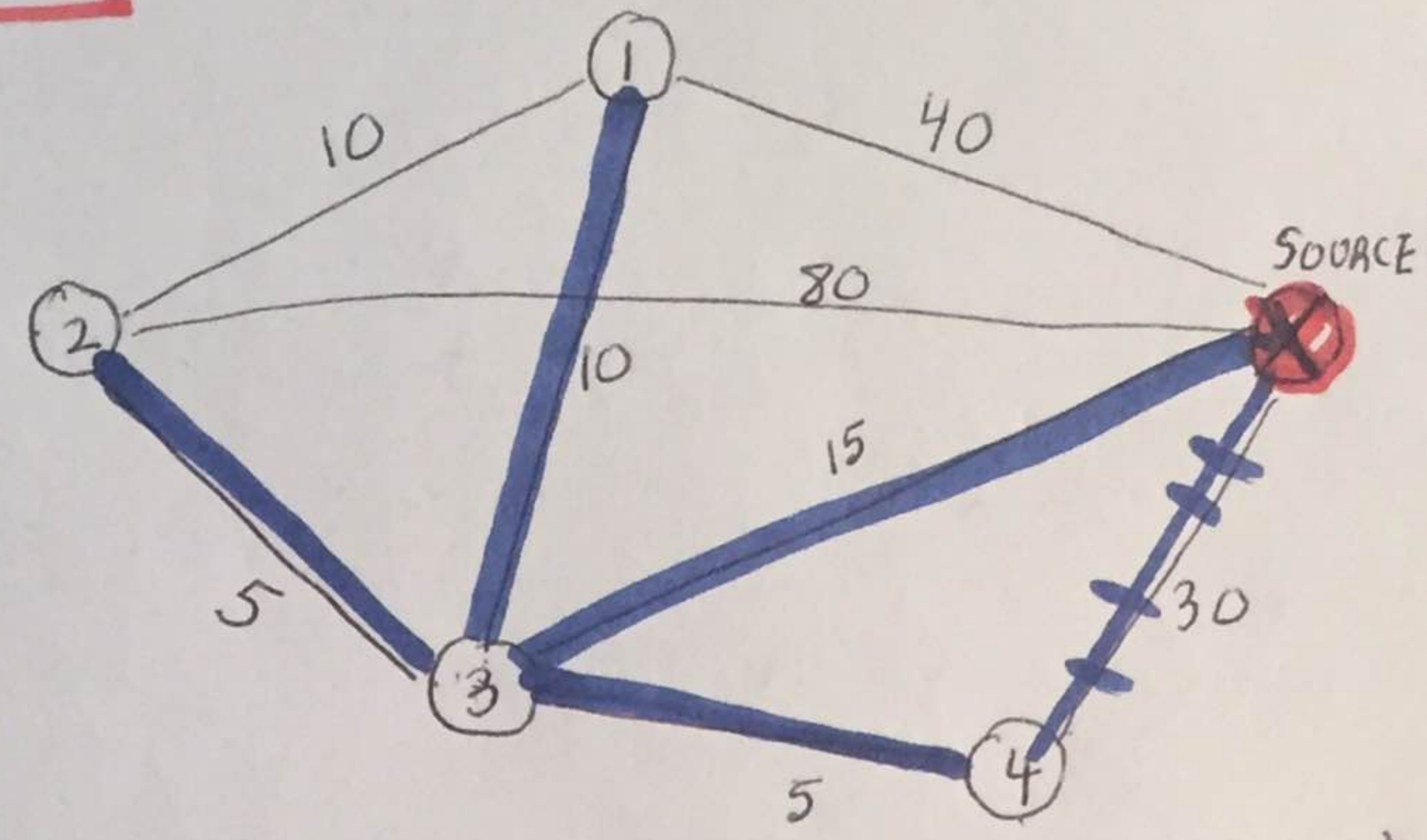
# Notes on Shapley Value

## EXAMPLES

(1) POWER GRID




(COST OF MISSING ARCS = $\infty$)

$v(1) = -40$
$v(2) = -80$
$v(3) = -15$
$v(4) = -30$

$v(12) = -50$
$v(13) = -25$
$v(23) = -17$
$v(14) = -70$
etc.

$v(123) = -30$
$v(234) = -25$
etc.

$v(1234) = -35$

**Q: HOW TO SPLIT −35 AMONG 1, 2, 3, 4 ?**

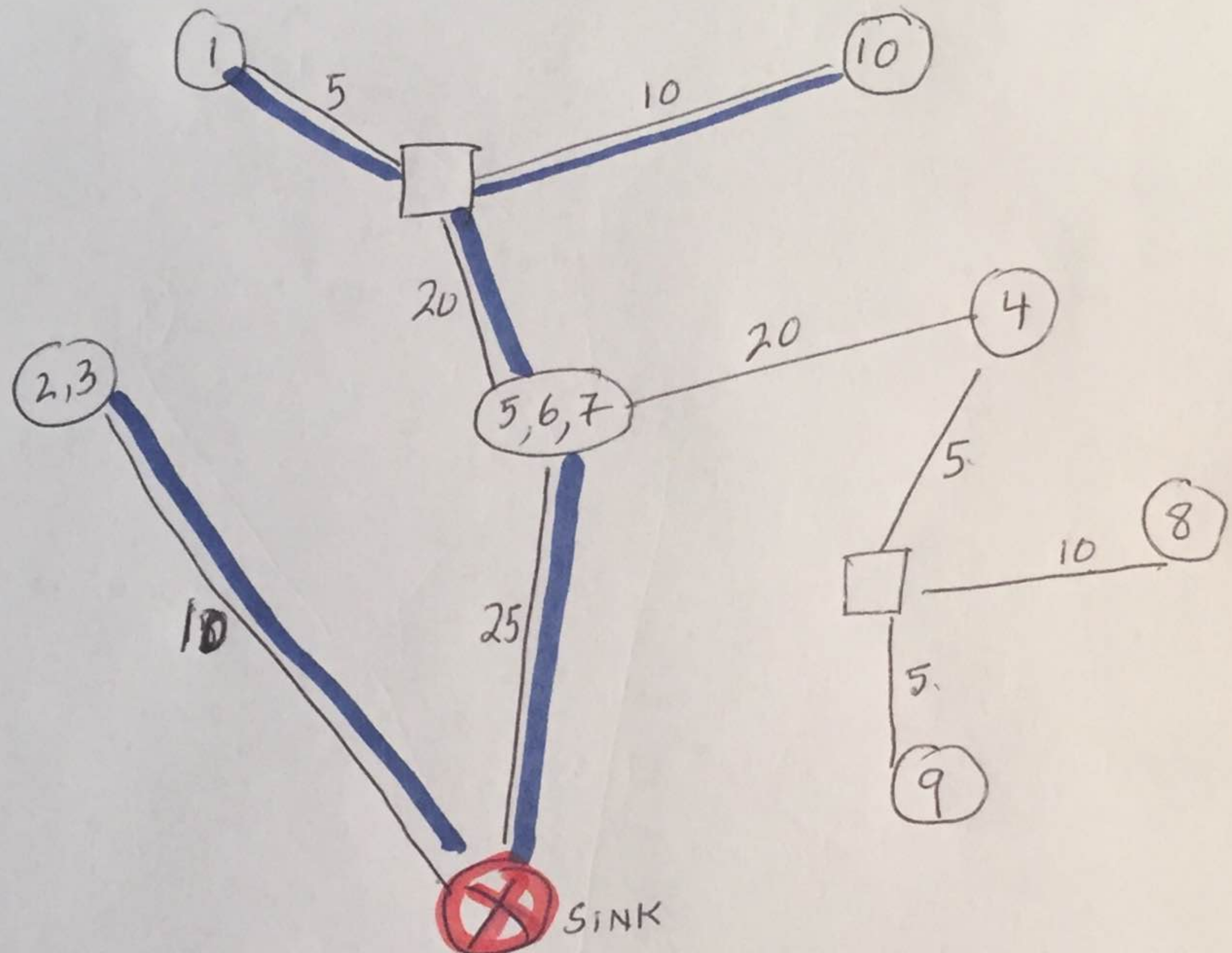
Note

$$v(S \cup T) \geq v(S) + v(T)$$
if
$$S \cap T = \phi$$

**SUPERADDITIVITY**

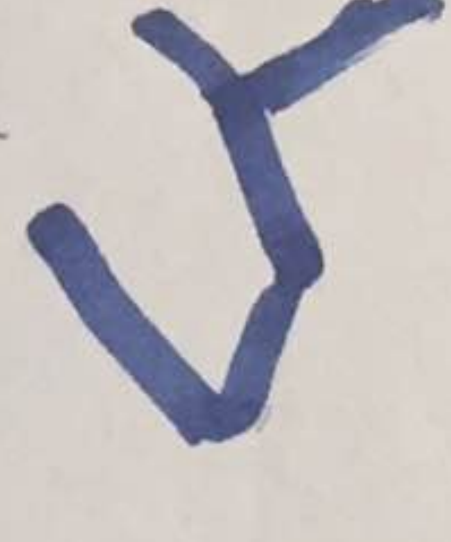
DATA TRANSMISSION






$N = \{1, 2, \cdots, 10\}$.

$v(1, 2, 10\}$ = −(Cost of Subtree connecting 1, 2, 10 to sink)

= − Cost of 

= − 70

**Q: How to split 110 among 1, 2, ..., 10 users**

1 → garage 30
2 → gas station 12
3 → restaurant 6

12 → together buy auto-accesories but owe mortgage

23 → gas station e restaurant boost each other
123 → buy together very profitable auto-parts store

$v(1) = 30$, $v(2) = 12$, $v(3) = 6$
$v(12) = 36$ = value of garage + gas st + auto-accesories — MORTGAGE
$v(23) = 30$ = gas station + restaurant + positive externality
$v(123) = 90$

$v(12) < v(1) + v(2)$

FAILURE OF SUPERADDITIVITY

## GAME

$N = \{1, \ldots, n\}$ = set of **players**

For $S \subset N$, **coalition**

$v(S)$ = **worth of coalition** $S$

[$v(\phi) = 0$ : convention]

Formally, denote $2^N$ = set of all coalitions (subsets) of $N$

$$\boxed{2^N \xrightarrow{v} \mathbb{R} \quad \text{s.t } v(\phi) = 0}$$ **GAME**

**QUESTION:** How to divide $v(N)$ among the players in $N$?

$$x = (x_1, \ldots, x_n) = (x_i)_{i \in N} \in \mathbb{R}^N$$

BETTER NOTATION

# Core $v$ (Coalitional Stability)

$$\text{Core } v = \left\{ x \in \mathbb{R}^N : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subset N \text{ and } \sum_{i \in N} x_i = v(N) \right\}$$

**Example** (Unanimity Game)

$$v(N) = 1$$
$$v(S) = 0 \text{ if } S \neq N$$

Then $(1, 0, \dots, 0) \in \text{Core } v$

In fact $(x_1, \dots, x_n) \in \text{Core } v$

$$\iff x_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in N} x_i = v(N)$$

Is this **FAIR?**

6

$v(123) = v(23) = 1$

$v(S) = 0$ if $S \neq \{123\}$ or $\{23\}$

Then any split of 1 between 2,3 is in Core $v$

eg $(1,0,0) \in$ Core $v$

FAIR ?

# SHAPLEY VALUE $\varphi(v)$ of GAMES $v$

Let $G_N$ = space of **ALL** games on $N$

Any $v \in G_N$ may be viewed as a vector with $2^{|N|}$ components, ~~whose coor~~ one for each $S \subset N$

(# subsets is $2^{|N|} \equiv 2^n$ if $|N| = n$).

eg.

$$
\begin{array}{l}
\\
\phi \rightarrow \\
\{1\} \rightarrow \\
\{2\} \rightarrow \\
\{3\} \rightarrow \\
\{12\} \rightarrow \\
\{13\} \rightarrow \\
\{23\} \rightarrow \\
\{123\} \rightarrow
\end{array}
\begin{array}{c}
v \\
\left(\begin{array}{c} 0 \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array}\right)
\end{array}
\begin{array}{c}
w \\
\left(\begin{array}{c} 0 \\ \tilde{a} \\ \tilde{b} \\ \tilde{c} \\ \tilde{d} \\ \tilde{e} \\ \tilde{f} \\ \tilde{g} \end{array}\right)
\end{array}
\quad \text{etc.}
$$

Axes of $G_N$ are indexed by coalitions

$\dim G_N = 2^n - 1$ (since $v(\phi) = 0$ always).

Consider

$$G_N \xrightarrow{\varphi} \mathbb{R}^N$$

$$v \longmapsto \varphi(v) = (\varphi_1(v), \dots, \varphi_n(v))$$
$$= (\varphi_i(v))_{i \in N}$$

$\varphi_i(v) =$ "value" of player $i$ in the game $v$.

For $v, w \in G_N$

Define $\mathbf{v+w}$ by

$$\boxed{(v+w)(S) = v(S) + w(S)} \text{ for all } S \subset N$$

& define, for any real number $c$, the game $\mathbf{cv}$ by

$$\boxed{(cv)(S) = c\,v(S)} \text{ for all } S \subset N$$

## AXIOMS ON $\varphi$

AX I (EFFICIENCY) $\displaystyle\sum_{i\in N} \varphi_i(v) = v(N)$

Def$^n$ Player $i$ is a "dummy" in $v$ if

$$v(S\cup i) = v(S) \text{ for all } S\subset N$$

NOTE: TAKING $S=\phi$, we see $v(i)=v(\phi)=0$

AX II (DUMMY) If $i$ is a dummy in $v$, then $\varphi_i(v) = 0$

<u>Def$^n$</u> Players $i$ and $j$ are called <u>substitutes</u> in $v$ if

$v(S \cup i) = v(S \cup j)$ for all $S \subset N$ such that $i \notin S$ and $j \notin S$

<u>Ax III</u> (<u>SUSTITUTES</u>)

If $i$ and $j$ are substitutes in $v$ then $\varphi_i(v) = \varphi_j(v)$

AXIOMS I, II, III are "local"
$\downarrow$
talk about a single game $v$

Only "global" axiom is

Ax IV (ADDITIVITY)

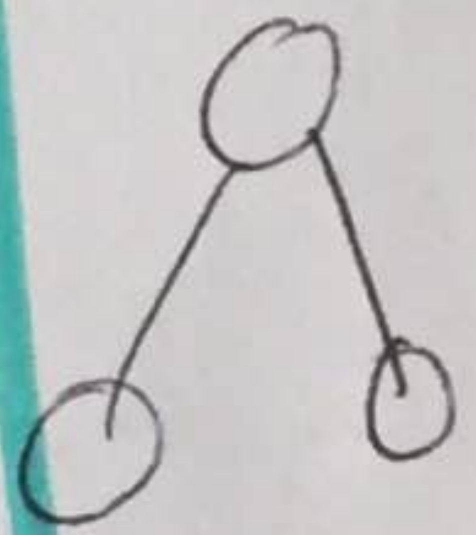
$$\varphi(v+w) = \varphi(v) + \varphi(v)$$

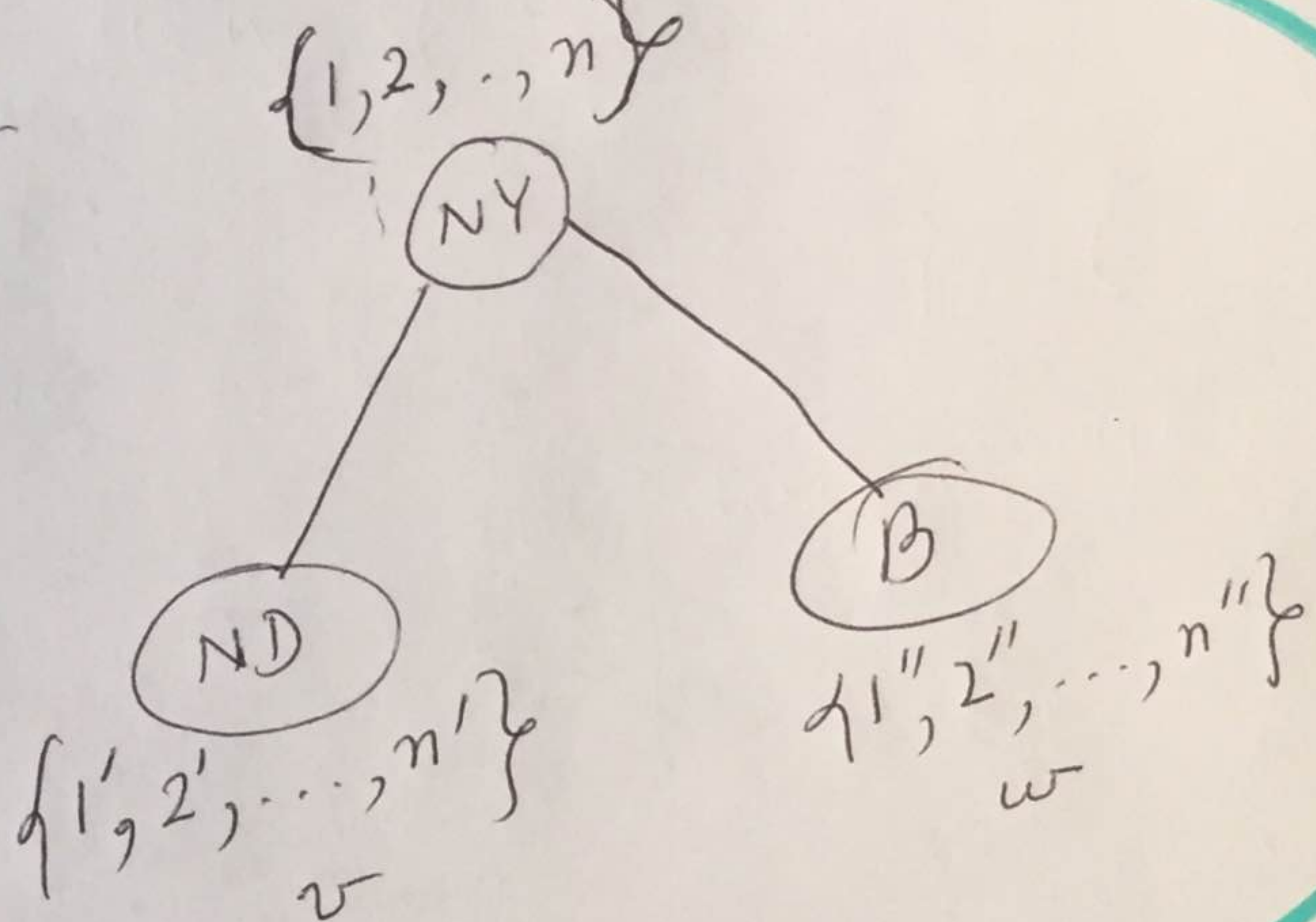
↓

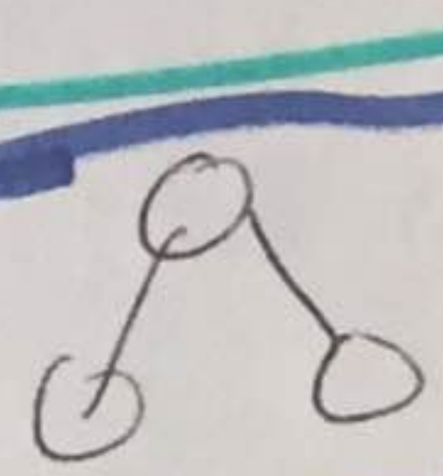
i.e. $\varphi_i(v+w) = \varphi_i(v) + \varphi_i(w)$ for all $i \in N$

CONSISTENCY MAY BE A BETTER WORD

WHAT IS THIS COMPOUND GAME?






 $= v + w$

THM (SHAPLEY, 1953)

There is a unique map

$$G_N \xrightarrow{\varphi} \mathbb{R}^N$$

satisfying axioms I, II, III, IV

and it is given by

$\varphi_i(v) =$ average marginal contribution of $i$ in the game $v$ (across all equiprobable random orders)

[RHS is a mental story to remember formula of $\varphi(v)$ .... the justification of $\varphi$ is in the axioms].

| Order | 1 | Marginal Cont of 2 | 3 |
|---|---|---|---|
| 1 2 3 | .. | $v(12) - v(1)$ | .. |
| 1 3 2 | .. | $v(123) - v(13)$ | .. |
| 2 1 3 | .. | $v(2) - v(\phi)$ | .. |
| 2 3 1 | .. | $v(2) - v(\phi)$ | .. |
| 3 1 2 | .. | $v(123) - v(13)$ | .. |
| 3 2 1 | .. | $v(23) - v(3)$ | .. |

$$\varphi_2(v) = \frac{\text{Sum of all entries of Column 2}}{6}$$

# In general

$s = |S|$

$$\underbrace{\cdot\ \cdot\ \cdot\ \cdots\ \cdot}_{S}\ \overset{i}{\downarrow}\ \underbrace{\cdot\ \cdot\ \cdot\ \cdots\ \cdot}_{N \setminus (S \cup i)}$$

How many such orderings? (In each of them $i$'s contribution is the same: $v(S \cup i) - v(S)$)

$$s!\,(n-1-s)!$$

So

$$\varphi_i(v) = \frac{1}{n!} \sum_{S \subset N \setminus \{i\}} s!\,(n-1-s)!\,[v(S \cup i) - v(S)]$$

i.e

$$\varphi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!\,(n-1-s)!}{n!} [v(S \cup i) - v(S)]$$

($0! = 1$ by DEFINITION)

For any $\underbrace{S \subset N}_{S \neq \emptyset}$ and $c \in \mathbb{R}$, define $cv_S \in G_N$ by

$$(cv_S)(T) = \begin{cases} c & \text{if } S \subset T \\ 0 & \text{otherwise} \end{cases}$$

Now all players in $N \setminus S$ are dummies in $cv_S$
and - - - - - in $S$ are substitutes in $cv_S$

Therefore (using efficiency)

$$\varphi_i(cv_S) = \begin{cases} \frac{c}{|S|} & \text{if } i \in S \\ 0 & \text{otherwise} \end{cases}$$

(Denoting $1v_S = v_S$, we have PROVED

$$\varphi(cv_S) = c\,\varphi(v_S)$$
)

We claim that the games

$$\{v_S\}_{\emptyset \neq S \subset N}$$

form a basis of $G_N$. Since

$$\# \{v_S\}_{\emptyset \neq S \subset N} = 2^{n-1} \quad (\text{recall } n = |N|)$$

it suffices to show that these games are linearly independent (in order to establish the claim).

Suppose, to the contrary, there is a non-trivial linear combination

$$\sum_S c_S v_S = 0 \qquad (*)$$

By dropping those $S$ for which $c_S = 0$ we ~~assu~~ may take $c_S \neq 0$ for all the terms in the LHS of $(*)$.

$$\sum_{S} c_S v_S = 0 \quad (*) \quad (\text{All } c_S \neq 0)$$

Let $T$ be $\underline{\underline{a}}$ set of smallest cardinality in the above expression $(*)$, then by rearranging terms we may write

$$v_T = \sum_{S} \frac{c_S}{c_T} v_S$$

(The blue $\Sigma$ has $T$ missing compared to $\Sigma$ of $(*)$)

Hence

$$1 = v_T(T) = \sum_{S} \frac{c_S}{c_T} v_S(T) = 0$$

But $v_S(T) = 0$ for all $S$ in $\Sigma$ because (i) if $|S| > |T|$, then it is impossible that $S \subset T$, so $v_S(T) = 0$ and (ii) if $|S| = |T|$, then $S$ and $T$ must be different sets of the same size, and again it is impossible that $S \subset T$ so $v_S(T) = 0$

This contradiction proves that the claim is true.

Now suppose there are two functions $\psi$ and $\varphi$ that satisfy Ax I, II, III, IV

We claim that $\psi = \varphi$

To see this, note that efficiency, symmetry and dummy imply that $\psi$ and $\varphi$ must be the same on all games of the type $cv_S$ (for they both give 0 to the dummies in $cv_S$, i.e. to players in $N \setminus S$; and they both give the same to all players in $S$ since they are substitutes, hence they both split the total $c$ and give $c/|S|$ to each player in $S$).

$$\boxed{\text{So } \varphi(cv_S) = \psi(cv_S)} \quad **$$

$$\begin{pmatrix} \forall \ \phi \neq S \subset N \\ \forall \ c \in \mathbb{R} \end{pmatrix}$$

Now take any $v \in G_N$

Since $\{v_S\}_{\phi \neq S \subset N}$ is a basis of $G_N$ there is a unique expression

$$v = \sum_S c_S v_S$$

Then

$$\varphi(v) \underset{\substack{\uparrow \\ \text{ADDITIVITY} \\ \text{OF } \varphi}}{=} \sum_S \varphi(c_S v_S) \underset{\substack{\uparrow \\ **}}{=} \sum_S \psi(c_S v_S) \underset{\substack{\uparrow \\ \text{ADDITIVITY} \\ \text{OF } \psi}}{=} \psi(v)$$

So: there is AT MOST ONE function satisfying axioms I, II, III, IV

Now we show that the function

$$\varphi_i(v) = \sum_{S \subset N \setminus i} \frac{s!\,(n-s-1)}{n!} \left[ v(S \cup i) - v(S) \right]$$

satisfies the axioms.

It is obvious that dummy & symmetry hold.

Also obvious that additivity holds since

$$(v+w)(S \cup i) - (v+w)(S) = \left[ v(S \cup i) + w(S \cup i) \right] - \left[ v(S) + w(S) \right]$$

$$= \left[ v(S \cup i) - v(S) \right] + \left[ w(S \cup i) - w(S) \right].$$

The only Axiom that is not obvious is Axiom IV (ADDITIVITY)

# CHECKING ADDITIVITY

| Order | Cont 1 | Cont 2 | . . . | Cont n |
|---|---|---|---|---|
| | | | | |
| ⋮ | | | . . . . | |
| | | | | |

Consider the matrix ($n!$ rows, $n$ columns) of marginal contributions.

$$\sum_{i \in N} \varphi_i(v) = \frac{1}{n!}\left[\text{sum of Col } 1 + \cdots + \text{sum of Col } n\right]$$

$$= \frac{1}{n!}\left[\text{sum of Row } 1 + \cdots + \text{sum of Row } n!\right]$$

$$= \frac{1}{n!}\left[v(N) + \cdots + v(N)\right]$$

$$= \frac{1}{n!} \; n! \, v(N) = v(N)$$

REASON

Sum of row corr to 321 $= v(3) - v(\phi) + v(23) - v(3) + v(123) - v(23)$

$= v(123)$ etc.