When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

Consider a random sample $\{X_1, \dots, X_n\}$ drawn from a Poisson distribution with parameter λ. The pmf of a Poisson distribution is

$$P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \dots.$$

We consider below the estimation of $g(\lambda) = \exp(-\lambda)$.

a) Let us first recall that the Moment Generating Function (MGF) of a Poisson distribution is given by

$$M_X(t) = \exp\left(\lambda(e^t - 1)\right), \quad t \in \mathbb{R},$$

and let

$$S = \sum_{i=1}^{n} X_i.$$

Show that S is a Poisson random variable with parameter $n\lambda$.

b) We now turn to the estimation of $g(\lambda) = \exp(-\lambda)$. Verify that the estimator

$$\hat{g} = \mathbb{1}(X_1 = 0) = \begin{cases} 1 & \text{if } X_1 = 0 \\ 0 & \text{otherwise} \end{cases},$$

is unbiased for $g(\lambda)$.

c) Determine the conditional distribution of X_1 given $S = \sum_{i=1}^{n} X_i$. That is, find

$$P\left(X_1 = k | S = s\right).$$

Notice that you need the result in *a*) to complete this part.

d) Find

$$\tilde{g} = E\left(\mathbb{1}(X_1 = 0) | S = s\right).$$

Show that $\tilde{g}$ is also an unbiased estimator of $g(\lambda)$.

e) Show that the variance of $\tilde{g}$ is smaller than the variance of $\hat{g}$. *Hint*: Use the law of total variance

$$Var(\hat{g}) = Var\left(E\left(\hat{g}|S\right)\right) + E\left(Var\left(\hat{g}|S\right)\right).$$

f) Let $g(\hat{\lambda})$, where $\hat{\lambda}$ is the maximum likelihood estimator of λ. Derive $g(\hat{\lambda})$ and its asymptotic distribution. Use it to explain why $\tilde{g}$ is not efficient. *Hint*: You may want to use the continuous mapping theorem which states that all types of convergence are preserved under continuous transformation.