From the book *Networks, Crowds, and Markets: Reasoning about a Highly Connected World.*
By David Easley and Jon Kleinberg. Cambridge University Press, 2010.
Complete preprint on-line at http://www.cs.cornell.edu/home/kleinber/networks-book/

Chapter 8

Modeling Network Traffic using Game Theory

Among the initial examples in our discussion of game theory in Chapter 6, we noted that traveling through a transportation network, or sending packets through the Internet, involves fundamentally game-theoretic reasoning: rather than simply choosing a route in isolation, individuals need to evaluate routes in the presence of the congestion resulting from the decisions made by themselves and everyone else. In this chapter, we develop models for network traffic using the game-theoretic ideas we've developed thus far. In the process of doing this, we will discover a rather unexpected result — known as *Braess's Paradox* [76] — which shows that adding capacity to a network can sometimes actually slow down the traffic.

8.1 Traffic at Equilibrium

Let's begin by developing a model of a transportation network and how it responds to traffic congestion; with this in place, we can then introduce the game-theoretic aspects of the problem.

We represent a transportation network by a directed graph: we consider the edges to be highways, and the nodes to be exits where you can get on or off a particular highway. There are two particular nodes, which we'll call A and B, and we'll assume everyone wants to drive from A to B. For example, we can imagine that A is an exit in the suburbs, B is an exit downtown, and we're looking at a large collection of morning commuters. Finally, each edge has a designated travel time that depends on the amount of traffic it contains.

To make this concrete, consider the graph in Figure 8.1. The label on each edge gives the

Draft version: June 10, 2010

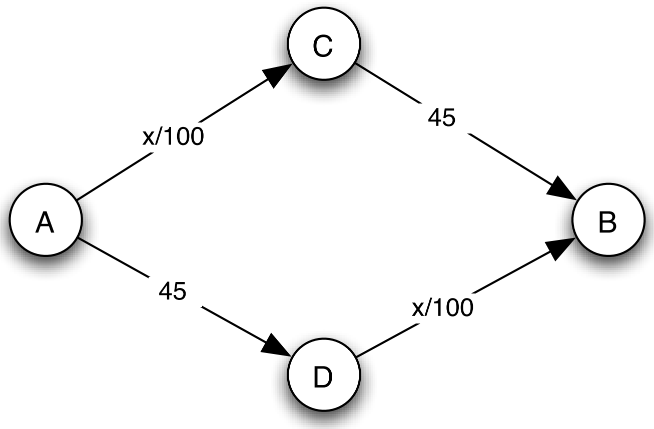


Figure 8.1: A highway network, with each edge labeled by its travel time (in minutes) when there are x cars using it. When 4000 cars need to get from A to B, they divide evenly over the two routes at equilibrium, and the travel time is 65 minutes.

travel time (in minutes) when there are x cars using the edge. In this simplified example, the A-D and C-B edges are insensitive to congestion: each takes 45 minutes to traverse regardless of the number of cars traveling on them. On the other hand, the A-C and D-B edges are highly sensitive to congestion: for each one, it takes $x/100$ minutes to traverse when there are x cars using the edge.[1]

Now, suppose that 4000 cars want to get from A to B as part of the morning commute. There are two possible routes that each car can choose: the upper route through C, or the lower route through D. For example, if each car takes the upper route (through C), then the total travel time for everyone is 85 minutes, since $4000/100 + 45 = 85$. The same is true if everyone takes the lower route. On the other hand, if the cars divide up evenly between the two routes, so that each carries 2000 cars, then the total travel time for people on both routes is $2000/100 + 45 = 65$.

Equilibrium traffic. So what do we expect will happen? The traffic model we've described is really a game in which the players correspond to the drivers, and each player's possible strategies consist of the possible routes from A to B. In our example, this means that each player only has two strategies; but in larger networks, there could be many strategies for each player. The payoff for a player is the negative of his or her travel time (we use the negative since large travel times are bad).

[1]The travel times here are simplified to make the reasoning clearer: in any real application, each road would have both some minimum travel time, and some sensitivity to the number of cars x that are using it. However, the analysis here adapts directly to more intricate functions specifying the travel times on edges.

This all fits very naturally into the framework we've been using. One thing to notice, of course, is that up to now we have focused primarily on games with two players, whereas the current game will generally have an enormous number of players (4000 in our example). But this poses no direct problem for applying any of the ideas we've developed. A game can have any number of players, each of whom can have any number of available strategies, and the payoff to each player depends on the strategies chosen by all. A Nash equilibrium is still a list of strategies, one for each player, so that each player's strategy is a best response to all the others. The notions of dominant strategies, mixed strategies and Nash equilibrium with mixed strategies all have direct parallels with their definitions for two-player games.

In this traffic game, there is generally not a dominant strategy; for example, in Figure 8.1 either route has the potential to be the best choice for a player if all the other players are using the other route. The game does have Nash equilibria, however: as we will see next, any list of strategies in which the drivers balance themselves evenly between the two routes (2000 on each) is a Nash equilibrium, and these are the only Nash equilibria.

Why does equal balance yield a Nash equilibrium, and why do all Nash equilibria have equal balance? To answer the first question, we just observe that with an even balance between the two routes, no driver has an incentive to switch over to the other route. For the second question, consider a list of strategies in which x drivers use the upper route and the remaining $4000-x$ drivers use the lower route. Then if x is not equal to 2000, the two routes will have unequal travel times, and any driver on the slower route would have an incentive to switch to the faster one. Hence any list of strategies in which x is not equal to 2000 cannot be a Nash equilibrium; and any list of strategies in which $x = 2000$ is a Nash equilibrium.

8.2 Braess's Paradox

In Figure 8.1, everything works out very cleanly: self-interested behavior by all drivers causes them — at equilibrium — to balance perfectly between the available routes. But with only a small change to the network, we can quickly find ourselves in truly counterintuitive territory.

The change is as follows: suppose that the city government decides to build a new, very fast highway from C to D, as indicated in Figure 8.2. To keep things simple, we'll model its travel time as 0, regardless of the number of cars on it, although the resulting effect would happen even with more realistic (but small) travel times. It would stand to reason that people's travel time from A to B ought to get better after this edge from C to D is added. Does it?

Here's the surprise: there is a unique Nash equilibrium in this new highway network, but it leads to a worse travel time for everyone. At equilibrium, every driver uses the route through both C and D; and as a result, the travel time for every driver is 80 (since $4000/100 + 0 + 4000/100 = 80$). To see why this is an equilibrium, note that no driver can

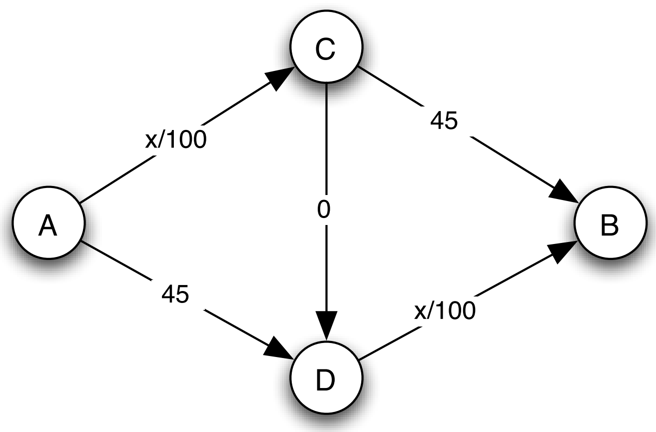


Figure 8.2: The highway network from the previous figure, after a very fast edge has been added from C to D. Although the highway system has been "upgraded," the travel time at equilibrium is now 80 minutes, since all cars use the route through C and D.

benefit by changing their route: with traffic snaking through C and D the way it is, any other route would now take 85 minutes. And to see why it's the only equilibrium, you can check that the creation of the edge from C to D has in fact made the route through C and D a dominant strategy for all drivers: regardless of the current traffic pattern, you gain by switching your route to go through C and D.

In other words, once the fast highway from C to D is built, the route through C and D acts like a "vortex" that draws all drivers into it — to the detriment of all. In the new network there is no way, given individually self-interested behavior by the drivers, to get back to the even-balance solution that was better for everyone.

This phenomenon — that adding resources to a transportation network can sometimes hurt performance at equilibrium — was first articulated by Dietrich Braess in 1968 [76], and it has become known as Braess's Paradox. Like many counterintuitive anomalies, it needs the right combination of conditions to actually pop up in real life; but it has been observed empirically in real transportation networks — including in Seoul, Korea, where the destruction of a six-lane highway to build a public park actually improved travel time into and out of the city (even though traffic volume stayed roughly the same before and after the change) [37].

Some reflections on Braess's paradox. Having now seen how Braess's paradox works, we can also appreciate that there is actually nothing really "paradoxical" about it. There are many settings in which adding a new strategy to a game makes things worse for everyone. For example, the Prisoner's Dilemma from Chapter 6 can be used to illustrate this point: if the only strategy for each player were Not-Confess (an admittedly very simple game), then both players would be better off compared with the game where Confess is added as an option. (Indeed, that's why the police offer Confess as an option in the first place.)

Still, it's reasonable to view the analogous phenomenon at the heart of the Braess Paradox as more paradoxical, at an intuitive level. We all have an informal sense that "upgrading" a network has to be a good thing, and so it is surprising when it turns out to make things worse.

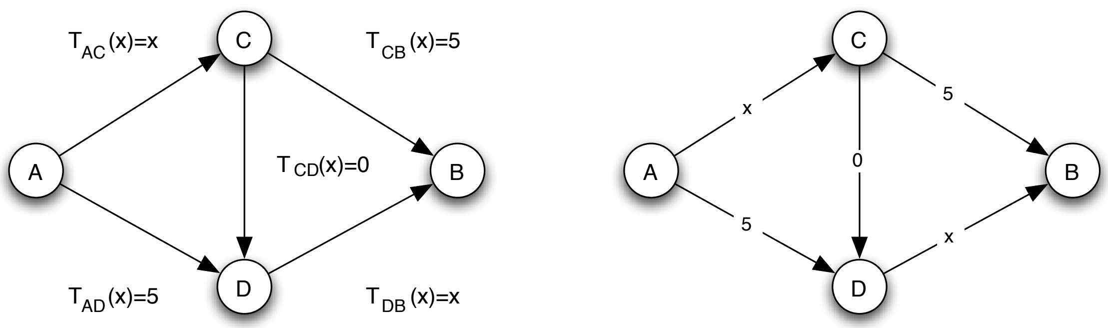
The example in this section is actually the starting point for a large body of work on game-theoretic analysis of network traffic. For example, we could ask how bad Braess's Paradox can be for networks in general: how much larger can the equilibrium travel time be after the addition of an edge, relative to what it was before? Suppose in particular that we allow the graph to be arbitrary, and we assume that the travel time on each edge depends in a linear way on the number of cars traversing it — that is, all travel times across edges have the form $ax + b$, where each of a and b is either 0 or a positive number. In this case, elegant results of Tim Roughgarden and Éva Tardos can be used to show that if we add edges to a network with an equilibrium pattern of traffic, there is always an equilibrium in the new network whose travel time is no more than 4/3 times as large [18, 353]. Moreover, 4/3 is the factor increase that we'd get in the example from Figures 8.1 and 8.2, if we replace the two travel times of 45 with 40. (In that case, the travel time at equilibrium would jump from 60 to 80 when we add the edge from C to D.) So the Roughgarden-Tardos result shows that this simple example is as bad as the Braess Paradox can get, in a quantitative sense, when edges respond linearly to traffic. (When edges can respond non-linearly, things can be much worse.)

There are many other types of questions that can be pursued as well. For example, we could think about ways of designing networks to prevent bad equilibria from arising, or to avoid bad equilibria through the judicious use of tolls on certain parts of the network. Many of these extensions, as well as others, are discussed by Tim Roughgarden in his book on game-theoretic models of network traffic [352].

8.3 Advanced Material: The Social Cost of Traffic at Equilibrium

The Braess Paradox is one aspect of a larger phenomenon, which is that network traffic at equilibrium may not be socially optimal. In this section, we try to quantify how *far* from optimal traffic can be at equilibrium.

We would like our analysis to apply to any network, and so we introduce the following general definitions. The network can be any directed graph. There is a set of drivers, and different drivers may have different starting points and destinations. Now, each edge e has



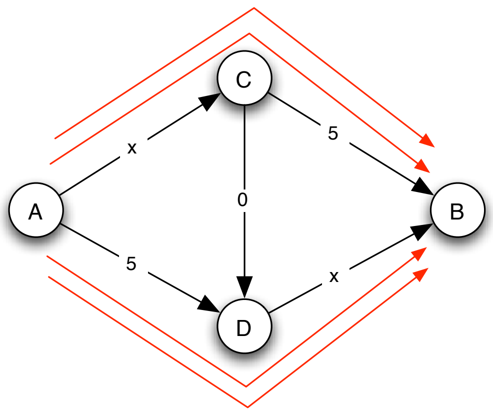
(a) *Travel times written as explicit functions of x.* (b) *Travel times written as annotations on the edges.*

Figure 8.3: A network annotated with the travel-time function on each edge.

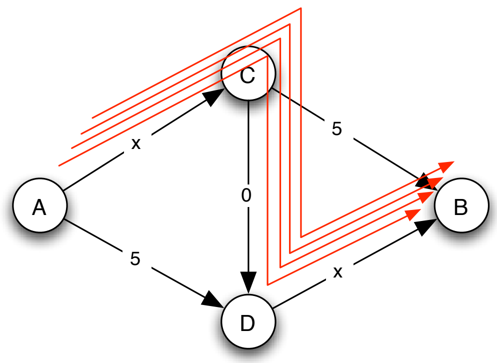
a *travel-time function* $T_e(x)$, which gives the time it takes all drivers to cross the edge when there are x drivers using it. These travel times are simply the functions that we drew as labels inside the edges in the figures in Section 8.1. We will assume that all travel-time functions are linear in the amount of traffic, so that $T_e(x) = a_e x + b_e$ for some choice of numbers a_e and b_e that are either positive or zero. For example, in Figure 8.3 we draw another network on which Braess's Paradox arises, with the travel-time functions scaled down to involve smaller numbers. The version of the drawing in Figure 8.3(a) has the travel-time functions explicitly written out, while the version of the drawing in Figure 8.3(b) has the travel-time functions written as labels inside the edges.

Finally, we say that a *traffic pattern* is simply a choice of a path by each driver, and the *social cost* of a given traffic pattern is the sum of the travel times incurred by all drivers when they use this traffic pattern. For example, Figure 8.4 shows two different traffic patterns on the network from Figure 8.3, when there are four drivers, each with starting node A and destination node B. The first of these traffic patterns, in Figure 8.4(a), achieves the minimum possible social cost — each driver requires 7 units of time to get to their destination, and so the social cost is 28. We will refer to such a traffic pattern, which achieves the minimum possible social cost, as *socially optimal*. (There are other traffic patterns on this network that also achieve a social cost of 28; that is, there are multiple traffic patterns for this network that are socially optimal.) Note that socially optimal traffic patterns are simply the social welfare maximizers of this traffic game, since the sum of the drivers' payoffs is the negative of the social cost. The second traffic pattern, Figure 8.4(b), is the unique Nash equilibrium, and it has a larger social cost of 32.

The main two questions we consider in the remainder of this chapter are the following. First, in any network (with linear travel-time functions), is there always an equilibrium traffic



(a) *The social optimum.*



(b) *The Nash equilibrium.*

Figure 8.4: A version of Braess's Paradox: In the socially optimal traffic pattern (on the left), the social cost is 28, while in the unique Nash equilibrium (on the right), the social cost is 32.

pattern? We have seen examples in Chapter 6 of games where equilibria do not exist using pure strategies, and it is not *a priori* clear that they should always exist for the traffic game we've defined here. However, we will find in fact that equilibria always do exist. The second main question is whether there always exists an equilibrium traffic pattern whose social cost is not much more than the social optimum. We will find that this is in fact the case: we will show a result due to Roughgarden and Tardos that there is always an equilibrium whose social cost is at most *twice* that of the optimum [353].[2]

A. How to Find a Traffic Pattern at Equilibrium

We will prove that an equilibrium exists by analyzing the following procedure that explicitly searches for one. The procedure starts from any traffic pattern. If it is an equilibrium, we are done. Otherwise, there is at least one driver whose best response, given what everyone else is doing, is some alternate path providing a strictly lower travel time. We pick one such driver and have him switch to this alternate path. We now have a new traffic pattern and we again check whether it is an equilibrium — if it isn't, then we have some driver switch to his best response, and we continue in this fashion.

This procedure is called *best-response dynamics*, since it dynamically reconfigures the

[2]In fact, stronger results of Roughgarden and Tardos, supplemented by subsequent results of Anshelevich et al. [18], establish that in fact every equilibrium traffic pattern has social cost at most 4/3 times the optimum. (One can show that this implies their result on the Braess Paradox cited in the previous section — that with linear travel times, adding edges can't make things worse by a factor of more than 4/3.) However, since it is harder to prove the bound of 4/3, we limit ourselves here to proving the easier but weaker factor of 2 between the social optimum and some equilibrium traffic pattern.

players' strategies by constantly having some player perform his or her best response to the current situation. If the procedure ever stops, in a state where everyone is in fact playing their best response to the current situation, then we have an equilibrium. So the key is to show that in any instance of our traffic game, best-response dynamics must eventually stop at an equilibrium.

But why should it? Certainly for games that lack an equilibrium, best-response dynamics will run forever: for example, in the Matching Pennies game from Chapter 6, when only pure strategies are allowed, best-response dynamics will simply consist of the two players endlessly switching their strategies between H and T. It seems plausible that for some network, this could happen in the traffic game as well: one at a time, drivers shift their routes to ones that are better for them, thus increasing the delay for another driver who then switches and continues the cascade.

In fact, however, this cannot happen in the traffic game. We now show that best-response dynamics must always terminate in an equilibrium, thus proving not only that equilibria exist but also that they can be reached by a simple process in which drivers constantly update what they're doing according to best responses.

Analyzing Best-Response Dynamics Via Potential Energy. How should we go about proving that best-response dynamics must come to a halt? When you have a process that runs according to some set of instructions like, "Do the following ten things and then stop," it's generally obvious that it will eventually come to an end: the process essentially comes with its own guarantee of termination. But we have a process that runs according to a different kind of rule, one that says, "Keep doing something until a particular condition happens to hold." In this case, there is no *a priori* reason to believe it will ever stop.

In such cases, a useful analysis technique is to define some kind of *progress measure* that tracks the process as it operates, and to show that eventually enough "progress" will be made that the process must stop. For the traffic game, it's natural to think of the social cost of the current traffic pattern as a possible progress measure, but in fact the social cost is not so useful for this purpose. Some best-response updates by drivers can make the social cost better (for example, if a driver leaves a congested road for a relatively empty one), but others can make it worse (as in the sequence of best-response updates that shifts the traffic pattern from the social optimum to the inferior equilibrium in the Braess Paradox). So in general, as best-response dynamics runs, the social cost of the current traffic pattern can oscillate between going up and going down, and it's not clear how this is related to our progress toward an equilibrium.

Instead, we're going to define an alternate quantity that initially seems a bit mysterious. However, we will see that it has the property that it strictly decreases with each best-response update, and so it can be used to track the progress of best-response dynamics [303]. We will refer to this quantity as the *potential energy* of a traffic pattern.

The potential energy of a traffic pattern is defined edge-by-edge, as follows. If an edge e currently has x drivers on it, then we define the potential energy of this edge to be

$$\text{Energy}(e) = T_e(1) + T_e(2) + \cdots + T_e(x).$$

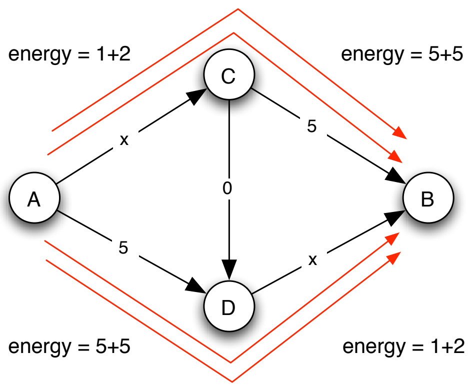
If an edge has no drivers on it, its potential energy is defined to be 0. The potential energy of a traffic pattern is then simply the sum of the potential energies of all the edges, with their current number of drivers in this traffic pattern. In Figure 8.5, we show the potential energy of each edge for the five traffic patterns that best-response dynamics produces as it moves from the social optimum to the unique equilibrium in the Braess-Paradox network from Figure 8.4.

Notice that the potential energy of an edge e with x drivers is not the total travel time experienced by the drivers that cross it. Since there are x drivers each experiencing a travel time of $T_e(x)$, their total travel time is $xT_e(x)$, which is a different number. The potential energy, instead, is a sort of "cumulative" quantity in which we imagine drivers crossing the edge one by one, and each driver only "feels" the delay caused by himself and the drivers crossing the edge in front of him.

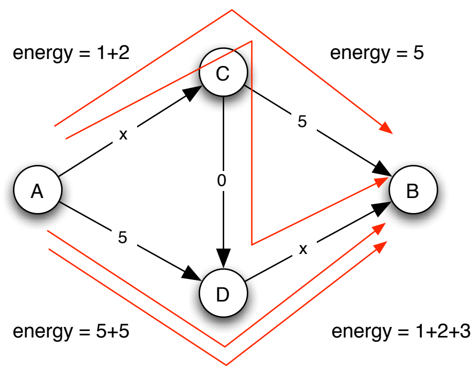
Of course, the potential energy is only useful for our purpose if it lets us analyze the progress of best-response dynamics. We show how to do this next.

Proving that Best-Response Dynamics Comes to an End. Our main claim is the following: each step of best-response dynamics causes the potential energy of the current traffic pattern to strictly decrease. Proving this will be enough to show that best-response dynamics must come to an end, for the following reason. The potential energy can only take a finite number of possible values — one for each possible traffic pattern. If it is strictly decreasing with each step of best-response dynamics, this means that it is "consuming" this finite supply of possible values, since it can never revisit a value once it drops below it. So best-response dynamics must come to a stop by the time the potential energy reaches its minimum possible value (if not sooner). And once best-response dynamics comes to a stop, we must be at an equilibrium — for otherwise, the dynamics would have a way to continue. Thus, showing that the potential energy strictly decreases in every step of best-response dynamics is enough to show the existence of an equilibrium traffic pattern.

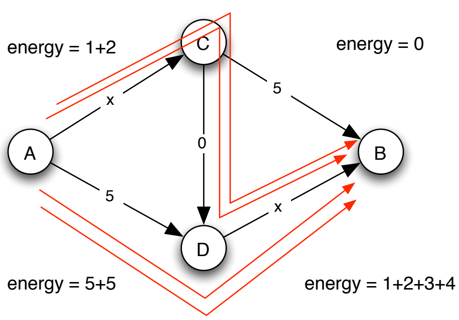
As an example, let's return to the sequence of best-response steps from Figure 8.5. Although the social cost is rising through the five traffic patterns (increasing from 28 to 32), the potential energy decreases strictly in each step (in the sequence 26, 24, 23, 21, 20). In fact, it is easy to track the change in potential energy through this sequence as follows. From one traffic pattern to the next, the only change is that one driver abandons his current path and switches to a new one. Suppose we really view this switch as a two-step process: first the



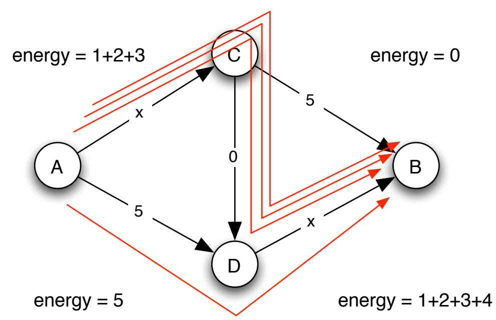
(a) *The initial traffic pattern. (Potential energy is 26.)*



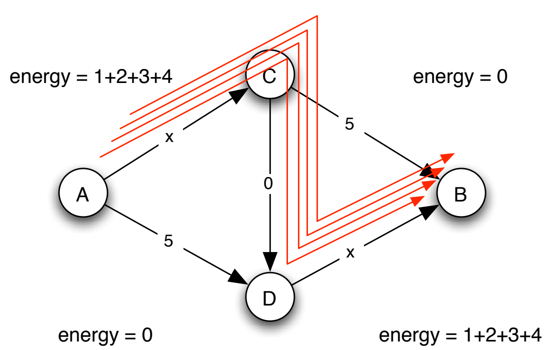
(b) *After one step of best-response dynamics. (Potential energy is 24.)*



(c) *After two steps. (Potential energy is 23.)*

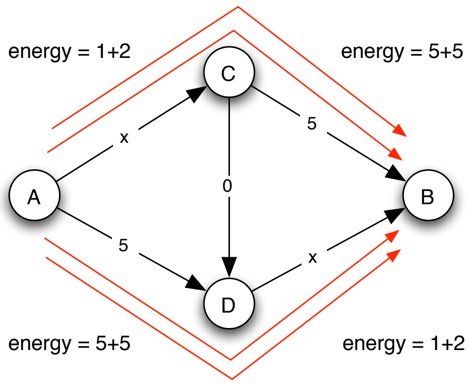


(d) *After three steps. (Potential energy is 21.)*

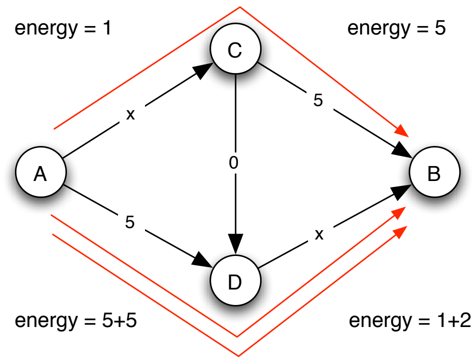


(e) *After four steps: Equilibrium is reached. (Potential energy is 20.)*

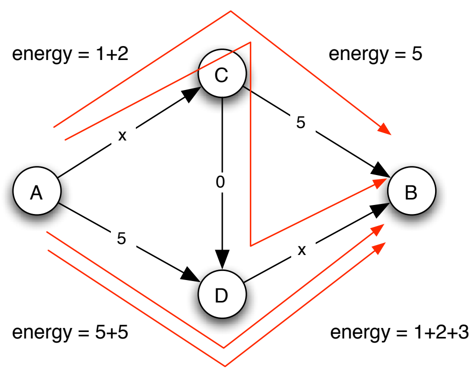
Figure 8.5: We can track the progress of best-response dynamics in the traffic game by watching how the potential energy changes.



(a) *The potential energy of a traffic pattern not in equilibrium.*



(b) *Potential energy is released when a driver abandons their current path.*



(c) *Potential energy is put back into the system when the driver chooses a new path.*

Figure 8.6: When a driver abandons one path in favor of another, the change in potential energy is exactly the improvement in the driver's travel time.

drivers abandons his current path, temporarily leaving the system; then, the driver returns to the system by adopting a new path. This first step releases potential energy as the driver leaves the system, and the second step adds potential energy as he re-joins. What's the net change?

For example, the transition from Figure 8.5(a) to 8.5(b) occurs because one driver abandons the upper path and adopts the zigzag path. As shown in Figure 8.6, abandoning the upper path releases $2 + 5 = 7$ units of potential energy, while adopting the zigzag path puts $2 + 0 + 3$ units of potential energy back into the system. The resulting change is a decrease of 2.

Notice that the decrease of 7 is simply the travel time the driver was experiencing on the path he abandoned, and the subsequent increase of 5 is the travel time the driver now experiences on the path he has adopted. This relationship is in fact true for any network and any best response by a driver, and it holds for a simple reason. Specifically, the potential energy of edge e with x drivers is

$$T_e(1) + T_e(2) + \cdots + T_e(x-1) + T_e(x),$$

and when one of these drivers leaves it drops to

$$T_e(1) + T_e(2) + \cdots + T_e(x-1).$$

Hence the change in potential energy on edge e is $T_e(x)$, exactly the travel time that the driver was experiencing on e. Summing this over all edges used by the driver, we see that *the potential energy released when a driver abandons his current path is exactly equal to the travel time the driver was experiencing.* By the same reasoning, when a driver adopts a new path, the potential energy on each edge e he joins increases from

$$T_e(1) + T_e(2) + \cdots + T_e(x)$$

to

$$T_e(1) + T_e(2) + \cdots + T_e(x) + T_e(x+1),$$

and the increase of $T_e(x+1)$ is exactly the new travel time the driver experiences on this edge. Hence, *the potential energy added to the system when a driver adopts a new path is exactly equal to the travel time the driver now experiences.*

It follows when a driver switches paths, the net change in potential energy is simply his new travel time minus his old travel time. But in best-response dynamics, a driver only changes paths when it causes his travel time to decrease — so the change in potential energy is negative for any best-response move. This establishes what we wanted to show: that the potential energy in the system strictly decreases throughout best-response dynamics. As argued above, since the potential energy cannot decrease forever, best-response dynamics must therefore eventually come to an end, at a traffic pattern in equilibrium.

B. Comparing Equilibrium Traffic to the Social Optimum

Having shown that an equilibrium traffic pattern always exists, we now consider how its travel time compares to that of a socially optimal traffic pattern. We will see that the potential energy we've defined is very useful for making this comparison. The basic idea is to establish a relationship between the potential energy of an edge and the total travel time of all drivers crossing the edge. Once we do this, we will sum these two quantities over all the edges to compare travel times at equiibrium and at social optimality.

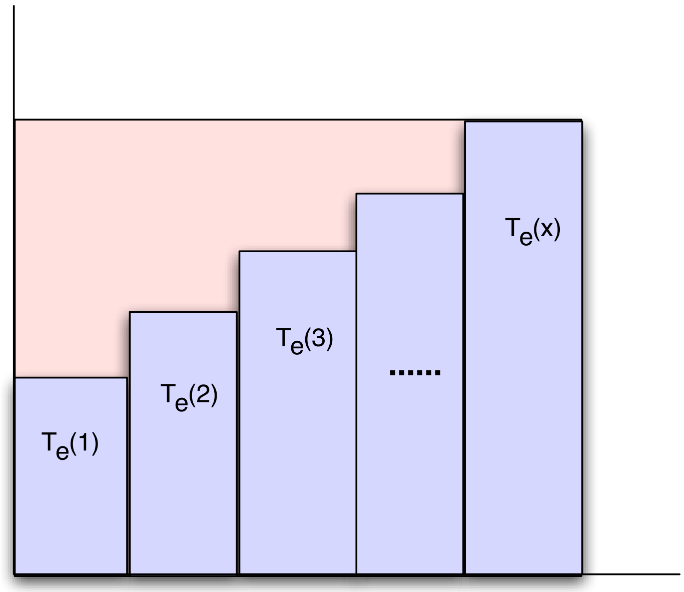


Figure 8.7: The potential energy is the area under the shaded rectangles; it is always at least half the total travel time, which is the area inside the enclosing rectangle.

Relating Potential Energy to Travel Time for a Single Edge. We denote the potential energy of an edge by Energy(e), and we recall that when there are x drivers, this potential energy is defined by

$$\text{Energy}(e) = T_e(1) + T_e(2) + \cdots + T_e(x).$$

On the other hand, each of the x drivers experiences a travel time of $T_e(x)$, and so the total travel time experienced by all drivers on the edge is

$$\text{Total-Travel-Time}(e) = xT_e(x).$$

For purposes of comparison with the potential energy, it is useful to write this as follows:

$$\text{Total-Travel-Time}(e) = \underbrace{T_e(x) + T_e(x) + \cdots + T_e(x)}_{x \text{ terms}}.$$

Since the potential energy and the total travel time each have x terms, but the terms in the latter expression are at least as large as the terms in the former, we have

$$\text{Energy}(e) \leq \text{Total-Travel-Time}(e).$$

Figure 8.7 shows how the potential energy and the total travel time compare when T_e is a linear function: the total travel time is the shaded area under the horizontal line with

y-value $T_e(x)$, while the potential energy is the total area under all the unit-width rectangles of heights $T_e(1), T_e(2), \ldots, T_e(x)$. As this figure makes clear geometrically, since T_e is a linear function, we have

$$T_e(1) + T_e(2) + \cdots + T_e(x) \geq \frac{1}{2} x T_e(x).$$

Alternately, we can see this by a bit of simple algebra, recalling that $T_e(x) = a_e x + b_e$:

$$\begin{aligned} T_e(1) + T_e(2) + \cdots + T_e(x) &= a_e(1 + 2 + \cdots + x) + b_e x \\ &= \frac{a_e x(x+1)}{2} + b_e x \\ &= x\left(\frac{a_e(x+1)}{2} + b_e\right) \\ &\geq \frac{1}{2} x(a_e x + b_e) \\ &= \frac{1}{2} x T_e(x). \end{aligned}$$

In terms of energies and total travel times, this says

$$\text{Energy}(e) \geq \frac{1}{2} \cdot \text{Total-Travel-Time}(e).$$

So the conclusion is that the potential energy of an edge is never far from the total travel time: it is sandwiched between the total travel time and half the total travel time.

Relating the Travel Time at Equilibrium and Social Optimality. We now use this relationship between potential energy and total travel to relate the equilibrium and socially optimal traffic patterns.

Let Z be a traffic pattern; we define Energy(Z) to be the total potential energy of all edges when drivers follow the traffic pattern Z. We write Social-Cost(Z) to denote the social cost of the traffic pattern; recall that this is the sum of the travel times experienced by all drivers. Equivalently, summing the social cost edge-by-edge, Social-Cost(Z) is the sum of the total travel times on all the edges. So applying our relationships between potential energy and travel time on an edge-by-edge basis, we see that the same relationships govern the potential energy and social cost of a traffic pattern:

$$\frac{1}{2} \cdot \text{Social-Cost}(Z) \leq \text{Energy}(Z) \leq \text{Social-Cost}(Z).$$

Now, suppose that we start from a socially optimal traffic pattern Z, and we then allow best-response dynamics to run until they stop at an equilibrium traffic pattern Z'. The social cost may start increasing as we run best-response dynamics, but the potential energy can only go down — and since the social cost can never be more than twice the potential energy, this shrinking potential energy keeps the social cost from ever getting more than twice as high as where it started. This shows that the social cost of the equilibrium we reach is at most twice the cost of the social optimum we started with — hence there is an equilibrium with at most twice the socially optimal cost, as we wanted to show.

Let's write this argument out in terms of the inequalities on energies and social costs. First, we saw in the previous section that the potential energy decreases as best-response dynamics moves from Z to Z', and so

$$\text{Energy}(Z') \leq \text{Energy}(Z).$$

Second, the quantitative relationships between energies and social cost say that

$$\text{Social-Cost}(Z') \leq 2 \cdot \text{Energy}(Z')$$

and

$$\text{Energy}(Z) \leq \text{Social-Cost}(Z).$$

Now we just chain these inequalities together, concluding that

$$\text{Social-Cost}(Z') \leq 2 \cdot \text{Energy}(Z') \leq 2 \cdot \text{Energy}(Z) \leq 2 \cdot \text{Social-Cost}(Z).$$

Note that this really is the same argument that we made in words in the previous paragraph: the potential energy decreases during best-response dynamics, and this decrease prevents the social cost from every increasing by more than a factor of two.

Thus, tracking potential energy is not only useful for showing that best-response dynamics must reach an equilibrium; by relating this potential energy to the social cost, we can use it to put a bound on the social cost of the equilibrium that is reached.

8.4 Exercises

1. There are 1000 cars which must travel from town A to town B. There are two possible routes that each car can take: the upper route through town C or the lower route through town D. Let x be the number of cars traveling on the edge AC and let y be the number of cars traveling on the edge DB. The directed graph in Figure 8.8 indicates that travel time per car on edge AC is $x/100$ if x cars use edge AC, and similarly the travel time per car on edge DB is $y/100$ if y cars use edge DB. The travel time per car on each of edges CB and AD is 12 regardless of the number of cars on these edges. Each driver wants to select a route to minimize his travel time. The drivers make simultaneous choices.

   (a) Find Nash equilibrium values of x and y.

   (b) Now the government builds a new (one-way) road from town C to town D. The new road adds the path ACDB to the network. This new road from C to D has a travel

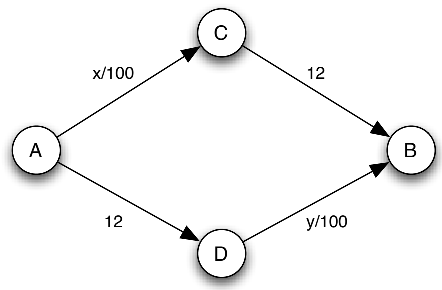


Figure 8.8: Traffic Network.

time of 0 per car regardless of the number of cars that use it. Find a Nash equilibrium for the game played on the new network. What are the equilibrium values of x and y? What happens to total cost-of-travel (the sum of total travel times for the 1000 cars) as a result of the availability of the new road?

(c) Suppose now that conditions on edges CB and AD are improved so that the travel times on each edge are reduced to 5. The road from C to D that was constructed in part (b) is still available. Find a Nash equilibrium for the game played on the network with the smaller travel times for CB and AD. What are the equilibrium values of x and y? What is the total cost-of-travel? What would happen to the total cost-of-travel if the government closed the road from C to D?

2. There are two cities A and B joined by two routes. There are 80 travelers who begin in city A and must travel to city B. There are two routes between A and B. Route I begins with a highway leaving city A, this highway takes one hour of travel time regardless of how many travelers use it, and ends with a local street leading into city B. This local street near city B requires a travel time in minutes equal to 10 plus the number of travelers who use the street. Route II begins with a local street leaving city A, which requires a travel time in minutes equal to 10 plus the number of travelers who use this street, and ends with a highway into city B which requires one hour of travel time regardless of the number of travelers who use this highway.

   (a) Draw the network described above and label the edges with the travel time needed to move along the edge. Let x be the number of travelers who use Route I. The network should be a directed graph as all roads are one-way.

   (b) Travelers simultaneously chose which route to use. Find the Nash equilibrium value of x.

   (c) Now the government builds a new (two-way) road connecting the nodes where local streets and highways meet. This adds two new routes. One new route consists of the

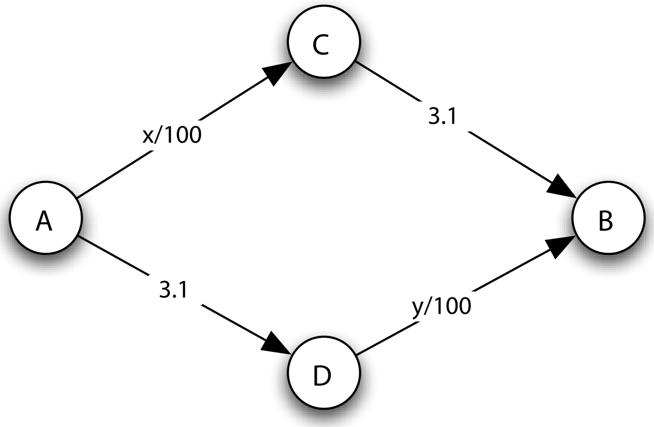


Figure 8.9: Traffic Network

local street leaving city A (on Route II), the new road and the local street into city B (on Route I). The second new route consists of the highway leaving city A (on Route I), the new road and the highway leading into city B (on Route II). The new road is very short and takes no travel time. Find the new Nash equilibrium. *(Hint: There is an equilibrium in which no one chooses to use the second new route described above.)*

(d) What happens to total travel time as a result of the availability of the new road?

(e) If you can assign travelers to routes, then in fact it's possible to reduce total travel time relative to what it was before the new road was built. That is, the total travel time of the population can be reduced (below that in the original Nash equilibrium from part (b)) by assigning travelers to routes. There are many assignments of routes that will accomplish this. Find one. Explain why your reassignment reduces total travel time. *(Hint: Remember that travel on the new road can go in either direction. You do not need find the total travel time minimizing assignment of travelers. One approach to this question is to start with the Nash equilibrium from part (b) and look for a way to assign some travelers to different routes so as to reduce total travel time.)*

3. There are 300 cars which must travel from city A to city B. There are two possible routes that each car can take. The upper route through city C or the lower route through city D. Let x be the number of cars traveling on the edge AC and let y be the number of cars traveling on the edge DB. The directed graph in Figure 8.9 indicates that total travel time per car along the upper route is $(x/100) + 3.1$ if x cars use the upper route, and similarly the total travel time per car along the lower route is $3.1 + (y/100)$ if y cars take the lower route. Each driver wants to select a route to

minimize his total travel time. The drivers make simultaneous choices.

(a) Find Nash equilibrium values of x and y.

(b) Now the government builds a new (one-way) road from city A to city B. The new route has a travel time of 5 per car regardless of the number of cars that use it. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel (the sum of total travel times for the 300 cars) as a result of the availability of the new road?

(c) Now the government closes the direct route between city A and city B and builds a new one-way road which links city C to city D. This new road between C and D is very short and has a travel time of 0 regardless of the number of cars that use it. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel as a result of the availability of the new road?

(d) The government is unhappy with the outcome in part (c) and decides to reopen the road directly linking city A and city B (the road that was built in part (b) and closed in part (c)). The route between C and D that was constructed in part (c) remains open. This road still has a travel time of 5 per car regardless of the number of cars that use it. Draw the new network and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way. Find a Nash equilibrium for the game played on the new network. What happens to total cost-of-travel as a result of re-opening the direct route between A and B?

4. There are two cities, A and B, joined by two routes, I and II. All roads are one-way roads. There are 100 travelers who begin in city A and must travel to city B. Route I links city A to city B through city C. This route begins with a road linking city A to city C which has a cost-of-travel for each traveler equal to $0.5 + x/200$, where x is the number of travelers on this road. Route I ends with a highway from city C to city B which has a cost-of-travel for each traveler of 1 regardless of the number of travelers who use it. Route II links city A to city B through city D. This route begins with a highway linking city A to city D which has a cost-of-travel for each traveler of 1 regardless of the number of travelers who use it. Route II ends with a road linking city D to city B which has a cost-of-travel for each traveler equal to $0.5 + y/200$, where y is the number of travelers on this road.

   These costs of travel are the value that travelers put on the time lost due to travel plus the cost of gasoline for the trip. Currently there are no tolls on these roads. So the government collects no revenue from travel on them.

(a) Draw the network described above and label the edges with the cost-of-travel needed to move along the edge. The network should be a directed graph as all roads are one-way.

(b) Travelers simultaneously chose which route to use. Find Nash equilibrium values of x and y.

(c) Now the government builds a new (one-way) road from city C to city D. The new road is very short and has 0 cost-of-travel. Find a Nash equilibrium for the game played on the new network.

(d) What happens to total cost-of-travel as a result of the availability of the new road?

(e) The government is unhappy with the outcome in part (c) and decides to impose a toll on users of the road from city A to city C and to simultaneously subsidize users of the highway from city A to city D. They charge a toll of 0.125 to each user, and thus increase the cost-of-travel by this amount, for users of the road from city A to city C. They also subsidize travel, and thus reduce the cost-of-travel by this amount, for each user of the highway from city A to city D by 0.125. Find a new Nash equilibrium. [If you are curious about how a subsidy could work you can think of it as a negative toll. In this economy all tolls are collected electronically, much as New York State attempts to do with its E-ZPass system. So a subsidy just reduces the total amount that highway users owe.]

(f) As you will observe in solving part (e) the toll and subsidy in part (e) were designed so that there is a Nash equilibrium in which the amount the government collects from the toll just equals the amount it loses on the subsidy. So the government is breaking even on this policy. What happens to total cost-of-travel between parts (c) and (e)? Can you explain why this occurs? Can you think of any break-even tolls and subsidies that could be placed on the roads from city C to city B, and from city D to city B, that would lower the total cost-of-travel even more?