Assignment 9

Haixiang Zhu

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1. Period-by-period budget constraint for each consumer in period t

$$c_t + q_t b_{t+1} = b_t + w_t$$

Given a sequence of endowments $\{\{w_{i,t}\}_{t=0}^\infty\}_{i=1}^2$, a competitive equilibrium with sequential trade consists of sequences of allocations $\{\{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^\infty\}_{i=1}^2$ and a sequence of prices $\{(q_t^b)^*\}_{t=0}^\infty$ such that

   (a) Given the price system, the allocation solves each consumer's problem. For $i = 1, 2$

$$\{c_{i,t}^*, b_{i,t+1}^*\}_{t=0}^\infty = \arg \max_{\{c_{i,t}, b_{i,t+1}\}_{t=0}^\infty} \sum_{t=0}^\infty \beta^t \frac{c_{i,t}^{1-\sigma}}{1-\sigma}$$

$$s.t. \qquad c_{i,t} + (q_t^b)^* b_{i,t+1} = b_{i,t} + w_{i,t} \quad \forall t$$

$$c_{i,t} \geq 0 \qquad \forall t$$

$$b_{i,0} = 0$$

$$\lim_{T \to \infty} b_{i,T+1}^* \prod_{t=0}^T (q_t^b)^* \geq 0$$

   (b) All markets clear. For goods market

$$\sum_i c_{i,t}^* = \sum_i w_{i,t} \qquad \forall t$$

   For asset market

$$\sum_i b_{i,t+1}^* = 0 \qquad \forall t$$

2. All conditions for equilibrium (FOC+B.C+M.C+TVC+nPg) $\forall t, i$

$$\beta^t(c_{i,t}^*)^{-\sigma} = \mu_{i,t} \tag{$c_{i,t}$}$$

$$(q_t^b)^*\mu_{i,t} = \mu_{i,t+1} \tag{$b_{i,t+1}$}$$

$$c_{i,t} + (q_t^b)^*b_{i,t+1} = b_{i,t} + w_{i,t} \tag{B.C}$$

$$c_{1,t}^* + c_{2,t}^* = w_{1,t} + w_{2,t} \tag{goods}$$

$$b_{1,t+1}^* + b_{2,t+1}^* = 0 \tag{asset}$$

$$\lim_{T \to \infty} \beta^T(c_{i,T}^*)^{-\sigma}b_{i,T+1}^* \leq 0 \tag{TVC}$$

$$\lim_{T \to \infty} b_{i,T+1}^* \prod_{t=0}^{T}(q_t^b)^* \geq 0 \tag{nPg}$$

where $\mu_{i,t}$ is the multiplier on consumer i's budget constraint for each period t. The non-negativity constrants is ignored because of the Inada condition for utility function.

3. *Proof.* The equivalence between the date-0 equilibrium and the sequential equilibrium.
Recall the characterization of the date-0 trade equilibrium $\forall t, i$

$$\begin{cases} \beta^t(c_{i,t}^*)^{-\sigma} = \lambda_i p_t^* \\ \displaystyle\sum_{t=0}^{\infty} p_t^* c_{i,t} = \sum_{t=0}^{\infty} p_t^* w_{i,t} \\ \displaystyle\sum_{i} c_{i,t}^* = \sum_{i} w_{i,t} \end{cases}$$

where λ_i is the multiplier on consumer i's budget constraint.

(a) Necessity (only if)
Consider any consumption choice $\{c_{i,t}^*\}_{t=0}^{\infty}$ that is feasible in the sequential trade equilibrium. Then, by rolling forward the sequential trade budget and using TVC and nPg it can shown that it satisfies the date-0 trade budget

$$\begin{aligned} b_{i,0} &= c_{i,0}^* - w_{i,0} + (q_0^b)^* b_{i,1}^* \\ &= c_{i,0}^* - w_{i,0} + (q_0^b)^*(c_{i,1}^* - w_{i,1}) + (q_0^b)^*(q_1^b)^* b_{i,2}^* \\ &= \ldots \\ &= \sum_{t=0}^{\infty}\prod_{s=0}^{t-1}(q_s^b)^*(c_{i,t}^* - w_{i,t}) + \lim_{T \to \infty} b_{i,T+1}^* \prod_{t=0}^{T}(q_t^b)^* \end{aligned}$$

Now note that in equilibrium $\prod_{s=0}^{t-1}(q_s^b)^* = \beta^t\left(\frac{c_{i,t}^*}{c_{i,0}^*}\right)^{-\sigma}$ and this also

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corresponds to $\dfrac{p_t^*}{p_0^*}$ in the date-0 trade equilibrium.

$$b_{i,0} = \sum_{t=0}^{\infty} \frac{p_t^*(c_{i,t}^* - w_{i,t})}{p_0^*} + \lim_{T \to \infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*}\right)^{-\sigma}$$

The last term is 0 (by using the nPg and TVC conditions) and using the zero initial wealth assumption and the normalization $p_0^* = 1$ we obtain

$$0 = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t})$$

(b) Sufficiency (if)

To show the opposite statement, that if $\{c_{i,t}^*\}_{t=0}^{\infty}$ satisfies the date-0 trade budget then it is feasible in the sequential trade economy, one can proceed by constructing the asset trades required to ensure the same consumption. Since in equilibrium

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*}\right)^{-\sigma}$$

then the asset choices can be constructed recursively

$$b_{i,t+1}^* = \frac{(c_{i,t}^*)^{-\sigma}}{\beta(c_{i,t+1}^*)^{-\sigma}}(b_{i,t}^* + w_{i,t} - c_{i,t}^*) \quad \forall t$$

With these choices for assets, and given that goods' markets clear, the asset market clears in every period (simply add $b_{i,t+1}$ across agents and show it equals zero). We can also show in a manner identical to before that

$$0 = b_{i,0} = \sum_{t=0}^{\infty} p_t^*(c_{i,t}^* - w_{i,t}) + \lim_{T \to \infty} b_{i,T+1}^* \beta^T \left(\frac{c_{i,T}^*}{c_{i,0}^*}\right)^{-\sigma}$$

and since the date-0 budget constraint is satisfied for this consumption sequence, this implies that

$$\lim_{T \to \infty} b_{i,T+1}^* \beta^T (c_{i,T}^*)^{-\sigma} = 0$$

that is, the nPg and TVC conditions are satisfied.

$\square$

4. Special cases

Since date-0 equilibrium and sequential equilibrium are equivalent, from FOC

$$(q_t^b)^* = \beta \left(\frac{c_{i,t+1}^*}{c_{i,t}^*}\right)^{-\sigma} = \frac{p_{t+1}^*}{p_t^*}$$

Then
$$c_{i,t}^* + \frac{p_{t+1}^*}{p_t^*} b_{i,t+1}^* = b_{i,t}^* + w_{i,t} \tag{1}$$

(a) $w_{1,t} = 2y, w_{2,t} = y \ \forall t$

Since $c_{i,t}^* = w_{i,t} \ \forall t, i$, no trade happens in this case, which implies $b_{i,t} = 0 \ \forall t$

(b) $w_{1,t} = \{2y, y, 2y, y, \dots\}, w_{2,t} = \{y, 2y, y, 2y, \dots\} \ \forall t$

$$\begin{cases} p_t^* = \beta^t \\ c_{1,t}^* = \dfrac{2+\beta}{1+\beta} y \\ c_{2,t}^* = \dfrac{1+2\beta}{1+\beta} y \end{cases} \forall t \tag{2}$$

Plugging (2) into (1)

$$b_{1,1}^* = \frac{1}{\beta}\left(0 + 2y - \frac{2+\beta}{1+\beta}y\right) = \frac{1}{1+\beta}y = -b_{2,1}^*$$

$$b_{1,2}^* = \frac{1}{\beta}\left(b_{1,1}^* + y - \frac{1+2\beta}{1+\beta}y\right) = 0 = -b_{2,2}^*$$

$$\dots$$

Therefore,

$$\begin{cases} b_{1,2t}^* = 0 \\ b_{1,2t+1}^* = \dfrac{1}{1+\beta}y \\ b_{2,2t}^* = 0 \\ b_{2,2t+1}^* = -\dfrac{1}{1+\beta}y \end{cases} \forall t$$

(c) $w_{1,t} = 2y, w_{2,t} = \{y, 3y, y, 3y, \dots\} \ \forall t$

$$\begin{cases} p_{2t}^* = \beta^{2t} \\ p_{2t+1}^* = \beta^{2t+1}\left(\dfrac{5}{3}\right)^{-\sigma} \\ c_{1,2t}^* = 6y\dfrac{1 + \beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\ c_{1,2t+1}^* = 10y\dfrac{1 + \beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\ c_{2,2t}^* = 3y\dfrac{1 + 3\beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\ c_{2,2t+1}^* = 5y\dfrac{1 + 3\beta\left(\frac{5}{3}\right)^{-\sigma}}{3 + 5\beta\left(\frac{5}{3}\right)^{-\sigma}} \end{cases} \forall t \tag{3}$$

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Plugging (3) into (1)

$$b_{1,1}^* = \frac{1}{\beta}\left(\frac{5}{3}\right)^\sigma\left(0+2y-6y\frac{1+\beta\left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}}\right) = \frac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}} = -b_{2,1}^*$$

$$b_{1,2}^* = \frac{1}{\beta}\left(\frac{5}{3}\right)^{-\sigma}\left(b_{1,1}^*+2y-10y\frac{1+\beta\left(\frac{5}{3}\right)^{-\sigma}}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}}\right) = 0 = -b_{2,2}^*$$

$\cdots$

Therefore,

$$\begin{cases} b_{1,2t}^* = 0 \\ b_{1,2t+1}^* = \dfrac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}} \\ b_{2,2t}^* = 0 \\ b_{2,2t+1}^* = -\dfrac{4y}{3+5\beta\left(\frac{5}{3}\right)^{-\sigma}} \end{cases} \quad \forall t$$

5. c, b, q, w donote the comsumption, bonds agent bought, bond price and exogenous endowment in current period respectively.
q', w' denote the bond price and endowment in the next period respectively.
b^{-1} donotes the amount of bond agent bought in the last period.
Bellman Equation is

$$V(b^{-1}, w, q) = \max_{c,b}\left\{\frac{c^{1-\sigma}}{1-\sigma} + \beta V(b, w', q')\right\}$$

$$s.t. \quad c+qb = b^{-1}+w$$

$$c \geq 0$$

$$q' = f^q(q)$$

$$w' = f^w(w)$$

$$\text{nPg holds}$$

$$b^{-1}, w \text{ given}$$

where state variables are b^{-1}, w, q and control variables are c, b.