Growth

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1 The Issues and their Importance

See Chapter 1 of Acemoglu's excellent book "Introduction to Modern Economic Growth" (Princeton University Press, 2009).

1.1 Questions

- What makes some countries rich and some poor?
- How do income levels of a given country vary across time?
- What makes some countries grow faster than others?
- What, if anything, can poor countries do to become rich?
- Will rich countries keep on growing forever and what can make them grow faster?
- Is inequality across countries innate or just an accident? Is it possible for all countries to be rich?

2 Growth in the Cass-Koopmans Model

The Cass-Koopmans model we have seen is often referred to as the neoclassical growth model. The reason is that it sheds some light on the question of growth and has interesting implications regarding observed growth rates.[1] Within that model, growth occurs when the capital stock

*These notes are largely based on Per Krusell's notes who should get all the credit for the interesting and correct parts. My contribution lies in the numerous errors that have undoubtedly crept in. When you find one, let me know!

[1] Note that Solow's exogenous savings model already provides most of the insights discussed here. The only difference lies in thinking about savings rates as exogenous in Solow's model and endogenous in Cass-Koopmans.

is below steady state. The idea is straightforward: economies with low capital per worker provide better opportunities for investment. This is because of the assumptions on the production function and specifically on the marginal product of capital, which can be thought of as the return to investment. This return is high at low levels of capital per worker (diminishing marginal returns). It is therefore optimal to forego some current consumption in order to invest, because the increase in future consumption is large enough to justify this. On the other hand, diminishing marginal returns imply that the return to investment will progressively fall as the economy grows, until at some point it is no longer optimal to forego current consumption. This idea explains why a steady state is eventually reached.

At a steady state, no more growth takes place, a prediction that runs against the empirical observation of *sustained* growth. This counterfactual implication will lead us to think about how to augment the model to incorporate sustained growth in section 3. For now, we delve deeper into the prediction of growth while in transition that the Cass-Koopmans model provides.

2.1 The Convergence Hypothesis

One of the implications of the growth model considered until here is that the further away a country is from steady state, the faster it will grow. Put differently, countries with lower levels of capital (and hence GDP) are expected to grow at faster rates. Circumstantial evidence seems to support this model prediction. For example, Germany and Japan, who suffered a large destruction of their capital stock during World War II did subsequently grow at substantially faster rates than most countries. Similarly, developing economies like India, China or Brazil also tend to have faster growth rates than developed economies like the US or Germany. These observations are consistent with the model's prediction. Finding themselves at a low level capital, countries offer ample opportunities for investment with high returns and this investment hike sparks particularly high growth rates. Following this argument to its logical conclusion would imply that income/output across countries should have a tendency to converge, i.e. the gap in income between rich and poor should be closing, as the poor invest and build up their capital stock.

The above hypothesis is known as the 'Convergence Hypothesis'. It can be explicitly derived as follows: Let $g(.)$ denote the policy function for capital so that

$$k_{t+1} = g(k_t)$$

This is a representation of the solution to the infinite horizon Cass-

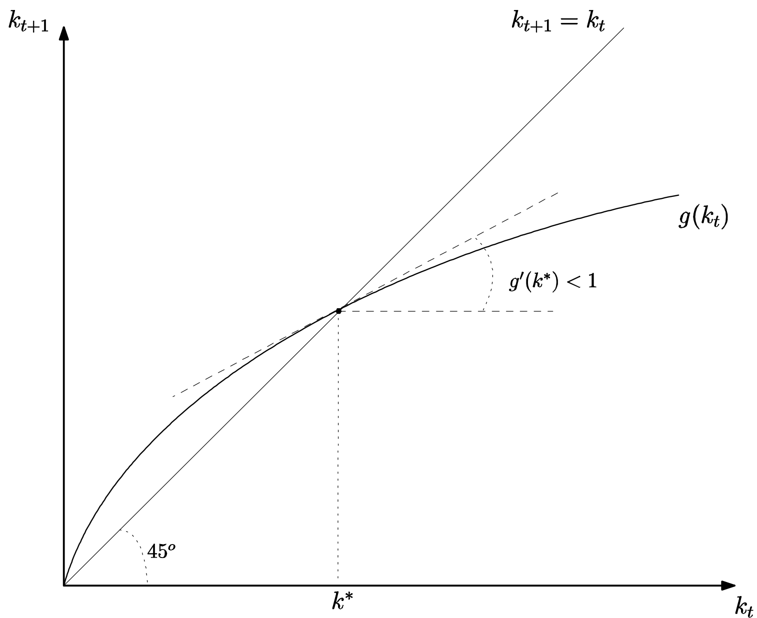

Figure 1: The policy function $g\left(k\right)$ and its slope at the steady state k^*.

Koopmans model. Under certain conditions, g is increasing, concave and crosses the 45 degree line from above at the steady state, i.e. $g'(k^*) < 1$, as shown in Figure 1.

Although the convergence result can be derived for a general production function (see Acemoglu Chapter 3.2), we illustrate the idea in the special case of a Cobb-Douglas production function

$$y_t = k_t^\alpha$$

Writing these two expressions in logs gives[2]

$$\begin{aligned} \log(k_{t+1}) &= \log(g(k_t)) \\ \log(y_t) &= \alpha \log(k_t) \end{aligned}$$

First, we can relate output growth to the growth in the capital stock using the production function

$$\begin{aligned} \log(y_{t+1}) - \log(y_t) &= \alpha \log(k_{t+1}) - \alpha \log(k_t) \Rightarrow \\ \log \frac{y_{t+1}}{y_t} &= \alpha \log(\frac{k_{t+1}}{k_t}) \end{aligned}$$

[2]Unless otherwise stated, we use *log* to denote the natural logarithm *ln* from here onwards.

Now the growth of k on the RHS is a function of the current capital k_t because $k_{t+1} = g(k_t)$

$$\log(\frac{k_{t+1}}{k_t}) = \log(\frac{g(k_t)}{k_t})$$

We do not have the function g but we can obtain an approximation for g around the steady state. Specifically, we approximate the right hand side with a log-linear function using a Taylor expansion with respect to $\log(k_t)$ around $\log(k^*)$. Define the function F as follows

$$F(\log k_t) \equiv \log \frac{g(k_t)}{k_t} = \log g(k_t) - \log(k_t)$$

The first-order Taylor expansion of F is

$$F(\log k_t) \simeq F(\log k^*) + \left[\frac{\partial F(\log k_t)}{\partial \log k_t}\right]_{k_t=k^*} (\log k_t - \log k^*)$$

To compute the derivative, note that

$$\begin{aligned}
\frac{\partial F(\log k_t)}{\partial k_t} &= \frac{\partial F(\log k_t)}{\partial \log k_t}\frac{\partial \log k_t}{\partial k_t} \Rightarrow \\
\frac{\partial F(\log k_t)}{\partial \log k_t} &= \frac{\partial F(\log k_t)}{\partial k_t} k_t \Rightarrow \\
\frac{\partial F(\log k_t)}{\partial \log k_t} &= \left(\frac{g'(k_t)}{g(k_t)} - \frac{1}{k_t}\right) k_t = k_t \frac{g'(k_t)}{g(k_t)} - 1
\end{aligned}$$

Evaluating this derivative at $k_t = k^*$, and noting that $g(k^*) = k^*$ at steady state, we obtain

$$\left[\frac{\partial F(\log k_t)}{\partial \log k_t}\right]_{k_t=k^*} = (g'(k^*) - 1)$$

Also note $F(\log k^*) = \log g(k^*) - \log(k^*) = 0$ so

$$F(\log k_t) \simeq (g'(k^*) - 1)(\log k_t - \log k^*)$$

or, written in terms of the growth of the capital stock

$$\log(\frac{k_{t+1}}{k_t}) \approx (g'(k^*) - 1)(\log k_t - \log k^*)$$

Given that at steady state the slope of the policy function is less than 1, we can say the following: the growth rate of the capital stock is positive for $k_t < k^*$, but it is decreasing in k_t and reaches 0 at $k_t = k^*$. Multiplying both sides by α and using the production function we find

$$\log(\frac{y_{t+1}}{y_t}) \approx (g'(k^*) - 1)(\log y_t - \log y^*) \tag{1}$$

which makes explicit the prediction of convergence in this model. The further away y_t is from steady state, the larger the growth rate of y_t.

The prediction can now be formally tested using standard econometric techniques and using information about all the countries for which we have data. This is the essence of the huge literature on growth and convergence that emerged in the 80's and 90's (see Robert Barro's work and references therein). Formally, the model predicts a negative relationship between the *level* of GDP in some year and the subsequent *growth rate* of GDP. It is straightforward to use cross-country data to test this hypothesis. Suppose we use data on GDP per capita in 1960 and average GDP per capita growth rates between 1960 and 1990 for each country indexed by i. Econometrically, think of a simple OLS regression

$$g_i = \beta_0 + \beta_1 \log(GDP_i) + \varepsilon_i \tag{2}$$

where g_i is the average annual growth rate of GDP per capita in country i between 1960 and 1990 and GDP_i is the level of GDP per capita of country i in 1960. The statistical test would be

$H_0 : \beta_1 = 0$

$H_1 : \beta_1 < 0$

where H_0 is the null hypothesis and H_1 is the alternative hypothesis. If we can find that the null hypothesis can be rejected in favor of the alternative, we would have statistical evidence in favor of the convergence hypothesis.

It turns out that the null here cannot be rejected. That is, there does not seem to be evidence of 'unconditional' (or 'absolute') convergence. The result is illustrated in Figure 2, which is taken from Acemoglu (2009), and shows the data for the entire world.

However, there is some evidence of convergence if one only focuses on countries with similar characteristics. That is, conditional on countries sharing some key characteristics, it does tend to be the case that those countries that have lower GDP per capita tend to subsequently grow faster than those that have higher GDP per capita. An illustration of this 'conditional' convergence is provided in Figure 3 where only countries that are members of the OECD are included.

Choosing to focus on OECD countries is to some extent arbitrary. What is it that makes OECD countries 'similar' to each other? A more

Figure 2: No unconditional convergence for the entire world.

careful approach would use the traditional econometric approach to 'condition on' or 'control for' similar characteristics, namely by adding control (explanatory) variables in a regression. Controlling for other variables in the regression, one can ensure that the *partial* effect of the independent variable ($logGDP$) on the dependent (g_i) is isolated by *keeping these other variables fixed*. The econometric model for conditional convergence is

$$g_i = \beta_0 + \beta_1 \log(GDP_i) + \beta_2' X_i + \varepsilon_i$$

where β_2 is a vector of parameters and X_i a vector of control variables that capture the important country characteristics. In the literature on growth regressions, many variables have been used as controls, most typically proxies for education (e.g. levels of schooling) and for health (e.g. life expectancy) as well as measures of trade openness, financial openness and political institutions. When we control for at least some of these variables, we find that the null hypothesis of $\beta_1 = 0$ can be rejected in favour of $\beta_1 < 0$, i.e. there is evidence of conditional convergence. The idea is that these control variables correct an omitted variable bias for β_1 and allow us to find a statistically significant negative effect of initial GDP on subsequent growth. In addition, the control variables used can be informative as to the question of what are country characteristics that can be important for growth.

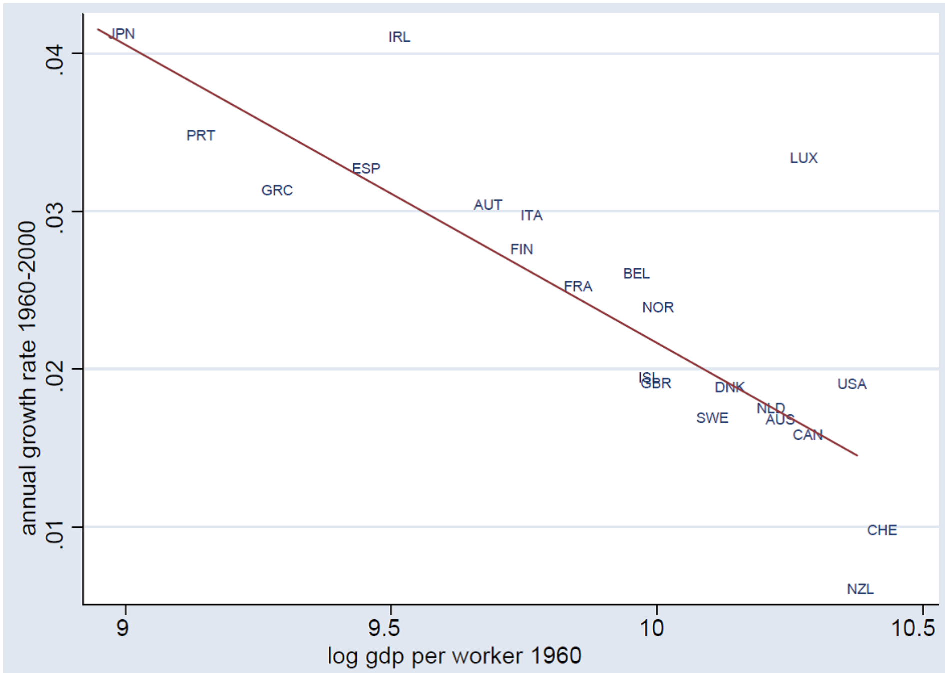

Figure 3: Conditional convergence for the OECD countries.

Although this discussion provides some success for the growth model as we have seen it, there is an important aspect that the model gets absolutely wrong: sustained growth seems to be a common phenomenon and the model predicts a steady state (zero growth) in the long run. In the following section, we will extend the standard growth model to incorporate sustained growth, but first let us introduce a labor/leisure choice to the Cass-Koopmans model.

2.2 Cass-Koopmans Model with labor-leisure choice

We have focused on the Cass-Koopmans model where all variables are interpreted as 'per worker' variables and implicitly we've assumed that labor is fixed and normalized to 1. Here, we briefly review the case of variable labor hours. First, let us clarify what is the labor input in the aggregate production function. This is intended to capture the total amount of hours worked in the economy, N_t. This N_t can be decomposed into the average number of hours worked per person times the number of people. In a yearly model, the total number of hours available is $365 * 24$ and each person has to divide their hours between labor and leisure, so that $laborhours_t + leisurehours_t = 365 * 24$. Since the level of variables is simply a normalization, we divide through by $365 * 24$. The right hand side is just 1, the left hand side now has the fraction of time spent working (we denote this n_t) and the fraction of time spent on leisure (we denote this l_t). Thus the time endowment restriction is

$$n_t + l_t = 1$$

This is the first component of the labor input. The second component has to do with population. Let M_t denote the population at time t. Then $N_t = n_t M_t$.[3] Until now, we have assumed that the population is constant $M_t = M$ and that $n_t = 1$. So we divided through by $N_t = M$ to write our model in per capita terms.

Maintaining the fixed population assumption,[4] we can introduce variable labor by assuming the time worked n_t is a choice variable and that leisure $(1 - n_t)$ is valued by agents and, hence, by the planner. The model is now

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

subject to

$$\begin{aligned} c_t + i_t &= F(k_t, n_t) \\ k_{t+1} &= (1 - \delta) k_t + i_t \\ c_t &\geq 0, k_{t+1} \geq 0 \\ 0 &\leq n_t \leq 1 \end{aligned}$$

The characterization of the optimum is left as an exercise.[5]

Most of the results and mechanics of this version work qualitatively in the same way as in the case of fixed labor. So we will be assuming that there exists a unique strictly positive steady state capital per worker k^* and, in addition, the policy function for capital starts at zero, is strictly increasing and concave and eventually crosses the 45 degree line only once from above (i.e. transition dynamics lead to the steady state).

[3] Note that a third component could be introduced relating the population to employment, which is really what should go in the labor input. We assume a constant employment to population ratio so that population and employment are just related by a constant, which we can normalize away.

[4] Introducing population growth can be dealt with in a similar fashion to the introduction of technological growth discussed in the following sections.

[5] See Assignments for the class. Note that the period utility depends on leisure $1 - n_t$. What happens if utility does not depend on leisure?

3 Sustained and Balanced Growth

3.1 Stylized Facts of Growth

The following facts about the growth experience of developed economies are often used as a starting reference for the study of growth. These are referred to as 'Kaldor's stylized facts of growth' after Nicholas Kaldor.

1. GDP (per capita) grows at constant rate
2. Capital to Labor ratio grows at constant rate
3. Capital to Output ratio is constant
4. Capital and labor shares of total income are constant
5. Real rates of return are constant
6. Growth rates vary persistently across countries

The word 'constant' in all those statements is used loosely. It is certainly the case that all these variables can vary from year to year, i.e. they are not really constant. What we mean by 'constant' is that they do not exhibit a clear upward or downward secular trend.[6]

The first observation asserts that growth is *sustained*. That is, even developed countries which have been growing for decades without experiencing catastrophic shocks to their capital stock, still maintain a positive growth rate. This is contrary to the Cass-Koopmans model prediction that a country will asymptotically approach a steady state where growth is zero. In the Cass-Koopmans model, when a country is poor, the returns to investment are high relative to when it is rich and thus investment and growth is relatively high. As the country grows richer, diminishing marginal returns reduce the return to the point where it is no longer optimal to keep growing. We will need to think about what makes returns to investment remain high enough so that new investment (over and above depreciation) persists. A widely offered explanation is new technology, i.e. technological growth.

Observations 1-5 have spurred the term *balanced growth*, referring to the fact that despite substantial variation in GDP across time and persistent growth, many features remain stable. Partly justified by those observations, we will build a model of sustained growth with the features of balanced growth. However, you should be aware that there are aspects of growth that are really not balanced. For example, we have

[6] To be more precise, these time series are stationary.

historically observed large sectoral shifts from agriculture, to manufacturing, to services. You should also be aware that there is some debate even about the stationarity of capital and labor shares and the capital-output ratio. Still, we will focus on balanced growth since this forms the backbone of the vast majority of modern macroeconomic research and it is also quite convenient from a methodological perspective. Specifically, within a balanced growth framework, we will be able to use all of the methodological machinery we have developed until here with only slight modifications.

Below we introduce technology growth in an otherwise standard Cass-Koopmans model. We call this an 'exogenous' growth model because growth in technology is assumed exogenously and leads directly to growth in GDP per capita. In this sense, sustained growth is essentially exogenous. Investigating the *causes* of technology growth is the subject of a large literature on "Endogenous Growth" which we only touch upon in section 5.

4 Exogenous Growth Model

4.1 Modelling Technology

Let us consider the Cass-Koopmans model with endogenous labor/leisure choice and introduce technology. The assumption is that the level of technology A_t at time t affects the productive capability of the economy so it enters as an additional argument in the production function

$$F(k_t, n_t, A_t)$$

The production function is increasing in the technology term A_t, which captures the idea that higher production can be obtained from the same amount of capital and labor inputs as long as technology is higher/better. We refer to A_t as technology, but perhaps the term "productivity" would be a more precise term, since the productive capability can be increased in a variety of different ways, not always directly related to technology in the narrow sense of the term. For example, processes that are more efficient are also captured by this. I use the terms technology and productivity interchangeably from now on.

The new factor A_t could affect production in many ways. Typical examples are

1. Total Factor Productivity (TFP, also called Hicks-Neutral)

$$A_t F(k_t, n_t)$$

2. Capital-Augmenting Technology (also called Solow-neutral)

$$F(A_t k_t, n_t)$$

3. Labor-Augmenting Technology (also called Harrod-neutral)

$$F(k_t, A_t n_t)$$

We will be assuming that technology A_t grows exponentially over time and try to build a model of balanced growth resulting from this.

It turns out that if we want the model to deliver balanced growth, this can only happen with technology growth that can be represented as labor-augmenting.[7] The assumption of labor augmenting technology essentially means that the production function depends on capital and *effective hours*. That is, we can think of the second argument of the production function, $A_t n_t$, as being measured in effective hours rather than just hours. The idea is that technology improvements increase the efficiency of a worker so that the same worker can produce more with one hour of work than they used to.

Rather than providing a proof of the necessity of labor-augmenting technology, it's more helpful to think about the intuition: Why does balanced growth require technology to augment labor and not capital? Looking at Kaldor's facts, we observe capital and output growing at the same constant rate but both grow faster than labor. What happens in the model? The labor input is bounded above, i.e. cannot be growing indefinitely.[8] Given the assumption of diminishing marginal returns to capital, as capital grows keeping labor fixed, the extra production will gradually fall, which means that output cannot be growing as fast as capital grows. This is the essence of the convergence to a steady state in the absence of technological growth. To ensure balanced growth is possible, any exogenous growth factor must augment the labor input at the same time in order to maintain the same increase in output as in the capital stock. Indeed, we will see that, with a constant returns to scale production function, as long as efficiency units grow at the same rate as the capital stock, we can conclude that output will also grow at the same rate. To put it differently, capital can grow indefinitely and at the same time the marginal product of capital (and thus real rates

[7] This is a theorem by Uzawa, you can see Proposition 2.6 of Acemoglu's textbook for a version of that theorem.

[8] The argument focuses on the case of no population growth for simplicity. Population growth simply increases the growth rates of all variables, but does not change the relative size of the growth rate, i.e. does not change the main idea.

of return) can remain constant. We formalize this intuition in the next section.

A couple of important points should be noted before we proceed with the technical details. First, the Cobb-Douglas production function is a particular case in which all of the types of technology growth are permitted. The reason is that all of these types *can be represented* as labor augmenting technologies

$$A_t k_t^\alpha n_t^{1-\alpha} = \left(A_t^{\frac{1}{\alpha}} k_t\right)^\alpha n_t^{1-\alpha} = k_t^\alpha \left(A_t^{\frac{1}{1-\alpha}} n_t\right)^{1-\alpha}$$

Second, we will first discuss what types of technology growth allow for balanced growth to be *feasible*, i.e. focus only on the constraints of the model. Since the feasibility constraints of the Cass-Koopmans model are the same as the ones in the Solow model, this discussion applies to both. Subsequently, we will also discuss restrictions on utility such that balanced growth can also be optimal in the long run. That discussion obviously applies to the micro-founded Cass-Koopmans model only.

4.2 Balanced Growth Path

Assume that efficiency grows at rate $\gamma - 1$ (the technology growth rate), i.e. $\frac{A_{t+1} - A_t}{A_t} = \gamma - 1$ or $A_{t+1} = \gamma A_t$, for all t.[9] The production function is written as

$$F(k_t, A_t n_t)$$

where $F : R_+^2 \to R_+$ satisfies the usual conditions. We focus our attention on a *balanced growth path*, henceforth BGP, where all variables grow at a constant rate. The BGP is the analog of the steady state of our previous model, but in an economy where all variables are growing forever. In the absence of technology growth we saw that starting from a given initial k_0, the economy will experience some transition period and eventually converge to a steady state. If initial capital is exactly equal to the steady state, then there is no transition and the economy is at steady state forever. With technological growth, an analogous situation will occur. Starting from a given initial k_0, the economy will again experience a period of transition and eventually converge to the BGP. But if the initial condition k_0 puts the economy *on the BGP to start with*, then the economy will be on the BGP from then on.

[9] Strictly speaking, the growth rate is $\gamma - 1$ and γ is the growth factor. In what follows, I am not very careful with this terminologyand sometimes refer to γ as the growth rate.

Also, we will sometimes replace A_t directly by γ^t based on the normalization $A_0 = 1$.

For notational simplicity, and without loss of generality, in the following derivations I will assume that the economy starts from the beginning on the BGP. Subsequently, we will return to the more general case where the BGP is reached after a period of transition. Let the growth factor of a variable x at time t be denoted by g_{xt}, i.e. define for any variable x

$$g_{xt} = \frac{x_{t+1}}{x_t}$$

and let g_x be the (constant) BGP growth factor of x. Assuming the economy starts at the BGP will let us write $x_t = x_0 g_x^t$.

4.3 Feasibility of Balanced Growth

Claim: Suppose all variables grow at constant rates, then $g_c = g_y = g_i = g_k = \gamma$ and $g_n = 1$ (i.e. n_t does not grow)

In words, this says that if consumption, output, investment and capital all grow at a constant rate, then they have to grow at the same rate and, furthermore, that rate has to equal the technology growth rate.

Heuristic Proof: The fraction of time devoted to labor market activities, n_t, cannot be growing indefinitely since it would eventually violate the time endowment restriction.[10] However, the effective units of labor $A_t n_t$ will grow at the technological growth rate by assumption.

Next, from the capital accumulation equation

$$\frac{k_{t+1}}{k_t} = (1-\delta) + \frac{i_t}{k_t}$$

we can conclude that investment and capital must grow at the same rate. Why? Because the LHS is constant and equal to the growth of capital g_k, $(1-\delta)$ is also constant so $\frac{i_t}{k_t}$ must be constant (i.e. independent of t). Since both i_t and k_t grow at constant rates, it must be the same rate

$$\frac{i_t}{k_t} = \frac{i_0 g_i^t}{k_0 g_k^t} = \frac{i_0}{k_0}\left(\frac{g_i}{g_k}\right)^t$$

This is because if $g_i > g_k$ then $\frac{i_t}{k_t}$ would grow over time and if $g_i < g_k$ then $\frac{i_t}{k_t}$ would fall over time. Therefore $g_i = g_k$.

Next, by the resource constraint, all three variables c, i and y (and thus also k) must have the same growth rate. To see this, use the resource

[10] It could, in principle, be falling exponentially to zero, but this is not the case in the data and we ignore this possibility.

constraint

$$
\begin{aligned}
c_t + i_t &= y_t \\
\frac{c_0 g_c^t}{y_0 g_y^t} + \frac{i_0 g_i^t}{y_0 g_y^t} &= 1 \\
\frac{c_0}{y_0}\left(\frac{g_c}{g_y}\right)^t + \frac{i_0}{y_0}\left(\frac{g_i}{g_y}\right)^t &= 1
\end{aligned}
$$

Consider cases

1. Suppose $g_c < g_y$ and $g_i < g_y$ then as t increases this will be falling (to 0).
2. Suppose $g_c > g_y$ and $g_i > g_y$ then as t increases this will be rising (to ∞).
3. Suppose $g_c < g_y$ and $g_i > g_y$ then as t increases this will be rising (to ∞).
4. Suppose $g_c > g_y$ and $g_i < g_y$ then as t increases this will be rising (to ∞).

In all those cases, the LHS would vary with t but the RHS is constant and equal to 1. Therefore, for the resource constraint to hold in a growing economy, none of those cases are possible and it can only be that $g_c = g_y = g_i \left(= g_k\right)$.

Finally, it remains to be shown that the common growth factor is exactly equal to γ, the technology growth factor. Using the production function and the fact that it has constant returns to scale we obtain

$$
\frac{y_t}{k_t} = F(1, \frac{A_t n_t}{k_t})
$$

Since y_t and k_t grow at equal rates, the LHS is constant so the RHS must also be constant. Therefore $\frac{A_t n_t}{k_t}$ must be constant and so k_t grows as fast as A_t.

Thus, at the BGP it must be that $g_c = g_y = g_i = g_k = \gamma$.

4.4 Optimality of Balanced Growth

We now turn to the question of whether balanced growth will be chosen as a result of utility maximization of the planner. The planner's problem is

$$
\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)
$$

subject to

$$c_t + k_{t+1} - (1-\delta)k_t = F(k_t, A_t n_t)$$

and first order conditions can be summarized in an Euler equation

$$u_c(c_t, 1-n_t) = \beta u_c(c_{t+1}, 1-n_{t+1})(1-\delta + F_k(k_{t+1}, A_{t+1}n_{t+1}))$$

and an intratemporal condition

$$\begin{aligned} -u_n(c_t, 1-n_t) &= u_c(c_t, 1-n_t) F_n(k_t, A_t n_t) \\ &= u_c(c_t, 1-n_t) A_t F_2(k_t, A_t n_t) \end{aligned}$$

where F_n denotes the partial derivative of the function F with respect to n_t whereas F_2 denotes the partial derivative of the function F with respect to the second argument of the function $A_t n_t$. So the question is whether a constant growth factor of c and k equal to γ can be consistent with these conditions in the long run. Looking at the Euler equation, note that F_k is homogeneous of degree zero since F is homogeneous of degree 1. In math,

$$\begin{aligned} F(zk, zAn) &= zF(k, An) \Rightarrow \\ zF_1(zk, zAn) &= zF_1(k, An) \Rightarrow \\ F_1(zk, zAn) &= F_1(k, An) \end{aligned}$$

Put differently, the marginal product of capital ($F_1 = F_k$) depends on the ratio $\frac{k_t}{A_t n_t}$ and this remains constant in the long run.[11] For the Euler equation to hold it must be that $\frac{u_c(c_t, 1-n_t)}{\beta u_c(c_{t+1}, 1-n_{t+1})}$, the intertemporal marginal rate of substitution (IMRS), remains constant in the long run.[12] This should be the case, despite growth in consumption which means that consumption levels should not matter for the IMRS, only the ratio $\frac{c_t}{c_{t+1}}$ should matter. Another way to say this is that we need a constant elasticity of substitution between c_t and c_{t+1} ($\frac{d\log\left(\frac{c_{t+1}}{c_t}\right)}{d\log\left(\frac{u_{c,t+1}}{u_{ct}}\right)}$).

Following the same reasoning as we did for F_1, we can show that F_2 is also homogeneous of degree zero and, hence, constant in the long run (it also depends on the ratio $\frac{k_t}{A_t n_t}$). However, $F_n = A_t F_2$ grows at the

[11] The marginal product of capital is the real rate of return mentioned in the Kaldor facts.

[12] The *intertemporal* marginal rate of substitution gives the marginal rate of substitution between two goods, the consumption good at time t and the consumption good at time $t+1$.

rate of technology.[13] Accordingly, the MRS between consumption and leisure $-\frac{u_n(c_t,1-n_t)}{u_c(c_t,1-n_t)}$ must grow at the same rate as technologyA_t.

The following period utility satisfies both requirements

$$\frac{[c_t v(1-n_t)]^{1-\sigma}-1}{1-\sigma}, \sigma > 0$$

where $v(.)$ is an increasing function of leisure.[14] Assuming as always that $n_t = n^*$ is stable in the balanced growth path, the IMRS depends on consumption growth $\frac{c_t}{c_{t+1}}$ only

$$\frac{u_c(c_t, 1-n_t)}{\beta u_c(c_{t+1}, 1-n_{t+1})} = \frac{c_t^{-\sigma}[v(1-n_t)]^{1-\sigma}}{\beta c_{t+1}^{-\sigma} v(1-n_{t+1})^{1-\sigma}} = \frac{1}{\beta}\frac{c_t}{c_{t+1}}\frac{[v(1-n_t)]^{1-\sigma}}{[v(1-n_{t+1})]^{1-\sigma}}$$

and the MRS between consumption and leisure grows at the consumption growth rate which is the same as the technology growth rate

$$-\frac{u_n(c_t, 1-n_t)}{u_c(c_t, 1-n_t)} = \frac{c_t^{1-\sigma}[v(1-n_t)]^{-\sigma} v'(1-n_t)}{c_t^{-\sigma}[v(1-n_t)]^{1-\sigma}} = c_t \frac{v'(1-n_t)}{v(1-n_t)}$$

The intuition for this is easier to understand if we introduce some market equilibrium terminology that will become clearer when we get to competitive equilibria (remember we are still consider a planner's maximization problem, no explicit markets yet). The idea is that optimization is consistent with constant labor supply in the long run, because consumption and the marginal product of labor (which equals the real wage rate from the point of view of a market equilibrium) grow at the same rate in the BGP. An interpretation that will be useful when we get to real business cycles is the following: The intratemporal first order condition essentially describes labor supply as a function of wages. A change in the wage rate has both income and substitution effects: an increase in the real wage rate would induce more work through the substitution effect since it makes leisure more costly, but less work through the income effect since a richer agent has higher consumption and thus lower marginal utility. The utility used here ensures that the two effects exactly cancel out for a permanent increase in wages so hours worked remain constant despite secular growth in real wages.

[13] This is the "return" to labor. We will relate it to the real wage rate when we introduce markets, prices and trade.

[14] Additional assumptions need to be imposed on $v(.)$ to ensure concavity, see King, Plosser and Rebelo (1988).

4.5 Is this economy consistent with Kaldor's facts?

After a period of transition, the economy converges to the Balanced Growth Path where many of the observations of Kaldor are replicated. GDP per capita grows at a constant rate $\gamma - 1$, the ratio of capital to output is constant since capital and output grow at the same rate and the ratio of capital to labor grows at a constant rate. In addition, the shares of capital and labor income are constant. Although we have not discussed general equilibrium yet, a brief preview will illustrate this. With a CRS production function it must be (by Euler's Theorem)

$$F(k_t, A_t n_t) = F_1 k_t + F_2 A_t n_t$$

and in equilibrium the real return to capital r_t must equal the marginal product of capital and the real wage rate w_t must equal the marginal product of labor. That is,

$$\begin{aligned} r_t &= \frac{\partial F(k_t, A_t n_t)}{\partial k_t} = F_1 \\ w_t &= \frac{\partial F(k_t, A_t n_t)}{\partial n_t} = F_2 A_t \end{aligned}$$

Note that these imply that real wages grow at the BGP whereas the real return to capital remains constant even though capital keeps growing. (Contrast this with the no-technology-growth case where the return falls as capital grows and, as a result, capital converges to a steady state.) The income share of capital is

$$\frac{r_t k_t}{y_t} = \frac{F_1 k_t}{F(k_t, A_t n_t)} = \frac{F_1 k_t}{y_t}$$

and the income share of labor is

$$\frac{w_t n_t}{y_t} = \frac{F_2 A_t n_t}{F(k_t, A_t n_t)} = \frac{F_2 A_t n_t}{y_t}$$

Both of these are constant in the BGP. For the specific case of the Cobb-Douglas production function, and replacing $A_t = \gamma^t$, we can express all these as follows

$$\begin{aligned} r_t &= \alpha \left(\frac{k_t}{\gamma^t n_t} \right)^{\alpha - 1} = \text{constant} \\ w_t &= (1 - \alpha) \left(\frac{k_t}{\gamma^t n_t} \right)^{\alpha} \gamma^t = \text{constant} * \gamma^t \\ \frac{r_t k_t}{y_t} &= \alpha \\ \frac{w_t n_t}{y_t} &= 1 - \alpha \end{aligned}$$

In fact, for the Cobb-Douglas production function constant factor income shares (α, $1-\alpha$) obtain even during the transition.

4.6 Transforming to a stationary model

At this point we know that at a BGP, $g_k = g_y = g_c = g_i = \gamma$ and $g_n = 1$. We use this knowledge to transform our model to a stationary one. This involves redefining variables in a way that will ensure those variables do not grow indefinitely. The transformation is

$$\begin{aligned}
\hat{c}_t &= \frac{c_t}{A_t} \\
\hat{\imath}_t &= \frac{i_t}{A_t} \\
\hat{k}_t &= \frac{k_t}{A_t} \\
\hat{y}_t &= \frac{y_t}{A_t}
\end{aligned}$$

Note that there is no need to transform n_t since it does not grow in the long run anyway. The hatted ('^') variables will be constant at the BGP. When transforming a model in practice, there are two ways one can follow: 1. Transform the objective and constraints and rewrite them in terms of the new, hatted variables or, 2. obtain optimality conditions for the model before transformation and then transform those conditions. Since it is important to know if utility is bounded before going ahead and using standard optimization techniques, we follow the first approach here. The transformation of the feasibility conditions (constraints) is in any case identical in the two cases. The resource constraint can be divided through by A_t

$$\begin{aligned}
c_t + i_t &= F(k_t, A_t n_t) \Rightarrow \\
\frac{c_t}{A_t} + \frac{i_t}{A_t} &= \frac{F(k_t, A_t n_t)}{A_t} \Rightarrow \\
\frac{c_t}{A_t} + \frac{i_t}{A_t} &= F(\frac{k_t}{A_t}, n_t) \Rightarrow \\
\hat{c}_t + \hat{\imath}_t &= F(\hat{k}_t, n_t)
\end{aligned}$$

The capital accumulation constraint can also be divided through by A_t, but we need to be careful with replacing k_{t+1} with $\hat{k}_{t+1}A_{t+1}$

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + i_t \Rightarrow \\ \frac{k_{t+1}}{A_t} &= (1-\delta)\frac{k_t}{A_t} + \frac{i_t}{A_t} \Rightarrow \\ \frac{k_{t+1}}{A_{t+1}}\frac{A_{t+1}}{A_t} &= (1-\delta)\frac{k_t}{A_t} + \frac{i_t}{A_t} \Rightarrow \\ \gamma\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + \hat{\imath}_t \end{aligned}$$

As for the utility, it is summing up the following terms (using $A_t = \gamma^t$)

$$\begin{aligned} \beta^t \frac{c_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - 1}{1-\sigma} &= \beta^t \frac{\gamma^{t(1-\sigma)}\hat{c}_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - 1}{1-\sigma} \\ &= \beta^t \frac{\gamma^{t(1-\sigma)}\left[\hat{c}_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - \gamma^{-t(1-\sigma)}\right]}{1-\sigma} \\ &= \beta^t \frac{\gamma^{t(1-\sigma)}\left[\hat{c}_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - 1 + 1 - \gamma^{-t(1-\sigma)}\right]}{1-\sigma} \\ &= \left(\beta\gamma^{(1-\sigma)}\right)^t \left[\frac{\hat{c}_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - 1}{1-\sigma} + \frac{1-\gamma^{-t(1-\sigma)}}{1-\sigma}\right] \end{aligned}$$

The second term is irrelevant from a maximization point of view (γ, β, t and σ are not choice variables) which means the optimal choice will be the same if we ignore it. Thus the transformed problem looks like

$$\max_{\{\hat{\imath}_t,\hat{c}_t,\hat{k}_{t+1},\hat{n}_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \left(\beta\gamma^{(1-\sigma)}\right)^t \frac{\hat{c}_t^{1-\sigma}\left[v(1-n_t)\right]^{1-\sigma} - 1}{1-\sigma}$$

subject to

$$\begin{aligned} \hat{c}_t + \hat{\imath}_t &= F(\hat{k}_t, n_t) \\ \gamma\hat{k}_{t+1} &= (1-\delta)\hat{k}_t + \hat{\imath}_t \\ &\hat{k}_0 \text{ given} \end{aligned}$$

This is very similar to our original Cass-Koopmans model. There are only two, minor differences. First, there is a γ appearing in the capital accumulation equation. This will not affect any of the qualitative dynamics we described before. Second, the discount factor is slightly changed, it is now a composite of three underlying parameters, β, γ and σ. It is important for the discount factor to be less than 1. Given that $\beta < 1$ and $\gamma > 1$, if $\sigma \geq 1$ then clearly $\beta\gamma^{(1-\sigma)} < 1$, so there is no problem. If $0 < \sigma < 1$, this might be a problem. We will need to ensure that parameters are such that $\beta\gamma^{(1-\sigma)} < 1$. As long as this is the case, the behavior of this model is really the same as the original growth model. In particular, $\hat{k}_t$ should reach a steady state $\hat{k}^*$ in the long run. If $\hat{k}_0 < \hat{k}^*$ we will observe growth of $\hat{k}_t$ to the steady state and if $\hat{k}_0 > \hat{k}^*$ we will observe reduction of $\hat{k}_t$ towards the steady state. This is the benefit of the transformation we have implemented: we do not need to work through the implications of this model because we have already done it. All the solution methodologies we have learned for the Cass-Koopmans model can be applied here to solve for the optimal sequences $\{\hat{\imath}_t, \hat{c}_t, \hat{k}_{t+1}, \hat{n}_t)_{t=0}^{\infty}$ (or policy functions for the hatted variables). However, we are ultimately interested in the behavior of y_t, c_t, i_t, k_t, i.e. the variables before transformation. So once the hatted variables are solved for, we will want to transform back and obtain the actual variables by multiplying the hatted variables by γ^t.

4.7 Summary of exogenous growth model dynamics

The overall behavior of the exogenous growth model can be summarized as follows: There is a period of transition to the balanced growth path that resembles the transition to steady state in the Cass-Koopmans model. The transition can be thought of in terms of the transformed variables, $\hat{c}_t$, $\hat{k}_t$, $\hat{\imath}_t$, and $\hat{y}_t$. Thus, starting from a low capital stock, the transformed capital stock will grow towards the steady state, but its growth will slow down until it reaches the steady state where growth is zero. The actual capital stock, k_t (and the rest of the variables, except n_t), has an added underlying growth factor γ. Thus, its growth rate will be more than $\gamma - 1$ initially during the transition, but will fall progressively over time down to $\gamma - 1$ when the economy reaches its balanced growth path. Therefore, this model is consistent with both the transitional dynamics of growth (conditional convergence) and the long run observation of sustained, balanced growth.

Overall, the growth models we've seen give two insights:

1. Countries grow because productivity improves over time (due to technology or otherwise) and this maintains the return to investment high even as the capital-labor ratio grows.

2. During a transition period, that is when a country has its capital stock depleted or very low due to historical reasons, the country is expected to experience even faster growth because returns to investment are unusually high.

Although this is a good starting point for our understanding of growth patterns across different countries, it suffers from a serious shortcoming: namely that long run (sustained) growth is fully dependent on technological growth which we have exogenously assumed. This is why the model is called an "Exogenous Growth" model. It sheds light on the effects of technological growth on GDP per capita growth, but it does not really answer the question of why does technology grow and what determines how fast it grows. Put differently, we would ultimately like our model to produce this technological growth endogenously, so that we can understand why there might be differentials in technological, or more accurately, productivity growth rates across time and/or countries.

5 Endogenous Growth Models

Recall that, in the absence of exogenous technological growth, the Cass-Koopmans model implied convergence to a steady state with zero growth. The reason could be traced back to an assumption on the production function, namely the assumption of diminishing marginal returns to capital. Given that the labor input (fraction of time spent on market activities) remained bounded, this implied that as capital grows, the capital-labor ratio grows and the returns to investing will gradually fall.

An almost trivial way to allow for sustained growth in a standard Cass-Koopmans model is to simply do away with the assumption of diminishing marginal returns to capital. This is essentially the so-called AK model, which we describe briefly below. We also touch briefly upon a model of human capital accumulation which will share very similar characteristics with the AK model in terms of its resulting dynamics.

The literature on 'Endogenous Growth' is large[15], you can find some well-known examples briefly mentioned in the lecture notes by Per Krusell:

- Human Capital accumulation: Productivity grows 'by choice' through investment in human capital. This works in similar ways to physical capital, but captures the idea that the labor force can become more productive if an economy invests in its education, skills and training.

- Externalities cause the *aggregate* production function to exhibit increasing returns to scale, even though each *individual firm*'s production function has constant returns to scale.

- Imperfect competition implies that firms have an incentive to invest in Research and Development. This leads them to come up with new innovative ideas and enjoy the increased profits resulting from them.

[15] Well-known contributors include Romer, Lucas and Aghion-Howitt.

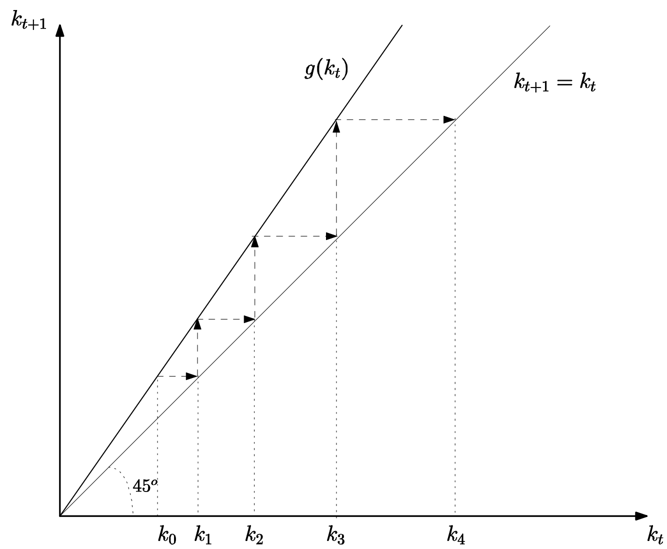

Figure 4: AK model dynamics from an initial k_0

5.1 AK model

The most basic endogenous growth model is the AK model. This is just a standard neoclassical growth model a la Cass-Koopmans, with the added twist that the production function is linear in capital

$$F(K) = AK$$

hence the name of this model. The idea in the AK model is as follows. Provided this assumption on production leads to a linear optimal capital policy function (which is true for the HARA class of preferences), there are different situations that can emerge: If the policy function has a slope less than 1 (more precisely if it crosses the 45 line from above), you will always find convergence to a steady state. But if the optimal policy has a slope larger than 1, then it can remain above the 45 degree line and the model will predict perpetual growth.

To be more concrete, we can write the AK model as

$$\max_{\{c_t, K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$c_t + K_{t+1} - (1-\delta)K_t = AK_t$$
$$K_0 \text{ given}$$

and obtain the first order condition for K_{t+1} (the Euler equation)

$$
\begin{aligned}
-c_t^{-\sigma} + \beta(1-\delta+A)c_{t+1}^{-\sigma} &= 0 \\
c_{t+1} &= [\beta(1-\delta+A)]^{\frac{1}{\sigma}} c_t
\end{aligned}
$$

which makes it clear that the consumption growth factor is $[\beta(1-\delta+A)]^{\frac{1}{\sigma}}$ at the optimum. Substituting consumption from the resource constraint, we can obtain a linear difference equation in capital which can be solved analytically. However, we can already see that consumption grows at a constant rate and it will have to be that capital and output grow at the same rate, otherwise the resource constraint cannot be satisfied.

Things to note about the AK model:

1. As long as $\beta(1-\delta+A) > 1$, consumption (and capital and output) grow perpetually at a constant rate.

2. The growth rate depends on parameters of model. Different countries will have different characteristics β, δ, A and σ and hence different growth rates.

Although the AK model delivers sustained and balanced growth and can shed some light on the determinants of growth, it does have some counterfactual aspects such as

1. There is no transition, there is a constant growth rate starting from period $t = 0$. But we have observed countries in transition growing faster than their long run average.

2. The production function is not very realistic in the sense that labor hours do not matter and the capital income share is 1.

3. It is hard to explain the level of dispersion in growth rates across countries using reasonable values for β, δ, A and σ.[16]

5.2 Human Capital Accumulation

Let h_t be the stock of human capital capturing skills and knowledge of the labor force acquired through education or other training. If this can capture the efficiency (which we denoted by A_t previously) then effective hours are now $h_t n_t$. To keep things simple, we assume no disutility

[16]Krusell argues that introducing cross-country variation in capital income tax rates can generate some improvement in this dimension.

of work so that $n_t = 1$. The production function is assumed to have constant returns to scale so

$$y_t = F(k_t, h_t) = Ak_t^{\alpha} h_t^{1-\alpha}$$

We model human capital accumulation completely symmetrically to physical capital accumulation. Specifically, the stock of human capital depreciates at a constant rate δ^h but can be replenished by investing i_t^h. The two stocks accumulate according to

$$\begin{aligned} k_{t+1} &= \left(1 - \delta^k\right) k_t + i_t^k \\ h_{t+1} &= \left(1 - \delta^h\right) h_t + i_t^h \end{aligned}$$

We are still assuming a one-good economy where the production can be consumed or invested but there are now two types of investment to choose from. The resource constraint is thus

$$c_t + i_t^h + i_t^k = y_t$$

Obtaining optimality conditions for the planner's problem in this case yields two Euler equations

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1}) \left(1 - \delta^k + \alpha A k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha}\right) \\ u'(c_t) &= \beta u'(c_{t+1}) \left(1 - \delta^h + (1-\alpha) A k_{t+1}^{\alpha} h_{t+1}^{-\alpha}\right) \end{aligned}$$

which, together with the resource constraint and the two capital accumulation equations as well as two transversality conditions, characterize the optimum. Notice that the Euler equations imply

$$\alpha A k_{t+1}^{\alpha-1} h_{t+1}^{1-\alpha} - \delta^k = (1-\alpha) A k_{t+1}^{\alpha} h_{t+1}^{-\alpha} - \delta^h \tag{3}$$

i.e. that the returns of the two types of investment have to be equalized at the optimum. Otherwise, it would make sense to take resources away from the capital with low returns and put them to the one with high returns. This condition also implies that the ratio of capital to human capital is optimally constant (because the condition relates $\frac{k_{t+1}}{h_{t+1}}$ to things that are independent of t and it has to hold for all t). This implies that at the optimum physical and human capital should increase at the same rate. The important implication of this optimal allocation of resources is that the return to investment in either of the stock will be constant along any path. Contrast this with the case of no human capital. In that case, as the economy grows the ratio of capital to labor increases and the return to investment in capital decreases. As a result, there comes a point where increasing capital further does not yield enough returns to cover the increase in depreciation costs. That point is the steady state where the economy stops growing. With human capital accumulation, as the economy grows, both physical and human capital grow and the returns remain constant. As long as that return is high enough to cover the extra depreciation costs, it is optimal to keep investing in both of those factors forever and the result is sustained growth.

This is very similar to the AK model where again the return to capital is constant *by assumption*. Here this is obtained as an endogenous result. Solving for the constant return requires solving the non-linear equation (3) for the ratio $\frac{k}{h}$. But for the special case $\delta^k = \delta^h = \delta$, we can obtain a closed form solution analytically, because in that case equation (3) becomes

$$\alpha A \left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha-1} = (1-\alpha) A \left(\frac{k_{t+1}}{h_{t+1}}\right)^{\alpha} \Rightarrow$$
$$\frac{k_{t+1}}{h_{t+1}} = \frac{\alpha}{1-\alpha}$$

so the return is $\alpha A \left(\frac{\alpha}{1-\alpha}\right)^{\alpha-1}$. We can also obtain the growth rate of the economy using the Euler equation and assuming a standard CRRA utility:

$$\frac{c_{t+1}}{c_t} = \left[\beta\left(1-\delta+(1-\alpha) A \left(\frac{\alpha}{1-\alpha}\right)^{\alpha}\right)\right]^{\frac{1}{\sigma}}$$

The growth rate of consumption is obtained as a function of underlying parameters. It is straightforward to show that all endogenous variables will have to grow at the same rate and the economy will be on the BGP from the beginning.