Cass-Koopmans: Infinite Horizon

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1 Cass-Koopmans Model - Infinite Horizon

It is common to consider sequential problems with an infinite horizon. The idea is that we do not expect the economy to stop existing after some finite period of time. Any finite horizon model will exhibit behavior that crucially depends on the finite horizon assumption and will thus be sensitive to the choice of the length of the horizon (i.e. the choice of T). This sensitivity to the parameter choice can be avoided by considering an infinite horizon. One can think of this as a "dynasty" that rules the country and cares about the infinite future because the same family will rule the economy forever. This, of course, is implicitly assuming that the current decision maker cares equally about the future generations (once the discount factor has been taken into account). The assumption of an infinite horizon will also buy us "stationarity", a concept that will become important when thinking about dynamic programming and will therefore allow us to use extra machinery in solving our models.

An infinite horizon problem also brings with it additional complications. The maximization problems will involve choices out of infinite dimensional spaces. That is, we will be searching for infinite sequences of consumption and capital. Strictly speaking, the Lagrangian approach that was discussed before only works in finite dimensional spaces. Rigorously extending the machinery to infinite dimensional spaces would take too much time for a first course in macroeconomics. Instead we will simply look at the practical side of things, i.e. what extra conditions will be needed for maximization on top of the list discussed in the previous section.

We take two approaches: First, we look at the solution of the finite horizon problem discussed previously and let $T \to \infty$. This is helpful in building intuition and is very straightforward to do. Unfortunately, this is not really a rigorous approach since the limit of the finite horizon solution does not necessarily have to coincide with the solution of the infinite horizon problem (although it will turn out it does in this case). We then prove a theorem on sufficient conditions for the maximization of the infinite horizon problem.

1.1 A Heuristic Approach

The conditions for maximization of the finite horizon problem are summarized below

$$\begin{aligned} u'(c_t) &= \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right] \text{ for } t = 0, 1, ...T-1 \\ c_t + k_{t+1} &= (1-\delta) k_t + f\left(k_t\right) \quad \text{ for } t = 0, 1, ..., T \end{aligned}$$

In addition, for the last period we have complementary slackness(using the fact that, at the optimum in the last period, the marginal value of relaxing capital non-negativity μ_T equals the marginal value of resources λ_T which in turn equals $\beta^T u'\left(c_T\right)$)

$$\begin{aligned} \beta^T u'\left(c_T\right) k_{T+1} &= 0 \\ \beta^T u'\left(c_T\right) &\geq 0 \\ k_{T+1} &\geq 0 \end{aligned}$$

In the previous section we argued that $\mu_T = \beta^T u'\left(c_T\right) > 0$ and hence $k_{T+1} = 0$ by complementary slackness. When $T \to \infty$, this does not work since $\beta^T \to 0$ and hence $\beta^T u'\left(c_T\right) k_{T+1}$ could go to zero even with k_{T+1} remaining bounded above zero. The "non-rigorous" approach to the infinite horizon problem is to simply impose the same condition as a limiting condition. So, overall, we are left with an infinite number of conditions for capital (the Euler equation) together with a limiting condition

$$\lim_{T \to \infty} \beta^T u'\left(c_T\right) k_{T+1} = 0$$

This limiting condition has replaced the terminal condition of the finite horizon problem which stated $k_{T+1} = 0$. Here there is no last period, so it is never optimal to run down the capital stock completely. If that were done, there would be no resources left for the future and that would have a huge negative effect on utility by restricting consumption to be zero thereafter. The limiting condition, known as a "transversality condition" (TVC), incorporates the idea of restricting the amount of savings in the distant future without requiring that these go to zero. The idea in the TVC is that it cannot be optimal to save so much in the future that the term $\beta^T u' c(_T) k_{T+1}$ remains bounded above zero.[1] Recall

[1]The TVC is a sufficient condition, we have not talked about necessity, which is why the statement here says "the idea is...".

that $\beta^T u'(c_T)$ is the value of the multiplier on the resource constraint at the optimum which has the interpretation of a 'shadow price' i.e. it gives the marginal value of resources. It is the value (in utility terms) of one unit of the consumption/investment good at T, from the point of view of time 0, which is the time at which the decisions are being made. Therefore, the transversality condition requires that the *value of* savings approaches zero as time goes to infinity. Note that the TVC is easily satisfied if capital and marginal utility are bounded, for example if c and k approach a constant, positive value.

As mentioned earlier, this is not a rigorous approach and we will now proceed to show that it is indeed correct to use the previous conditions with a TVC added as sufficient conditions for maximization. From a practical perspective, what should be clear is that

1. The TVC is a condition for maximization that you should include whenever you are asked for optimality conditions in an infinite horizon problem.

2. The TVC has a clear economic interpretation, in the current setting it sets a limit on the amount of savings in the distant future.

3. One can usually "heuristically derive" the TVC condition(s) in any problem by writing down the terminal conditions in the associated finite horizon problem and taking limits on the complementary slackness condition for the non-negativity on state variables in the last period.

1.2 A Theorem on Sufficient Conditions for Maximization

The theorem and proof can be found in Krusell's notes (see Proposition 4.4). A more general proof (but ignoring corner solutions) of the same result can also be found in Stokey and Lucas Ch. 4.5.

1.3 Steady State

In analyzing the solution to an infinite horizon problem, we will need to find ways to solve the system of difference equations described by the Euler Equation and the resource constraint

$$\begin{aligned}
u'(c_t) &= \beta u'(c_{t+1}) \left[f'(k_{t+1}) + (1-\delta) \right] \text{ for all } t \\
c_t + k_{t+1} - (1-\delta)k_t &= f(k_t) \text{ for all } t \\
\lim_{T\to\infty} \beta^T u'(c_T) k^*_{T+1} &= 0 \\
k_0 &\text{ given}
\end{aligned}$$

Although we will defer these computational methods to later in the course, the first step to analyzing the dynamics of a difference equation is to look at steady states. A steady state consists of values for the endogenous variables k^*, c^*, i^*, y^* such that if $k_0 = k^*$, then the sequence of optimal choices for capital consists of constant values for capital for all periods i.e. $k_t = k^*$ for all t. Clearly, constant capital implies that the rest of the endogenous variables will also be constant. The values k^*, c^*, i^*, y^* are called the steady state. It is a helpful starting point in the analysis of the dynamics because we know that, if the economy ever reaches k^* it will stay there forever. Given such a steady state, we can proceed to discussing stability of this point and the dynamics away from it (will capital converge toward it or diverge?).[2]

Computing the steady state is straightforward. Clearly, all conditions for maximization have to hold at a steady state too and, in particular, the Euler equation has to hold for a constant capital k^* and consumption c^*

$$\begin{aligned} u'(c^*) &= \beta u'(c^*) \left[f'(k^*) + (1-\delta) \right] \Rightarrow \\ f'(k^*) &= \frac{1}{\beta} - 1 + \delta \end{aligned} \tag{1}$$

This provides an (non-linear) equation in k^*. Once k^* is found, one can use the steady state versions of the capital accumulation equation and the resource constraint to find i^* and c^*

$$\begin{aligned} i^* &= \delta k^* \\ c^* &= f(k^*) - i^* \end{aligned}$$

Remember that we also require our choice to be non-negative, which needs to be shown for the above solutions (i.e. to conclude a proof of existence of a steady state). Along the way, we can also prove uniqueness of a *strictly positive* steady state.[3] To prove that a solution to equation (1) exists and is unique one needs to use the Inada conditions on the production function.

As shown in Figure 1, the marginal product of capital is ∞ at zero and zero at ∞. Furthermore it is continuous and monotonically decreasing, due to f being continuously differentiable and strictly concave, so it crosses the horizontal line $\frac{1}{\beta} - 1 + \delta (> 0)$ only once. In addition, the crossing point is at a strictly positive $k^* > 0$. To conclude the proof

[2] In engineering this stable point of a dynamic system is sometimes referred to as an "equilibrium". We want to avoid this terminology because, in economics, we reserve the term equilibrium to mean something entirely different.

[3] Note that another steady state is the trivial one $k^* = c^* = i^* = y^* = 0$.

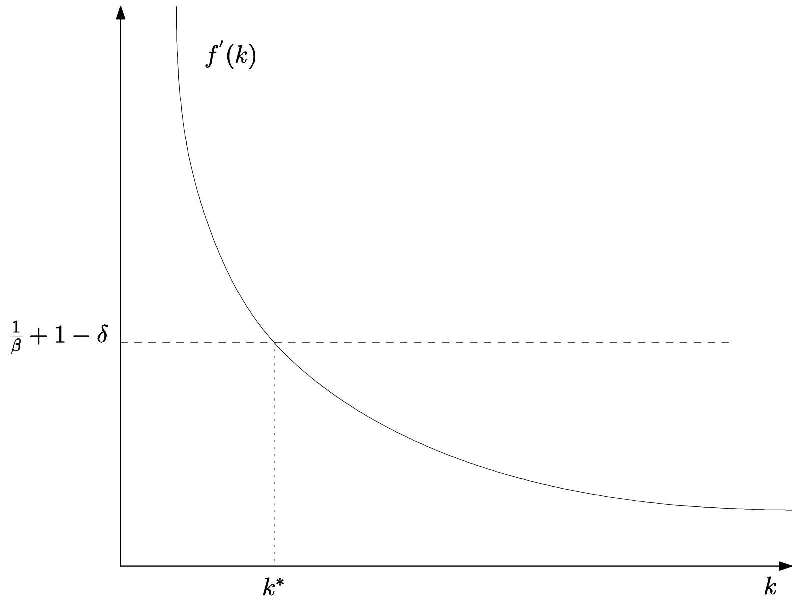

Figure 1: The steady state capital k^* in the Cass-Koopmans model.

that this is indeed a constant solution to the optimization problem we need to show that $c^* = f(k^*) - \delta k^* \geq 0$. We can see this is the case by comparing to the Golden Rule steady state of the Solow model. This will also provide some intuition for the differences between the Solow model's solution and the Cass-Koopmans model's solution.

How does the steady state condition here relate to Solow's model? The Golden Rule condition in Solow's model required

$$f'\left(k^*_{gold}\right) = \delta \tag{2}$$

Comparing with (1), notice that the difference is the extra term $\frac{1}{\beta} - 1$ which is called the 'time preference rate'. In the limiting case where $\beta = 1$ and the planner cares equally about current and future consumption, the steady state coincides (i.e. $k^* = k^*_{gold}$). But since we require $\beta < 1$, the Golden Rule condition is modified (hence the term "Modified Golden Rule") to take into account impatience. Note that with $\beta < 1$, it follows that $f'(k^*) > f'\left(k^*_{gold}\right)$ which implies $k^*_{gold} > k^*$. So, in the long run, the optimal amount of savings with impatience in the utility is lower than without, as expected. Furthermore, looking at the Golden rule graph of Solow's model (see Figure 2) we can see that consumption at k^* is positive since $0 \leq k^* \leq k^*_{gold}$

It can be shown (see Krusell's notes or Stokey and Lucas) that this steady state is stable in the sense that the dynamics of the optimal

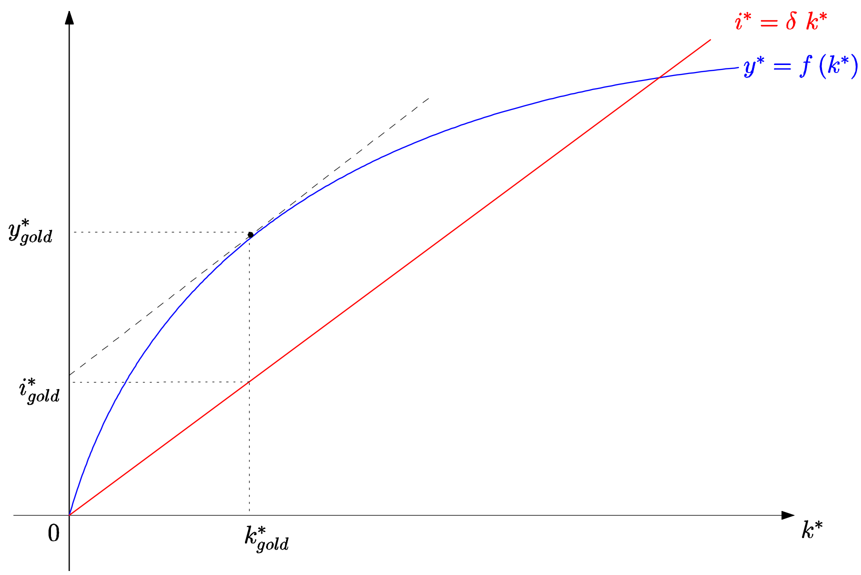

Figure 2: The Golden Rule in Solow's model.

solution starting from any $k_0 \neq k^*$ will converge to k^* (in fact, will do so monotonically).

An interesting interpretation of the modified golden rule steady state comes from looking at the special case of linear period utility u, i.e. a case where the planner maximizes discounted consumption. In this case the Euler equation becomes

$$f'(k_{t+1}) = \frac{1}{\beta} - 1 + \delta$$

for all t. Put differently, starting at any k_0, the optimal choice is to immediately jump to steady state and set $k_t = k^*$ for $t = 1, 2, ...$ To be more precise, a qualification on the values of k_0 that can lead to this is needed. Although the general idea above is correct, note that we have assumed that the non-negativity on consumption will not bind because of the Inada conditions. But with linear u, the Inada conditions are not satisfied and the constraint could bind. If in trying to set $k_1 = k^*$ the planner is forced to set negative consumption c_0 then the inequality is violated. Put differently, c_0 is given by

$$\begin{aligned} c_0 &= (1-\delta)k_0 + f(k_0) - k_1 \\ &= (1-\delta)k_0 + f(k_0) - k^* \end{aligned}$$

If k_0 is large enough, then $(1-\delta)k_0 + f(k_0) - k^* \geq 0$ and we can jump to steady state immediately (potentially suffering a low but still non-negative consumption at $t = 0$). But if $(1 - \delta)k_0 + f(k_0) - k^* < 0$, choosing $k_1 = k^*$ would imply $c_0 < 0$ which is not feasible. The optimal choice would then require setting consumption to 0 and saving the maximum possible. This would continue until capital can reach k^* with consumption being non-negative.

Note that the solution with linear utility differs from the one with a concave utility because concavity implies a preference for smooth consumption. To jump immediately to k^*, consumption will be very low temporarily. Concavity will imply that the planner will prefer doing this more gradually.