Competitive Equilibrium: A Review of a Static Economy

Alexis Anagnostopoulos
Stony Brook University

From now on, we will leave centrally planned economies behind and focus on market economies. The models considered up to this point were always maximization problems where a social planner had to maximize utility subject to technological and feasibility constraints. The planner chose aggregate consumption and aggregate investment and dictatorially imposed these choices on the population. In addition, there was no mention of how these aggregates are to be distributed among different individuals in the economy.

A feature of these centrally planned economies was that there were no prices to be considered; after all, if goods are not traded in well specified markets, one cannot talk about prices. We now want to begin introducing markets where the goods are traded by individuals. It is important to be clear about the main components of an equilibrium model. In particular, for any model considered, we should be able to identify the players or the *agents* of the model, the goods that are being traded and the corresponding markets in which they are traded. We need to have a clear idea of who supplies the goods in each market and who demands them. An important concept that we should be clear about is the concept of *market clearing*. We say a market clears if the supply and demand in that market are equal, i.e. there are no goods left over in the hands of the suppliers that nobody wants (no excess supply) and there are no individuals that would like more goods after all the supply has been exhausted (no excess demand).

Under certain assumptions, the allocations chosen by a social planner will also arise as equilibrium allocations in a market economy. In these cases we will say we have *decentralized* the planner's allocation.

In a dynamic economy, the rules that prevail in the markets, notably *when* the markets open and *what* can be traded, is something that will need to be decided. There are alternative choices regarding those market rules and different choices will lead to different dynamic equilibrium concepts. But before embarking upon our analysis of competitive equilibria in a dynamic economy, let us consider a simple static example. This is intended to:

1. Refresh your memory about basic equilibrium concepts.
2. Introduce the basic components of an equilibrium definition.
3. Explain how we can re-interpret a simple two-agent (or even a one-agent!) model as a model of the macroeconomy as a whole.
4. Introduce the idea of relating equilibrium model outcomes to planner maximization problem outcomes.

1 A Simple Exchange (or Endowment) Economy

Suppose that the economy is composed of two individuals, $i = 1, 2$, that live for only one period. You can think of the two individuals living in an island and each owning a part of the island. There are two kinds of trees in the island, apple trees and orange trees, producing two goods, apples and oranges. Each individual is endowed with an allocation of apples and oranges, that is, the trees that are on their side of the island produce a certain number of each good. Each individual i has an endowment w_{iA} of apples and an endowment w_{iO} of oranges. An extreme example would be that the side belonging to $i = 1$ only has apple trees ($w_{1O} = 0$) and the side belonging to $i = 2$ only has orange trees ($w_{2A} = 0$). We will allow the two agents to trade and seek to determine how many apples and oranges each agent ends up with (allocations) and what are the prevailing market prices.

There are two things that need to be specified in order to do this: The preferences and constraints of each individual and how the market operates.

After trade, each agent consumes their allocation of apples and oranges and derives utility denoted by $u_i\left(c_{iA}, c_{iO}\right)$. For the purposes of illustration, let us assume Constant Elasticity of Substitution (CES) utility functions

$$u_1(c_{1A}, c_{1O}) = \frac{c_{1A}^{1-\sigma_1}}{1-\sigma_1} + \frac{c_{1O}^{1-\sigma_1}}{1-\sigma_1}$$
$$u_2(c_{2A}, c_{2O}) = \frac{c_{2A}^{1-\sigma_2}}{1-\sigma_2} + \frac{c_{2O}^{1-\sigma_2}}{1-\sigma_2}$$

where $\sigma_1 > 0$ and $\sigma_2 > 0$ are parameters that control the elasticity of substitution.[1]

The utilities are strictly increasing in both goods, i.e. the two consumers strictly prefer more apples (and oranges) to less. If given the opportunity they would choose to consume an infinite amount of each, but they have to respect budget constraints. In specifying the budget constraints, we need to express income and spending in the *same units*, units of the numeraire good. Suppose for the moment that the numeraire is money (currency)[2] and that the prices of apples and oranges p_A and p_O are expressed in \$ per unit. Then the budget constraint for $i = 1$ is

$$p_A c_{1A} + p_O c_{1O} = p_A w_{1A} + p_O w_{1O}$$

One way to think of this, is that agent 1 sells some $(w_{1A} - c_{1A})$ of his apples to receive $p_A(w_{1A} - c_{1A})$ and uses this money to buy some $(c_{1O} - w_{1O})$ additional oranges by paying $p_O(c_{1O} - w_{1O})$. The proceeds from the sale of apples must equal the payment for the purchases of oranges. The agent could, of course, prefer to sell $w_{1O} - c_{1O}$ oranges and buy $c_{1A} - w_{1A}$ apples. The above representation of the budget captures both scenarios and tends to be more standard. For $i = 2$ the budget constraint is

$$p_A c_{2A} + p_O c_{2O} = p_A w_{2A} + p_O w_{2O}$$

It is important to notice that there is a single price for apples that both of these consumers pay and a single price of oranges (i.e. there is no p_{1A} and p_{2A}, just p_A).

We now have a complete characterization of the preferences and constraints for each agent. All that is left is to decide on the rules of trade. We will consider here competitive equilibria. A competitive equilibrium in this case is an equilibrium where none of the two agents have any market power and, therefore, they cannot affect prices. At first sight, this sounds unreasonable in this example because, for example, $i = 1$ could have all the apples and could charge whatever she wants. To put it differently, with only two agents it's hard to rationalize price taking

[1] For agent i the elasticity of substitution between apples and oranges is

$$\frac{d\left(\ln \frac{c_{iA}}{c_{iO}}\right)}{d\left(\ln \frac{u'_A(c_{iA},c_{iO})}{u'_O(c_{iA},c_{iO})}\right)} = \frac{d\left(\ln \frac{c_{iA}}{c_{iO}}\right)}{d\left(-\sigma_i \ln \frac{c_{iA}}{c_{iO}}\right)} = -\frac{1}{\sigma_i}$$

so it is *constant* in the sense that it is independent of the quantity of apples and oranges consumed c_{iA}, c_{iO}.

[2] This is just for discussion purposes. There is no money in this economy so we will ultimately need to choose one of the goods to be the numeraire by normalizing its price to 1.

behavior. However, the underlying idea is that there is a large number of consumers and $i = 1, 2$ just indexes their type. Thus, we write the model as a two agent model, but we are thinking of each agent as a representative of a whole class of agents all of whom are identical. In that case, we can assume perfect competition amongst apple (orange) owners which implies that each individual apple (orange) owner, when making decisions, does not have the power to influence prices and just takes them as given. Taking prices as given is the essence of a competitive equilibrium. The idea that the two agents here really correspond to a large number of agents that can be divided in two types is something that we will use very often. It allows one to think of these economies as macroeconomies. This means we can even consider equilibria where only one 'representative' agent appears (see below). Again, the underlying idea will be that there are many individuals, but they are all identical and we can thus focus on the behavior of one of them, the representative agent.

The economy will be in equilibrium when markets clear. *Market clearing* refers to a situation where supply is equal to demand. The mechanism through which market clearing occurs is by movements in prices. To put if differently, we will need to find market clearing prices, i.e. prices that ensure supply equals demand.

We need to specify what markets operate in this model. The number of markets operating is necessarily equal to the number of different goods. That is, there is a market for apples and a market for oranges. Often, in describing equilibria, we can clearly identify who is supplying the good and who demands the good. But the current example makes it clear that this distinction between suppliers and demanders is not as clear cut as it sounds. Here, each consumer can be either a supplier or a demander of the goods depending on their preferences and initial endowments. The total supply of apples is

$$W_A = w_{1A} + w_{2A}$$

and of oranges

$$W_O = w_{1O} + w_{2O}$$

The demand for these goods is given by the desired levels of consumption of the two agents. For apples it is $c_{1A}+c_{2A}$ and for oranges it is $c_{1O}+c_{2O}$. An equilibrium must involve market clearing, i.e. supply must equal demand. Market clearing conditions are therefore

$$c_{1A} + c_{2A} = w_{1A} + w_{2A} \tag{1}$$

$$c_{1O} + c_{2O} = w_{1O} + w_{2O} \tag{2}$$

This is a good point to raise a note of caution. Students sometimes confuse market clearing conditions with budget constraints. The two are very different conceptually. One fundamental practical difference is that *market clearing conditions involve no prices.* They are expressed in terms of goods. For example, equation (1) states that in the apple market the number of apples demanded must equal the number of apples supplied. There is no need for prices because everything is in terms of the same unit: apples. Prices appear in budget constraints because we are adding apples to oranges and that requires an exchange rate, i.e. a way to convert everything to the same units.

A competitive equilibrium in this economy will consist of allocations (i.e. how many goods does each agent end up with) and prices. Allocations and prices will have to satisfy certain conditions for them to be equilibrium prices and allocations. The idea is that the quantities will have to be chosen optimally by each individual *given* the equilibrium price and the equilibrium price will be determined by the requirement that markets clear. Put differently, each agent decides how much they want to consume for any given price (their demand function). Picking any arbitrary price, the demand level of the two consumers will not equal the available supply. The competitive equilibrium price (or the market clearing price) is the one that generates aggregate demand that is exactly equal to aggregate supply. In all the models we will see this price will be unique, i.e. the models considered will have a unique equilibrium.

Notation in defining equilibria is important. Here is a precise definition of a competitive equilibrium in the above setting:

DEFINITION: Given exogenous endowments $\{w_{1A}, w_{1O}, w_{2A}, w_{2O}\}$, a competitive equilibrium is a collection of allocations $\{c^*_{1A}, c^*_{1O}, c^*_{2A}, c^*_{2O}\}$ and a collection of prices $\{p^*_A, p^*_O\}$ such that

1. Allocations are optimal for each agent given the equilibrium prices $\{p^*_A, p^*_O\}$. Mathematically

$$\{c^*_{1A}, c^*_{1O}\} = \arg \max_{\{c_{1A}, c_{1O}\}} \left[\frac{c_{1A}^{1-\sigma_1}}{1-\sigma_1} + \frac{c_{1O}^{1-\sigma_1}}{1-\sigma_1}\right]$$

$$s.t.$$

$$p^*_A c_{1A} + p^*_O c_{1O} = p^*_A w_{1A} + p^*_O w_{1O}$$

and

$$\{c^*_{2A}, c^*_{2O}\} = \arg \max_{\{c_{2A}, c_{2O}\}} \left[\frac{c_{2A}^{1-\sigma_2}}{1-\sigma_2} + \frac{c_{2O}^{1-\sigma_2}}{1-\sigma_2}\right]$$

$$s.t.$$

$$p^*_A c_{2A} + p^*_O c_{2O} = p^*_A w_{2A} + p^*_O w_{2O}$$

2. Prices $\{p_A^*, p_O^*\}$ are such that markets clear

$$c_{1A}^* + c_{2A}^* = w_{1A} + w_{2A}$$
$$c_{1O}^* + c_{2O}^* = w_{1O} + w_{2O}$$

A couple of noteworthy points: Sometimes the endowments are omitted from the definition of equilibria since they are just exogenously given, but I include them for this first semester course in order to be as transparent as possible. Conceptually, notice that an equilibrium in this case is a collection of numbers, broadly categorized into quantities and prices. Also, there is good reason for why p_A^* and p_O^* appear in the consumers' maximization problems (and not p_A and p_O). It means that the equilibrium allocations $\{c_{1A}^*, c_{1O}^*, c_{2A}^*, c_{2O}^*\}$ are optimal for the agents *at the equilibrium prices*, which we denote using stars.

We can proceed to characterizing the equilibrium prices and allocations in this economy by collecting all conditions that are necessary and sufficient for equilibrium. This involves obtaining sufficient conditions for each one of the maximization problems and adding to those the market clearing conditions. Counting equations and unknowns there is always a structure that ensures the number of equations and unknowns is equal. In particular, every quantity variable will need to be a choice of an agent and, therefore, there will be an associated first order condition. Similarly, for every price there is a market in which the good is being traded and so a market clearing condition. Finally, in solving the maximization problems, we will need to introduce multipliers and there will be one multiplier for each constraint.

Equilibrium characterization in the above example is as follows. Supposing the multiplier on the budget constraint of agent i is λ_i (indexed by i because there is one for each agent's problem), the first order conditions for maximization in the two problems are[3]

$$c_{1A}^{-\sigma_1} = \lambda_1 p_A$$
$$c_{1O}^{-\sigma_1} = \lambda_1 p_O$$
$$c_{2A}^{-\sigma_2} = \lambda_2 p_A$$
$$c_{2O}^{-\sigma_2} = \lambda_2 p_O$$

The multipliers here have similar but not exactly identical interpretations with the planner's problem we saw before. Here λ_i will give the increase in maximized utility that would arise from a marginal increase in agent i's income, i.e. the marginal value of income. The first order

[3]All variables should have stars since these conditions will hold for equilibrium prices and allocations, but I suppress the star notation from here onwards.

condition for c_{1A} states that, at an optimum, the marginal benefit and marginal cost of an increase in c_{1A} must be equalized. The marginal benefit comes from the corresponding increase in utility $c_{1A}^{-\sigma_1}$. The marginal cost comes in terms of the decrease in income p_A times the marginal value of income λ_i. A similar interpretation can be provided for the other first order conditions (try it). We can substitute out multipliers to write

$$\frac{c_{1A}^{-\sigma_1}}{c_{1O}^{-\sigma_1}} = \frac{p_A}{p_O}$$
$$\frac{c_{2A}^{-\sigma_2}}{c_{2O}^{-\sigma_2}} = \frac{p_A}{p_O}$$

This says that the ratio of marginal utilities with respect to the two goods (the marginal rate of substitution) has to equal the relative price of the two goods. Conditions for maximization of each agent's problem also include the budget constraints

$$p_A c_{1A} + p_O c_{1O} = p_A w_{1A} + p_O w_{1O}$$
$$p_A c_{2A} + p_O c_{2O} = p_A w_{2A} + p_O w_{2O}$$

Conditions for an equilibrium include all the conditions for maximization and, in addition, market clearing

$$c_{1A} + c_{2A} = w_{1A} + w_{2A}$$
$$c_{1O} + c_{2O} = w_{1O} + w_{2O}$$

A version of Walras' Law will imply that one of the market clearing conditions is redundant. Loosely speaking, Walras' Law states that if there are n markets and $n-1$ of them clear, then the last market necessarily clears. Mathematically, it is easy to see that from the 4 preceding equations, using any 3 you can show that the fourth is satisfied. This implies that, together with the first order conditions, we have only five independent equations in $c_{1A}, c_{1O}, c_{2A}, c_{2O}, p_A, p_O$. This is a mathematical illustration of the fact that *only relative prices can be determined.* Put differently, one needs to choose a numeraire, a unit of account, otherwise prices have no meaning. In this economy there is no currency, so the numeraire will have to be one of the two commodities. Without loss of generality, we pick apples to be the numeraire and seek to determine the price of oranges in terms of apples $\frac{p_O}{p_A}$. Walras' Law will always apply so it is often the case that this choice of numeraire happens from the beginning when we set up the model. That is, often, instead of explicitly writing a price for the numeraire, we simply normalize the price of the numeraire to be 1 from the beginning. The implication is that all prices can be thought of as relative to the numeraire. In the example above, normalizing $p_A = 1$, means that p_O is a relative price (how many apples do you need to pay for one orange). At this point we can, in principle, solve the five equations for the five unknowns and obtain the competitive equilibrium allocations and the (relative) price of oranges.

2 The representative agent and the macroeconomy

The above analysis can be given a 'macroeconomic' flavour by making the assumption that there are two 'types' of agents, $i = 1, 2$, with a large number of agents of each type. To be concrete, suppose there are N_1 agents of type 1 all of them identical in the sense of having the same preferences and endowments. Similarly, suppose there are N_2 agents of type 2. Each individual agent's problem is exactly the one described in the previous section and all agents of the same type make the exact same decisions. We only need to be careful when aggregating across agents to write market clearing conditions, which in this case would be

$$\begin{aligned} N_1 c_{1A} + N_2 c_{2A} &= N_1 w_{1A} + N_2 w_{2A} \\ N_1 c_{1O} + N_2 c_{2O} &= N_1 w_{1O} + N_2 w_{2O} \end{aligned}$$

In the previous section we implicitly assumed the same number of agents for the two types, $N_1 = N_2 = N$, in which case the N cancels out and we obtain the market clearing conditions in $(1) - (2)$. So the model of the previous section could be thought of as a macroeconomic model with two goods and a large number of agents of two different types (with equal numbers for the two types).

Following this idea, many macroeconomic models make an even more extreme assumption, namely that there is only one type and everyone in the economy is identical. By the same logic as before, identical agents will make identical choices and we can focus on the *representative agent*. However, it is important to realize that we can still define a competitive equilibrium. This equilibrium will have some strange, although entirely standard, features. For example, given that everyone is identical and has the same preferences, there will be no trade in equilibrium; everyone will simply consume their endowments. However, competitive prices can still be defined as the prices that ensure the allocations in this no trade equilibrium are optimal for the representative agent. The equilibrium looks as follows:

DEFINITION (representative agent): Given a collection of endowments $\{w_A, w_O\}$ for the representative agent, a competitive equilibrium is a collection of allocations for the representative agent $\{c_A^*, c_O^*\}$, and a collection of prices $\{p_A^*, p_O^*\}$ such that

1. Allocations are optimal for the agent given the prices $\{p_A^*, p_O^*\}$. Mathematically

$$\begin{gathered}\{c_A^*, c_O^*\} = \arg \max_{\{c_A, c_O\}} u(c_A, c_O) \\ s.t. \\ p_A^* c_A + p_O^* c_O = p_A^* w_A + p_O^* w_O\end{gathered}$$

2. Prices adjust so that markets clear

$$\begin{aligned} c_A^* &= w_A \\ c_O^* &= w_O \end{aligned}$$

Market clearing simply says the agent consumes his own endowment. One could explicitly write down the market clearing condition for N agents

$$\sum_{i=1}^{N} c_{iA}^* = \sum_{i=1}^{N} w_{iA}$$

but given identical endowments and identical choices $c_{iA}^* = c_A^*$ and $w_{iA}^* = w_A^*$ so the market clearing condition is written as above. Note also that in this case, allocations are trivially given by the market clearing condition, but (relative) prices need to be computed

$$\frac{p_A^*}{p_O^*} = \frac{u'(c_A^*)}{u'(c_O^*)} = \frac{u'(w_A)}{u'(w_O)}$$

The relative price of the two goods is given by the relative value (in terms of utility) of the two goods. We will see that relative prices will have this form even in more complicated models. For this simple example, at this equilibrium price, the representative agent does not wish to trade, he is happy with the endowment he has.

3 Efficiency of equilibrium and relation to social planner problem

Suppose we consider the same economy but instead of allowing agents to trade and choose their consumption, we instead let a social planner dictate allocations. Suppose the planner's objective is to maximize a weighted sum of the utilities of all the agents subject to resource constraints. The social welfare function is thus

$$\sum_{i=1}^{2} \xi_i u\left(c_{iA}, c_{iO}\right)$$

where, without loss of generality, we can assume that the weights attached to each agent's utility add up to 1, i.e. $\sum_{i=1}^{2} \xi_i = 1$. Resource constraints say that the total consumption of apples (oranges) has to be less than the total amount of available apples (oranges)

$$\sum_{i=1}^{2} c_{iA} \leq \sum_{i=1}^{2} w_{iA}$$
$$\sum_{i=1}^{2} c_{iO} \leq \sum_{i=1}^{2} w_{iO}$$

Given strictly increasing utility, the resource constraints will be optimally satisfied as equalities. The allocation the planner chooses will depend on how the weights ξ_i are specified. For different weights, different Pareto optimal allocations can be obtained. One of those Pareto optimal allocations will be exactly the competitive equilibrium allocation.[4]

Exercise: What do the weights need to be in order for the planner to choose the competitive equilibrium allocations?

[4] Recall the fundamental welfare theorems.