Solow Model

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1 The Solow Model

We begin with a standard undergraduate textbook model, the Solow growth model. As the name suggests, the model is interested in analyzing and understanding economic growth. In particular, we will be looking at the growth rate of Real Gross Domestic Product (Real GDP) per capita.

Remember that we use Real GDP as a measure of total economic activity, in particular it provides a measure for Production, Income and Expenditure all at the same time. Real GDP is defined in such a way as to make all these three variables exactly equal. Because of this, we often use the terms production, output and income interchangeably.

Figure 1 shows Real GDP per capita in the US over the last century. The example of the US points to an important feature of Real GDP in modern, capitalist societies: it tends to grow over time. There are periods of recession, where it actually falls, and periods of expansion where it grows faster than normal, but on average (in the long run) Real GDP per capita is growing. It is this average long run behaviour that the Solow model attempts to explain leaving the business cycle fluctuations for other models.

In attempting to model and understand growth, it is important to realize there is a time dimension. That is true in a trivial sense (since growth refers to growth over time) but also in a much more important sense: the economy is *dynamic*, current performance and current behaviour by participants is affected by past performance and will in turn affect future performance. This intricate connection between past and future is what we mean by the term "dynamic" in macroeconomics. Modern macroeconomic theory takes seriously the dynamic aspects of the economy and does so by building up dynamic models. The Solow growth model is, perhaps, the only undergraduate textbook model that is dynamic and therefore we will build on this.

The main idea of the Solow model is in what causes the economy to be dynamic. The answer given by Solow is: the capital stock. In particular, investment today implies new productive capabilities tomorrow as the capital stock accumulates. That is why past choices will affect today's performance - that is why the economy is dynamic. When we refer to the capital stock, we are

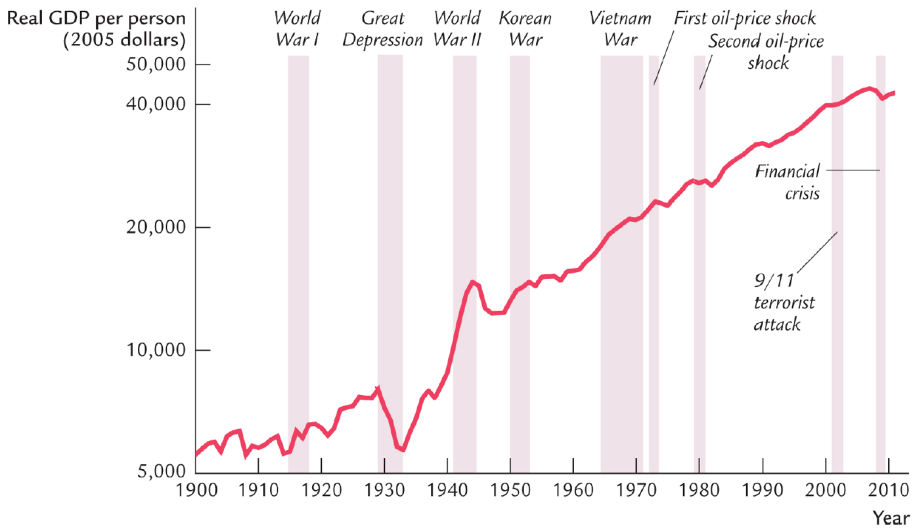

Figure 1: US Real GDP per capita (from the textbook "Macroeconomics" by Mankiw)

thinking of physical capital such as machines and buildings which are used as factors of production. The accumulation of capital stock is modelled in a very intuitive way. How is the capital stock in an economy augmented? What adds to the capital stock? The answer was given above and it is investment. Is there anything that reduces the capital stock over time? The answer is yes, the capital stock depreciates over time. You can think of this as a depreciation in the value of the machines over time, as they become used and their technology becomes obsolete. Or you can think of it in physical terms as ultimately a certain fraction of the machines in an economy will break down and become unusable. Either way, we can start modelling capital accumulation using these two basic forces.

Let time be discrete and t denote a period (say a year). Let K_t denote the capital stock at time t and I_t denote investment in year t. Also, denote by δ the depreciation rate, i.e. the percentage of the capital stock K_t that depreciates in any year. Then the above discussion says that the change in capital stock from one year to the other must equal investment minus depreciation

$$K_{t+1} - K_t = I_t - \delta K_t$$

This is the celebrated capital accumulation equation. The importance of tracking capital accumulation arises from the fact that capital is a factor of production. In particular,we will introduce a neoclassical production function according to which, production is the result of the combination of two inputs: capital and labor. Exactly how these inputs are combined is a reflection of the available technology in the economy and will be represented by the production function

$F : R_+^2 \to R_+$. So $F(.)$ will tell us how inputs capital K_t and labor N_t are transformed to output Y_t

$$Y_t = F(K_t, N_t)$$

Keep in mind that this is an aggregate model, so we think of this economy as having only one good - the aggregate good Y_t. The term "neoclassical" used to characterize the production function essentially refers to some assumptions that we will be imposing on the available technology. Those are as follows

1. Positive and Diminishing Marginal products of K and N

$$\begin{aligned} F_K(K,N) &> 0, F_K(K,N) > 0 \quad \text{all } K, N > 0 \\ F_{KK}(K,N) &< 0, F_{NN}(K,N) < 0 \quad \text{all } K, N > 0 \end{aligned}$$

2. Constant Returns to Scale (CRS)

$$\text{For any } z > 0,\ F(zK, zN) = zF(K,N)$$

3. Inada Conditions

$$\begin{aligned} \lim_{K \to 0} F_K(K,N) &= \lim_{n \to 0} F_n(K,N) = \infty \\ \lim_{K \to \infty} F_K(K,N) &= \lim_{N \to \infty} F_N(K,N) = 0 \end{aligned}$$

4. Essentiality:

$$F(0,N) = 0, F(K,0) = 0$$

Assumption 4 says that both factors of production are needed to produce, if no capital (labor) is used, then no production can take place (this assumption can easily be relaxed). Assumption 1 should be familiar from undergraduate microeconomics, using more labor (capital) leads to more output, but the returns to hiring additional factors of production are diminishing. CRS is also familiar from undergraduate economics, the idea is simple: if you double the amount of workers and double the amount of machines, your production will double. Note that, contrary to Assumption 1 which is intuitive and realistic, there is no a priori reason why returns to scale should be constant and not increasing or decreasing. For the moment, we make this assumption for simplicity (more justification to be provided later in the course).

The most widely used example of a neoclassical production function is the Cobb-Douglas production function

$$F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$$

where α is a parameter. Given the output Y_t in the economy (the resources available), what remains to be determined is how these resources are distributed. The National Income Accounts Identity breaks up GDP in four expenditure components: Consumption, Investment, Government Spending and Net

Exports. Here we will assume a closed economy so net exports are 0 and we will ignore the government so there is no government spending. The identity then reads

$$C_t + I_t = Y_t$$

We call this a resource constraint because it constrains consumption C_t and investment I_t to be (less than or) equal to the resources available for consumption and investment. Note again that this is a one good economy, so Y_t, C_t and I_t are all in terms of the same unit. Also, at this stage we do not allow for free disposal, i.e. we do not allow for resources to be thrown away (we will see this is optimal later on).

As if all the above simplifications were not enough, we will simplify further by assuming that the population (and hence labor since this is a full employment economy) is constant, $N_t = N$ for all t. This allows us to get rid of the labor variable and write our model in "per capita" or "per person" terms. We use small letters to denote per capita variables

$$c_t = \frac{C_t}{N_t}, k_t = \frac{K_t}{N_t}, y_t = \frac{Y_t}{N_t}, i_t = \frac{I_t}{N_t}$$

The three main building blocks of the Solow model until now can easily be transformed to "per capita" terms thanks to the constant returns to scale assumption (here is one of the nice things CRS buys us). This is achieved by dividing throughout by N

$$\begin{aligned} k_{t+1} &= (1-\delta)\, k_t + i_t \\ c_t + i_t &= y_t \\ y_t &= f(k_t) \end{aligned}$$

where we use the notation $f(k_t)$ instead of $F(k_t, 1)$ and call $f(.)$ the per capita production function. This says that output per capita only depends on capital per capita. All the assumptions on F are inherited by f, its shape is illustrated in Figure 2.

The distribution of output between c_t and i_t is what remains to be determined here to close the model. This distribution is often termed the consumption/savings choice. The reason for the name is straightforward. In this closed economy, national savings are all funneled into investment. So, once we know what portion of income y_t is saved, we know investment and therefore consumption. The assumption about the consumption/savings choice is the fundamental difference between undergraduate macroeconomics (the Solow model) and graduate macroeconomics (starting with the Cass-Koopmans model). We first complete the description and analysis of the Solow model and will proceed to "graduate" macroeconomics in the next section.

According to Solow, an exogenous and fixed fraction s of output is saved every period. Formally, the behavioral equations are given by

$$i_t = sy_t$$

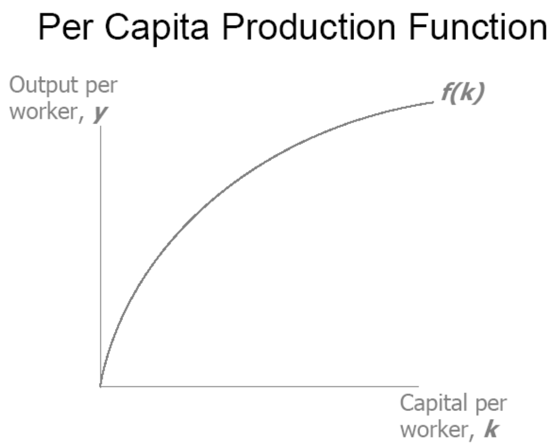

Figure 2: The per capita produnction function $f(.)$.

and as a result of the resource constraint

$$c_t = (1 - s) y_t$$

Given these equations, we can now go ahead and use the model. In particular, given an initial capital stock k_0, one can iteratively produce time series for y_t and compare it with the actual time series for real GDP per capita. The same is true for all the variables in this model. The iterative procedure is simple: given k_0, find y_0. Then use y_0 to find c_0 and i_0 and finally, from the capital accumulation equation, find k_1. At this point we can start again and find all $t = 1$ variables and so on. In this way, one can produce artificial time series data, i.e. sequences $\{c_t\}_{t=0}^{T}$, $\{i_t\}_{t=0}^{T}$, $\{k_{t+1}\}_{t=0}^{T}$,$\{y_t\}_{t=0}^{T}$ of any desirable length T. As with any dynamic system, this process requires "initial conditions" to get started; here we need k_0 to be given.

At this point an accounting exercise is helpful in order to keep track of all the parameters and variables in this model. The Solow model's equation are summarized below

$$\begin{aligned} k_{t+1} &= (1 - \delta) k_t + i_t \\ c_t + i_t &= y_t \\ y_t &= f(k_t) \\ i_t &= s y_t \end{aligned}$$

The model consists of some parameters, the depreciation rate δ, the savings rate s and the production functional form $f(.)$. Parameters are fixed over time and are chosen ex ante - we will see later in the course how to choose parameters in

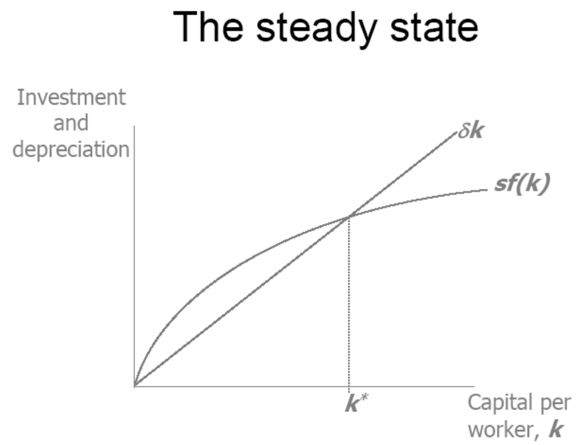

Figure 3: The steady state capital per worker k^* occurs when investment equals depreciation.

a manner consistent with actual economies. The model also consists of variables c_t, i_t, k_t and y_t all of which are endogenous (except for initial capital k_0). The model is dynamic so variables can change over time. This is in contrast to static models, which could also involve many periods, but where each period is unconnected to the other and there is no potential for change of the variables across time.

There are different approaches to analyzing the Solow model, approaches that are in many ways complementary. Iterative computation of paths for the endogenous variables as above is one way (typically using a computer), graphical qualitative analysis is another and explicit algebraic solution for the stationary points and dynamics of the main difference equation is another. Here we touch on some of those methods and their results.

We begin with the graphical approach (see Figure 3). The diagram shows a plot of investment and depreciation, where all variables are in per capita terms. Depreciation as a function of k is clearly linear with a slope of δ. The shape of the investment line is dictated by the assumptions on the production function: it is increasing and concave, starting at the origin (essentiality). The Inada conditions ensure that after the origin the investment line is above the depreciation line, but eventually it falls below. This is basically a graphical proof of the fact that there is only one crossing point (except the origin). We call that point the steady state for reasons that will soon become obvious and denote the value of capital at that point by k^*. At any point $k_t < k^*$, $sf(k_t) > \delta k_t$ which by the capital accumulation equation implies $k_{t+1} > k_t$. In words, when the capital stock is less than k^*, new additions to the capital stock through investment are higher than the reduction caused by depreciation and thus the capital stock grows. Conversely, to the right of k^*, the capital stock falls. When $k_t = k^*$, $sf(k^*) = \delta k^*$ and so $k_{t+1} = k^*$, i.e. when investment equals depreciation the capital stock remains constant. Furthermore, once such a level for the capital stock is reached, then the capital will forever remain at this level - this is known as a steady state. The previous discussion completely characterized the dynamics of the capital stock in the Solow model. Since the relationship between k_t and y_t is monotonic the same can be said about real GDP per capita y_t. If an economy begins with low y_t, then it will grow until it reaches the steady state level $y^* = f(k^*)$. If the economy begins with a high level of y_t, then it will shrink until it reaches y^*.

Conclusion 1 *A positive steady state exists, it is unique and the dynamics converge to it.*

Note the qualifier "positive" since $k = 0$ is also a steady state, but clearly not stable, i.e. the dynamics diverge from it. The steady state can be found with the iterative procedure described above. But it is much easier to compute it algebraically as the solution to the steady state equation

$$sf(k^*) = \delta k^*$$

Given k^*, the rest of the variables at steady state are also easily computed

$$\begin{aligned} y^* &= f(k^*) \\ i^* &= sy^* \\ c^* &= y^* - i^* \end{aligned}$$

One feature of the Solow model economy that should be pointed out here is that, given exogenous parameters (and k_0), the economy naturally takes its course without any intervention, i.e. without any choice being made individuals or any policy decision being taken. This is largely because of the behavioral assumption that consumption and savings are mechanically decided: an exogenous, fixed fraction is saved and consumed. The Solow model is an "Exogenous Savings model". In the next section we will begin introducing choice by endogenizing savings, but we can already say some things about policy. In particular, we can think of the savings rate s as being controlled by a government and try to analyze what the government should do. The government can affect the savings rate in the economy in a number of ways. It can do so directly, by deciding how much to save (or borrow) itself, i.e. by choosing its fiscal policy and controlling the budget deficit. It can also induce the private sector to save more or less by providing tax incentives for saving. For example, by lowering taxes on capital gains or corporate profits, offering tax breaks on individual retirement accounts or giving subsidies to investment, the government can provide incentives to agents to save/invest more and consume less. These channels through which the government can affect savings are not modelled explicitly here. Instead, we take it as a given that a planner/government can control s and try to

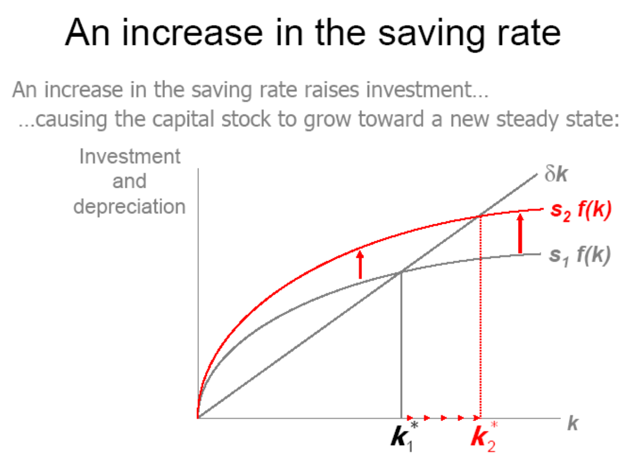

Figure 4: The effect of an increase in the savings rate s

answer the question: should the government increase or decrease savings and by how much?

To answer such a question we need to address two questions:

- How does a change in s affect the economy?
- What should the objective be?

The first question can be answered easily using the graph in Figure 4. Suppose the economy is at a steady state k_1^*. An increase in s from s_1 to $s_2 > s_1$, tilts the investment line upwards. With the capital stock being at k_1^*, depreciation is the same as before, but investment is now higher (since savings have increased). As a result, the capital stock starts growing and keeps growing until it reaches a new steady state $k_2^* > k_1^*$.

Clearly, at the new steady state $y_2^* = f(k_2^*) > f(k_1^*) > y_1^*$ so real GDP per capita is higher. Similarly, $i_2^* = s_2 f(k_2^*) > s_1 f(k_1^*) = i_1^*$ since both income and the savings rate have increased. On the other hand, the effect on consumption is ambiguous: the total income per capita available for consumption is higher than before, but we now consume a smaller fraction of it. So is the economy in better shape or not? Here we have to address the second question above, i.e. determine what we mean by "better". We follow the (consumerist) approach that individuals in the economy are better-off if they consume more. This is by no means uncontroversial, but it is a reasonable starting point. After all, what good is a huge income if households cannot use it to buy the goods they want?

Given this objective, the best policy will be the one that leads consumers to the highest possible consumption. If the savings rate is $s_2 = 0$, depreciation

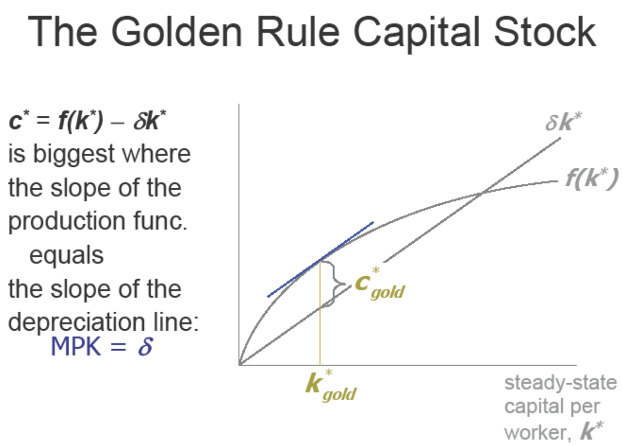

Figure 5: The golden rule level of capital per worker k^*_{gold} occurs when the marginal product of capital equals the depreciation rate.

will eventually deplete all of the capital stock so the long run steady state will be one where $k_2^* = y_2^* = i_2^* = c_2^* = 0$. On the other hand, if all of the income is saved, i.e. if the savings rate is $s_2 = 1$, then by the argument above we will obtain the highest possible k_2^*, y_2^* and i_2^*. But notice that again $c_2^* = 0$ since all of the income is saved. Clearly, if we are looking to maximize steady state consumption then we will need to choose an intermediate value for the savings rate $s_{gold} \in (0,1)$. We use the notation s_{gold}, the "Golden Rule" savings rate, to denote the best savings rate. Finding s_{gold} can be achieved as follows: In principle, we could pick a whole range of values for $s \in (0,1)$. For each of those values, compute the steady state $k^*(s), y^*(s), i^*(s)$ and $c^*(s)$. Note that we write the steady states as functions of s, since for every different s we find a different steady state. Then compare $c^*(s)$ across all possible s and find the highest one. This situation can be represented graphically as below.

Figure 5 plots GDP per capita and investment per capita at all possible steady states as functions of capital per capita at steady state. At any steady state it must be that

$$\begin{aligned} y^* &= f(k^*) \\ i^* &= \delta k^* \end{aligned}$$

so the GDP per capita graph has the shape of the production function and the investment line is straight with slope δ. Each k^* on the x-axis corresponds to one choice for s. For example, $k^* = 0$ corresponds to the choice $s = 0$ and the corresponding y^* and i^* are also 0. The point where the two lines cross each other corresponds to the choice $s = 1$. At that point, $y^* = i^*$ and so $c^* = 0$.

Our problem is reduced to finding k^*_{gold} that gives the maximum consumption, i.e. the maximum distance between the two lines. This is achieved where the two lines have the same slope. In terms of math, this is achieved when

$$f'(k^*_{gold}) = \delta$$

Solving that equation for k^*_{gold} will give us our answer. This "Golden Rule" condition can also be derived using simple calculus.

$$\max_{k^*} c^* = y^* - i^* = f(k^*) - \delta k^*$$

The first order condition gives the Golden Rule condition. The idea is that capital should keep increasing as long the resulting marginal increase in output (the marginal product of capital) is bigger than the resulting marginal increase in required investment. As long as this is the case, increasing capital increases consumption.

To conclude on the Golden Rule, remember that the original question was formulated in terms of the best savings rate, but we only found the best steady state. Once we know the best steady state, it is straightforward to find which savings rate will lead us there

$$\begin{aligned} y^*_{gold} &= f(k^*_{gold}) \\ i^*_{gold} &= \delta k^*_{gold} \end{aligned}$$

so the savings rate prevailing in this economy is

$$s_{gold} = \frac{i^*_{gold}}{y^*_{gold}}$$

The conclusion is that if an economy is saving $s < s_{gold}$ ($s > s_{gold}$) the government should follow policies to increase (decrease) savings. It is an empirical fact that actual economies tend to save less than the Golden Rule, which is possibly different across countries, and therefore political discussions tend to focus on efforts to increase savings.

Although the approach described here provides a good starting point for thinking about optimal consumption/savings choices, it is a simplified approach in more than one ways. First, notice that we focused on comparing across steady states, but the economy spends a large amount of time out of steady state, in transition. When the savings rate is increased to s_{gold} we learned that *in the long run the economy* will enjoy higher consumption. But in the short run, given that the capital stock and thus GDP per capita will take time to start increasing, consumption is reduced since we have to save more and consume less! In deriving the golden rule, we have ignored these short run effects.

Second, the original assumption is that the savings rate is fixed across time. Even when we consider a change in the savings rate, we are assuming that the change is once and for all. There is no reason why the optimal choice for the savings rate would not be varying with time. For example, as the economy grows richer, it might be optimal to save less (or more). Once we start thinking about the transition period and allowing the savings rate to change there's another question that arises: is it better for consumers to consume 0 today and a lot tomorrow or a lot today and zero tomorrow? And are consumers equally happy consuming the following stream $10, 0, 10, 0, 10, \ldots$, i.e. alternating every year between zero and positive consumption or $5, 5, 5, 5, \ldots$? The two consumption streams give the same total consumption but are they equally enjoyed? These questions refer to consumer preferences. Here we have ignored preferences by directly assuming an ad hoc behavioral rule. The next step is to *derive* the behavioral rule from first principles by assuming preferences and rational, optimizing agents.

Summarizing the limitations of the Solow model with regard to the optimal savings choice we have the following issues:

1. Only looking at steady states, not transition.
2. Assuming fixed savings rate whereas optimally it might vary.

For all these reasons, we proceed in the next couple of sections to introducing a model that addresses these issues and, along the way, highlights a number of important economic mechanisms absent from the Solow model.

As a concluding remark, it should be noted that the basic Solow model presented here gives us some important insights but is ultimately unsatisfactory as a model of *sustained* economic growth. On the positive side, it tells us that an economy which experiences a large sudden depletion of its capital stock should be expected to grow back to its steady state. In that sense, it provides a good starting point to understand the experiences of countries whose capital stock was depleted during the Second World War (e.g.Germany and Japan), who went on to a long period of fast growth after the war. On the negative side, the prediction that ultimately countries should reach a steady state where there is no growth (real GDP per capita stops changing) is not borne out of the data. Industrialized countries have enjoyed more than a century of continued economic growth (business cycle fluctuations aside). For this reason, the Solow model has been extended to include population growth and, importantly, technological growth. These two features have helped to bring the model's prediction more closely in line with the empirical observation. Instead of introducing these aspects in the context of the Solow model, we proceed to the Cass-Koopmans model where the savings choice is endogenous and we will introduce these aspects there.

1.0.1 Useful Reading

Mankiw, Gregory. "Macroeconomics", 8th Edition, Chapters 8 and 9.(easy, undergrad level)