

# Assignment 2

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## 1. C-K Model: 3-period

### (a) Lagrangian Function:

$$L = \sum_{t=0}^2 \beta^t u(c_t, 1 - n_t) + \sum_{t=0}^2 \lambda_t (k_t^\alpha n_t^{1-\alpha} - c_t - k_{t+1} + (1 - \delta)k_t) \\ + \sum_{t=0}^2 \mu_t k_{t+1} + \sum_{t=0}^2 \theta_t n_t + \sum_{t=0}^2 \eta_t (1 - n_t) + \sum_{t=0}^2 \nu_t c_t$$

Necessary Conditions:

#### i. FOC:

$$\begin{cases} \frac{\partial L}{\partial c_t} = \beta^t \frac{\partial u(c_t, 1 - n_t)}{\partial c_t} - \lambda_t + \nu_t = 0, \text{ for } t = 0, 1, 2 \\ \frac{\partial L}{\partial k_{t+1}} = -\lambda_t + \lambda_{t+1}(\alpha k_{t+1}^{\alpha-1} n_{t+1}^{1-\alpha} + 1 - \delta) + \mu_t = 0, \text{ for } t = 0, 1 \\ \frac{\partial L}{\partial k_3} = -\lambda_2 + \mu_2 = 0 \\ \frac{\partial L}{\partial n_t} = \beta^t \frac{\partial u(c_t, 1 - n_t)}{\partial n_t} + \lambda_t(1 - \alpha)k_t^\alpha n_t^{-\alpha} + \theta_t - \eta_t = 0, \text{ for } t = 0, 1, 2 \end{cases}$$

#### ii. Equality Constraints:

$$k_t^\alpha n_t^{1-\alpha} - c_t - k_{t+1} + (1 - \delta)k_t = 0, \text{ for } t = 0, 1, 2$$

#### iii. Inequality Constraints:

$$\begin{cases} k_{t+1} \geq 0, \text{ for } t = 0, 1, 2 \\ n_t \geq 0, \text{ for } t = 0, 1, 2 \\ 1 - n_t \geq 0, \text{ for } t = 0, 1, 2 \\ c_t \geq 0, \text{ for } t = 0, 1, 2 \end{cases}$$

iv. Multipliers are non-negative:

$$\begin{cases} \lambda_t \geq 0, & \text{for } t = 0, 1, 2 \\ \mu_t \geq 0, & \text{for } t = 0, 1, 2 \\ \theta_t \geq 0, & \text{for } t = 0, 1, 2 \\ \eta_t \geq 0, & \text{for } t = 0, 1, 2 \\ \nu_t \geq 0, & \text{for } t = 0, 1, 2 \end{cases}$$

v. Complementary Slackness:

$$\begin{cases} \mu_t k_{t+1} = 0, & \text{for } t = 0, 1, 2 \\ \theta_t n_t = 0, & \text{for } t = 0, 1, 2 \\ \eta_t (1 - n_t) = 0, & \text{for } t = 0, 1, 2 \\ \nu_t c_t = 0, & \text{for } t = 0, 1, 2 \end{cases}$$

(b) Properties of production function  $f(k_t, n_t) = k_t^\alpha n_t^{1-\alpha}$ , for  $t = 0, 1, 2$ :

i. Positive and diminishing marginal returns of capital and labor:

*Proof.*

$$\begin{aligned} f_1 &= \alpha k_t^{\alpha-1} n_t^{1-\alpha} > 0, f_{11} = -\alpha(1-\alpha) k_t^{\alpha-2} n_t^{1-\alpha} < 0 \\ f_2 &= (1-\alpha) k_t^\alpha n_t^{-\alpha} > 0, f_{22} = -\alpha(1-\alpha) k_t^\alpha n_t^{-\alpha-1} < 0 \end{aligned}$$

□

ii. Inada Condition:

*Proof.*

$$\begin{aligned} \lim_{k_t \rightarrow 0} f_1(k_t, n_t) &= \lim_{k_t \rightarrow 0} \alpha k_t^{\alpha-1} n_t^{1-\alpha} = \infty \\ \lim_{n_t \rightarrow 0} f_2(k_t, n_t) &= \lim_{n_t \rightarrow 0} (1-\alpha) k_t^\alpha n_t^{-\alpha} = \infty \end{aligned}$$

□

Properties of utility function  $u(c_t, 1 - n_t) = \frac{c_t^{1-\sigma}}{1-\sigma} + A \frac{(1-n_t)^{1-\gamma}}{1-\gamma}$ ,

for  $t = 0, 1, 2$ :

i. Positive and diminishing marginal utility of consumption and leisure:

*Proof.*

$$\begin{aligned} u_1 &= c_t^{-\sigma} > 0, u_{11} = -\sigma c_t^{-\sigma-1} < 0 \\ u_2 &= A(1-n_t)^{-\gamma} > 0, u_{22} = -\gamma A(1-n_t)^{-\gamma-1} < 0 \end{aligned}$$

□

ii. Inada Condition:

*Proof.*

$$\begin{aligned}\lim_{c_t \rightarrow 0} u_1(c_t, 1 - n_t) &= \lim_{c_t \rightarrow 0} c_t^{-\sigma} = \infty \\ \lim_{(1-n_t) \rightarrow 0} u_2(c_t, 1 - n_t) &= \lim_{(1-n_t) \rightarrow 0} A(1 - n_t)^{-\gamma} = \infty\end{aligned}$$

□

Consider the consumption non-negative constraints  $c_t \geq 0$ , for  $t = 0, 1, 2$ . These constraints cannot be binding because of the Inada condition  $\lim_{c_t \rightarrow 0} u_1(c_t, 1 - n_t) = \infty$  and FOC of  $[c_t]$ . A very small increase of consumption would lead to a huge increase in utility. Therefore  $c_t > 0$ , for  $t = 0, 1, 2$ . By the complementary slackness conditions, it must be that  $\nu_t = 0$ , for  $t = 0, 1, 2$ .

Similarly, the labor constraints  $n_t \geq 0$  and  $1 - n_t \geq 0$  for  $t = 0, 1, 2$  will not bind in any period thanks to the Inada condition  $\lim_{n_t \rightarrow 0} f_2(k_t, n_t) = \infty$ ,  $\lim_{(1-n_t) \rightarrow 0} u_2(c_t, 1 - n_t) = \infty$  and FOC of  $[n_t]$ . By the complementary slackness conditions, it must be that  $\theta_t = \eta_t = 0$ , for  $t = 0, 1, 2$ .

Finally, consider the capital non-negative constraints.  $k_1 \geq 0$  and  $k_2 \geq 0$  are not binding due to the Inada condition  $\lim_{k_t \rightarrow 0} f_1(k_t, n_t) = \infty$  and FOC of  $[k_{t+1}]$ . In the last period, however,  $k_3 \geq 0$  is binding. By FOC, we have  $\mu_2 = \lambda_2 = \beta^2 u_1(c_2, 1 - n_2) > 0$ . By the complementary slackness condition,  $k_3 = 0$ .

(c) Properties of utility function  $u(c_t, 1 - n_t) = \frac{c_t^{1-\sigma}}{1-\sigma}$ , for  $t = 0, 1, 2$ :

i. Positive and diminishing marginal utility of consumption:

*Proof.*

$$\begin{aligned}u_1 &= c_t^{-\sigma} > 0 \\ u_{11} &= -\sigma c_t^{-\sigma-1} < 0\end{aligned}$$

□

ii. The utility is irrelevant to the working hour.

*Proof.*

$$\frac{\partial u(c_t, 1 - n_t)}{\partial n_t} = 0$$

□

The optimal choice of labor is  $n_t = 1$ , for  $t = 0, 1, 2$ . The only difference between (b) and (c) is whether  $1 - n_t \geq 0$  is binding or not. Intuitive explanation: In part (b), there is a trade-off between consumption and working. However, the new utility function in part (c) is irrelevant to the working hour, which implies working increases production without reducing utility. Thus, the optimum is utilizing labor as much as possible.

## 2. C-K Model: finite horizon

### (a) Lagrangian Function:

$$L = \sum_{t=0}^T \beta^t \{u(c_t) + \lambda_t [f(k_t) - c_t - i_t] + \gamma_t [(1-\delta)k_t + i_t - k_{t+1}] + \mu_t k_{t+1} + \nu_t c_t\}$$

Necessary Conditions:

#### i. FOC:

$$\begin{cases} \frac{\partial L}{\partial c_t} = \beta^t [u'(c_t) - \lambda_t + \nu_t] = 0, \text{ for } t = 0, 1, \dots, T \\ \frac{\partial L}{\partial i_t} = \beta^t (-\lambda_t + \gamma_t) = 0, \text{ for } t = 0, 1, \dots, T \\ \frac{\partial L}{\partial k_{t+1}} = \beta^t [-\gamma_t + \mu_t + \beta \lambda_{t+1} f'(k_{t+1}) + \beta \gamma_{t+1} (1 - \delta)] = 0, \text{ for } t = 0, 1, \dots, T-1 \\ \frac{\partial L}{\partial k_T} = \beta^T (-\lambda_T + \mu_T) = 0 \end{cases}$$

#### ii. Inequality Constraints:

$$\begin{cases} f(k_t) - c_t - i_t \geq 0, \text{ for } t = 0, 1, \dots, T \\ (1 - \delta)k_t + i_t - k_{t+1} \geq 0, \text{ for } t = 0, 1, \dots, T \\ k_{t+1} \geq 0, \text{ for } t = 0, 1, \dots, T \\ c_t \geq 0, \text{ for } t = 0, 1, \dots, T \end{cases}$$

#### iii. Multipliers are non-negative:

$$\begin{cases} \beta^t \lambda_t \geq 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \gamma_t \geq 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \mu_t \geq 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \nu_t \geq 0, \text{ for } t = 0, 1, \dots, T \end{cases}$$

#### iv. Complementary Slackness:

$$\begin{cases} \beta^t \lambda_t (f(k_t) - c_t - i_t) = 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \gamma_t [(1 - \delta)k_t + i_t - k_{t+1}] = 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \mu_t k_{t+1} = 0, \text{ for } t = 0, 1, \dots, T \\ \beta^t \nu_t c_t = 0, \text{ for } t = 0, 1, \dots, T \end{cases}$$

- (b) Consider consumption non-negative constraints  $c_t \geq 0, \forall t = 0, 1, \dots, T$ . By Inada condition  $\lim_{c_t \rightarrow 0} u'(c_t) = \infty, c_t = 0, \forall t = 0, 1, \dots, T$  can never be optimal and it has to be the case that  $c_t > 0, \forall t = 0, 1, \dots, T$ . By Complementary slackness, this implies that  $\nu_t = 0, \forall t = 0, 1, \dots, T$ . For the resource constraints, by FOC and  $u'(c_t) > 0$ , we have  $\lambda_t = u'(c_t) > 0, \forall t = 0, 1, \dots, T$ . Thus, by complementary slackness, the resource constraints are binding, i.e.  $f(k_t) - c_t - i_t = 0, \forall t = 0, 1, \dots, T$ . For the capital accumulation constraints, we have  $\lambda_t = \gamma_t > 0, \forall t = 0, 1, \dots, T$  from FOC. Therefore, by complementary slackness, the capital accumulation constraints are binding, i.e.  $(1 - \delta)k_t + i_t - k_{t+1} = 0, \forall t = 0, 1, \dots, T$ . Eventually, consider the capital non-negative constraints.  $k_{t+1} \geq 0, \forall t = 0, 1, \dots, T - 1$  is not binding because  $f(\cdot)$  satisfies Inada condition and FOC of  $[k_{t+1}]$  goes to infinity at  $k_{t+1} = 0$ . Thus, by complementary slackness,  $\mu_t = 0, \forall t = 0, 1, \dots, T - 1$ . However, in the last period, we have  $\mu_T = \lambda_T > 0$  from FOC. Therefore, by complementary slackness,  $k_{T+1} = 0$ .

- (c) Rewriting the remaining equations

$$\begin{cases} \lambda_t = u'(c_t), \text{ for } t = 0, 1, \dots, T \\ \lambda_t = \gamma_t, \text{ for } t = 0, 1, \dots, T \\ \gamma_t = \beta\lambda_{t+1}f'(k_{t+1}) + \beta\gamma_{t+1}(1 - \delta), \text{ for } t = 0, 1, \dots, T - 1 \\ f(k_t) - c_t - i_t = 0, \text{ for } t = 0, 1, \dots, T \\ (1 - \delta)k_t + i_t - k_{t+1} = 0, \text{ for } t = 0, 1, \dots, T \\ k_{T+1} = 0 \end{cases}$$

Eliminating multipliers gives

$$\begin{cases} c_t = f(k_t) + (1 - \delta)k_t - k_{t+1}, \text{ for } t = 0, 1, \dots, T \\ u'(c_t) = \beta[f'(k_{t+1}) + (1 - \delta)]u'(c_{t+1}), \text{ for } t = 0, 1, \dots, T - 1 \\ k_{T+1} = 0 \end{cases} \Rightarrow \begin{cases} u'[f(k_t) + (1 - \delta)k_t - k_{t+1}] = \beta[f'(k_{t+1}) + (1 - \delta)] \\ \quad \cdot u'[f(k_{t+1}) + (1 - \delta)k_{t+1} - k_{t+2}], \forall t = 0, 1, \dots, T - 1 \\ k_{T+1} = 0 \end{cases}$$