Problem Set 2

1. Consider the following 3-period version of the Cass-Koopmans model which includes a labor/leisure choice. In particular, the social planner has to choose how much work n_t to put in every period. Working increases production but reduces leisure l_t (free time) and the planner may value leisure. Let the total amount of time in a period be normalized to 1 (so that $l_t = 1 - n_t$) and let the period utility be denoted by $u(c, 1-n)$

$$\max_{\{c_t, k_{t+1}, n_t\}_{t=0}^{2}} \sum_{t=0}^{2} \beta^t u(c_t, 1-n_t)$$

subject to

$$\begin{aligned}
c_t + k_{t+1} - (1-\delta)k_t &= k_t^{\alpha} n_t^{1-\alpha} && \text{for } t = 0,1,2 \\
k_{t+1} &\geq 0 && \text{for } t = 0,1,2 \\
0 &\leq n_t \leq 1 && \text{for } t = 0,1,2 \\
c_t &\geq 0 && \text{for } t = 0,1,2 \\
& k_0 \quad \text{given}
\end{aligned}$$

(a) Setup the Lagrangian and obtain necessary conditions for an optimum. (Note that there are five constraints, four inequality constraints and an equality constraint for each period t)

(b) Assume

$$u(c_t, 1-n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + A\frac{(1-n_t)^{1-\gamma}}{1-\gamma}$$

where $\sigma, \gamma, A > 0$ are given parameters. Determine which inequality constraints bind and which do not. (If you need to use any properties of the utility and production functions you will have to prove them first)

(c) Assume instead

$$u(c_t, 1-n_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$$

Find the optimal choice of labor n_t, for every $t = 0,1,2$ and explain intuitively why this differs from your answer in part b.

2. Consider the finite horizon version of the Cass-Koopmans model

$$\max_{\{c_t, i_t, k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t u(c_t)$$

subject to

$$\begin{aligned} c_t + i_t &\leq f(k_t) && \text{for all } t = 0, 1, ..., T \\ k_{t+1} &\leq (1-\delta)\, k_t + i_t && \text{for all } t = 0, 1, ..., T \\ k_{t+1} &\geq 0 && \text{for all } t = 0, 1, ..., T \\ c_t &\geq 0 && \text{for all } t = 0, 1, ..., T \\ & k_0 \quad \text{given} \end{aligned}$$

where $0 < \beta < 1$. Assume that $u : R_+ \to R$ and $f : R_+ \to R_+$ are both continuously differentiable, strictly increasing, concave and satisfy the Inada conditions.

(a) Construct the Lagrangian by attaching a multiplier $\beta^t \lambda_t$ to each resource constraint, a multiplier $\beta^t \gamma_t$ to each capital accumulation constraint, $\beta^t \mu_t$ to each non-negativity constraint for capital and $\beta^t \nu_t$ to each non-negativity constraint for consumption. Obtain ALL conditions for maximization.

(b) Use complementary slackness to determine which constraints bind and which multipliers are zero.

(c) Using the remaining equations, show that the optimal capital sequence for the above problem can be characterized by the following two equations

$$u'\left(f(k_t) + (1-\delta)\, k_t - k_{t+1}\right) = \beta\left[(1-\delta) + f'(k_{t+1})\right] u'\left(f(k_{t+1}) + (1-\delta)\, k_{t+1} - k_{t+2}\right) \tag{1}$$

for $t = 0, 1, ..., T-1$ and

$$k_{T+1} = 0 \tag{2}$$

NOTE: Most of this is covered in the Lecture Notes, the objective is to practice setting up Lagrangians and dealing with multipliers. The only differences are that we impose the capital accumulation constraint as an inequality and we use a standard re-normalization of multipliers.