PART II: THE COOPERATIVE THEORY

3.6 TU and NTU Feasible Sets

The cooperative theory assumes that the players are permitted to communicate freely and collude—if they wish—with the intention of manipulating the game to their mutual advantage. An initial period of open discussion and pre-play negotiation is postulated, from which the players may emerge with a specific agreement spelling out the strategies they will play. Since this is an irrevocable decision, the actual game that follows is just a formality and the final payoff vector (or expected payoff vector, if mixed strategies are employed) is a foregone conclusion.[1]

They may also fail to agree, however. In that case, the theory requires that before they break off their talks, they must choose and announce to each other the individual strategies that they intend to use if no agreement is reached; these declarations (called "threats") are also irrevocable. So the final payoff vector of the game, called the "disagreement point," is foreordained in this case too. But since such an outcome is usually worse for both parties than some agreement they could have reached, it will turn out that in the cooperative *solution* the threats remain just threats—i.e., strategy declarations that might influence the players' thinking but are never actually carried out.

Apart from formalizing the *output* of the pre-play negotiation, the theory imposes virtually no conditions on its form—this is a matter for the players to work out for themselves. (In particular, the order in which they announce their threats makes no difference at all, as it turns out.) But we must still carefully define the range of possible agreements, i.e., the specific set of payoff vectors in $\mathbf{R}^2$ that is available to the players if they cooperate. We call this the *cooperative feasible set*, in analogy to the noncooperative feasible set we defined earlier.

[1]This is not unlike the practice of settling a lawsuit "out of court," just before the trial begins. The litigants, having taken one last, realistic look at their prospects, put their heads together and work out a compromise that avoids further uncertainty and expense, as well as the possibility that the judge or jury may hand down a suboptimal verdict, i.e., one that is worse for *both* parties than what theycould have settled for.

There is a complication, however. When players are free to collude, the resources available to them from outside the game the game cannot be ignored, and there is one resource whose presence or absence lead to two significantly different feasible sets and associated solution theories, called "TU" and "NTU" (for *transferable* and *non-transferable utility*, respectively. The resource we are referring to is of course *money*.

The NTU approach is based on the assmuption that the negotiations are concerned *only* with the choice of a joint strategy. Recall that to obtain the noncooperative ("NC") feasible set we ran through all pairs of mixed strategies $\langle p, q\rangle$. But with cooperation there are more possibilities, since the probability mixes do not have to be independent. The pure strategies of the "coalition" $\{\mathrm{I}, \mathrm{II}\}$ are all the pairs $\langle i, j\rangle$ of individual pure strategies, so a "coalitional" mixed strategy is any probability distribution over this set of pairs. This makes for a larger range of strategy choices[2] and often makes available better payoffs.

The NTU feasible set that results is easily defined:

> DEFINITION. The *NTU feasible set* of an $m \times n$ bimatrix game $[A, B]$ is the convex hull of the set of payoff vectors (a_{ij}, b_{ij}), $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Stated another way, the NTU feasible set is the smallest convex polygon in $\mathbf{R}^2$ that contains all the payoff vectors in the bimatrix.[3]

The TU feasible set is larger, but in the end easier to work with. It is based on a broader and perhaps more realistic view of the negotiatory possibilities but requires the existence of a form of money (called by some authors *u-money*), having certain ideal properties that are not always found in real money. Let us enumerate them. (1) The *pure* game outcomes can be assigned "u-values" by each player, so that one outcome is preferred to another if and only if its u-value is higher. (2)

[2]In an $m \times n$ game the dimension of the set of *pairs of individual mixed strategies* is only $(m-1)+(n-1)$, while the dimension of the set of *joint mixed strategies* is $mn-1$.

[3]To visualize the convex hull of a finite point-set in the plane, imagine inserting a thumb tack at each point and stretching a rubber band around the outside. Note that there may be points that do not touch the rubber band, and so do not appear as vertices of the resulting polygon.

Each player is indifferent between two *mixed* outcomes having the same u-value. (3) U-money exists in a tangible, infinitely divisible form, so that arbitrary quantities of it can be transfered without cost between the players. (4) Each player starts the game with a supply of u-money sufficient for any side payment the solution might call for. We have been using (1) and (2) implicitly all along, but (3) and (4) are specific to the intoduction of transferable utility.

The TU approach, then, envisages that *side payments* as well as joint strategy choices are proper subjects for negotiation and can be included in the final, irrevocable agreement. This greatly increases the opportunities for manipulating the outcome and obviously has a major effect on the feasible set. Indeed, we have:

DEFINITION. The *TU feasible set* of an $m \times n$ bimatrix game $[A, B]$ is the convex hull of the set of vectors of the form $(a_{ij} + s, b_{ij} - s)$, where $i = 1, \ldots, m$, $j = i, \ldots, n$, and $s \in \mathbf{R}$.

Here the number s, if positive, represents a payment from II to I, and if negative, a payment from I to II. Thus, we draw a 45° line from "northwest" to "southeast" across $\mathbf{R}^2$ through each payoff vector (a_{ij}, b_{ij}) and then, by taking the convex hull, fill in the spaces between the lines. The final result is that the TU feasible set is an infinite diagonal band, stretching across $\mathbf{R}^2$ and just covering the NTU feasible set. The band is bounded on the "northeast" by the line

$$u_{\mathrm{I}} + u_{\mathrm{II}} = \max_i \max_j (a_{ij} + b_{ij})$$

and on the "southwest" by the line

$$u_{\mathrm{I}} + u_{\mathrm{II}} = \min_i \min_j (a_{ij} + b_{ij}).$$

In the extreme, constant-sum case, these lines coincide.

In Figure 3–13 the three types of feasible set are illustrated for a simple 2×2 game.

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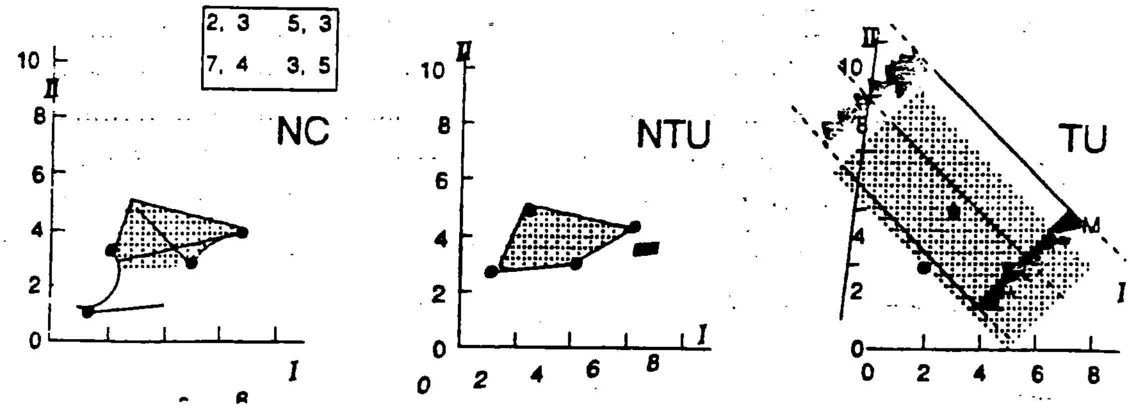

Figure 3–13. **Three kinds of feasible set.**

As the players bargain their way toward an agreement they have every reason to collaborate to make the *sum* of their payoffs as large as possible. This puts the outcome on the upper edge of the TU feasible set. Indeed, failure to achieve this joint maximum would leave a situation in which both players could improve simultaneously. It may happen, however, that this joint maximum is only achievable by some very lop-sided outcome, overly advantageous to one player, like the point M in Figure 3–14 which is far below II's safety level of 6. In this case it would be surprising if some compensatory side payment could not be agreed upon as a *quid pro quo* for II's loss in the actual game, since both players stand to gain in the end.

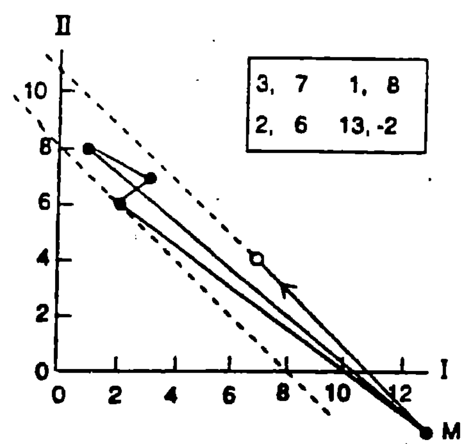

Figure 3–14. **Side payment as compensation.**

But even when a side payment is clearly desirable, how can the players possibly agree on the amount? On this question their interests are directly opposed,[4] and yet both must agree. John Nash, in a famous 1953 paper (written when he was a student), introduced the key notion of a *threat game*, which takes place during the pre-play negotiations and resolves the seeming impasse. It provides a reasonable way to evaluate and balance off the opposing threats, and so provide a compromise that saves the day. Our treatment in these notes is a modern adaptation of Nash's fundamental idea.

Before we define the solution concept in a precise way, let's talk our way through an example.

Example.

> THE APARTMENT. Players I and II own a small apartment, room enough for only one. Because of the terms of joint ownership, however, neither can move in unless both agree. But the apartment, because of its location (or some other circumstance), is worth more to I than to II—say, \$600/mo. compared to \$500/mo.

The following game in strategic form seems to capture the essential options open to the players:

	R	G	N
R	–, –	600, 0	0, 0
G	0, 500	–, –	0, 0
N	0, 0	0, 0	0, 0

Here the letters R, G, N signify "Request permission to move in," "Give permission if requested," and "No deal." If both choose R or both choose G then the decision is deferred, and the blanks (–,–) stand for undetermined positive numbers totalling 600 or less, representing the anticipated value to each player of the settlement they may reach later. It turns out that these numbers are not important for the solution.

[4] Recall Figure 3–4 in §3.1.

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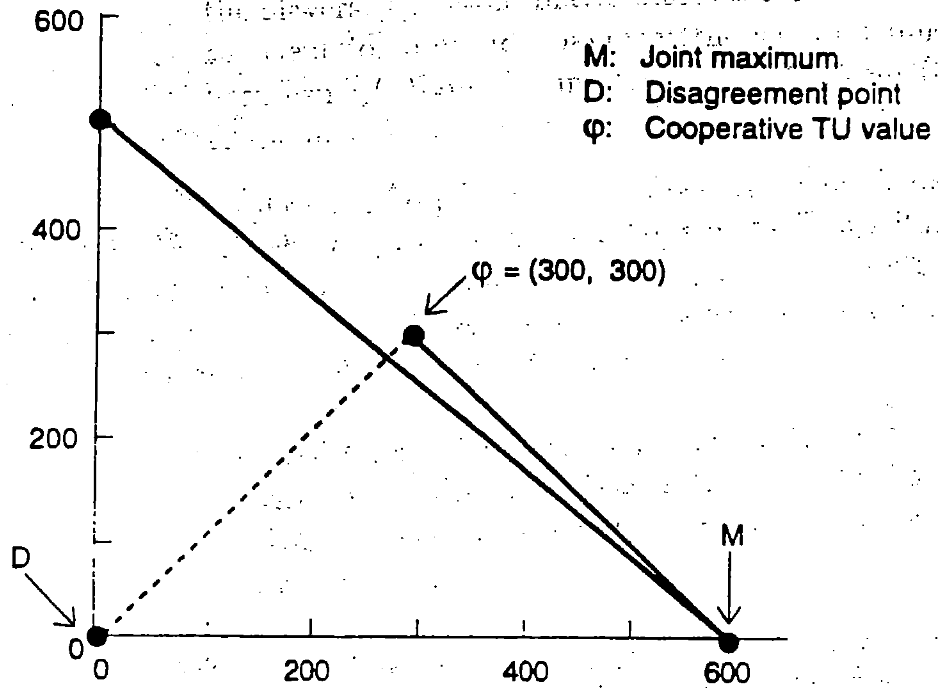

Figure 3–15. "The Apartment"

As we shall see presently, the cooperative TU solution says that I gets the apartment and pays \$300/mo. to II. The rationale for this is that both players suffer equally if the agreement is broken.

EXERCISE

Exercise 3.17 Find all PSEs of "The Apartment," with the blanks "–" replaced by arbitrary numbers satisfying the stated constraints. Are there any other SEs?

3.7 Definition of the TU Solution

Given the bimatrix game $[A, B]$, we define two auxiliary matrices as follows:

(3.11) $\quad$ <u>sum</u>: $\Sigma = A + B$, $\qquad$ <u>difference</u>: $\Delta = A - B$.

The TU value of $[A, B]$ will be based on certain numerical "values" σ and δ that we shall assign to these two matrices.

Since Σ expresses a common interest of the players, it is natural to treat it as a joint maximization problem and define its "value" to be its largest entry:

$$\sigma = \max_i \max_j (a_{ij} + b_{ij}). \tag{3.12}$$

On the other hand, Δ is more an expression of the rivalry between the players.[5] In addition to the joint objective represented by (3.12), each player also has a private objective of maximizing his share of σ at the expense of his rival. Thus, I also wants to maximize the difference $a_{ij} - b_{ij}$ while II wants to minimize it. So it is natural to regard Δ as the matrix of a zero sum game and define its value δ to be its *minimax value* in the sense of Chapter II, thus:

$$\delta = \max_{p \in P} \min_{q \in Q} (a(p,q) - b(p,q)) = \min_{q \in Q} \max_{p \in P} (a(p,q) - b(p,q)). \tag{3.13}$$

The p and q of (3.13) are the *threat strategies* mentioned in the previous section: unilateral declarations by the players *which they players are bound to put into effect if negotiations break down.* With their aid we can determine the *disagreement point* $D(p,q) \in \mathbf{R}^2$, as follows:

$$D_{\mathrm{I}}(p,q) = a(p,q) = \sum_{i,j} p_i q_j a_{ij}, \quad D_{\mathrm{II}}(p,q) = b(p,q) = \sum_{i,j} p_i q_j b_{ij}.$$

To this point the players will add their agreed shares of the surplus which their cooperation creates. If the agreed joint strategy is optimal in the sense of (3.12), this surplus will amount to exactly

$$\sigma - (D_{\mathrm{I}}(p,q) + D_{\mathrm{II}}(p,q)).$$

[5]See Figure 3–4 in §3.1.

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The TU solution requires that they split this equally, since once the threats p and q have been announced there is no further room for maneuver and the game is reduced to a perfectly symmetrical, almost trivial bargaining problem.

Accordingly, we define the *resolution point* associated with the strategy selection $\langle p, q\rangle$ by

$$R_{\mathrm{I}}(p,q) = D_{\mathrm{I}}(p,q) + \frac{\sigma - D_{\mathrm{I}}(p,q) - D_{\mathrm{II}}(p,q)}{2}$$

and

$$R_{\mathrm{II}}(p,q) = D_{\mathrm{II}}(p,q) + \frac{\sigma - D_{\mathrm{I}}(p,q) - D_{\mathrm{II}}(p,q)}{2},$$

which reduce to the twin conditions

$$\begin{cases} R_{\mathrm{I}}(p,q) + R_{\mathrm{II}}(p,q) = \sigma \\ R_{\mathrm{I}}(p,q) - R_{\mathrm{II}}(p,q) = D_{\mathrm{I}}(p,q) - D_{\mathrm{II}}(p,q). \end{cases}$$

The relationship between $D(p,q)$ and $R(p,q)$ is illustrated in the figure below. The figure also shows the jointly maximal point M as well as the side payment vectors π', π'', π''' that lead from M to various typical resolution points.

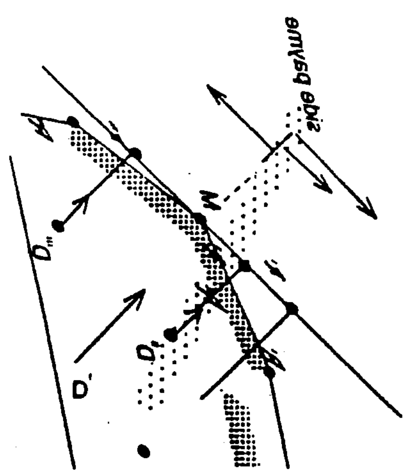

Figure 3-16. Resolution of disagreements.

Anticipating this split-the-difference resolution process, the players will want to make threats p and q that are optimal in the constant-sum game with payoffs given (in mixed strategies) by $[R_{\mathrm{I}}, R_{\mathrm{II}}]$. This is, of course, strategically equivalent to the zero-sum *difference game* Δ, so we solve that game instead.

If $p^{\#}$ and $q^{\#}$ are optimal mixed strategies of Δ, then the resulting "optimal disagreement point" satisfies

$$D_{\mathrm{I}}(p^{\#}, q^{\#}) - D_{\mathrm{II}}(p^{\#}, q^{\#}) = \delta,$$

by (3.12). The associated resolution point $R(p^{\#}, q^{\#})$ (whose payoff difference is also δ) is then taken as the definition of the *TU value*, denoted φ or $\varphi([A, B])$, and we see that φ is characterized by the easy-to-remember pair of equations

$$\varphi_{\mathrm{I}} + \varphi_{\mathrm{II}} = \sigma \quad \text{and} \quad \varphi_{\mathrm{I}} - \varphi_{\mathrm{II}} = \delta, \tag{3.14}$$

which in turn yield the direct formulas

$$\varphi_{\mathrm{I}} = \frac{\sigma + \delta}{2} \quad \text{and} \quad \varphi_{\mathrm{II}} = \frac{\sigma - \delta}{2}. \tag{3.15}$$

Remarks.

1. We have approached this definition through a rather lengthy and cautious argument. But the end result is not only mechanically very simple, but also intuitively quite plausible. And when we translate (3.14) into words it even sounds good: The *sum* of the values is the *max-max* value of the *sum* of $A+B$. The *difference* of the values is the *minimax* value of the *difference* of $A - B$.

2. If the optimal threats $p^{\#}$ or $q^{\#}$ are not unique, then in general neither is the disagreement point $D(p^{\#}, q^{\#})$. But the value of the game Δ *is* unique, so the optimal disagreement points must all lie on the same 45° line and determine the same TU value φ.

3. The fact that Δ is a zero-sum matrix game enables us to ignore the question of who makes the first threat. The minimax theorem for zero-sum games implies that there is no advantage or disadvantage in having to declare first. Of course, the randomization for a mixed-strategy threat should not be carried out until the game is actually played.

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Examples.

As at the beginning of this chapter, we first consider two extreme bimatrix types:

1. Suppose first that $[A, B]$ happens to be zero-sum—i.e., that $B = -A$. Then Σ is a matrix of zeroes, so $\sigma = 0$. On the other hand, Δ is $2A$, which is strategically equivalent to A. So the optimal threats are just the optimal strategies of A and $\delta = 2v(A)$. By (3.15), the TU value is $\varphi = (v(A), -v(A))$.

This is as it should be. It is to be expected that a cooperative solution will reduce to the antagonistic minimax solution in a game where there is nothing to be gained from cooperation.

2. The opposite effect is produced by a "game of pure coordination" with $A = B$. In this case Σ is $2A$ and Δ is all zeroes. So $\delta = 0$ and σ is double the maximum element of A. There are no meaningful threats and no reason for any side payment, so φ gives each player the largest number in the matrix. Again, this is as it should be.

3. We next dispose of "The Apartment" (see p. 41). The auxiliary matrices are

$$\Sigma = \begin{array}{c|c|c|c|} & \text{R} & \text{G} & \text{N} \\ \hline \text{R} & — & 600 & 0 \\ \hline \text{G} & 500 & — & 0 \\ \hline \text{N} & 0 & 0 & 0 \\ \hline \end{array}\,, \qquad \Delta = \begin{array}{c|c|c|c|} & \text{R} & \text{G} & \text{N} \\ \hline \text{R} & — & 600 & 0 \\ \hline \text{G} & -500 & — & 0 \\ \hline \text{N} & 0 & 0 & 0 \\ \hline \end{array}\,,$$

where the undetermined entries "—" are all ≤ 600. So $\sigma = 600$ and $\delta = 0$, giving us the following, detailed TU solution:

- cooperative strategy : $\langle$R, G$\rangle$
- optimal threats : N and N
- disagreement point : (0, 0)
- TU cooperative value : (300, 300)
- side payment($\varphi - M$) : I pays 300 to II,

confirming what we said at the end of §3.6.

4. This example is more typical of the general case than the preceding three, and has a more interesting pre-play:

$$A = \boxed{\begin{matrix} 30 & 50 \\ 60 & 0 \end{matrix}}, \qquad B = \boxed{\begin{matrix} 20 & 30 \\ 10 & 10 \end{matrix}}.$$

The auxiliary matrices are:

$$\Sigma = \boxed{\begin{matrix} 50 & 80 \\ 70 & 10 \end{matrix}}, \qquad \Delta = \boxed{\begin{matrix} 10 & 20 \\ 50 & -10 \end{matrix}}.$$

We calculate that $\sigma = 80$ and $\delta = \frac{110}{7}$, and using (3.15), we obtain the TU value:

$$\varphi = (\tfrac{335}{7}, \tfrac{225}{7}) \approx (47.86, 32.14).$$

To achieve this result, I and II agree (a) to use the pure strategy pair $\langle 1, 2\rangle$ that yields $M = (50, 30)$, and (b) to transfer the amount $\frac{15}{7} \approx 2.14$ from I to II. Figure 3–17. illustrates.

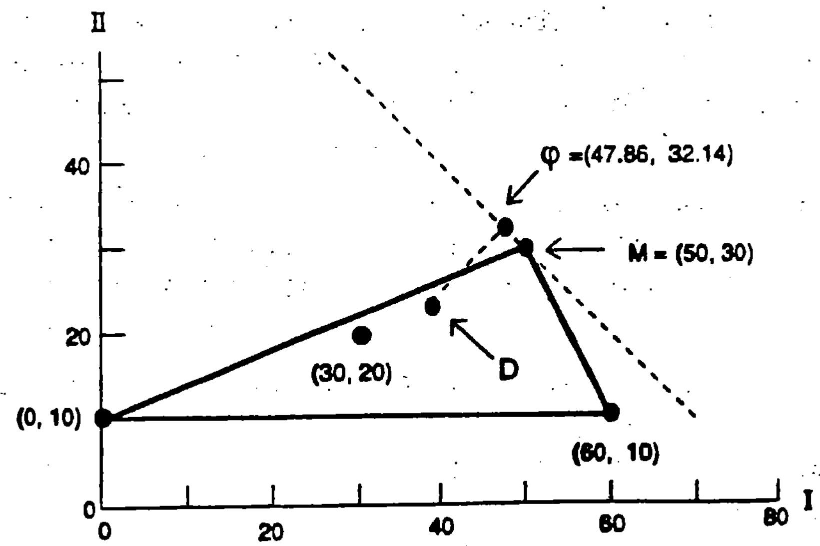

Figure 3-17. Example 4.

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Moreover, the optimal threats are unique:

$$p^{\#} = (\tfrac{6}{7}, \tfrac{1}{7}), \qquad q^{\#} = (\tfrac{3}{7}, \tfrac{4}{7})$$

and yield the disagreement point

$$D = (\tfrac{1920}{49}, \tfrac{1150}{49}) \approx (39.18, 23.47).$$

As a check, we observe that the surplus of M over D is $(50+30) - (39.18 - 23.47) = 17.35$ and that the TU value does indeed split it equally.

5. Our final example illustrates two different types of non-uniqueness:

$$A = \begin{bmatrix} 8 & -1 & 6 \\ 2 & 7 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 4 & 4 \end{bmatrix}.$$

$$\Sigma = \begin{bmatrix} 11 & -2 & 10 \\ 6 & 11 & 9 \end{bmatrix} \qquad \Delta = \begin{bmatrix} 5 & 0 & 2 \\ -2 & 3 & 1 \end{bmatrix}.$$

$$\sigma = 11 \qquad \delta = 1.5$$

$$\begin{aligned} \varphi &= (6.25, 4.75) \\ p^{\#} &= (.5, .5) \\ q^{\#} &= (.3, .7, 0) \quad \underline{\text{or}} \quad (0, .25, .75) \\ D &= (3.6, 2.1) \quad \underline{\text{or}} \quad (4.875, 3.375). \end{aligned}$$

First note that there are two points marked D in the figure, each claiming to be the optimal disagreement point. In fact, every point on the segment $\overline{DD}$ is an optimal disagreement point.[6] Nevertheless, all lead to the same resolution. In general, I's optimal threats force the outcome to lie ***on or below*** the 45° line through $\overline{DD}$, no matter what II does, while II's optimal threats force it to lie ***on or above*** that line, no matter what I does. Since φ depends only on that line, not on D's position in it, φ is unique even though D is not.

[6]Why? See Proposition 3 in §2.6.

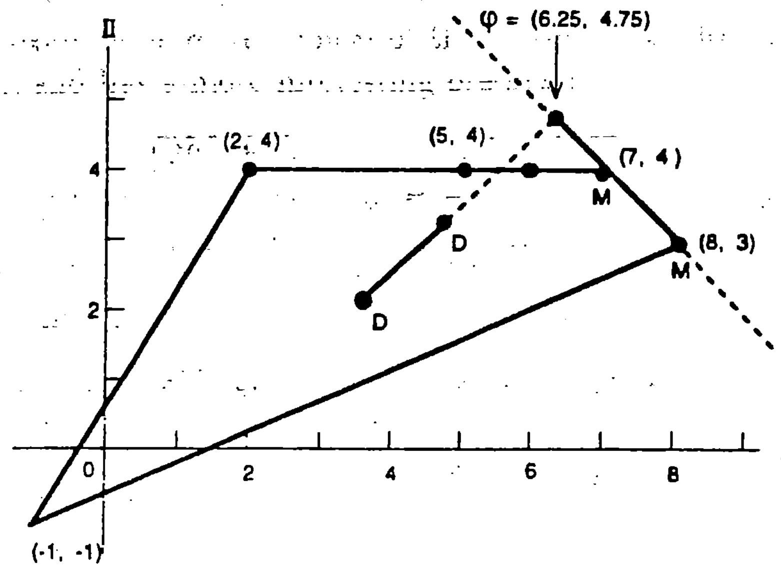

Figure 3-18. Nonunique D and M.

Now note that there are also two points marked M in the figure. This merely means is that there are two $\langle i, j\rangle$ pairs in the bimatrix that maximize the total payoff, namely $\langle 1, 1\rangle$ and $\langle 2, 2\rangle$. They are equally effective as joint strategies, but note that the *side payment*, which is defined to be the vector $\varphi - M$, depends on which one is chosen. At the upper M the side payment is $(-.75, .75)$ (i.e., I promises to pay .75 to II), while at the lower M it is $(-1.75, 1.75)$.

Actually, all the points on the segment $\overline{MM}$ are equally effective, but in general they require a joint mixed strategy instead of a simple $\langle i, j\rangle$. If, however, φ had happened to lie between the two M's, we could reduce the side payment to zero by using the right joint mixed strategy. This useful observation will be important later, when it is time to consider the NTU case.

Another point of interest in this example (along with Example 4) is that II's negotiated payoff of 4.75 (or 47.86 in Example 4), is strictly higher than any payoff in her matrix, B! This may seem strange at first, but II's advantage is due to her control over I's situation. She is in a position to perform—or refuse to perform—a service of some value, and so is entitled to something in return.[7]

[7]For another, more drastic example of this effect, see Exercise 3–18(d).

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EXERCISES

Exercise 3.18 Find the TU values, the associated side payments and the optimal threats for the following bimatrix games:

(*a*)

3, 2	5, 1
0, 0	1, 4

(*b*)

3, 2	1, 4
0, 0	5, 1

(*c*)

7, 4	2, 8
4, 5	5, 6

(*d*)*

−1, 0	3, 0
−3, 0	7, 0

*Cf. Exercise 3.3 in §3.2.

Exercise 3.19 Solve the following bimatrix game, and make a diagram for it in the style of Figures 3–17 and 3·18.

1, 3	3, 1	2, 2
2, 1	2, 3	1, 2
1, 1	3, 2	3, 3

Exercise 3.20 Prove or disprove: If the maximum elements of both A and B occur at the same place, say $\langle i^*, j^*\rangle$, then the TU value of $[A,\ B]$ is $\varphi = (a_{i^*j^*},\ b_{i^*j^*})$.

Exercise 3.21 Show that all optimal disagreement points lie on the 45° line through φ. (Cf. Remark 2 above.)