Competitive Equilibrium over time: Production

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Our discussion of competitive equilibria until now has focused on exchange economies where the supply of goods is exogenous. In this section, we consider decentralizations of the Cass-Koopmans growth model where the supply of goods is endogenously determined. We will restrict ourselves to the case of identical households and focus on the representative household.

In a production economy, consumers are not the only agents. There are also firms which are responsible for the production of goods. We assume that households own the factors of production, labor and capital. They supply these factors to the firms through factor markets. Firms hire these factors for a period at the market clearing rate, use them for production and return them to the households at the end of the period. To put it differently, firms buy capital *services* and labor *services*, but the ownership of the capital stock and of labor remains with the households.

The assumption that households own their labor is natural. The assumption that households own the capital stock in the economy might seem less natural. In practice, many firms own their capital instead of renting it every period. However, firms are themselves owned by households. It turns out that the ownership assumption is innocuous; one can construct the equilibrium under the assumption that capital is owned by the firm which then pays dividends to its owners (the households) and the equilibrium will be the same (at least for the complete markets, frictionless economies studied here).

Firms produce consumption/investment goods Y_t by combining capital and labor inputs using a standard neoclassical production function with the usual properties. Even though the underlying idea is that there are many firms (hence their competitive, price taking behavior), the assumption of constant returns to scale in the production technology will allow us to equivalently think of one big firm. This is because, with constant returns to scale, one big firm is equivalent to m small firms of size $\frac{1}{m}$. As a result, in what follows we often refer to "firms" or "the firm" interchangeably. The firm uses as inputs to production aggregate capital K_t and aggregate labor N_t

$$Y_t = F(K_t, N_t)$$

We use upper case letters to denote the aggregates and to distinguish the aggregates from an individual household's capital k_t and hours worked n_t. This is often referred to as *big-K, small-k* notation. The distinction is important conceptually, but is sometimes avoided altogether in a representative agent framework for the following reason. If agents are indexed by i then their capital at t can be denoted k_{it}. Assuming all agents are identical $k_{it} = k_t$. Now suppose there are M agents, then the aggregate supply of capital is $\sum_{i=1}^{M} k_{it} = \sum_{i=1}^{M} k_t = Mk_t$. Instead of assuming M households, we will assume there is a *continuum of households of measure 1.* This means the index i takes an infinite number (a continuum) of possible values in the interval from zero to one, $i \in [0, 1]$. In this case, the aggregate supply of capital at t is $\int_0^1 k_{it} di = \int_0^1 k_t di = k_t \int_0^1 di = k_t$. That is, the assumption of a continuum of identical households of measure 1 implies that k_t represents both the individual household's supply of capital and the *aggregate* supply of capital. This means that, at least for some purposes, the distinction between k and K is not necessary. But the conceptual distinction is always there and it sometimes becomes crucial to keep that distinction clear.[1] We will maintain the distinction in section 1 for educational purposes, but will drop the distinction from then on unless it is necessary.

The goods traded in this economy are: (dated) consumption/investment goods, (dated) capital services and (dated) labor services. As a result, there are three markets for every t. As in the case of dynamic exchange economies, different assumptions about when the markets open and what can be traded will lead to different equilibrium concepts: the date-0 trade equilibrium and sequential trade equilibrium.

1 Equilibrium with date-0 trade

Similarly to the case of an exchange economy, in the date-0 trade equilibrium we assume all markets open once at the beginning of time. Let us define the prices for each good. The price of the consumption/investment good available at t is denoted by p_t. Recall that all prices are relative so we will need to choose a numeraire good. We typically assume this is the

[1] An important example is when we discuss recursive competitive equilibrium, see later chapters.

$t = 0$ consumption/investment good and use the normalization $p_0 = 1$. Thus, p_t expresses the price of the time-t good in terms of the time-0 good. The rental price of capital in period t, in terms of the consumption good in period t, is denoted by r_t. The real wage rate (the price of labor services) in period t, in terms of the consumption/investment good in period t, is w_t. To express these prices in terms of the time-0 good we can write $p_t r_t$ and $p_t w_t$ respectively. Each price p_t, r_t and w_t will be determined by market clearing in the respective market.

In the market for the consumption/investment good at t, households are on the demand side and firms on the supply side. Firms supply $Y_t^S = F(K_t, N_t)$. Each household demands $c_t + i_t = c_t + k_{t+1} - (1-\delta)k_t$ and aggregating across all (identical) households the aggregate demand is $Y_t^D = C_t + I_t = C_t + K_{t+1} - (1-\delta)K_t$. Market clearing requires that for every t, aggregate demand equals aggregate supply $Y_t^D = Y_t^S$ or

$$C_t + K_{t+1} - (1-\delta)K_t = F(K_t, N_t)$$

In the factor markets at time t, households are the suppliers and firms are on the demand side. Factor demand is determined by the firm's choices given prices (perfect competition). The firm's profits in period t can be expressed as

$$\pi_t = F(K_t, N_t) - w_t N_t - r_t K_t$$

where the first term captures revenues and the next two terms are the labor and capital costs respectively. This expression gives profits in units of the consumption/investment good in period t. We will assume that the firm's objective is to maximize overall profits

$$\sum_{t=0}^{\infty} p_t \left[F(K_t, N_t) - w_t N_t - r_t K_t\right]$$

where we have used the relative prices p_t to express period profits π_t in terms of the numeraire in order to sum them up. Notice that the firm's problem is purely static, there is no connection between the choice in period t and the choice in period $t+1$. This is because we have left the dynamic decision of capital accumulation to the households. It is therefore equivalent to think of the firm as maximizing profits π_t period-by-period and we will use this formulation in what follows. Maximization of π_t (given prices) for each t will lead to expressions that can be thought of as capital demand and labor demand. Clearly, once the firm has chosen their capital and labor they have also implicitly chosen total production $Y_t = F(K_t, N_t)$. To summarize, the firm's maximization of profits will give: capital and labor demand and consumption/investment goods supply.

The other side of each market will be given by the household's maximization problem. That is, households will demand consumption/investment goods and supply capital and labor in order to maximize their utility given prices and subject to a budget constraint. The budget constraint for the representative household requires that expenditure is less than income. Given that we are considering a date-0 trade equilibrium, this will include expenditure and income on all dated goods. The household spends on buying consumption/investment goods. We write the total goods bought as a sum of consumption and investment, since the household will also have to choose between these two uses of those goods. Expressing these expenditures in units of the numeraire and adding them up gives

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta)k_t\right]$$

Income is obtained through renting labor and capital to the firm.[2] Expressing labor income and capital income in terms of the numeraire and adding them up gives

$$\sum_{t=0}^{\infty} \left[p_t w_t n_t + p_t r_t k_t\right]$$

Therefore the date-0 trade budget constraint of the representative household is

$$\sum_{t=0}^{\infty} p_t \left[c_t + k_{t+1} - (1-\delta)k_t\right] \leq \sum_{t=0}^{\infty} p_t \left[w_t n_t + r_t k_t\right]$$

We are now in a position to define a competitive equilibrium with date-0 trade for this production economy.

Definition: Given initial capital holdings k_0, a competitive equilibrium with date-0 trade is a set of price sequences $\{p_t^*\}_{t=0}^{\infty}$, $\{r_t^*\}_{t=0}^{\infty}$ and $\{w_t^*\}_{t=0}^{\infty}$ and a set of individual and aggregate quantity sequences (allocations) $\{c_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty}$ and $\{C_t^*, K_t^*, N_t^*\}_{t=0}^{\infty}$ such that

1. Given prices, the allocations are optimal for the household

$$\left\{c_t^*, k_{t+1}^*, n_t^*\right\}_{t=0}^{\infty} = \arg \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

[2] In principle, we should also include any profits the firm makes since the households are the owners of the firm. We ignore this because we know that equilibrium profits with constant returns to scale will be zero.

s.t

$$\sum_{t=0}^{\infty} p_t^* \left[c_t + k_{t+1} - (1-\delta)k_t\right] = \sum_{t=0}^{\infty} p_t^* \left[w_t^* n_t + r_t^* k_t\right]$$
$$c_t \geq 0,\ 0 \leq n_t \leq 1,\ k_{t+1} \geq 0$$
$$k_0 \text{ given}$$

2. Given prices, the allocations are optimal for the firm. For every t[3]

$$\{K_t^*, N_t^*\} = \arg\max_{K_t, N_t} \left[F(K_t, N_t) - w_t^* N_t - r_t^* K_t\right]$$

3. All markets clear. In particular the market for the consumption/investment good for every t

$$C_t^* + K_{t+1}^* - (1-\delta)K_t^* = F(K_t^*, N_t^*)$$

the market for labor services for every t

$$N_t^S = N_t^*$$

the market for capital services for every t

$$K_t^S = K_t^*$$

where $C_t^* = c_t^*$, $N_t^S = n_t^*$ and $K_t^S = k_t^*$ by the assumption of a continuum of identical households.

The definition provided helps in making the economic structure of the environment transparent. In every market, there is an aggregate demand function and an aggregate supply function, both being functions of the prices. Market clearing provides the way to determine the equilibrium price. Of course these markets are interconnected. This has an impact on the mechanical solution of the model: we cannot solve for quantities and prices in each market independently, we have to solve for everything simultaneously. Still, it is important to keep in mind the underlying structure as you go through mechanical solution methods.

On the other hand, writing the definition in this way introduces a lot of notation and is not standard. We have used N_t^S and K_t^S to

[3]Notice this implies the firm chooses K_0 even though each household's initial capital k_0 (and hence the aggregate capital) is given. This is consistent when you realize that the *supply* of capital is inelastic at $t = 0$, but the firm still has a capital *demand* function. The equilibrium level of capital will be determined by the supply whereas demand will pin down the market clearing price r_0.

denote the aggregate supply of factors and we have distinguished small k, n, c with big K, N, C even though in equilibrium the distinction will not be needed. A more standard approach to defining equilibria with a representative agent is to notice that $k_t^* = K_t^*$, $n_t^* = N_t^*$ and $c_t^* = C_t^*$ and completely avoid these distinctions in the above definition. This might confuse a student the first time they see this definition but it does reduce notation significantly and we will follow that approach from now on, so we provide that version below.

DEFINITION: Given initial capital holdings k_0, a competitive equilibrium with date-0 trade is a set of price sequences $\{p_t^*\}_{t=0}^{\infty}$, $\{r_t^*\}_{t=0}^{\infty}$ and $\{w_t^*\}_{t=0}^{\infty}$ and a set of quantity sequences (allocations) $\{c_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty}$ such that

1. Given prices, the allocations are optimal for the household

$$\left\{c_t^*, k_{t+1}^*, n_t^*\right\}_{t=0}^{\infty} = \arg \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

s.t

$$\begin{aligned} \sum_{t=0}^{\infty} p_t^* \left[c_t + k_{t+1} - (1-\delta)k_t\right] = & \sum_{t=0}^{\infty} p_t^* \left[w_t^* n_t + r_t^* k_t\right] \\ & c_t \geq 0,\ 0 \leq n_t \leq 1,\ k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

2. Given prices, the allocations are optimal for the firm. For every t

$$\{k_t^*, n_t^*\} = \arg \max_{k_t, n_t} \left[F(k_t, n_t) - w_t^* n_t - r_t^* k_t\right]$$

3. All markets clear. In particular the market for the consumption/investment good for every t

$$c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, n_t^*)$$

as well as the markets for labor services and capital services for every t[4]

In general, solving for the equilibrium will require the solution of a system of non-linear difference equations that admits no closed form solution. What we can do is at least characterize the equilibrium, i.e.

[4] Already implicitly imposed by writing k_t^*, n_t^* both for the supply coming from the household problem in point 1 and for the demand coming from the firm problem in point 2.

collect all conditions resulting from points 1-3 of the definition and make any statements that these allow us regarding the properties of the equilibrium. This is done in what follows.

First, consider the household's problem. This is a standard infinite horizon maximization problem. Sufficient conditions can be obtained by forming the Lagrangian, obtaining conditions and remembering to add TVC conditions.[5] Attaching a multiplier λ on the budget constraint we get

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t) + \lambda \left(\sum_{t=0}^{\infty} p_t^* \left[w_t^* n_t + r_t^* k_t \right] - \sum_{t=0}^{\infty} p_t^* \left[c_t + k_{t+1} - (1-\delta) k_t \right] \right)$$

Sufficient conditions are

$$\begin{aligned} \beta^t u_c(c_t) &= \lambda p_t^* \\ p_t^* &= p_{t+1}^* (1 - \delta + r_{t+1}^*) \\ \beta^t u_n(c_t, 1-n_t) + \lambda p_t^* w_t^* &= 0 \\ \lim_{T \to \infty} \beta^T u_c(c_T) k_{T+1} &= 0 \end{aligned}$$

Note that, at the optimum, the (gross) interest rate $\frac{p_t}{p_{t+1}}$ is equalized to the (gross) return on capital.

The firm's problem is much easier in the sense that it is static since we can think of the firm as choosing capital and labor demand given prices in every period, without regard to previous or future periods. Given the real wage rate and real rental price of capital, the optimal choice of capital and labor in period t is described by

$$\begin{aligned} r_t &= F_k(k_t, n_t) \\ w_t &= F_n(k_t, n_t) \end{aligned}$$

These define implicitly capital and labor demand functions. With a constant returns to scale production function, these also imply zero profits for the firm since by Euler's Theorem

$$F = F_k k + F_n n$$

2 Equilibrium with sequential trade

Consider now competitive equilibrium in a production economy where markets are allowed to open every period but only goods available at

[5] The inequality constraints $c_t \geq 0$, $0 \leq n_t \leq 1$ and $k_{t+1} \geq 0$ are ignored in what follows. As an exercise, show under what conditions these will not bind. This requires adapting the arguments we used in the social planner version of the economy to this equilibrium model. In particular, be careful with any arguments relating to marginal returns to factors of production. Contrary to the planner, the household takes these returns as given.

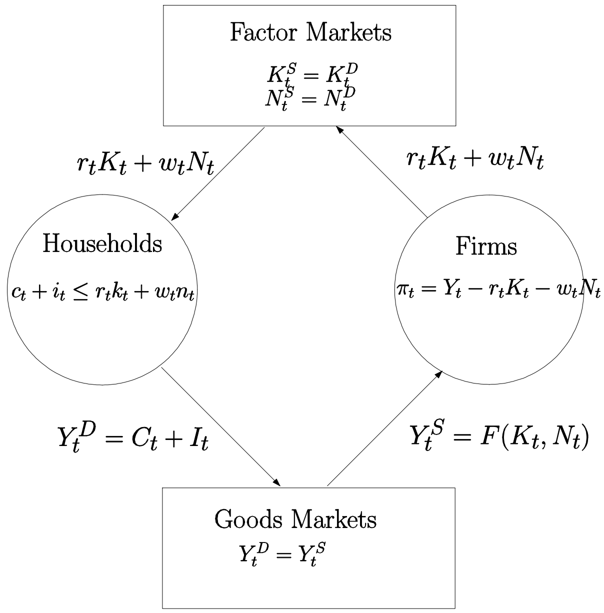

Figure 1: Circular Flow Diagram

t can be traded. In a sequential trade economy, where current goods cannot be explicitly traded with future goods, some financial assets are typically needed to allow agents to transfer funds across time in order to obtain a complete markets allocation equivalent to the date-0 trade allocation. In general, this would also be true here. Even though households have an asset (physical capital), which they can use as a savings instrument, the restriction that physical capital cannot be negative means that borrowing is not possible using that asset. However, we have assumed identical households and we know this will imply no trade in financial assets anyway, so we will ignore financial assets in the definition here. As soon as heterogeneity is introduced, one would need to allow for the possibility to borrow by introducing financial assets, otherwise markets would be incomplete.

Thus, we rely on the representative household assumption and define the equilibrium with sequential trade without additional assets. Subsequently, we show that introducing additional assets is redundant.

DEFINITION: Given an initial allocation of capital k_0, an equilibrium with sequential trade consists of price sequences: $\{r_t^*\}_{t=0}^{\infty}$ and $\{w_t^*\}_{t=0}^{\infty}$ and quantity sequences (allocations) $\{c_t^*\}_{t=0}^{\infty}$, $\{k_{t+1}^*\}_{t=0}^{\infty}$ and $\{n_t^*\}_{t=0}^{\infty}$ such that

1. Given prices, allocations are optimal for the household

$$\{c_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty} = \arg \max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

   s.t.

$$\begin{aligned} c_t + k_{t+1} - (1-\delta)k_t &= w_t^* n_t + r_t^* k_t \\ c_t \geq 0,\ 0 \leq n_t &\leq 1,\ k_{t+1} \geq 0 \\ k_0 &\text{ given} \end{aligned}$$

2. Given prices, allocations are optimal for the firm. That is, for every t

$$\{k_t^*, n_t^*\} = \arg \max_{k_t, n_t} F(k_t, n_t) - w_t^* n_t - r_t^* k_t$$

3. All markets clear. In particular, the markets for (dated) consumption/investment goods

$$c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, n_t^*) \quad \text{for all } t$$

   as well as markets for (dated) labor services and (dated) capital services.

Some things one should notice about this equilibrium: First, there is a whole sequence of budget constraints (one for every t) and, therefore, a whole sequence of multipliers will need to be used when solving the household's maximization problem. Second, there is a whole sequence of firm problems, again one for every t. Third, there is no p_t involved here since agents cannot trade directly the consumption/investment goods of different periods. Finally, Walras' Law will still hold here. Suppose the capital and labor markets clear, then the market will clear automatically for the consumption/investment good. To see this, notice that the households' income is equal to production $F(k_t, n_t)$ and so the budget constraint will imply the market clearing condition. This is true for every period t, which explains why we have normalized the price of consumption/investment goods to 1 *in every period*.

Characterization of equilibrium will include all conditions required for household maximization. These boil down to two first order conditions (ignoring inequality constraints)

$$u_c(c_t, 1 - n_t) = \beta u_c(c_{t+1}, 1 - n_{t+1})(1 - \delta + r_{t+1}) \tag{1}$$

$$-u_n(c_t, 1 - n_t) = w_t u_c(c_t, 1 - n_t) \tag{2}$$

together with a TVC. In addition, equilibrium conditions include firm factor demand, which equates factor prices to marginal products, as well as the market clearing condition.

2.1 Introducing an additional asset

We introduce financial assets like the ones we used for the exchange economy. A unit of the asset at time t costs one unit of the consumption good at time t and pays out R_{t+1} units of the consumption good at $t+1$. Let a_{t+1} denote the number of assets bought at time t.

DEFINITION: Given initial allocations of capital and assets, k_0 and $R_0 a_0$, an equilibrium with sequential trade is a set of prices $\{R_t^*\}_{t=1}^{\infty}$, $\{r_t^*\}_{t=0}^{\infty}$ and $\{w_t^*\}_{t=0}^{\infty}$ and set of quantities (allocations) $\{c_t^*\}_{t=0}^{\infty}$, $\{a_{t+1}^*\}_{t=0}^{\infty}$, $\{k_{t+1}^*\}_{t=0}^{\infty}$ and $\{n_t^*\}_{t=0}^{\infty}$ such that

1. Given the price sequences, allocations are optimal for households

$$\left\{c_t^*, a_{t+1}^*, k_{t+1}^*, n_t^*\right\}_{t=0}^{\infty} = \arg \max_{\{c_t, a_{t+1}, k_{t+1}, n_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-n_t)$$

s.t

$$\begin{aligned} c_t + a_{t+1} + k_{t+1} - (1-\delta)k_t &= R_t^* a_t + w_t^* n_t + r_t^* k_t \quad \text{for all } t \\ c_t \geq 0,\ 0 \leq n_t \leq 1,\ k_{t+1} &\geq 0 \quad \text{for all } t \\ \lim_{T\to\infty} \left(\prod_{t=1}^{T} R_t^*\right)^{-1} a_{T+1} &= 0 \\ k_0 \text{ and } R_0 a_0\ (=0) & \text{ given} \end{aligned}$$

2. Given the price sequences, allocations are optimal for the firm. That is, for all t

$$\{k_t^*, n_t^*\} = \arg \max_{k_t, n_t} F(k_t, n_t) - w_t^* n_t - r_t^* k_t$$

3. All markets clear. In particular, asset markets for all t

$$a_{t+1}^* = 0$$

goods markets for all t

$$c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, n_t^*)$$

and also labor and capital markets for all t.

Household maximization leads to the usual intratemporal condition describing the within-period consumption/leisure choice

$$-u_n(c_t, 1-n_t) = w_t u_c(c_t, 1-n_t)$$

as well as two Euler equations, one for physical capital and one for the financial asset

$$\begin{aligned} u_c(c_t, 1-n_t) &= \beta u_c(c_{t+1}, 1-n_{t+1})(1-\delta+r^*_{t+1}) \\ u_c(c_t, 1-n_t) &= \beta u_c(c_{t+1}, 1-n_{t+1})R^*_{t+1} \end{aligned}$$

Together, the two Euler equations imply that the return on capital must equal the return on the asset

$$R^*_{t+1} = (1-\delta+r^*_{t+1})$$

which is reasonable since otherwise there would exist an arbitrage possibility. One implication is that the choice of k_{t+1} versus a_{t+1} as a saving instrument for the household is indeterminate, all that matters is the total savings $a_{t+1}+k_{t+1}$. To put it differently, if we use the equalization of returns, the budget of a household *in equilibrium* can be written as

$$c_t + (a_{t+1}+k_{t+1}) = R^*_t\,(a_t+k_t) + w^*_t n_t$$

which makes the point that a_t and k_t are perfectly substitutable for the household (up to the non-negativity of capital).

Given the assumption of a representative agent, there will be no borrowing/lending between households in equilibrium ($a^*_{t+1} = 0$). To put it differently, the availability of the new asset a is redundant in the sense that it does not affect allocations and prices.

3 Date-0 equilibrium, sequential trade equilibrium and efficiency

Equilibrium allocations satisfy

$$\begin{aligned} u_c(c_t, 1-n_t) &= \beta u_c(c_{t+1}, 1-n_{t+1})(1-\delta+r_{t+1}) \\ -u_n(c_t, 1-n_t) &= w_t u_c(c_t, 1-n_t) \\ \lim_{T\to\infty} \beta^T u_c(c_T)k_{T+1} &= 0 \\ c_t + k_{t+1} - (1-\delta)k_t &= F(k_t, n_t) \end{aligned}$$

Since this is true for both the date-0 trade equilibrium and the sequential trade equilibrium, the two equilibrium concepts imply the same allocations. Notice the relation between the date-0 prices and the return to an asset (whether it is capital or another financial asset)

$$\frac{p_t}{p_{t+1}} = \frac{u_c\,(c_t, 1-n_t)}{\beta u_c\,(c_{t+1}, 1-n_{t+1})} = 1-\delta+r_{t+1} = R_{t+1}$$

In addition, simply replacing factor prices

$$r_t = F_k(k_t, n_t)$$
$$w_t = F_n(k_t, n_t)$$

in the Euler equation and the intratemporal consumption/leisure condition, shows that the allocations also coincide with the planner allocations we derived in the Cass-Koopmans model. That is, the equilibrium is efficient along both the intertemporal margin and the intratemporal margin.

Although we have only focused on a representative household version of the economy, these equivalences also hold for heterogeneous households, as long as a financial asset is available to complete the markets in the sequential trade economy.