Assignment 9

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1. Period-by-period budget constraint for each consumer in period t

$$c_t + q_t b_{t+1} = b_t + w_t$$

Given a sequence of endowments $\{\{w_{i,t}\}_{t=0}^{\infty}\}_{i=1}^{2}$, a competitive equilibrium with sequential trade consists of sequences of allocations $\{\{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a sequence of prices $\{(q_t^b)^*\}_{t=0}^{\infty}$ such that

(a) Given the price system, the allocation solves each consumer's problem. For $i = 1, 2$

$$\{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^{\infty} = \arg \max_{\{c_{i,t}, b_{i,t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\sigma}}{1-\sigma}$$

$$\begin{aligned} s.t. \qquad c_{i,t} + (q_t^b)^* b_{i,t+1} &= b_{i,t} + w_{i,t} \quad && \forall t \\ c_{i,t} &\geq 0 && \forall t \\ b_{i,0} &= 0 \\ \lim_{T\to\infty} b_{i,T+1}^* \prod_{t=0}^{T} (q_t^b)^* &\geq 0 \end{aligned}$$

(b) All markets clear. For goods market

$$\sum_i c_{i,t}^* = \sum_i w_{i,t} \qquad \forall t$$

For asset market

$$\sum_i b_{i,t+1}^* = 0 \qquad \forall t$$

2. All conditions for equilibrium (FOC+B.C+M.C+TVC+nPg) $\forall t, i$

$$\beta^t (c_{i,t}^*)^{-\sigma} = \mu_{i,t} \tag{$c_{i,t}$}$$

$$(q_t^b)^* \mu_{i,t} = \mu_{i,t+1} \tag{$b_{i,t+1}$}$$

$$c_{i,t} + (q_t^b)^* b_{i,t+1} = b_{i,t} + w_{i,t} \tag{B.C}$$

$$c_{1,t}^* + c_{2,t}^* = w_{1,t} + w_{2,t} \tag{goods}$$

$$b_{1,t+1}^* + b_{2,t+1}^* = 0 \tag{asset}$$

$$\lim_{T\to\infty} \beta^T (c_{i,T}^*)^{-\sigma} b_{i,T+1}^* \leq 0 \tag{TVC}$$

$$\lim_{T\to\infty} b_{i,T+1}^* \prod_{t=0}^{T} (q_t^b)^* \geq 0 \tag{nPg}$$

where $\mu_{i,t}$ is the multiplier on consumer i's budget constraint for each period t. The non-negativity constrants is ignored because of the Inada condition for utility function.

3. *Proof.* The equivalence between the date-0 equilibrium and the sequential equilibrium.
Recall the characterization of the date-0 trade equilibrium $\forall t, i$

$$\begin{cases} \beta^t (c_{i,t}^*)^{-\sigma} = \lambda_i p_t^* \\ \sum_{t=0}^{\infty} p_t^* c_{i,t} = \sum_{t=0}^{\infty} p_t^* w_{i,t} \\ \sum_i c_{i,t}^* = \sum_i w_{i,t} \end{cases}$$

where λ_i is the multiplier on consumer i's budget constraint.

   (a) Necessity (only if)
   Consider any consumption choice $\{c_{i,t}^*\}_{t=0}^{\infty}$ that is feasible in the sequential trade equilibrium. Then, by rolling forward the sequential trade budget and using TVC and nPg it can shown that it satisfies the date-0 trade budget

$$\begin{aligned} b_{i,0} &= c_{i,0}^* - w_{i,0} + (q_0^b)^* b_{i,1}^* \\ &= c_{i,0}^* - w_{i,0} + (q_0^b)^* (c_{i,1}^* - w_{i,1}) + (q_0^b)^* (q_1^b)^* b_{i,2}^* \\ &= \dots \\ &= \sum_{t=0}^{\infty} \prod_{s=0}^{t-1} (q_s^b)^* (c_{i,t}^* - w_{i,t}) + \lim_{T\to\infty} b_{i,T+1}^* \prod_{t=0}^{T} (q_t^b)^* \end{aligned}$$

   Now note that in equilibrium $\prod_{s=0}^{t-1} (q_s^b)^* = \beta^t \left(\frac{c_{i,t}^*}{c_{i,0}^*} \right)^{-\sigma}$ and this also corresponds to $\frac{p_t^*}{p_0^*}$ in the date-0 trade equilibrium.

$$b_{i,0} = \sum_{t=0}^{\infty} \frac{p_t^*(c_{i,t}^* - w_{i,t})}{p_0^*} + \lim_{T\to\infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*}\right)^{-\sigma}$$

The last term is 0 (by using the nPg and TVC conditions) and using the zero initial wealth assumption and the normalization $p_0^* = 1$ we obtain

$$0 = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t})$$

(b) Sufficiency (if)
To show the opposite statement, that if $\{c_{i,t}^*\}_{t=0}^{\infty}$ satisfies the date-0 trade budget then it is feasible in the sequential trade economy, one can proceed by constructing the asset trades required to ensure the same consumption. Since in equilibrium

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*}\right)^{-\sigma}$$

then the asset choices can be constructed recursively

$$b_{i,t+1}^* = \frac{(c_{i,t}^*)^{-\sigma}}{\beta(c_{i,t+1}^*)^{-\sigma}}(b_{i,t}^* + w_{i,t} - c_{i,t}^*) \quad \forall t$$

With these choices for assets, and given that goods' markets clear, the asset market clears in every period (simply add $b_{i,t+1}$ across agents and show it equals zero). We can also show in a manner identical to before that

$$0 = b_{i,0} = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t}) + \lim_{T\to\infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*}\right)^{-\sigma}$$

and since the date-0 budget constraint is satisfied for this consumption sequence, this implies that

$$\lim_{T\to\infty} b_{i,T+1}^* \beta^T (c_{i,T}^*)^{-\sigma} = 0$$

that is, the nPg and TVC conditions are satisfied.

□

4. Special cases
Since date-0 equilibrium and sequential equilibrium are equivalent, from FOC

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*}\right)^{-\sigma} = \frac{p_{t+1}^*}{p_t^*}$$

Then

$$c^*_{i,t} + \frac{p^*_{t+1}}{p^*_t} b^*_{i,t+1} = b^*_{i,t} + w_{i,t} \tag{1}$$

(a) $w_{1,t} = 2y, w_{2,t} = y \ \ \forall t$
Since $c^*_{i,t} = w_{i,t} \ \ \forall t, i$, no trade happens in this case, which implies $b_{i,t} = 0 \ \ \forall t$

(b) $w_{1,t} = \{2y, y, 2y, y, \dots\}, w_{2,t} = \{y, 2y, y, 2y, \dots\} \ \ \forall t$

$$\begin{cases} p^*_t = \beta^t \\ c^*_{1,t} = \dfrac{2+\beta}{1+\beta} y \\ c^*_{2,t} = \dfrac{1+2\beta}{1+\beta} y \end{cases} \quad \forall t \tag{2}$$

Plugging (2) into (1)

$$\begin{aligned} &b^*_{1,1} = \frac{1}{\beta}(0 + 2y - \frac{2+\beta}{1+\beta} y) = \frac{1}{1+\beta} y = -b^*_{2,1} \\ &b^*_{1,2} = \frac{1}{\beta}(b^*_{1,1} + y - \frac{1+2\beta}{1+\beta} y) = 0 = -b^*_{2,2} \\ &\dots \end{aligned}$$

Therefore,

$$\begin{cases} b^*_{1,2t} = 0 \\ b^*_{1,2t+1} = \dfrac{1}{1+\beta} y \\ b^*_{2,2t} = 0 \\ b^*_{2,2t+1} = -\dfrac{1}{1+\beta} y \end{cases} \quad \forall t$$

(c) $w_{1,t} = 2y, w_{2,t} = \{y, 3y, y, 3y, \dots\} \ \ \forall t$

$$\begin{cases} p^*_{2t} = \beta^{2t} \\ p^*_{2t+1} = \beta^{2t+1} \left(\dfrac{5}{3}\right)^{-\sigma} \\ c^*_{1,2t} = 6y \dfrac{1 + \beta \left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c^*_{1,2t+1} = 10y \dfrac{1 + \beta \left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c^*_{2,2t} = 3y \dfrac{1 + 3\beta \left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3}\right)^{-\sigma}} \\ c^*_{2,2t+1} = 5y \dfrac{1 + 3\beta \left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta \left(\frac{5}{3}\right)^{-\sigma}} \end{cases} \quad \forall t \tag{3}$$

Plugging (3) into (1)

$$
\begin{aligned}
b_{1,1}^* &= \frac{1}{\beta}\left(\frac{5}{3}\right)^{\sigma}\left(0+2y-6y\frac{1+\beta\left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}}\right) = \frac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}} = -b_{2,1}^* \\
b_{1,2}^* &= \frac{1}{\beta}\left(\frac{5}{3}\right)^{-\sigma}\left(b_{1,1}^*+2y-10y\frac{1+\beta\left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}}\right) = 0 = -b_{2,2}^* \\
&\dots
\end{aligned}
$$

Therefore,

$$
\begin{cases}
b_{1,2t}^* = 0 \\
b_{1,2t+1}^* = \dfrac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\
b_{2,2t}^* = 0 \\
b_{2,2t+1}^* = -\dfrac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}}
\end{cases} \qquad \forall t
$$

5. c, b, q, w donote the comsumption, bonds agent bought, bond price and exogenous endowment in current period respectively.
   q', w' denote the bond price and endowment in the next period respectively.
   b^- denotes the amount of bond agent bought in the last period.
   Bellman Equation is

$$
\begin{aligned}
V(b^-, w, q) = \max_{c,b} &\left\{\frac{c^{1-\sigma}}{1-\sigma} + \beta V(b, w', q')\right\} \\
s.t. \quad c + qb &= b^- + w \\
c &\geq 0 \\
q' &= f^q(q) \\
w' &= f^w(w) \\
\text{nPg}&\text{ holds} \\
b^-, w&\text{ given}
\end{aligned}
$$

   where state variables are b^-, w, q and control variables are c, b.