Solution to Hwk 1

Q I(a)

MPP

	1	2	3
a	A, ~~D~~, ~~E~~	A	A
b		E	~~E~~, B
c		C	C
d	B, ~~C~~	~~B~~, D	D

$\Rightarrow \langle Aa, Bb, Cc, Dd, E \rangle$

WPP

	1	2
A	d	d
B	a	a
C	b	b
D		c
E	~~d~~	

$\Rightarrow \langle Ad, Ba, Cb, Dc, E \rangle$

Q I(b) After a replaces BCADE by BCDEA, the MPP is:

a	~~A~~, D, ~~E~~	D	D	D	D	~~D~~, C	C	C
b		A, ~~E~~	A	A, ~~B~~	A	A	A	A
c		C	C	C	~~C~~, B	B	B	B
d	B, ~~C~~	B	~~B~~, E	E	E	E	~~E~~, D	D

(E ou

yielding the matching $\langle Ab, Bc, Ca, Dd, E \rangle$;
and so a gets C whom she prefers to A (in her true preference), thus profiting by the misrepresentation.

Q II

(1c) (Shapley's notes)

MPP

	1	2	3	4	5	6	7
a	~~U~~, Y	Y	Y	~~Y~~, ~~X~~, V	V	V	V
b	V	~~V~~, U	U, ~~X~~	U	U, ~~Y~~	U	~~U~~, W
c	X	~~X~~, W	W	W	W	~~W~~, Y	Y
d	~~W~~, Z	Z	Z, ~~V~~	Z	Z, ~~X~~	Z	Z

	8	9
a	V	V
b	W	W
c	Y, ~~U~~	Y
d	Z	Z, ~~U~~

MPP matching ⟨U, Va, Wb, X, Yc, Zd⟩

Q II

1(c) of Shapley

WPP

	1	2	3
U			
V	a, ~~c~~	a	a
W	b	b	b
X			
Y			c
Z	d	d, ~~c~~	d

$WPP_{matching} = \langle U, Va, Wb, X, Yc, Zd \rangle$

(4(a)) of Shapley

(Q II) 4(a) $\langle X : acdf,\ Y : eg,\ Z : hik \rangle$. b and j are left out.

X:	a, b, c	a, b, c, d	a, b, c, d, f, j	a, c, d, f	a, c, d, f	a, c, d, f
Y:	d, e, f, g	e, g, j	e, g	e, g, b	e, g	e, g
Z:	h, i, j, k	h, i, k, f	h, i, k	h, i, k	h, i, k, b	h, i, k
Rej:	d, f, g	j, f	b, j	b	b	
Out:			j		b	

not asked, (but here it is!) ← 4(b) $\langle X : ijklmn,\ Y : abcdef \rangle$. g and h are left out.

X:	$a, b, c, e, f, g, h, i, j$	e, f, g, h, i, j	e, f, g, h, i, j, k, l
Y:	k, l, m, n	k, l, m, n, a, b, c, d	m, n, a, b, c, d
Rej:	a, b, c, d	k, l	e, f

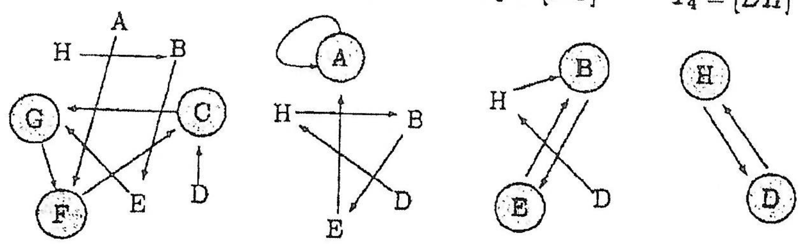
X:	g, h, i, j, k, l	g, h, i, j, k, l, m, n	i, j, k, l, m, n
Y:	m, n, a, b, c, d, e, f	a, b, c, d, e, f	a, b, c, d, e, f, g, h
Rej:	m, n	g, h	g, h
Out:			g, h

X:	i, j, k, l, m, n
Y:	a, b, c, d, e, f
Rej:	

(Q V) Ex 6(a) of Shapley

Top Trading Cycles

$T_1 = [CGF]$ $T_2 = [A]$ $T_3 = [BE]$ $T_4 = [DH]$

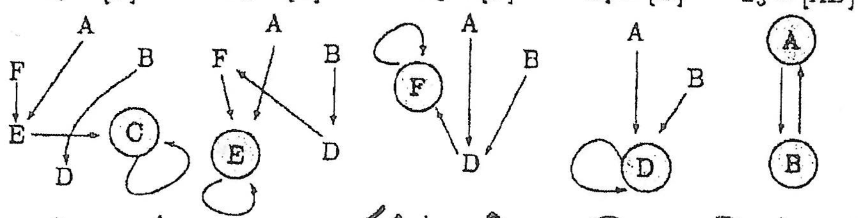


(b) Strongly stable allocation = $\langle Ab, Ba, Cc, Dd, Ee, Ff \rangle$.

Ex 6(b) of Shapley

Top Trading Cycles

$T_1 = [C]$ $T_2 = [E]$ $T_3 = [F]$ $T_4 = [D]$ $T_5 = [AB]$



Strongly stable allocation = $\langle Ab, Ba, Cc, Dd, Ee, Ff \rangle$

Q III (a) Suppose Bob & John both point to Mary. Then they must be married to Mary in different matchings, say:

Bob – Mary in μ & John – Mary in $\tilde{\mu}$

Without loss of generality (w.l.o.g.) suppose Mary likes John more than Bob. Then, in the matching μ, John likes Mary more than his wife (because Mary is the better of his two wives in μ and $\tilde{\mu}$) and Mary likes John more than her husband Bob. This shows that μ is unstable, a contradiction.

(b) Let λ denote the matching obtained from μ and $\tilde{\mu}$ by the pointing in part (a) above.

$$\underset{\lambda}{\begin{pmatrix} X \text{ or } Y - \text{Carol} \\ \text{John} - \text{Mary} \end{pmatrix}} \qquad \underset{\mu}{\begin{pmatrix} X - \text{Carol} \\ \text{John} - \text{Mary} \end{pmatrix}}, \quad \underset{\tilde{\mu}}{\begin{pmatrix} Y - \text{Carol} \\ \text{John} - ? \end{pmatrix}}$$

Using the notation of the above display, argue as follows:

Suppose Carol $\succ_{John}$ Mary (in λ) .., this is just shorthand for "John likes Carol more than Mary"

Then, denoting(*) the husbands of Carol in μ and $\tilde{\mu}$ by X and Y, we must have

$X \succ_{Carol}$ John (otherwise μ is unstable)

and

$Y \succ_{Carol}$ John (otherwise $\tilde{\mu}$ is unstable)

(recalling that John likes Carol more than Mary and Mary more than ?, unless ? = Mary)

This proves that λ is stable, since Carol likes her husband in λ more than John.

(c) Similar argument as above.

(*) we allow for the possibility that X = Y.

Q IV

Let μ denote the matching obtained from MPP and WPP

Suppose $\hat{\mu}$ is another matching which is stable, and different from μ

Then there is a couple, say John-Mary in μ which is missing in $\hat{\mu}$.

Say John-Carol and Bob-Mary in $\hat{\mu}$. So Carol $\in$ Poss (John) and Bob $\in$ Poss (Mary). By Thm 2 of Gale-Shapley, John is Mary's top choice in Poss (Mary), so

$\boxed{\text{Mary prefers John to Bob}}$

Again, by Thm 2, Mary is John's top choice in Poss (John), so

$\boxed{\text{John prefers Mary to Carol}}$

The two boxed displays show that $\hat{\mu}$ is not stable, a contradiction, proving there is no stable matching other than μ.

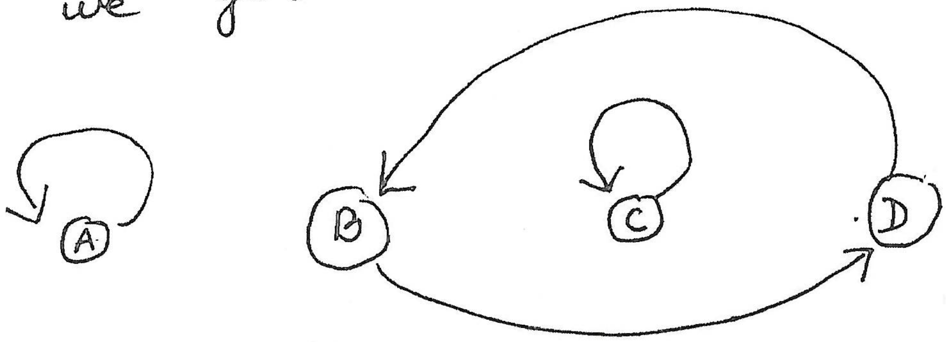
QV

Ex 8(a) A and C have their top choices in $\mathcal{A}'$. So, for the core, the only coalitions that can consider objecting to $\mathcal{A}'$ are $\{B\}$, $\{D\}$, $\{B, D\}$; and these can achieve allocations $\langle Bb \rangle$, $\langle Dd \rangle$, $\langle Bb, Dd \rangle$ + $\langle Bd, Db \rangle$ respectively. B is worse-off (compared to $\mathcal{A}'$) in $\langle Bb \rangle$; D is worse off in $\langle Dd \rangle$; and B is worse-off in both $\langle Bb, Dd \rangle$ and $\langle Bd, Db \rangle$.

Thus $\mathcal{A}' \in$ Core.

Ex 8 (b) But $\mathcal{A} = \langle Ab, Bc, Cd, Da \rangle$ makes B and D both better off compared to $\mathcal{A}'$; leaving A and C as before. So $\mathcal{A}' \notin$ Strict core.

Ex 8 (c) Note now that A owns b, B owns a
C owns d, and D owns c.
So we get



So the strict core is

$$\langle Ab,\ Bc,\ Cd,\ Da\rangle$$

This is also the core, because A and D own their top choices already and will not part from them. The only remaining possibility is that B & D also keep their own houses, i.e $\langle Ba, Dc\rangle$ occurs in the core besides $\langle Ab\rangle$ and $\langle Cd\rangle$. But, by swapping $\{B, D\}$ get $\langle Bc, Da\rangle$ and can both be better off, a contradiction.