ECO 520 - Fall 2020
Midterm Exam

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1. Let P,C be permutation and combination respectively.

   (a) Drawing with replacement

      i.
      $$Prob = \frac{6 \cdot 25^5}{26^6} = 0.1897$$

      ii.
      $$Prob = \frac{6^6}{26^6} = 1.5103 \times 10^{-4}$$

      iii.
      $$Prob = \frac{1}{26^6} = 3.2371 \times 10^{-9}$$

   (b) Drawing without replacement

      i.
      $$Prob = \frac{6 \cdot P_{25}^5}{P_{26}^6} = \frac{6 \cdot \frac{25!}{20!}}{\frac{26!}{20!}} = \frac{3}{13}$$

      ii.
      $$Prob = \frac{P_6^6}{P_{26}^6} = \frac{6!}{\frac{26!}{20!}} = \frac{1}{230230}$$

      iii. Since there are two "r" and two "e" in "reader", it cannot happen in drawing without replacement.
      $$Prob = 0$$

2. Let X, Y be the number of boys and girls respectively.

(a) *Proof.*

$$\because\ P(A) = P(X=0) + P(Y=0) = 2 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{4}$$

$$P(B) = P(Y \le 1) = P(Y=0) + P(Y=1) = \left(\frac{1}{2}\right)^3 + 3 \cdot \left(\frac{1}{2}\right)^3 = \frac{1}{2}$$

$$P(A \cap B) = P(Y=0) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$\therefore\ P(A \cap B) = P(A) \cdot P(B) = \frac{1}{8}$$

$$\because\ P(C) = P(X \ge 1, Y \ge 1) = 1 - P(X=0) - P(Y=0) = \frac{3}{4}$$

$$P(B \cap C) = P(X \ge 1, Y = 1) = P(X=2, Y=1) = 3 \cdot \left(\frac{1}{2}\right)^3 = \frac{3}{8}$$

$$\therefore\ P(B \cap C) = P(B) \cdot P(C) = \frac{3}{8}$$

□

(b)

$$\because\ P(A \cap C) = 0$$

$$\therefore\ P(A \cap C) \neq P(A) \cdot P(C) = \frac{3}{16}$$

Hence, A is not independent of C

(c)

$$\because\ P'(A) = P'(X=0) + P'(Y=0) = \left(\frac{2}{3}\right)^3 + \left(\frac{1}{3}\right)^3 = \frac{1}{3}$$

$$P'(B) = P'(Y=0) + P'(Y=1) = \left(\frac{1}{3}\right)^3 + 3 \cdot \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{7}{27}$$

$$P'(C) = 1 - P'(A) = \frac{2}{3}$$

$$P'(A \cap B) = P'(Y=0) = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

$$P'(B \cap C) = P'(X=2, Y=1) = 3 \cdot \frac{2}{3} \cdot \left(\frac{1}{3}\right)^2 = \frac{2}{9}$$

$$P'(A \cap C) = 0$$

$$\therefore\ P'(A \cap B) \neq P'(A) \cdot P'(B)$$

$$P'(B \cap C) \neq P'(B) \cdot P'(C)$$

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Hence, A is still not independent of C. But A is not independent of B and B is not independent of C.

3. *Proof.* Sufficiency(if)

   (a) $P(G) = P(T) = 0$
   Obviously, $P(T|G) = P(G|T) = 0$

   (b) $P(G) = P(T) \neq 0$

$$
\begin{aligned}
&P(G) = P(T) \\
\Rightarrow &\frac{P(T \cap G)}{P(G)} = \frac{P(G \cap T)}{P(T)} \\
\Rightarrow &P(T|G) = P(G|T)
\end{aligned}
$$

   Necessity(only if)

$$
\begin{aligned}
&P(G|T) = P(T|G) \\
\Rightarrow &\frac{P(G \cap T)}{P(T)} = \frac{P(T \cap G)}{P(G)} \\
\Rightarrow &P(G) = P(T)
\end{aligned}
$$

□

4. *Proof.* Let X be the total number of heads. $X \sim B(N, p)$, $N \sim P(\lambda)$

$$
\begin{aligned}
P(X = k) &= \sum_{n=0}^{\infty} P(X = k|N = n)P(N = n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \cdot \frac{\lambda^n e^{-\lambda}}{n!} \\
&= \frac{(\lambda p)^k e^{-\lambda p}}{k!} \cdot \sum_{n=k}^{\infty} \frac{[\lambda(1-p)]^{n-k} e^{-\lambda(1-p)}}{(n-k)!} \\
&= \frac{(\lambda p)^k e^{-\lambda p}}{k!}
\end{aligned}
$$

□

5. Define the following events:

   - A: Picking a tough-guy
   - B_1: Opponent's winning a fight on the first day
   - B_2: Opponent's winning a fight on the second day
   - C_1: Fighting on the first day
   - C_2: Fighting on the second day

Then

$$
\begin{aligned}
&P(A) = P(\overline{A}) = \frac{1}{2} \\
&P(C_1|A) = P(C_2|A) = t \\
&P(C_1|\overline{A}) = P(C_2|\overline{A}) = w \\
&P(B_1|AC_1) = P(B_2|AC_2) = 0.7, P(\overline{B_1}|AC_1) = P(\overline{B_2}|AC_2) = 0.3 \\
&P(B_1|\overline{A}C_1) = P(B_2|\overline{A}C_2) = 0.4, P(\overline{B_1}|\overline{A}C_1) = P(\overline{B_2}|\overline{A}C_2) = 0.6,
\end{aligned}
$$

(a)

$$
\begin{aligned}
P(A|\overline{B_1}C_1) &= \frac{P(A\overline{B_1}C_1)}{P(\overline{B_1}C_1)} \\
&= \frac{P(A)P(C_1|A)P(\overline{B_1}|AC_1)}{P(A)P(C_1|A)P(\overline{B_1}|AC_1) + P(\overline{A})P(C_1|\overline{A})P(\overline{B_1}|\overline{A}C_1)} \\
&= \frac{0.5 \cdot t \cdot 0.3}{0.5 \cdot t \cdot 0.3 + 0.5 \cdot w \cdot 0.6} \\
&= \frac{0.15t}{0.15t + 0.3w}
\end{aligned}
$$

(b) Updated beliefs on the distribution of types in the population

$$
P(A) = \frac{0.15t}{0.15t + 0.3w}, P(\overline{A}) = \frac{0.3w}{0.15t + 0.3w}
$$

therefore, under same condition in (a)

$$
\begin{aligned}
P(B_2C_2) &= P(A)P(C_2|A)P(B_2|AC_2) + P(\overline{A})P(C_2|\overline{A})P(B_2|\overline{A}C_2) \\
&= \frac{0.15t}{0.15t + 0.3w} \cdot 0.7t + \frac{0.3w}{0.15t + 0.3w} \cdot 0.4w \\
&= \frac{0.105t^2 + 0.12w^2}{0.15t + 0.3w}
\end{aligned}
$$

(c)

$$
\begin{aligned}
\text{expected payoff} &= 10P(\overline{B_2}C_2) - 10P(B_2C_2) \\
&= 10[P(A)P(C_2|A)P(\overline{B_2}|AC_2) + P(\overline{A})P(C_2|\overline{A})P(\overline{B_2}|\overline{A}C_2)] - 10P(B_2C_2) \\
&= \frac{0.15t}{0.15t + 0.3w} \cdot 3t + \frac{0.3w}{0.15t + 0.3w} \cdot 6w - \frac{1.05t^2 + 1.2w^2}{0.15t + 0.3w} \\
&= \frac{0.6(w^2 - t^2)}{0.15t + 0.3w}
\end{aligned}
$$

Hence, if $w < t$, the sailor will back down; if $w > t$, the sailor will re-match; if $w = t$, there is no difference between two choices.

6. (a)

$$\begin{aligned}
&\int_{\infty}^{\infty} f_X(x)\mathrm{d}x = 1 \\
\Rightarrow &\int_0^3 cx\mathrm{d}x + \int_3^6 c(6-x)\mathrm{d}x = 1 \\
\Rightarrow &c \cdot \left.\frac{x^2}{2}\right|_0^3 + 18c - c \cdot \left.\frac{x^2}{2}\right|_3^6 \\
\Rightarrow &c = \frac{1}{9}
\end{aligned}$$

(b) i.

$$\begin{aligned}
P(X > 3) &= \int_3^{\infty} f_X(x)\mathrm{d}x \\
&= \frac{1}{9}\int_3^6 (6-x)\mathrm{d}x \\
&= 2 - \frac{1}{9} \cdot \left.\frac{x^2}{2}\right|_3^6 \\
&= \frac{1}{2}
\end{aligned}$$

ii.

$$\begin{aligned}
P(1.5 < X < 4.5) &= \frac{1}{9}\int_{1.5}^{3} x\mathrm{d}x + \frac{1}{9}\int_3^{4.5} (6-x)\mathrm{d}x \\
&= \frac{1}{9} \cdot \left.\frac{x^2}{2}\right|_{1.5}^{3} + 1 - \frac{1}{9} \cdot \left.\frac{x^2}{2}\right|_3^{4.5} \\
&= \frac{3}{4}
\end{aligned}$$

(c)

$$\begin{aligned}
P(A \cap B) &= P(3 < X < 4.5) \\
&= \frac{1}{9}\int_3^{4.5} (6-x)\mathrm{d}x \\
&= 1 - \frac{1}{9} \cdot \left.\frac{x^2}{2}\right|_3^{4.5} \\
&= \frac{3}{8} = \frac{1}{2} \cdot \frac{3}{4} \\
&= P(A) \cdot P(B)
\end{aligned}$$

Thus, A and B are independent.

7.
$$F_X(x) = \int_0^x \lambda e^{-\lambda t} \mathrm{d}t = -e^{-\lambda t}\Big|_0^x = 1 - e^{-\lambda x}, \quad \forall x \geq 0$$

When $0 \leq y \leq 1$,

$$\begin{aligned}
F_Y(y) &= P(Y \leq y) = P(X \leq y, X \leq 1) + P(\frac{1}{X} \leq y, X > 1) \\
&= P(X \leq y) + P(X \geq \frac{1}{y}) \\
&= F_X(y) + 1 - F_X(\frac{1}{y}) \\
&= 1 - e^{-\lambda y} + e^{-\frac{\lambda}{y}}
\end{aligned}$$

Hence,

$$f_Y(y) = F_Y'(y) = \begin{cases} \lambda e^{-\lambda y} + \dfrac{\lambda}{y^2} e^{-\frac{\lambda}{y}} & 0 \leq y \leq 1 \\ 0 & otherwise \end{cases}$$

8. (a)

$$\begin{aligned}
f_X(x) &= \int_0^\infty y e^{-y(x+1)} \mathrm{d}y \\
&= -\frac{1}{x+1} \cdot y e^{-y(x+1)}\Big|_0^{+\infty} + \frac{1}{x+1} \int_0^{+\infty} e^{-y(x+1)} \mathrm{d}y \\
&= -\frac{1}{(x+1)^2} \cdot e^{-y(x+1)}\Big|_0^{+\infty} \\
&= \frac{1}{(x+1)^2} \\
f_Y(y) &= \int_0^\infty y e^{-y(x+1)} \mathrm{d}x \\
&= -\, e^{-y(x+1)}\Big|_0^{+\infty} \\
&= e^{-y}
\end{aligned}$$

(b)

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{XY}(x,y)}{f_Y(y)} = y e^{-yx} \\
\Rightarrow F_{X|Y}(x|y) &= \int_0^x y e^{-yt} \mathrm{d}t = 1 - e^{-yx}, \quad \forall x, y > 0
\end{aligned}$$

9. Denote by $\Phi(\cdot)$ and $\phi(\cdot)$ the cdf and pdf of $N(0,1)$.

$$\begin{aligned}
F_X(x) &= P(X \le x) = P(YZ \le x) \\
&= P(YZ \le x|Y=0)P(Y=0) + P(YZ \le x|Y=1)P(Y=1) \\
&= \frac{1}{2}P(0 \le x) + \frac{1}{2}P(Z \le x) \\
\Rightarrow F_X(x) &= \begin{cases} \frac{1}{2}\Phi(x) & x < 0 \\ \frac{1}{2} + \frac{1}{2}\Phi(x) & x \ge 0 \end{cases} \\
\Rightarrow f_X(x) &= \frac{1}{2}\phi(x) = \frac{1}{2\sqrt{2\pi}}e^{-\frac{x^2}{2}}, \quad x \in \mathbb{R}
\end{aligned}$$

10.

$$\because \; E(X_i) = \mu, Var(X_i) = \sigma^2, E(N) = n, Var(N) = \nu$$

$$\begin{aligned}
\therefore \; Var(S) &= E[Var(S|N)] + Var[E(S|N)] \\
&= E[Var(X_i)N] + Var[E(X_i)N] \\
&= Var(X_i)E[N] + [E(X_i)]^2 Var[N] \\
&= n\sigma^2 + \nu\mu^2
\end{aligned}$$