Assignment 5

Haixiang Zhu

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1. (a) Define $V(\cdot)$ to be the value function, k' to be the capital stock chosen today and available for production in next period and c, k, respectively, to be the consumption and capital stock available for production in current period. k is the state variable and c, k' are the choice variables.
Bellman equation:

$$\begin{aligned}
V(k) &= \max_{c,k'}[u(c) + \beta V(k')] \\
&= \max_{c,k'}[\frac{c^{1-\sigma}-1}{1-\sigma} + \beta V(k')] \\
s.t.\quad & c + k' = Ak \\
& c \geq 0 \\
& k' \geq 0 \\
& \sigma > 0 \\
& k \text{ given}
\end{aligned}$$

(b) Inada conditions of the period utility function ($\sigma > 0$):

$$\begin{cases} \lim_{c\to 0} u'(c) = \lim_{c\to 0} c^{-\sigma} = \infty \\ \lim_{c\to\infty} u'(c) = \lim_{c\to\infty} c^{-\sigma} = 0 \end{cases}$$

Guess:
Consider a one-period problem:
Since $V_0(k) = 0$,

$$\begin{aligned}
V_1(k) &= \max_{c,k'}[\frac{c^{1-\sigma}-1}{1-\sigma} + \beta \cdot 0] \\
s.t.\quad & c + k' = Ak \\
& c \geq 0 \\
& k' \geq 0 \\
& \sigma > 0 \\
& k \text{ given}
\end{aligned}$$

Because $u(\cdot)$ satisfy the Inada conditions, the consumption constraint will not bind, i.e. $c > 0$. By the FOC of $[k']$ and complementary slackness, one can show that the policy functions

$$\begin{aligned}
&k' = 0 \equiv g_1(k) \\
\Rightarrow &c = Ak \equiv g_1^c(k) \\
\Rightarrow &V_1(k) = \frac{(Ak)^{1-\sigma} - 1}{1-\sigma}
\end{aligned}$$

Consider a two-period problem:

$$\begin{aligned}
V_2(k) &= \max_{c,k'} [\frac{c^{1-\sigma} - 1}{1-\sigma} + \beta V_1(k')] \\
&= \max_{c,k'} [\frac{c^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(Ak')^{1-\sigma} - 1}{1-\sigma}] \\
s.t. \quad & c + k' = Ak \\
& c \geq 0 \\
& k' \geq 0 \\
& \sigma > 0 \\
& k \text{ given}
\end{aligned}$$

Lagrangian Function:

$$L = \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(Ak')^{1-\sigma} - 1}{1-\sigma} - \lambda(c + k' - Ak) + \nu c + \mu k'$$

Necessary Conditions:

$$\begin{cases}
\dfrac{\partial L}{\partial c} = c^{-\sigma} - \lambda + \nu = 0 \\
\dfrac{\partial L}{\partial k'} = \beta A^{1-\sigma} (k')^{-\sigma} - \lambda + \mu = 0 \\
c + k' - Ak = 0 \\
c \geq 0 \\
k' \geq 0 \\
\nu \geq 0 \\
\mu \geq 0 \\
\nu c = 0 \\
\mu k' = 0
\end{cases}$$

Similar to one-period problem, $c > 0, \nu = 0$. If $k' \to 0$, $\beta A^{1-\sigma}(k')^{-\sigma} \to \infty$, which contradicts the FOC of $[k']$. Hence, $k' > 0, \mu = 0$.

Reduced form:

$$\begin{aligned}
&\beta A^{1-\sigma}(k')^{-\sigma} = (Ak-k')^{-\sigma} \\
\Rightarrow & k' = \frac{(\beta A)^{\frac{1}{\sigma}}}{A+(\beta A)^{\frac{1}{\sigma}}} Ak \equiv g_2(k) \\
\Rightarrow & c = Ak - k' = \frac{A}{A+(\beta A)^{\frac{1}{\sigma}}} Ak \equiv g_2^c(k) \\
\Rightarrow & V_2(k) = \frac{(\frac{A}{A+(\beta A)^{\frac{1}{\sigma}}} Ak)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{(A\frac{(\beta A)^{\frac{1}{\sigma}}}{A+(\beta A)^{\frac{1}{\sigma}}} Ak)^{1-\sigma} - 1}{1-\sigma}
\end{aligned}$$

From $V_1(k)$ and $V_2(k)$, we guess the stationary value has the form

$$V(k) = \frac{Xk^{1-\sigma} + Y}{1-\sigma}$$

Verify:
The infinite horizon Bellman equation we are trying to solve is:

$$\begin{aligned}
V(k) = \max_{k'} & [\frac{(Ak-k')^{1-\sigma}-1}{1-\sigma} + \beta\frac{X(k')^{1-\sigma}+Y}{1-\sigma}] \\
s.t. \quad & c = Ak - k' \geq 0 \\
& k' \geq 0 \\
& \sigma > 0 \\
& k \text{ given}
\end{aligned}$$

Lagrangian Function:

$$L = \frac{(Ak-k')^{1-\sigma}-1}{1-\sigma} + \beta\frac{X(k')^{1-\sigma}+Y}{1-\sigma} + \nu(Ak-k') + \mu k'$$

Necessary Conditions:

$$\begin{cases}
\frac{\partial L}{\partial k'} = -(Ak-k')^{-\sigma} + \beta X(k')^{-\sigma} - \nu + \mu = 0 \\
Ak - k' \geq 0 \\
k' \geq 0 \\
\nu \geq 0 \\
\mu \geq 0 \\
\nu(Ak-k') = 0 \\
\mu k' = 0
\end{cases}$$

Similar to two-period problem, $k' > 0, \mu = 0$. If $(Ak - k') \to 0$, $-(Ak-k')^{-\sigma} \to -\infty$, which contradicts the FOC of $[k']$. Hence,

$Ak - k' > 0, \nu = 0$.
Reduced form:

$$\begin{aligned}
&\beta X(k')^{-\sigma} = (Ak - k')^{-\sigma} \\
\Rightarrow& k' = \frac{(\beta X)^{\frac{1}{\sigma}}}{1 + (\beta X)^{\frac{1}{\sigma}}} Ak \equiv g(k) \\
\Rightarrow& c = Ak - k' = \frac{1}{1 + (\beta X)^{\frac{1}{\sigma}}} Ak \equiv g^c(k) \\
\Rightarrow& V(k) = \frac{\left(\frac{1}{1+(\beta X)^{\frac{1}{\sigma}}} Ak\right)^{1-\sigma} - 1}{1-\sigma} + \beta \frac{X\left(\frac{(\beta X)^{\frac{1}{\sigma}}}{1+(\beta X)^{\frac{1}{\sigma}}} Ak\right)^{1-\sigma} + Y}{1-\sigma} \\
&= \frac{\left(\frac{A}{1+(\beta X)^{\frac{1}{\sigma}}}\right)^{1-\sigma} [1 + (\beta X)^{\frac{1}{\sigma}}] k^{1-\sigma} + \beta Y - 1}{1-\sigma} \\
&= \frac{A^{1-\sigma}[1 + (\beta X)^{\frac{1}{\sigma}}]^{\sigma} k^{1-\sigma} + \beta Y - 1}{1-\sigma}
\end{aligned}$$

For the Bellman equation to be satisfied, it must be that this is equal to our guess. We do this by equating coefficients

$$\begin{cases} A^{1-\sigma}[1 + (\beta X)^{\frac{1}{\sigma}}]^{\sigma} = X \\ \beta Y - 1 = Y \end{cases}$$

$$\Rightarrow \begin{cases} X = (A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}})^{-\sigma} \\ Y = \dfrac{1}{\beta - 1} \end{cases}$$

Hence, the value function is

$$V(k) = \frac{(A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}})^{-\sigma} k^{1-\sigma}}{1-\sigma} + \frac{1}{(1-\sigma)(\beta-1)}$$

and the optimal policy function of capital is

$$\begin{aligned}
g(k) &= \frac{(\beta X)^{\frac{1}{\sigma}}}{1 + (\beta X)^{\frac{1}{\sigma}}} Ak \\
&= \frac{Ak}{1 + (\beta X)^{-\frac{1}{\sigma}}} \\
&= \frac{Ak}{1 + [(A^{\frac{\sigma-1}{\sigma}} \beta^{-\frac{1}{\sigma}} - 1)^{-\sigma}]^{-\frac{1}{\sigma}}} \\
&= \frac{Ak}{A^{\frac{\sigma-1}{\sigma}} \beta^{-\frac{1}{\sigma}}} \\
&= k(A\beta)^{\frac{1}{\sigma}}
\end{aligned}$$

The optimal policy function of consumption is

$$g^c(k) = Ak - k' = Ak(1 - A^{\frac{1-\sigma}{\sigma}}\beta^{\frac{1}{\sigma}})$$

(c) *Proof.*

$$\begin{aligned}
\lim_{\sigma\to 1} u(c_t) &= \lim_{\sigma\to 1} \frac{c_t^{1-\sigma} - 1}{1-\sigma} \\
&\overset{t=1-\sigma}{=\!=\!=} \lim_{t\to 0} \frac{c^t - 1}{t} \\
&\overset{L'Hopital's\ \ rule}{=\!=\!=\!=\!=\!=} \lim_{t\to 0} c^t \ln(c_t) \\
&= \ln(c_t)
\end{aligned}$$

□

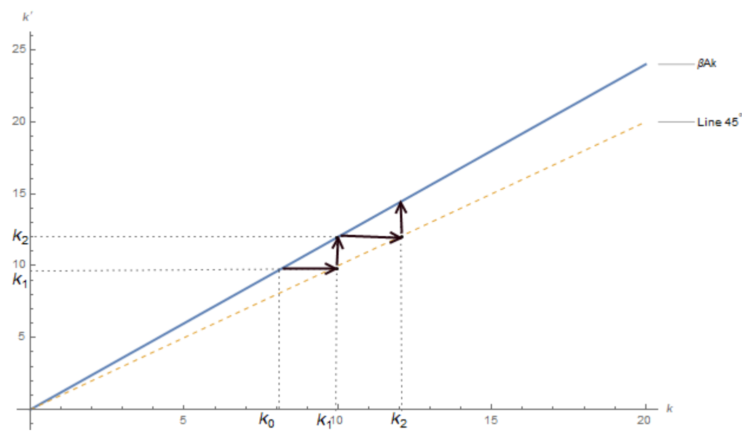
When $\sigma = 1$, the value function is

$$\begin{aligned}
\lim_{\sigma\to 1} V(k) &= \lim_{\sigma\to 1} \frac{(A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}})^{-\sigma} k^{1-\sigma} + (\beta-1)^{-1}}{1-\sigma} \\
&= (1-\beta)^{-1} \lim_{\sigma\to 1} \frac{k^{1-\sigma} - 1}{1-\sigma} + \lim_{\sigma\to 1} \frac{(A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}})^{-\sigma} + (\beta-1)^{-1}}{(1-\sigma)} \\
\overset{L'Hopital's\ \ rule}{=\!=\!=\!=\!=\!=} & \frac{\ln k}{1-\beta} + \lim_{\sigma\to 1} (A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}})^{-\sigma} \\
&\cdot \lim_{\sigma\to 1} \left[\ln(A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}}) + \sigma \frac{A^{\frac{\sigma-1}{\sigma}} \ln(A)\sigma^{-2} + \beta^{\frac{1}{\sigma}} \ln(\beta)\sigma^{-2}}{A^{\frac{\sigma-1}{\sigma}} - \beta^{\frac{1}{\sigma}}} \right] \\
&= \frac{\ln k}{1-\beta} + \frac{1}{1-\beta} \left[\ln(1-\beta) + \frac{\ln A + \beta \ln \beta}{1-\beta} \right]
\end{aligned}$$

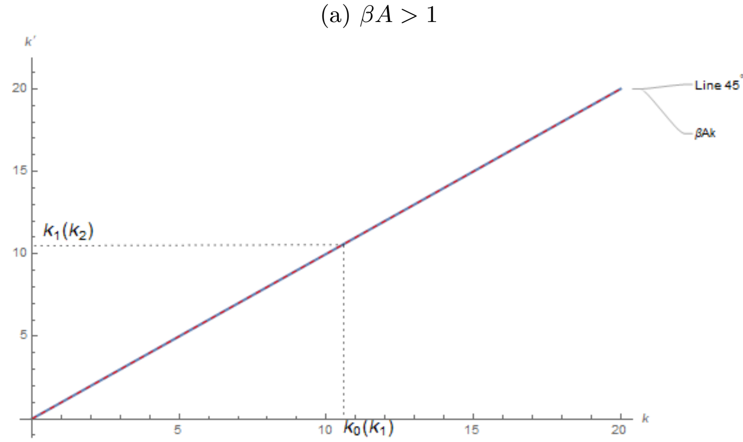
and, plugging $\sigma = 1$, the optimal policy functions are

$$\begin{cases} g(k) = \beta A k \\ g^c(k) = (1-\beta) A k \end{cases}$$

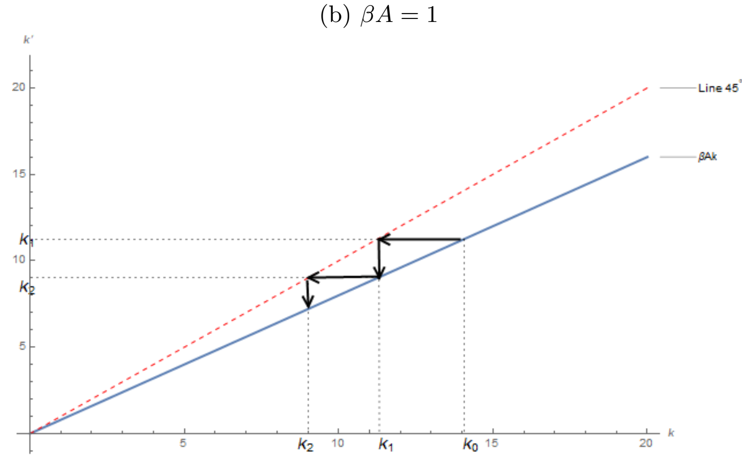
(d) As shown in Figure 1, the dynamic of capital depends on the product of β and A. If $\beta A > 1$, $\forall k_0 > 0$, the capital stock will keep increasing by $\beta A - 1$. If $\beta A = 1$, $\forall k_0 > 0$, the capital stock will remain unchanged. If $\beta A < 1$, $\forall k_0 > 0$, the capital stock will keep decreasing by $1 - \beta A$ until $k = 0$.



(a) $\beta A > 1$



(b) $\beta A = 1$



(c) $\beta A < 1$

Figure 1: The Dynamics of Capital

2. (a) Budget constraint:

$$b_t + s_t = Rs_{t-1} + w, \quad \forall t = 0, 1, 2, \dots$$

where $s_{-1} = 0, b_t, s_t \geq 0$

(b) Dynamic programming problem:
Define $u(\cdot)$ to be the utility function, $V(\cdot)$ to be the value function, s^- to be the bananas saved in last period and s, b, respectively, to be the bananas saved and consumed in current period.
s^- is the state variable and s, b are the choice variables.
Bellman equation:

$$V(s^-) = \max_{b,s}[u(b) + \beta V(s)]$$

$$\begin{aligned} s.t. \quad & b + s - Rs^- = w \\ & b \geq 0 \\ & s \geq 0 \\ & s^- \text{ given} \end{aligned}$$

(c) WLOG, assume that $t = 2k, \ \forall k = 1, 2, \dots$

$$\begin{cases} b_t + s_t = Rs_{t-1} + w_H \\ b_{t-1} + s_{t-1} = Rs_{t-2} + w_L \end{cases}$$
$$\Rightarrow b_t + s_t + b_{t-1} + (1-R)s_{t-1} - Rs_{t-2} = w_H + w_L$$

Dynamic programming problem:
Define $u(\cdot)$ to be the utility function, $V(\cdot)$ to be the value function, $s^=$ to be the bananas saved two period's ago, s^-, b^-, respectively, to be the bananas saved and consumed in last period and s, b, respectively, to be the bananas saved and consumed in current period.
$b^-, s^-, s^=$ are the state variables and b, s are the choice variables.
Bellman equation:

$$V(b^-, s^-, s^=) = \max_{b,s}[u(b) + \beta V(b, s, s^-)]$$

$$\begin{aligned} s.t. \quad & b + s + b^- + (1-R)s^- - Rs^= = w_H + w_L \\ & b \geq 0 \\ & s \geq 0 \\ & b^-, s^-, s^= \text{ given} \end{aligned}$$