When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

1. Let X_n denote a random variable with mean μ and variance σ^2/n^p, where $p > 0$, μ, and σ^2 are constants (not functions of n). Show that X_n converges in probability to μ. (*Hint*: Use Chebyshev?s inequality.)

2. Let $W_n \sim \chi^2_n$. Then the moment generating function of W_n is given by

$$M_{W_n}(t) = (1-2t)^{-n/2}, \text{ for } t < 0.5.$$

We would like to investigate the limiting distribution of the random variable

$$Y_n = \frac{W_n - n}{\sqrt{2n}}.$$

Follow these steps

a) Derive the Moment Generating Function of Y_n. Show that this is equal to

$$M_{Y_n}(t) = \left(e^{t\sqrt{2/n}} - t\sqrt{\frac{2}{n}}e^{t\sqrt{2/n}}\right)^{-n/2}, \text{ for } t < \sqrt{\frac{n}{2}}.$$

b) Use a Taylor expansion of the exponential function up to the third order to finally show that

$$\lim_{n\to\infty} M_{Y_n}(t) = e^{t^2/2}.$$

c) What is then the asymptotic distribution of the random variable Y_n?

3. Consider the following random variable $X^* \sim N(\mu, 1)$, with $\{X_i^*, i = 1, \dots, n\}$, an IID sample from this distribution. In some cases, it is not possible to directly observe X_i^*, and we only have access to a (nonlinear) transformation of X_i^*, which we denote X_i. Derive the asymptotic properties of the maximum likelihood estimator of μ in the three following cases, where $\mathbb{1}$ is the indicator function,

a) $X_i = X_i^*$. That is, we directly observe the random variable X_i^*.

b) $X_i = X_i^* \mathbb{1}(X_i^* > 0)$. In this case, we only observe X_i^* when it is positive, and 0 otherwise. The distribution is a truncated normal distribution at 0. Its pdf is

$$f_X(x;\mu) = \frac{\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(X-\mu)^2}{2}\right)}{1-\Phi(-\mu)},$$

where Φ is the CDF of a standard normal random variable. (*Hint*: You are not going to be able to find a closed form solution here. Obtain directly the second derivative and infer the properties of the asymptotic distribution. Let $\phi(\cdot)$ be the pdf of a standard normal distribution. The mean and variance of a truncated normal at 0 are given by

$$E(X) = \mu + \frac{\phi(-\mu)}{1-\Phi(-\mu)}$$

$$Var(X) = 1 - \frac{\mu\phi(-\mu)}{1-\Phi(-\mu)} - \left(\frac{\phi(-\mu)}{1-\Phi(-\mu)}\right)^2)$$

c) $X_i = \mathbb{1}(X_i^* > 0)$. (*Hint*: You should know what the distribution of X_i is in this case.)

d) Are you able to say anything about the relative asymptotic efficiency of these estimators? What happens when our information about X_i^* decreases?

4. Consider the uniform distribution with density function $f_X(x|\theta) = 1/\theta$, $0 \leq x \leq \theta$, and θ unknown.

a) Show that the Pareto distribution,

$$\pi_\Theta(\theta) = \begin{cases} a k^a \theta^{-(a+1)}, & \theta \geq k, a > 0 \\ 0, & \text{otherwise} \end{cases},$$

is a conjugate prior for the uniform distribution.

b) Show that $\hat{\theta} = \max\{X_1, \ldots, X_n\}$ is the Maximum Likelihood Estimator of θ, where $\{X_1, \ldots, X_n\}$ is an IID sample from $f_X(x;\theta)$.

c) Find the posterior distribution. (*Hint*: It is convenient in this case to find the exact expression of the posterior, so you may not want to ignore the denominator in the Bayes' formula this time.)

d) Find the bayesian point estimator for the quadratic cost function.

e) Find the MAP estimator and compare it to the MLE.

The End.