ECO 510 - Fall 2020 Midterm Exam

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1. (a) The planner's maximization problem

$$\max_{\{c_t,n_t,k_{t+1},h_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [\log c_t + B\log(1-n_t)]$$

$$\begin{aligned} s.t. \qquad y_t &= c_t + i_t^k + i_t^h \\ y_t &= Ak_t^{\alpha}(h_t n_t)^{1-\alpha} \\ k_{t+1} &= (1-\delta)k_t + i_t^k \\ h_{t+1} &= (1-\delta)h_t + i_t^h \\ c_t &\geq 0 \\ 0 \leq n_t &\leq 1 \\ k_{t+1} &\geq 0 \\ h_{t+1} &\geq 0 \\ k_0, h_0 & \ \ given \end{aligned}$$

Rewriting the equality constraints

$$c_t + k_{t+1} + h_{t+1} - (1-\delta)(k_t + h_t) = Ak_t^{\alpha}(h_t n_t)^{1-\alpha} \quad \forall t$$

Assume that inequality constraints never bind, i.e.$c_t > 0, 0 < n_t < 1, k_{t+1} > 0, h_{t+1} > 0 \quad \forall t$.
Lagrangian function

$$L = \sum_{t=0}^{\infty} \beta^t [\log c_t + B\log(1-n_t)] - \lambda_t [c_t + k_{t+1} + h_{t+1} - (1-\delta)(k_t+h_t) - Ak_t^{\alpha}(h_t n_t)^{1-\alpha}]$$

Necessary conditions
Equality constraint

$$c_t + k_{t+1} + h_{t+1} - (1-\delta)(k_t + h_t) = Ak_t^{\alpha}(h_t n_t)^{1-\alpha} \quad \forall t$$

FOC, $\forall t$

$$\frac{\beta^t}{c_t} - \lambda_t = 0 \tag{1}$$

$$-\frac{B\beta^t}{1-n_t} + A(1-\alpha)\lambda_t(k_t^\alpha(h_t)^{1-\alpha}n_t^{-\alpha}) = 0 \tag{2}$$

$$-\lambda_t + \lambda_{t+1}[A\alpha k_{t+1}^{\alpha-1}(h_{t+1}n_{t+1})^{1-\alpha} + (1-\delta)] = 0 \tag{3}$$

$$-\lambda_t + \lambda_{t+1}[A(1-\alpha)k_{t+1}^{\alpha}(n_{t+1})^{1-\alpha}h_{t+1}^{-\alpha} + (1-\delta)] = 0 \tag{4}$$

TVC

$$\lim_{T\to\infty} \lambda_T k_{T+1} = 0$$
$$\lim_{T\to\infty} \lambda_T h_{T+1} = 0$$

(b) Plugging (1) into (2),(3) and (4)

$$\frac{B}{1-n_t} = A(1-\alpha)(k_t^\alpha(h_t)^{1-\alpha}n_t^{-\alpha}) \tag{n_t}$$

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}[A\alpha k_{t+1}^{\alpha-1}(h_{t+1}n_{t+1})^{1-\alpha} + (1-\delta)] \tag{k_{t+1}}$$

$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}[A(1-\alpha)k_{t+1}^{\alpha}(n_{t+1})^{1-\alpha}h_{t+1}^{-\alpha} + (1-\delta)] \tag{h_{t+1}}$$

All the LHSs and RHSs above are cost and benefit respectively.
In specific, for FOC of (n_t) ,a marginal increase in production needs marginal increase in labor n_t and decrease in leisure $(1-n_t)$.
For FOC of (k_{t+1}) ,a marginal increase in k_{t+1} needs marginal increase in i_t^k and decrease in consumption c_t.
For FOC of (h_{t+1}) ,a marginal increase in h_{t+1} needs marginal increase in i_t^h and decrease in consumption c_t.

(c) Assume that $n_t = n$, the constant growth rates of all variables at BGP is g. From FOC of (k_{t+1}) and (h_{t+1})

$$\begin{aligned} \alpha h_{t+1} =& (1-\alpha)k_{t+1} \\ g_c =& \beta\left[A\alpha\left(\frac{\alpha}{1-\alpha}\right)^{\alpha-1} n_{t+1}^{1-\alpha} + (1-\delta)\right] \end{aligned}$$

Then we can conclude that $\frac{k_{t+1}}{h_{t+1}}$ is a constant, which implies $g_h = g_k$.
From capital accumulation equation $k_{t+1} = (1-\delta)k_t + i_t^k$ and $h_{t+1} = (1-\delta)h_t + i_t^h$, we have $g_k = g_{i^k}$ and $g_h = g_{i^h}$.
Since production function $y_t = Ak_t^\alpha(h_t n_t)^{1-\alpha}$ is homogeneous of degree 1 and $g_n = 1$, we have $g_y = g_k = g_h$.
After that, from resource constraint $y_t = c_t + i_t^k + i_t^h$, we have $g_c = g_y = g_{i^k} = g_{i^h}$.
Finally, we have $g_h = g_k = g_c = g_y = g_{i^k} = g_{i^h} = g, g_n = 1$, which is consistent with optimality and feasibility at BGP.

(d) Kaldor's facts

i. Real GDP per capita grows at a constant rate.
This is not true because $g_y = g_k \Rightarrow \frac{y}{k}$ is a constant.

ii. Capital to labor ratio grows at a constant rate.
This is not true because $g_k = g_h, g_n = 1 \Rightarrow \frac{k}{hn}$ is a constant.

iii. Capital to output ratio is constant.
This is true because $g_k = g_y \Rightarrow \frac{k}{y}$ is a constant.

iv. Real rates of reture are constant.
Let $y_t = F(k_t, h_t n_t)$. Since F is homogeneous of degree 1, F_1 and F_2 are homogeneous of degree 0. Therefore $r_t = F_1$ and $w_t = F_2$ are constant.

v. Capital and labor shares of total income are constant.
Since $g_k = g_y, r$ is constant, capital shares of total income $\frac{rk}{y}$ is constant.
Since $g_k = g_y, g_n = 1, w$ is constant, labor shares of total income $\frac{whn}{y}$ is constant.

vi. Growth rates vary persistently across countries.
This fact cannot be verified because other countries may have different macro models in terms of untility functions and constraints.

(e) Denote by k, h physical and human capital in current period respectively. Denote by k', h' physical and human capital in the next period respectively. Denote by c consumption in current period. Denote by n the fraction of time spent working.
Bellman equation

$$V(k,h) = \max_{c,n,k',h'} [\log c + B\log(1-n) + \beta V(k',h')]$$

$$\begin{aligned} s.t. \quad c + k' + h' - (1-\delta)(k+h) &= Ak^\alpha (hn)^{1-\alpha} \\ c &\geq 0 \\ n &\geq 0 \\ k' &\geq 0 \\ h' &\geq 0 \\ k, h \ \ &given \end{aligned}$$

where state variables are k, h and choice variables are c, n, k', h'.

2. (a) The farmer's maximization problem

$$\max_{\{b_t, w_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (b_t w_t)^{\frac{1}{2}}$$

$$\begin{aligned} s.t. \quad b_t + w_{t+1} &= T = 1 \\ b_t &\geq 0 \\ w_{t+1} &\geq 0 \\ w_0 \ \ &given \end{aligned}$$

Non-negativity constraints never bind because the utility function satisfy the Inada Condition. Lagrangian function

$$L = \beta^t (b_t w_t)^{\frac{1}{2}} - \lambda_t (b_t + w_{t+1} - 1)$$

Equality constraint

$$b_t + w_{t+1} = 1 \quad \forall t$$

FOC, $\forall t$

$$\frac{1}{2}\beta^t (w_t)^{\frac{1}{2}} b_t^{-\frac{1}{2}} - \lambda_t = 0$$
$$-\lambda_t + \frac{1}{2}\beta^{t+1} (b_{t+1})^{\frac{1}{2}} w_{t+1}^{-\frac{1}{2}} = 0$$

Euler equation

$$\frac{w_t}{1 - w_{t+1}} = \beta^2 \frac{1 - w_{t+2}}{w_{t+1}}$$

Intutively, the farmer will save wine for considering two period later. In addition $\frac{b_{t+1}}{w_{t+1}}$ grows at a constant rate$\frac{1}{\beta^2}$.

(b) Denote by b, w consumption of bread and wine in current period respectively. Denote by w' consumption of wine in the next period. Bellman equation

$$V(w) = \max_{b, w'} [(bw)^{\frac{1}{2}} + \beta V(w')]$$

$$\begin{aligned} s.t. \quad b + w' &= 1 \\ b &\geq 0 \\ w' &\geq 0 \\ w \ \ &given \end{aligned}$$

where state variable is w and choice variables are b, w'.

(c) Guess $V_0 = 0$.

$$
\begin{aligned}
&V_1(w) = \max[(1-w')w]^{\frac{1}{2}} \\
\Rightarrow &w' = 0 \equiv g_1(w) \\
\Rightarrow &V_1(w) = w^{\frac{1}{2}}
\end{aligned}
$$

Then

$$
\begin{aligned}
&V_2(w) = \max_{w'} \left\{ [(1-w')w]^{\frac{1}{2}} + \beta(w')^{\frac{1}{2}} \right\} \\
\Rightarrow &-\frac{1}{2}(1-w')^{-\frac{1}{2}} w^{\frac{1}{2}} + \frac{1}{2}\beta(w')^{-\frac{1}{2}} = 0 \\
\Rightarrow &w' = \frac{\beta^2}{w+\beta^2} \equiv g_2(w) \\
\Rightarrow &V_2(w) = (w+\beta^2)^{\frac{1}{2}}
\end{aligned}
$$

After that, a more general guess

$$
V(w) = \alpha(w+\gamma)^{\frac{1}{2}}
$$

Verify

$$
V(w) = \max_{w'} \left\{ [(1-w')w]^{\frac{1}{2}} + \beta\alpha(w'+\gamma)^{\frac{1}{2}} \right\} \tag{5}
$$

FOC

$$
\begin{aligned}
&-\frac{1}{2}(1-w')^{-\frac{1}{2}} w^{\frac{1}{2}} + \frac{1}{2}\alpha\beta(w'+\gamma)^{-\frac{1}{2}} = 0 \\
\Rightarrow &\frac{w}{1-w'} = \frac{\alpha^2\beta^2}{w'+\gamma} \\
\Rightarrow &w' = \frac{\alpha^2\beta^2 - \gamma w}{w' + \alpha^2\beta^2}
\end{aligned}
$$

Plugging w' into (5)

$$
\begin{aligned}
\alpha(w+\gamma)^{\frac{1}{2}} &= \left(1 - \frac{\alpha^2\beta^2 - \gamma w}{w' + \alpha^2\beta^2}\right)^{\frac{1}{2}} w^{\frac{1}{2}} + \alpha\beta\left(\frac{\alpha^2\beta^2 - \gamma w}{w' + \alpha^2\beta^2} + \gamma\right)^{\frac{1}{2}} \\
&= \frac{(1+\gamma)^{\frac{1}{2}} w}{(w+\alpha^2\beta^2)^{\frac{1}{2}}} + \frac{(1+\gamma)^{\frac{1}{2}}\alpha^2\beta^2}{(w+\alpha^2\beta^2)^{\frac{1}{2}}} \\
&= (1+\gamma)^{\frac{1}{2}}(w+\alpha^2\beta^2)^{\frac{1}{2}}
\end{aligned}
$$

Equating coefficients

$$
\begin{cases}
\alpha = (1+\gamma)^{\frac{1}{2}} \\
\gamma = \alpha^2\beta^2
\end{cases}
$$

$$
\Rightarrow \begin{cases}
\alpha = \dfrac{1}{(1-\beta^2)^{\frac{1}{2}}} \\
\gamma = \dfrac{\beta^2}{1-\beta^2}
\end{cases}
$$

Plugging α, γ into w'

$$w' = \frac{\beta^2(1-w)}{w + \beta^2(1-w)}$$

Plugging α, γ into (5)

$$V(w) = \frac{[w + \beta^2(1-w)]^{\frac{1}{2}}}{1-\beta^2}$$