Assignment 8

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1. Given a collection of endowments $\{\{w_{i,t}\}_{t=0}^{\infty}\}_{i=1}^{2}$, a competitive equilibrium with date-0 trading is a collection of allocations $\{\{c_{i,t}^{*}\}_{t=0}^{\infty}\}_{i=1}^{2}$ and a collection of prices $\{p_t^*\}_{t=0}^{\infty}$ such that

   (a) Given the price system, the allocation solves each consumer's problem. For $i = 1, 2$

   $$\{c_{i,t}^{*}\}_{t=0}^{\infty} = \arg\max_{\{c_{i,t}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_{i,t}^{1-\sigma}}{1-\sigma}$$

   $$s.t. \quad \sum_{t=0}^{\infty} p_t^* c_{i,t} \le \sum_{t=0}^{\infty} p_t^* w_{i,t}$$

   $$c_{i,t} \ge 0 \qquad \forall t$$

   (b) The price are such that all markets clear.

   $$\sum_i c_{i,t}^{*} = \sum_i w_{i,t} \quad \forall t$$

2. All equilibrium conditions

   $$\beta^t (c_{i,t}^{*})^{-\sigma} = \lambda_i p_t^* \qquad\qquad \forall t, i \qquad\qquad (1)$$

   $$\sum_i c_{i,t}^{*} = \sum_i w_{i,t} \qquad\qquad \forall t$$

   $$\sum_{t=0}^{\infty} p_t^* c_{i,t} = \sum_{t=0}^{\infty} p_t^* w_{i,t} \qquad\qquad \forall i \qquad\qquad (2)$$

   where λ_i is the multiplier on consumer i's budget constraint. The non-negativity constrants is ignored because utility function satisfies the Inada condition .

3. Solution
   Assume that $p_0 = 1$, from (1), we have

   $$\left(\frac{c_{1,t}^{*}}{c_{2,t}^{*}}\right)^{-\sigma} = \frac{\lambda_1}{\lambda_2} \quad \forall t \qquad\qquad (3)$$

and
$$(c_{i,0}^*)^{-\sigma} = \lambda_i \quad \forall i \tag{4}$$

Therefore, together with the fact that $c_{1,t}^* + c_{2,t}^* = w_{1,t} + w_{2,t}$, we have

$$\begin{cases} c_{1,t}^* = \dfrac{\lambda_2^{\frac{1}{\sigma}}}{\lambda_1^{\frac{1}{\sigma}} + \lambda_2^{\frac{1}{\sigma}}}(w_{1,t} + w_{2,t}) \\[4mm] c_{2,t}^* = \dfrac{\lambda_1^{\frac{1}{\sigma}}}{\lambda_1^{\frac{1}{\sigma}} + \lambda_2^{\frac{1}{\sigma}}}(w_{1,t} + w_{2,t}) \end{cases} \quad \forall t \tag{5}$$

Rewriting (5)
$$c_{i,t}^* = \frac{c_{1,0}^*}{w_{1,0} + w_{2,0}}(w_{1,t} + w_{2,t}) \quad \forall t, i \tag{6}$$

Again, from (1), for any two periods s and t

$$\frac{\beta^s(c_{i,s}^*)^{-\sigma}}{\beta^t(c_{i,t}^*)^{-\sigma}} = \frac{p_s^*}{p_t^*} = \frac{\beta^s(c_{j,s}^*)^{-\sigma}}{\beta^t(c_{j,t}^*)^{-\sigma}} \quad \forall i, j, s, t \tag{7}$$

Then
$$\frac{c_{1,s}^*}{c_{1,t}^*} = \frac{c_{2,s}^*}{c_{2,t}^*} = \frac{c_{1,s}^* + c_{2,s}^*}{c_{1,t}^* + c_{2,t}^*} = \frac{w_{1,s} + w_{2,s}}{w_{1,t} + w_{2,t}}$$

Therefore, assume that $p_0^* = 1$

$$\frac{p_s^*}{p_t^*} = \beta^{s-t}\left(\frac{w_{1,s} + w_{2,s}}{w_{1,t} + w_{2,t}}\right)^{-\sigma} \quad \forall t \tag{8}$$

$$\Rightarrow p_t^* = \beta^t\left(\frac{w_{1,t} + w_{2,t}}{w_{1,0} + w_{2,0}}\right)^{-\sigma} \quad \forall t \tag{9}$$

From (7)

$$c_{i,t}^* = \left(\frac{p_t^*}{\beta^t}\right)^{-\frac{1}{\sigma}} c_{i,0}^* \tag{10}$$

Let $w_{1,0} + w_{2,0} = W_0$ and $w_{1,t} + w_{2,t} = W_t$. Plugging (9) and (10) into (2)

$$\sum_{t=0}^{\infty} p_t^* w_{i,t} = \sum_{t=0}^{\infty} p_t^*\left(\frac{p_t^*}{\beta^t}\right)^{-\frac{1}{\sigma}} c_{i,0}^*$$

$$\Rightarrow c_{i,0}^* = \frac{\sum_{t=0}^{\infty}\left[\beta^t\left(\frac{W_t}{W_0}\right)^{-\sigma} w_{i,t}\right]}{\sum_{t=0}^{\infty}\left[\beta^t\left(\frac{W_t}{W_0}\right)^{1-\sigma}\right]} \tag{11}$$

Plugging (11) into (5) and (4)

$$c_{i,t}^* = \frac{W_t}{W_0} c_{i,0}^* = \frac{\sum_{t=0}^{\infty} \left(\beta^t W_t^{-\sigma} w_{i,t} \right)}{\sum_{t=0}^{\infty} \left(\beta^t W_t^{1-\sigma} \right)} W_t$$

$$\lambda_i = \left[\frac{\sum_{t=0}^{\infty} \left(\beta^t W_t^{-\sigma} w_{i,t} \right)}{\sum_{t=0}^{\infty} \left(\beta^t W_t^{1-\sigma} \right)} W_0 \right]^{-\sigma} \tag{12}$$

4. Intuitive explanation

   (a) From (8), WLOG, assume that $s > t$.

      i. The smaller the ratio of aggregate endowments $\frac{w_{1,s} + w_{2,s}}{w_{1,t} + w_{2,t}}$ in period s, the more scarce the good, the higher the relative value of the good in period s and the higher its relative price.

      ii. The larger β is, the less discounted the good in period s, the greater the relative value of the good in period s and the greater its relative price.

      iii. The larger σ is, the less willing is the consumer to substitute consumption across time, the lower the relative price of the good in period s with more aggregate endowment.

   (b) From (5), this constant fraction is an illustration of consumption smoothing. Variation in individual endowments does not translate to variation in consumption because agents smooth their consumption through trade. Other things equal, the larger endowments or earlier endowments or smoother endowments, the larger consumption fraction.

5. Special cases

   (a) In this case, the relative price and consumption are all constant across periods. Perfect consumption smoothing is feasible. Denote constant consumption $\bar{c}_i^*$ for each i. From (9), we have

   $$p_t^* = \beta^t \quad \forall t$$

   From budget constraint, we have

   $$\sum_{t=0}^{\infty} \beta^t (\bar{c}_i^* - w_{i,t}) = 0 \tag{13}$$

   where $w_{i,t} = w_i$. Therefore, the solution is

   $$\begin{cases} c_{1,t}^* = 2y \\ c_{2,t}^* = y \end{cases} \quad \forall t$$

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(b) Again, aggregate endowment is constant, therefore, the relative price and consumption for each consumer are constant. From (9), we have

$$p_t^* = \beta^t \quad \forall t$$

From (13), we have

$$\begin{cases} \displaystyle\sum_{t=0}^{\infty} \beta^t \bar{c}_1^* = 2y \sum_{t=0}^{\infty} \beta^{2t} + y \sum_{t=0}^{\infty} \beta^{2t+1} \\ \displaystyle\sum_{t=0}^{\infty} \beta^t \bar{c}_2^* = y \sum_{t=0}^{\infty} \beta^{2t} + 2y \sum_{t=0}^{\infty} \beta^{2t+1} \end{cases}$$

Hence, the soulution is

$$\begin{cases} c_{1,t}^* = \dfrac{2+\beta}{1+\beta} y \\ c_{2,t}^* = \dfrac{1+2\beta}{1+\beta} y \end{cases} \quad \forall t$$

(c) Since aggregate endowment is constant in even and odd periods, therefore, the relative price and consumption for each consumer are constant in even and odd periods. From (9), we have

$$\begin{cases} p_{2t}^* = \beta^{2t} \\ p_{2t+1}^* = \beta^{2t+1} \left(\dfrac{5}{3} \right)^{-\sigma} \end{cases} \quad \forall t$$

From (5), consumption in any period is a constant fraction of aggregate endowment in that period. Therefore, we have

$$\begin{cases} c_{i,2t}^* = \mu_i (w_{1,2t} + w_{2,2t}) \\ c_{i,2t+1}^* = \mu_i (w_{1,2t+1} + w_{2,2t+1}) \end{cases} \quad \forall t$$

where μ_i is a constant fraction that depends on the endowments and the utility function parameters β and σ. Then

$$\frac{\bar{c}_{i,2t}^*}{\bar{c}_{i,2t+1}^*} = \frac{3}{5} \quad \forall i, t \tag{14}$$

From (13), we have

$$\begin{cases} \displaystyle\sum_{t=0}^{\infty} \beta^{2t} \bar{c}_{1,2t}^* + \sum_{t=0}^{\infty} \beta^{2t+1} \left(\frac{5}{3} \right)^{-\sigma} \bar{c}_{1,2t+1}^* = 2y \sum_{t=0}^{\infty} \beta^{2t} + 2y \sum_{t=0}^{\infty} \beta^{2t+1} \left(\frac{5}{3} \right)^{-\sigma} \\ \displaystyle\sum_{t=0}^{\infty} \beta^{2t} \bar{c}_{2,2t}^* + \sum_{t=0}^{\infty} \beta^{2t+1} \left(\frac{5}{3} \right)^{-\sigma} \bar{c}_{2,2t+1}^* = y \sum_{t=0}^{\infty} \beta^{2t} + 3y \sum_{t=0}^{\infty} \beta^{2t+1} \left(\frac{5}{3} \right)^{-\sigma} \end{cases} \tag{15}$$

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Plugging (14) into (15), the solution is

$$
\begin{cases}
c_{1,2t}^* = 6y\dfrac{1 + \beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\[3mm]
c_{1,2t+1}^* = 10y\dfrac{1 + \beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\[3mm]
c_{2,2t}^* = 3y\dfrac{1 + 3\beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\[3mm]
c_{2,2t+1}^* = 5y\dfrac{1 + 3\beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}}
\end{cases}
\quad \forall t
$$

6. FOC for this planner's problem

$$
\xi_i \beta^t (c_{i,t}^*)^{-\sigma} = \theta_t \quad \forall t, i
$$

where θ_t is the multiplier on the resource constraint at t. Then

$$
\left(\frac{c_{1,t}^*}{c_{2,t}^*}\right)^{-\sigma} = \frac{\xi_2}{\xi_1} \tag{16}
$$

From (3) in competitive equilibrium and (16), the solution is

$$
\begin{cases}
\xi_1 = \dfrac{\lambda_2}{\lambda_1 + \lambda_2} \\[3mm]
\xi_2 = \dfrac{\lambda_1}{\lambda_1 + \lambda_2}
\end{cases}
\tag{17}
$$

where λ_i is the multiplier on consumer i's budget constraint.
Plugging (12) into (17)

$$
\begin{cases}
\xi_1 = \dfrac{\left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{1,t}\right)^{\sigma}}{\left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{1,t}\right)^{\sigma} + \left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{2,t}\right)^{\sigma}} \\[4mm]
\xi_2 = \dfrac{\left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{2,t}\right)^{\sigma}}{\left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{i,t}\right)^{\sigma} + \left(\sum_{t=0}^{\infty} \beta^t W_t^{-\sigma} w_{i,t}\right)^{\sigma}}
\end{cases}
$$

According to the specific cases from part 5, intuitively, these weights depend on the endowments and the utility function parameters β and σ. Other things equal, higher or earlier endowment and smoother endowment will imply a higher weight.