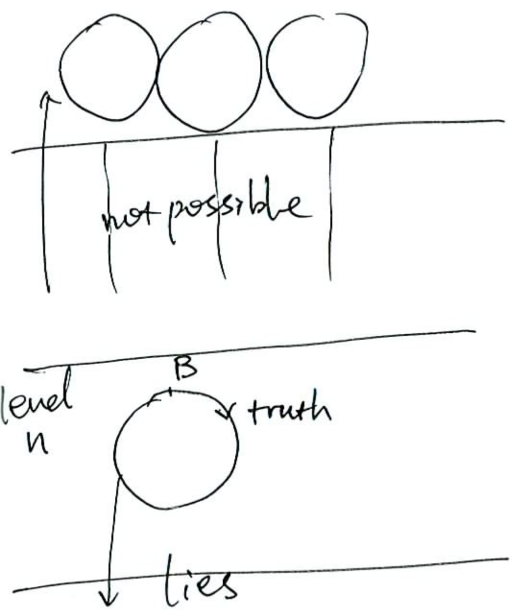
IV Incentive Compatibility

Theorem: Reporting true preference is a dominant strategy.

Pf. Assume only one person B considers telling a lie. Others' preferences are held fixed.



1. Suppose B is originally at level 1. If he tells the truth he will get his top choice among ALL houses. So he can never do better by telling a lie.



2. Suppose B is originally at level n. If B wants to get better off, he will want to get some house from cycles of levels $1, 2, \ldots, n-1$, because by telling the truth he already gets his top choice in $Y \equiv$ set of all houses of levels $n, n+1, \ldots$ But cycles of levels $1, 2, \ldots, n-1$ form regardless of B's preference. They are dictated solely by the preferences of the players in cycles of levels $1, \ldots, n-1$. So, no matter which preference B submits, he will

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only get a house from Y. But telling the truth already gives him the best ~~house~~ choice in Y. So he cannot get better off by telling a lie either.