Assignment 4

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Sepetember 19, 2020

1. (a) Since α is the capital share of output and β is the discout factor,$\alpha \in (0,1)$ and $\beta \in (0,1)$. Thus $\alpha\beta \in (0,1)$.

$$\begin{aligned}\lim_{T\to\infty} k_{t+1} &= \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha} \\ &= \alpha\beta k_t^{\alpha}, \qquad \forall t = 0, 1, \dots, T\end{aligned}$$

(b) i. Arbitrary value is given to α and β in Figure 1. Similar to Solow

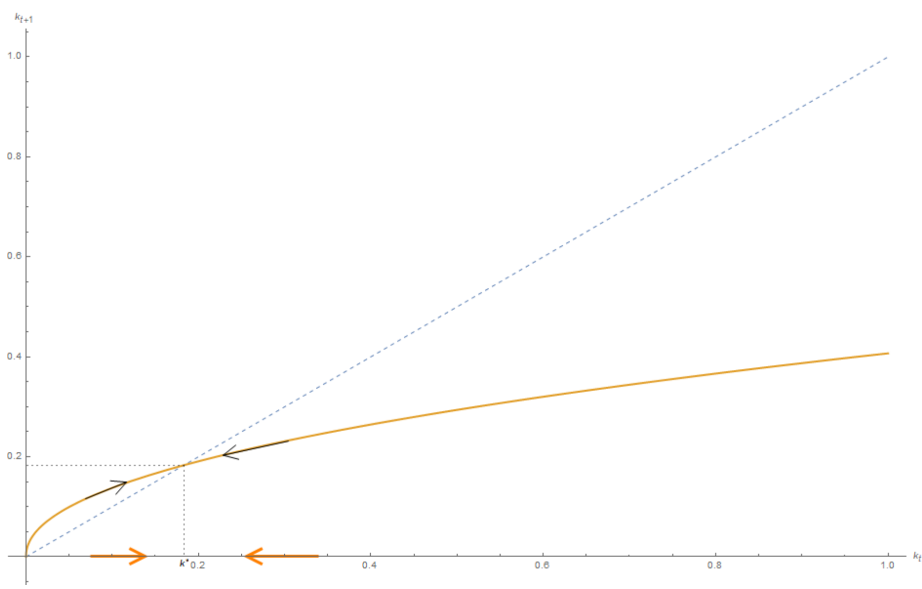

Figure 1: Optimal k_{t+1} versus k_t ($\alpha = 0.47, \beta = 0.866$)

Model, there exists a steady state k^*. It is unique if the trivial solution ($k = f(k) = 0$) is not included. The dynamics of capital outside the steady state will follow the direction of arrows in the Figure 1. If k_0 is greater than k^*, the capital stock will keep decreasing until the steady state fulfills; if k_0 is less than k^*, the capital stock will keep increasing until the steady state meets.

ii. At the steady state,

$$\begin{aligned}
&k^* = \alpha\beta(k^*)^\alpha \\
\Rightarrow &k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} \\
\Rightarrow &i^* = k^* = (\alpha\beta)^{\frac{1}{1-\alpha}} \\
\Rightarrow &c^* = f(k^*) - i^* = (\alpha\beta)^{\frac{1}{1-\alpha}}\left[(\alpha\beta)^{-1} - 1\right]
\end{aligned}$$

By the Solow model's golden rule,

$$\begin{aligned}
&f'(k^{gold}) = \delta \\
\Rightarrow &k^{gold} = \alpha^{\frac{1}{1-\alpha}} \\
\Rightarrow &i^{gold} = k^{gold} = \alpha^{\frac{1}{1-\alpha}} \\
\Rightarrow &c^{gold} = f(k^{gold}) - i^{gold} = \alpha^{\frac{1}{1-\alpha}}\left[\alpha^{-1} - 1\right]
\end{aligned}$$

Thus, at Solow model's golden rule steady state, the capital stock, investment, consumption and production aren't related to time preference rate β, while the C-K model are. Plus, the golden saving rate is α in Solow model, while in C-K model, the saving rate is $\alpha\beta$.

iii. According to the question 2 of Problem Set 3,

$$\begin{aligned}
\lim_{T\to\infty} z_t &= \lim_{T\to\infty} \alpha\beta\frac{1-(\alpha\beta)^{T-t+1}}{1-(\alpha\beta)^{T-t+2}} \\
&= \alpha\beta, \qquad\qquad \forall t = 1, 2, \ldots, T+1
\end{aligned}$$

Economic interpretation

$$\begin{aligned}
&\because \delta = 1 \\
&\therefore z_t = \frac{k_{t+1}}{k_t^\alpha} = \frac{i_t}{k_t^\alpha} = s
\end{aligned}$$

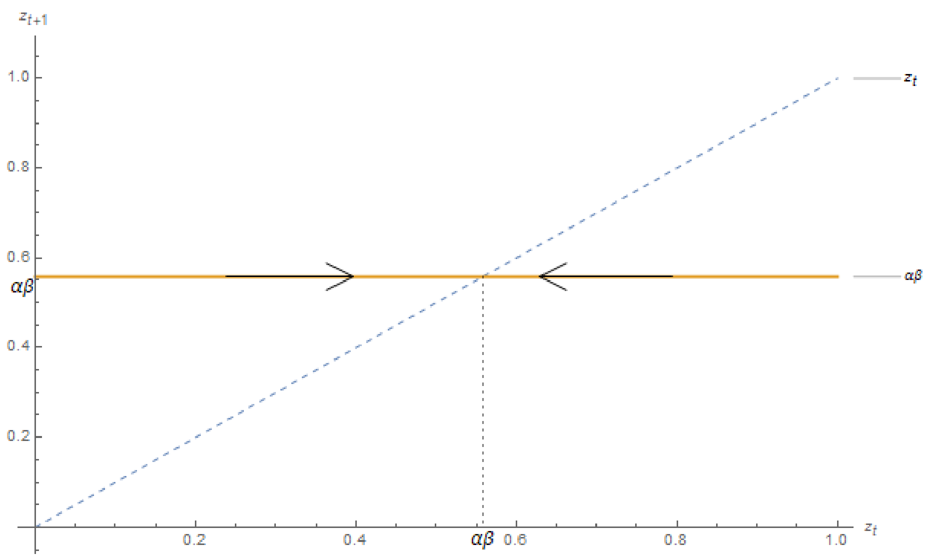

Figure 2: Optimal z_{t+1} versus z_t

Hence, Z_t is the savings rate. As shown in Figure 2, the optimal saving rate (the steady state) equals to $\alpha\beta$. For any z_0 not equal to $\alpha\beta$, the savings rate z_t will jump to $\alpha\beta$ and remain unchanged in the following periods.

2. Maximizing problem

$$
\begin{aligned}
\max \quad & \sum_{t=0}^{\infty}(\frac{1}{1+r})^t(D_t - E_t) \\
s.t. \quad & D_t + I_t = F(K_t) + E_t - C(E_t) \\
& K_{t+1} = (1-\delta)K_t + I_t \\
& D_t \geq 0 \\
& E_t \geq 0 \\
& K_{t+1} \geq 0 \\
& I_t \geq 0 \\
& K_0 \text{ is given}
\end{aligned}
$$

Lagrangian Function

$$
\begin{aligned}
L = \sum_{t=0}^{\infty}\{(\frac{1}{1+r})^t(D_t - E_t) + \lambda_t[F(K_t) + E_t - C(E_t) - D_t - I_t] \\
+ \eta_t[(1-\delta)K_t + I_t - K_{t+1}] + \nu_t D_t + \mu_t E_t + \theta_t K_{t+1} + \gamma_t I_t\}
\end{aligned}
$$

(a) Necessary Conditions

FOC

$$\begin{cases} \dfrac{\partial L}{\partial D_t} = (\dfrac{1}{1+r})^t - \lambda_t + \nu_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \dfrac{\partial L}{\partial E_t} = -(\dfrac{1}{1+r})^t + \lambda_t(1 - C'(E_t)) + \mu_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \dfrac{\partial L}{\partial I_t} = -\lambda_t + \eta_t + \gamma_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \dfrac{\partial L}{\partial K_{t+1}} = \lambda_{t+1} F'(K_{t+1}) + \eta_{t+1}(1-\delta) - \eta_t + \theta_t = 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

Equality Constraints

$$\begin{cases} F(K_t) + E_t - C(E_t) - D_t - I_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ (1-\delta)K_t + I_t - K_{t+1} = 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

Inequality Constraints

$$\begin{cases} D_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ E_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ K_{t+1} \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ I_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

Multipliers are non-negative

$$\begin{cases} \nu_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ \mu_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ \theta_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \\ \gamma_t \geq 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

Complementary Slackness

$$\begin{cases} \nu_t D_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \mu_t E_t = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \theta_t K_{t+1} = 0, \text{ for } t = 0, 1, \ldots, \infty \\ \gamma_t I_t = 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

TVC

$$\lim_{T \to \infty} \lambda_T k_{T+1} = 0$$

(b) $C(E_t) = 0, \forall E_t$

i. Based on the FOC of $[D_t]$ and $[E_t]$, we have

$$\begin{cases} \nu_t = \lambda_t - (\dfrac{1}{1+r})^t \\ \mu_t = (\dfrac{1}{1+r})^t - \lambda_t \end{cases} \Rightarrow \nu_t + \mu_t = 0, \forall t$$

Since multipliers are non-negative,

$$\begin{cases} \nu_t \geq 0 \\ \mu_t \geq 0 \end{cases} \Rightarrow \nu_t = \mu_t = 0, \forall t$$

By the complementary slackness, two non-negativity constraints will never bind.

ii. K_{t+1} will never bind because $F(K_t)$ satifies the Inada Conditions and if $K_t = 0$, $D_t - E_t = -I_t \leq 0$, which implies the firm's objective cannot be optimal. Thus, $K_{t+1} > 0, \forall t$ and by complementary slackness, $\theta_t = 0, \forall t$.
As for investment non-negativity constraints, if $I_t = 0$, $\lim_{t\to\infty} K_{t+1} = 0$, a contradiction. Hence, $I_t > 0, \forall t$ and by complementary slackness, $\gamma_t = 0, \forall t$.
Plugging $\lambda_t, \eta_t, \mu_t, \nu_t, \theta_t, \gamma_t, I_t$, the necessary conditions can be reduced as follows.

$$\begin{cases} F'(K_{t+1}) = \delta + r, \text{ for } t = 0, 1, \ldots, \infty \\ F(K_t) + E_t - D_t + (1-\delta)K_t - K_{t+1} = 0, \text{ for } t = 0, 1, \ldots, \infty \end{cases}$$

Since $F' > 0$ and $F'' < 0$, F' must be a injective function, which implies that there exists a unique steady stade $K^* = (F')^{-1}(\delta + r)$, given δ and r. If initial $K_0 < K^*$ and $F'(K_0) > \delta + r$, K_{t+1} will keep increasing until $K_{t+1} = K^*$ and $F'(K_{t+1})$ will keep decreasing until $F'(K_{t+1}) = \delta + r$; if initial $K_0 > K^*$ and $F'(K_0) < \delta + r$, K_{t+1} will keep decreasing until $K_{t+1} = K^*$ and $F'(K_{t+1})$ will keep increasing until $F'(K_{t+1}) = \delta + r$.

iii. In standard Cass-Koopmans Model,

$$F'(K^*) = \frac{1}{\beta} + \delta - 1$$

In this model,

$$F'(K^*) = r + \delta$$

Since the time preference β can be also wriiten as $\frac{1}{1+r}$, the two model are equivalent in terms of steady state.
Plus, in C-K model, utility function is concave, individual will smooth their consumption. Given K_0, K_t will converge to the steady state gradually. Analogously, given K_0 in this model, K_1 will jump to K^* by adjusting $(D - E)$.

iv. Given a K^*, we have

$$\begin{cases} F'(K^*) = \delta + r \\ F(K^*) + E_t - D_t - \delta K^* = 0 \end{cases}$$

$$\Rightarrow D_t - E_t = F(K^*) - \delta K^*, \forall t$$

From the equation system above, we cannot determine D_t and E_t separately.

(c) $C(0) = 0, 0 \leq C(E) < E, 0 < C'(E) < 1, C''(E) > 0$

Proof. From the FOC of $[D_t]$ and $[E_t]$, we have

$$\begin{cases} \nu_t = \lambda_t - (\frac{1}{1+r})^t \\ \mu_t = (\frac{1}{1+r})^t - \lambda_t(1 - C'(E_t)) \end{cases}$$
$$\Rightarrow \nu_t + \mu_t = \lambda_t C'(E_t), \forall t$$

Since $\lambda_t > 0$ (multiplier of a equality constraint) and $0 < C'(E) < 1$, the RHS is positive. Thus $\nu_t + \mu_t > 0, \forall t$. Because multipliers are non-negative, three possible situations are as follows.

(i) $\nu_t > 0, \mu_t > 0$
By complementary Slackness, $D_t = E_t = 0, \forall t$.

(ii) $\nu_t > 0, \mu_t = 0$
By complementary Slackness, $D_t = 0, E_t > 0, \forall t$.

(iii) $\nu_t = 0, \mu_t > 0$
By complementary Slackness, $D_t > 0, E_t = 0, \forall t$.

□