Midterm Exam

INSTRUCTIONS:

- *Allocate your time efficiently - do not spend too much time on one small part. You should allocate approximately 1h30min to question 1 and the same to question 2.*
- *Make sure you answer ALL questions. In particular, if you cannot solve one part of the question, always attempt the following parts too. Most subparts are independent of each other. If you need to assume something from a previous subpart, just state the assumption and continue.*
- *You need to show all your steps. The final answer alone will not earn you points. Also, do not skip parts of the exam that ask you to provide interpretations or discussions! These constitute a large fraction of the points allocated.*

GOOD LUCK!

1. (50 pts) Consider a planner's problem in an economy where the production function is given by

$$y_t = Ak_t^{\alpha} \left(h_t n_t\right)^{1-\alpha}$$

where y_t denotes output, n_t denotes the fraction of time spent working and k_t, h_t denote physical capital and human capital respectively. The parameters $\alpha \in (0,1)$ and $A > 0$ are given. The planner can invest in both physical and human capital, the accumulation equations are given by

$$\begin{aligned} k_{t+1} &= (1-\delta) k_t + i_t^k \\ h_{t+1} &= (1-\delta) h_t + i_t^h \end{aligned}$$

where $\delta \in [0,1]$ is the common depreciation rate and i_t^k, i_t^h denote investment rates for physical and human capital respectively. The planner aims to maximize utility given by

$$\sum_{t=0}^{\infty} \beta^t \left[\log c_t + B \log\left(1-n_t\right)\right]$$

where $\beta \in (0,1)$ is the discount factor, $B > 0$ is a preference parameter and c_t denotes consumption. The usual resource constraint applies

$$c_t + i_t^k + i_t^h = y_t$$

(a) Carefully setup a Lagrangian and obtain conditions for maximization (you do not need to deal with inequality constraints, you may assume they never bind).

(b) Substitute out multipliers and reduce the first order conditions to three equations describing the optimal choices of physical capital k_{t+1}, human capital h_{t+1} and labor n_t. Provide **careful**, **detailed interpretations** for **each** of these conditions.

(c) Assuming that n_t is constant in the balanced growth path, determine the constant growth rates of all variables that are consistent with optimality and feasibility.

(d) State Kaldor's stylized facts of growth and determine whether this economy satisfies these facts.

(e) Provide a dynamic programming formulation of this problem. Make sure you define variables carefully and show the Bellman equation indicating clearly which are state and which are choice variables.

2. (50 pts) A farmer in Tuscany produces bread and wine for his own consumption. The farmer is endowed with T hours of time each period, where you can assume $T = 1$ for simplicity. He allocates his time between two activities: baking bread and pressing grapes to make grape juice. The farmer does not value leisure. The only input required to produce bread and grape juice is labor. Moreover, the production technology is linear: each unit of time devoted to baking bread produces one unit of bread and each unit of time devoted to pressing grapes produces one unit of grape juice. Each unit of grape juice becomes (via the process of fermentation) one unit of wine in the next period. Both bread and wine are perishable (i.e. non-storable) goods. The farmer allocates his time between baking bread and pressing grapes in order to maximize the lifetime utility of his own consumption of bread and wine; this utility is given by $\sum_{t=0}^{\infty} \beta^t (b_t w_t)^{\frac{1}{2}}$, where b_t is the amount of bread consumed in period t, w_t is the amount of wine consumed in period t, and $\beta \in (0, 1)$.

   (a) Write the farmer's problem in sequence form, obtain the Euler equation describing the optimal choice and provide a careful, intuitive interpretation of the Euler equation.

   (b) Formulate the farmer's optimization problem as a dynamic programming problem, i.e. identify state and choice variables and display the Bellman equation.

   (c) Solve the Bellman equation using the guess-and-verify approach.