Homework 2

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1. Restricted Least Squares

   Lagrangian function:

   $$L(\beta, \lambda) = (y - X\beta)'(y - X\beta) + 2\lambda'(R\beta - c)$$

   FOC:

   $$\begin{aligned} &\beta : X'X\beta + R'\lambda = X'y \\ &\lambda : R\beta = c \end{aligned}$$

   Then

   $\hat{\beta}_{RLS} = (X'X)^{-1}X'y - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}[R(X'X)^{-1}X'y - r]$

   Since $E(\hat{\beta}_{OLS}) = \beta$

   $$\begin{aligned} E(\hat{\beta}_{RLS}) &= E(\hat{\beta}_{OLS}) - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R \cdot E(\hat{\beta}_{OLS} - \beta) \\ &= E(\hat{\beta}_{OLS}) = \beta \end{aligned}$$

   So the RLS estimator is unbiased if the OLS estimator is unbiased.

   Proof. Let $D = I - (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R$ and $C = (X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1}$

   $$\begin{aligned} Var(\hat{\beta}_{RLS}) &= E[(\hat{\beta}_{RLS} - \beta)'(\hat{\beta}_{RLS} - \beta)] \\ &= DE[(\hat{\beta}_{OLS} - \beta)'(\hat{\beta}_{OLS} - \beta)]D' \\ &= \sigma^2 D(X'X)^{-1} \\ &= \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1} \\ &= Var(\hat{\beta}_{OLS}) - \sigma^2 C \end{aligned}$$

   Since C is a positive semi-definite matrix, $Var(\hat{\beta}_{RLS}) < Var(\hat{\beta}_{OLS})$ □

   Thus, its asymptotic distribution

   $$\sqrt{n}(\hat{\beta}_{RLS} - \beta) \sim N(0, Var(\hat{\beta}_{RLS}))$$

   where Var$(\hat{\beta}_{RLS}) = \sigma^2(X'X)^{-1} - \sigma^2(X'X)^{-1}R'[R(X'X)^{-1}R']^{-1}R(X'X)^{-1}$

2. Properties of R^2 without constant

   In the case with no constant, instead of fitting a line through the mean values, we need to fit the line through the origin

$$
\begin{aligned}
(y_i - 0) &= (\hat{y}_i - 0) + (y_i - \hat{y}_i) \\
\sum_{i=1}^{n}(y_i - 0)^2 &= \sum_{i=1}^{n}(\hat{y}_i - 0)^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2 + 2\sum_{i=1}^{n}(\hat{y}_i - 0)(y_i - \hat{y}_i)
\end{aligned}
$$

   And

$$
\begin{aligned}
\sum_{i=1}^{n}(\hat{y}_i - 0)(y_i - \hat{y}_i) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\beta_j x_{ij}\left(y_i - \sum_{j=1}^{m}\beta_j x_{ij}\right) \\
&= \sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n}x_{ij}y_i - b_j\sum_{i=1}^{n}x_{ij}^2\right) \\
&= \sum_{j=1}^{m}\beta_j\left(\sum_{i=1}^{n}x_{ij}y_i - \frac{\sum_{i=1}^{n}x_{ij}y_i}{\sum_{i=1}^{n}x_{ij}^2}\sum_{i=1}^{n}x_{ij}^2\right) \\
&= 0
\end{aligned}
$$

   Then

$$
\sum_{i=1}^{n}(y_i - 0)^2 = \sum_{i=1}^{n}(\hat{y}_i - 0)^2 + \sum_{i=1}^{n}(y_i - \hat{y}_i)^2
$$

   Again, we have

$$
TSS = RSS + ESS
$$

   where $TSS = \sum_{i=1}^{n} y_i^2, RSS = \sum_{i=1}^{n}\hat{y}_i^2, ESS = \sum_{i=1}^{n}(y_i - \hat{y}_i)^2$ So

$$
R^2 = \frac{RSS}{TSS} = \frac{\sum_{i=1}^{n}\hat{y}_i^2}{\sum_{i=1}^{n}y_i^2} = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}y_i^2}
$$

3. Asymptotic Distribution

   (a)

$$
\begin{aligned}
plim\left(\frac{\hat{u}'\hat{u}}{n}\right) &= plim\frac{u'(I - P_x)u}{n} \\
&= [n - (k+1)]\sigma^2 \\
Var\left(\frac{\hat{u}'\hat{u}}{n}\right) &= \frac{\sigma^4}{n^2}\cdot 2[n - (k+1)]
\end{aligned}
$$

asymptotic distribution

$$\sqrt{n}\left(\frac{\hat{u}'\hat{u}}{n} - [n-(k+1)]\sigma^2\right) \sim N\left(0, 2[n-(k+1)]\frac{\sigma^4}{n}\right)$$

(b)

$$plim\left(\frac{\hat{u}'\hat{y}}{n}\right) = plim(\frac{u(I-P_x)X\hat{\beta}}{n}) = 0$$

Therefore, $\frac{\hat{u}'\hat{y}}{n}$ is a constant.

(c) The result is same as part (a)

$$\begin{aligned} plim\left(\frac{\hat{u}'u}{n}\right) &= plim\frac{u'(I-P_x)u}{n} \\ &= [n-(k+1)]\sigma^2 \\ Var\left(\frac{\hat{u}'u}{n}\right) &= \frac{\sigma^4}{n^2}\cdot 2[n-(k+1)] \end{aligned}$$

asymptotic distribution

$$\sqrt{n}\left(\frac{\hat{u}'u}{n} - [n-(k+1)]\sigma^2\right) \sim N\left(0, 2[n-(k+1)]\frac{\sigma^4}{n}\right)$$