Endogenous Savings: The Lagrangian Approach to Sequence Problems

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1 Endogenous Savings: The Lagrangian Approach to Sequence Problems

We depart from the Solow model by introducing an explicit model for the choice between consumption and savings. The bulk of the Solow model is kept intact so that the technological constraints are the same as before. Until we start discussing competitive equilibria, we will keep our focus on"planned" economies. That means, there will be a social planner that owns all of the resources and makes decisions about investment and consumption. Decisions will be made rationally with the objective of maximizing the utility of consumption streams $\{c_t\}_{t=0}^T$, where T can be thought of as the lifetime of the planner and is known at the time of maximization $t = 0$.

1.1 Finite Horizon: 2-period problem

The easiest starting point, that is enough to present all the methodological complications arising from endogenous choice, is to consider a two period problem, i.e. assume $T = 1$. The technological constraints facing the planner are

$$\begin{aligned} c_t + i_t &\leq f(k_t) \\ k_{t+1} &= (1-\delta)k_t + i_t \end{aligned}$$

for $t = 0, 1$, where k_0 is given. Note that we write the resource constraint in its more general version as an inequality. That is, the planner is allowed to throw away some resources if they find this optimal. We will show that this cannot be optimal under standard assumptions. Notice that the above constraints involve four restrictions, one resource constraint and one capital accumulation constraint for each t. In addition, we will also take seriously the economic meaning of the variables c_t and k_{t+1} and impose the restriction that they cannot be negative.

The problem of the planner is to maximize his/her utility subject to the above constraints. The utility function $U : R_+^2 \rightarrow R$ takes as inputs the consumptions c_0 and c_1 and gives as output the utility that the planner derives from this consumption stream. We provide further structure to the utility function and assume it is additively separable over time. We also assume that earlier consumption is preferred to later consumption, the parameter governing this preference for early consumption being the discount factor $\beta \in [0, 1]$. The functional form is thus

$$U(c_0, c_1) = u(c_0) + \beta u(c_1)$$

When the discount factor β is equal to one, then there is no preference for early consumption. If $\beta = 0$, then the preference for early consumption is so strong that second period consumption (c_1) is not valued at all. We assume $0 < \beta < 1$ so that we focus on the intermediate cases. The "period" utility function $u : R_+ \rightarrow R$ is assumed to be strictly increasing, strictly concave and satisfying Inada conditions for very low and very high consumption levels. Mathematically,

$$\begin{aligned} u'(c) &> 0 \\ u''(c) &< 0 \\ \lim_{c \to 0} u'(c) &= \infty \\ \lim_{c \to \infty} u'(c) &= 0 \end{aligned}$$

The first one says that more consumption is always *strictly* preferred to less consumption. The second one ensures that marginal utility is decreasing with consumption. The idea is that at very low levels of consumption, an additional unit makes a big difference in the level of satisfaction, but as consumption increases, the extra unit of consumption gives a smaller improvement. The Inada conditions put stronger restrictions on the curvature of the period utility, namely that marginal utility tends to infinity close to zero and tends to zero close to infinity. Summarizing, the constrained optimization problem facing the planner is

$$\max_{k_1, k_2, c_0, c_1} [u(c_0) + \beta u(c_1)]$$

subject to (s.t. henceforth)

$$\begin{aligned}
c_0 + k_1 - (1-\delta)k_0 &\leq f(k_0) \\
c_1 + k_2 - (1-\delta)k_1 &\leq f(k_1) \\
k_1 \geq 0, k_2 \geq 0, c_0 &\geq 0, c_1 \geq 0 \\
k_0 &\text{ given}
\end{aligned}$$

Note that for each period, the investment variable has been substituted out in the resource constraint using the capital accumulation equation. This is simply for convenience and it is equivalent to keep all the original constraints in the problem (try it as an exercise).

Given that we will be considering constrained optimization problems repeatedly throughout the course, it is important to learn at the outset how to set up the problem precisely.[1] All variables have to be accounted for in any maximization problem, meaning that they must either be given (exogenous) or chosen (endogenous). Here, k_1, k_2, c_0 and c_1 are all endogenous variables, they are chosen by the planner. This is indicated by listing them under the max operator. On the other hand, k_0 is given exogenously and this is stated explicitly in the constraints.

The mechanical approach to solving this constrained maximization problem consists of formulating the Lagrangian and using it to obtain conditions (equations and inequalities) that are sufficient for maximization. Subsequently, the resulting set of conditions is solved for the unknown variables, i.e. the endogenous variables.

To set up the Lagrangian, one can write all inequality constraints as "≥ 0" constraints. For example, the first constraint above can be written as

$$f(k_0) - c_0 - k_1 + (1-\delta)k_0 \geq 0$$

Then a new variable (a multiplier) is introduced and attached to each constraint. Here, I use the following multipliers corresponding to each constraint in order of appearance $\lambda_0, \lambda_1, \mu_0, \mu_1, \nu_0, \nu_1$. Note that I use the letter λ for the multipliers on the resource constraint and distinguish between the multiplier on the resource constraint for the first period and that for the second period by using the period as a subscript. This is especially helpful when we start considering models with more periods. The Lagrangian is constructed by adding to the objective function a series of terms, each consisting of the left hand side of each inequality constraint multiplied by its associated multiplier.[2] For the problem at

[1] For optimization with inequality constraints you can also refer back to your math camp notes as well as Chapter 18 of Simon and Blume.

[2] It is customary to specify the Lagrangian so that multipliers turn out to

hand, the Lagrangian is

$$L = u(c_0) + \beta u(c_1) + \lambda_0 \left(f(k_0) - k_1 + (1-\delta)k_0 - c_0\right) + \lambda_1 \left(f(k_1) - k_2 + (1-\delta)k_1 - c_1\right) + \mu_0 k_1 + \mu_1 k_2 + \nu_0 c_0 + \nu_1 c_1$$

This Lagrangian is used to obtain conditions necessary for maximization (also sufficient as long as objective is concave and constraints are convex). The conditions include:

1. First order conditions, i.e. setting the partial derivative of the Lagrangian with respect to each choice variable equal to zero.
2. All equality constraints must hold.
3. All inequality constraints must hold.
4. Complementary Slackness conditions must hold. That is, for each *inequality* constraint, either the constraint holds as equality or the multiplier is zero (or both in rare cases).
5. Multipliers have to be non-negative (as long as the Lagrangian has been set up as described above).

For the above problem, first order conditions (FOC) with respect to c_0, c_1, k_1 and k_2 respectively are

$$\begin{aligned} u'(c_0) - \lambda_0 + \nu_0 &= 0 \\ \beta u'(c_1) - \lambda_1 + \nu_1 &= 0 \\ -\lambda_0 + \lambda_1 \left(1 - \delta + f'(k_1)\right) + \mu_0 &= 0 \\ -\lambda_1 + \mu_1 &= 0 \end{aligned}$$

Since we substituted out investment from the capital accumulation equation, there are no equality constraints left in the problem. Inequality constraints are

$$\begin{aligned} c_0 + k_1 - (1-\delta)k_0 &\le f(k_0) \\ c_1 + k_2 - (1-\delta)k_1 &\le f(k_1) \\ k_1 \ge 0, k_2 \ge 0, c_0 &\ge 0, c_1 \ge 0 \end{aligned}$$

be non-negative. If the Lagrangian were written as $L = u(c_0) + \beta u(c_1) - \lambda_0 \left(f(k_0) - k_1 + (1-\delta)k_0 - c_0\right) + ...$
then λ_0 would turn out to be negative. You should try to avoid this.

Complementary slackness conditions are

$$\begin{aligned}
\lambda_0\left(f(k_0) - k_1 + (1-\delta)k_0 - c_0\right) &= 0 \\
\lambda_1\left(f(k_1) - k_2 + (1-\delta)k_1 - c_1\right) &= 0 \\
\mu_0 k_1 &= 0 \\
\mu_1 k_2 &= 0 \\
\nu_0 c_0 &= 0 \\
\nu_1 c_1 &= 0
\end{aligned}$$

and non-negativity of multipliers gives

$$\lambda_0 \geq 0, \lambda_1 \geq 0, \mu_0 \geq 0, \mu_1 \geq 0, \nu_0 \geq 0, \nu_1 \geq 0$$

When considering any constrained optimization problem there is a clear structure that you can keep in mind and use to help you ensure you have not forgotten anything. There should be one first order condition for each choice variable. There should be an equality constraint corresponding to each multiplier on equality constraints. Finally, there should be two inequalities AND a complementary slackness condition corresponding to each multiplier on inequality constraints (for example, $k_1 \geq 0$, $\mu_0 \geq 0$ and $\mu_0 k_1 = 0$).

In the absence of inequality constraints, this would be a straightforward algebra problem with N equations in N unknowns. With inequality constraints, solving the system of equalities and inequalities for the unknowns requires the use of a standard procedure which essentially consists in considering different cases. The idea is as follows: each complementary slackness condition consists of a product of two variables being equal to zero. If we can show that one of the variables cannot be zero (because it would not be optimal), then we can conclude that the other has to be. This process is illustrated for this example:

Consider first the two consumption non-negativity constraints, $c_t \geq 0$ for $t = 0, 1$. These constraints cannot be binding. The reason is that the Inada condition $\lim_{c \to 0} u'(c) = \infty$ ensures that, if we chose c_0 or c_1 equal to zero, there would be a way to improve the value of the objective. A very small increase of consumption would lead to a huge increase in utility (marginal utility is infinite). Therefore $c_0 = 0$ and $c_1 = 0$ can never be optimal and it has to be the case that $c_0 > 0$ and $c_1 > 0$. Although this does not help in finding exactly how much consumption should actually be, it does help in finding the exact value of the multipliers on those constraints. By the complementary slackness conditions, it must be that $\nu_0 = \nu_1 = 0$ (see below for economic interpretation of multipliers).

Moving on to the resource constraints, it should be straightforward to see intuitively that these constraints must bind. After all, if the resource constraint is satisfied as an inequality at the optimum it means that it is optimal to throw away some resources. This cannot be optimal because resources are valuable: one could instead use those to increase consumption and thus increase utility. This argument relies on the assumption $u'(c) > 0$. If $u'(c)$ could be zero at some points, then adding the extra resources to consumption would not lead to higher utility. The argument can be put in math: Since $\nu_0 = 0$, the first FOC says $\lambda_0 = u'(c_0)$. Since $u'(c_0) > 0$ this means $\lambda_0 > 0$. By complementary slackness

$$f(k_0) - k_1 + (1-\delta)k_0 - c_0 = 0$$

Similarly, $\lambda_1 > 0$ and thus

$$f(k_1) - k_2 + (1-\delta)k_1 - c_1 = 0$$

Can the non-negativity of the capital stock be binding in any period? That is, can it be optimal for k_1 and/or k_2 to be set to zero? Intuitively, it sounds like a good idea to set $k_2 = 0$. After all, a positive k_2 would mean saving for the future, but the planner dies in the second period (or at least does not care about the future). In the last period of life, it must be optimal to consume all of the resources and leave nothing for later. This intuition is confirmed mathematically by looking at the last first order condition

$$\mu_1 = \lambda_1 > 0$$

The multiplier on $k_2 \geq 0$ is strictly positive which means (using the complementary slackness condition $\mu_1 k_2 = 0$) that $k_2 = 0$.

Finally, can $k_1 = 0$? The answer is no. Saving nothing in the first period would mean that there would be no resources left in the second period (using here the assumption that $f(0) = 0$) and therefore consumption c_1 would have to be zero which cannot be optimal because of the Inada condition.[3] Using math, assuming $k_1 = 0$ and plugging in the resource constraint for the second period

$$\begin{aligned} f(k_1) &= k_2 - (1-\delta)k_1 + c_1 \Rightarrow \\ c_1 + k_2 &= 0 \Rightarrow c_1 = 0 \end{aligned}$$

which is a contradiction. So $k_1 > 0$ and thus $\mu_0 = 0$.

[3] A different argument can also be used for more general cases where k is not essential for resources to be positive. This alternative argument relies on the Inada condition ensuring that the marginal product of capital is infinite at zero (see later in the course).

The above arguments have made use of complementary slackness conditions to conclude that $k_2 = \nu_0 = \nu_1 = \mu_0 = 0$ and that the resource constraints must be satisfied as equalities. What remains is a set of six equations in six unknowns $c_0, c_1, k_1, \lambda_0, \lambda_1, \mu_1$

$$c_0 + k_1 - (1-\delta)k_0 = f(k_0)$$
$$c_1 + k_2 - (1-\delta)k_1 = f(k_1)$$

$$
\begin{aligned}
u'(c_0) &= \lambda_0 \\
\beta u'(c_1) &= \lambda_1 \\
\lambda_1 \left(1 - \delta + f'(k_1)\right) &= \lambda_0 \\
\mu_1 &= \lambda_1
\end{aligned}
$$

One could further reduce the problem by substituting out the auxiliary multiplier variables and focusing on the consumption and savings variables

$$
\begin{aligned}
c_0 + k_1 - (1-\delta)k_0 &= f(k_0) \\
c_1 - (1-\delta)k_1 &= f(k_1) \\
\beta u'(c_1) \left(1 - \delta + f'(k_1)\right) &= u'(c_0)
\end{aligned}
$$

Indeed, we could also substitute out the consumption variables in the last equation and remain with one non-linear equation in k_1. Once this is solved, the rest of the variables can be recovered.

1.1.1 Multipliers and the Euler Equation

In any given model, the ultimate objective is to obtain a good, intuitive understanding of the economic mechanisms that are at play and not get lost in the long mathematical derivations. To this end, this section discusses two extremely important concepts and their economic interpretation: multipliers and the Euler equation.

Let us begin with the multipliers. Multipliers play the role of "shadow prices". For example, the multiplier on the resource constraint for the first period, λ_0, is the shadow price of resources at $t = 0$. *At the optimum*, if the planner were given one more unit of resources at $t = 0$, this would increase his objective (utility) by λ_0. More precisely, λ_0 is the *marginal* increase in utility that would result from a marginal increase in resources at $t = 0$. In that sense, the multiplier measures the value of resources from the point of view of the planner. One could think of this as a price: it is the price that the planner would be willing to pay for one unit of resources. It is called a shadow price because there are no markets where resources can be traded in this model.

More generally, the multiplier will give the value of relaxing the associated constraint. So, for example, the fact that $\nu_0 = 0$ means that the value of allowing for negative consumption is zero. In that case, relaxing the constraint would have no effect on the objective. This is because the planner finds it optimal to choose strictly positive consumption anyway. To give another example, we found that in the last period there is optimally no saving $k_2 = 0$ and the associated multiplier is positive $\mu_1 > 0$. Here, relaxing the constraint could actually improve the utility of the planner. If the planner were allowed to choose negative savings (i.e. $k_2 < 0$), then he would find it optimal to do so in the last period of his life and use the extra resources for consumption. That would increase the value of the utility and the multiplier μ_1 gives the exact quantitative value of such a relaxation at the margin.

Armed with an understanding of the meaning of multipliers, we can proceed to interpreting the first order conditions associated with maximization and obtaining an economic understanding of how the optimal consumption/savings choice is made. Note that, at the optimum, it must be the case that $\lambda_t = \beta^t u'(c_t)$. This says that the marginal value of resources (i.e. of relaxing the resource constraint) must equal the marginal utility at the optimal choice c_t. This is intuitive since, a marginal increase in resources at $t = 0$ could be used to increase consumption c_0 marginally and the effect of this on the objective is exactly equal to $u'(c_0)$. Similarly, marginal increase in resources at $t = 1$ could be used to increase consumption c_1 marginally and the effect of this on the objective is exactly equal to $\beta u'(c_1)$. In the case $t = 1$, the discount term arises because the resources come in the future and thus are valued less.

Perhaps the most important condition in modern, dynamic macroeconomics is the first order condition with respect to savings k_1

$$\lambda_0 = \lambda_1 \left(1 - \delta + f'(k_1)\right)$$

or after substitution of the multipliers

$$u'(c_0) = \beta u'(c_1) \left(1 - \delta + f'(k_1)\right)$$

This is known as an Euler Equation and it describes the optimal consumption/savings choice (at $t = 0$). It is an *intertemporal* condition since it relates current ($t = 0$) and future ($t = 1$) variables and it is where the dynamic aspect of the model really starts to bite. The consumption/savings choice is a dynamic choice in the sense that saving entails a current cost which has to be weighed against a future benefit. The Euler equation describes exactly how this trade-off should be resolved optimally. A marginal increase in savings at $t = 0$ (i.e. in

k_1) requires a utility cost at $t = 0$. This means that there must be a marginal decrease in current consumption and the utility cost of this is equal to $u'(c_0)$, the left hand side of the Euler equation. The benefit of this extra saving, arises at $t = 1$. Notice that the resources available at $t = 1$ are equal to $(1-\delta)\,k_1 + f(k_1)$ so that a marginal increase in k_1 implies a marginal increase in resources at $t = 1$ equal to $(1 - \delta + f'(k_1))$. But remember that the shadow price of resources at $t = 1$ is equal to $\beta u'(c_1)(= \lambda_1)$. Therefore, the overall benefit of the marginal increase in k_1 is $\beta u'(c_1)\,(1 - \delta + f'(k_1))$, the right hand side of the Euler Equation. The Euler Equation states that the optimal choice of k_1 occurs when the marginal loss is exactly equal to the marginal benefit. To understand this, it is often helpful to consider what would happen if that were not true. Suppose the planner chooses k_1 in a way that implies

$$\lambda_1\,(1 - \delta + f'(k_1)) > \lambda_0$$

In this situation, the marginal benefit of increasing k_1 is greater than the marginal cost. Put differently, a marginal increase in k_1 would lead to an increase in lifetime utility (equal to $\lambda_1\,(1 - \delta + f'(k_1)) - \lambda_0$). This level of savings is too low to be optimal, the planner can do better.

Although the discussion above focuses on the specific model that we are considering here, the approach to interpreting first order conditions in any economic model is the same. The first order conditions with respect to any dynamic variable (i.e. any variable that affects the decision maker at two or more different periods) can always be interpreted as setting current marginal cost equal to future marginal benefit. It is important to be able to provide such interpretations for any specific context you might encounter.

Clearly, the assumption of only two periods is unrealistic, especially if those periods are thought of as years or quarters as we want to think of them. Lack of realism is not a problem per se, but it becomes a problem if the assumption clearly affects the conclusions, i.e. if it is restrictive. Here, the conclusion would be that, within two years, savings and therefore production would disappear. This is an undesirable feature of the model and it is clearly driven by the assumption of two periods. We will therefore move on to considering longer time horizons. Although the concepts are exactly the same, it is helpful to present the model with an arbitrarily long time horizon in order to introduce some useful notation.

1.1.2 Finite Horizon: $T + 1$- period problem

The extension to more than two periods is relatively easy. Most of the derivations and arguments can be carried over from the two period case, so this section will be brief. A longer finite horizon will suffer from the same problem as the two-period problem: at some arbitrary (exogenously chosen) point in time, the level of the capital stock will be optimally run down to zero. This is one of the reasons why we ultimately want to consider a model with an infinite horizon that will not suffer from this problem. Considering the $T+1$ - period problem will help in building an intuitive understanding of some of the peculiarities of the infinite horizon problem as well as a good starting point to start thinking about dynamic programming.

There are now $T+1$ periods, $t = 0, 1, ..., T$. The problem can be stated as

$$\max_{\{c_t\}_{t=0}^T, \{k_{t+1}\}_{t=0}^T} \sum_{t=0}^{T} \beta^t u(c_t)$$

s.t.

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &\leq f(k_t) \quad \text{for } t = 0, 1, ...T \\ k_{t+1} &\geq 0 \quad \text{for } t = 0, 1, ...T \\ c_t &\geq 0 \quad \text{for } t = 0, 1, ...T \\ k_0 &\text{ given} \end{aligned}$$

Notice the use of the summation notation in the utility which provides for a more concise presentation. In addition, notice that there are $T+1$ resource constraints which we summarize by writing the resource constraint for a generic period t and spelling out all the values of t for which this must hold (similarly for the non-negativity constraints). But do not forget that there $3(T+1)$ constraints! We also make heavy use of the summation notation in constructing the Lagrangian

$$L = \sum_{t=0}^{T} \beta^t u(c_t) + \sum_{t=0}^{T} \lambda_t \left[f(k_t) - k_{t+1} + (1-\delta)k_t - c_t\right] + \sum_{t=0}^{T} \mu_t k_{t+1} + \sum_{t=0}^{T} \nu_t c_t$$

or

$$L = \sum_{t=0}^{T} \left[\beta^t u(c_t) + \lambda_t \left[f(k_t) - k_{t+1} + (1-\delta)k_t - c_t\right] + \mu_t k_{t+1} + \nu_t c_t\right]$$

In obtaining first order conditions, keep in mind that there are $2(T+1)$ choice variables so there should be an equal number of first order conditions. Fortunately, there is a common structure across (almost) all periods and once we obtain the first order conditions for one generic period t, the rest of the conditions follow the same structure.[4] For any period $t = 0, 1, ...T-1$, the FOC with respect to c_t and k_{t+1} are

$$\begin{aligned} \beta^t u'(c_t) + \nu_t &= \lambda_t \text{ for } t = 0, 1, ...T-1 \\ \lambda_t - \mu_t &= \lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] \text{ for } t = 0, 1, ...T-1 \end{aligned}$$

We have obtained the FOC up to the one but last period and deliberately left the last period $t = T$ to be treated separately. Looking at the Lagrangian, the FOC with respect to c_T has the same form as the ones before (this is because consumption is not the dynamic variable in this model).

$$\beta^T u'(c_T) + \nu_T = \lambda_T$$

but the FOC for k_{T+1} is

$$\frac{\partial L}{\partial k_{T+1}} = -\lambda_T + \mu_T = 0$$

To all these FOC we have to add the constraints, non-negativity of multipliers and complementary slackness conditions for $t = 0, 1, ..., T$. Thus, the optimum in this problem is completely characterized by the following set of equations

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t \text{ for } t = 0, 1, ...T \\ \lambda_t - \mu_t &= \lambda_{t+1}\left[f'(k_{t+1}) + (1-\delta)\right] \text{ for } t = 0, 1, ...T-1 \\ -\lambda_T + \mu_T &= 0 \\ c_t + k_{t+1} - (1-\delta)k_t &\leq f(k_t) \quad \text{for } t = 0, 1, ...T \\ c_t &\geq 0, k_{t+1} \geq 0 \text{ for } t = 0, 1, ...T \\ \left[f(k_t) - c_t - k_{t+1} + (1-\delta)k_t\ \right]\lambda_t & \text{ for } t = 0, 1, ...T \\ \mu_t k_{t+1} &= 0 \text{ for } t = 0, 1, ...T \\ \nu_t c_t &= 0 \text{ for } t = 0, 1, ...T \\ \lambda_t &\geq 0, \mu_t \geq 0, \nu_t \geq 0 \text{ for } t = 0, 1, ...T \end{aligned}$$

Proceeding in a manner parallel to the previous section, we can use complementary slackness together with the assumptions on the period utility and on the production function to show that $\nu_t = 0$ for all t (consumption non-negativity will not bind in any period thanks to the assumption $\lim_{c \to 0} u'(c) = \infty$), $\lambda_t > 0$ (the resource constraint will bind in every period because of the assumption $u'(c) > 0$) and $\mu_t = 0$ for all

[4] It is instructive to write down a 4 or 5 period problem and convince yourself of this, I strongly suggest you try it.

$t < T$ (capital non-negativity will not bind in any period except the last one). In addition, we can show that in the last period it is not optimal to save anything since $\mu_T = \lambda_T > 0$ so $k_{T+1} = 0$.

Using these results, the system can be reduced to

$$
\begin{aligned}
\beta^t u'(c_t) &= \lambda_t \quad \text{for } t = 0, 1, ..., T \\
\lambda_t &= \lambda_{t+1} \left[f'(k_{t+1}) + (1-\delta) \right] \text{ for } t = 0, 1, ...T-1 \\
c_t + k_{t+1} - (1-\delta)k_t &= f(k_t) \quad \text{for } t = 0, 1, ...T \\
k_{T+1} &= 0 \\
k_0 & \text{ given}
\end{aligned}
$$

or, even further, substituting out multipliers and consumptions

$$
\begin{aligned}
u'(f(k_t) - k_{t+1} + (1-\delta)k_t) &= \beta u'(f(k_{t+1}) - k_{t+2} + (1-\delta)k_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right] \text{ for } t = 0, 1, ...T \\
k_{T+1} &= 0 \\
k_0 & \text{ given}
\end{aligned}
$$

This is a set of T equations to solve for T unknowns, namely $\{k_{t+1}\}_{t=0}^{T-1}$. For a large value of T, solving T simultaneous equations can be daunting. But note that these equations have a lot of structure. In fact, what we are faced with is a second order difference equation. A first order difference equation would involve only k_t and k_{t+1}. If you go back to the Solow model (exogenous savings), you will notice that it involved a first order difference equation. Remember that we used different approaches to discuss the predictions of the model in that setup. One of them was qualitative, graphical analysis. A similar approach can be taken here albeit with some additional complications. We also looked at an iterative approach, computing k_t recursively beginning at k_0. For a first order difference equation with one initial condition (i.e. the given value for k_0) that is feasible.

Here the difference equation involves k_t, k_{t+1} AND k_{t+2} which makes this a second order difference equation. To use the iterative procedure we would need two initial conditions. In this case we only have one initial condition. To obtain a particular solution to a second order difference equation one needs two boundary conditions. The second boundary condition here is a terminal (as opposed to an initial) condition, namely $k_{T+1} = 0$. The iterative procedure used in the Solow model is not feasible in this case. Although we have other, relatively straightforward methods to solve for linear difference equations, this is a non-linear difference equation and typically needs to be solved numerically (more on this later). What you should keep in mind as we move on to the infinite horizon version is that a second order difference equation will require the imposition of two boundary conditions. The 'terminal' condition for the infinite horizon problem cannot be about k_{T+1} (since $T \to \infty$). It will arise naturally as a limiting condition known as the transversality condition.

1.1.3 Useful Reading

Simon and Blume. "Mathematics for economists", Chapters 18 and 19.

Chiang. "Fundamental Methods of Mathematical Economics", Third Edition, Chapter 21.