Midterm ECO 521

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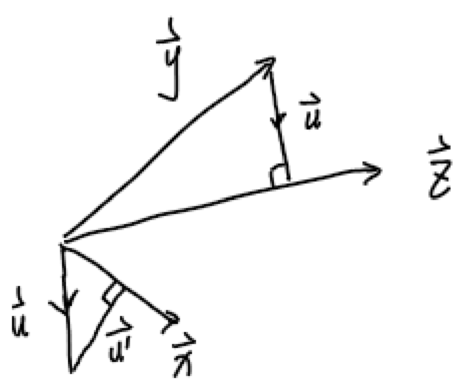
1. Omitted Variables

$$\begin{aligned}
\hat{b} &= (X'X)^{-1}X'y \\
&= \beta + (X'X)^{-1}X'Z\gamma + (X'X)^{-1}X'u \\
plim\hat{b} &= \beta + plim\left(\frac{X'X}{n}\right)^{-1} plim\left(\frac{X'Z}{n}\right)\gamma + plim\left(\frac{X'X}{n}\right)^{-1} plim\left(\frac{X'u}{n}\right) \\
&= \beta + plim\left(\frac{X'X}{n}\right)^{-1} plim\left(\frac{X'Z}{n}\right)\gamma \\
Var(\hat{b}) &= E(\hat{b} - plim\hat{b})(\hat{b} - plim\hat{b})' \\
&= E\left[plim\left(\frac{X'X}{n}\right)^{-1} plim\left(\frac{X'u}{n}\right) plim\left(\frac{u'X}{n}\right) plim\left(\frac{X'X}{n}\right)^{-1}\right] \\
&= \frac{1}{n}\sigma^2 plim\left(\frac{X'X}{n}\right)^{-1}
\end{aligned}$$

Its asymptotic properties

$$\sqrt{n}(\hat{b} - plim\hat{b}) \sim N\left(0, \sigma^2 plim\left(\frac{X'X}{n}\right)^{-1}\right)$$

2. Projection of residuals



3. Dummy Variables

$$e_i = \sum_j X_{ij}\beta_j + A_i\gamma + A_i\sum_j X_{ij}\delta_j + u_i$$

where X_{ij} is a dummy variable, which equals to 1 if person i injected with vaccine j, β_j is the efftiveness of the vaccine holding everthing else constant and δ_j is the interaction of age with different common vaccines.

4. DID Model

$$\begin{aligned}
&\log w_i = X_i\beta + u_i \\
&Var(u_i) = e^{\lambda + Age_i\delta + Female_i\alpha + Age_iFemale_i\theta + e_i} \\
\Rightarrow &\log(u_i^2) = \lambda + Age_i\delta + Female_i\alpha + Age_iFemale_i\theta + e_i
\end{aligned}$$

5. ARMA(2,2)

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2}$$

Let $\gamma(k) = E(u_t u_{t-k})$. For simplicity, we first calculate $E(u_t e_t), E(u_t e_{t-1})$ and $E(u_t e_{t-2})$ respectively.

$$\begin{aligned}
E(u_te_t) &= \rho_1 E(u_{t-1}e_t) + \rho_2 E(u_{t-2}e_t) + a_0 Ee_t^2 + a_1 E(e_{t-1}e_t) + a_2 E(e_{t-2}e_t) \\
&= a_0\sigma_e^2 \qquad (1) \\
&= E(u_{t-1}e_{t-1}) = E(u_{t-2}e_{t-2}) \\
E(u_te_{t-1}) &= \rho_1 E(u_{t-1}e_{t-1}) + \rho_2 E(u_{t-2}e_{t-1}) + a_0 E(e_te_{t-1}) + a_1 E(e_{t-1}^2) + a_2 E(e_{t-2}e_{t-1}) \\
&= (\rho_1 a_0 + a_1)\sigma_e^2 \qquad (2) \\
&= E(u_{t-1}e_{t-2}) \\
E(u_te_{t-2}) &= \rho_1 E(u_{t-1}e_{t-2}) + \rho_2 E(u_{t-2}e_{t-2}) + a_0 E(e_te_{t-2}) + a_1 E(e_{t-1}e_{t-2}) + a_2 E(e_{t-2}^2) \\
&= \rho_1(\rho_1 a_0 + a_1)\sigma_e^2 + \rho_2 a_0\sigma_e^2 + a_2\sigma_e^2 \\
&= (\rho_1^2 a_0 + \rho_1 a_1 + \rho_2 a_0 + a_2)\sigma_e^2 \qquad (3)
\end{aligned}$$

The Yule-Walker equations for an ARMA(2,2) process is

$$\begin{aligned}
\gamma(0) &= Eu_t^2 \\
&= E(\rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2})^2 \\
&= (\rho_1^2 + \rho_2^2)\gamma(0) + 2\rho_1\rho_2\gamma(1) + (a_0^2 + a_1^2 + a_2^2)\sigma_e^2 + 2\rho_1 a_1 E(u_{t-1}e_{t-1}) \\
&\quad + 2\rho_2 a_2 E(u_{t-2}e_{t-2}) + 2\rho_1 a_2 E(u_{t-1}e_{t-2}) \\
&= (\rho_1^2 + \rho_2^2)\gamma(0) + 2\rho_1\rho_2\gamma(1) + (a_0^2 + a_1^2 + a_2^2 + 2\rho_1 a_1 a_0 + 2\rho_2 a_2 a_0 + 2\rho_1 a_2 a_1 + 2\rho_1^2 a_2 a_0)\sigma_e^2
\end{aligned}$$

Alternatively

$$
\begin{aligned}
\gamma(0) &= Eu_t^2 \\
&= E[u_t(\rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2})] \\
&= \rho_1 E(u_t u_{t-1}) + \rho_2 E(u_t u_{t-2}) + a_0 E(u_t e_t) + a_1 E(u_t e_{t-1}) + a_2 E(u_t e_{t-2}) \\
&\xlongequal{\text{plugging in (1),(2) and (3)}} \rho_1\gamma(1) + \rho_2\gamma(2) + (a_0^2 + a_1^2 + a_2^2 + a_1\rho_1 a_0 + a_2\rho_2 a_0 + a_2\rho_1 a_1 + a_2\rho_1^2 a_0)\sigma_e^2 \\
\gamma(1) &= E(u_t u_{t-1}) \\
&= E[u_{t-1}(\rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2})] \\
&= \rho_1 Eu_{t-1}^2 + \rho_2 E(u_{t-1} u_{t-2}) + a_0 E(u_{t-1} e_t) + a_1 E(u_{t-1} e_{t-1}) + a_2 E(u_{t-1} e_{t-2}) \\
&\xlongequal{\text{plugging in (1) and (2)}} \rho_1\gamma(0) + \rho_2\gamma(1) + (a_1 a_0 + a_2 a_1 + a_2\rho_1 a_0)\sigma_e^2 \\
\gamma(2) &= E(u_t u_{t-2}) \\
&= E[u_{t-2}(\rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2})] \\
&= \rho_1 E(u_{t-2} u_{t-1}) + \rho_2 Eu_{t-2}^2 + a_0 E(u_{t-2} e_t) + a_1 E(u_{t-2} e_{t-1}) + a_2 E(u_{t-2} e_{t-2}) \\
&\xlongequal{\text{plugging in (1)}} \rho_1\gamma(1) + \rho_2\gamma(0) + a_2 a_0 \sigma_e^2 \\
\gamma(k) &= E(u_t u_{t-k}) \\
&= E[u_{t-k}(\rho_1 u_{t-1} + \rho_2 u_{t-2} + a_0 e_t + a_1 e_{t-1} + a_2 e_{t-2})] \\
&= \rho_1\gamma(k-1) + \rho_2\gamma(k-2) \quad k \geq 3
\end{aligned}
$$

6. Random Effect Model

Rewrite the error as

$$v = u + e$$

where e is a AR(1) process and u is a random effect.

$$
\begin{aligned}
\Omega &= E(vv') \\
&= \begin{pmatrix} A & 0_T & \cdots & 0_T \\ 0_T & A & \cdots & 0_T \\ \vdots & \vdots & \ddots & \vdots \\ 0_T & 0_T & \cdots & A \end{pmatrix}
\end{aligned}
$$

where

$$
A_{T\times T} = \begin{pmatrix}
\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \rho\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \cdots & \rho^{T-1}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 \\
\rho\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \cdots & \rho^{T-2}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 \\
\vdots & \vdots & \ddots & \vdots \\
\rho^{T-1}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \rho^{T-2}\frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2 & \cdots & \frac{\sigma_\varepsilon^2}{1-\rho^2} + \sigma_u^2
\end{pmatrix}
$$

There are three unknown parameters: $\rho, \sigma_\varepsilon, \sigma_u$. Using MOM, we comptute first three moments to estimate them and get $\hat{\Omega}$. Therefore,

$$\beta_{FGLS} = \left(X'\hat{\Omega}^{-1}X\right)^{-1} X'\hat{\Omega}^{-1}y$$

Alternatively, following Andrews (1991) and Newey & West (1987,1994), we can construct a heteroskedasticity and autocorrection consistent variance-covariance matrix estimator to estimate Ω.