Problem Set 5

1. Consider the following planner's problem

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= Ak_t \\ c_t &\geq 0 \\ k_{t+1} &\geq 0 \\ & k_0 \text{ given} \end{aligned}$$

where $0 < \beta < 1$ and $A > 0$ are constants. Note that this is a growth model with a linear production function. Suppose the period utility function is given by

$$u(c_t) = \frac{c_t^{1-\sigma} - 1}{1 - \sigma}, \quad \sigma > 0$$

(a) Write down the recursive formulation of the problem (that is, write down the Bellman equation and clearly define states and control variables).

(b) Find the value function V and the optimal policy function for capital g^k using the "guess and verify" approach. (You need to explicitly show how you come up with a guess)

(c) What happens when $\sigma = 1$? Prove that the period utility $u(.)$ converges to $\ln c_t$ as σ tends to 1. Use this result to obtain the value function and optimal policy function for capital in this case by taking limits as $\sigma \to 1$ of your results in part b.

(d) Plot the policy function for capital together with the 45 degree line. Use this plot to describe the dynamics of capital starting from any initial k_0.

2. Consider an agent that wants to maximize his lifetime utility $\sum_{t=0}^{\infty} \beta^t u(b_t)$, where b_t represents the amount of bananas that he eats every period. The agent can choose to eat his current stock of bananas right away, or place some of them in the fridge and eat them tomorrow for breakfast. Let s_t denote the number of bananas he puts in the fridge in period t and assume that for each banana that he puts in the fridge today, he gets $R < 1$ units tomorrow (because a part of them ripens fast, so you have to throw some pieces away). In addition to the bananas saved, the agent also has a tree in his backyard that delivers w bananas every period. At each point in time, he must decide how many bananas to consume and how many to store for next period's consumption.

   (a) Write down his budget constraint in sequence form

   (b) Write the problem as a dynamic programming problem. This means: Setup the Bellman Equation, clearly indicating which are the state variables and which are the choice variables. *Make sure you use timeless notation and define precisely (in words!) all variables you use.*

   (c) Suppose that the tree does not deliver a constant amount of bananas every period, but instead it oscillates deterministically between two values w_H and w_L, where $w_H > w_L$. In particular, in period t the tree delivers w_t bananas, with $w_t = w_H$ if t is even and $w_t = w_L$ if t is odd. Repeat b for this setup (remember there should be no t anywhere!).