Problem Set 3 - Finite Horizon Cass-Koopmans

This problem set is a continuation of question 2 from Problem Set 2. You may use the results from there directly without proof. Assume

$$\begin{aligned} f(k) &= k^{\alpha}, \ \ 0 < \alpha < 1 \\ u(c) &= \frac{c^{1-\sigma} - 1}{1 - \sigma}, \ \sigma > 0 \end{aligned}$$

1. Assume $\delta = 0.1$, $\alpha = 0.3$, $\beta = 0.96$, $\sigma = 2$ and also that $k_0 = 0.01$.

   (a) Suppose $T = 1$ (a two-period model).

      i. List all conditions that will be needed to solve for allocations $\{c_0, c_1, k_1, k_2\}$.

      ii. Conditions from part i can be reduced to one non-linear equation for k_1. Use Matlab to solve that equation numerically. Report the solution for all variables $\{c_0, c_1, k_1, k_2\}$

   (b) Suppose $T = 2$ (a three-period model).

      i. List all conditions that will be needed to solve for allocations $\{c_0, c_1, c_2, k_1, k_2, k_3\}$

      ii. Solve for those allocations (Note you now need to solve two non-linear equations for k_1 and k_2 *simultaneously*) and report your solution for all variables

      iii. Report the savings rate in each period and comment on how it differs from the Solow model

   (c) Now suppose $T = 200$. Solve the model numerically as above and report plots of the sequences of capital, consumption and savings rate over time. Hint: You will need to write a Matlab code that can solve this for any value of T. Use vectors and 'for' loops to avoid having to define 200 variables and to specify 200 equations in Matlab.

2. Now let $\delta = 1$(full depreciation) and also $\sigma = 1$ (α, β and k_0 left unspecified)

   (a) Show that when $\sigma \to 1$ the utility becomes $u(c) = \ln(c)$

   (b) Show that the Inada conditions are satisfied for both $f(.)$ and $u(.)$.

   (c) Write the capital Euler equation (equation 1 in problem set 2) for this case and use the change of variable $z_t = \frac{k_t}{k_{t-1}^{\alpha}}$ to convert the result into a first order difference equation in z_t (that is an equation that involves z_{t+1} and z_t). Plot z_{t+1} against z_t and plot the 45^o line on the same graph.

   (d) Use the fact that $z_{T+1} = 0$ to show that

   $$z_t = \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}} \text{ for all } t = 1, 2, ..., T+1$$

   HINT: Work backwards to solve for z_T, z_{T-1} etc. until you notice a pattern. You will need to use the following result

   $$\sum_{i=0}^{M} x^i = \frac{1-x^{M+1}}{1-x}$$

   (e) Substitute back for z_t in terms of capital to obtain the following first order difference equation in capital.

   $$k_{t+1} = \alpha\beta\frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}}k_t^{\alpha} \text{ for } t = 0, 1, ..., T$$