When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

1. Given $P(A) = 1/3$, $P(B) = 1/4$, and $P(A \cap B) = 1/6$, find the following probabilities

$$P(A^c), \quad P(A^c \cup B), \quad P(A \cap B^c), \quad P(A \cup B^c), \quad P(A^c \cup B^c).$$

2. For any three events A, B, and C, show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

3. Three boxes contain two rings each, but in one of them they are both gold, in the second both silver, and in the third one of each type. You have the choice of randomly extracting a ring from one of the boxes, the content of which is unknown to you. You look at the extracted ring, and then you have the possibility to extract a second ring again from any of the three boxes. Suppose you can select each box with equal probability. Let us assume the first ring you extract is a gold one. Is it preferable to extract the second one from the same or from a different box?

4. Recall that the pmf of the Poisson distribution can be written as

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Show that the pmf satisfies

$$p_X(k) = \frac{\lambda}{k} p_X(k-1),$$

and hence determine the values of k for which the terms $p_X(k)$ reach their maximum (for a given λ).

5. Suppose that the duration (in minutes) of long-distance telephone calls follows an exponential density function

$$f_X(x) = \frac{1}{5}e^{-x/5}, \text{ for } x > 0.$$

Find the probability that the duration of a conversation

a) Will exceed 5 minutes.
b) Will be between 5 and 6 minutes.
c) Will be less than 3 minutes.
d) Will be less than 6 minutes given that it was greater than 3 minutes.

6. Let (X, Y) be *uniform* in the triangle $x \geq 0$, $y \geq 0$, and $x + y \leq 2$ (*Hint*: Uniform means it is constant over the entire support). Find:

a) The joint pdf of (X, Y).
b) The marginal pdf of X.

c) The conditional pdf of Y given $X = x$. Is this conditional distribution defined for every $x \in [0, 2]$? Explain.

d) $E[Y|X = x]$.

7. Let X and Z be independent, each following a standard normal distribution. Let $a, b \in \mathbb{R}$ (not simultaneously equal to 0), and let $Y = aX + bZ$.

   a) Compute $Corr(X, Y)$.

   b) Show that $|Corr(X, Y)| \leq 1$ in this case.

   c) Give necessary and sufficient conditions on the values of a and b such that $Corr(X, Y) = 1$.

8. Show that for a continuous random variable X with density function f_X and cumulative distribution function F_X

$$\mu = E(X) = \int_0^\infty (1 - F_X(x))\, dx - \int_{-\infty}^0 F_X(x) dx.$$

   (*Hint*: Use the definition of the mean and integration by parts).

The End.