Homework 1

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1. The test statistic is
$$\frac{\bar{X}-1}{\sqrt{\frac{2}{n}}} \sim N(0,1)$$

2. *Proof.* $g(\cdot)$ is a differentiable function $\Rightarrow g(\cdot)$ is continuous at point θ. Continuity at θ means that $\forall \varepsilon > 0$, we can find a δ such that $|S_n - \theta| < \delta$ implies that $|g(S_n) - g(\theta)| < \varepsilon$. Therefore
$$P(|S_n - \theta| < \delta) \leq P(|g(S_n) - g(\theta)| < \varepsilon)$$
Because the LHS will converge to one by assumption ($plim S_n = \theta$), the result follows. $\square$

3. *Proof.* Let $X_T = (a_1, a_2, a_3), f(x) = x, g(x) = \log(x)$. Then $a_1 = (1, 1, \ldots, 1)', a_2 = (f(1), f(2), \ldots, f(T))', a_3 = (g(1), g(2), \ldots, g(T))'$. By inspection, (a_1, a_2, a_3) are not orthonormal columns. By calculation
$$\frac{X_T' X_T}{T} = \begin{pmatrix} 1 & \frac{1}{2}(T+1) & E(\log i) \\ \frac{1}{2}(T+1) & \frac{1}{6}(T+1)(2T+1) & E(i \log i) \\ E(\log i) & E(i \log i) & E(\log^2 i) \end{pmatrix}$$
Obviously, if $T \to \infty$, the matrix above won't converge to a finite matrix. $\square$

   Let $Q = X_T D_T' = (\eta_1, \eta_2, \eta_3)$. Since $Q'Q$ converges to a finite full-rank matrix as $T \to \infty$, it must be the case that Q is orthogonal matrix. Applying Gram-Schmidt Orthogonalization
$$\begin{aligned} b_1 &= a_1 \\ b_2 &= a_2 - k_0 b_1 = a_2 - k_0 a_1 \\ b_3 &= a_3 - k_1 b_1 - k_2 b_2 = a_3 + (k_0 k_2 - k_1) a_1 - k_2 a_2 \\ \eta_1 &= \frac{b_1}{\|b_1\|} \\ \eta_2 &= \frac{b_2}{\|b_2\|} \\ \eta_3 &= \frac{b_3}{\|b_3\|} \end{aligned}$$

Then

$$
\begin{aligned}
(\eta_1, \eta_2, \eta_3) &= (b_1, b_2, b_3) \begin{pmatrix} \frac{1}{\|b_1\|} & 0 & 0 \\ 0 & \frac{1}{\|b_2\|} & 0 \\ 0 & 0 & \frac{1}{\|b_3\|} \end{pmatrix} \\
&= (a_1, a_2, a_3) \begin{pmatrix} 1 & -k_0 & k_0 k_2 - k_1 \\ 0 & 1 & -k_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\|b_1\|} & 0 & 0 \\ 0 & \frac{1}{\|b_2\|} & 0 \\ 0 & 0 & \frac{1}{\|b_3\|} \end{pmatrix}
\end{aligned}
$$

Therefore

$$
D'_T = \begin{pmatrix} \dfrac{1}{\|b_1\|} & \dfrac{-k_0}{\|b_2\|} & \dfrac{k_0 k_2 - k_1}{\|b_3\|} \\ 0 & \dfrac{1}{\|b_2\|} & \dfrac{-k_2}{\|b_3\|} \\ 0 & 0 & \dfrac{1}{\|b_3\|} \end{pmatrix}
$$

Finally

$$
D_T = \begin{pmatrix} \dfrac{1}{\|b_1\|} & 0 & 0 \\ \dfrac{-k_0}{\|b_2\|} & \dfrac{1}{\|b_2\|} & 0 \\ \dfrac{k_0 k_2 - k_1}{\|b_3\|} & \dfrac{-k_2}{\|b_3\|} & \dfrac{1}{\|b_3\|} \end{pmatrix}
$$

where

$$
\begin{aligned}
\|b_1\| &= \langle b_1, b_1 \rangle = \sqrt{T} \\
\|b_2\| &= \langle b_2, b_2 \rangle = \sqrt{\frac{T(T+1)(2T+1)}{6} + \frac{\sqrt{T}(\sqrt{T} - 2T)(T+1)^2}{4}} \\
\|b_3\| &= \langle b_3, b_3 \rangle \\
k_0 &= \frac{\langle a_2, b_1 \rangle}{\langle b_1, b_1 \rangle} \\
&= \frac{\sqrt{T}(T+1)}{2} \\
k_1 &= \frac{\langle a_3, b_1 \rangle}{\langle b_1, b_1 \rangle} \\
&= \frac{1}{\sqrt{T}} \sum_{i=1}^{T} \log i \\
k_2 &= \frac{\langle a_3, b_2 \rangle}{\langle b_2, b_2 \rangle} \\
&= \frac{\sum_{i=1}^{T} (i - \frac{\sqrt{T}(T+1)}{2}) \log i}{\|b_2\|}
\end{aligned}
$$