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Compound voting and the Banzhaf index

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Abstract

We present three axioms for a power index defined on the domain of simple (voting) games. *Positivity* requires that no voter has negative power, and at least one has positive power. *Transfer* requires that, when winning coalitions are enhanced in a game, the change in voting power depends only on the change in the game, i.e., on the set of new winning coalitions. The most crucial axiom is *composition*: the value of a player in a compound voting game is the product of his power in the relevant first-tier game and the power of his delegate in the second-tier game. We prove that these three axioms categorically determine the Banzhaf index.
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1. Introduction

The Banzhaf index has had a reputable run as a concept of voting power, not just in academic circles, but in courts of law during debates on fair electoral representation and

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the "one man, one vote" principle. For an excellent overview, see the book by Felsenthal and Machover (1998); and, of course, the papers by Banzhaf (1965, 1966, 1968) himself.

Indeed, as brought out in this book, the main points of Banzhaf's analysis were already present in a trail-blazing paper due to Penrose (1946), even though it went largely unnoticed. The basic scenario considered by Penrose (and later reinvented by Banzhaf) is as follows. There are k constituencies with disjoint populations of sizes $n_1, \ldots, n_k$ and with delegates $1, \ldots, k$ appointed to represent them. Each citizen is, on average, equally likely to vote Yes (Y) or No (N) on the passage of a proposed bill. Once the citizens cast their votes, the outcome is decided in any constituency by the simple majority rule. Decision-making then moves to the council of delegates. Every delegate votes Y or N in accordance with the plebiscite in his constituency. The final outcome of the vote in the council is settled in more subtle manner, via an abstract simple (voting) game v defined on the set of delegates $1, \ldots, k$.

Both Penrose and Banzhaf felt that the true measure of "voting power" of a citizen is given by the probability that his single vote can swing the balance in the passage of the bill in this two-tiered process. A little reflection reveals that his power, so defined, must be: (the probability that he can swing the balance in his constituency) $\times$ (the probability that his delegate can swing the balance in the council). When the population sizes are large, the voting power of a citizen in constituency j is approximated by $\sqrt{2}(\sqrt{\pi n_j})^{-1} \cdot (2^{1-k}\eta_j(v))$, where $\eta_j(v)$ denotes the number of coalitions in the council that are converted from losing to winning by the admission of delegate j. This is the famous "square root rule" of voting theory. (See Chapter 3 of Felsenthal and Machover, 1998.)

To put it in its proper perspective, we must turn to the general description of voting, developed in terms of simple games in the seminal paper of Shapley (1962). Two-tiered voting processes of the Penrose–Banzhaf kind then become special instances of what Shapley called "compound games." And the square root rule follows from a principle of much greater generality: (the power of a voter i in a compound game) $\equiv$ (i's power in the first-tier game that he participates in) $\times$ (the power of i's delegate in the second-tier game). This *composition* principle is what the square root rule—at bottom—comes from. But it lay hidden from sight until brought into prominence by Owen (1978). Owen elevated it to an axiom which, in conjunction with other standard requirements, yielded the Banzhaf value on the space of all constant-sum games (together with the null and dictatorial values). While an undoubted break-through, Owen's analysis would perhaps have even more relevance for voting theory, if done "in the context of simple games alone." The quote here is from Shapley (1954), who first emphasized the importance of this domain restriction in order to achieve an axiomatic foundation for the concept of voting power.

This is the point of entry for our analysis. We consider the domain of all simple games and show that three straightforward axioms categorically determine the Banzhaf power index:

- The *composition* axiom has been spelled out already. It is postulated—as in Owen (1978)—for the scenario in which all the first-tier games are decisive. (Outside of this scenario, composition in fact does not hold.)
- Our second axiom is *positivity*. It states that no voter has negative power and at least one has positive power. To justify it, note that since simple games are monotonic by

definition, a player can never convert a winning coalition to losing when he joins it, and it stands to reason that his power must be non-negative as he has no negative impact in the game; at the same time, there are always some minimal winning coalitions, and so there exist players who have positive impact and therefore non-zero power. (It is evident that the positivity axiom immediately rules out both the null and the dictatorial values.)

- The third, and last, axiom, is called *transfer*, and was introduced in Dubey (1975). It is equivalent to the requirement that when a simple game is enhanced by adding more winning coalitions, the *change* in voting power depends only on the *change* in the game (i.e., on the collection of the new winning coalitions).

It is worth noting that we do *not* assume either the anonymity axiom or the dummy axiom, both of which were needed in Owen (1978). We allow for all possible designations of delegates in our version of composition, and this—in conjunction with positivity—implies the anonymity axiom (see Lemma 2 in Section 4). In turn, anonymity—in conjunction with composition—implies[1] the dummy axiom (see Lemma 3 in Section 4).

The paper is organized as follows. In Section 2 we introduce simple games and the definition of the Banzhaf power index. Section 3 describes our three axioms. The main result on the uniqueness of the Banzhaf index is stated and proved in Section 4. Finally, in Section 5 we give a brief summary of other axiomatizations of the Banzhaf index.

2. Simple games and the Banzhaf index

Let U be an infinite[2] (underlying) set of *players*. Denote the collection of all *coalitions* (subsets of U) by 2^U, and the empty coalition by $\emptyset$. Then a *game* on U is given by a map $v : 2^U \to R$ with $v(\emptyset) = 0$. A coalition $N \subset U$ is called a *carrier* of v if $v(S) = v(S \cap N)$ for any $S \in 2^U$. We say that v is a *finite game* if it has a finite carrier. The space of all finite games on U is denoted by $\mathcal{G}$.

The domain $\mathcal{SG} \subset \mathcal{G}$ of *simple games* on U consists of all $v \in \mathcal{G}$ such that

(i) $v(S) \in \{0, 1\}$ for all $S \in 2^U$;
(ii) $v(U) = 1$;
(iii) v is *monotonic*, i.e., if $S \subset T$ then $v(S) \leqslant v(T)$.

As discussed in Shapley (1962), simple games are ideally suited to capture the essence of many voting processes. In the interpretation, a coalition S is *winning* in $v \in \mathcal{SG}$ if $v(S) = 1$, and *losing* otherwise. If $v \in \mathcal{SG}$ is constant-sum, i.e., S is winning in v if and only if $U \backslash S$ is losing in v, then v is called *decisive*.

[1] We use here the fact that there is no upper bound on the size of our player-sets. Otherwise, one would need to *assume* dummy alongside positivity. (See Remark 2.)

[2] Here we follow in the footsteps of Shapley (1953). Again, see Remark 2 for the analysis with a fixed, finite player-set.

It will be technically convenient to describe players' payoffs in terms of certain trivial games. A game v is called *additive* if $v(S \cup T) = v(S) + v(T)$ whenever $S \cap T = \emptyset$. Let $\mathcal{AG} \equiv \{v \in \mathcal{G} \mid v \text{ is additive}\}$. Any $w \in \mathcal{AG}$ with (finite) carrier N can be identified with the vector[3] $\{w(i) \mid i \in N\}$, and so one may think of such a w as a *payoff vector* to the players in N.

The *Banzhaf value* $\beta v \in \mathcal{AG}$ of the game $v \in \mathcal{G}$ with finite carrier N is given by

$$(\beta v)(i) = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} v\big(S \cup \{i\}\big) - v(S)$$

if $i \in N$, where $n = |N|$; and

$$(\beta v)(i) = 0$$

if $i \in U \setminus N$. It is easy to see that βv is invariant of the choice of the carrier N of v.

If v is a simple game, we can interpret $(\beta v)(i)$ to be the voting power of i, and so βv is often called the *Banzhaf index* (of power) when the domain of β is restricted to $\mathcal{SG}$. Indeed Banzhaf defined β only on $\mathcal{SG}$. Its extension to $\mathcal{G}$ was first mentioned much later in Dubey and Shapley (1979).

3. Axioms

A *power index* φ is a map $\varphi : \mathcal{SG} \to \mathcal{AG}$, with the interpretation that $\varphi v(i) \equiv$ the voting power of player i in the game v. We shall show that β is the unique power index on $\mathcal{SG}$ which satisfies three straightforward (and, familiar) axioms.

First, some definitions. For $v, w \in \mathcal{SG}$ define $v \vee w, v \wedge w \in \mathcal{SG}$ by:

$$(v \vee w)(S) = \max\big\{v(S), w(S)\big\},$$

$$(v \wedge w)(S) = \min\big\{v(S), w(S)\big\},$$

for all $S \in 2^U$. (It is evident that $\mathcal{SG}$ is closed under operations $\vee, \wedge$.) Thus a coalition is winning in $v \vee w$ if, and only if, it is winning in at least one of v or w; and it is winning in $v \wedge w$ if, and only if, it is winning in both v and w.

Next, we come to the notion of compound games (see Shapley, 1962). Consider games $v, w_1, \dots, w_k \in \mathcal{SG}$, such that:

(i) v has a carrier $N \subset U$ with $|N| = k$;
(ii) $w_1, \dots, w_k$ have disjoint finite carriers $M_1, \dots, M_k$.

Let $\alpha : \{1, \dots, k\} \to N$ be a bijection. Then the game $u \in \mathcal{SG}$ is said to be the *compounding* of v with $w_1, \dots, w_k$ via α, and we write $u = v_\alpha[w_1, \dots, w_k]$, if

$$u(S) = v\big(\big\{\alpha(j) \mid w_j(S) = 1\big\}\big).$$

Notice that $\bigcup_{j=1}^{k} M_j$ is a carrier for u.

[3] We abbreviate $(\{i\})$ to (i).

As was said, one can think of u as a two-tiered voting process. First there is a simultaneous vote among the citizens of constituencies $1, \ldots, k$. The outcome of the vote in the population M_j of constituency j is determined via w_j. The vote then moves to the council of delegates. The delegate of each constituency j is identified with $\alpha(j) \in N$, and his vote must concur with the plebiscite in his constituency. Outcomes of the council vote among the delegates in N are given by v.

We are ready to state our axioms.

Axiom 1 (*Positivity*). φv is monotonic and non-zero for each $v \in \mathcal{SG}$.

Axiom 2 (*Transfer*). $\varphi(v \vee w) + \varphi(v \wedge w) = \varphi v + \varphi w$ for all $v, w \in \mathcal{SG}$.

Axiom 3 (*Composition*). Suppose $u = v_\alpha[w_1, \ldots, w_k]$, where $v, w_1, \ldots, w_k$ have finite carriers $N, M_1, \ldots, M_k$, with $M_1, \ldots, M_k$ disjoint; and where $\alpha : \{1, \ldots, k\} \to N$ is a bijection. Further, suppose that each w_j is a decisive game. Then

$$(\varphi u)(i) = (\varphi v)\big(\alpha(j)\big) \cdot (\varphi w_j)(i)$$

if $i \in M_j$, for all $i \in \bigcup_{j=1}^{k} M_j$.

The axioms have already been discussed in the introduction. Axiom 2 has sometimes been called "somewhat opaque."[4] It can be restated in a seemingly stronger (but equivalent) form, which might help to render its meaning more clear. To be precise, consider two pairs of games v, v' and w, w' in $\mathcal{SG}$. Suppose that the transitions from v' to v and w' to w entail adding the *same* set of winning coalitions (i.e., $v \geqslant v'$, $w \geqslant w'$, and $v - v' = w - w'$). Then we require

$$\varphi v - \varphi v' = \varphi w - \varphi w'. \tag{1}$$

This says that the *change* in power depends only on the *change* in the voting game.[5] Note that this requirement is equivalent to the transfer axiom. Indeed, since $v \vee w - v = w - v \wedge w$ for any $v, w \in \mathcal{SG}$, (1) implies the transfer axiom. Conversely, suppose that v', v, w', and w are as in the premiss for (1). Then $v = v' \vee u$ and $w = w' \vee u$, where $u \in \mathcal{SG}$ is the game whose minimal winning coalitions are precisely those that are converted from maximal losing to minimal winning in the transition from v' to v (or, equivalently, from w' to w), i.e., $u(S) = 1$ if and only if there is $T \subset S$ such that $v(T) - v'(T) = w(T) - w'(T) = 1$. Thus, using the transfer axiom,

$$\varphi v - \varphi v' = \varphi(v' \vee u) - \varphi v' = \varphi u - \varphi(v' \wedge u). \tag{2}$$

However, since $v - v' = w - w'$, it is clear that $v' \wedge u = w' \wedge u$; and so (using transfer again) (2) may be rewritten as

$$\varphi u - \varphi(w' \wedge u) = \varphi(w' \vee u) - \varphi w' = \varphi w - \varphi w'. \tag{3}$$

Now (2) and (3) together imply (1).

[4] See Straffin (1982 p. 298).

[5] This version of transfer was described verbally in Dubey and Shapley (1979) and a special case of it was formally stated in Laruelle and Valenciano (2001).

4. Uniqueness of the Banzhaf index

Theorem. *There exists one, and only one, power index satisfying Axioms* 1–3 *and it is the Banzhaf index* β.

Proof. That β satisfies Axioms 1 and 2 is obvious. Axiom 3 is checked in Owen (1982), Theorem X.3.7 (see also Owen, 1978).

We now show that the axioms uniquely imply β. Fix a power index φ which satisfies Axioms 1–3.

Lemma 1. *Denote by* u_i *the unanimity game on the set* $\{i\}$, *i.e.,*

$$u_i(S) = \begin{cases} 1, & \textit{if } i \in S, \\ 0, & \textit{otherwise}. \end{cases}$$

Then $(\varphi u_i)(i) = 1$.

Proof of Lemma 1. Consider a compounding $u_i[u_i]$ of u_i with itself (here $k = 1$, $M_1 = \{i\}$, $w_1 = u_i$), via the trivial bijection α with $\alpha(1) = i$. Clearly, $u_i[u_i] = u_i$, and thus

$$\big(\varphi\big(u_i[u_i]\big)\big)(i) = (\varphi u_i)(i).$$

However, by the composition axiom,

$$\big(\varphi\big(u_i[u_i]\big)\big)(i) = \big((\varphi u_i)(i)\big)^2,$$

and so

$$(\varphi u_i)(i) \in \{0, 1\}. \tag{4}$$

Suppose $(\varphi u_i)(i) = 0$. Consider any $j \in U \backslash \{i\}$. Then (with $k = 1$, $M_1 = \{i, j\}$ viewed as a carrier for $w_1 = u_i$, and α as before, i.e., $\alpha(1) = i$) the composition axiom implies that

$$(\varphi u_i)(j) = \big(\varphi\big(u_i[u_i]\big)\big)(j) = (\varphi u_i)(i) \cdot (\varphi u_i)(j) = 0,$$

and thus $\varphi(u_i[u_i]) \equiv 0$, contradicting the positivity axiom. We conclude: $(\varphi u_i)(i) = 1$. □

For any finite[6] permutation $\pi : U \to U$ and a game $v \in \mathcal{SG}$, define $\pi v \in \mathcal{SG}$ by

$$(\pi v)(S) = v\big(\pi(S)\big)$$

for all $S \in 2^U$. The game πv is the same as v except that players are relabeled according to π.

Lemma 2. φ *satisfies the* anonymity *axiom:*[7] $\varphi(\pi v) = \pi(\varphi v)$ *for all* $v \in \mathcal{SG}$ *and all finite permutations* π *of* U.

[6] I.e., one that satisfies $\pi(i) = i$ for all but finitely many $i \in U$.

[7] I.e., if players are relabeled in a game, this has the effect of relabeling their power indices accordingly: irrelevant characteristics of the players, outside of their role in the game v, have no influence on the power index.

Proof of Lemma 2. Let $N = \{i_1, \ldots, i_k\}$ be a finite coalition that is a carrier of v, and (by taking N large enough) outside which the permutation π is the identity function. Now consider the bijection $\alpha : \{1, \ldots, k\} \to N$ given by $\alpha(j) = \pi(i_j)$ for every $1 \leqslant j \leqslant k$, and the compound game $u = v_\alpha[u_{i_1}, \ldots, u_{i_k}]$. Here constituency j consists of just the player i_j, and his delegate is $\pi(i_j)$. It is therefore evident that $u = \pi v$. Thus, by the composition axiom and Lemma 1, for any $1 \leqslant j \leqslant k$

$$\big(\varphi(\pi v)\big)(i_j) = (\varphi u)(i_j) = (\varphi v)\big(\alpha(j)\big) \cdot (\varphi u_{i_j})(i_j) = (\varphi v)\big(\pi(i_j)\big).$$

Since N can be taken arbitrarily large, this argument actually shows that

$$\big(\varphi(\pi v)\big)(i) = \pi(\varphi v)(i)$$

for any $i \in U$, and this establishes the anonymity of φ. □

Lemma 3. *φ satisfies the* dummy *axiom: if $v \in \mathcal{SG}$, and i is a dummy player in v, i.e., $v(S \cup \{i\}) = v(S) + v(i)$ for every $S \subset U \backslash \{i\}$, then $(\varphi v)(i) = v(i)$.*

Proof of Lemma 3. First, assume that $v(i) = 0$. Since i is also a dummy player, v has a carrier that excludes i. Both v and φv are finite games, and, consequently, there exists $j \in U$ which is outside some common carrier of v and φv. Thus, $\pi_{ij} v = v$ (where π_{ij} is the permutation of U that interchanges i and j), and $(\varphi v)(j) = 0$. But, from the anonymity axiom established in Lemma 2, $(\varphi v)(i) = (\varphi v)(j)$, and we deduce that $(\varphi v)(i) = 0 = v(i)$.

It remains to consider the case when $v(i) = 1$. However, since the game v is simple, the existence of such a dummy i implies that $v = u_i$, and we have already shown in Lemma 1 that $(\varphi u_i)(i) = 1 = u_i(i)$. □

Completion of the proof. Since φ satisfies the anonymity, dummy, positivity and transfer axioms, Theorem 2.4 of Einy (1987) applies:[8] there is a unique extension of φ to a linear map $\bar{\varphi} : \mathcal{G} \to \mathcal{AG}$. By Theorem X.3.11 of Owen (1982), $\bar{\varphi}$ is either the Banzhaf value, or the null value ($\bar{\varphi} \equiv 0$), or the dictatorial value ($\bar{\varphi}(v)(i) = v(i)$ for all $i \in U$ and $v \in \mathcal{G}$) on the domain $\mathcal{CG} \subset \mathcal{G}$ of all constant-sum[9] games.[10] Our positivity axiom rules out the latter two possibilities, since the null and the dictatorial values vanish on all decisive simple games v in which $v(i) = 0$ for all $i \in U$. Thus, $\bar{\varphi}$ coincides with the Banzhaf value on $\mathcal{CG}$. However, by Theorem X.3.18 of Owen (1982), the linear extension of the Banzhaf value from $\mathcal{CG}$ to $\mathcal{G}$ is unique, provided it satisfies the dummy, anonymity and composition axioms on $\mathcal{SG}$. Since both $\bar{\varphi}$ and the Banzhaf value are such extensions, $\bar{\varphi}$ coincides with the Banzhaf value on $\mathcal{G}$ and, consequently, φ is the Banzhaf index. □

Theorems X.3.11 and X.3.18 of Owen (1982) are proved by establishing bases for the linear spaces $\mathcal{CG}$ and $\mathcal{G}$, such that $\bar{\varphi}$ is uniquely determined by the axioms for the games in

[8] Einy (1987) required anonymity for *all* permutations π of U, not just for finite permutations. But it is easy to check that his theorem, and its proof, hold if we restrict to finite permutations.

[9] A game v is constant-sum if $v(S) + v(U \backslash S) = v(U)$ for all $S \in 2^U$.

[10] Theorem X.3.11 of Owen (1982) is stated under the assumption that the linear map satisfies the dummy, anonymity, and composition axioms on $\mathcal{CG}$. However, its proof remains unchanged even when these axioms are postulated to hold only on $\mathcal{SG} \cap \mathcal{CG}$.

these bases. For better perspective, we now give an alternative way to complete the proof of our theorem. It uses some of the machinery from the theory of semivalues,[11] which makes it sufficient to pinpoint the value of φ for three-person majority games only.

Alternative completion of the proof. We showed that φ satisfies the anonymity, dummy, positivity, and transfer axioms. Thus, according to Theorem 2.5 of Einy (1987),[12] there exists a unique probability measure ξ on $[0,1]$ such that for any $v \in \mathcal{SG}$ with finite carrier N

$$(\varphi v)(i) = \sum_{S \subset N \setminus \{i\}} p_{|S|}^{|N|} \big[v\big(S \cup \{i\}\big) - v(S)\big] \tag{5}$$

if $i \in N$, where

$$p_s^n = \int_0^1 x^s (1-x)^{n-s-1} \, \mathrm{d}\xi(x); \tag{6}$$

and

$$(\varphi v)(i) = 0 \tag{7}$$

if $i \in U \setminus N$. Note that equalities (5) and (7) allow us to extend φ linearly to the space $\mathcal{G}$ of all finite games.[13] This, in turn, leads to:

Lemma 4. *For any $i, j, k \in U$ denote by $v_{i,j,k}$ a three-person majority game in $\mathcal{G}$, given by*

$$v_{i,j,k}(S) = \begin{cases} 1, & \text{if } |S \cap \{i,j,k\}| \geqslant 2, \\ 0, & \text{otherwise} \end{cases}$$

for all $S \in 2^U$. Then there is $q \in \{0, \frac{1}{2}\}$, such that for all $i, j, k \in U$

$$(\varphi v_{i,j,k})(i) = (\varphi v_{i,j,k})(j) = (\varphi v_{i,j,k})(k) = q.$$

Proof of Lemma 4. See the proof of Lemma X.3.13 in (Owen, 1982), which uses the dummy, anonymity, and composition axioms satisfied by φ, together with the linearity of its extension to sums of simple games. □

Now consider the game $v_{i,j,k}$ for some $i, j, k \in U$. Since $\{i, j, k\}$ is a carrier of this game, and $\varphi v_{i,j,k}$ vanishes outside the carrier by Lemma 3, positivity of φ implies

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[11] A semivalue on $\mathcal{G}$ (see Dubey et al., 1981) is a map $\varphi : \mathcal{G} \to \mathcal{AG}$ which satisfies linearity, anonymity, (weak) positivity (i.e., φv is monotonic if v is monotonic), and projection (i.e., $\varphi v = v$ if $v \in \mathcal{AG}$). A semivalue on $\mathcal{SG}$ is defined similarly, replacing $\mathcal{G}$ by $\mathcal{SG}$, "linearity" by "transfer," and "projection" by the dummy axiom.

[12] The main point of Einy (1987) is that there is a unique linear extension to $\mathcal{G}$ of any semivalue on $\mathcal{SG}$; and that this extension is, in fact, a semivalue on $\mathcal{G}$. The explicit representation of the semivalue in terms of the measure ξ then follows from Dubey et al. (1981).

[13] The extended φ is, in fact, a semivalue on $\mathcal{G}$.

that $\varphi v_{i,j,k}$ should be positive for at least one player in $\{i, j, k\}$. We conclude that $q = 1/2$, and thus

$$(\varphi v_{i,j,k})(i) = \frac{1}{2}. \tag{8}$$

It also follows from (5) and (6) that

$$(\varphi v_{i,j,k})(i) = 2p_1^2 = \int_0^1 2x(1-x)\,\mathrm{d}\xi(x). \tag{9}$$

However, the function $2x(1-x)$ attains its unique maximum $1/2$ at $x = 1/2$. Therefore, (8) and (9) can be consistent only if ξ is the distribution supported on the point $1/2$. Substituting this into (6), equalities (5) and (7) show that φ coincides with the Banzhaf power index. This completes the proof of the theorem. □

Remark 1 (*Independence of the axioms*). It is easy to see that our three axioms are independent:

(i) The null index, $\varphi v \equiv 0$ for all $v \in \mathcal{SG}$, satisfies all the axioms except positivity.
(ii) Given $v \in \mathcal{SG}$, let $N(v)$ be its minimal carrier, and define

$$(\varphi v)(i) = \begin{cases} 1, & \text{if } i \in N(v), \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that φ satisfies all the axioms except transfer.
(iii) The Shapley (1953) value, restricted to simple games, satisfies all the axioms except composition.

Remark 2 (*Finite set of players*). If the set of players U is finite, our theorem breaks down. Indeed, defining $(\varphi v)(i) \equiv 1$ for every $v \in \mathcal{SG}$ and $i \in U$, we see that φ satisfies the positivity, transfer, and composition axioms, but $\varphi \neq \beta$.

One way to amend the situation is to add the dummy axiom to our current three axioms. The theorem is then reinstated, and its proof goes through. It even becomes simpler since the need for Lemma 3 is eliminated.

Remark 3 (*Representing payoffs by additive games with unrestricted carriers*). We defined a power index as a map $\varphi : \mathcal{SG} \to \mathcal{AG}$. By definition, additive games in $\mathcal{AG}$ have finite carriers. Thus, we implicitly assumed that only finitely many players in $v \in \mathcal{G}$ receive non-zero payoffs according to φ. This is a plausible assumption, since only finitely many players (in the minimal carrier $N(v)$ of v) have any influence in v.[14] However, this assumption also plays an important role in the proof of the theorem: it allows to deduce the dummy axiom from anonymity in Lemma 3. Without restricting the range of φ to additive games with finite carriers, our theorem breaks down just as in Remark 2 (with the same counter-example). The way to restore the validity of the theorem is, again, by explicitly postulating the dummy axiom in addition to the other three.

[14] And we did not even require that a carrier of $\varphi(v)$ be contained in $N(v)$. It only has to be finite.

5. Other axiomatizations of the Banzhaf index

We present a brief summary[15] of some other axiomatizations of the Banzhaf index on the domain of simple games. The standard axioms assumed in these approaches are suppressed, and we highlight the new axioms introduced in them.

(1) In Dubey and Shapley (1979),[16] it was postulated that if N is a carrier of $v \in \mathcal{SG}$, then

$$\sum_{i\in N}(\varphi v)(i) = 2^{1-|N|}\sum_{i\in N}\eta_i(v) = \sum_{i\in N}\beta_i(v),$$

where

$$\eta_i(v) \equiv \sum_{S\subset N\setminus\{i\}}\bigl[v\bigl(S\cup\{i\}\bigr) - v(S)\bigr],$$

which unfortunately put the notion of "swings"[17] (hence of the Banzhaf index) already into the axioms.

(2) Albizuri and Ruiz (2001) required $\varphi_i(v) \geqslant \varphi_j(v) \Leftrightarrow W_i(v) \geqslant W_j(v)$ (in addition to using a weakened version of the composition axiom), where $W_l(v) \equiv$ the number of winning coalitions in v that contain l and are contained in N, for some fixed finite carrier N of v. However, as noted in Dubey and Shapley (1979), $\eta_i(v) = 2W_i(v) - W(v)$, where $W(v) \equiv$ the total number of winning coalitions in v. Thus, their axiom is tantamount to $\varphi_i(v) \geqslant \varphi_j(v) \Leftrightarrow \beta_i(v) \geqslant \beta_j(v)$, and so, again, the notion of the Banzhaf index is embedded in it.

(3) Lehrer (1988) had an alternative axiomatization of the Banzhaf index in which he uses an amalgamation axiom. This says, in brief, that if two players are amalgamated into one, the value of the new player is at least as much as the sum of the values of the original two. (Incidentally, this axiom is not valid if more than two players are amalgamated.) In contrast, our approach has been to analyze the Banzhaf index through its behavior on compound games, where the set of players is expanded rather than shrunk.

(4) Laruelle and Valenciano (2001) introduce an "average gain-loss balance" axiom, which says that if a minimal winning coalition S is deleted from $v \in \mathcal{SG}$ with finite carrier N, (the average loss in power of players in S) $\equiv$ (the average gain in power of players in $N\setminus S$). They obtain the Banzhaf index up to a symmetric affine transformation.

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[15] If our summary is incomplete, it is only because of our lack of awareness.

[16] See footnote 21, p. 105 therein.

[17] (S,i) is a swing for i if $v(S\cup\{i\}) - v(S) = 1$. Thus $\eta_i(v)$ is the total number of swings of player i in the game v.

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References

Albizuri, J.M., Ruiz, L., 2001. A new axiomatization of the Banzhaf semivalue. Spanish Econ. Rev. 3, 97–109.

Banzhaf, J.F., 1965. Weighted voting doesn't work: a mathematical analysis. Rutgers Law Rev. 19, 317–343.

Banzhaf, J.F., 1966. Multi-member electoral districts—Do they violate the "one man, one vote" principle? Yale Law J. 75, 1309–1338.

Banzhaf, J.F., 1968. One man, 3.312 votes: A mathematical analysis of the electoral college. Vilanova Law Rev. 13, 304–332.

Dubey, P., 1975. On the uniqueness of the Shapley value. Int. J. Game Theory 4, 131–139.

Dubey, P., Shapley, L.S., 1979. Mathematical properties of the Banzhaf power index. Math. Operations Res. 4, 99–131.

Dubey, P., Neyman, A., Weber, R.J., 1981. Value theory without efficiency. Math. Operations Res. 6, 122–128.

Einy, E., 1987. Semivalues of simple games. Math. Operations Res. 12, 185–192.

Felsenthal, D.S., Machover, M., 1998. The Measurement of Voting Power: Theory and Practice, Problems and Paradoxes. Edward Elgar, London, UK.

Laruelle, A., Valenciano, F., 2001. Shapley–Shubik and Banzhaf indices revisited. Math. Operations Res. 26, 89–104.

Lehrer, E., 1988. An axiomatization of the Banzhaf value. Int. J. Game Theory 17, 89–99.

Owen, G., 1978. Characterization of the Banzhaf–Coleman index. SIAM J. Appl. Math. 35, 315–327.

Owen, G., 1982. Game Theory, 2nd Edition. Academic Press, New York.

Penrose, L.S., 1946. The elementary statistics of majority voting. J. Royal Statist. Soc. 109, 53–57.

Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II. Princeton Univ. Press, Princeton. Also in: Ann. Math. Stud. 28.

Shapley, L.S., 1954. Lecture Notes on Game Theory. Princeton University.

Shapley, L.S., 1962. Simple games: an outline of the descriptive theory. Behavioral Science 7, 59–66.

Straffin, P.D., 1982. Power indices in politics. In: Brahms, S.J., Lucas, W.F., Straffin, P.D. (Eds.), Political and Related Models. In: In: Lucas, W.F. (Ed.), Models in Applied Mathematics, vol. 2. Springer, New York.