Problem Set 4

1. (Continuation of question 2 of Problem Set 3) For the Cass-Koopmans model with full depreciation ($\delta = 1$), Cobb-Douglas aggregate production ($f(k) = k^{\alpha}$) and logarithmic utility ($u(c) = \ln c$), suppose the horizon goes to infinity $T \to \infty$.

   (a) Find the limit as $T \to \infty$ of the policy function for capital

   $$k_{t+1} = \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^{\alpha} \text{ for } t = 0, 1, ..., T$$

   (b) Assuming that the limit of the solutions to the finite horizon problem is the solution to the infinite horizon problem we now focus on the infinite horizon case:

      i. Plot the optimal policy function of capital (i.e. k_{t+1} versus k_t). Plot the 45 degree line on the same graph. Is there a steady state? Is it unique? What are the dynamics of capital outside the steady state?

      ii. Solve for the steady state (k^*, i^*, c^*) in terms of parameters. How does it compare to the Solow model's golden rule steady state?

      iii. Go back to the variable $z_t = \frac{k_{t+1}}{k_t^{\alpha}}$ we used as a transformation. What is the economic meaning of this variable? Repeat part i for z instead of k, i.e. plot the optimal policy for z_{t+1} as a function of z_t, plot also the 45 degree line and describe the steady state and the dynamics of this variable starting from any z_0.

2. A firm owns capital K_t and uses it to generate revenue according to the production function $F(K_t)$, where $F(0) = 0$, $F' > 0$, $F'' < 0$ and $\lim_{K \to 0} F'(K) = \infty$. The firm decides on investment I_t and on how much dividend D_t to pay to its shareholders. Investment and dividends can be financed using current revenue $F(K_t)$ or by issuing new equity E_t, but equity issuance is costly; the cost of issuing equity is given by a function $C(E_t)$. The firm's constraints are thus

   $$\begin{aligned} D_t + I_t &= F(K_t) + E_t - C(E_t) \\ K_{t+1} &= (1 - \delta) K_t + I_t \end{aligned}$$

   In addition, the firm cannot pay negative dividends or issue negative equity, i.e. $D_t \geq 0$ and $E_t \geq 0$. The firm's objective is

   $$\sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t (D_t - E_t)$$

   where $r > 0$ is an exogenously given interest rate.

(a) Obtain necessary conditions for maximization of the firm's objective.

(b) Suppose first that $C(E_t) = 0$ for all E_t.

i. Show that the two non-negativity constraints $D_t \geq 0$ and $E_t \geq 0$ will never bind.

ii. Characterize the dynamics of capital, i.e. what is the steady state K^*, what happens in the transition from any initial K_0?

iii. Explain how the capital path differs from the standard Cass-Koopmans model and provide intuition for the difference.

iv. Given a K^*, find the payout $D_t - E_t$ for all t. Can you determine D_t and E_t separately?

(c) Now let $C(E_t)$ be such that $0 \leq C(E) < E$ with $C(0) = 0$. In addition, assume $0 < C'(E) < 1$ and $C''(E) > 0$. Show that the firm will never issue equity and pay dividends in the same period.