Assignment 10

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1. (a) Homogeneous HH

i. Representative HH

Given initial allocations of capital and assets, k_0 and b_{-1}, a competitive equilibrium with sequential trade consists of sequences of allocations $\{c_t^*, b_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty}$ and sequences of prices $\{(q_t^b)^*, r_t^*, w_t^*\}_{t=0}^{\infty}$ such that

1) Given prices, allocations are optimal for the household

$$\{c_t^*, b_t^*, k_{t+1}^*, n_t^*\}_{t=0}^{\infty} = \underset{\{c_t, b_t, k_{t+1}, n_t\}_{t=0}^{\infty}}{\arg\max} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t)$$

$$\begin{aligned}
s.t. \quad c_t + (q_t^b)^* b_t + k_{t+1} - (1-\delta)k_t &= b_{t-1} + w_t^* n_t + r_t^* k_t && \forall t \\
c_t &\geq 0 && \forall t \\
k_{t+1} &\geq 0 && \forall t \\
0 \leq n_t &\leq 1 && \forall t \\
b_{-1} &= 0 \\
\lim_{T\to\infty} b_T^* \prod_{t=0}^{T} (q_t^b)^* &\geq 0 \\
k_0 & \text{ given}
\end{aligned}$$

2) Given prices, allocations are optimal for the firm

$$\{k_t^*, n_t^*\} = \underset{k_t, n_t}{\arg\max}\, F(k_t, n_t) - w_t^* n_t - r_t^* k_t \quad \forall t$$

3) All markets clear. For goods market

$$c_t^* + k_{t+1}^* - (1-\delta)k_t^* = F(k_t^*, n_t^*) \qquad \forall t$$

For asset market

$$b_t^* = 0 \qquad \forall t$$

ii. *Proof.* If $k_{t+1} = 0$, then return of capital $r^*_{t+1} = F_k(k_{t+1}, n_{t+1})$ goes to infinity because of the Inada condition of production function. So it is never optimal for HH to choose $k_{t+1} = 0$, i.e. non-negativity of capital will not bind in equilibrium. □

iii. FOC

$$\beta^t u_c(c_t, 1 - n_t) = \lambda_t \tag{c_t}$$

$$(q^b_t)^* \lambda_t = \lambda_{t+1} \tag{b_t}$$

$$\beta^t u_n(c_t, 1 - n_t) = -w^*_t \lambda_t \tag{n_t}$$

$$\lambda_t = (1 - \delta + r^*_{t+1})\lambda_{t+1} \tag{k_{t+1}}$$

where λ_t is the multiplier on HH's budget constraint for each period t.

From FOC of $(c_t), (b_t)$ and (k_{t+1})

$$(q^b_t)^* = \frac{\beta u_c(c_{t+1}, 1 - n_{t+1})}{u_c(c_t, 1 - n_t)} = \frac{1}{1 - \delta + r^*_{t+1}} \tag{1}$$

Therefore, the bond price is inverse to the return on capital.

Incorporating date-0 CE

$$(q^b_t)^* = \frac{1}{1 - \delta + r^*_{t+1}} = \frac{\beta u_c(c_{t+1}, 1 - n_{t+1})}{u_c(c_t, 1 - n_t)} = \frac{p^*_{t+1}}{p^*_t}$$

iv. From (1), we have proved that in the perfect competitive market, we only need one asset to achieve optimum, all the other assets are redundant, and the redundant assets can be priced using the single asset. Thus, the effect of introducing additional assets is indifferent.

(b) Heterogeneous HH

i. Given initial allocations of capital and assets, $k_{i,0}$ and $b_{i,-1}$, a competitive equilibrium with sequential trade consists of sequences of prices $\{(q^b_t)^*, r^*_t, w^*_t\}^\infty_{t=0}$, HH's allocations $\{\{c^*_{i,t}, b^*_{i,t}, k^*_{i,t+1}, n^*_{i,t}\}^\infty_{t=0}\}^2_{i=1}$ and firm's choices $\{K^*_t, N^*_t\}^\infty_{t=0}$ such that

1) Given prices, allocations are optimal for the household. For

$i = 1, 2$

$$
\begin{aligned}
\{c^*_{i,t}, b^*_{i,t}, k^*_{i,t+1}, n^*_{i,t}\}_{t=0}^{\infty} = &\underset{\{c_{i,t}, b_{i,t}, k_{i,t+1}, n_{i,t}\}_{t=0}^{\infty}}{\arg\max} \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, 1 - n_{i,t}) \\
s.t. \quad c_{i,t} + (q^b_t)^* b_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t} &= b_{i,t-1} + w^*_t n_{i,t} + r^*_t k_{i,t} \quad \forall t \\
c_{i,t} &\geq 0 \quad \forall t \\
k_{i,t+1} &\geq 0 \quad \forall t \\
0 \leq n_{i,t} &\leq 1 \quad \forall t \\
b_{i,-1} &= 0 \\
\lim_{T \to \infty} b^*_{i,T} \prod_{t=0}^{T} (q^b_t)^* &\geq 0 \\
k_{i,0} &\text{ given}
\end{aligned}
$$

2) Given prices, allocations are optimal for the firm

$$
\{K^*_t, N^*_t\} = \underset{K_t, N_t}{\arg\max}\, F(K_t, N_t) - w^*_t N_t - r^*_t K_t \quad \forall t
$$

3) All markets clear. For goods market

$$
\sum_i [c^*_{i,t} + k^*_{i,t+1} - (1-\delta)k^*_{i,t}] = F(K^*_t, N^*_t) \qquad \forall t
$$

For asset market

$$
\sum_i b^*_{i,t} = 0 \qquad \forall t
$$

For labor market

$$
\sum_i n^*_{i,t} = N^*_t \qquad \forall t
$$

For capital market

$$
\sum_i k^*_{i,t} = K^*_t \qquad \forall t
$$

ii. *Proof.* If $k_{1,t+1} = k_{2,t+1} = 0$, then $k_{t+1} = 0$. Return of capital $r^*_{t+1} = F_k(k_{t+1}, n_{t+1})$ goes to infinity because of the Inada condition of production function. So it is never optimal for HH to choose $k_{1,t+1} = k_{2,t+1} = 0$.
Let$(r^b_{t+1})^*$be the return of bond. In equilibrium, no-arbitrage condition

$$
r^*_{t+1} + 1 - \delta = (r^b_{t+1})^* \qquad \forall t
$$

which implies the capital multiplier $\nu_{i,t} = 0$, i.e. non-negativity of capital will not bind for one of them only. $\square$

This would not be true if there were no financial asset available because heterogeneous HH would smooth their consumption via bond market in equilibrium. Then the best each HH could do is $k_{i,t+1} = 0$.

iii. Given initial allocations of capital holdings $k_{i,0}$, a competitive equilibrium with date-0 trade consists of sequences of prices $\{p_t^*, r_t^*, w_t^*\}_{t=0}^{\infty}$, HH's allocations $\{\{c_{i,t}^*, k_{i,t+1}^*, n_{i,t}^*\}_{t=0}^{\infty}\}_{i=1}^{2}$ and firm's choices $\{K_t^*, N_t^*\}_{t=0}^{\infty}$ such that

1) Given prices, allocations are optimal for the household. For $i = 1, 2$

$$\{c_{i,t}^*, k_{i,t+1}^*, n_{i,t}^*\}_{t=0}^{\infty} = \underset{\{c_{i,t}, k_{i,t+1}, n_{i,t}\}_{t=0}^{\infty}}{\arg\max} \sum_{t=0}^{\infty} \beta^t u(c_{i,t}, 1 - n_{i,t})$$

$$\begin{aligned} s.t. \quad \sum_{t=0}^{\infty} p_t^*[c_{i,t} + k_{i,t+1} - (1-\delta)k_{i,t}] &= \sum_{t=0}^{\infty} p_t^*[w_t^* n_{i,t} + r_t^* k_{i,t}] \\ c_{i,t} &\geq 0 \qquad \forall t \\ k_{i,t+1} &\geq 0 \qquad \forall t \\ 0 \leq n_{i,t} &\leq 1 \qquad \forall t \\ k_{i,0} &\text{ given} \end{aligned}$$

2) Given prices, allocations are optimal for the firm

$$\{K_t^*, N_t^*\} = \underset{K_t, N_t}{\arg\max}\, F(K_t, N_t) - w_t^* N_t - r_t^* K_t \quad \forall t$$

3) All markets clear.

$$\begin{aligned} \sum_i [c_{i,t}^* + k_{i,t+1}^* - (1-\delta)k_{i,t}^*] &= F(K_t^*, N_t^*) \qquad \forall t \\ \sum_i n_{i,t}^* &= N_t^* \qquad \forall t \\ \sum_i k_{i,t}^* &= K_t^* \qquad \forall t \end{aligned}$$

iv. *Proof.* Equilibrium conditions $\forall t, i$

$$\begin{aligned} \beta^t u_c(c_{i,t}, 1 - n_{i,t}) &= \mu_i p_t^* && (c_{i,t}) \\ p_{t+1}^*(1 - \delta + r_{t+1}^*) &= p_t^* && (k_{i,t+1}) \\ \beta^t u_n(c_{i,t}, 1 - n_{i,t}) + \mu_i p_t^* w_t^* &= 0 && (n_{i,t}) \\ \lim_{T\to\infty} \beta^T u_c(c_{i,T}, 1 - n_{i,T}) k_{i,T+1} &= 0 && \text{(TVC)} \end{aligned}$$

where μ_i is the multiplier on HH i's budget constraint.
Assume that $p_0^* = 1$. From FOC of $(c_{i,t})$

$$p_t^* = \beta^t \frac{u_{c_{i,t}}}{u_{c_{i,0}}} \tag{2}$$

Rewrite budget constraint

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t^* c_{i,t} &= \sum_{t=0}^{\infty} p_t^* w_t^* n_{i,t} + \sum_{t=0}^{\infty} p_t^* [(1-\delta+r_t^*)k_{i,t} - k_{i,t+1}] \\
&= \sum_{t=0}^{\infty} p_t^* w_t^* n_{i,t} + (1-\delta+r_0^*)k_{i,0} - p_0^* k_{i,1} + p_0^* k_{i,1} - \\
&\quad \cdots + \frac{1}{\mu_i} \lim_{T\to\infty} \beta^T u_c(c_{i,T}, 1-n_{i,T}) k_{i,T+1} \\
&\qquad \text{(using FOC of } (k_{i,t+1}), (c_{i,t}))
\end{aligned}$$

Substituting p_t^* with (2) and using (TVC) condition, we obtain

$$\sum_{t=0}^{\infty} \beta^t \frac{u_{c_{i,t}}}{u_{c_{i,0}}} c_{i,t} = \sum_{t=0}^{\infty} \beta^t \frac{u_{c_{i,t}}}{u_{c_{i,0}}} w_t^* n_{i,t} + (1-\delta+r_0^*)k_{i,0} \tag{3}$$

□

v. *Proof.* if $\{c_t^*, n_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ satisfies the date-0 trade budget then it is feasible in the sequential trade economy. Since in equilibrium

$$(q_t^b)^* = \beta \frac{u_{c_{i,t+1}^*}}{u_{c_{i,t}^*}}$$

then the asset choices can be constructed recursively

$$b_{i,t}^* = \frac{u_{c_{i,t}^*}}{\beta u_{c_{i,t+1}^*}} [b_{i,t-1}^* + w_t^* n_{i,t}^* + (1-\delta+r_t^*)k_{i,t}^* - k_{i,t+1}^* - c_{i,t}^*] \quad \forall t$$

Together with $b_{1,t} + b_{2,t} = 0$

$$0 = b_{i,-1} = \sum_{t=0}^{\infty} p_t^* [c_{i,t}^* + k_{i,t+1}^* - (1-\delta)k_{i,t}^* - w_t^* n_{i,t}^* - r_t^* k_{i,t}^*] + \lim_{T\to\infty} b_{i,T}^* \beta^T \frac{u_{c_{i,T}^*}}{u_{c_{i,0}^*}}$$

and since the date-0 budget constraint is satisfied for this consumption sequence, this implies that

$$\lim_{T\to\infty} b_{i,T}^* \beta^T u_{c_{i,T}^*} = 0$$

that is, the nPg and TVC conditions are satisfied. □

If there were no financial asset available, the proof will fail in that there exists some time t when expenditure is larger than income in date-0 trade, which cannot be achieved in sequential trade.