Comps ECO 520

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Poisson Distribution

a) *Proof.*

$$\begin{cases} X_i \sim iid\, Poisson(\lambda) \\ S = \sum_{i=1}^{n} X_i \\ M_X(t) = \exp\{\lambda(e^t - 1)\}, \;\; t \in \mathbb{R} \end{cases}$$

Then

$$\begin{aligned} M_S(t) &= E[e^{tS}] \\ &= \prod_{i=1}^{n} E[e^{tX_i}] \\ &= (M_X(t))^n \\ &= \exp\{n\lambda(e^t - 1)\}, \;\; t \in \mathbb{R} \end{aligned}$$

$$\Rightarrow E(S) = M_S'(0) = n\lambda e^t e^{n\lambda(e^t-1)}\Big|_{t=0} = n\lambda$$

Therefore, S is a Poisson random variable with parameter $n\lambda$. □

b)

$$\begin{aligned} E(\hat{g}) &= E[\mathbb{1}(X_1 = 0)] \\ &= P(X_1 = 0) \cdot 1 \\ &= \exp(-\lambda) \\ &= g(\lambda) \end{aligned}$$

Thus, the estimator $\hat{g}$ is unbiased for $g(\lambda)$.

c)

$$
\begin{aligned}
P(X_1 = k \mid S = s) &= \frac{P(X_1 = k, S = s)}{P(S = s)} \\
&= \frac{P(X_1 = k)P(X_1 = k, \sum_{i=2}^{n} X_i = s - k)}{P(S = s)} \\
&= \frac{\frac{\lambda^k \exp(-\lambda)}{k!} \cdot \frac{[(n-1)\lambda]^k \exp\{-(n-1)\lambda\}}{(s-k)!}}{\frac{(n\lambda)^k \exp(-n\lambda)}{s!}} \\
&= \binom{s}{k} \frac{(n-1)^{s-k}}{n^s}
\end{aligned}
$$

d) *Proof.*

$$
\begin{aligned}
E(\tilde{g}) &= E[E[\mathbb{1}(X_1 = 0 \mid S = s)]] \\
&= E[P(X_1 = 0 \mid S = s)] \\
&= E\left[\left(\frac{n-1}{n}\right)^s\right] \\
&= \sum_{k=0}^{\infty} \left(\frac{n-1}{n}\right)^k e^{-n\lambda} \frac{(n\lambda)^k}{k!} \\
&= \exp(-\lambda) \\
&= g(\lambda)
\end{aligned}
$$

□

e) *Proof.* Note that $\tilde{g} = E(\hat{g} \mid S)$. By the law of total variance

$$
\begin{aligned}
Var(\hat{g}) &= Var(E(\hat{g} \mid S)) + E(Var(\hat{g} \mid S)) \\
&= Var(\tilde{g}) + E(Var(\hat{g} \mid S))
\end{aligned}
$$

Therefore, $Var(\tilde{g}) < Var(\hat{g})$. □

f) Applying MLE, $\lambda_{MLE} = \hat{\lambda} = \bar{X} = \frac{S}{n}$ and $g(\hat{\lambda}) = \exp(-\bar{X})$. Then $E(\hat{\lambda}) = \lambda, Var(\hat{\lambda}) = \frac{\lambda}{n}$.

Applying delta method, the asymptotic distribution of $g(\hat{\lambda})$ is

$$\sqrt{n}[g(\hat{\lambda}) - \exp(-\lambda)] \sim N(0, \exp(-2\lambda)\lambda)$$

Since $\bar{X}$ attains the Cramer-Rao lower bound, by the generalization of Cramer-Rao inequality,

$$Var(\tilde{g}) \geq \frac{\exp(-2\lambda)\lambda}{n}$$

Thus, $\tilde{g}$ is not efficient.