Assignment 11

Haixiang Zhu

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Let $S_0 = \{s_0\}, \pi(s_0) = 1$

1. Date-0 trade

   (a) Given endowment processes and their corresponding probability distributions $\{\{w_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^{1}$ for each i, a competitive equilibrium with date-0 trade is a set of allocations $\{\{c^*_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^{1}$ for each i and prices $\{\{p^*_t(s_t)\}_{s_t \in S_t}\}_{t=0}^{1}$ such that

      1) Given prices, allocations are optimal for each consumer i

$$\{c^*_{i,t}(s_t)\}_{t=0}^{1} = \underset{\{c_{i,t}(s_t)\}_{t=0}^{1}}{\arg\max} \sum_{t=0}^{1} \sum_{s_t \in S_t} \beta^t \pi(s_t) \frac{[c_{i,t}(s_t)]^{1-\sigma}}{1-\sigma}$$

$$s.t. \quad \sum_{t=0}^{1} \sum_{s_t \in S_t} p^*_t(s_t) c_{i,t}(s_t) \leq \sum_{t=0}^{1} \sum_{s_t \in S_t} p^*_t(s_t) w_{i,t}(s_t)$$

$$c_{i,t}(s_t) \geq 0 \qquad \forall t$$

      2) The price are such that all markets clear.

$$\sum_i c^*_{i,t}(s_t) = \sum_i w_{i,t}(s_t) \qquad \forall t, i$$

   (b) All equilibrium conditions

$$\beta^t \pi(s_t)[c^*_{i,t}(s_t)]^{-\sigma} = \lambda_i p^*_t(s_t) \qquad \forall t, i \qquad (c_{i,t}(s_t))$$

$$\sum_{t=0}^{1} \sum_{s_t \in S_t} p^*_t(s_t) c_{i,t}(s_t) = \sum_{t=0}^{1} \sum_{s_t \in S_t} p^*_t(s_t) w_{i,t}(s_t) \qquad \forall i \qquad \text{(B.C.)}$$

$$\sum_i c^*_{i,t}(s_t) = \sum_i w_{i,t}(s_t) \qquad \forall t, i \qquad \text{(M.C.)}$$

   where λ_i is the multiplier on consumer i's budget constraint.
   From FOC of $(c_{i,t}(s_t))$, and note that, $\pi(s_0) = 1, p^*_0 = 1, c^*_{i,0}(s_0) =$

$c^*_{i,0}$

$$p^*_t(s_t) = \beta^t \pi(s_t) \left[\frac{c^*_{i,t}(s_t)}{c^*_{i,0}}\right]^{-\sigma} \tag{1}$$

$$c^*_{i,t}(s_t) = \left[\frac{p^*_t(s_t)}{\beta^t \pi(s_t)}\right]^{-\frac{1}{\sigma}} c^*_{i,0} \tag{2}$$

From (M.C.) and FOC of $(c_{i,t}(s_t))$

$$\frac{c^*_{A,t}(s_t)}{c_{A,0}} = \frac{c^*_{B,t}(s_t)}{c_{B,0}} = \frac{c^*_{A,t}(s_t) + c^*_{B,t}(s_t)}{c_{A,0} + c_{B,0}} = \frac{w_{A,t}(s_t) + w_{B,t}(s_t)}{w_{A,0} + w_{B,0}} \tag{3}$$

Plugging (3) into (1), and let $w_{A,t}(s_t) + w_{B,t}(s_t) = W_t(s_t), w_{A,0} + w_{B,0} = W_0$

$$p^*_t(s_t) = \beta^t \pi(s_t) \left[\frac{W_t(s_t)}{W_0}\right]^{-\sigma}, s_t \in S_t \tag{4}$$

Plugging $(2), (4)$ into (M.C.)

$$\begin{aligned}
\sum_t \sum_{s_t} p^*_t(s_t) \left[\frac{p^*_t(s_t)}{\beta^t \pi(s_t)}\right]^{-\frac{1}{\sigma}} c^*_{i,0} &= \sum_t \sum_{s_t} p^*_t(s_t) w_{i,t}(s_t) \\
c^*_{i,0} &= \frac{\sum_t \sum_{s_t} \beta^t \pi(s_t) \left[\frac{W_t(s_t)}{W_0}\right]^{-\sigma} w_{i,t}(s_t)}{\sum_t \sum_{s_t} \beta^t \pi(s_t) \left[\frac{W_t(s_t)}{W_0}\right]^{1-\sigma}} \\
c^*_{i,t}(s_t) &= \frac{W_t(s_t)}{W_0} c^*_{i,0} \\
c^*_{i,t}(s_t) &= \frac{\sum_t \sum_{s_t} \beta^t \pi(s_t) \left[W_t(s_t)\right]^{-\sigma} w_{i,t}(s_t)}{\sum_t \sum_{s_t} \beta^t \pi(s_t) \left[W_t(s_t)\right]^{1-\sigma}} W_t(s_t)
\end{aligned} \tag{5}$$

where $s_t \in S_t$

(c) Specific case 1
From (4) and (5)

$$\begin{cases} p_1(s_1) = \dfrac{\beta}{2} \\ c^*_{i,0} = c^*_{i,1}(s_1) = 5 \end{cases} \quad i \in (A, B), s_1 \in S_1$$

Intuitively, consumer is indifferent between state s_1 and state s'_1 because there is no uncertainty in terms of aggregate endowment.

(d) The coeffcient of relative risk aversion is

$$\begin{aligned}\gamma &= -\frac{cu''(c)}{u'(c)} \\ &= -\frac{c(-\sigma)c^{-\sigma-1}}{c^{-\sigma}} \\ &= \sigma\end{aligned}$$

The elasticity of intertemporal substitution is $\frac{1}{\sigma}$, which can be viewed as the inverse of the coeffcient of relative risk aversion. The larger σ is, the less willing is the household to substitute consumption across time. Therefore, the elasticity of intertemporal substitution measures the smoothing incentive over time; the coefficient of relative risk aversion measures the smoothing incentive over state.

(e) Specific case 2
B is better off because increasing probability $\pi(s_1')$ yields high benefit to one who owns larger endowment at state s_1', i.e. B.
From (4) and (5)

$$\begin{cases} p_1(s_1) = \dfrac{\beta}{3} \\ p_1(s_1') = \dfrac{2\beta}{3} \\ c^*_{A,0} = c^*_{A,1}(s) = \dfrac{15+14\beta}{3(1+\beta)} & s \in S_1 \\ c^*_{B,0} = c^*_{B,1}(s) = \dfrac{15+16\beta}{3(1+\beta)} \end{cases}$$

From the results above, B gets a higher consumption fraction, which is consistent with the previous arguement.

2. Sequential trade

(a) Given initial distribution of assets $b_{i,-1}$, endowment processes and their corresponding probability distributions $\{\{w_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^1$ for each i, a competitive equilibrium with sequential trade is a set of allocations $\{\{c^*_{i,t}(s_t), b^*_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^1$ for each i and contingent claims prices $\{\{q^*_t(s_t)\}_{s_t \in S_t}\}_{t=0}^1$ such that

i. Given the price system, the allocation solves each consumer's problem. For $i = A, B$

$$\{c^*_{i,t}(s_t), b^*_{i,t}(s_t)\}^1_{t=0} = \underset{\{c_{i,t}(s_t), b_{i,t}(s_t)\}^1_{t=0}}{\arg\max} \sum_{t=0}^{1} \sum_{s_t \in S_t} \beta^t \pi(s_t) \frac{[c_{i,t}(s_t)]^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} s.t. \quad c_{i,t}(s_t) + \sum_{s_t} q^*_t(s_t) b_{i,t}(s_t) &= b_{i,t-1}(s_t) + w_{i,t}(s_t) \quad && \forall t \\ c_{i,t}(s_t) &\geq 0 && \forall t \\ b_{i,-1} &= 0 \\ b_{i,1}(s_t) &= 0 \end{aligned}$$

ii. All markets clear. For goods market

$$\sum_i c^*_{i,t}(s_t) = \sum_i w_{i,t}(s_t) \qquad \forall t, \forall s_t \in S_t$$

For asset market

$$\sum_i b^*_{i,t}(s_t) = 0 \qquad \forall t, \forall s_t \in S_t$$

(b) Since date-0 equilibrium and sequential equilibrium are equivalent, from (4) and (5)

$$q^*_{t-1}(s_t) = p^*_t(s_t) = \beta^t \pi(s_t) \left[\frac{W_t(s_t)}{W_0}\right]^{-\sigma} \tag{6}$$

$$\begin{aligned} b^*_{i,t-1}(s_t) &= c^*_{i,t}(s_t) - w_{i,t}(s_t) \\ &= \frac{\sum_t \sum_{s_t} \beta^t \pi(s_t) [W_t(s_t)]^{-\sigma} w_{i,t}(s_t)}{\sum_t \sum_{s_t} \beta^t \pi(s_t) [W_t(s_t)]^{1-\sigma}} W_t(s_t) - w_{i,t}(s_t) \end{aligned} \tag{7}$$

where $s_t \in S_t$

(c) Specific cases

i. A. 1c

From (6) and (7)

$$\begin{cases} q_0(s) = \dfrac{\beta}{2} \\ b^*_{A,0}(s_1) = b^*_{B,0}(s'_1) = -1 \\ b^*_{A,0}(s'_1) = b^*_{B,0}(s_1) = 1 \end{cases} \quad s \in S_1$$

The cost of the portfolio bought by each consumer at time 0 is 0, they are neither net borrower or lender.

B. 1e

From (6) and (7)

$$\begin{cases} q_0(s_1) = \dfrac{\beta}{3} \\ q_0(s_1') = \dfrac{2\beta}{3} \\ b^*_{A,0}(s_1) = -\dfrac{3+4\beta}{3(1+\beta)} \\ b^*_{A,0}(s_1') = \dfrac{3+2\beta}{3(1+\beta)} \\ b^*_{B,0}(s_1) = \dfrac{3+4\beta}{3(1+\beta)} \\ b^*_{B,0}(s_1') = -\dfrac{3+2\beta}{3(1+\beta)} \end{cases}$$

The cost of the portfolio bought by consumer A at time 0 is $\dfrac{\beta}{3(1+\beta)} > 0$, which turns out to be net lender.

The cost of the portfolio bought by consumer B at time 0 is $-\dfrac{\beta}{3(1+\beta)} < 0$, which turns out to be net borrower.

ii. For both specific cases in 1c and 1e, the equilibrium price of a discount bond is

$$p_b = \sum_{s_1} \pi(s_1) q_0(s_1) = \beta \qquad\qquad s_1 \in S_1$$

(d) Given initial distribution of asset $b_{i,-1}$, endowment processes and their corresponding probability distributions $\{\{w_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^{1}$ for each i, a competitive equilibrium with sequential trade is a set of allocations $\{b^*_{i,t}, \{c^*_{i,t}(s_t)\}_{s_t \in S_t}\}_{t=0}^{1}$ for each i and risk free bond prices $q^*_b)$ at $t = 0$ such that

i. Given the price system, the allocation solves each consumer's problem. For $i = A, B$

$$\{c^*_{i,t}(s_t), b^*_{i,t}\}_{t=0}^{1} = \underset{\{c_{i,t}(s_t), b_{i,t}\}_{t=0}^{1}}{\arg\max} \sum_{t=0}^{1} \sum_{s_t \in S_t} \beta^t \pi(s_t) \frac{[c_{i,t}(s_t)]^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} s.t. \quad c_{i,0} + q^*_b b_{i,0} &= w_{i,0} \\ c_{i,1}(s_1) &= w_{i,1}(s_1) + b_{i,0} \\ c_{i,t}(s_t) &\geq 0 \qquad\qquad \forall t \\ b_{i,-1} &= 0 \end{aligned}$$

ii. All markets clear. For goods market

$$\sum_i c^*_{i,t}(s_t) = \sum_i w_{i,t}(s_t) \qquad \forall t, \forall s_t \in S_t$$

For asset market

$$\sum_i b^*_{i,t} = 0 \qquad \forall t$$

FOC

$$\begin{cases} (c^*_{i,0})^\sigma = \lambda_{i,0} \\ \beta\pi(s_1)[c^*_{i,1}(s_1)]^\sigma = \lambda_{i,1}(s_1) \\ \lambda_{i,0} q^*_b = \sum_{s_1} \lambda_{i,1}(s_1) \end{cases} \quad s_1 \in S_1$$

where $\lambda_{i,0}, \lambda_{i,1}(s_1)$ are multipliers on budget constraints.
Intuitively, the bond price will be different because it cannot be traded over state to smooth each consumer's consumption.