LANDLORD L + n workers

A

$$v(S) = \begin{cases} 0 & \text{if } L \notin S \\ f(k) & \text{if } L \in S \text{ and } \underbrace{|S| = k+1}_{S \text{ has } k \text{ workers}} \end{cases}$$

$$\underbrace{\_\ \_\ \_\ \_}_{\substack{k \\ \text{workers}}}\ \ \overset{L}{\downarrow}\ \ \underbrace{\_\ \_\ \_\ \_\ \_}_{\substack{n-k \\ \text{workers}}}$$

- In every such order, cont of L = $f(k)$

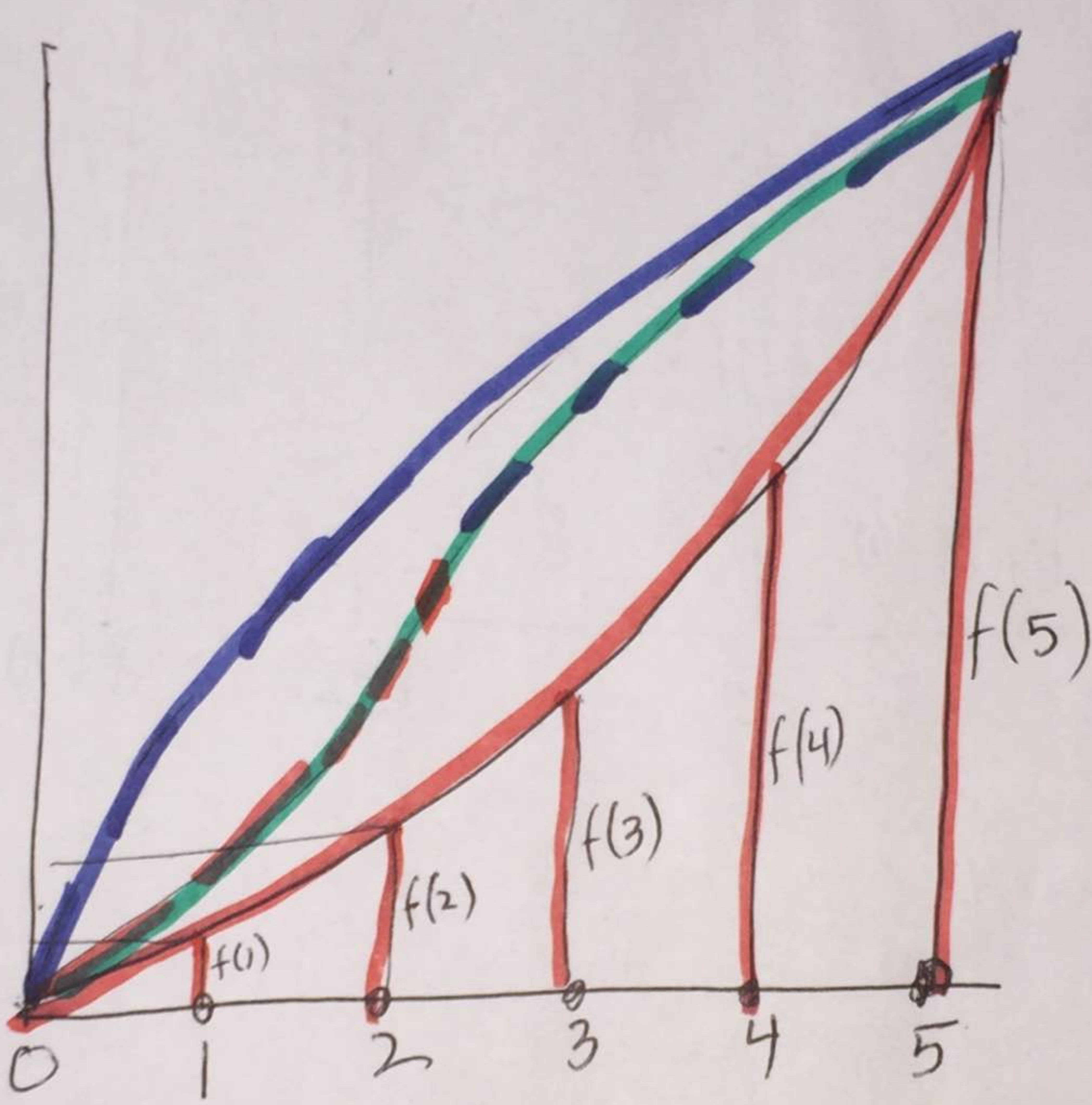
Q HOW MANY SUCH orders?

A $n!$

So $$\varphi_L(v) = \frac{n!}{(n+1)!}\left[f(0) + f(1) + \cdots + f(n)\right]$$

$$= \frac{1}{n+1}\left[f(0) + f(1) + \cdots + f(n)\right]$$

$$\left(= \frac{1}{11}\left[0^2 + 1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2\right] \right)$$



$\varphi_L(v) =$ Average value of f on $0, 1, 2, 3, 4, 5$

$$\varphi_W(w) = \frac{f(5) - \varphi_L(v)}{5}$$

(c)

2 Landlords L_1, L_2 + n workers.

$$v(S) = \begin{cases} f(k) & \text{if } L_1 \in S,\ L_2 \in S,\ \#\text{workers in } S = k \\ 0 & \text{if } L_1 \notin S \text{ or } L_2 \notin S. \end{cases}$$

Cont of L_2 $= f(k)$

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{L_1 + k \text{ workers}}\ L_2\ \underbrace{\_\ \_\ \_\ \_\ \_\ \_}_{n-k \text{ workers}}$$

HOW MANY SUCH ORDERS?

$n!$ $L_1\ \_\ \_\ \_\ \_\ \_\ L_2\ \_\ \_\ \_\ \_\ \_\ \_$

$n!$ $\_\ L_1\ \_\ \_\ \_\ \_\ L_2\ \_\ \_\ \_\ \_\ \_\ \_$

$n!$ $\_\ \_\ L_1\ \_\ \_\ \_\ L_2\ \_\ \_\ \_\ \_\ \_\ \_$

etc.

$\vdots$

$n!$ $\_\ \_\ \_\ \_\ \_\ L_1\ L_2\ \_\ \_\ \_\ \_\ \_\ \_$

$(k+1)\,n!$ such orders.

D

$$\frac{\sum_{k=0}^{n} n!\,(k+1)\,f(k)}{(n+2)!} = \frac{n!}{(n+2)!}\sum_{k=0}^{n}(k+1)f(k)$$

$$= \frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}(k+1)f(k)$$

$$\frac{1}{(11)(12)}\left[1f(0) + 2f(1) + \cdots + 11f(10)\right] = \cdots$$

$$\frac{1}{11 \times 12}\left[1.0^2 + 2.1^2 + 3.2^2 + 4.3^2 + \cdots + 11.10^2\right]$$

E

$$v(S) = \begin{cases} f(k) & \text{if } k + \text{workers} + \text{at least one landlord} \\ 0 & \text{if } L_1 \notin S \text{ and } L_2 \notin S. \end{cases}$$

$$\underbrace{\_\ \_\ \_\ \_}_{k \text{ workers}}\ L_2\ \underbrace{\_\ \_\ \_\ \_\ \_}_{n-k \text{ workers } + L_1}$$

Cont of L_2 is $f(k)$

$\_\ \_\ \_\ L_2 L_1 \_\ \_\ \_\ \_$

$\_\ \_\ \_\ L_2 \_\ L_1 \_\ \_\ \_\ \_$

$\_\ \_\ \_\ L_2 \_\ \_\ L_1 \_\ \_$

etc

\# such orders

$n!\,(n-k+1)$

$$\varphi_i(L_1) = \frac{1}{(n+2)!} \sum_{k=0}^{n} n!\,(n-k+1)\,f(k)$$

$$= \frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} (n-k+1)\,f(k)$$

$$= \frac{1}{11 \times 12}\left[11 \cdot 0^2 + 10 \cdot 1^2 + 9 \cdot 2^2 + \cdots + 2 \cdot 9^2 + 1 \cdot 10^2\right]$$

$$\left.\begin{array}{l} 2^N \xrightarrow{v} \mathbb{R} \\ S \longrightarrow v(S) \end{array}\right\} \text{GAME}$$

DEF^n v is called a SIMPLE GAME (or, voting game) if

(i) $v(S) = 0$ or 1 $\left(\text{i.e. } 2^N \xrightarrow{v} \{0,1\}\right)$

(ii) $\left.\begin{array}{l} S \subset T \\ v(S)=1 \end{array}\right\} \Rightarrow v(T)=1$

(iii) $v(N) = 1$

ALTERNATIVE DEF^n

$$2^N \xrightarrow{v} \{W, L\}$$

(ii)$'$ $\left.\begin{array}{l} v(S)=W \\ S \subset T \end{array}\right\} \Rightarrow v(T)=W$

(iii) $v(N) = W$

W and L not numbers here

$\varphi(v)$ is defined as before

Order	Cont 1	Cont 2	Cont 3	Cont 4	Cont 5	Con 6
.						
.						
2 6 5 4 3 1	0	0	0	1	0	0

4 IS PIVOTAL in 265431

(HE CHANGES A LOSING COALITION $\{2,6,5\}$ to WINNING BY JOINING IT)

8.

So

$$\boxed{\begin{aligned}\varphi_i(v) &= \frac{\#\text{ times } i \text{ is pivotal (across all orders)}}{n!} \\ &= \text{probability that } i \text{ is pivotal}\end{aligned}}$$

EXAMPLES

Weighted voting games

$$\left[51\,;\ \overset{1}{49},\ \overset{2}{48},\ \overset{3}{3}\right] \Rightarrow \begin{array}{l} v(i)=0 \\ v(ij)=1 \\ v(123)=1 \end{array}$$

$$\left[3\,;\ \overset{1}{2},\ \overset{2}{2},\ \overset{3}{1}\right]$$

$$\left[4\,;\ \overset{1}{2},\ \overset{2}{2},\ \overset{3}{2},\ \overset{4}{1}\right]$$

Gen^l notation $\{q\,;\ w_1, \ldots, w_n\}$

(WEIGHTED VOTING GAME)

For $S \subset \{1, \ldots, n\}$

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

SECURITY COUNCIL

5 permanent members

10 non-permanent members

~~A~~

$$v(S) = \begin{cases} W, & \text{if (i) all five permanent members are in } S \\ & \text{(ii) at least four non-permanent members are in } S \\ L, & \text{otherwise} \end{cases}$$

Each of the 5 permanent members is a VETO player

DEF^n Player i is a veto player in the simple game v if:

$\boxed{i \notin S \implies v(S)}$ $\Longleftrightarrow$ $\boxed{\text{i is NECESSARY for S to WIN}}$

Exercise A simple gave v has a non-empty core if, and only if, it has a nonempty set V of veto players; and in this case

$$\text{Core } v = \{ x \in \mathbb{R}^N_+ : \sum_{i \in V} x_i = 1, \ x_j = 0 \text{ for } j \in N \setminus V \}$$

Interpretation "MOST" voting games have empty cores.

SECURITY COUNCIL

All Permanent (P) members are symmetric to each other

Similarly All non-permanent members are symmetric to each other

So, by symmetry & efficiency axioms,

$$5\varphi_P + 10\varphi_{NP} = 1$$

· We can know everyone's value from just one player's value

<u>CHOICE</u> compute φ_P

OR

compute φ_{NP}.

SECURITY COUNCIL

6B

$$\underbrace{-\ -\ -\ -\ -}_{4\ \text{PER} + k\ \text{Non-per}}\ \overset{\text{CHINA}}{\downarrow}\ \underbrace{-\ -\ -}_{10-k}\ \Bigg\}\ \begin{matrix}\text{CHINA}\\ \text{PIVOTAL}\end{matrix}$$

$k \geq 4$

So # such orders

$$\binom{10}{k}(4+k)!\,(10-k)!$$

So

$$\varphi_{CH}(v) = \frac{\sum_{k=4}^{10}\binom{10}{k}(4+k)!\,(10-k)!}{15!}$$

= TOO LONG TO COMPUTE !!

ALTERNATIVE CALCULATION for φ_{NP}

INDIA (non-per)

$\downarrow$

_ _ _ _ _ _ _ _ _ _ _ _ _ _

5 per + 3 non per 6 non-per

HOW MANY SUCH ORDERS

$$\binom{9}{3}\, 8!\, 6!$$

So

$$\varphi_{INDIA} = \left(\frac{9!}{3!\,6!}\, 8!\, 6!\right) \frac{1}{15!}$$

$$\varphi_{CHINA} = \frac{1 - 10\varphi_{INDIA}}{5}$$

$$\text{Calculated} = \varphi_{INDIA} = \frac{4}{15 \times 13 \times 11} \approx 0.00186$$

$$\Longrightarrow \left.\begin{aligned} \varphi_{NP} &= 0.00186 \\ \varphi_P &\approx 0.1963 \end{aligned}\right\} \varphi_P > 10\varphi_{NP}$$

VARY RULES

eg.

Need 5 permanent + k Non-per

(a) $k \geq 3$

(b) $k \geq 5$

etc.

How does ρ_{NP} vary with k?

(<u>Intuition:</u> If $k = 10$
then $\rho_P = \rho_{NP}$?)

Does $\rho_{NP} \uparrow$ as $k \uparrow$?

Fix $N = \{1, \cdots, n\}$

Let $\mathcal{C} \equiv$ set of all simple games on N.

NOTE If $v \in \mathcal{C}$ and $w \in \mathcal{C}$, then

$$v + w \notin \mathcal{C}$$

(e.g. $(v+w)(N) = v(N) + w(N) = 1 + 1 = 2$)

SO ADDITIVITY

$$\varphi(v+w) = \varphi(v) + \varphi(w)$$

HAS NO MEANING IF WE LOOK AT

$$\mathcal{C} \xrightarrow{\varphi} \mathbb{R}^N$$

$$v \longrightarrow \varphi(v) = (\underbrace{\varphi_1(v)}_{\text{voting power of 1 in } v}, \cdots, \underbrace{\varphi_n(v)}_{\text{voting power of } n \text{ in } v})$$

REPLACE ADDITIVITY AXIOM BY

- CHANGE IN VALUE DEPENDS ONLY ON THE CHANGE IN THE GAME

= • $\Delta[\varphi(v)] = F[\Delta v]$ for some function F

= $(*)$ $\left.\begin{array}{l} v \geq v' \\ w \geq w' \\ v - v' = w - w' \end{array}\right\} \Rightarrow \varphi(v) - \varphi(v') = \varphi(w) - \varphi(w')$

RE-EXPRESS $*$ AS FOLLOWS.

For $v, w \in \mathcal{C}$
define $v \vee w \in \mathcal{C}$ and $v \wedge w \in \mathcal{C}$ by

$$(v \vee w)(S) = \begin{cases} W & \text{if } v(S) = W \text{ OR } w(S) = W \text{ or both} \\ L & \text{otherwise} \end{cases}$$

$$(v \wedge w)(S) = \begin{cases} W & \text{if } v(S) = W \text{ AND } w(S) = W \\ L & \text{otherwise} \end{cases}$$

MODULARITY

$$\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$$

for all $v, w \in \mathcal{C}$

$\downarrow$

REPLACES ADDITIVITY

<u>Thm</u> There is a unique map

$$\mathcal{C} \xrightarrow{\varphi} \mathbb{R}^N$$

satisfying Dummy, Symmetry, Efficiency & Modularity and it is the Shapley value.

See my 1975 paper

$$\mathcal{C}_s = \{ v \in \mathcal{C} : v \text{ is superadditive} \}$$

$$v(S \cup T) \geq v(S) + v(T)$$

i.e. two disjoint coalitions cannot both win.

Now $v \in \mathcal{C}_s$, $w \in \mathcal{C}_s \not\Rightarrow v \vee w \in \mathcal{C}_s$

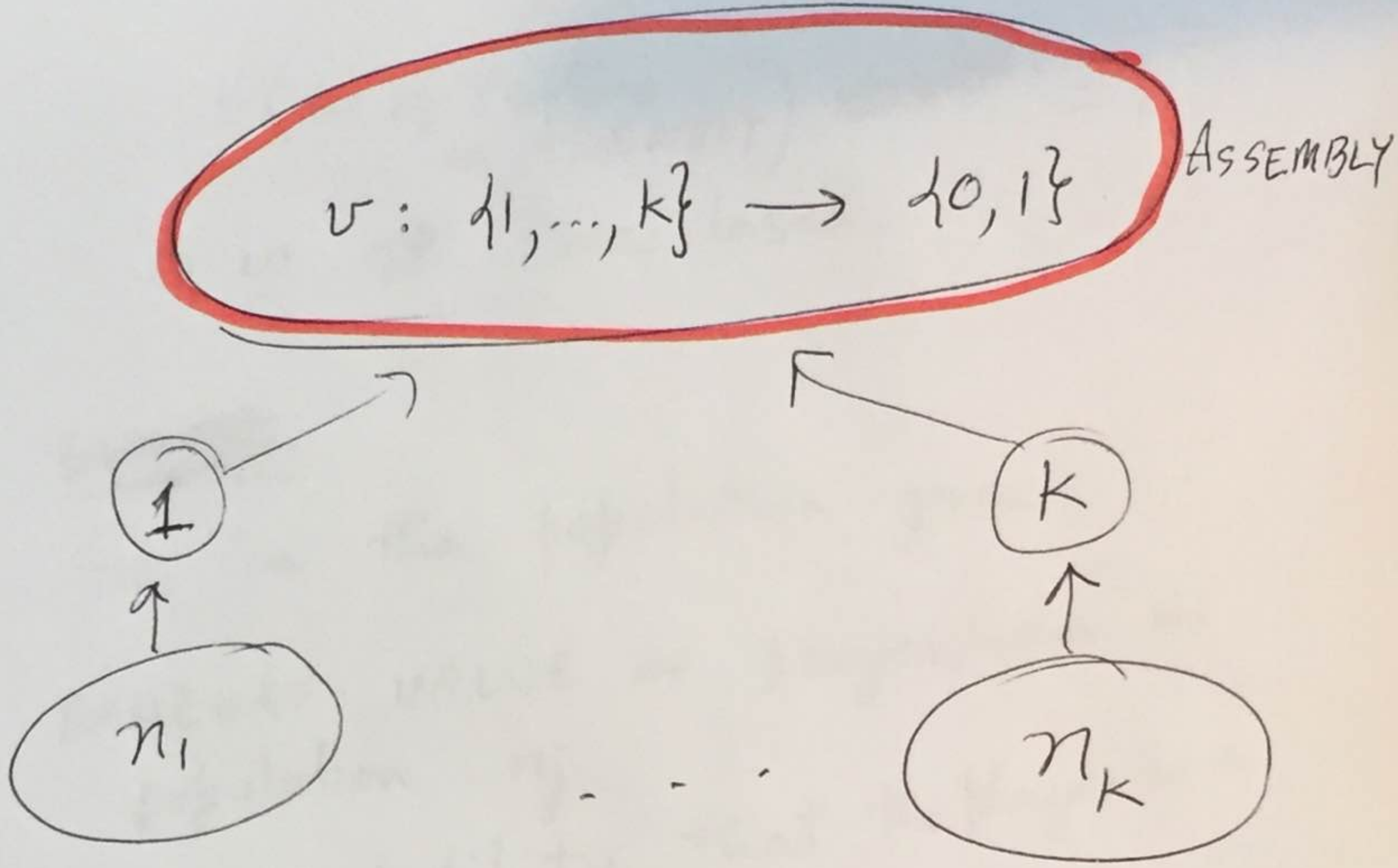
So write

IF $v \in \mathcal{C}_s$, $w \in \mathcal{C}_s$ and $v \vee w \in \mathcal{C}_s$

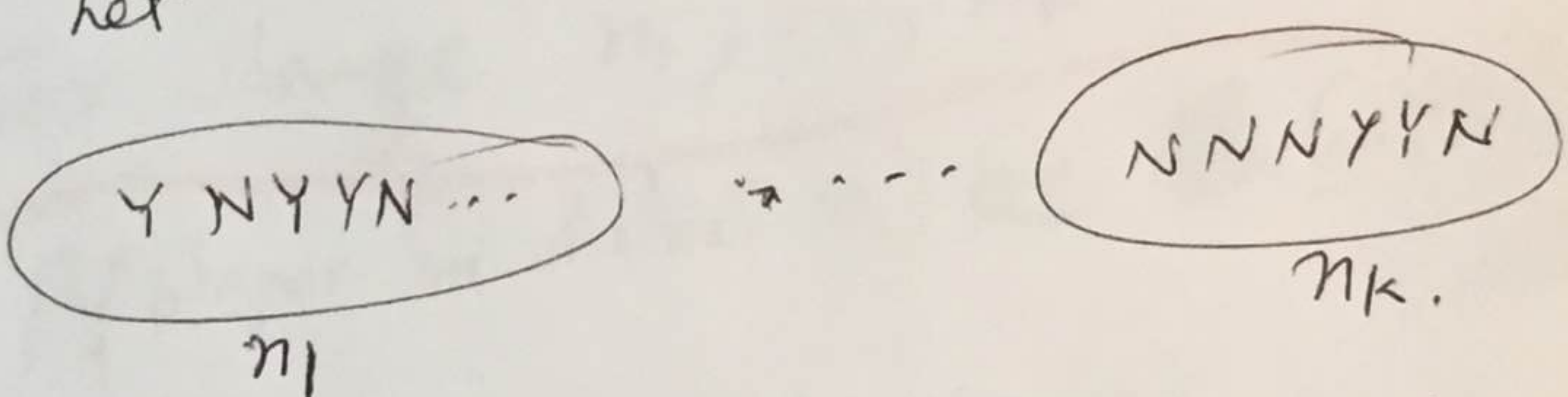
** **THEN** $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$

Thm There is a unique map etc (as before), replacing * by **

BANZHAF



Let



If 50% or more n_j = Y $\Rightarrow$ rep j votes Y
otherwise he votes no.

Issue wins if

v (set of Y voters in ASSEMBLY) ~~is~~ $= 1$

o.w ~~it~~ issue loses.

~~WANT~~

So in the population game

BANZHAF VALUE OF player/voter in population n_j

= probability that the player is a "swinger"

For large $n_1, \ldots, n_K$

SQUARE ROOT RULE

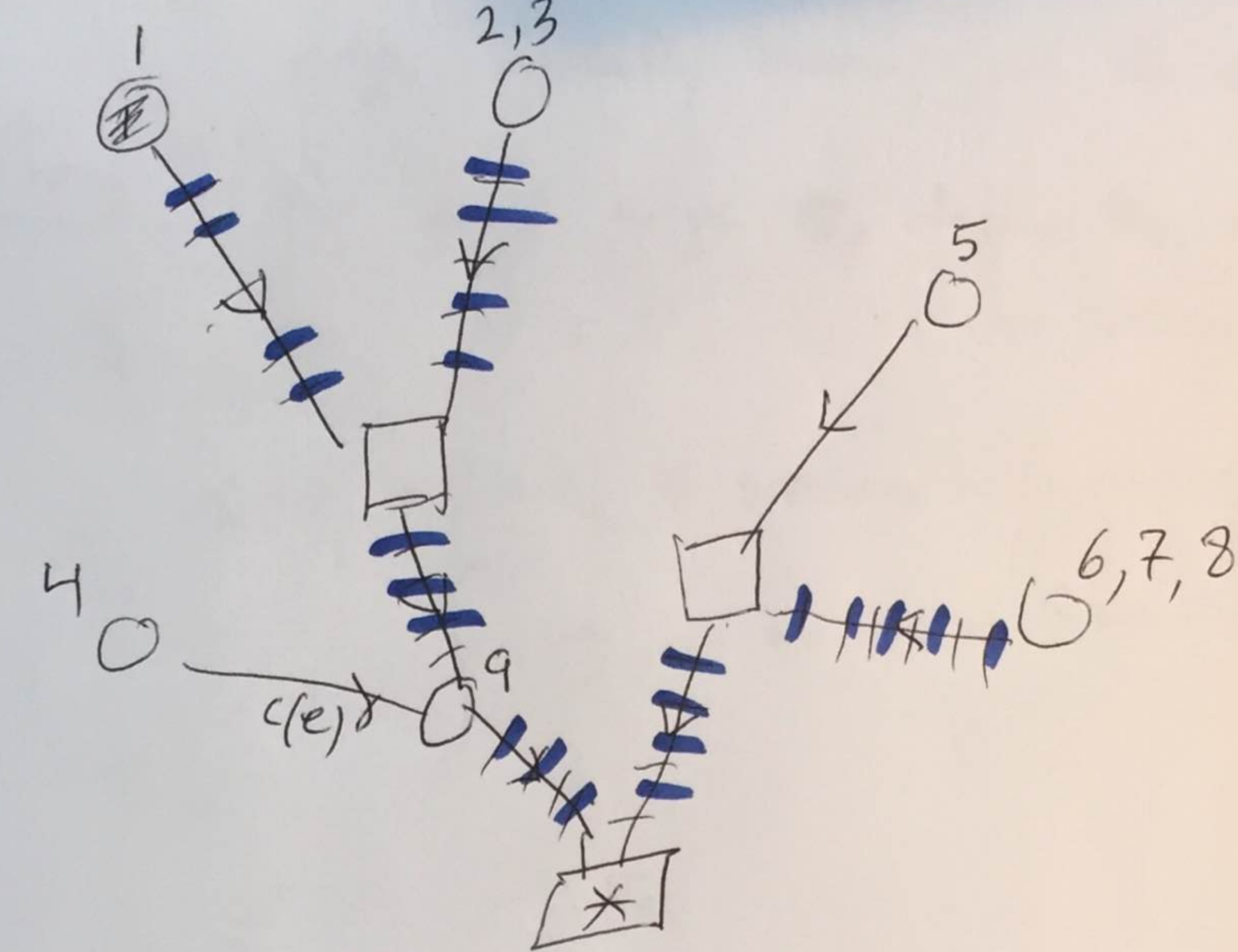
β_A(player in district n_i) $\approx$ ~~Const~~

$\approx$ Constant $\dfrac{\beta_i(v)}{\sqrt{n_i}}$

where β_i = Banzhaf value of representative i in the game v

$$\boxed{\beta_i(v) \approx \sum_{S \subset N \setminus \{i\}} v(S \cup i) - v(S)}$$

DATA TRANSMISSION GAME



$(4,9) = e$

$c(e) =$ cost of using the edge 4,9

$v(1,2,3,8) =$ cost of tree connecting 1, 2, 3, 8 to $\boxed{*}$

$=$ Sum of the cost of all edges marked

Claim Shapley value = divide cost of each edge equally among all its users

Proof For each edge **e**, define the game v_e on $N = \{1, \dots, 14\}$ as follows:

$$v_e(S) = \begin{cases} -c_e & \text{if someone in } S \text{ uses } e \\ 0 & \text{if no one in } S \text{ uses } e \end{cases}$$

Then

$$v = \sum_{e \in E} v_e$$

$$\Rightarrow \varphi(v) \underset{\text{additivity axiom}}{=} \sum_{e \in E} \varphi(v_e) \qquad (*)$$

But users of e are ~~sym~~ substitutes in V_e
And non-users of e are dummies in V_e
Therefore, denoting users of e by $S(e)$,

$$\varphi_i(v_e) = \begin{cases} \frac{c_e}{|S_e|} & \text{if } i \in S_e \\ 0 & \text{o.w} \end{cases}$$

The claim now follows from $(*)$.

A3

SPECIAL CASE

AIRPORT GAME (Littlechild & Thompson)