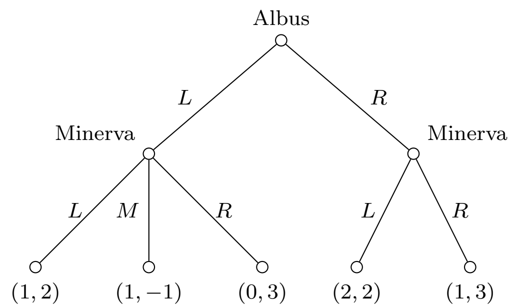
Game theory practice problems

Eco500/Shmaya

These problems are *not* sample exam problems

Problem 1

Find the backward induction solution of the following game (in each pair of payoffs, the first number is Albus's payoff and the second is Minerva's payoff)



Problem 2

Consider the following simultaneous move game.

	L	C	R
T	5,2	1,6	9,4
M	8,1	2,7	0,5
B	4, 10	0,0	1,7

a. Explain why strategy M of the row player is *not* a dominant strategy.

b. Explain why the configuration (T, R) is *not* a Nash equilibrium.

c. Which strategies survive sequential elimination of dominated strategies ?

d. What are the Nash Equilibria in the game ?

Problem 3

The five Dukes of Earl are scheduled to arrive at the royal palace on each of the first five days of May. Duke One is scheduled to arrive on the first day of May, Duke Two on the second, etc. Each Duke, upon arrival, can either kill the king or support the king. If he kills the king, he takes the king's place, becomes the new king, and awaits the next Duke's arrival. If he supports the king, all subsequent Dukes cancel their visits. A Duke's first priority is to remain alive, and his second priority is to become king. Who is king on May 6?

Problem 4

Mark Twain and Stephan King are visiting an italian restaurant, and the owner offers both of them a free eight-slice pizza under the following condition. Each of them must simultaneously decide how many slices to request, between 0 and 8. Suppose Twain requests t slices and King requests k slices. If $t + k > 8$ the owner will call them greedy scoundrels and kick them out of his restaurant with no slices. If $t + k \leq 8$ then each player gets his request.

a. Explain why requesting 5 slices in *not* a dominant strategy.

b. Explain why requesting 5 slices in *not* a dominated strategy.

c. What are the Nash Equilibria in the game ?

Problem 5

Two firms, Harbor Mattresses (HM) and Bravos Carpets (BC), each has one job opening. HM offers wage 8 and BC offers wage 6.

Consider a simultaneous move game between two workers. Each worker can apply to a position in one of the two firms. If only one worker applies to a firm, that worker gets the job; if both workers apply to a firm, the firm hires one worker at random, and the other worker is unemployed (which gives the worker a payoff of zero).

a. Write down the game matrix between the workers.

b. Which game from the canonical games does this game resemble?

c. Find all the Nash equilibria in the game (pure and mixed).

Problem 6

The following table shows the probability of the server winning the point in tennis conditioned on the server's type of service (forehand (SF) or backhand (SB)) and the receiver's defense (forehand (DF) or backhand (DB)).

	DF	DB
SF	70%	90%
SB	80%	70%

a. Arthur Dent is a tennis champion who always serves backhand. Which defense should Arthur's opponent use? What would be the probability that Dent will win the point?

b. Ford Perfect always tosses a fair coin to decide on his type of service. Which defense should Ford's opponent use? What would be the probability of Perfect will win the point?

c. Trillian Astra asks your advice for the mixing probability she should use when choosing between SF and SB. Assuming that her mixing probabilities become known, and optimally responded to by the receiver, how should she mix?

Problem 7 (The Volunteer's Dilemma)

Your company has embarked on a major project. The CEO loves the project, but everybody else knows it's a gigantic waste of time and money. If somebody tells the CEO about the mistake each member of the company gets payoff 1000. However a person who delivers the bad news the CEO incurs a cost 400 so that person's payoff is 1000-400=600. If nobody tells the CEO the project continues as planned and everbody gets 0.

a. Assume first there are $n = 2$ workes in the firm (in addition the CEO). Write dowen the game matrix in a simultaneous move game between the two workers, each can either tell the CEO or keep silent.

b. What are the Nash Equilibria in pure strategies ?

c. Find a Nash equilibria in mixed strategies under which each worker tell with some probability $0 < p < 1$.

d. Do the same questions where the number of workers is $n = 10$.

e. What are the probabilities that the project will happen under the mixed Nash Equilibrium in the case of $n = 2$ and $n = 10$?

Problem 8 (Prisoner's dilemma+stag hunt)

Consider a two-player game that is played in two rounds. In the first round, the players play prisoner's dilemma game. In the second round, they play a stag hunt game. The payoff matrices are given below. The payoff to each player in the two-round game is the sum of the payoffs she received in each game.

a. Is there a subgame-perfect Nash Equilibrium in the two-rounds game under which both players play Cooperate in the first round?

b. Answer the same question for the two-rounds game in which the players don't observe opponent's action in the first round when they choose their action in the second round.

	Cooperate	Defect
Cooperate	3,3	0,4
Defect	4,0	1,1

	Stag	Hare
Stag	5,5	0,3
Hare	3,0	3,3

Problem 9

Two firms have to make a simultaneous decision whether to lobby or not. If both firm lobby or neither firm lobbies then the government will make a decision that yields a payoff 20 to each firm. If one firm lobbies and the other firm does not lobby then the government will make a decision that yields a payoff 30 to the firm that lobbied and 0 to the firm that did not lobby.

The cost of lobbying of each firm is random between 0 and 60 with all numbers equally likely. Each firm knows its own cost but not the other firm's cost. Therefore, this is a game of asymmetric information.

a. You are one of these firms and you know that your opponent's strategy is to lobby if its cost is below 30. How will you play?

b. Find the Nash Equilibrium in the game.

Problem 10

Rivendell is a small town in middle earth with two bars. Each bar can charge a price per drink of $2, $4, or $5. There is a pool of 1000 customers in town, 600 are tourists who will pick a bar randomly, and 400 are natives who select the bar with the lowest price (or selects randomly if both bars charge the same price). Both bars have zero production cost.

a. Write down the game matrix for the one-shot, simultaneous move game between the bars

b. Which strategies survive iterated elimination of dominated strategies in this game?

Problem 11

A firm has two divisions, each of which has its own manager. Managers of these divisions are paid according to their effort in promoting productivity in their divisions. The payment scheme is based on a comparison of the two outcomes. If both managers have expended the same level of effort, each earns $100,000 a year. But if one of the two managers shows "high effort" whereas the other shows "low effort," the "high effort" manager is paid $130,000, and the second ("low effort") manager gets a reduced salary of $60,000. A manager that expends high effort bears a cost equivalent to $20,000. Managers make their effort decisions independently and without knowledge of the other manager's choice.

a. Write the game matrix between the managers.

b. Which game from the following games does this game resemble?

   (a) Prisoner's Dilemma
   (b) Stag Hunt
   (c) Battle of the sexs
   (d) Matching pennies

c. Find all the Nash Equilibria (mixed and pure) in the one-shot game

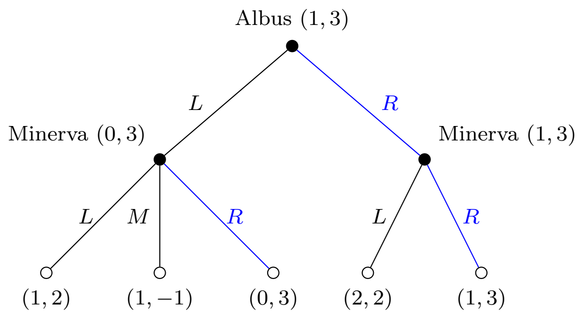
d. Write explicitly, in the context of the problem, what grim trigger strategy means in the repeated game in which the managers play the one-shot game every year.

e. Consider the infinitely repeated game with interest rate r, so that \$1 next year equals $\$1-r$ today. For which values of r is it in equilibrium for both players to play grim trigger?

Solutions

Solution 1

The blue edges represents the choices of the players in each node. According to the solution, Albus chooses R and then Minerva chooses R.



Solution 2

a. A strategy is dominant if it is better than any other strategy, regardless of what the oponent does. In this case, if the column player plays R then playing T is better than playing M for the row player.

b. The elimination process goes as follow:

   (a) Eliminate B (Dominated by T)

   (b) Eliminate R and L (they became dominated by C)

   (c) Eliminate T (became dominated by M)

   Now the only remaining strategies are M and C.

c. The Unique Nash equilibrium in the game is (M,C). (It is always the case that when there is only one strategy remaining for each player after sequential elimination of dominant strategies, the resulting configuration of strategies is a Nash equilibrium)

Solution 3

This is a sequential move game and we solve it using backward induction:

- If duke 5 arrives to the palace, he will king the king and become the new king
- If duke 4 arrives to the palace, he will support the king (since duke 4 knows that otherwise duke 5 will kill him)
- If duke 3 arrives to the palace, he will kill the king (since duke 3 knows that duke 4 will support him)

- If duke 2 arrives to the palace, he will support the king (since duke 2 knows that otherwise duke 3 will kill him)
- When duke 1 arrives to the palace, he will kill the king (since duke 1 knows that duke 2 will support him)

Solution 4

a. A strategy is dominant if it is better than any other strategy, regardless of what the oponent does. In this case, if for example the oponent requests 6 slices then it is better to request 2 than 5.

b. A strategy A is dominated by a strategy B if B gives higher payoff than A regardless of what the opponent does. But in this game, if the opponent requests 3 slices then there is no strategy that is better than requesting 5.

c. There are 9 Nash Equilibrium configurations: $(0,8),(1,7),(2,6),(3,5),(4,4),(5,3),(6,2),(7,1),(8,0)$. For example $(1,7)$ means that Twain requests 1 and King requests 7. Given the opponent's strategy, no player can gain from deviating.

Solution 5

a.

	HM	BC
HM	4,4	8,6
BC	6,8	3,3

b. Battle of the sexes

c. There are two pure strategies Nash equilibria: (HM,BC) and (BC,HM). There is also a mixed strategy Nash equilibrium in which each worker goes to HM with probability 5/7 and to BC with probability 2/7.

Explanation: In a mixed strategy equilibrium each player randomizes in a way that makes the opponent indifferent between his actions.

To find the mixture probability, suppose that the row player goes to HM with probability p and for BC with probability $1-p$. If the opponent goes to HM then opponent's payoff is $4p+8(1-p)$. If opponent goes to BC then the opponent's payoff is $6p+3(1-p)$. In equilibrium these quantities are the same, $4p+8(1-p)=6p+3(1-p)$, which implies $p=5/7$.

Solution 6

a. The receiver will use DB. The probability that Dent will win is 70%.

Explanation: If the receiver would use DF then the probability that Dent would win would be 80%. So the receiver prefers DB over DF

b. The receiver will use DF. The probability that Dent will win is $\frac{1}{2} * 70 + \frac{1}{2} * 80 = 75\%$.

   Explanation: If the receiver would use DB then the probability that Dent would win would be $\frac{1}{2} * 90 + \frac{1}{2} * 70 = 80\%$. So the receiver prefers DF over DB.

c. She should use SF with probability 1/3. If she does this, then her success probability is 76.6%.

   Explanation: This is a zero-sum game and we know that players should play their equilibrium strategy. Therefore, she should randomize in a way that will make the receiver indifferent between his actions.

   To find the optimal mixture probability, suppose that she goes to SF with probability p and for SB with prob $1 - p$. If the receiver chooses DF then Trillian's success probability is $70p + 80(1 - p)$. If the receiver goes to DB then Trillian's success probability is $90p + 70(1 - p)$. In equilibrium these quantities are the same, $70p + 80(1 - p) = 90p + 70(1 - p)$, which implies $p = 1/3$.

Solution 7

a. When $n = 2$ there are two pure strategy Nash Equilibria when one player tells and the other keeps silent. In the mixed strategy Nash Equilibrium, each player tells with probabilit 3/5. The probability that somebody tells is $1 - (2/5)^2 = 0.84$.

b. For arbitrary n the mixed strategy Nash Equilibrium is such that each player tells with probability p. Since in equilibrium each player is indifferent we get that $600 = 1000 * (1 - (1 - p)^{n-})$ or $p = 1 - (2/5)^{1/(n-1)}$.

c. The probability that somebody will tell is

$$1 - (1 - p)^n = 1 - (2/5)^{n/n-1}.$$

   For $n = 10$ the probability that somebody will tell is 0.64. (When n increases the probability that somebody will tell goes down to 0.6.)

Solution 8

a. Yes. Assume both players play the following strategy

   - At round 1 play cooperate
   - At round 2, if everobody cooperated at round 1 play stag, if somebody defected play hare.

   This strategy pair gives 3+5=8 to each player. Nobody will deviate: if I defect at round 1 then I know that opponent will play hare next round so my payoff will be 4+1=5. Therefore, this is an equilibrium.

b. In this case there is no equilibrium according to which we cooperate at the first round. When the opponent cannot condition his action in the second round on what I did in the first round, it is a dominant strategy for me to defect in the first round because this will increase my payoff in the first round and not change my payoff in the second round.

Solution 9

a. My strategy would be to lobby if my cost is less than 15.

   Explanation: The probability that the opponent will lobby is 1/2. If I lobby then my payoff will be $\frac{1}{2} * 20 + \frac{1}{2} * 30 = 25$. If I don't lobby then my payoff will be $\frac{1}{2} * 0 + \frac{1}{2} * 20 = 10$. Therefore it is worth lobbying if the cost is less than 15.

b. In the equilibrium strategy each firm lobbies if its cost is less than 12.

   Explanation: Suppose that firm 1 chooses a threshold strategy to lobby if its cost is less than c_1, and that firm 2 chooses a threshold strategy to lobby if its cost is less than c_2.

   Then the probability that firm 1 lobbies is $p_1 = c_1/60$. The best response of firm 2 is to lobby if its cost is at most

   $$\big(p_1 * 20 + (1 - p_1) * 30\big) - \big(p_1 * 0 + (1 - p_1) * 20\big) = 10 + 10p_1 = 10 + c_1/6.$$

   Therefore, in equilibrium $c_2 = 10 + c_1/6$ and $c_1 = 10 + c_2/6$ which implies that $c_1 = c_2 = 12$

Solution 10

The game matrix is given by

	\$2	\$4	\$5
\$2	1000,1000	1400,1200	1400, 1500
\$4	1200,1400	2000,2000	2800, 1500
\$5	1500, 1400	1500,2800	2500, 2500

For every player, the only strategy pair that survives sequential elimination of dominated strategies is \$4.
Explanation: Each player eliminates \$2 (dominated by \$4 and \$5) and then each player eliminates \$5 (becomes dominated by \$4).

Solution 11

a.

	Low	High
Low	100,100	60,110
High	110,60	80,80

b. Prisoner's dilemma

   Explanation: High is a dominant strategy, and both players playing High yields a bad payoff to both.

c. The unique equilibrium is (High, High).

d. Grim Trigger means: I start with Low (cooperate), and continue playing Low as long as opponent plays Low. If at some point the opponent defects (plays High) then I will play High forever.

e. $r \leq 66\%$.

   Explanation: If both players play grim trigger, each player can either follow the equilibrium strategy and get 100 every day or deviate and get 110 today and then 80 every day. So following the equilibrium gives $PV(100, 100, 100, \dots) = 100/r$. Deviating gives $PV(110, 80, 80, \dots) = 110 + (1-r)80/r$. In order for grim trigger to be an equilibrium deviation should not worth it. This means that $100/r \geq 110 + (1-r)80/r$ or $r \leq 2/3$.