Homework 3

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1. Wald statistic and the Likelihood Ratio statistic

   Proof. Assume that error variance σ^2 is known. Denote M_A and P_A by the usual residual maker and projection matrices on A. The Wald statistic is given by

$$
\begin{aligned}
W &= n\hat{\gamma}'[n\sigma^2(Z'M_XZ)^{-1}]^{-1}\hat{\gamma} \\
&= \frac{\hat{\gamma}'Z'M_XZ\hat{\gamma}}{\sigma^2} \\
&= \frac{y'M_XZ(Z'M_XZ)^{-1}Z'M_XZ(Z'M_XZ)^{-1}Z'M_Xy}{\sigma^2} \\
&= \frac{y'M_XZ(Z'M_XZ)^{-1}Z'M_Xy}{\sigma^2} \\
&= \frac{y'P_{M_XZ}y}{\sigma^2} \\
&= \frac{y'P_{Z\perp X}y}{\sigma^2}
\end{aligned}
$$

   where the third equality follows from the Frisch-Waugh-Lovell theorem.

   Let $Q = (X \vdots Z)$. Denote $\hat{\beta}_U$ and $\hat{\beta}_R$ by the unrestricted and restricted estimators.

$$
\begin{aligned}
L(\hat{\beta}_U) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{(y-Q\hat{\beta}_U)'(y-Q\hat{\beta}_U)}{2\sigma^2} \\
L(\hat{\beta}_R) &= -\frac{n}{2}\log(2\pi\sigma^2) - \frac{(y-Q\hat{\beta}_R)'(y-Q\hat{\beta}_R)}{2\sigma^2}
\end{aligned}
$$

The Likelihood ratio statistic under known error variance is given by

$$
\begin{aligned}
LR &= 2[L(\hat{\beta}_U) - L(\hat{\beta}_R)] \\
&= \frac{(y - Q\hat{\beta}_R)'(y - Q\hat{\beta}_R) - (y - Q\hat{\beta}_U)'(y - Q\hat{\beta}_U)}{\sigma^2} \\
&= \frac{y'(I - P_X)y - y'(I - P_U)y}{\sigma^2} \\
&= \frac{y'(P_U - P_X)y}{\sigma^2} \\
&= \frac{y'P_{Z\perp X}y}{\sigma^2}
\end{aligned}
$$

Therefore $W = LR$ $\square$

2. Hypothesis Testing

   (a) By the Slutsky theorem

   $$\text{plim } g(\hat{\beta}) = g(\text{plim } \hat{\beta}) = g(\beta)$$

   So a consistent estimator of $g(\beta)$ is $g(\hat{\beta})$, where $\hat{\beta} = (X'X)^{-1}X'y$

   (b) Applying the delta method, its asymptotic distribution

   $$\sqrt{n}[g(\hat{\beta}) - g(\beta)] \sim N(0, \Gamma Var(\hat{\beta})\Gamma')$$

   where $Var(\hat{\beta}) = \sigma^2 \left(\frac{X'X}{n}\right)^{-1}, \Gamma = \frac{\partial g(\beta)}{\partial \beta'}$

   (c) Since the $H_0 : g(\beta) = 0$, the Wald(normalized) distance measure

   $$z = \frac{\sqrt{n}g(\hat{\beta}) - g(\beta)}{\sigma\sqrt{\Gamma\left(\frac{X'X}{n}\right)^{-1}\Gamma'}}$$

   where $\Gamma = \frac{\partial g(\beta)}{\partial \beta'}$

   Decision rule: If z is large (larger than a critical value), reject H_0.

3. Transformed Model

   (a) Since

   $\log W_i = \beta_0 + \beta_1 Educ_i + \beta_2(1 - Male_i) + \beta_3(1 - White_i - Asian_i) + \beta_4 Asian_i + u_{1i}$

   Then

   $$\begin{cases} \alpha_0 = \beta_0 + \beta_2 + \beta_3 \\ \alpha_1 = \beta_1 \\ \alpha_2 = -\beta_2 \\ \alpha_3 = -\beta_3 \\ \alpha_4 = \beta_4 - \beta_3 \end{cases}$$

Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4)', B = (b_0, b_1, b_2, b_3, b_4)'$

$$\alpha = B\beta = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix} \beta$$

(b) *Proof.* Let $Z = \log W, X = (1, Educ, Male, White, Asian)$.

$$Z = X\alpha + u_2$$

Then

$$\hat{\alpha} = (X'X)^{-1}X'Z$$

From (a), we have

$$Z = XB\beta + u_2$$

Then

$$\hat{\beta} = [(XB)'XB]^{-1}(XB)'Z$$

Multiplying by B both sides

$$\begin{aligned} B\hat{\beta} &= B[(XB)'XB]^{-1}(XB)'Z \\ &= BB^{-1}(X'X)^{-1}(B')^{-1}B'X'Z \\ &= (X'X)^{-1}X'Z \\ &= \hat{\alpha} \end{aligned}$$

□

(c) *Proof.* From (a), we have $u_1 = u_2$. Then

$$R_1^2 = 1 - \frac{\sum u_1^2}{\sum (Z - \bar{Z})^2} = 1 - \frac{\sum u_2^2}{\sum (Z - \bar{Z})^2} = R_2^2$$

Therefore, R^2 for both equations are identically the same. □