When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

Consider a random sample $\{X_1, \ldots, X_n\}$ drawn from an exponential distribution with parameter θ. The pdf of an exponential distribution is given by

$$f_X(X_i|\theta) = \begin{cases} \theta e^{-\theta X_i} & \text{if } X_i \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

a) For $t < \theta$, find the moment generating function (MGF) of the exponential distribution in (1).

b) Find the mean and the variance of the exponential distribution in (1).

c) Let $\bar{X}$ be the sample mean. Derive the MGF of $\bar{X}$.

d) The Gamma distribution has density

$$f(\bar{x}; \alpha, \beta) = \frac{\beta^\alpha \bar{x}^{\alpha-1} e^{-\beta\bar{x}}}{\Gamma(\alpha)},$$

for $\bar{x} > 0$, and $\alpha, \beta > 0$, where α is the shape parameter, β is the rate parameter and $\Gamma(\alpha) = (\alpha - 1)!$ is the so-called Gamma function. The MGF of a Gamma distribution is equal to

$$MGF(t) = \left(\frac{\beta}{\beta - t}\right)^\alpha, \text{ for } t < \beta.$$

Show that the MGF derived in c is the one of a Gamma distribution, upon appropriate choices of α and β.

e) Compute $E\left(1/\bar{x}\right)$.

f) Derive the maximum likelihood (ML) estimator of θ for the one parameter exponential distribution in (1) and its asymptotic distribution.

g) Using your computation in e), show that the maximum likelihood estimator is biased, and derive a bias corrected estimator of θ. Show that the bias corrected estimator is asymptotically equivalent to the ML estimator.