When a question asks to *show* a result or to *find* an object, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

Recall that the probability density function of a normal random variable X with mean μ and variance σ^2 is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

1. Let Y is a strictly positive random variable which follows a normal distribution truncated at 0, with location parameter equal to 0, and scale parameter equal to σ_Y^2. That is, the support of Y is $[0, \infty)$, and its density is equal to

$$f_Y(y) = \frac{2}{\sqrt{2\pi}\sigma_Y} e^{-\frac{y^2}{2\sigma_Y^2}} \mathbb{1}(y \geq 0)$$

   a) Show that the MGF of Y is equal to

$$M_Y(t) = 2e^{\frac{\sigma_Y^2 t^2}{2}}(1 - \Phi(-\sigma_Y t)),$$

   where Φ is the cdf of a standard normal random variable.

   b) Find the mean and the variance of Y.

2. Let $X_1, \ldots, X_n$ be an IID sample with pdf equal to

$$f_X(x, \theta_0) = \theta_0 x^{\theta_0 - 1}, \quad x \in [0, 1], \text{ and } \theta_0 > 0.$$

   a) Find the MLE of θ_0, $\hat{\theta}_n$.
   b) Show that $\hat{\theta}_n$ is a consistent estimator of θ_0.
   c) Find the asymptotic distribution of $\hat{\theta}_n$.

3. Let $X_1 \sim N(\mu_1, \sigma_1^2)$ and $X_2 \sim N(\mu_2, \sigma_2^2)$ be two independent random variables, and $Y = X_1 + X_2$

   a) Find the distribution of Y.
   b) Would the result in (a) still hold true if X_1 and X_2 were correlated? Explain.