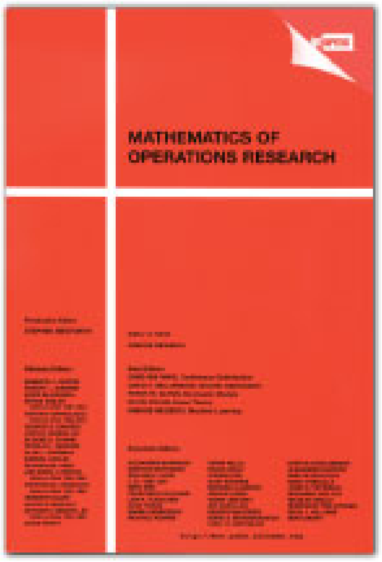

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MATHEMATICAL PROPERTIES OF THE BANZHAF POWER INDEX*

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The Banzhaf index of power in a voting situation depends on the number of ways in which each voter can effect a "swing" in the outcome. It is comparable—but not actually equivalent—to the better-known Shapley-Shubik index, which depends on the number of alignments or "orders of support" in which each voter is pivotal. This paper investigates some properties of the Banzhaf index, the main topics being its derivation from axioms and its behavior in weighted-voting models when the number of small voters tends to infinity. These matters have previously been studied from the Shapley-Shubik viewpoint, but the present work reveals some striking differences between the two indices. The paper also attempts to promote better communication and less duplication of mathematical effort by identifying and describing several other theories, formally equivalent to Banzhaf's, that are found in fields ranging from sociology to electrical engineering. An extensive bibliography is provided.

1. Introduction and background. The use of game theory to study the distribution of power in voting systems can be traced back to the invention of "simple games" by John von Neumann and Oskar Morgenstern in their 1944 classic, *Theory of Games and Economic Behavior* [91]. Speaking intuitively, a simple game is a cooperative/competitive enterprise in which the only goal is "winning" and the only rule is a specification of which coalitions are capable of doing so. This abstract definition covers most of the familiar examples of constitutional political machinery, among them direct majority rule, weighted voting, direct or indirect election of a President, bicameral or multicameral legislatures, committees and subcommittees, veto situations, etc.[1] Moreover, simple games often make for clearer modelling and neater mathematical proofs than some of the more restricted classes of voting rules commonly considered by political scientists, who have sometimes adopted an unfortunately narrow view of the logical possibilities for systems of representation and governance.[2]

After some exploration of the mathematical structure of simple games, von Neumann and Morgenstern set out to apply to them the "stable set" concept that they had already developed for a more general class of coalitional games.[3] The logic of that solution concept led them to construct a little economic vote-selling model, in which the equilibrium prices describe the share of the spoils that each player might expect to receive if he ends up on the winning side. Only a small (but significant) class of simple games ultimately proved to be solvable in this way, but these N-M price vectors can nevertheless be regarded as an early form of *power index*, representing an important

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[1] See Shapley [74].

[2] See note 4 in §12 below.

[3] In [91], the term "simple game" refers only to what we would now call a *decisive* simple game; see §2 below. The term "stable set" was not used in [91].

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step forward in the quantitative analysis of the power of voters in abstract political systems.[4]

The next step was taken in a 1954 paper by Shapley and Shubik [79], who proceeded to specialize another general-purpose solution concept, the so-called "Shapley value" [73], to the case of simple games. This approach yielded numerical indices which can be interpreted directly in terms of the *a priori* ability of the players to affect the outcome; moreover they have the advantage over the N-M equilibrium prices of being well defined for all simple games. The Shapley-Shubik power index has become widely known and applied in game theory and political science.[5]

An unexpected practical turn was given to the problem of measuring voting power when the U.S. Supreme Court in the 1960s handed down a series of "one person one vote" decisions, setting forth new standards of constitutional fairness for systems of electoral representation at the state and local levels.[6] As a result, many existing voting systems had to be revised or at least re-examined, and the advocates or opponents of proposed reforms had frequent occasion to invoke either the Shapley-Shubik index or a rather similarly-conceived index due to John F. Banzhaf III, a young lawyer and reformer with a mathematical background [4]–[7]. Laborious calculations using real data were carried out on the computers of the day and presented as evidence in the courtroom or at legislative hearings.[7] The main ideas underlying the game-theoretic approach to power eventually found wide legal acceptance; indeed, in New York State today, some of the county supervisorial boards are constituted according to a form of Banzhaf's index, in an attempt to equalize the representation of citizens living in municipalities of different size.[8]

The actual numerical values that issue from the Banzhaf (Bz) and Shapley-Shubik (S-S) models are often quite similar, and the two can be regarded as equivalent for many practical purposes if we grant that law and politics are far from being exact sciences.[9] Nevertheless, there are significant differentiating features that have not yet been explored mathematically to any great depth. The Bz index seems to have had the greater appeal to the legal mind, perhaps because of its more straightforward verbal definition. But up to now the S-S index has attracted the lion's share of attention from game theorists, partly because of a certain perceived naturalness in its mathematical foundations and partly as a by-product of research devoted to its parent solution concept, the "Shapley value" for general cooperative games.[10] With the idea of

[4] Op. cit., pp. 435–443. For some later work on these "main simple solutions," as they are called, see Gurk and Isbell [26], Vickrey [90], and Wilson [93]. Other game-theoretic approaches to vote selling will be found in Young [99] and Shubik and Weber [82]; see also Wilson [92].

[5] Published political science applications include Riker [65] (French National Assembly); MacRae and Price [44] (U.S. Senate); Riker and Niemi [67] (House of Representatives); David, Goldman, and Bain [18] (U.S. party conventions); Mann and Shapley [45], Owen [58], [60], Spatt [83], and Merrill [47] (Electoral College); Riker and Shapley [69] and Nozick [56] (one-man-one-vote); Owen [57] (Israeli Knesset); Junn [36] and Monjardet [50] (UN Security Council); Miller [48] and Straffin [87] (Canadian constitution); Brams and Riker [12] and Straffin [86] (bandwagon effect); Krislov [39], Riker [66], Riker and Ordeshook [68], and Brams [10] (other topics); and Lucas [43] (a general survey and mathematical text).

[6] *Baker v. Carr*, 369 U.S. 186 (1962); *Grey v. Sanders*, 372 U.S. 368 (1963); *Wesberry v. Sanders*, 376 U.S. 1 (1964); *Reynolds v. Sims*, 377 U.S. 533 (1964). The late Justice Frankfurter, dissenting in *Baker v. Carr*, cautioned the Court against dragging the law into "political thickets and mathematical quagmires." We wonder how he would feel about our present work!

[7] One court went so far as to find that the very expense and complexity of the power computations made the question of the fairness of a certain weighted voting proposal legally undecidable. (*Ianucci v. Board of Supervisors of Washington County*, N.Y. State Court of Appeals, 1967.)

[8] Imrie [33]; Lucas [43].

[9] For some comparisons, see Brams [10], Owen [60], Shapley [78], Straffin [85], [87].

[10] See [68], or [2] where further references will be found. The Bz index can also be related to a solution for general cooperative games: see note 8 in §12 below and also Owen [59], [61] and Roth [70].

redressing this imbalance, we have undertaken here to investigate the Bz index from a mathematical standpoint. Much of our work parallels earlier studies of the S-S index, but the conclusions reached are often quite different.

Let us briefly outline the contents. §2 defines the Bz index and relates it to an explanatory probability model, which is then compared with the corresponding S-S model with the help of a small example. §3 then shows how to derive the Bz index from a set of axioms, comparable to a set Dubey has recently given for the S-S case [21]. The Bz axioms make special use of a certain number, denoted $\bar{\eta}$, which has no counterpart in the S-S theory; intuitively it represents the "total power" available in the game. §4 derives some bounds for this number under various assumptions on the set of winning coalitions.

The most substantial portion of the paper, mathematically speaking, deals with the asymptotic properties of *weighted majority games* when there are many small voters. §5 prepares for this by describing some of the elementary properties of weighted majority games and their power indices. §6 then formulates the passage to the limit and the next five sections investigate it in detail. This is a problem that was studied many years ago from the S-S standpoint [72], but the Bz version reveals several surprising new features. For a further discussion of these results, see the introductory paragraphs of §6.

Finally, in §12 we present a series of supplementary notes, dealing with various extensions and applications of the Bz index. Included are brief summaries of several other theories, both in and out of political science, which in their mathematical aspects can be shown to be equivalent to the Banzhaf theory; they are associated with the names of Coleman, Rae, Dahl, Chow, and others. For the most part, these theories have been developed independently, without reference either to each other or to the game-theory literature. While these supplementary notes do add to the length of an already long paper, we feel that they promote the cause of better communication and less duplication among parallel lines of research having a common mathematical core. In the same spirit, we have assembled an extensive set of references from many sources.

Much of §§3, 6–11 appears in chapters I and II of Dubey's doctoral dissertation, *Some Results on Values of Finite and Infinite Games*, Cornell University, 1975.

2. Preliminaries. A *game*, or *game on* N, is a real-valued function v defined on the subsets of a nonempty, finite set, N, and vanishing on the empty set. The elements of N are called "players," and we shall often identify them with the integers $1, 2, \ldots, n$, where $n = |N|$. The symbol $\mathcal{G}(N)$ will denote the set of all games on N, and $\mathcal{G}_{sa}(N)$ the set of *superadditive* games on N, i.e., those obeying the condition

$$v(S \cup T) \geqslant v(S) + v(T) \quad \text{whenever } S \cap T = \emptyset.$$

Superadditivity is quite important in most applications since it permits one to regard the number $v(S)$ as the total amount of something—of money, say, or "transferable utility"—that the members of S can be sure of getting if they form a coalition. Non-superadditive games are sometimes called *improper* because they pose problems in interpretation, but they are nevertheless useful in the mathematical theory.

A game v is said to be *simple* if it assumes only the values 0 and 1, obeys the condition of monotonicity:

$$v(S) \geqslant v(T) \quad \text{whenever } S \supset T,$$

and is not identically 0. Thus, a simple game on N always has $v(N) = 1$. The symbol $\mathcal{C}(N)$ will denote the set of all simple games on N and $\mathcal{C}_{sa}(N)$ the set of superadditive or *proper* simple games on N. In the context of simple games, propriety is equivalent to the condition

$$v(S) + v(N - S) \leqslant 1 \quad \text{for all } S.$$

If equality holds here for every S, the function v is *self-dual* (see §5) and the game is said to be *decisive*; the latter term refers to the interpretation of v as a political constitution or other group decision rule.[11] Pursuant to this interpretation, sets S with $v(S) = 1$ are called *winning coalitions* and sets with $v(S) = 0$ *losing coalitions*. Sets whose complements lose are called *blocking coalitions*. It is not hard to see that in a proper game winning implies blocking, while in a decisive game winning and blocking are equivalent. In an improper game, however, there will be at least two winning coalitions that do not block. In other words there will be at least one pair of nonintersecting winning coalitions. The number of simple games on a fixed N is finite, of course, but it grows very rapidly with increasing n since we are dealing with sets of sets. (There are already 180 five-person simple games, not counting permutations.) Indeed, every family of pairwise independent subsets of N can serve as the set of minimal winning coalitions defining a simple game.[12]

We now define a *swing*, or *swing for player* i: this is a pair of sets of the form $(S, S - \{i\})$ such that S is winning and $S - \{i\}$ is not. For each $i \in N$, we denote by $\eta_i(v)$ the *number of swings for* i in the game $v \in \mathcal{G}(N)$. We shall write $\bar{\eta}(v)$ for the total number of swings, i.e., $\bar{\eta}(v) = \sum_{i \in N} \eta_i(v)$. A player with $\eta_i(v) = 0$ is called a *dummy* because, intuitively, he can never help a coalition to win. At the other end of the scale, a player with $\eta_i(v) = \bar{\eta}(v)$ is called a *dictator*, for obvious reasons.[13]

The swing numbers $\eta_i(v)$ will be called the "raw" Banzhaf indices. These are the indices that Banzhaf actually defined and used in his work.[14] But since the principal interest in these numbers lies in their ratios rather than their magnitudes, it has been common practice to normalize them to add up to 1:

$$\beta_i(v) = \eta_i(v)/\bar{\eta}(v), \qquad i = 1, \ldots, n. \tag{1}$$

As we shall see, however, this normalization is not as innocent as it seems. We shall call the numbers (1) the *normalized Banzhaf indices*. For convenience, we shall write $\eta(v)$, $\beta(v)$, etc., for the vectors $(\eta_1(v), \ldots, \eta_n(v))$, $(\beta_1(v), \ldots, \beta_n(v))$, etc., and we shall sometimes omit the "(v)."

Another normalization is in many respects more natural:

$$\beta_i'(v) = \eta_i(v)/2^{n-1}, \qquad i = 1, \ldots, n. \tag{2}$$

These numbers we shall call the *swing probabilities* of the players. This term arises from the following probability model. Suppose that a bill is to be decided by an assembly, and that each member of the assembly randomly votes "yea" or "nay" on

[11] See [74].

[12] Two subsets are *independent* if neither contains the other. Families of independent subsets are sometimes called "Sperner families," "coherent systems," or "clutters," and their enumeration and classification have occupied mathematicians since Dedekind in the 19th century. An account of this work will be found in [27, pp. 1030–1032], or see [80, pp. 23–24]. See also Sperner [84], Isbell [34], Golomb [25], the proof of lemma 2 in §8 below, and note 7 in §12 below.

[13] Equivalently, we could define η_i as the difference $\omega_i - \tilde{\omega}_i$ between the number of winning coalitions containing i and the number not containing i. *Proof*: Take any winning coalition that contains i, and remove i from it. This either produces a swing or leaves the coalition winning. Moreover, every swing for i and every winning coalition not containing i is obtained in this way. So $\omega_i = \eta_i + \tilde{\omega}_i$. (Cf. note 6 in §12 below.)

[14] See [4, p. 331].

the toss of a coin.[15] The set Y of yea-voters is then a random variable, assigning probability $1/2^n$ to each subset of N. Of course the bill passes if and only if Y is a winning coalition, under the prescribed voting rule of the assembly. Call member i a *swinger* for Y if changing his vote would affect the passage of the bill. That is, i is a swinger for Y if the pair $(Y \cup \{i\}, Y - \{i\})$ is a swing for i, as previously defined. Since for any given swing for i there are exactly two sets Y that yield it, its probability is $1/2^{n-1}$. So the probability that i swings in this model is exactly $\beta_i'(v)$.

The above makes an interesting contrast with the familiar probability model for the S-S index.[16] There, the bill or issue under consideration is assumed to *align* the players in order of their enthusiasm for the proposal, with the most fervid supporter coming first and the most stubborn opponent last. Given any such alignment, there will be a unique marginal player, i.e., one who, by joining with his more enthusiastic colleagues, brings the coalition up to winning strength. This player is called the *pivot* of the ordering. Intuitively, he is the one whom the others must try to persuade or dissuade, or who perhaps determines how strong a law will be enacted, or how much money will actually be appropriated for some purpose, or how hard a candidate will have to campaign, etc. If we now assume *a priori* that all $n!$ orderings are equiprobable, then the S-S index for each player is precisely his probability of being pivotal.

We see, then, that mathematically the S-S index rests on equiprobable *permutations* of N while the Bz index rests on equiprobable *combinations* of N.[17] Since each permutation produces exactly one pivot, the S-S index is inherently an additive measure of power, applicable to sets as well as to individuals. Thus, if φ_i denotes the S-S index for i, then the sum $\sum_{i \in S} \varphi_i$ is a plausible measure of coalitional power, because it represents exactly the probability that S contains the pivot. On the other hand, a single *combination* only rarely produces exactly one swinger. Typically there will be either many swingers, as in a very close election, or none at all, as when a candidate wins by a comfortable margin. The sum $\sum_{i \in S} \beta_i'$ therefore does not represent the probability that S contains a swinger, nor does it even represent the probability that the set S as a whole, throwing its weight one way or the other, could swing the outcome. Rather, this sum represents the *expected number of individual swingers in S*, and only in this rather strange sense do we get an additive measure of "coalitional power" out of the Banzhaf approach.[18]

To illustrate these remarks, consider a 9-person tricameral assembly having three "chambers" of players, A, B, C, with 1, 3, 5 members respectively. The winning coalitions are those that include a majority of every chamber. The lone member of A therefore has veto power, but he will swing only if Y includes majorities of both B and C. This can happen in 64 ways. A typical member of B will swing if and only if Y includes A, exactly one other member of B, and a majority of C; this can happen in 32 ways. A typical member of C will swing if Y includes A, a majority of B, and

[15] The case of probabilities other than 1/2 will be considered in §12, note 1; see also Blair [9], Dubey [22], and Straffin [85].

[16] See [72], [73], [79], etc.

[17] The S-S index has the effect of making the probability of a combination depend on its size, with the total probability of each *size* being the same.

Banzhaf himself, apparently seeking to disarm criticism, asserts that "no assumptions are made as to the relative likelihood of any combination" [4, p. 326]. His formal definitions, however, speak otherwise. In another place, he argues that "because *a priori* all voting combinations are equally possible, any objective measure of voting power must treat them as equally significant" [5, p. 1316].

[18] See [78]. There may be a philosophical difficulty in simultaneously considering more than one player as a potential swinger (as we must do if we wish to add up swing probabilities over a set of players). Our basic explanatory model for β_i' gives player i freedom of choice while making behavioristic assumptions about the other players, so we are being asked to accept n different subjective views of the voting process, artificially fused into a single model.

exactly two other members of C; this can happen in 24 ways. Hence

$$\eta = (64, 32, 32, 32, 24, 24, 24, 24, 24),$$

$$\beta = \left(\tfrac{8}{35}, \tfrac{4}{35}, \tfrac{4}{35}, \tfrac{4}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}, \tfrac{3}{35}\right),$$

$$\beta' = \left(\tfrac{8}{32}, \tfrac{4}{32}, \tfrac{4}{32}, \tfrac{4}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}, \tfrac{3}{32}\right),$$

$$\bar{\eta} = 280, \quad \bar{\beta} = 1, \quad \bar{\beta}' = 35/32 = 1.09375.$$

For comparison, the S-S indices are

$$\varphi = \left(\tfrac{32}{84}, \tfrac{9}{84}, \tfrac{9}{84}, \tfrac{9}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84}, \tfrac{5}{84}\right);$$

see [79] for the details of this not-too-difficult calculation.

By adding up these indices within the sets A, B, and C, Brams [10, p. 193] arrived at the interesting observation that "Bz" and "S-S" rank these sets in opposite order. In fact, the ratios of cameral power are, respectively,

$$\text{Bz:} \quad 8 : 12 : 15, \qquad \text{S-S:} \quad 32 : 27 : 25.$$

So which chamber is really most powerful? The apparent conflict between the two theories is largely a matter of semantics. What is meant by the "power" of a *set* of individuals? As noted in our previous remarks on additivity, the sum $\sum_S \varphi_i$ represents the probability that S contains the pivot, while the sum $\sum_S \beta_i'$ represents the expected number of swingers in S. If we ask instead for the probability that each chamber contains at least one swinger, Bz gives us the ratio 8 : 6 : 5, which agrees fairly well with the S-S ratio. On the other hand, if we compute the probabilities that each chamber could swing if it voted as a bloc, the other individuals still voting randomly and independently, then we get the ratio 1 : 1 : 1, or equal bloc-swinging power to each house.[19]

3. Axioms for the Banzhaf index. For any game $v \in \mathcal{G}(N)$, if π is a permutation of N we define πv by

$$(\pi v)(S) = v(\pi^{-1}(S)).$$

For simple games $v, w \in \mathcal{C}(N)$, we define the operations $v \vee w$ and $v \wedge w$ by[20]

$$(v \vee w)(S) = \max(v(S), w(S)), \quad (v \wedge w)(S) = \min(v(S), w(S)).$$

It is clear that $\mathcal{C}(N)$ is closed under the operations π, $\vee$, and $\wedge$, and that $\mathcal{C}_{sa}(N)$ is closed under π and $\wedge$.

THEOREM 1. *There is a unique function φ: $\mathcal{C}(N) \to R^n$ that satisfies the following four axioms:*

A1: *If i is a dummy in v then $\varphi_i(v) = 0$.*

A2: $\sum_{i \in N} \varphi_i(v) = \bar{\eta}(v)$.

A3: *For any permutation π of N, $\varphi_{\pi(i)}(\pi v) = \varphi_i(v)$.*

A4: *For any $v \in \mathcal{C}(N)$ and $w \in \mathcal{C}(N)$, $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi(v) + \varphi(w)$.*

Moreover, $\varphi(v) = \eta(v)$ for all v in $\mathcal{C}(N)$.

PROOF. For any $S \subset N$, $S \neq \emptyset$, define the game v_S by

$$v_S(T) = \begin{cases} 0 & \text{if } T \not\supset S, \\ 1 & \text{if } T \supset S. \end{cases}$$

[19] The reader is referred to [78] for a more extended discussion of this example and the whole question of the additivity of coalitional power indices.

[20] Compare the "sum" and "product" operations on simple games, defined in [74], [76], [77].

Each $i \in N - S$ is a dummy in v_S, therefore, by A1, $\varphi_i(v_S) = 0$ for such i. Also if π is the permutation that interchanges i and j (for any $i \in S$ and $j \in S$) and leaves the other players fixed, then $\pi v_S = v_S$ and thus, by A3,

$$\varphi_i(v_S) = \varphi_j(v_S).$$

Therefore $\varphi(v_S)$ is uniquely determined, if φ exists, and is given by

$$\varphi_i(v_S) = \begin{cases} 0, & \text{if } i \in N - S, \\ \bar{\eta}(v_S)/|S| = 2^{|N-S|}, & \text{if } i \in S, \end{cases} \tag{3}$$

using A2.

Now every v in $\mathcal{C}(N)$ has a finite number of *minimal* winning coalitions $S_1, \ldots, S_m$, and they completely determine v, since $v(T) = 1$ if and only if $T \supset S_j$ for at least one $j = 1, \ldots, m$. Clearly, we have $v = v_{S_1} \vee v_{S_2} \vee \ldots \vee v_{S_m}$, where the right-hand side is defined associatively. Now if $v \in \mathcal{C}(N)$ is not of the form v_S, then $m > 1$, so v can be written as $v' \vee v''$, where v' and v'' are games with *fewer* winning coalitions than v. For example, let $v' = v_{S_1}$ and $v'' = v_{S_2} \vee \ldots \vee v_{S_m}$. Of course, the game $v' \wedge v''$ has even fewer winning coalitions. So we can perform an induction on the number of winning coalitions, using A4:

$$\varphi(v) = \varphi(v' \vee v'') = \varphi(v') + \varphi(v'') - \varphi(v' \wedge v''), \tag{4}$$

and it follows that $\varphi(v)$ is uniquely determined.

We must still prove existence. The foregoing proof of uniqueness has implicit in it a recursive construction of φ that establishes existence; however, it is simpler to check directly that the function η as already defined satisfies A1–A4. In fact, A1–A3 are obvious, while A4 follows from the equation

$$\eta_i(v) = \sum_{S\,:\, i \in S \subset N} [v(S) - v(S - \{i\})],$$

showing that $\eta(v)$ can be extended to a linear function on $\mathcal{G}(N)$. Since

$$(v \vee w) + (v \wedge w) \equiv v + w,$$

A4 is now obvious. ∎

It may be worth pointing out that if v is superadditive, then so are the other three games used in the inductive step (4). Hence the same proof shows that η is the unique power index for $\mathcal{C}_{\text{sa}}(N)$, if we understand A1–A3 to be restricted to v in $\mathcal{C}_{\text{sa}}(N)$ and A4 to be restricted to those v and w in $\mathcal{C}_{\text{sa}}(N)$ such that $v \vee w$ is also in $\mathcal{C}_{\text{sa}}(N)$.

Theorem 1 reveals that the Bz index is fundamentally very similar to the S-S index. By changing A2 to:

$$A2'\!: \sum_{i \in N} \varphi_i(v) = n!,$$

we obtain a "raw" S-S index, while changing $n!$ here to 1 gives us the more usual form that corresponds to the probability model in §2.[21] The proof of this was given in [21].

Thus, it is only in the second axiom that the difference between the two indices is reflected. In this axiom we make an a priori judgment concerning the meaning of

[21] Changing the right side to $\bar{\eta}(v)/2^{n-1}$ would yield the index $\beta'(v)$. But note that we cannot get the normalized Bz index in this way, since $\beta(v)$ does not obey A4. This may be taken as an initial sign of trouble with the normalization (1).

"power," by specifying the extent to which the game is responsive to the combined "powers" of the individual players.[22]

We do not attempt here a heuristic justification of A4. This axiom can be presented in a variety of mathematical guises. For example, one can adopt a "marginal" viewpoint and consider the effect of changing the status of just one coalition from winning to losing—more precisely, from minimal winning to maximal losing. One may then replace A4 by the assumption that under such a one-coalition change the variation in power for each player is independent of the rest of the game, i.e., is the same for all voting rules in which the given coalition is minimal winning. The proof of this depends on the fact that any sequence of one-coalition changes that takes us from v to $v \wedge w$ will also take us from $v \vee w$ to w.

Another approach would be to introduce lotteries over sets of simple games. Call two lotteries *equivalent* if they yield the same probabilities for each coalition to be winning. Then in place of A4 we could assume that equivalent lotteries yield the same expected power index. Formally, let $L(v)$ denote the probability of the game $v \in \mathcal{C}(N)$ in the lottery L. The new axiom would state that if $\sum L(v)v = \sum L'(v)v$ then $\sum L(v)\varphi(v) = \sum L'(v)\varphi(v)$. To derive A4 from this, let L give probability $1/2$ to each of v and w and let L' give probability $1/2$ to each of $v \wedge w$ and $v \vee w$: the rest is straightforward.

Other equivalent forms of A4 can also be described. Our purpose in this section has only been to clarify the underlying logical structure of the Bz index. But perhaps in some of the varied interpretations and applications of the index (see e.g., §12, notes 4, 5, and 6), it may be possible to attach a "story" to one of these versions of A4 that will bolster its intuitive plausibility.

4. Counting swings. We have seen how the total number of swings in a simple game v, denoted $\bar{\eta}(v)$, plays an important role in the formal axiomatization of the Banzhaf power index. Intuitively speaking, $\bar{\eta}(v)$ reflects the "volatility" or "degree of suspense" in the decision rule. It gives an indication of the likelihood of a close decision, i.e., one so close that a single voter could tip the scales. As discussed later,[23] it is also a kind of democratic participation index, measuring the decision rule's sensitivity to the desires of the "average voter" or to the "public will." The swing total for a given number of players can vary widely. For example, if there are ten voters, then direct majority rule produces a total of 1260 swings while the rule of "unanimous consent" produces only 10 swings.

We shall now examine some mathematical properties of the swing total. Theorem 2 gives the absolute upper bound as a function of n. Theorems 3 and 4 and corollaries 1 and 2 give lower bounds, based on different assumptions about the size and number of winning coalitions.[24]

THEOREM 2. *Let N be a fixed set and define $n = |N|$ and $m = [n/2] + 1$, i.e., m is the next integer after $n/2$. Then for any simple game $v \in \mathcal{C}(N)$, we have*

$$\bar{\eta}(v) \leqslant m\binom{n}{m}.$$

Moreover, equality holds if and only if all coalitions with more than $n/2$ members win and all coalitions with less than $n/2$ members lose.

In particular, if n is odd then theorem 2 tells us that the maximum number of swings in $\mathcal{C}(N)$ is achieved *uniquely* by direct majority rule.[25]

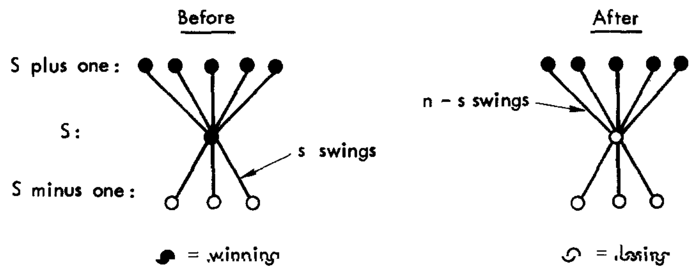
[22] Cf. note 4 in §12.

[23] Note 4 in §12.

[24] See also note 3 in §12.

[25] A special case of this theorem was conjectured by Rae [64] and proved by Taylor [89]; see also Curtis [16]. The present simple proof is based on a suggestion of Michael Todd: see also Straffin [88].

PROOF. Let S be a minimal winning coalition of v. There are just $s = |S|$ swings involving S. Changing S from winning to losing would destroy those s swings, but would create $n - s$ new swings, as illustrated here for $n = 8$, $s = 3$:



Thus, the swing total $\bar{\eta}$ can be increased if $n - s > s$, i.e., if $s < n/2$. Hence, at the maximum, all coalitions smaller than $n/2$ must lose. By the dual argument starting with a maximal losing coalition, all coalitions larger than $n/2$ must win. If n is odd, then the game is determined completely by these conditions and there are precisely $\binom{n}{m}$ marginally winning coalitions, each contributing m swings to the total. If n is even, then we note first that the total number of swings is not changed if we make all the coalitions of size $n/2$ lose. Then there are again $\binom{n}{m}$ marginally winning coalitions, each contributing m swings. ∎

THEOREM 3. *Let v be a simple game with n players, and let m be an integer such that $n/2 < m \leqslant n$ and*

$$|S| \geqslant m \Rightarrow v(S) = 1, \qquad |S| \leqslant n - m \Rightarrow v(S) = 0.$$

That is, every coalition with at least m members both wins and blocks. Then we have

$$\bar{\eta}(M) \geqslant m\binom{n}{m}.$$

PROOF. Consider the edge-capacitated network whose nodes are the 2^n subsets of N and whose edges are the $n2^{n-1}$ pairs $(S, S - \{i\})$, $i \in S \subset N$, each edge having the same capacity $c > 0$. By the "rank" of a node S we shall mean the integer $|S|$. Let the nodes of rank m be called *sources* and those of rank $n - m$ *sinks*. Since $m > n/2 > n - m$, the sources are higher than the sinks. There are $m\binom{n}{m}$ edges descending from nodes of rank m to nodes of rank $m - 1$. Similarly, we may count $(m - 1)\binom{n}{m-1}$ edges from rank $m - 1$ to $m - 2$, and so on. Finally, there are $(n - m + 1)\binom{n}{n-m+1}$ edges entering the set of sinks, this last number being equal to $m\binom{n}{m}$. Since these counts first increase and then decrease, it is obviously possible to send a total flow of $cm\binom{n}{m}$ units from the source set to the sink set without exceeding the capacity at any edge. For example, we can distribute the flow evenly among all edges at each level.

Now the swings of the game are just the edges of the network that connects a "winning" node to a "losing" node. Since the sources all win and the sinks all lose, removing all swings from the network necessarily cuts off *all* flow between sources and sinks. The total capacity of the swinging edges is by definition $c\bar{\eta}(v)$. This obviously must be at least as great[26] as the amount $cm\binom{n}{m}$ we were able to send through the network before removing them. Hence $\bar{\eta}(v) \geqslant m\binom{n}{m}$. ∎

[26] We are using here the Ford-Fulkerson "max flow equals min cut" theorem [23], but only in the trivial direction which asserts that the capacity of any cut is greater than or equal to the amount of any flow.

THEOREM 4. *Let v be a simple game with n players, ω winning coalitions, and λ losing coalitions (so that $\omega + \lambda = 2^n$). Then*

$$\bar{\eta}(v) \geqslant \sum_{k=0}^{\omega-1} (n - 2g(k)) = \sum_{k=0}^{\lambda-1} (n - 2g(k)). \tag{5}$$

where $g(k)$ is the sum of the digits in the binary representation of the integer k. Moreover, for each n and ω there is a simple game for which equality is attained.

This theorem was proved by Sergiu Hart [29];[27] for an application see §12, note 3. The following corollary gives a weaker but less complicated bound.

COROLLARY 1. *Under the conditions of Theorem 4 we have*

$$\bar{\eta}(v) \geqslant \mu[n - \log_2 \mu], \tag{6}$$

where $\mu = \min(\omega, \lambda)$, and $[x]$ denotes here the greatest integer $\leqslant x$.

PROOF. If we write out a list of the integers $0, 1, \ldots, \mu - 1$ in binary form:

$$\begin{array}{c} 0 \\ 1 \\ 10 \\ 11 \\ 100 \\ \cdots \end{array}$$

we see that no more than $\mu/2$ of the digits in each column can be 1's. As the number of columns is $\langle \log_2 \mu \rangle$, where $\langle x \rangle$ denotes the least integer $\geqslant x$, we have at once

$$\sum_{k=0}^{\mu-1} g(k) \leqslant \frac{\mu}{2} \langle \log_2 \mu \rangle.$$

With the aid of this inequality, (6) follows directly from (5). ∎

COROLLARY 2. *If v is a decisive simple game with n players, then $\bar{\eta}(v) \geqslant 2^{n-1}$.*

PROOF. Apply corollary 1 with $\mu = 2^{n-1}$. ∎

To see that this bound is sharp, it suffices to consider the game where one player is a dictator. As far as we know, corollary 2, which seems intuitively obvious, can be proved only by way of theorem 4.

5. Weighted majority games. The duality principle. We now turn to a special class of simple games called *weighted majority games*. The symbol

$$[c; w_1, \ldots, w_n] \tag{7}$$

will be used, where c and $w_1, \ldots, w_n$ are real numbers that are nonnegative and satisfy $0 < c \leqslant \sum_{i \in N} w_i$. We may think of w_i as the number of votes, or *weight* of player i, and c as the threshold or *quota* needed for a coalition to win. Thus, (7) represents the simple game v defined by

$$v(S) = \begin{cases} 1 & \text{if } w(S) \geqslant c, \\ 0 & \text{if } w(S) < c, \end{cases} \tag{8}$$

where $w(S)$ means $\sum_{i \in S} w_i$. The class of all games in $\mathcal{S}(N)$ having such a representation will be denoted $\mathfrak{M}(N)$. Obviously the representation (7) is never unique; in fact,

[27] An equivalent result was previously established by Bernstein [8], following Harper [28].

the set of vectors (7) that represent any given game in $\mathfrak{M}(N)$ forms a full-dimensional convex cone in R^{n+1}.[28]

In place of (8) we may sometimes prefer the criterion $w(S) > c$ for winning; in this case we shall use the symbol

$$\langle c; w_1, \ldots, w_n \rangle. \tag{9}$$

in place of (7) and restrict c by $0 \leqslant c < \sum_{i \in N} w_i$. Obviously, any particular weighted majority game expressed in the form (7) can be put into the form (9) and vice versa, by making a small adjustment in the quota c.

The *dual* of any game $\Gamma = (N, v)$ in $\mathcal{G}(N)$ is defined to be the game $\Gamma^* = (N, v^*)$, where

$$v^*(S) = v(N) - v(N - S), \quad \text{all } S \subset N. \tag{10}$$

Obviously, $\Gamma^{**} = \Gamma$. For $\mathcal{C}(N)$, duality interchanges the ideas of winning and blocking, and a simple game is decisive if and only if it is self-dual: $v = v^*$. For games in $\mathfrak{M}(N)$ we have

$$\begin{aligned} [c; w_1, \ldots, w_n]^* &= \langle w(N) - c; w_1, \ldots, w_n \rangle, \\ \langle c; w_1, \ldots, w_n \rangle^* &= [w(N) - c; w_1, \ldots, w_n]. \end{aligned} \tag{11}$$

It follows that of any mutually dual pair of weighted majority games, at least one is proper.[29]

It is easily seen that any simple game and its dual have the same raw swing count for each player, since (S, T) is a swing for i in v if and only if $(N - T, N - S)$ is a swing for i in v^*. Hence we have

THEOREM 5. *If v is any simple game, then*

$$\eta(v) = \eta(v^*), \quad \beta(v) = \beta(v^*), \quad \textit{and} \quad \beta'(v) = \beta'(v^*).$$

As might be expected, the voting weights and the power indices are closely related. Both w and β (or β' or η) induce the same ranking on the players, except that two players with unequal weights may have equal power. Also, it is easily shown that a player's power is a nondecreasing function of his weight if either (a) the rest of the weights and the quota are held fixed, or (b) the rest of the weights are fixed and the quota is kept a fixed fraction of the total weight. Quite often the vectors β and w are even found to be roughly proportional, but the following simple examples from [61] show how "rough" the proportionality can be:

$$\begin{aligned} &[8; 3, 5, 7]: && \beta = (1/3, 1/3, 1/3), \\ &[51; 49, 48, 3]: && \beta = (1/3, 1/3, 1/3), \\ &[4; 2, 2, 2, 1]: && \beta = (1/3, 1/3, 1/3, 0), \\ &[5; 2, 2, 2, 2, 1]: && \beta = (1/5, 1/5, 1/5, 1/5, 1/5). \end{aligned}$$

[28] Somewhat surprisingly, there is not always a unique minimal representation in integers. Muroga et al. [52] in their exhaustive enumeration of threshold functions uncovered several cases with as few as eight players in which two symmetric players must be given different weights in a minimal integer representation; e.g., [12; 7, 6, 6, 4, 4, 4, 3, 2] = [12; 7, 6, 6, 4, 4, 4, 2, 3]. Previously, Isbell [34] had exhibited a remarkable 12-player example in which the affected players are not symmetric.

[29] This is not true for simple games in general. An example is the improper four-person game with minimal winning coalitions $\{\{1, 2\}, \{2, 3\}, \{3, 4\}\}$. Its dual, also improper, has minimal winning coalitions $\{\{1, 3\}, \{2, 3\}, \{2, 4\}\}$ (see [74]). This also illustrates how a game can be isomorphic to its dual without being self-dual: we have here $\pi v = v^*$ for a certain permutation π but not $v = v^*$.

If we let the quota vary, however, a smooth relationship can be obtained by considering the *average* power, as the following theorem reveals.[30]

THEOREM 6. *Let $w_1, \ldots, w_n$ be fixed nonnegative numbers and let c be a random variable uniformly distributed over the interval* $(0, w(N)]$. *Then for each player i in the weighted majority game* (7) (*or* (9)), *we have*

$$\mathrm{E}\{\beta_i'\} = w_i/w(N). \tag{12}$$

PROOF. Fix i and S, with $i \in S$, and note that

$$\mathrm{Prob}\{(S, S - \{i\}) \text{ is a swing}\} = \mathrm{Prob}\{w(S) \geqslant c > w(S) - w_i\} = w_i/w(N).$$

The expected number of swings for i is therefore

$$\mathrm{E}\{\eta_i\} = \sum_{S \ni i} w_i/w(N) = 2^{n-1}w_i/w(N),$$

and the result follows. ∎

COROLLARY 3. *Choose c only from $(w/2, w]$ (i.e., superadditive case only). Then again* (12) *holds.*

This follows from theorem 6 with the aid of (11) and theorem 5. Of course, (12) implies that $\mathrm{E}\{\overline{\beta'}\} = 1$ and $\mathrm{E}\{\bar{\eta}\} = 2^{n-1}$.

It is obvious that the weights w_i can always be made integers. If this is done, then it is sufficient in theorem 6 to average over just the integers $c = 1, 2, \ldots, w(N)$, or in corollary 3, over the integers $c = \langle w(N)/2\rangle, \ldots, w(N)$, except that in the latter case we must give the first term half weight if $w(N)$ is odd. The following table illustrates these remarks by showing the raw swing counts for the games $[c;\ 1, 2, 3, 4, 5]$, $c = 1, 2, \ldots, 15$.

c	η_1	η_2	η_3	η_4	η_5	$\bar{\eta}$
1, 15	1	1	1	1	1	5
2, 14	0	2	2	2	2	8
3, 13	1	1	3	3	3	11
4, 12	1	1	3	5	5	15
5, 11	1	3	3	5	7	19
6, 10	2	4	4	6	8	24
7, 9	1	3	5	7	9	25
8	2	2	6	6	10	26
Total	16	32	48	64	80	240

Note that in this set of games there is no choice of quota that makes power exactly proportional to voting weight; indeed there is only one proper quota ($c = 9$) that even succeeds in making all the powers different. Nevertheless, the totals are proportional to the weights.

6. Passage to the limit. Most of the rest of this paper takes its cue from a 1960 Rand report of Shapiro and Shapley, recently published in this Journal [72], which considers the following weighted majority situation: There is a fixed quota, a fixed

[30] A similar result holds for the S-S index (see e.g., [75]), as well as for the generalized Bz probability index discussed in §12, note 1.

total weight, and a fixed set of "major" players with fixed individual weights. But there is also a population of "minor" players whose number is allowed to grow to infinity while their individual weights go to zero, keeping the total constant. The question now asked is: What happens to the power indices of the major players in the limit?

In contrast to the result in [72] for the S-S index, we shall find that the normalized Bz indices for the major players do not necessarily converge to a limit if no regularity conditions are imposed on the manner in which the minor weights go to zero. Further, the limiting Bz indices, when we do have convergence, depend rather strangely on the quota c; in fact, for much of the domain $R = (0, w(N)]$ of c, they are identically zero. But there is an interior region in R in which the major players are not "destroyed," and we are pleasantly surprised to find that in this region their limiting indices can be computed easily from another weighted majority game, involving just themselves and having a suitably reduced quota. This region consists of a certain open interval I in R from which a finite number of points have been deleted; these are the curious "pitfall points" that crop up in I where the major players are all "destroyed" simultaneously and have no power at all, in the "β" sense. At such values of c, the minor-player swings suddenly become so numerous that the *relative* number of major-player swings goes to zero.

Specifically, let $\{\Gamma^\nu : \nu = 1, 2, \dots\}$ be a sequence of weighted majority games, as follows:

$$\Gamma^\nu = [c; w_1, \dots, w_l, \alpha_1^\nu, \dots, \alpha_{m^\nu}^\nu], \tag{13}$$

where l is a fixed positive integer. We require that

$$\sum_{i=1}^{m^\nu} \alpha_i^\nu = \alpha, \quad \text{for each } \nu, \tag{14}$$

where $\alpha > 0$, and that

$$\alpha_{\max}^\nu \to 0, \quad \text{as } \nu \to \infty, \tag{15}$$

where $\alpha_{\max}^\nu$ denotes the maximum of the α_i^ν, $i = 1, \dots, m^\nu$. The set of *major players*, indexed $1, 2, \dots, l$, is denoted L; the set of *minor players*, indexed for convenience[31] $1, 2, \dots, m^\nu$, is denoted M^ν. Note that necessarily $m^\nu \to \infty$, since $m^\nu \alpha_{\max}^\nu \geqslant \alpha$. Note also that the minor players do not retain their identity from one game Γ^ν to the next.

Define

$$\begin{aligned} R &= \{c : 0 < c \leqslant w(L) + \alpha\}, \\ I &= \{c : \alpha/2 < c < w(L) + \alpha/2\}, \\ Z &= R - I, \\ P &= \{c : c = w(S) + \alpha/2 \text{ for some } S \subseteq L\}. \end{aligned}$$

Thus, R is the range of possible values for the quota c, while I, Z, and P are various special subsets of that range; the letters chosen are meant to suggest "interior," "zero," and "pitfall," respectively.

Let Γ_0 denote the weighted majority game that would result if the total weight of the minor players were distributed exactly half "yea" and half "nay," thus

$$\Gamma_0 = [c - \alpha/2; w_1, \dots, w_l]. \tag{16}$$

[31] L and M^ν have no members in common, despite the numbering convention. That is, the jth minor player is really the $(l + j)$th player of the game.

The relevance of this game to the limiting situation should be apparent from the probability model discussed in §2, since we should expect that the votes of a continuous "ocean" of coin-tossing minor players would be equally divided. Of course, for Γ_0 to be well defined, we must have $0 < c - \alpha/2 \leqslant w(L)$; in other words, c must lie in the half-closure of I.

We shall first tackle the symmetric case, where all the minor players have equal weight. The following theorem will be proved in the next two sections:

THEOREM 7. *In the sequence of weighted majority games defined by* (13)–(15), *if* $\alpha_j^\nu = \alpha/m^\nu$ *for all* $j = 1, \ldots, m^\nu$ *and all* ν, *then*

$$\lim_{\nu\to\infty} \beta_i^\nu = \begin{cases} \beta_i(\Gamma_0) & \text{if } c \in I - P, \\ 0 & \text{if } c \in Z \cup P, \end{cases} \tag{17}$$

for each major player $i \in L$.

7. The symmetric case: One major player. The argument for theorem 7 will be best conveyed if we consider first the case where there is only one major player, i.e., $l = 1$. To determine the number of swings for that player we must count the number of subsets of minor players having weight less than c but not less than $c - w_1$. Since each minor player has individual weight[32] α/m, these subsets are just those having k members, where $0 \leqslant k \leqslant m$ and

$$\frac{(c - w_1)m}{\alpha} \leqslant k < \frac{cm}{\alpha}. \tag{18}$$

It is convenient to define r_1, r_0 by[33]

$$r_1 = \left\langle \frac{(c - w_1)m}{\alpha} \right\rangle - 1, \qquad r_0 = \left\langle \frac{cm}{\alpha} \right\rangle - 1.$$

The number of major swings is then given by

$$\eta_1 = \sum_{k = r_1 + 1}^{r_0} \binom{m}{k}, \tag{19}$$

it being understood that $\binom{m}{k} = 0$ if $k > m$ or $k < 0$.

Minor swings, on the other hand, can arise in two ways: either from a subset of r_0 minor players (i.e., just short of a winning coalition), or from a subset of r_1 minor players together with the lone major player. The former type of situation produces $m - r_0$ minor swings, the latter $m - r_1$ minor swings. Hence the total number of minor swings is

$$\sum_{j \in M} \eta_j = (m - r_0)\binom{m}{r_0} + (m - r_1)\binom{m}{r_1}. \tag{20}$$

Since we are working with the normalized Banzhaf index, everything depends on the ratio of (19) to (20) as we pass to the limit. If (19) dominates, then the limit of $\beta_1 = \eta_1/\bar{\eta}$ will be 1; if (20) dominates, it will be 0. Only if (19) and (20) are of the same order of magnitude could we obtain intermediate values between 0 and 1. The following lemma gives us the necessary test for dominant terms; the crucial question proves to be whether or not the summation in (19) includes a "neighborhood of the center" of the binomial sequence $\binom{m}{0}, \binom{m}{1}, \ldots, \binom{m}{m}$.

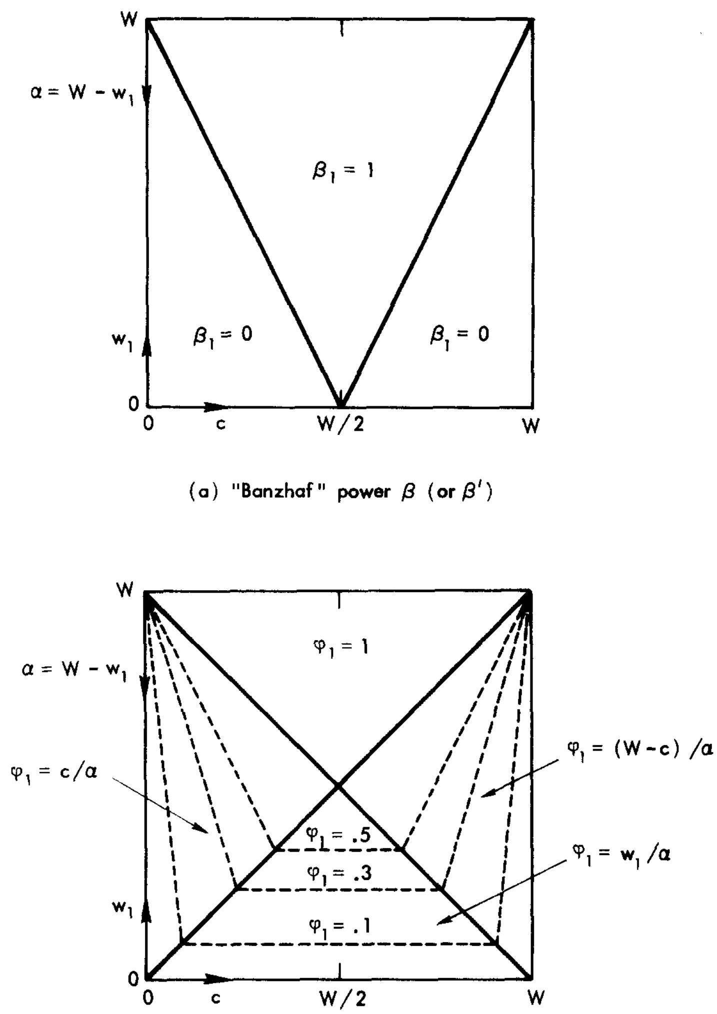
[32] We are omitting the superscript "ν" until it is needed again.

[33] Recall that $\langle x \rangle$ means the least integer $\geqslant x$.

LEMMA 1. *Let $0 \leqslant r < s \leqslant 1$. Let $m \to \infty$. Then*

$$\frac{\sum_{k=\langle rm\rangle}^{\langle sm\rangle}\binom{m}{k}}{m\binom{m}{\langle sm\rangle}} \to \begin{cases} \infty & \text{if } s > 1/2, \\ 0 & \text{otherwise.} \end{cases} \tag{21}$$

PROOF. If p and q are fixed real numbers between 0 and 1 with $p \neq q$ and $p \neq 1 - q$, and if m is large, then the ratio $\binom{m}{\langle pm\rangle}/\binom{m}{\langle qm\rangle}$ behaves exponentially in m.[34] It then becomes a simple exercise to verify (21). ∎



(a) "Banzhaf" power β (or β')

(b) "Shapley-Shubik" power φ

FIGURE 1. Limiting power indices with one major player.

[34] In fact, Stirling's approximation yields

$$\binom{m}{pm}\Big/\binom{m}{qm} \cong \sqrt{\frac{q(1-q)}{p(1-p)}}\left(\frac{q^q(1-q)^{1-q}}{p^p(1-p)^{1-p}}\right)^m$$

Returning now to the task of comparing (19) and (20), we see from lemma 1 that if $r_1/m \geqslant 1/2$ in the limit, then (20) eventually dominates (19) and we have $\beta_1^{\nu} \to 0$. The same happens if $r_0/m \leqslant 1/2$ in the limit. These cases correspond to $c \geqslant \alpha/2 + w_1$ and $c \leqslant \alpha/2$, respectively, or in other words, to $c \in Z$. In the remaining case, $r_1 < m/2 < r_0$; however, it is (19) that finally dominates, so the minor players are "wiped out" and $\beta_1^{\nu} \to 1$. This case corresponds to $\alpha/2 < c < \alpha/2 + w_1$, or in other words, to $c \in I$. Since the pitfall points do not matter with only one major player, i.e., since $P \cap I = \emptyset$, we have now verified theorem 7 for the case $l = 1$.

Let us pause to compare the Banzhaf power with the Shapley-Shubik power for the case of one major player and a limiting "ocean" of minor players [49]. There are essentially just two free parameters, since the voting unit is arbitrary. If we hold the total vote W constant, the allowed values of c and w_1 describe a square, with α determined by $w_1 + \alpha = W$ (figure 1). The Bz indices[35] make the major player out to be either powerless or all-powerful. The S-S indices give him a smaller dictatorial region, but no region where he is completely "wiped out." The sharpest discrepancy occurs near $c = W/2$ with w_1 small. For example, the main stockholder in a company might have 10% of the shares with the other individual holdings very small. Then under majority rule he has virtually all the Bz power but only about one ninth of the S-S power. On the other hand, if a three-fifths majority is required, the big man's Bz power drops to 0 while his S-S power remains at $1/9$.

8. The symmetric case: Many major players. We now remove the restriction $l = 1$, but retain the symmetry of the minor players: $\alpha_j = \alpha/m$. Let i be a major player, let S be a set of major players including i, and let $\eta_{i,S}$ denote the number of swings for i of the form $(T, T - \{i\})$ with $T \cap L = S$. Clearly

$$\eta_i = \sum_{S : i \in S \subset L} \eta_{i,S}. \tag{22}$$

For any such swing, the number $k = |T - S|$ of minor players in T must satisfy:

$$\frac{(c - w(S))m}{\alpha} \leqslant k < \frac{(c - w(S - \{i\}))m}{\alpha},$$

in analogy to (18) above. Accordingly, if we define

$$r_S = \left\langle \frac{(c - w(S))m}{\alpha} \right\rangle - 1,$$

we have

$$\eta_{i,S} = \sum_{k = r_S + 1}^{r_{S-\{i\}}} \binom{m}{k}, \tag{23}$$

in analogy to (19) above. With (22) this gives the number of swings for the major player i.

The total number of minor swings, on the other hand, is easily seen to be just the following generalization of (20):

$$\sum_{j \in M} \eta_j = \sum_{S \subset L} (m - r_S)\binom{m}{r_S}. \tag{24}$$

[35] Figure 1a serves equally well for the probability index β_1'. The only difference occurs along the lines $c = (W \pm w_1)/2$, where we have $\lim \beta_1^{\nu} = 0$ and $\lim \beta_1'^{\nu} = 1/2$. (See theorem 8.)

We must now search for the dominant terms among (23) and (24). Note that as m goes to the limit the number of expressions (23), i.e., the range of i and S, is fixed. Also, the number of terms in the summation on S in (24) is fixed.

First, recalling lemma 1, we observe that (24) will surely dominate in the limit if one of the terms $\binom{m}{r_S}$ is "central" in the sequence of binomial coefficients $\binom{m}{0}, \ldots, \binom{m}{m}$, that is, if $r_S/m \to 1/2$ for at least one S in L. This will happen if and only if we have $(c - w(S))/\alpha = 1/2$ for at least one S in L, and the reader will recognize this as the definition of a "pitfall." Hence we have

$$c \in P \Rightarrow \lim_{\nu \to \infty} \beta_i^\nu = 0 \quad \text{for each } i \in L. \tag{25}$$

Next, if none of the $\binom{m}{r_S}$ in (24) is "central," lemma 1 tells us it is still possible for (24) to dominate in the limit if at least one $\binom{m}{r_S}$ is as near to the "center" as any of the terms $\binom{m}{k}$ that appear in the expressions (23). But these terms (considering all values of i and S) cover the entire range of integers k from r_L up to $r_\varnothing$. In order to avoid covering the "center" we must have either $\lim(r_L/m) > 1/2$ or $\lim(r_\varnothing/m) < 1/2$. These conditions define our region Z, minus the boundary points $c = \alpha/2$ and $c = w(L) + \alpha/2$. The latter are members of P, however, so we can assert

$$c \in Z - P \Rightarrow \lim_{\nu \to \infty} \beta_i^\nu = 0 \quad \text{for each } i \in L, \tag{26}$$

which completes the proof of the bottom line of (17) in theorem 7.

In all the remaining cases, i.e., whenever $c \in I - P$, at least one of the sums (23) will include a "central" term and (24) will not. Hence the proportion of *minor* swings will be negligible in the limit in these cases, and it remains only to establish the relative distribution of swings among the major players. (This task was unnecessary in the case $l = 1$.)

An easy way to do this is to adopt the probability viewpoint and observe that for $0 \leqslant p < q \leqslant 1$, the expression

$$\frac{1}{2^m} \sum_{k = \langle pm \rangle}^{\langle qm \rangle - 1} \binom{m}{k}$$

represents the probability that the fraction of successes in m fair coin tosses lies between p and q. The limit of this probability, as $m \to \infty$, is of course 1 if $p < 1/2 < q$ and 0 if $p > 1/2$ or $q < 1/2$. Hence, returning to (23), we see that $\eta_{i,S}/2^m$ converges to 1 if

$$\frac{c - w(S)}{\alpha} < \frac{1}{2} < \frac{c - w(S - \{i\})}{\alpha} \tag{27}$$

and converges to 0 if

$$\frac{c - w(S)}{\alpha} > \frac{1}{2} \quad \text{or} \quad \frac{c - w(S - \{i\})}{\alpha} < \frac{1}{2}.$$

Since we are now assuming $c \in I - P$, we do not have to worry about equality here, so by (22) the relative number of swings for i is proportional in the limit to *exactly the number of sets S in L that satisfy* (27). But (27), still assuming $c \notin P$, is equivalent to

$$w(S) \geqslant c - \alpha/2 > w(S - \{i\}),$$

which is just the condition for swinging in the game Γ_0. This, with (25) and (26), completes the proof of theorem 7. ∎

Returning to the example of the stockholders, suppose that there are now *two* major interests owning nonnegligible fractions of the company, the rest of the shares being scattered.[36] Under majority rule (figure 2a), if they are different in size the larger of the two gets all the Bz power. But if they are the same size—say, 5% each—then both are "destroyed" by a pitfall (dotted line), despite the fact that each has a 50% chance of swinging (see theorem 8). Raising the winning threshold (figure 2b) brings in two new cases: a "middle" region E, surrounded by pitfalls, where the big players split the power equally, and a "zero" region Z, defined by $\alpha/2 > W - c$, where the minor players are in control because they can form blocking coalitions by merely voting randomly.

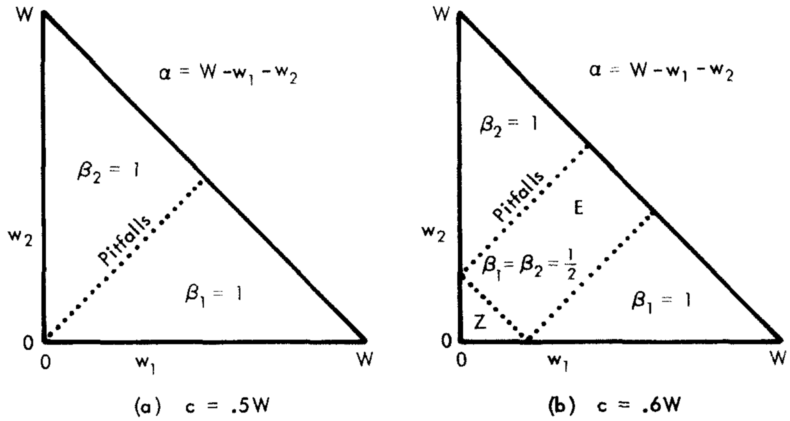


FIGURE 2. Limiting Bz indices with two major players.

9. Convergence of β'. We return to the general setting of §6 in which the minor weights are restricted only by the conditions $\sum \alpha_j^\nu = \alpha > 0$ and $\alpha_{\max}^\nu \to 0$. First we shall consider the probability index β', as it is better behaved. Let Γ_0, as before, denote the game

$$[c - \alpha/2; w_1, \dots, w_l],$$

and let Γ_0' denote the game

$$\langle c - \alpha/2; w_1, \dots, w_l \rangle.$$

These games are well defined for all $c \in I$ and they are identical unless $c \in P$. Thus, we could have used either one in theorem 7, but now we must distinguish between them.[37]

THEOREM 8. *In the sequence of games described by* (13)–(15), *we have*

$$\lim_{\nu \to \infty} \beta_i'^{\nu} = \begin{cases} \frac{1}{2}\beta_i'(\Gamma_0) + \frac{1}{2}\beta_i'(\Gamma_0') & \text{if } c \in I, \\ 0 & \text{if } c \in Z \end{cases} \tag{28}$$

for each major player $i \in L$.

The following two lemmas form the heart of the proof. Let us write $\alpha^\nu(S)$ for $\sum_{i \in S} \alpha_i^\nu$.

LEMMA 2. *Let $0 \leqslant q \leqslant \alpha$, and choose a subset S "at random" from M^ν, i.e., with probability $1/2^{m^\nu}$. Then*

$$\lim_{\nu \to \infty} \operatorname{Prob}\{\alpha^\nu(S) = q\} = 0. \tag{29}$$

[36] Cf. [49] for the S-S indices in this case.

[37] Compare (28) with (17). For generalizations of this result, see notes 1 and 2 in §12.

PROOF. Let $\mathbb{S}^{\nu}(q)$ denote the collection of $S \subset M^{\nu}$ such that $\alpha^{\nu}(S) = q$; we must show that $|\mathbb{S}^{\nu}(q)|/2^{m^{\nu}}$ goes to zero as $m^{\nu} \to \infty$. The members of $\mathbb{S}^{\nu}(q)$ are pairwise independent, that is, if $S \in \mathbb{S}^{\nu}(q)$, $T \in \mathbb{S}^{\nu}(q)$ and $S \subset T$, then $S = T$. Such a collection of sets is called a "clutter" or "Sperner family," and Sperner's well-known lemma [84][38] asserts that

$$|\mathbb{S}^{\nu}(q)| \leqslant \binom{m^{\nu}}{\langle m^{\nu}/2 \rangle}.$$

When divided by $2^{m^{\nu}}$, this goes to zero like $1/\sqrt{m^{\nu}}$. ∎

LEMMA 3. *Let $0 \leqslant p < q \leqslant \alpha$, and choose $S \subset M^{\nu}$ as in Lemma 2. Then*

$$\lim_{\nu \to \infty} \operatorname{Prob}\{ p \leqslant \alpha^{\nu}(S) < q \} = \begin{cases} 0 & \text{if } p > \dfrac{\alpha}{2} \text{ or } q < \dfrac{\alpha}{2}, \\ 1 & \text{if } p < \dfrac{\alpha}{2} < q, \\ \dfrac{1}{2} & \text{if } p = \dfrac{\alpha}{2} \text{ or } q = \dfrac{\alpha}{2}. \end{cases} \tag{30}$$

PROOF. The random variable $\alpha^{\nu}(S)$ may be written $\sum_1^{m^{\nu}} \alpha_i^{\nu} X_i$, where the X_i are independent random variables taking the values 1 and 0 with probability 1/2. Then

$$E\{\alpha^{\nu}(S)\} = \frac{1}{2} \sum_{i=1}^{m^{\nu}} \alpha_i^{\nu} = \frac{\alpha}{2}$$

and

$$\operatorname{Var}\{\alpha^{\nu}(S)\} = \sum_{i=1}^{m^{\nu}} (\alpha_i^{\nu})^2 \operatorname{Var}\{X_i\} = \frac{1}{4} \sum_{i=1}^{m^{\nu}} (\alpha_i^{\nu})^2 \leqslant \frac{\alpha_{\max}^{\nu} \alpha}{4}.$$

Chebyshev's inequality now yields

$$\operatorname{Prob}\{|\alpha^{\nu}(S) - \alpha/2| \geqslant t\} \leqslant \frac{\alpha_{\max}^{\nu} \alpha}{4t^2}$$

for any $t > 0$. Since $\alpha_{\max}^{\nu} \to 0$ as $\nu \to \infty$, we have

$$\lim_{\nu \to \infty} \operatorname{Prob}\{\alpha^{\nu}(S) < q\} = 0 \quad \text{if } q < \alpha/2$$

and

$$\lim_{\nu \to \infty} \operatorname{Prob}\{\alpha^{\nu}(S) \geqslant p\} = 0 \quad \text{if } p > \alpha/2,$$

giving us the first line of (30). The second line follows from the fact that the total probability is 1, for each finite ν and hence in the limit. Finally, line 3 of (30) follows from the fact that symmetric open intervals of the form $(r, \alpha/2)$ and $(\alpha/2, \alpha - r)$, with $r < \alpha/2$, must have equal probability in the limit, while their combined probability in the limit must be 1, since the single point between them is negligible by lemma 2. Hence their separate probabilities in the limit are 1/2, 1/2. ∎

PROOF OF THEOREM 8. The definition of $\beta_1'^{\nu}$ may be written

$$\frac{1}{2^{l-1}} \sum_{T : i \in T \subset L} \operatorname{Prob}\{c - w(T) \leqslant \alpha^{\nu}(S) < c - w(T - \{i\})\},$$

[38] Not to be confused with his equally famous lemma on labelled triangulations of the simplex. The proof of theorem 2 could have been based instead on a form of the Central Limit Theorem (see e.g. [24, pp. 101–102]); the present proof focuses attention on the possible "bunching up" of the partial sums $\alpha^{\nu}(S)$ when the α_i^{ν} are uncontrolled—a theme that will recur throughout §§10, 11.

where S is a random subset of M^{ν} as above. As we pass to the limit, the summand approaches 0, 1/2, or 1, by lemma 3. If $c \in Z$, the limit is always 0. If $c \in I$ the games Γ_0 and Γ_0' are well defined, and for any i and T for which the limit is 1 we have a swing for i in both games. When the limit is 1/2, we have a swing in exactly one of these games, and when the limit is 0 we have a swing in neither. The proof is completed by the observation that $\beta'(\Gamma_0) = \eta(\Gamma_0)/2^{l-1}$ and $\beta'(\Gamma_0') = \eta(\Gamma_0')/2^{l-1}$. ∎

10. The asymmetric case: A counterexample. Unfortunately, it is not always true that the normalized Banzhaf indices converge as in theorem 7 when the minor weights are allowed to go to zero in arbitrary fashion, even if we rule out the "pitfall" quotas $c \in P$. The major problem is caused by the minor swings. We have just seen that the probability indices $\beta_i'^{\nu}$ *do* converge in general, so the trouble can be traced to the behavior of the denominator. In fact, for the major β indices to converge (when c is in $I - P$), the sum of the minor β' indices must go to zero. In other words (taking out the constant 2^{l-1}), convergence when $c \in I - P$ requires that

$$\lim_{\nu \to \infty} \frac{\sum_{j \in M^{\nu}} \eta_j^{\nu}}{2^{m^{\nu}}} = 0. \tag{31}$$

In this section we shall show by a construction that if the distribution of minor weights is not controlled in some way, the ratio in (31) may instead go to infinity, or perhaps not converge at all. The idea will be to set up two populations of minor players, with one group very much more numerous than the other. A "pitfall" type situation will then be created in which there are inordinately many swings by the smaller minor players.

Consider first a weighted majority game with $m = 3m_1 + 2m_2$ players, as follows:

$$\Gamma_{m_1 m_2} = \Big[2\gamma + \delta;\ \underbrace{\frac{\gamma}{m_1}, \ldots, \frac{\gamma}{m_1}}_{3m_1},\ \underbrace{\frac{\delta}{m_2}, \ldots, \frac{\delta}{m_2}}_{2m_2}\Big]. \tag{32}$$

Here γ and δ are positive constants, and the total voting weight is $3\gamma + 2\delta$. We shall focus on the coalitions that have exactly $2m_1$ members of the first type and m_2 of the second. The number of such coalitions is

$$\binom{3m_1}{2m_1}\binom{2m_2}{m_2},$$

which works out by Stirling's approximation to

$$\frac{\sqrt{3}\,2^m}{2\pi\sqrt{m_1 m_2}}\left(\frac{27}{32}\right)^{m_1}\left(1 - O\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right).$$

Such a coalition is minimal winning and so contributes $2m_1 + m_2$ swings, one for each player in it. We therefore have an asymptotic lower bound for the total number of swings:

$$\bar{\eta}(\Gamma_{m_1 m_2}) \geq (2m_1 + m_2)\frac{2^m J}{\sqrt{m_1 m_2}\,K^{m_1}}\left(1 - O\left(\frac{1}{m_1} + \frac{1}{m_2}\right)\right), \tag{33}$$

where J and K are constants with $J > 0$ and $K > 1$.

We now send m_1 and m_2 to infinity in such a way that the second population grows much more rapidly than the first. The idea is to make the ratio of m_2 to $m_1 K^{2m_1}$ go to infinity so that the denominator in (33) will be $o(m_2)$, giving us the desired "explosion" in the expected number of swings. More precisely, we shall require that m_1 go to infinity satisfying

$$m_1 < \frac{h \log m}{\log K} \tag{34}$$

for some $0 < h < 1/2$. Then we have $K^{m_1} < m^h$, $m_1 = o(m)$, and $m_2 = O(m)$, so (33) yields

$$\bar{\beta}(\Gamma_{m_1 m_2}) = \bar{\eta}(\Gamma_{m_1 m_2})/2^{m-1} \geqslant O\big(m^{1/2-h}/\sqrt{\log m}\,\big) \to \infty. \quad \blacksquare \tag{35}$$

In order to apply this construction to our general set-up (13) with both major and minor players, we introduce major weights $w_1, \ldots, w_l$, a total minor weight $\alpha > 0$, and a quota $c \in I - P$. Let R be any coalition of major players satisfying

$$\alpha/2 < c - w(R) < 2\alpha/3. \tag{36}$$

If R votes "yea" and $L - R$ votes "nay" then the minor players in effect face a reduced quota of $c - w(R)$. Fix the values of γ and δ by

$$\begin{cases} 3\gamma + 2\delta = \alpha \\ 2\gamma + \delta = c - w(R) \end{cases} \quad \text{i.e.,} \quad \begin{cases} \gamma = 2(c - w(R)) - \alpha \\ \delta = 2\alpha - 3(c - w(R)). \end{cases}$$

By (36), γ and δ are positive. If we now group the minor players into two populations as in (32) and let them grow satisfying (34), then the "explosion" (35) will overwhelm the major players' swings, causing their proportion of the total to converge to 0 and so wiping out their normalized Bz indices.

Moreover, since some or all of the major β indices converge to positive numbers when the minor weights are symmetric as in theorem 7, it is clear that a sequence of games could be constructed for which they do not converge at all.

Condition (36) is quite unexceptional, but if the weights w_i are such that no coalition $R \subset L$ satisfies (36) then some simple modification of this construction, like changing the "3" in "$3m_1$" to a higher integer or using duality, may do the trick. In fact, it can be shown that convergence for *any* set of major weights can be spoiled by a counterexample of this type, provided only that the numbers α and c are such that c is in $I - P$; we omit the details.

11. Convergence of β. §10 shows that the assignment of votes to the minor players must be restricted somehow if the normalized Bz indices of the major players are to behave well in the limit. Several ways of doing this might be considered. The important thing seems to be to make the distribution of minor-player votes reasonably smooth, in some sense, so as to keep the "pitfall" situation under control. One device might be to require that the ratio of $\alpha^{\nu}_{\max}$ to $\alpha^{\nu}_{\min}$ stay bounded as the number of players increases, or at least that it not grow too rapidly. Another way would be to ignore $\alpha^{\nu}_{\min}$ (i.e., permit some players to be very small), but require that the ratio of $\alpha^{\nu}_{\max}$ to the *mean* minor weight α/m^{ν} stay bounded, or at least not grow too fast. A third way might be to insist that the standard deviation of the minor weights go to zero sufficiently rapidly. Of course, these approaches are not entirely independent, being interlinked through certain inequalities.

We shall concentrate on the first and second approaches, i.e., on conditions for the asymptotic behavior of $\alpha^{\nu}_{\max}/\alpha^{\nu}_{\min}$ and $m^{\nu}\alpha^{\nu}_{\max}$ as $\nu \to \infty$ (see (44) and (37) below). The latter will yield a weaker result of the third kind as a corollary. We shall repeatedly refer to the two-population example of §10, in order to indicate how much room there might be for sharper bounds.

We remark that the proofs in this section make use of theorem 8 as well as some advanced tools of probability theory, but are independent of theorem 7 (the symmetric case), which they to some extent generalize.

THEOREM 9. *In the sequence of weighted majority games defined by* (13)–(15), *if* $c \in I - P$ *and if*

$$m^{\nu}\alpha^{\nu}_{\max} = o\left(\sqrt{m^{\nu}/\log m^{\nu}}\right), \tag{37}$$

then $\beta_i^{\nu} \to \beta_i(\Gamma_0)$ *for each* $i \in L$.

We do not know if (37) can be sharpened. It is easily verified from (34) that in §10 we had $\alpha^{\nu}_{\max} = O(1/\log m^{\nu})$, so the uncertainty about convergence is reduced to the gap between $1/\sqrt{m^{\nu}\log m^{\nu}}$ and $1/\log m^{\nu}$.

The proof of theorem 9 is based on the following theorem of Kemperman's, which we quote here directly from [36, p. 113]:

LEMMA 4. *Let* $\{Z_k : k = 0, 1, \dots\}$ *be a sequence of independent real-valued random variables such that* $|Z_k| \leqslant 1$ *for all* k. *Let further* c_k *be a sequence of real constants such that*

$$s^2 = \sum_{k=0}^{\infty} c_k^2 < \infty \qquad (s \geqslant 0).$$

Then

$$S = \sum_{k=0}^{\infty} c_k(Z_k - m_k) \qquad (m_k = \mathrm{E}\{Z_k\}),$$

satisfies

$$\mathrm{Prob}\{S > \delta s\} \leqslant e^{-\delta^2/2}, \qquad \mathrm{Prob}\{S < -\delta s\} \leqslant e^{-\delta^2/2}, \tag{38}$$

for each number $\delta > 0$.

This lemma in effect estimates the "tails" of a sum of random variables. To apply it to our problem, fix ν and let Z_k for each $k = 1, \dots, m^{\nu}$ be the random variable that takes on the value $\alpha_k^{\nu}/\alpha^{\nu}_{\max}$ with probability $1/2$ and is otherwise 0. For $k > m^{\nu}$ we let $Z_k \equiv 0$. For the constants c_k, we set the first m^{ν} of them equal to $\alpha^{\nu}_{\max}$ and the rest equal to 0; this makes the "s" of the lemma equal to $\sqrt{m^{\nu}}\,\alpha^{\nu}_{\max}$. The "$S$" of the lemma is just the difference between our random variable $\alpha^{\nu}(T)$ and its mean, $\alpha/2$, where T is a randomly chosen subset of M^{ν} like the "S" in lemmas 2 and 3. With these identifications, (38) enables us to conclude that for any positive number ϵ,

$$\mathrm{Prob}\{\alpha^{\nu}(T) > \alpha/2 + \epsilon\} = \mathrm{Prob}\{S > \epsilon\} \leqslant e^{-\delta^2/2}. \tag{39}$$

where δ is an abbreviation for $\epsilon/s = \epsilon/(\sqrt{m^{\nu}}\,\alpha^{\nu}_{\max})$. For convenience, define $A_{\nu} = \sqrt{m^{\nu}\log m^{\nu}}\,\alpha^{\nu}_{\max}$. Then (39) reduces to

$$\mathrm{Prob}\{\alpha^{\nu}(T) - \alpha/2 > \epsilon\} \leqslant (m^{\nu})^{-\epsilon^2/2A_{\nu}^2}. \tag{40}$$

Since our hypothesis is that A_{ν} goes to zero, (40) will enable us to prove theorem 9.

Indeed, we have for each ν,

$$\beta_i^{\nu} = \frac{\beta_i'^{\nu}}{\sum_{j \in L} \beta_j'^{\nu} + \sum_{k \in M^{\nu}} \beta_k'^{\nu}}. \tag{41}$$

We must show that the second term in the denominator of (41), i.e., the expected number of minor swings, goes to zero. Note that no minor swing can occur unless $\alpha^\nu(T)$ is within the distance $\alpha^\nu_{\max}$ of $c - w(R)$ for some $R \subset L$. But for any fixed quota c in the open set $I - P$, there is an $\epsilon > 0$ such that for every $R \subset L$,

$$|c - w(R) - \alpha/2| > 2\epsilon.$$

For sufficiently large ν we have $\alpha^\nu_{\max} < \epsilon$, and so our necessary condition for a minor swing, namely

$$|\alpha^\nu(T) - (c - w(R))| \leqslant \alpha^\nu_{\max},$$

makes it necessary that:

$$|\alpha^\nu(T) - \alpha/2| > \epsilon.$$

But there are at most m^ν minor swings for any particular $R \subset L$, and at most 2^l possible choices of R. Thus the expected total number of minor swings must satisfy

$$\sum_{k \in M^\nu} \beta_k'^\nu \leqslant 2^l m^\nu \operatorname{Prob}\{|\alpha^\nu(T) - \alpha/2| > \epsilon\}. \tag{42}$$

By (40), we therefore have

$$\sum_{k \in M^\nu} \beta_k'^\nu \leqslant 2^{l+1}(m^\nu)^{1-\epsilon^2/2A_\nu^2}. \tag{43}$$

(There is an extra factor of 2 here to take care of the absolute value in (42).) Now, as A_ν approaches zero, the exponent of m^ν eventually becomes negative, so the right hand side of (43) goes to zero. Hence we may pass to the limit in (41) and, with the aid of theorem 8, obtain

$$\lim_{\nu \to \infty} \beta_i^\nu = \frac{\beta_i'(\Gamma_0)}{\sum_{j \in L} \beta_j'(\Gamma_0)} = \beta_i(\Gamma_0).$$

This completes the proof of theorem 9. ∎

We now consider the ratio of $\alpha^\nu_{\max}$ to $\alpha^\nu_{\min}$. Since $\alpha^\nu_{\min} \leqslant \alpha/m^\nu$, theorem 9 allows a modest rate of growth, namely $\alpha^\nu_{\max}/\alpha^\nu_{\min} = o(\sqrt{m^\nu/\log m^\nu}\,)$, without disrupting convergence. The next theorem improves on this.

THEOREM 10. *In the sequence of weighted majority games defined by* (13)–(15), *if* $c \in I - P$ *and if*

$$\alpha^\nu_{\max}/\alpha^\nu_{\min} = o(m^\nu/\log m^\nu), \tag{44}$$

then $\beta_i' \to \beta_i(\Gamma_0)$ *for each major player* $i \in L$.

PROOF. A result of W. Hoeffding [32][39] states that if $\alpha_1, \ldots, \alpha_m$ are positive numbers totalling α, and if a subset T of $\{1, \ldots, m\}$ is chosen at random, then for any $\epsilon > 0$,

$$\operatorname{Prob}\left\{\alpha(T) > \frac{\alpha}{2} + \epsilon\right\} \leqslant \exp\left\{-\frac{8m\epsilon^2\theta}{\alpha^2(1+\theta)^2}\right\}, \tag{45}$$

where θ denotes $\alpha_{\max}/\alpha_{\min}$. The right-hand side of (45) may be rewritten as m to the

[39] Or see (4.8) in [37]. We are grateful to J. H. B. Kemperman for pointing out this result.

power

$$\left\{-\frac{8\epsilon^2/\alpha^2}{(\log m)(\theta+2+1/\theta)/m}\right\}.$$

If (44) holds, the denominator of this exponent goes to zero. (Note that $1/\theta$ is bounded.) So we are in the same position as at (40) in the proof of theorem 9 and can finish the proof in the same way. ∎

To illustrate theorem 10, consider a two-population situation in which the number of "bigger" minor players goes to ∞ no faster than $O(\log m^\nu)$ (compare (34)). In order for the major Bz indices β_i to converge, (44) tells us that it is sufficient that $\alpha_{\max} = o(1/\log m^\nu)$, since clearly $\alpha_{\min} = O(1/m^\nu)$. On the other hand, our example of nonconvergence has $\alpha_{\max} = O(1/\log m^\nu)$, so for this type of situation the gap of uncertainty about convergence is almost closed.

COROLLARY 4. *Let $\sigma(\nu)$ denote the standard deviation of the set of numbers $\{\alpha_j^\nu : j \in M^\nu\}$. If $c \in I - P$, and if*

$$\sigma(\nu) = o\left(1/m^\nu\sqrt{\log m^\nu}\right), \tag{46}$$

then $\beta_i^\nu \to \beta_i(\Gamma_0)$ for each $i \in L$.

PROOF. By considering the "worst case" (one weight equal to $\alpha_{\max}$, all others equal to $\alpha_{\min}$), we find that $\alpha_{\max} \leqslant \alpha/m + \sqrt{m-1}\,\sigma$. Hence (46) implies

$$\alpha_{\max}^\nu \leqslant O(1/m^\nu) + o\left(1/\sqrt{m^\nu \log m^\nu}\right),$$

which implies (37). ∎

The construction in §10 shows that it is possible to have nonconvergence with $\sigma(\nu) = O(1/\sqrt{m^\nu \log m^\nu})$.

This concludes our study of the finer points of the troublesome behavior of the normalized Bz index. It may be remarked that somewhat similar convergence problems arise in the study of the "asymptotic μ-values" that have recently been defined for certain general classes of games with a continuum of players: see Hart [30].

12. Remarks and extensions. This final section is a kind of appendix, presenting a miscellany of refinements, extensions, and alternative interpretations of the mathematical theory developed in the preceding pages. The reader's attention is particularly directed to notes 4 through 7, which make connections to the literature of political science and electrical engineering where several concepts formally equivalent to the Banzhaf index have been introduced.

Note 1. It is only natural to try and extend the probability model in §2 by introducing parameters p_i, $0 < p_i < 1$, $i \in N$, and making the players vote "yea" and "nay" with probability p_i and $1 - p_i$, respectively, rather than 1/2, 1/2.[40] The generalized Bz probability index is then given by

$$\beta_i'[p] = \sum_{S\,:\,i \in S \subset N} P_{S,i}\left[v(S) - v(S - \{i\})\right], \tag{47}$$

where $P_{S,i}$ denotes the probability that $Y = S - \{i\}$:

$$P_{S,i} = \left(\prod_{j \in S - \{i\}} p_j\right)\left(\prod_{j \in N - S}(1 - p_j)\right). \tag{48}$$

[40] Since these probabilities are essentially subjective, existing only in the minds of the other players, it might be better to go at once to doubly-indexed parameters p_{ij}, $i \neq j$, representing *j's estimate of i's probability of voting "yea."* Another generalization, where the voting probabilities are themselves random variables, not necessarily independent, has been considered by Straffin [85]; see also Blair [9] and Dubey [22].

These indices can of course be normalized, if desired, by defining $\beta_i[p] = \beta'_i[p] / \sum_N \beta'_j[p]$.

We note that $\beta'[p]$ satisfies axiom A4 (see §3), and that it also satisfies axiom A3 when all the p_i's are equal. It is interesting also that theorem 6 holds for $\beta'[p]$ (see §5), although the accompanying corollary 3, which depends on duality, does not.

Turning to the weighted majority games Γ^ν of §6, suppose that each major player $i \in L$ belongs to Y with probability p_i^ν and each minor player $j \in M^\nu$ belongs to Y with probability q_j^ν. Then the swing probability for $i \in L$ is given by

$$\beta'_i[p^\nu, q^\nu] = \sum_{S : i \in S \subset L} \sum_{T \subset M^\nu} P^\nu_{S \cup T, i}[v^\nu(S \cup T) - v^\nu((S - \{i\}) \cup T)],$$

where v^ν is the characteristic function of the game Γ^ν and $P_{S \cup T, i}$ (compare (48)) is given by

$$P_{S \cup T, i} = \prod_{k \in S - \{i\}} p_i^\nu \prod_{k \in L - S} (1 - p_i^\nu) \prod_{j \in T} q_j^\nu \prod_{j \in M^\nu - T} (1 - q_j^\nu).$$

In order to make the indices $\beta'_i[p^\nu, q^\nu]$ converge, i.e., in order to generalize theorem 8, we shall need some conditions on the parameters p^ν, q^ν as $\nu \to \infty$. For each major player i, we obviously must assume that p_i^ν converges to some limit, say p_i^*. For the minor players, what we need to know is that the weight $\alpha^\nu(T)$ of the random subset $T = M^\nu \cap Y$ will *converge in probability* to its mean, which is simply

$$\mu^\nu = \sum_{j \in M^\nu} q_j^\nu \alpha_j^\nu;$$

moreover this mean must converge in turn to some limit, say μ^*. By a theorem of Gnedenko and Kolmogorov,[41] a necessary and sufficient condition for the convergence in probability is that

$$\sum_{j \in M^\nu} \operatorname{Prob}\{|\alpha^\nu - \mu^\nu| > 1\} \to 0 \tag{49}$$

and

$$\sum_{j \in M^\nu} q_j^\nu (1 - q_j^\nu)^2 (\alpha_j^\nu)^2 \to 0. \tag{50}$$

Both (49) and (50) are easily seen to be satisfied in our case, because of the assumption that $\alpha^\nu_{\max} \to 0$. Indeed, the first expression is identically zero for $\alpha_{\max} < 1$, while the second expression is bounded by $\sum (\alpha_j^\nu)^2$, and hence by $\sum \alpha_j^\nu \alpha_{\max} = \alpha \alpha^\nu_{\max}$.

So we need to assume only that the parameters $p_1^\nu, \ldots, p_l^\nu$ and μ^ν, namely, the probability of each major player voting "yea" and the mean weight of the set of yea-voting minor players, all converge to their respective limits as $\nu \to \infty$. The same methods as in §9 then lead us to the desired generalization of theorem 8, namely,

$$\lim_{\nu \to \infty} \beta'_i[p^\nu, q^\nu] = \begin{cases} \frac{1}{2}\beta'_i[p^*](\Delta_0) + \frac{1}{2}\beta'_i[p^*](\Delta'_0), & \text{if } \mu^* < c < w(L) + \mu^*, \\ 0, & \text{if otherwise.} \end{cases}$$

Here Δ_0, Δ'_0 are the weighted majority games $[c - \mu^*; w_1, \ldots, w_l]$ and $\langle c - \mu^*; w_1, \ldots, w_l \rangle$ respectively. Note that the set of "pitfall points" (i.e., values of c for which Δ_0 and Δ'_0 are actually different) now depends on μ^*.

[41] See [24, p. 105].

Note 2. There are some fairly obvious extensions of §§6–11 to the case where any or all of the parameters $c, w_1, \ldots, w_l$, and α also depend on ν, converging to stated limits as $\nu \to \infty$. We omit the details.[42]

Note 3. Sergiu Hart in [29] has defined the *swing probability* of a simple game to be the probability of getting a swing when two adjacent coalitions are chosen at random.[43] If there are n players, each such coalition-pair (i.e., each edge of the n-cube) has probability $1/(n2^{n-1})$, and so Hart's swing probability in our notation is just $\bar{\beta}'/n$. Applying his bound on the total number of swings (our theorem 4), Hart shows that in any sequence of n^ν-player games with $n^\nu \to \infty$, if the *fraction of coalitions that are winning* converges to a limit other than 0 or 1, then the swing probabilities π^ν of these games satisfy

$$\liminf_{\nu \to \infty} n^\nu \pi^\nu > 0. \tag{51}$$

In other words, the number $\bar{\beta}'^\nu$, which represents the *expected number of swingers*, does not go to zero in such a sequence of games.[44] On the other hand, it is possible for this number to go to infinity, as we can see, e.g., from theorem 3 with $m = [n/2]$.

Note 4. Douglas Rae in a 1969 paper [64] addressed the problem of comparing the *responsiveness* of different voting systems to the general will of the electorate. His basic idea was to count the number of ways in which the average voter can find his vote in agreement with the outcome of the voting. Assuming symmetry, Rae considered only a single, generic voter, but it is natural to extend his approach by defining an "index of agreement" for each voter:[45]

$$\rho_i = \#\{ Y \subset N : i \in Y \in \mathcal{W} \text{ or } i \notin Y \notin \mathcal{W} \}, \tag{52}$$

where $\mathcal{W}$ denotes the set of all winning coalitions in a simple game $v \in \mathcal{S}(N)$. The overall responsiveness of the voting system may then be measured by the sum $\bar{\rho}$, or, if we prefer, by the average $\bar{\rho}/n$ or by the average probability of agreement $\bar{\rho}/(n2^n)$.

It was not noticed for several years that this "Rae index" is nothing but the Banzhaf index in disguise. In fact, the following identity holds:

$$\rho_i \equiv 2^{n-1} + \eta_i. \tag{53}$$

This may be seen readily from the following little table, in which the possible "yea-voting" sets Y are grouped into six classes according to how Y relates to $\mathcal{W}$ and i:

	$Y - \{i\} \in \mathcal{W}$	$Y \cup \{i\} \in \mathcal{W}$	$Y \ni i$	i agrees	i swings
1	true	true	true	√	×
2	true	true	false	×	×
3	false	true	true	√	√
4	false	true	false	√	√
5	false	false	true	×	×
6	false	false	false	√	×

[42] Cf. the analogous extension in [72].

[43] More generally, he considers the class of arbitrary (0, 1)-games with $v(\emptyset) = 0$, which he unconventionally calls the simple games.

[44] Actual convergence of the winning fraction is not essential; it is sufficient to assume that 0 and 1 are not limit points.

[45] See Straffin [88]. The term "satisfaction index" has also been proposed [11], [53], but we feel that this term quite unnecessarily injects considerations of *utility* (not to mention voting sincerity!) into what is otherwise a purely structural analysis. Having his vote agree with the outcome may or may not "satisfy" the voter.

Note that classes 1 and 2 are of equal size, being based on the same collection of subsets of $N - \{i\}$; similarly classes 3 and 4 and classes 5 and 6. Hence, the number of times player i is "in agreement" is just half the total (i.e., 2^{n-1} times), plus half the number of Y's that make him a swinger. As there are exactly $2\eta_i$ Y's that make i a swinger, (53) is proved.

In view of (53), it is a simple matter to translate results about $\bar{\eta}$ (see §4) into results about $\bar{\rho}$. Rae originally conjectured, and Taylor [89] proved that for any fixed number of voters *responsiveness is maximized by direct majority rule*. However, Rae and Taylor considered only the narrow class of symmetric games[46] $M_{n,k} = [k;\ 1, 1, \ldots, 1]$. Our theorem 2 shows that the conclusion holds over the much larger domain of *all* simple games. Moreover, the proof is very easy in the broader context, since one can proceed by altering just one coalition at a time. Theorem 2 also identifies the other games of maximum responsiveness, namely those obtained from majority rule (when n is even) by designating some of the $n/2$-player sets to be winning. These could be of practical value in committee design, as they enable the decisiveness of the voting rule to be enhanced without loss of responsiveness, superadditivity, or even symmetry.[47]

Note 5. James S. Coleman [14], [15] considered two kinds of power exercised by a member of a "collectivity" (his term for a simple game), namely, a power to *prevent* action and a power to *initiate* action. The former he took to be the fraction of all winning coalitions in which the player in question is essential, in that without him the coalition would lose. The latter he took to be the fraction of all losing coalitions which the player in question, by joining, could convert to winning. In our notation, these two "Coleman indices" come down to

$$\gamma_i = \eta_i/\omega, \qquad \gamma_i^* = \eta_i/\lambda. \tag{54}$$

(Recall our use of ω and λ to denote the total number of winning and losing coalitions, respectively.) There is an obvious duality here, given by $\gamma^*(v) = \gamma(v^*)$; that is, the "power to initiate" in any simple game is the same as the "power to prevent" in its dual (see §5). The two indices obviously give the same *relative* distribution of power, and in proper games the "prevent" power is always $\geqslant$ the "initiate" power, since $\lambda \geqslant \omega$. Both indices lie between 0 and 1, and we have $\gamma_i = 1$ if and only if i has a veto, $\gamma_i^* = 1$ if and only if i is a dictator, and $\gamma_i = \gamma_i^* = 0$ if and only if i is a dummy.

Of the three forms of the Banzhaf index defined in §2 the Coleman indices relate

[46] These are by no means the only symmetric simple games. For example, arrange the players $N = (1, \ldots, n)$ in a circle, select any $S \subset N$, and let the minimal winning coalitions be S and all its "rotations" (i.e., images under the cyclic permutations of N). The resulting game is symmetric in the players but is not an $M_{n,k}$ game if $1 < |S| < n - 1$. For another, more realistic example, group the players into d equal districts and let winning be defined as having a majority of district majorities. Again the result is player-symmetric but is not of the form $M_{n,k}$ if $n > 4$ and $1 < d < n$.

Rae and Taylor did generalize their model to the extent of using different voting probabilities, somewhat in the manner of note 1 above; and others [3], [16], [71] have followed this lead. But, surprisingly, none of these investigations look beyond $M_{n,k}$ for their underlying voting rule.

[47] For example, if n is even and > 4, we can find a set S with $n/2$ members whose complement is *not* one of its rotations (see above). By declaring S and all its rotations to be winning, we eliminate some of the possible deadlocks and so increase decisiveness—without sacrificing either player-symmetry or responsiveness. Such a game, however, will not be fully decisive.

The search for full decisiveness with symmetry proves to be a delicate problem in the theory of finite groups. Roughly speaking, such a "homogeneous game" (Isbell's term) exists only if n has a sufficiently large *odd* divisor. In his definitive paper [35], he settled a great many cases and observed in effect that the simple test: "Does n have an odd divisor greater than $\sqrt{n}$?" would cover all known results. Thus, homogeneous games exist of size 6, 10, 14, 18, 20, 22, etc., but not 2, 4, 8, 12, 16, 24, etc. At the time of [35], the only cases less than 100 still in doubt were 40, 56, 80, and 88. More recently, however, Peter Neumann (correspondence) has refuted the $\sqrt{n}$ criterion by producing a homogeneous game of size 56.

most closely to β'. Indeed, β_i' is just the harmonic mean of γ_i and γ_i^*:

$$\frac{1}{\beta_i'} = \frac{1}{2}\left(\frac{1}{\gamma_i} + \frac{1}{\gamma_i^*}\right),$$

as the reader may easily verify. It follows that for proper simple games we always have $\gamma_i^* \leqslant \beta_i' \leqslant \gamma_i$, with equality for all i if and only if the game is decisive.

Note 6. Robert A. Dahl proposed in a well-known 1957 paper [17] to define the *power of one individual over another* as the extent to which the first can get the other to do something he would not otherwise do, minus the extent that the second can similarly impose his will on the first. Mathematizing this "directed power" concept with the aid of correlations and conditional probabilities, Dahl proceeded to analyze some empirical political data with very interesting results.

More recently, Michael Allingham [1] set out to apply this notion to abstract voting systems. We quote from his explanation:

> "The group decision is binding on its members, so that the acts on which [the directed] power is defined are the favoring of a proposal (by the power applier) and the subjection to the proposal (by the power receiver—or indeed the whole group). Probability is introduced . . . so that there is equal probability of any player favoring or opposing the proposal."

Member i's power, over anyone or everyone else in the group, is then defined as the probability of a random proposal being carried when i favors it, less the probability of it being carried when i opposes it. The resulting "Dahlingham index" can be written

$$\delta_i = \omega_i/2^{n-1} - \tilde{\omega}_i/2^{n-1}, \tag{55}$$

where ω_i and $\tilde{\omega}_i = \omega - \omega_i$ denote the numbers of winning coalitions containing and not containing i, respectively. But this, as Allingham was quick to point out, is nothing but a re-definition of the Banzhaf index; indeed, we have $\delta_i \equiv \beta_i'$ by virtue of the identity $\eta_i \equiv \omega_i - \tilde{\omega}_i$ noted in §2, footnote 13. For further discussion of this re-interpretation of the Banzhaf index we refer the reader to [1].

Note 7. There is an extensive literature in electrical engineering on the subject of *threshold logic* and *switching functions*.[48] The latter are functions of the form

$$f : \{0, 1\}^n \to \{0, 1\},$$

which may be thought of as attaching labels "0" and "1" to the vertices of an n-cube. In switching theory, these labels would be interpreted as "off" and "on," just as in logic (where the functions are called Boolean functions) they are read as "false" and "true" and in simple games as "losing" and "winning." The simple games correspond formally to the *positive switching functions* (PSFs),[49] characterized by the condition that $x \geqslant y$ always implies $f(x) \geqslant f(y)$. The weighted majority games correspond to the *positive threshold functions* (PTFs), characterized geometrically by the existence of a hyperplane with positive normal that separates the "ons" from the "offs."

The electrical engineers seem to have a practical interest in discovering when a given switching function is a threshold function, and they have compiled exhaustive lists of PTFs for this purpose.[50] These lists are arranged for the user's convenience

[48] See for example [96] or [97], where more than a hundred references will be found.

[49] Also known as *monotonic Boolean functions*; see Golomb [25].

[50] Winder in [94] gives what amounts to a table of all decisive weighted majority games with eight or fewer players, together with their Banzhaf indices; there are 2470 of them not counting permutations of players. Muroga, Tsuboi, and Baugh in [52] take up the next case; their computations (10 hours on ILIAC II) indicate that there are 172,958 decisive weighted majority games with nine essential players. They do not actually list them in [52], but give a wealth of statistical information and a clear exposition of the methodology. See also [19], [34], [51], [95], and appendix B of [96].

according to certain numerical parameters, introduced by C. K. Chow [13], which are well defined for all switching functions. The user is meant to calculate these parameters for the PSF he is interested in and then enter the table in search of a PTF with the same parameters.

The reader will hardly be surprised to learn that the Chow parameters are related to the Banzhaf indices. In fact, in their original form [13] they are just the numbers we have denoted by $\omega_1, \ldots, \omega_n$, and ω, i.e., the number of winning coalitions each player belongs to and the total number of winning coalitions. Subsequent authors have found it convenient to redefine them as the differences $\omega_i - \tilde{\omega}_i$, together with ω, and they appear in the tables in that form. As we have seen, these differences are *exactly* the raw Banzhaf counts η_i, a happenstance that makes the tables especially convenient for the game theorist or political scientist interested in such tabulations.

In his original paper [13], Chow established the remarkable properties of his parameters that justify their role in the table look-up scheme. The central lemma can be stated as follows: If $f \in$ PSF and $g \in$ PTF, and if $\omega_i(f) = \omega_i(g)$ for all i, then either $f = g$ or $\omega(f) > \omega(g)$. From this, it follows that the n numbers $\omega_1(g), \ldots, \omega_n(g)$ characterize the function g among all PTFs, while the $n+1$ numbers $\omega_1(g), \ldots, \omega_n(g)$, $\omega(g)$ characterize it among all PSFs. In other words, no two PTFs have identical Chow parameters, even if we ignore the parameter ω, and no PSF that is not a PTF has all of its Chow parameters identical to those of any PTF.[51]

Chow's theorem tells us with the aid of the identity

$$\eta_i \equiv 2\omega_i - \omega \tag{56}$$

that the η_i together with ω serve to distinguish any weighted majority game from all other simple games. But the η_i alone, unlike the ω_i alone, do not suffice to distinguish the weighted majority games from all other weighted majority games. Indeed, this would be too much to expect, since we know that simple games and their duals have the same Banzhaf indices (theorem 5). But this turns out to be the only exception. It can be shown that if v and w are weighted majority games having the same raw swing counts η_i, then either $v = w$ or $v = w^*$. (The proof is a simple extension of the Chow-Lapidot argument.) It follows that there is at most one *proper* weighted majority game with a given set of η_i. The same holds for the β_i', but not for the β_i.

Many other properties of the Chow parameters have been described in the literature,[52] and they often translate into interesting facts about the Banzhaf indices. For example, it is apparent from (56) that the raw swing counts η_i are either *all even* or *all odd*; moreover, in a decisive game with $n > 1$ they are all even (since decisiveness implies $\omega = 2^{n-1}$). A less obvious result, attributed in [98] to Ichizo Ninomiya, is that in a decisive game the integers $\eta_i/2$ are themselves either all even or all odd; in other words, the differences $\eta_i - \eta_j$ are all divisible by 4.[53] Such remarks have a direct bearing on the problem of constructing a simple game having prescribed power indices.[54]

Note 8. It is natural to try to generalize from the Banzhaf power index to a *Banzhaf value*, defined for all games representable by numerical characteristic func-

[51] As too often happens in this field, Chow's theorem was independently discovered several years later by Lapidot [40], who gave substantially the same proof. Lapidot called $(\omega_1, \ldots, \omega_n)$ the *counting vector* of the simple game and used it to improve the upper bound on $|\mathfrak{M}(N)|$. He did not, however, make the connection either to switching theory or to the Banzhaf index.

[52] See Winder's survey [98].

[53] The S-S indices have a similar property. Let $\pi_i \equiv n!\varphi_i$ denote the raw pivot count for player i (see §2 above). Then in any simple game the differences $\pi_i - \pi_j$ are divisible by n, and in any decisive simple game they are divisible by $2n$. For some other simple arithmetic properties of the S-S indices, see Nozick [56].

[54] See Imrie [33].

tions $v : 2^N \to R$. The analogous generalization of the S-S index leads to the well-known Shapley value

$$\varphi_i[v] = \sum_{S : i \in S \subset N} \frac{|S - \{i\}|!|N - S|!}{|N|!} [v(S) - v(S - \{i\})]. \tag{57}$$

which may be regarded as an average of player i's marginal contribution to all possible coalitions. Note, however, that sets S of different size get unequal weight in forming this average. The Banzhaf philosophy of regarding all coalitions as equally likely[55] suggests the following variant of (57) as a candidate for a "Banzhaf value": note that it reduces directly to β_i' in the case of a simple game:

$$\beta_i'[v] = \sum_{S : i \in S \subset N} \frac{1}{2^{|N - \{i\}|}} [v(S) - v(S - \{i\})]. \tag{58}$$

This definition enjoys the symmetry, dummy, and linearity properties that are traditionally used to axiomatize the Shapley value.[56] Only the "efficiency" axiom fails, since we have in general $\overline{\beta}'[v] \neq v(N)$, just as we had in general $\overline{\beta}' \neq 1$ for simple games.[57]

When it is only a matter of measuring power, this failure of the indices to add up the "right" total can be tolerated. Indeed, we saw in Note 4 that the number $\overline{\beta}'$ (or $\overline{\eta}'$) can tell us something about the responsiveness of a voting system. But a *value* solution is generally supposed to represent some actual or possible outcome of the game, expressed as a vector of players' payoffs or utilities. Since usually $\overline{\beta}'[v] \neq v(N)$, the evaluation (58) will usually be either too pessimistic or too optimistic, i.e., will correspond to an outcome that is either subefficient and possibly infeasible, or hyperefficient and certainly infeasible.

One can of course recover efficiency by a change of scale, i.e., by extending to (58) the normalization (1) that led us to the index β in the case of simple games. But the conversion formula is rather messy:[58]

$$\beta_i[v] = \frac{\left[v(N) - \sum_j v(\{j\})\right]\beta_i'[v] + \left[\,\overline{\beta'[v]} - v(N)\right]v(\{i\})}{\overline{\beta'[v]} - \sum_j v(\{j\})}. \tag{59}$$

and if nothing else, its typographical appearance should warn us that a "normalized Banzhaf value" is not a very natural mathematical concept. Indeed, the few authors [59], [61], [70] who have tried to work with the Banzhaf value have tended to prefer the β' form, while soft-pedalling the problem of infeasibility.

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[55] See footnote 17 above.
[56] See [73], or appendix A in [2].
[57] Cf. the discussion of axioms A2 and A2′ in §3 above; also §5 of [61].
[58] Note that (58) must still be substituted into (59). Normalizing $v(\{i\}) = 0$ for all i would simplify the appearance of (59) but would not alter the underlying artificiality of the definition.

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