Comps ECO 520

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Exponential Distribution

a) The MGF of $Exp(\theta)$ is

$$\begin{aligned} M_X(t) &= E[e^{tx}] \\ &= \int_0^\infty e^{tx}\theta e^{-\theta x}\mathrm{d}x \\ &= \frac{\theta}{\theta - t} \quad \forall t < \theta \end{aligned}$$

b) The cumulant generating function (CGF) is

$$K(t) = \log\left(\frac{\theta}{\theta - t}\right)$$

Then

$$\begin{aligned} E(X) &= K'(0) \\ &= \left.\frac{1}{\theta - t}\right|_{t=0} \\ &= \frac{1}{\theta} \\ Var(X) &= K''(0) \\ &= \left.\frac{1}{(\theta - t)^2}\right|_{t=0} \\ &= \frac{1}{\theta^2} \end{aligned}$$

c) The MGF of $\bar{X}$ is

$$\begin{aligned}
M_{\bar{X}}(t) &= E\left[e^{t\bar{X}}\right] \\
&= \prod_{i=1}^{n} E\left[e^{\frac{t}{n}X_i}\right] \\
&= \left(M_X\left(\frac{t}{n}\right)\right)^n \\
&= \left(\frac{\theta}{\theta - \frac{t}{n}}\right)^n \quad \forall t < \theta
\end{aligned}$$

d) *Proof.* Let $\alpha = n, \beta = n\theta$.

$$\begin{aligned}
M_{\bar{X}}(t) &= \left(\frac{\theta}{\theta - \frac{t}{n}}\right)^n \\
&= \left(\frac{n\theta}{n\theta - t}\right)^n \\
&= \left(\frac{\beta}{\beta - t}\right)^\alpha \\
&= MGF(t)
\end{aligned}$$

Therefore, $\bar{X} \sim Gamma(n, n\theta)$. □

e)

$$\begin{aligned}
E(\frac{1}{\bar{x}}) &= \int_0^\infty \frac{1}{\bar{x}} \cdot \frac{(n\theta)^n \bar{x}^{n-1}\exp(-n\theta\bar{x})}{\Gamma(n)} \mathrm{d}\bar{x} \\
&= \frac{n\theta}{n-1}\int_0^\infty \frac{(n\theta)^{n-1}\bar{x}^{n-2}\exp(-n\theta\bar{x})}{\Gamma(n-1)} \mathrm{d}\bar{x} \\
&= \frac{n\theta}{n-1}
\end{aligned}$$

f) Applying MLE, $\theta_{MLE} = \hat{\theta} = \dfrac{1}{\bar{X}}$. Using results in e)

$$
\begin{aligned}
plim(\hat{\theta}) &= \lim_{n\to\infty} \frac{n\theta}{n-1} \\
&= \theta \\
Asy.Var(\hat{\theta}) &= \lim_{n\to\infty} n \left[\int_0^{\infty} \frac{1}{\bar{x}^2} \cdot \frac{(n\theta)^n \bar{x}^{n-1} \exp(-n\theta\bar{x})}{\Gamma(n)} \mathrm{d}\bar{x} - \left(\frac{n\theta}{n-1} \right)^2 \right] \\
&= \lim_{n\to\infty} n \left[\frac{(n\theta)^2}{(n-1)(n-2)} - \frac{(n\theta)^2}{(n-1)^2} \right] \\
&= \lim_{n\to\infty} \frac{n(n\theta)^2}{(n-1)^2(n-2)} \\
&= \theta^2
\end{aligned}
$$

Therefore, its asymptotic distribution

$$\sqrt{n}(\hat{\theta} - \theta) \sim N(0, \theta^2)$$

g) The bias of the maximum likelihood estimator is

$$
\begin{aligned}
Bias &= \theta - E(\hat{\theta}) \\
&= \theta - \frac{n\theta}{n-1} \\
&= -\frac{\theta}{n-1}
\end{aligned}
$$

The bias corrected estimator $\tilde{\theta}$ is $\dfrac{n-1}{n\bar{X}}$.

$$
\begin{aligned}
plim(\tilde{\theta}) &= \lim_{n\to\infty} \int_0^\infty \frac{n-1}{n\bar{x}} \cdot \frac{(n\theta)^n \bar{x}^{n-1} \exp(-n\theta\bar{x})}{\Gamma(n)} \mathrm{d}\bar{x} \\
&= \theta \\
Asy.Var(\tilde{\theta}) &= \lim_{n\to\infty} n \left[\int_0^\infty \frac{(n-1)^2}{(n\bar{x})^2} \cdot \frac{(n\theta)^n \bar{x}^{n-1} \exp(-n\theta\bar{x})}{\Gamma(n)} \mathrm{d}\bar{x} - \theta^2 \right] \\
&= \lim_{n\to\infty} n \left[\frac{(n-1)\theta^2}{n-2} - \theta^2 \right] \\
&= \lim_{n\to\infty} \frac{n\theta^2}{n-2} \\
&= \theta^2
\end{aligned}
$$

Therefore, its asymptotic distribution

$$\sqrt{n}(\tilde{\theta} - \theta) \sim N(0, \theta^2)$$

which is asymptotically equivalent to the ML estimator.