# Recursive Competitive Equilibrium

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We have seen that recursive representations of maximization problems can be useful both computationally and in terms of building intuition. Here, we want to introduce the recursive approach as applied to equilibrium setups and define a *recursive competitive equilibrium* (RCE). Our focus will be on a recursive representation of the production economy with a representative agent. Clearly the date-0 trade equilibrium does not lend itself to thinking recursively since everything takes place at date zero. So we'll start from the sequential trade equilibrium and re-formulate everything in recursive terms.

We begin by discussing the main ideas under the assumption of inelastic labor supply ($n_t = 1$). The extension to endogenous labor supply is provided in the last section.

## 1 Exogenous labor supply

In the competitive equilibria we have been discussing, households treat prices as exogenous to them, but then prices are endogenous to the model due to the requirement that markets clear. In order to build some intuition, it helps to start by focusing on the household problem where prices are taken as exogenous. You can think of this as a partial equilibrium version of the model, where the requirement of market clearing is dropped and prices are completely exogenous. Once this is understood, we can bring back market clearing and close the model.

So suppose that the household faces exogenous sequences of prices $\{w_t^*, r_t^*\}_{t=0}^{\infty}$. The household problem in sequence form is

$$\max_{\{c_t, k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

s.t.

$$
\begin{aligned}
c_t + k_{t+1} - (1-\delta)k_t &= w_t^* + r_t^* k_t \\
c_t &\geq 0,\ k_{t+1} \geq 0 \\
& k_0 \text{ given} \\
& \{w_t^*, r_t^*\}_{t=0}^{\infty} \text{ given}
\end{aligned}
$$

If the prices were fixed over time, i.e. $w_t^* = \bar{w}$ and $r_t^* = \bar{r}$ for all $t$, then we could write the household's problem recursively as

$$V(k) = \max_{k',c} \{u(c) + \beta V(k')\}$$

s.t.

$$
\begin{aligned}
c + k' - (1-\delta)k &= \bar{w} + \bar{r}k \\
c &\geq 0, k' \geq 0 \\
& k \text{ given}
\end{aligned}
$$

because we could treat $\bar{w}, \bar{r}$ as non-varying parameters. The only state variable would be the inherited level of capital $k$.

What if the prices vary over time? In that case, the household would need to keep track of the evolution of prices. That is, prices will need to be state variables. The household needs to know the current level of the prices $w, r$ in order to know their current level of income. As soon as $w, r$ are added as arguments to the value function on the LHS, it should be clear that next period's prices, denoted by $w', r'$, would need to be added as arguments to the value function on the RHS. In order for the maximization problem to be well-defined, we would then need to endow the agent with an understanding of how $w'$, $r'$ arise. Our assumption of a given time path for $w_t$, $r_t$ doesn't provide enough structure to do this in a recursive manner. So let us assume a Markovian structure for the exogenous price processes. For the sake of exposition, we'll make the simplifying assumption that prices evolve according to an exogenous law of motion that determines current prices as a function of last period's prices only. That is, assume that $w_t = f^w(w_{t-1}, r_{t-1})$ and $r_t = f^r(w_{t-1}, r_{t-1}.)$, for some exogenously given functions $f^w(.)$ and $f^r(.)$. We can now write the Bellman equation for the household as follows

$$V(k, w, r) = \max_{k',c} \{u(c) + \beta V(k', w', r')\}$$

s.t.

$$
\begin{aligned}
c + k' - (1-\delta)k &= w + rk \\
c \geq 0, k' &\geq 0 \\
r' &= f^r(w, r) \\
w' &= f^w(w, r) \\
k, w, r &\text{ given}
\end{aligned}
$$

Here, $k$ is an endogenous state variable and $w, r$ are exogenous state variables. Households do not choose or affect prices, but they do need to know the current level of prices $w, r$ as well as have a belief regarding how current state variables will induce future prices (this is provided by the functions $f^w$ and $f^r$). This problem will yield a policy function for $k' = g^k(k, w, r)$, which can be thought of as the individual household's supply of capital for tomorrow as a function of the states, i.e. current capital and prices.

The important takeaway message is that time varying prices need to be state variables for the household *and* households need to have some perceptions about how future prices arise given current prices.

Now let us consider the general equilibrium version of our economy, where factor prices are endogenous and determined by market clearing. It is still the case that prices are exogenous *from the point of view of the households*. Households cannot affect prices, but they will need to know current prices and they need to be able to make forecasts about them. From the sequential version of the problem, we know that the firm will choose aggregate capital $K_t$ so that factor prices are equal to marginal products

$$
\begin{aligned}
w_t &= F_2(K_t, 1) \\
r_t &= F_1(K_t, 1)
\end{aligned}
$$

This gives us an idea regarding the dynamic evolution of prices: because all that matters for factor prices is the aggregate capital, all that the household needs in order to know current prices is knowledge of the current level of aggregate capital. In timeless notation

$$
w = w(K) \equiv F_2(K, 1) \tag{1}
$$

$$
r = r(K) \equiv F_1(K, 1) \tag{2}
$$

where $K$ is inherited aggregate capital. In addition, all that is needed for the household to make forecasts about future prices is a forecast of future aggregate capital $K'$.

Notice that we have re-introduced the distinction between individual capital $k$ and aggregate capital $K$, the big-$K$, small-$k$ notation. This notation is crucial here because we want the household to understand how the existing aggregate capital affects prices but we certainly don't want the household to think it can affect those prices by choosing $k$. So it is important to maintain that distinction, even though with the assumption of a continuum of identical households it will ultimately be the case that $k = K$.

To write the household problem recursively, we will therefore need to use two state variables: the household's own wealth level $k$, which is needed to know the current level of wealth, *and* the aggregate capital $K$, which is needed in order to infer the current prices using $(1) - (2)$. In addition, the household will need to have some belief about how the current state of the aggregate economy $K$ will map into a future aggregate capital $K'$ (in order to infer future prices). Let us denote this belief about how current $K$ induces $K'$ using a function $G^B(.)$

$$K' = G^B(K)$$

The household's problem written recursively is then

$$V(k, K) = \max_{k',c} \{u(c) + \beta V(k', K')\} \qquad \text{(HH Bellman)}$$

s.t.

$$\begin{aligned} c + k' - (1-\delta)k &= w(K) + r(K)k \\ c &\geq 0, k' \geq 0 \\ K' &= G^B(K) \\ k, K &\text{ given} \end{aligned}$$

The maximization problem makes clear the reason why distinguishing between $K$ and $k$ is important. The household can choose $k$ but it operates in a competitive environment and cannot affect prices because it cannot choose $K$ (even though in equilibrium the two will be equalized). At this point, we have left the belief function $G^B$ unspecified and it is, indeed, possible to define a recursive competitive equilibrium for any *arbitrary, exogenously given* belief $G^B(.)$.

Definition (for arbitrary beliefs): A Recursive Competitive Equilibrium given beliefs $G^B(.)$ is a set of functions: a value function $V(k, K)$ and associated policy functions for the representative household $g^k(k, K), g^c(k, K)$, price functions $w(K)$ and $r(K)$ and a function $G(K)$ describing the actual law of motion of capital such that

1. Given the functions $G^B(.)$, $w(.)$ and $r(.)$, $V(k,K)$ solves the household problem in $(HHBellman)$ and $g^k(k,K), g^c(k,K)$ are the associated policy functions

2. $r(K)$ and $w(K)$ are given by

$$w(K) = F_2(K,1)$$
$$r(K) = F_1(K,1)$$

3. The function $G(.)$ arises from aggregating the individual capital policy functions (and noting $k = K$)

$$G(K) = g^k(K,K)$$

4. The goods market clears for all $K$

$$g^c(K,K) + g^k(K,K) - (1-\delta)K = F(K,1)$$

Notice the distinction between the function $G(.)$ describing the actual evolution of capital and the (different) function $G^B(.)$ describing the perceived evolution of capital according to households. Given any $K$, household think that $K'$ will be $G^B(K)$. They use this perception to choose their own individual $k' = g^k(k,K)$. If we aggregate across households their choices of $k'$ we will obtain the actual $K'$. Because of the representative agent assumption all of the agents have the same $k$ and because of the continuum of identical agents trick they all have the average/aggregate level $K$, i.e. $k = K$. So when we integrate the $g^k(k,K)$ across all individuals $(k)$ we are integrating a constant and hence the integral is simply $g^k(K,K)$. This is what condition 3 says: the actual evolution of capital is given by a function of $K$ which is given by $G(K) = g^k(K,K)$.

This equilibrium setup is somewhat problematic in the sense that it allows households to be making systematically wrong predictions about the evolution of capital. Assuming households can have wild beliefs $G^B(K)$ will, in general, imply that we can support a wide range of equilibria purely on these exogenously assumed beliefs. We would prefer these beliefs to be endogenous and we'll do that by adding the requirement of consistency. Consistency requires that households do not make systematic errors in their beliefs/expectations. If we add the consistency condition $G = G^B$, ensuring that the household beliefs about how capital evolves is consistent with the actual evolution of capital resulting from the aggregation of household choices, we will obtain the notion of equilibrium we are really interested in. In this case, rather than being arbitrary, the belief function will be determined endogenously to coincide with the equilibrium evolution of capital.

Definition (Rational Expectations): A Recursive Competitive Equilibrium (RCE) is a set of functions: a value function $V(k,K)$ and associated policy functions for the representative household $g^k(k,K), g^c(k,K)$, price functions $w(K)$ and $r(K)$ and a function $G(K)$ describing the law of motion of aggregate capital such that

1. Given functions $G(.)$, $w(.)$ and $r(.)$, $V(k,K)$ solves the household problem in $(HHBellman)$[1] and $g^k(k,K), g^c(k,K)$ are the associated policy functions

2. $r(K)$ and $w(K)$ are given by

$$w(K) = F_2(K,1)$$
$$r(K) = F_1(K,1)$$

3. Consistency

$$G(K) = g^k(K,K)$$

4. Market Clearing for goods[2]

$$g_c(K,K) + g_k(K,K) - (1-\delta)K = F(K,1)$$

The difference here is that households assume $K'$ is generated by the correct equilibrium function $G$ when making their choices. This is, of course, circular in some sense: households need to know $G$ to solve their problem but to know $G$ we need to first solve all the household problems and aggregate them. Computationally this calls for an iterative scheme where some $G$ is guessed, the household problems are solved, the new, implied $G$ is obtained by aggregation and then we need to update the guess and keep iterating until a fixed point is reached.

## 1.1 Endogenous labor supply

In the previous section, we simplified the environment by assuming households do not value leisure. This allowed to us to impose $n_t = 1$ to start with in the household problem as well as to impose aggregate labor demand equal to 1 by invoking labor market clearing. Here, we provide

[1] With $G^B$ replaced by $G$.

[2] This condition will be satisfied automatically by Walras' Law (since we have already implicitly imposed capital and labor market clearing), and is therefore sometimes omitted from the RCE definition.

a recursive competitive equilibrium definition for the case of endogenous labor supply. We only focus on the rational expectations equilibrium.

Let $g_n(k,K)$ denote the policy function for household labor supply. Because of the assumption of a continuum of identical households of measure one, $g_n(k,K)$ also captures aggregate labor supply. In equilibrium we'll have $k = K$, which implies that aggregate labor supply will only depend on aggregate capital $K$. Therefore it is still true that the prices are functions of aggregate capital

$$\begin{aligned} w &= w(K) = F_2(K, g_n(K,K)) \\ r &= r(K) = F_1(K, g_n(K,K)) \end{aligned}$$

The household problem is now

$$V(k,K) = \max_{k',c,n} \{u(c, 1-n) + \beta V(k', K')\} \qquad \text{(HH Bellman Labor)}$$

s.t.

$$\begin{aligned} c + k' - (1-\delta)k &= w(K)n + r(K)k \\ c &\geq 0, k' \geq 0, 0 \leq n \leq 1 \\ K' &= G(K) \\ k, K &\text{ given} \end{aligned}$$

and the recursive competitive equilibrium definition is as follows.

Definition: A Recursive Competitive Equilibrium is a set of functions, value functions and policy functions for individuals $V(k,K)$ and $g_k(k,K)$, $g_c(k,K)$ and $g_n(k,K)$, price functions $w(K)$ and $r(K)$ and a function $G(K)$ describing the aggregate evolution of capital such that

1. Given the functions $G(K)$, $w(K)$ and $r(K)$, $V(k,K)$ solves the household problem in $(HHBellmanLabor)$ and $g_k(k,K)$, $g_c(k,K)$ and $g_n(k,K)$ are the associated policy functions.

2. $r(K)$ and $w(K)$ are given by

$$\begin{aligned} w(K) &= F_2(K, g_n(K,K)) \\ r(K) &= F_1(K, g_n(K,K)) \end{aligned}$$

3. Consistency

$$G(K) = g_k(K,K)$$

4. The goods market clears

$$g_c(K,K) + g_k(K,K) - (1-\delta)K = F(K, g_n(K,K))$$

As always, the capital and labor markets also should clear, but we have already implicitly imposed those in the above formulation.