

Assignment 9

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1. Period-by-period budget constraint for each consumer in period t

$$c_t + q_t b_{t+1} = b_t + w_t$$

Given a sequence of endowments $\{\{w_{i,t}\}_{t=0}^{\infty}\}_{i=1}^2$, a competitive equilibrium with sequential trade consists of sequences of allocations $\{\{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^{\infty}\}_{i=1}^2$ and a sequence of prices $\{(q_t^b)^*\}_{t=0}^{\infty}$ such that

- (a) Given the price system, the allocation solves each consumer's problem. For $i = 1, 2$

$$\begin{aligned} \{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^{\infty} &= \arg \max_{\{c_{i,t}, b_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\sigma}}{1-\sigma} \\ \text{s.t.} \quad c_{i,t} + (q_t^b)^* b_{i,t+1} &= b_{i,t} + w_{i,t} \quad \forall t \\ c_{i,t} &\geq 0 \quad \forall t \\ b_{i,0} &= 0 \\ \lim_{T \rightarrow \infty} b_{i,T+1}^* \prod_{t=0}^T (q_t^b)^* &\geq 0 \end{aligned}$$

- (b) All markets clear. For goods market

$$\sum_i c_{i,t}^* = \sum_i w_{i,t} \quad \forall t$$

For asset market

$$\sum_i b_{i,t+1}^* = 0 \quad \forall t$$

2. All conditions for equilibrium (FOC+B.C+M.C+TVC+nPg) $\forall t, i$

$$\begin{aligned}
\beta^t (c_{i,t}^*)^{-\sigma} &= \mu_{i,t} & (c_{i,t}) \\
(q_t^b)^* \mu_{i,t} &= \mu_{i,t+1} & (b_{i,t+1}) \\
c_{i,t} + (q_t^b)^* b_{i,t+1} &= b_{i,t} + w_{i,t} & (\text{B.C}) \\
c_{1,t}^* + c_{2,t}^* &= w_{1,t} + w_{2,t} & (\text{goods}) \\
b_{1,t+1}^* + b_{2,t+1}^* &= 0 & (\text{asset}) \\
\lim_{T \rightarrow \infty} \beta^T (c_{i,T}^*)^{-\sigma} b_{i,T+1}^* &\leq 0 & (\text{TVC}) \\
\lim_{T \rightarrow \infty} b_{i,T+1}^* \prod_{t=0}^T (q_t^b)^* &\geq 0 & (\text{nPg})
\end{aligned}$$

where $\mu_{i,t}$ is the multiplier on consumer i 's budget constraint for each period t . The non-negativity constraints is ignored because of the Inada condition for utility function.

3. *Proof.* The equivalence between the date-0 equilibrium and the sequential equilibrium.

Recall the characterization of the date-0 trade equilibrium $\forall t, i$

$$\begin{cases}
\beta^t (c_{i,t}^*)^{-\sigma} = \lambda_i p_t^* \\
\sum_{t=0}^{\infty} p_t^* c_{i,t} = \sum_{t=0}^{\infty} p_t^* w_{i,t} \\
\sum_i c_{i,t}^* = \sum_i w_{i,t}
\end{cases}$$

where λ_i is the multiplier on consumer i 's budget constraint.

(a) Necessity (only if)

Consider any consumption choice $\{c_{i,t}^*\}_{t=0}^{\infty}$ that is feasible in the sequential trade equilibrium. Then, by rolling forward the sequential trade budget and using TVC and nPg it can shown that it satisfies the date-0 trade budget

$$\begin{aligned}
b_{i,0} &= c_{i,0}^* - w_{i,0} + (q_0^b)^* b_{i,1}^* \\
&= c_{i,0}^* - w_{i,0} + (q_0^b)^* (c_{i,1}^* - w_{i,1}) + (q_0^b)^* (q_1^b)^* b_{i,2}^* \\
&= \dots \\
&= \sum_{t=0}^{\infty} \prod_{s=0}^{t-1} (q_s^b)^* (c_{i,t}^* - w_{i,t}) + \lim_{T \rightarrow \infty} b_{i,T+1}^* \prod_{t=0}^T (q_t^b)^*
\end{aligned}$$

Now note that in equilibrium $\prod_{s=0}^{t-1} (q_s^b)^* = \beta^t \left(\frac{c_{i,t}^*}{c_{i,0}^*} \right)^{-\sigma}$ and this also

corresponds to $\frac{p_t^*}{p_0^*}$ in the date-0 trade equilibrium.

$$b_{i,0} = \sum_{t=0}^{\infty} \frac{p_t^*(c_{i,t}^* - w_{i,t})}{p_0^*} + \lim_{T \rightarrow \infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*} \right)^{-\sigma}$$

The last term is 0 (by using the nPg and TVC conditions) and using the zero initial wealth assumption and the normalization $p_0^* = 1$ we obtain

$$0 = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t})$$

(b) Sufficiency (if)

To show the opposite statement, that if $\{c_{i,t}^*\}_{t=0}^{\infty}$ satisfies the date-0 trade budget then it is feasible in the sequential trade economy, one can proceed by constructing the asset trades required to ensure the same consumption. Since in equilibrium

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*} \right)^{-\sigma}$$

then the asset choices can be constructed recursively

$$b_{i,t+1}^* = \frac{(c_{i,t}^*)^{-\sigma}}{\beta(c_{i,t+1}^*)^{-\sigma}} (b_{i,t}^* + w_{i,t} - c_{i,t}^*) \quad \forall t$$

With these choices for assets, and given that goods' markets clear, the asset market clears in every period (simply add $b_{i,t+1}$ across agents and show it equals zero). We can also show in a manner identical to before that

$$0 = b_{i,0} = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t}) + \lim_{T \rightarrow \infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*} \right)^{-\sigma}$$

and since the date-0 budget constraint is satisfied for this consumption sequence, this implies that

$$\lim_{T \rightarrow \infty} b_{i,T+1}^* \beta^T (c_{i,T}^*)^{-\sigma} = 0$$

that is, the nPg and TVC conditions are satisfied.

□

4. Special cases

Since date-0 equilibrium and sequential equilibrium are equivalent, from FOC

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*} \right)^{-\sigma} = \frac{p_{t+1}^*}{p_t^*}$$

Then

$$c_{i,t}^* + \frac{p_{t+1}^*}{p_t^*} b_{i,t+1}^* = b_{i,t}^* + w_{i,t} \quad (1)$$

- (a) $w_{1,t} = 2y, w_{2,t} = y \quad \forall t$
 Since $c_{i,t}^* = w_{i,t} \quad \forall t, i$, no trade happens in this case, which implies
 $b_{i,t} = 0 \quad \forall t$
- (b) $w_{1,t} = \{2y, y, 2y, y, \dots\}, w_{2,t} = \{y, 2y, y, 2y, \dots\} \quad \forall t$

$$\begin{cases} p_t^* = \beta^t \\ c_{1,t}^* = \frac{2+\beta}{1+\beta} y \\ c_{2,t}^* = \frac{1+2\beta}{1+\beta} y \end{cases} \quad \forall t \quad (2)$$

Plugging (2) into (1)

$$\begin{aligned} b_{1,1}^* &= \frac{1}{\beta} (0 + 2y - \frac{2+\beta}{1+\beta} y) = \frac{1}{1+\beta} y = -b_{2,1}^* \\ b_{1,2}^* &= \frac{1}{\beta} (b_{1,1}^* + y - \frac{1+2\beta}{1+\beta} y) = 0 = -b_{2,2}^* \\ &\dots \end{aligned}$$

Therefore,

$$\begin{cases} b_{1,2t}^* = 0 \\ b_{1,2t+1}^* = \frac{1}{1+\beta} y \\ b_{2,2t}^* = 0 \\ b_{2,2t+1}^* = -\frac{1}{1+\beta} y \end{cases} \quad \forall t$$

- (c) $w_{1,t} = 2y, w_{2,t} = \{y, 3y, y, 3y, \dots\} \quad \forall t$

$$\begin{cases} p_{2t}^* = \beta^{2t} \\ p_{2t+1}^* = \beta^{2t+1} \left(\frac{5}{3}\right)^{-\sigma} \\ c_{1,2t}^* = 6y \frac{1+\beta \left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c_{1,2t+1}^* = 10y \frac{1+\beta \left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c_{2,2t}^* = 3y \frac{1+3\beta \left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c_{2,2t+1}^* = 5y \frac{1+3\beta \left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta \left(\frac{5}{3}\right)^{-\sigma}} \end{cases} \quad \forall t \quad (3)$$

Plugging (3) into (1)

$$b_{1,1}^* = \frac{1}{\beta} \left(\frac{5}{3} \right)^\sigma \left(0 + 2y - 6y \frac{1 + \beta \left(\frac{5}{3} \right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3} \right)^{-\sigma}} \right) = \frac{4y}{3 + 5\beta \left(\frac{5}{3} \right)^{-\sigma}} = -b_{2,1}^*$$

$$b_{1,2}^* = \frac{1}{\beta} \left(\frac{5}{3} \right)^{-\sigma} \left(b_{1,1}^* + 2y - 10y \frac{1 + \beta \left(\frac{5}{3} \right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3} \right)^{-\sigma}} \right) = 0 = -b_{2,2}^*$$

...

Therefore,

$$\begin{cases} b_{1,2t}^* = 0 \\ b_{1,2t+1}^* = \frac{4y}{3 + 5\beta \left(\frac{5}{3} \right)^{-\sigma}} \\ b_{2,2t}^* = 0 \\ b_{2,2t+1}^* = -\frac{4y}{3 + 5\beta \left(\frac{5}{3} \right)^{-\sigma}} \end{cases} \quad \forall t$$

5. c, b, q, w denote the consumption, bonds agent bought, bond price and exogenous endowment in current period respectively.

q', w' denote the bond price and endowment in the next period respectively.

b^- denotes the amount of bond agent bought in the last period.

Bellman Equation is

$$V(b^-, w, q) = \max_{c, b} \left\{ \frac{c^{1-\sigma}}{1-\sigma} + \beta V(b, w', q') \right\}$$

$$s.t. \quad c + qb = b^- + w$$

$$c \geq 0$$

$$q' = f^q(q)$$

$$w' = f^w(w)$$

nPg holds

b^-, w given

where state variables are b^-, w, q and control variables are c, b .