Econometrics Comp Exam

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Part I

1. (a)

$$
\begin{aligned}
M_Y(t) &= E(e^{ty}) \\
&= \int_0^\infty \frac{2}{\sqrt{2\pi}\sigma_Y} \exp\{-\frac{y^2}{2\sigma_Y^2} + ty\}\mathrm{d}y \\
&= 2\exp\{\frac{\sigma_Y^2 t^2}{2}\} \int_0^\infty \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\{-\frac{(y-\sigma_Y^2 t)^2}{2\sigma_Y^2}\}\mathrm{d}y \\
&= 2\exp\{\frac{\sigma_Y^2 t^2}{2}\} \int_{-\sigma_Y^2 t}^\infty \frac{1}{\sqrt{2\pi}\sigma_Y} \exp\{-\frac{u^2}{2\sigma_Y^2}\}\mathrm{d}u \\
&= 2e^{\frac{\sigma_Y^2 t^2}{2}}[1-\Phi(-\sigma_Y^2 t)]
\end{aligned}
$$

(b) Note that $\Phi(0) = \frac{1}{2}, \varphi(0) = \frac{1}{\sqrt{2\pi}}$.

$$
\begin{aligned}
m_Y(t) &= \log(M_Y(t)) \\
&= \log(2) + \frac{\sigma_Y^2 t^2}{2} + \log[1 - \Phi(-\sigma_Y^2 t)] \\
E(Y) &= m_Y'(0) \\
&= \sigma_Y^2 \left. \frac{\varphi(-\sigma_Y^2 t)}{1 - \Phi(-\sigma_Y^2 t)} \right|_{t=0} \\
&= \sqrt{\frac{2}{\pi}} \sigma_Y^2 \\
Var(Y) &= m_Y''(0) \\
&= \sigma_Y^2 + \sigma_Y^2 \left. \frac{\varphi'(-\sigma_Y^2 t)[1 - \Phi(-\sigma_Y^2 t)] - [\varphi(-\sigma_Y^2 t)]^2 \sigma_Y^2}{[1 - \Phi(-\sigma_Y^2 t)]^2} \right|_{t=0} \\
&= \sigma_Y^2 \left(1 - \frac{2\sigma_Y^2}{\pi}\right)
\end{aligned}
$$

2. (a) The log-likelihood function is given by

$$
l(\theta_0) = n \log(\theta_0) + (\theta_0 - 1) \sum_{i=1}^{n} \log(x_i)
$$

FOC gives

$$
\frac{\partial l(\theta_0)}{\partial \theta_0} = \frac{n}{\theta_0} + \sum_{i=1}^{n} \log(x_i) = 0
$$

Thus, the MLE of θ_0

$$
\hat{\theta}_n = -\frac{n}{\sum\limits_{i=1}^{n} \log(x_i)}
$$

(b) *Proof.* Let $\tilde{x} = \frac{1}{n} \sum\limits_{i=1}^{n} \log(x_i), g(x) = -\frac{1}{x}$.

Note that

$$
\begin{aligned}
E(\tilde{x}) &= E(\log x) \\
&= \int_0^1 \log(x)\theta_0 x^{\theta-1}\mathrm{d}x \\
&= \log(x)x^{\theta_0}\Big|_0^1 - \int_0^1 x^{\theta-1}\mathrm{d}x \\
&= -\frac{1}{\theta_n} \\
Var(\tilde{x}) &= \frac{1}{n}Var(\log x) \\
&= \frac{1}{n}[E(\log^2 x) - E^2(\log x)] \\
&= \frac{1}{n}\left(\int_0^1 \log^2(x)\theta_0 x^{\theta-1}\mathrm{d}x - E^2(\log x)\right) \\
&= \frac{1}{n}\left(\log^2(x)x^{\theta_0}\Big|_0^1 - 2\int_0^1 \log(x)x^{\theta-1}\mathrm{d}x - E^2(\log x)\right) \\
&= \frac{1}{n}\left(-\frac{2}{\theta_0}E(\log x) - E^2(\log x)\right) \\
&= \frac{1}{n\theta_0^2}
\end{aligned}
$$

Then

$$
\begin{aligned}
E(\hat{\theta}_n) &= E[g(\tilde{x})] \\
&= g[E(\tilde{x})] \\
&= \theta_0
\end{aligned}
$$

Applying delta method

$$
\begin{aligned}
Var(\hat{\theta}_n) &= Var(\tilde{x})(g'(\theta_0))^2 \\
&= \frac{1}{n\theta_0^6}
\end{aligned}
$$

Since $\lim_{n\to\infty} Var(\hat{\theta}_n) = 0$, applying Chebyshev's inequality to $\hat{\theta}_n$

$$
\lim_{n\to\infty} P[|\hat{\theta}_n - \theta_0| \geq \epsilon] = 0 \quad \forall \epsilon > 0
$$

Therefore $\hat{\theta}_n$ is a consistent estimator of θ_0. $\square$

(c) Since $\log(X_i)$ has a finite mean and variance, applying CLT, the asymptotic distribution is

$$\sqrt{n}(\hat{\theta}_n - \theta_0) \sim N\left(0, \frac{1}{\theta_0^6}\right)$$

3. (a)

$$\left.\begin{array}{c} X_1, X_2 \text{ independent} \\ X_1 \sim N(\mu_1, \sigma_1^2), X_2 \sim N(\mu_2, \sigma_2^2) \\ Y = X_1 + X_2 \end{array}\right\} \Rightarrow Y \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

(b) Let

$$X = (X_1, X_2)^T$$

be a 2×1 multivariate normal random vector with mean

$$\mu = (\mu_1, \mu_2)^T$$

and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{21} & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$$

Let $A = (1, 1)$.

$$\begin{aligned} Y &= AX \\ E(Y) &= A\mu = \mu_1 + \mu_2 \\ Var(Y) &= A\Sigma A^T = \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2 \\ Y &\sim N(\mu_1 + \mu_2, \sigma_1^2 + 2\rho\sigma_1\sigma_2 + \sigma_2^2) \end{aligned}$$

Therefore, the result in (a) still hold true, i.e. Y is still normal distribution.

Part II

1. (a) *Proof.*

$$\begin{aligned} x'\hat{u} &= x'(y - x\hat{\beta}) \\ &= x'y - x'x(x'x)^{-1}x'y \\ &= x'y - x'y \\ &= 0 \end{aligned}$$

$\square$

(b) The OLS estimate of α, $\hat{\alpha} = 1$.
Since $ESS = 0$

$$R^2 = 1 - \frac{ESS}{TSS} = 1$$

2. (a) *Proof.*

$$P = \begin{pmatrix} 1 & -\beta_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ -\beta_2 & 1 & \gamma_1 & \gamma_2 & \gamma_3 \end{pmatrix}$$

with restrictions $\alpha_3 = \gamma_2 = 0$

$$\Phi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{aligned} P\Phi_1 &= \begin{pmatrix} \alpha_3 \\ \gamma_3 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ \gamma_3 \end{pmatrix} \end{aligned}$$

$$Rank(P\Phi_1) = 2 - 1 = 1$$

Therefore, the structural parameters in the first equation are identified. $\square$

(b) I use ILS here.

The reduced form is

$$
\begin{aligned}
y_{1i} &= \frac{1}{1-\beta_1\beta_2}[\beta_1\gamma_0 + \alpha_0 + (\beta_1\gamma_1 + \alpha_1)x_{i1} + \alpha_2 x_{2i} + \beta_1\gamma_3 x_{3i} + u_{1i} + \beta_1 u_{2i}] \\
&= \pi_{01} + \pi_{11}x_{1i} + \pi_{21}x_{2i} + \pi_{31}x_{3i} + v_{1i} \\
y_{2i} &= \frac{1}{1-\beta_1\beta_2}[\beta_2\alpha_0 + \gamma_0 + (\beta_2\alpha_1 + \gamma_1)x_{i1} + \beta_2\alpha_2 x_{2i} + \gamma_3 x_{3i} + \beta_2 u_{1i} + u_{2i}] \\
&= \pi_{02} + \pi_{12}x_{1i} + \pi_{22}x_{2i} + \pi_{32}x_{3i} + v_{2i}
\end{aligned}
$$

Equating corresonding coefficients with eight unknown parameters and eight equtions, we can estimate all the structural parameters.

3. MLE (refer to discrete choice notes pp.15)

Define

$$Pr(y_{it} = 1 \mid x_{it}, u_i) = \Phi(x_{it}\beta + u_i)$$

Then, the it-specific conditional likelihood contribution is

$$L_{it}(u_i) = \Phi(x_{it}\beta + u_i)^{y_{it}}[1 - \Phi(x_{it}\beta + u_i)]^{1-y_{it}}$$

Once we condition on u_i, the observations over t for i are independent. Therefore, the i-specific conditional likelihood contribution is

$$L_i(u_i) = \prod_{t=1}^{T} \Phi(x_{it}\beta + u_i)^{y_{it}}[1 - \Phi(x_{it}\beta + u_i)]^{1-y_{it}}$$

and the i-specific unconditional likelihood contribution is

$$L_i = \int L_i(u_i)\mathrm{d}F(u_i) = \int L_i(u_i)\frac{1}{\sigma_u}\phi\left(\frac{u_i}{\sigma_u}\right)\mathrm{d}u_i$$

The likelihood function is

$$
\begin{aligned}
L &= \prod_{i=1}^{n} L_i \\
&= \prod_{i=1}^{n} \int \prod_{t=1}^{T} \Phi(x_{it}\beta + u_i)^{y_{it}}[1 - \Phi(x_{it}\beta + u_i)]^{1-y_{it}} \frac{1}{\sigma_u}\phi\left(\frac{u_i}{\sigma_u}\right)\mathrm{d}u_i
\end{aligned}
$$

FOC of the log-likelihood function gives the estimator of (β, σ_u^2)

It is the covariation of ε_{ij} that identifies the σ_u^2.

4. (a)

$$P_{ij} = Pr(y_{ij} = 1) = \frac{\exp\{x_{ij}\beta\}}{\sum\limits_k \exp\{x_{ik}\beta\}}$$

(b) The covariance matrix of y_i is (refer to discrete choice notes pp.26)

$$\Omega = \begin{pmatrix} P_{i1}(1-P_{i1}) & -P_{i1}P_{i2} & \cdots & -P_{i1}P_{iJ} \\ -P_{i1}P_{i2} & P_{i2}(1-P_{i2}) & \cdots & -P_{i2}P_{iJ} \\ \vdots & \vdots & \ddots & \vdots \\ -P_{i1}P_{iJ} & -P_{i2}P_{iJ} & \cdots & P_{iJ}(1-P_{iJ}) \end{pmatrix}$$