Competitive Equilibrium over time: An Exchange Economy

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The main idea of equilibria is perfectly illustrated by the simple static example of the previous chapter. There are some technical and some conceptual details that need to be introduced when thinking of dynamic settings. The question is how to define and treat equilibria in a setting where a dynamic aspect is introduced, i.e. a setting where there are different periods in which choices have to be made. Even if the economy has only one "kind" of good, say apples, the dynamic setting implies that these goods are differentiated. They are differentiated in terms of the period in which they are available. To put it simply, an apple today and an apple tomorrow is not really the same good. So, even if the economy only has apples, in practice we should think of the economy as having many different goods. We refer to goods that are differentiated by the time period in which they are available as *dated goods*.

We will focus on economies that only produce one kind of good, but does so over time so there are many dated goods. A standard interpretation of the good in a given period is to think of a whole basket of goods like real GDP and its corresponding price as an index such as the GDP deflator. The important point to keep in mind is that, in an infinite horizon setting, there are an infinite number of dated goods. In terms of markets, there must be the same number of markets as goods. We will build on the two agent economy of the previous chapter and then also consider the representative agent economy.

Let w_{it} be the endowment of agent i in period t. The aggregate supply of the good at t is given exogenously by the sum of the endowments available to the two agents $w_{1t} + w_{2t}$ and aggregate demand can be obtained by adding up the demand of the two agents $c_{1t} + c_{2t}$. Therefore, there are an infinite number of markets with corresponding market clearing conditions given by

$$c_{1t} + c_{2t} = w_{1t} + w_{2t} \quad \text{for all } t$$

It is possible to proceed in complete analogy to the static case with the only substantial difference being that instead of two goods we now have an infinite number of (dated) goods. Conceptually that requires assuming that all the markets open simultaneously and all goods can be traded at the same time. But in terms of the methods, there is nothing really new here compared to the static economy. This is how we will approach this when we look at the date-0 trade equilibrium.

However, the date-0 trade equilibrium does seem to be making awkward assumptions, namely that a good that will be available in the future can be traded already today. An alternative approach would be to take the evolution of time seriously and assume that goods can only be traded when they become available. This will lead to an alternative equilibrium concept, equilibrium with sequential trade, and we will analyze the conditions under which it yields allocations that coincide with the date-0 trade equilibrium.

To summarize, we will distinguish between different equilibrium concepts depending on the assumption we make regarding how often markets open and what can be traded when they do.

1 Competitive equilibrium with date 0 trade

Even though the second arrangement (sequential trade) seems more natural, we begin with the first one because it corresponds more closely to the typical static equilibrium we saw in the simple example of the previous chapter. That is exactly the idea which leads to the date-0 trade equilibrium concept. The idea is simple and beautiful: We know very well how to define and analyze equilibria with multiple goods. All we need to do is apply this knowledge to a dynamic setting by treating goods available in different periods as different goods.

So consider the following institutional setup: At $t = 0$ all markets meet. Each consumer i has some amount w_{it} of each dated consumption good in hand, their endowment of the period t good. They all go to the markets and freely exchange their goods at market clearing prices. They leave the market with the amounts bought c_{it} of each good t. Subsequently, markets close and from then on nothing interesting happens, apart from the fact that consumers consume their goods when the appropriate time comes, i.e. when the goods become available.

Let p_t be the price of the time t good. Note that the price of each good is indexed by t, since goods are indexed by t. Then consider the budget constraint of each agent $i = 1, 2$

$$\sum_{t=0}^{\infty} p_t c_{it} \leq \sum_{t=0}^{\infty} p_t w_{it}$$

Since the agent makes all decisions and trades at one point in time, there is only one budget constraint just like in the static economy we saw previously. Given utility for each agent

$$\sum_{t=0}^{\infty} \beta^t u(c_{it})$$

where I assume for simplicity that utility is the same for the two agents, we can define a competitive equilibrium just as before.

DEFINITION: Given a collection of endowments $\{\{w_{it}\}_{t=0}^{\infty}\}_{i=1}^{2}$, a competitive equilibrium is a collection of allocations $\{\{c_{it}^*\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a collection of prices $\{p_t^*\}_{t=0}^{\infty}$ such that

1. Allocations are optimal for each agent given the prices $\{p_t^*\}_{t=0}^{\infty}$. Mathematically, for $i = 1, 2$

$$\{c_{it}^*\}_{t=0}^{\infty} = \arg \max_{\{c_{it}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

   s.t.

$$\sum_{t=0}^{\infty} p_t^* c_{it} \leq \sum_{t=0}^{\infty} p_t^* w_{it}$$
$$c_{it} \geq 0$$

2. Prices adjust so that markets clear for all t

$$c_{1t}^* + c_{2t}^* = w_{1t} + w_{2t} \quad \text{for all } t$$

One can also define a date-0 equilibrium for the case where all agents are identical and we can focus on the representative agent. In the above definition, we would drop the i's and market clearing would be simply $c_t = w_t$ for all t.

DEFINITION (representative agent): Given a collection of endowments $\{w_t\}_{t=0}^{\infty}$, a competitive equilibrium is a sequence of prices $\{p_t^*\}_{t=0}^{\infty}$ and a sequence of quantities $\{c_t^*\}_{t=0}^{\infty}$ such that

1. Given prices $\{p_t^*\}_{t=0}^{\infty}$, the household's optimal decision is to consume $\{c_t^*\}_{t=0}^{\infty}$

$$\{c_t^*\}_{t=0}^{\infty} = \arg \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

   s.t.

$$\sum_{t=0}^{\infty} p_t^* c_t \leq \sum_{t=0}^{\infty} p_t^* w_t$$
$$c_t \geq 0 \quad \text{for all } t$$

2. Prices $\{p_t^*\}_{t=0}^{\infty}$ are such that

$$c_t^* = w_t \quad \text{for all } t$$

1.1 Discussion and Characterization

It might sound strange that an agent can take their endowment of tomorrow's apples to the market today. After all, the apple tree will only produce the apples in the following period so these apples do not even exist when the market opens. One way to make sense of this is to think of trade in contracts, i.e. pieces of paper that entitle the holder to some future goods. At $t = 0$, agent i knows she is entitled to an endowment of w_{it} apples in period t.[1] She goes to the market and promises to give these future endowments in exchange for current apples (or other future endowments). For example, a contract might say "I promise to deliver to the holder of this contract 10 apples from my endowment in period $t = 20$" and this can be exchanged with someone else's piece of paper saying " I promise to deliver 2 apples in period 1". When all exchange (of pieces of paper) has finished, everyone has a 'promised' allocation in terms of contracts that they are holding. From then on, agents simply claim the apples that they are entitled to in every period t (and deliver the apples they have promised) and consume. That is, all trade happens at period 0 and, subsequently, all that happens is that contracts are carried out.

Given the definition, it is straightforward to characterize the equilibrium. If, in addition, we are given specific functional forms for utility, we can solve for the equilibrium prices and allocations. Characterization involves collecting all conditions. Optimality conditions for households are[2]

$$\beta^t u'(c_{it}^*) = \lambda_i p_t^* \text{ for all } t \text{ and } i = 1, 2$$
$$\sum_{t=0}^{\infty} p_t^* c_{it} = \sum_{t=0}^{\infty} p_t^* w_{it} \text{ for } i = 1, 2$$

where λ_i is the multiplier on agent i's budget constraint. Market clearing requires

$$c_{1t}^* + c_{2t}^* = w_{1t} + w_{2t} \text{ for all } t$$

[1] Note there is no uncertainty. The amount of apples produced by the tree in each and every future period is known at the beginning. We introduce uncertainty in subsequent chapters.

[2] As usual, we assume strictly increasing utility and an Inada condition at zero consumption so the budget constraint binds and consumption non-negativity does not bind.

As always, we will only be able to determine relative prices, so we will normalize one price, $p_0 = 1$. The following things can be proved about the equilibrium:

1. The relative price between two periods' goods is equal to the marginal rate of substitution between these two goods, i.e. the rate at which consumers are willing to give up units of one good in exchange for units of the other.

2. The marginal rate of substitution is equalized across agents (since they face the same relative prices).

With additional assumptions on utility, we can explicitly solve for equilibrium prices and allocations and discuss some further properties. Suppose

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 0$$

Then, it can be shown (left as an exercise) that

$$\begin{aligned} p_t^* &= \beta^t \left(\frac{w_{1t} + w_{2t}}{w_{10} + w_{20}} \right)^{-\sigma} \\ c_{it}^* &= \Omega_i \left(w_{1t} + w_{2t} \right) \end{aligned}$$

where Ω_i is a constant fraction that depends on the endowments and the utility function parameters β and σ. Things to note

1. Other things equal, goods available earlier will have a higher equilibrium price (because they are valued more).

2. The relative price between two periods' goods depends on the relative scarcity of the two goods. The smaller the aggregate endowment in a given period (in relative terms), the higher the price of the good of that period.

3. The price depends on the relative size of the *aggregate* endowment, but not on the distribution of the endowments across agents. The implication is that variability in prices arises only to the extent that there is variability in aggregate endowments.

4. Individual consumption in any period is a *constant* fraction Ω_i of the aggregate endowment in that period. This implies that variation in consumption will only result from variation in aggregate endowments. In particular, if aggregate endowments are fixed over time then individual consumption will also be fixed over time. This

is an illustration of consumption smoothing. Variation in individual endowments does not translate to variation in consumption because agents smooth their consumption through trade.

5. Each agent's fraction Ω_i depends on the sequence of endowments they start with. Other things equal, larger endowments or earlier endowments or smoother (less variable across periods) endowments for i will imply a higher consumption fraction Ω_i for i in equilibrium.

2 Competitive equilibrium with sequential trade

In the date-0 equilibrium, an agent's endowment in any period could be traded for the endowment of any other period. Here, we want to allow agents to trade only endowments that they already have available. Given that we have assumed there is only one kind of good per period, there is no reason why agents would ever trade (exchange apples with apples). Without adding anything else in the model, the equilibrium would trivially be one of autarky, i.e. $c_{it} = w_{it}$ for all i, t.

We introduce a financial asset that allows agents to carry over funds to the next period, that is we will allow them to borrow and lend by trading in one period bonds. The financial asset bought by agent i in period t is denoted by a_{it+1}. The index $t+1$ refers to the fact that they buy $a_{i,t+1}$ units of the asset that will pay in period $t+1$. To obtain one unit of asset $a_{i,t+1}$ in period t, agent i has to pay one unit of the available consumption good. Holding one unit of the asset entitles the owner to a return of R_{t+1} units of consumption next period. To put it differently, when an agent lends $a_{i,t+1}$ units of the consumption good at t, the borrower promises to pay them $R_{t+1}a_{i,t+1}$ units at $t+1$.[3] With this notation, $a_{i,t+1} < 0$ means that agent i is a borrower in period t.

In this setup, we have the same number of consumption goods as before (infinite) and so the same markets that will operate (albeit opening at different times). Goods market clearing will thus be as before

$$c_{1t} + c_{2t} = w_{1t} + w_{2t} \text{ for all } t$$

There is also a financial asset traded. The financial asset is in zero net supply, which means that for every consumer that issues the asset (the lender) there has to be a consumer that buys it (the borrower); there is

[3]Note that R_{t+1} will be the market clearing price for the asset market of period t, where $a_{i,t+1}$ is traded. It depends on information up to t (just like $a_{i,t+1}$) despite its subscript.

no outside entity issuing assets. Given that the total endowment of the assets is zero the market clearing condition at every t is given by

$$a_{1t} + a_{2t} = 0$$

The supply of consumption goods and of assets is exogenous. The demand will be determined again by optimal behavior of each agent that chooses their consumption and asset demand given prices. In making their choices, they are constrained by budget constraints. The budget constraint for agent i is

$$c_{it} + a_{i,t+1} = R_t a_{it} + w_{it} \text{ for all } t$$

Notice three things: First, contrary to the date-0 trade case, there are *many* budget constraints, one for each period t. Second, a full specification of the budget will require a statement about the initial wealth $R_0 a_{i0}$ which is exogenously given. Without loss of generality we will assume $R_0 a_{i0} = 0$.[4] Third, we will also need to impose a restriction on borrowing. We define the equilibrium first and discuss this restriction subsequently.

DEFINITION: Given a sequence of endowments $\{\{w_{it}\}_{i=1}^2\}_{t=0}^\infty$, a competitive equilibrium with sequential trade consists of sequences of allocations $\left\{\{c_{it}^*\}_{t=0}^\infty, \{a_{it+1}^*\}_{t=0}^\infty\right\}_{i=1}^2$ and a sequence of prices (returns) $\left\{R_{t+1}^*\right\}_{t=0}^\infty$ such that

1. Given the prices $\left\{R_{t+1}^*\right\}_{t=0}^\infty$, the allocations are optimal for each consumer. Formally, for $i = 1, 2$

$$\{c_{it}^*, a_{it+1}^*\}_{t=0}^\infty = \arg \max_{\{c_{it}, a_{i,t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t u(c_{it})$$

s.t

$$\begin{aligned} c_{it} + a_{i,t+1} &= R_t^* a_{it} + w_{it} \text{ for all } t \\ R_0 a_{i0} &= 0 \text{ given} \\ c_{it} &\geq 0 \text{ for all } t \\ \lim_{T\to\infty} a_{T+1}^* \left(\prod_{t=1}^T R_t^*\right)^{-1} &\geq 0 \end{aligned}$$

2. The asset market in each t clears

$$a_{1t+1}^* + a_{2t+1}^* = 0 \text{ for all } t$$

[4] Any non-zero values for intial wealth can be subsumed in the initial endowments w_{i0}.

3. The goods market clears for all t

$$c^*_{1t} + c^*_{2t} = w_{1t} + w_{2t} \text{ for all } t$$

Here we have as always a version of Walras' law implying that if one market clears, the other should also clear by default. This explains why we have normalized the price of the consumption good in period t to equal 1. Note that this happens in every period t. For each period, two markets open and if one clears, then the other also clears.[5]

2.1 The No-Ponzi Game condition

We have added the following constraint to the consumer's maximization problem

$$\lim_{T\to\infty} a_{T+1} \left(\prod_{t=1}^{T} R_t \right)^{-1} \geq 0$$

This is a restriction that needs to be added to make the equilibrium with sequential trade well defined. This does not restrict the agent's saving/lending ($a > 0$ is not restricted), rather it restricts the amount of borrowing. In particular, it restricts an agent by not allowing them to choose debt sequences (negative a) that grow as fast as, or faster than, $\prod_{j=1}^{t} R_j$ forever. The consumer is allowed to have growing debt as long as it grows at a slower rate than $\prod_{j=1}^{t} R_j$. The reason is that, if we allow debt to grow so fast, then agents could run a Ponzi-scheme and that would imply that no amount of borrowing can be supported as an equilibrium. This limiting condition is referred to as a "no Ponzi-scheme" or "no-Ponzi game" condition (NPG).

A Ponzi-scheme runs as follows: Borrow today in order to finance higher consumption. When loan repayment time comes, borrow enough to repay the loan with interest. When the new loan has to be repaid, borrow again to repay this new loan with interest. Continue like this and you essentially have increased consumption today for free since you never have to repay the loans. In terms of the model notation, suppose the agent chooses $a_{i1} < 0$ (borrows) and uses it to consume more than their endowment in period zero $c_{i0} > w_{i0}$. In the following period she repays the debt with interest $R_1 a_{i1}$ by borrowing again $a_{i2} = R_1 a_{i1} < 0$. In the period after that, she repays $R_2 a_{i2} = R_2 R_1 a_{i1}$ by borrowing $a_{i3} = R_2 R_1 a_{i1} < 0$ and so on.

[5] Prove this using the budget constraints.

In a finite horizon model this scheme fails because in the last period noone is willing to lend any funds at *any* positive return R_T. This, is turn, means that there can be no borrowing in this last period and, as a result, the agent running the Ponzi-scheme would need to repay all of the accumulated debt and his consumption would be too low (or even non-positive). This cannot be optimal from the agent's point of view so there is no need to worry about Ponzi-schemes. But in an infinite horizon model, there is no last period, so it is always a 'good' idea to run such a Ponzi scheme. An agent can always improve their utility by borrowing more. That leads to a situation where

1. There is no maximum to the utility for any given return. For any level of borrowing a_{i1} in the first period, we can always find an allocation that is even better for the agent. Borrow more than a_{i1} and then run the Ponzi scheme. That is, one could finance infinite consumption with infinite borrowing.

2. There can be no equilibrium with positive return. For any positive return, the demand for borrowing would be infinite (the demand for savings is minus infinity) for both agents and the asset market could not clear. The only possible equilibrium would have zero (or negative) returns and no borrowing.

That would defeat the purpose of introducing assets to complete the markets so we dispense with this peculiarity of infinite time by imposing the no Ponzi-Scheme condition as an institutional restriction on the agents' behavior. With this condition imposed, we will be able to show that the budget set of the consumer in this sequential trade equilibrium is equivalent to the one under date-0 trade.

It is also helpful to say a couple of words at this point regarding the relation, or rather the difference, between the NPG condition and the TVC condition we encountered in simple maximization problems. Recall that, in obtaining sufficient conditions for each agent's maximization problem, we will need to impose a transversality condition

$$\lim_{T\to\infty} \beta^T u'(c_{iT}) a_{i,T+1} \leq 0$$

It can be shown (this requires solving for the equilibrium prices and allocations, see the following section) that in equilibrium this can be expressed in terms of R_{t+1} as[6]

$$\lim_{T\to\infty}\left(\prod_{t=1}^{T} R_t\right)^{-1} a_{i,T+1} \le 0$$

So the transversality condition looks very similar to the no Ponzi-scheme condition. But it is important to clarify the differences:

1. The TVC is a condition for optimality whereas the NPG is an assumption we impose ex ante to ensure existence of an equilibrium with borrowing.

2. Loosely speaking, the TVC states that it cannot be optimal to *save* too much in the distant future. The NPG states that it is not allowed to *borrow* too much in the distant future. The previous discussion makes the point that it would actually be optimal to do so if allowed.

Taken together the two lead to the condition

$$\lim_{T\to\infty}\left(\prod_{t=1}^{T} R_t\right)^{-1} a_{i,T+1} = \lim_{T\to\infty}\beta^T u'(c_{iT})a_{i,T+1} = 0$$

2.2 The representative agent case

We can also define an equilibrium with a representative agent (i.e. where all agents identical) that will again feature no trade, trivial allocations but non-trivial prices.

DEFINITION (representative agent): Given a sequence of endowments $\{w_t\}_{t=0}^{\infty}$, a competitive equilibrium with sequential trade consists of sequences of allocations $\{c_t^*\}_{t=0}^{\infty}, \{a_{t+1}^*\}_{t=0}^{\infty}$ and a sequence of returns $\left\{R_{t+1}^*\right\}_{t=0}^{\infty}$ such that

1. Given the prices $\left\{R_{t+1}^*\right\}_{t=0}^{\infty}$, the quantities are optimal for the representative consumer. Formally,

$$\{c_t^*, a_{t+1}^*\}_{t=0}^{\infty} = \arg\max_{\{c_t, a_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty}\beta^t u(c_t)$$

[6] Note that we often use the transversality condition with equality, but strictly speaking, it only requires an inequality - check the proof of sufficiency to see that. In the growth model, the fact that $k_{t+1} \geq 0$ implies an equality. Here we do not impose $a_{it+1} \geq 0$ but the no Ponzi scheme condition will imply an equality.

s.t

$$
\begin{aligned}
c_t + a_{t+1} &= R_t^* a_t + w_t \text{ for all } t \\
c_t &\geq 0 \text{ for all } t \\
R_0 a_0 &= 0 \\
\lim_{T\to\infty} a_{T+1} \left(\prod_{t=0}^{T} R_t^*\right)^{-1} &\geq 0
\end{aligned}
$$

2. All markets clear. In particular, the asset market in each t clears

$$a_t^* = 0 \text{ for all } t$$

and the goods market clears for all t

$$c_t^* = w_t \text{ for all } t$$

As usual, in the representative agent version of an exchange economy quantities are trivially determined (no trade). Using the agent's first order condition we can obtain prices as

$$
\begin{aligned}
u'(c_t^*) &= \beta u'(c_{t+1}^*) R_{t+1}^* \Rightarrow \\
R_{t+1}^* &= \frac{u'(w_t)}{\beta u'(w_{t+1})}
\end{aligned}
$$

Notice that the (gross) interest rate R_{t+1} is simply the exchange rate between goods at time t and $t+1$. In the date-0 trade equilibrium, these goods could be directly traded at a relative price (exchange rate) $\frac{p_t}{p_{t+1}}$.

3 Equivalence of date-0 and sequential trade equilibria

The date-0 trade and sequential trade equilibria will deliver the same equilibrium allocations c_{it}^*. This is trivial to show in the case of the representative agent ($c_t^* = w_t$ for both equilibria). With heterogeneity, allocations and prices $\{c_{it}^*\}_{t=0}^{\infty}, \{p_t^*\}_{t=0}^{\infty}$ in the date-0 trade equilibrium satisfy

$$
\begin{aligned}
\beta^t u'(c_{it}^*) &= \lambda_i p_t^* \text{ for all } t \text{ and } i = 1, 2 && (1) \\
\sum_{t=0}^{\infty} p_t c_{it}^* &= \sum_{t=0}^{\infty} p_t^* w_{it} \text{ for } i = 1, 2 && (2) \\
c_{1t}^* + c_{2t}^* &= w_{1t} + w_{2t} \text{ for all } t && (3)
\end{aligned}
$$

and allocations and prices $\{c^*_{it}\}_{t=0}^{\infty}, \{a^*_{i,t+1}\}_{t=0}^{\infty}, \{R^*_{t+1}\}_{t=0}^{\infty}$ in the sequential trade equilibrium satisfy (γ_{it} is the multiplier on i's budget at t)

$$\beta^t u'(c^*_{it}) = \gamma_{it} \text{ for all } t \text{ and } i = 1,2 \tag{4}$$
$$\gamma_{it} = R^*_{t+1}\gamma_{i,t+1} \text{ for all } t \text{ and } i = 1,2 \tag{5}$$
$$c^*_{it} + a^*_{i,t+1} = R^*_t a^*_{it} + w_{it} \text{ for all } t \text{ and } i = 1,2 \tag{6}$$
$$R_0 a_{i0} = 0 \text{ for } i = 1,2 \tag{7}$$
$$c^*_{1t} + c^*_{2t} = w_{1t} + w_{2t} \text{ for all } t \tag{8}$$
$$a^*_{1t+1} + a^*_{2t+1} = 0 \text{ for all } t \tag{9}$$
$$\lim_{T\to\infty} \beta^T u'(c^*_{iT}) a^*_{i,T+1} \leq 0 \text{ (TVC) for } i = 1,2 \tag{10}$$
$$\lim_{T\to\infty} a^*_{i,T+1}\left(\prod_{t=1}^{T} R^*_t\right)^{-1} \geq 0 \text{ (NPG) for } i = 1,2 \tag{11}$$

To show the equivalence between date-0 and sequential trade equilibrium allocations[7]

1. Suppose $\{c^*_{it}\}_{t=0}^{\infty}, \{a^*_{i,t+1}\}_{t=0}^{\infty}, \{R^*_{t+1}\}_{t=0}^{\infty}$ is an equilibrium with sequential trade, i.e. they satisfy equations (4) - (11). Then show that $\{c^*_{it}\}_{t=0}^{\infty}$ also satisfies the equilibrium conditions for the date-0 trade economy $(1) - (3)$ by constructing the prices p^*_{it} and showing the date-0 trade budget holds for all i.

2. Now do the reverse. Suppose $\{c^*_{it}\}_{t=0}^{\infty}, \{p^*_t\}_{t=0}^{\infty}$ is an equilibrium with date-0 trade, i.e. they satisfy equations $(1) - (3)$. Then show that $\{c^*_{it}\}_{t=0}^{\infty}$ also satisfies the equilibrium conditions for the sequential trade economy (4) - (11) by constructing the returns R^*_{t+1} and asset trades a^*_{it+1} and showing that the asset markets clear and NPG and TVC are satisfied.

The crucial step is to show the equivalence of the choice sets for consumption in the two setups. Consider any consumption choice $\{c^*_{it}\}_{t=0}^{\infty}$ that is feasible in the sequential trade equilibrium. Then, by rolling forward the sequential trade budget and using TVC and NPG it can be

[7] I assume $R_0 a_{i0} = 0$ for simplicity, it is easy to adjust the arguments to the case of non-zero initial wealth.

shown that it satisfies the date-0 trade budget

$$\begin{aligned}
a_{i0} &= \frac{c_{i0}^{*}-w_{i0}}{R_0}+\frac{a_{i1}^{*}}{R_0} \\
&= \frac{c_{i0}^{*}-w_{i0}}{R_0}+\frac{c_{i1}^{*}-w_{i1}}{R_0R_1^{*}}+\frac{a_{i2}^{*}}{R_0R_1^{*}} \\
&= \ldots \\
&= \sum_{t=0}^{\infty}\frac{\left(c_{it}^{*}-w_{it}\right)}{\prod\limits_{s=0}^{t}R_s^{*}}+\lim_{T\to\infty}a_{i,T+1}^{*}\left(\prod_{t=0}^{T}R_t^{*}\right)^{-1} \\
R_0a_{i0} &= \sum_{t=0}^{\infty}\frac{\left(c_{it}^{*}-w_{it}\right)}{\prod\limits_{s=1}^{t}R_s^{*}}+\lim_{T\to\infty}a_{i,T+1}^{*}\left(\prod_{t=1}^{T}R_t^{*}\right)^{-1}
\end{aligned}$$

Now note that in equilibrium $\prod\limits_{s=1}^{t}R_t^{*}=\frac{u'(c_{i0}^{*})}{\beta^t u'(c_{it}^{*})}$ and this also corresponds to $\frac{p_0}{p_t}$ in the date-0 trade equilibrium.

$$R_0a_{i0}=\sum_{t=0}^{\infty}\frac{p_t\left(c_{it}^{*}-w_{it}\right)}{p_0}+\lim_{T\to\infty}a_{i,T+1}^{*}\frac{\beta^T u'(c_{iT}^{*})}{u'(c_{i0}^{*})}$$

The last term is 0 (by using the NPG and TVC conditions) and using the zero initial wealth assumption and the normalization $p_0=1$ we obtain

$$0=\sum_{t=0}^{\infty}p_t\left(c_{it}^{*}-w_{it}\right)$$

To show the opposite statement, that if $\{c_{it}^{*}\}_{t=0}^{\infty}$ satisfies the date-0 trade budget then it is feasible in the sequential trade economy, one can proceed by constructing the asset trades required to ensure the same consumption. Since in equilibrium

$$R_{t+1}^{*}=\frac{u'(c_{it}^{*})}{\beta u'(c_{i,t+1}^{*})}$$

then the asset choices can be constructed recursively using

$$\begin{aligned}
a_{i,1} &= w_{i0}-c_{i0}^{*} \\
a_{i,t+1} &= \frac{u'(c_{it-1}^{*})}{\beta u'(c_{i,t}^{*})}a_{it}+w_{it}-c_{it}^{*} \text{ for } t=1,2,3,\ldots
\end{aligned}$$

With these choices for assets, and given that goods' markets clear, the asset market clears in every period (simply add a_{it+1} across agents and show it equals zero). We can also show in a manner identical to before that

$$0 = R_0 a_{i0} = \sum_{t=0}^{\infty} p_t \left(c_{it}^* - w_{it}\right) + \frac{1}{u'(c_{i0}^*)} \lim_{T\to\infty} a_{i,T+1}^* \beta^T u'(c_{iT}^*)$$

and since the date-0 budget constraint is satisfied for this consumption sequence, this implies that

$$\lim_{T\to\infty} a_{i,T+1}^* \beta^T u'(c_{iT}^*) = 0$$

that is, the NPG and TVC conditions are satisfied.

4 Efficiency of date-0 and sequential trade equilibria

The equilibrium allocations, which coincide for the two equilibrium concepts, are Pareto efficient. Again, this is trivial for the case of a representative agent. With heterogeneity, one can construct weights ξ_i such that the equilibrium allocations are also solutions to the following planner's problem[8]

$$\max_{\{c_{1t}, c_{2t}\}_{t=0}^{\infty}} \sum_{i=1}^{2} \xi_i \sum_{t=0}^{\infty} \beta^t u(c_{it})$$

subject to

$$c_{1t} + c_{2t} \leq w_{1t} + w_{2t} \text{ for all } t$$

Notice that this is a simple application of the fundamental welfare theorems. These rely on the absence of frictions such as market incompleteness (recall what happens if no asset is introduced), incomplete information, market power, externalities etc. For this first course, we only consider frictionless economies. The assumption is quite unrealistic but provides a useful benchmark. Once this is clearly understood, relaxing any of the above assumptions, will give us a good idea of the effects of these various frictions on equilibrium allocations and prices.

[8] You should do this as an exercise. Simply obtain first order conditions for the planner's problem, compare them to the first order conditions in the competitive equilibria and conclude what the weights ξ_i need to be in order for consumptions to be the same.

5 Practice Examples

See if you can predict intuitively the properties of equilibrium prices and allocations for the following. Compute the equilibrium assuming a CES utility to check your intuition.

EXAMPLE 1: Suppose $w_{1t} = y$ and $w_{2t} = 2y$ for all t

What are equilibrium prices? Will there be trade?

EXAMPLE 2: Suppose $w_{1t} = y$ and $w_{2t} = 2y$ for $t = 0, 2, 4, 6, ...$ and $w_{1t} = 2y$ and $w_{2t} = y$ for $t = 1, 3, 5, 7, ...$

What are equilibrium prices? How does consumption c_{it} vary over time? Who consumes more in equilibrium?

EXAMPLE 3: Suppose $w_{1t} = y$ for all t and $w_{2t} = \frac{3}{2}y$ for $t = 0, 2, 4, 6...$ and $w_{2t} = \frac{1}{2}y$ for $t = 1, 3, 5, 7...$

How does consumption c_{it} vary over time? If there is no discounting ($\beta = 1$), who consumes more in equilibrium? What if there is discounting ($\beta < 1$)?