ECO 520 - FALL 2019 FINAL EXAM DECEMBER 6, 2019

When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

1. Suppose that if $X_n \xrightarrow{p} X$ and $Y_n \xrightarrow{p} Y$ with $X = Y$, almost surely. Show that $P(|X_n - Y_n| > \epsilon) \to 0$ as $n \to \infty$, for any $\epsilon > 0$.

2. The pair $\{X, Y\}$ has joint density function $f_{XY}(x, y) = x + y$, for $0 \le x \le 1$ and $0 \le y \le 1$.

   a) Find the Moment Generating Function and the Characteristic Function of this joint distribution.
   b) Find the marginal distribution of X, its Moment Generating function and its Characteristic function.

3. Recall that the Moment Generating function of the binomial distribution $B(x; n, p)$ is given by

   $$M_X(t) = (pe^t + 1 - p)^n.$$

   Use this result to show that for $n \to \infty$ and $p \to 0$, such that $np \to \lambda < \infty$, the MGF of a binomial distribution converges to the MGF of a Poisson distribution with parameter λ for every $t \in \mathbb{R}$, where the MGF of the Poisson distribution is given by $M_X(t) = e^{\lambda(e^t-1)}$.

4. Consider a random sample $\{(X_{11}, X_{21}), \ldots, (X_{1n}, X_{2n})\}$ of size n drawn from the bivariate normal distribution

   $$N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right).$$

   Recall that the density of the bivariate normal distribution is given by

   $$f_{X_1X_2}(x_1, x_2; \mu_1, \mu_2, \sigma_1, \sigma_2, \rho) = \frac{1}{2\pi\sqrt{(1-\rho^2)}\sigma_1\sigma_2} \exp\left(-\frac{(\sigma_2^2 x_1^2 - 2\rho\sigma_1\sigma_2 x_1 x_2 + \sigma_1^2 x_2^2)}{2(1-\rho^2)\sigma_1^2\sigma_2^2}\right)$$

   a) Determine the Maximum Likelihood estimator of the correlation coefficient ρ when the parameters μ_1, μ_2, σ_1, σ_2 are known.
   b) Find the information matrix and determine the asymptotic distribution of the estimator of ρ derived above.
   c) Determine the Maximum Likelihood estimator of the correlation coefficient ρ when the other parameters are unknown.

5. Consider the Poisson distribution with density function $f_{X|\theta}(x|\theta) = e^{-\theta}\theta^x/x!$, $x = 0, 1, 2, \ldots$, and θ unknown.

   a) We consider the following exponential prior distribution for θ,

   $$\pi_\Theta(\theta) = \begin{cases}\lambda e^{-\lambda\theta}, & \theta > 0\\ 0, & \text{otherwise}\end{cases}.$$

   Find the posterior distribution. Do the Poisson and exponential form a conjugate family of distributions?
   b) Find the Maximum Likelihood Estimator of θ, where $\{X_1, \ldots, X_n\}$ is an IID sample from $f_X(x; \theta)$.
   c) Find the mean of the posterior distribution.
   e) Find the MAP estimator and compare it to the MLE.