Assignment 3

Haixiang Zhu

Sepetember 15, 2020

1. From the second question of problem set 2, we have

$$\begin{cases} c_t = f(k_t) + (1-\delta)k_t - k_{t+1}, \text{ for } t = 0, 1, \dots, T \\ u'(c_t) = \beta[f'(k_{k+1}) + (1-\delta)]u'(c_{t+1}), \text{ for } t = 0, 1, \dots, T-1 \\ k_{T+1} = 0 \end{cases}$$

Setting up the model: $\delta = 0.1, \alpha = 0.3, \beta = 0.96, \sigma = 2, k_0 = 0.01$, $f(k) = k^{0.3}$, and $u(c) = 1 - c^{-1}$.

(a) T=1

i. All conditions for solving allocations$\{c_t, k_{t+1}\}_{t=0}^{1}$:

$$\begin{cases} c_0 = 0.01^{0.3} + (1-0.1) \times 0.01 - k_1 \\ c_1 = k_1^{0.3} + (1-0.1)k_1 - k_2 \\ c_0^{-2} = 0.96 \times (1 - 0.1 + 0.3k_1^{-0.7})c_1^{-2} \\ k_2 = 0 \end{cases}$$

ii. Reduced form

$$(0.01^{0.3} + 0.009 - k_1)^{-2} - 0.96(0.9 + 0.3k_1^{-0.7})(k_1^{0.3} + 0.9k_1)^{-2} = 0$$

Numerical results(See attached Matlab code "ps3_1a.m" and "fun_1a.m")

$$\begin{cases} c_0 = 0.2200 \\ c_1 = 0.4173 \\ k_1 = 0.0402 \\ k_2 = 0 \end{cases}$$

(b) T=2

i. All conditions for solving allocations$\{c_t, k_{t+1}\}_{t=0}^{2}$

$$\begin{cases} c_0 = 0.01^{0.3} + (1-0.1) \times 0.01 - k_1 \\ c_1 = k_1^{0.3} + (1-0.1)k_1 - k_2 \\ c_2 = k_2^{0.3} + (1-0.1)k_2 - k_3 \\ c_0^{-2} = 0.96 \times (1 - 0.1 + 0.3k_1^{-0.7})c_1^{-2} \\ c_1^{-2} = 0.96 \times (1 - 0.1 + 0.3k_2^{-0.7})c_2^{-2} \\ k_3 = 0 \end{cases}$$

ii. Reduced form

$$\begin{cases} (0.01^{0.3} + 0.009 - k_1)^{-2} - 0.96(0.9 + 0.3k_1^{-0.7})(k_1^{0.3} + 0.9k_1 - k_2)^{-2} = 0 \\ (k_1^{0.3} + 0.9k_1 - k_2)^{-2} - 0.96(0.9 + 0.3k_2^{-0.7})(k_2^{0.3} + 0.9k_2)^{-2} = 0 \end{cases}$$

Numerical results(See attached Matlab code "ps3_1b.m" and "fun_1b.m")

$$\begin{cases} c_0 = 0.2079 \\ c_1 = 0.3682 \\ c_2 = 0.5704 \\ k_1 = 0.0523 \\ k_2 = 0.0915 \\ k_3 = 0 \end{cases}$$

iii. Comparasion between Cass-Koopmans model and Solow model with respect to saving rate

$$\begin{cases} s_0 = 0.1724 \\ s_1 = 0.1077 \\ s_2 = -0.1688 \end{cases}$$

In Cass-Koopmans Model, the savings rate is endogenous and varies from time to time. The savings rate at the period before the last period must be negative. As for Solow Model, the steady-state savings rate is exogenous, while the savings rate given by the Golden Rule is endogenous.

(c) T=200
All conditions are as follows.

$$\begin{cases} [k_t^{\alpha} + (1-\delta)k_t - k_{t+1}]^{-\sigma} - \beta[(1-\delta) + \alpha k_{t+1}^{\alpha-1}] \\ \qquad \cdot [k_{t+1}^{\alpha} + (1-\delta)k_{t+1} - k_{t+2}]^{-\sigma} = 0, \forall t = 0, 1, \ldots, T-1 \\ k_{T+1} = 0 \end{cases}$$

The Matlab code is in attached file "ps3_1c.m" and "fun_1c.m".

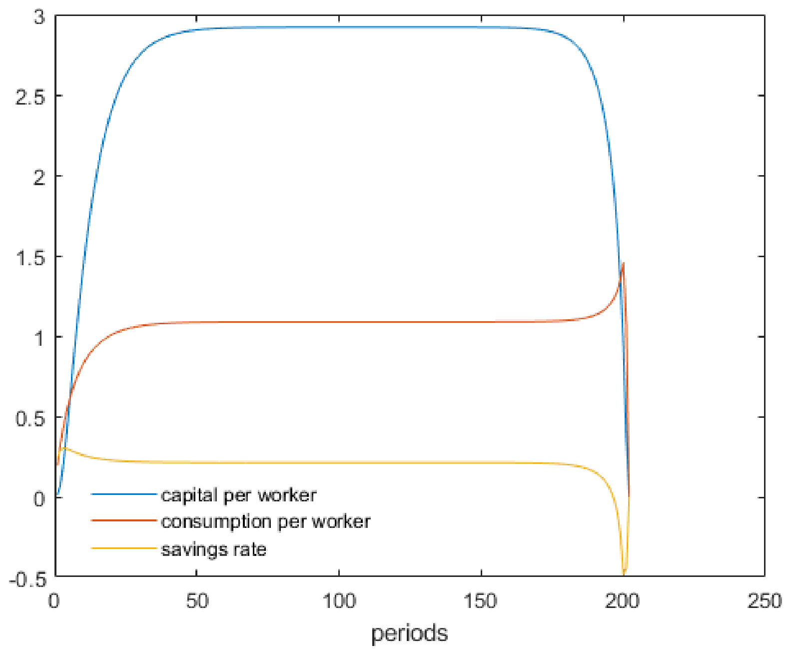

Figure 1: The sequences of capital, consumption and savings rate over time

2. Full Depreciation

   (a) Let $t = 1 - \sigma$. $\sigma \to 1 \Rightarrow t \to 0$
   By L'Hopital's rule,

$$\lim_{\sigma \to 1} u(c) = \lim_{t \to 0} \frac{c^t - 1}{t} = \lim_{t \to 0} c^t \ln(c) = \ln(c)$$

   (b) Inada conditions

   Proof.

$$f(\cdot) : \begin{cases} \lim\limits_{k \to 0} f'(k) = \lim\limits_{k \to 0} \alpha k^{\alpha - 1} = \infty & \alpha \in (0,1) \\ \lim\limits_{k \to \infty} f'(k) = \lim\limits_{k \to \infty} \alpha k^{\alpha - 1} = 0 & \alpha \in (0,1) \end{cases}$$

$$u(\cdot) : \begin{cases} \lim\limits_{c \to 0} u'(c) = \lim\limits_{c \to 0} c^{-1} = \infty \\ \lim\limits_{c \to \infty} u'(c) = \lim\limits_{c \to \infty} c^{-1} = 0 \end{cases}$$

□

(c) Euler equation

$$\begin{aligned} u'[f(k_t) + (1-\delta)k_t - k_{t+1}] =& \beta[f'(k_{t+1}) + (1-\delta)] \\ & \cdot u'[f(k_{t+1}) + (1-\delta)k_{t+1} - k_{t+2}], \forall t = 0, 1, \ldots, T-1 \end{aligned}$$

Plugging $u(c) = \ln(c), \delta = 1, f(k) = k^\alpha$, we have

$$\begin{aligned} \frac{1}{k_t^\alpha - k_{t+1}} &= \frac{\alpha\beta k_{t+1}^{\alpha-1}}{k_{t+1}^\alpha - k_{t+2}}, \forall t = 0, 1, \ldots, T-1 \\ \Rightarrow \frac{k_{t+1}^\alpha - k_{t+2}}{k_{t+1}^\alpha} &= \alpha\beta \frac{k_t^\alpha - k_{t+1}}{k_{t+1}}, \forall t = 0, 1, \ldots, T-1 \end{aligned}$$

Substituting $z_t = \frac{k_t}{k_{t-1}^\alpha}$, we obtain

$$\begin{aligned} 1 - z_{t+2} &= \alpha\beta\left(\frac{1}{z_{t+1}} - 1\right), \forall t = 0, 1, \ldots, T-1 \\ \Rightarrow z_{t+1} &= 1 + \alpha\beta - \frac{\alpha\beta}{z_t}, \forall t = 1, 2, \ldots, T \end{aligned}$$

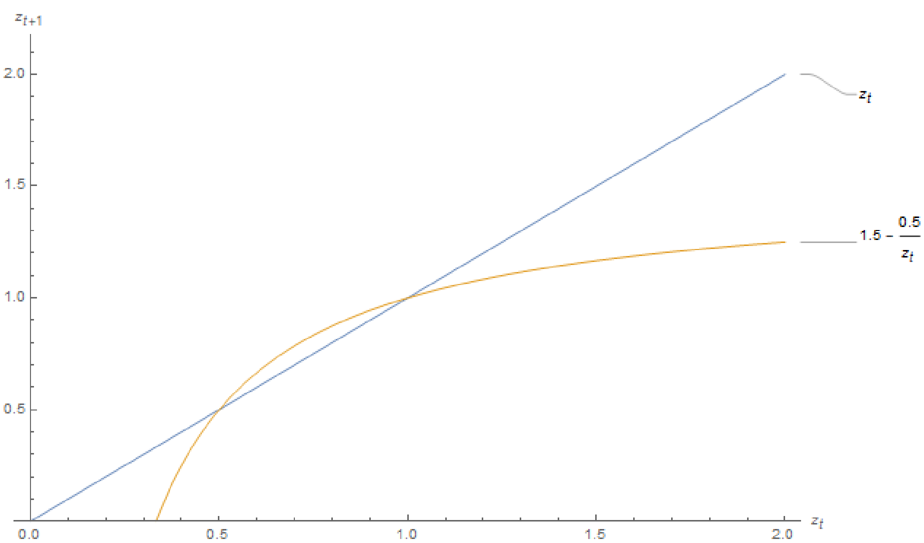

Figure 2: z_{t+1} against z_t ($\alpha\beta = 0.5$)

(d) *Proof.*

$$\begin{aligned}
&\because z_t = \frac{\alpha\beta}{1+\alpha\beta - z_{t+1}}, \forall t = 1, 2, \ldots, T \\
&z_{T+1} = 0 \\
&\therefore z_T = \frac{\alpha\beta}{1+\alpha\beta} \\
&z_{T-1} = \alpha\beta\frac{1+\alpha\beta}{1+\alpha\beta+(\alpha\beta)^2} \\
&z_{T-2} = \alpha\beta\frac{1+\alpha\beta+(\alpha\beta)^2}{1+\alpha\beta+(\alpha\beta)^2+(\alpha\beta)^3} \\
&\cdots \\
&z_{T-k} = \alpha\beta\frac{\sum_{i=0}^{k}(\alpha\beta)^i}{\sum_{i=0}^{k+1}(\alpha\beta)^i} = \alpha\beta\frac{1-(\alpha\beta)^{k+1}}{1-(\alpha\beta)^{k+2}}, \forall k = 0, 1, \ldots, T-1
\end{aligned}$$

Let $t = T - k$, then $k = T - t$.

$$z_t = \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}}, \forall t = 1, 2, \ldots, T$$

And since $z_{T+1} = 0$,

$$z_t = \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}}, \forall t = 1, 2, \ldots, T+1$$

$\square$

(e) Plugging $z_t = \frac{k_t}{k_{t-1}^{\alpha}}$,

$$\frac{k_t}{k_{t-1}^{\alpha}} = \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}}, \forall t = 1, 2, \ldots, T+1$$

Rewriting it as

$$k_{t+1} = \alpha\beta\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}k_t^{\alpha}, \forall t = 0, 1, \ldots, T$$