Final ECO 521

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1. AR(1) process

$$
\begin{aligned}
plim\hat{\rho} &= \frac{plim\frac{1}{T-1}\sum u_t u_{t-1}}{plim\frac{1}{T-1}\sum u_t^2} \\
&= \frac{\frac{\rho\sigma_\varepsilon^2}{1-\rho^2}}{\frac{\sigma_\varepsilon^2}{1-\rho^2}} \\
&= \rho \\
Asy.Var &= E\sqrt{T}(\hat{\rho} - plim\hat{\rho})\sqrt{T}(\hat{\rho} - plim\hat{\rho})' \\
&= T \cdot E\left(\frac{\sum u_{t-1}\varepsilon_t}{\sum u_t^2}\right)\left(\frac{\sum u_{t-1}\varepsilon_t}{\sum u_t^2}\right)' \\
&= T\sigma_\varepsilon^2
\end{aligned}
$$

   Its asymptotic properties

$$\sqrt{T}(\hat{\rho} - \rho) \sim N(0, T\sigma_\varepsilon^2)$$

2. RLS($J = 1$)

   Rewritting $H_0 : R\beta = c$

   where $R = (0, 0, 1, -1, 0, \dots, 0), c = 0$

   Let $e = y - X\hat{\beta}$.

   The test statistic is

$$\frac{\sqrt{T}(R\hat{\beta} - c)}{\sqrt{s^2 R\left(\frac{X'X}{T}\right)^{-1} R'}}$$

where $s^2 = \dfrac{e'e}{T-(n+1)}$

Its distribution

$$\frac{\sqrt{T}(R\hat{\beta}-c)}{\sqrt{s^2 R\left(\frac{X'X}{T}\right)^{-1}R'}} = \frac{\dfrac{\sqrt{T}(R\hat{\beta}-c)}{\sqrt{\sigma_u^2 R\left(\frac{X'X}{T}\right)^{-1}R'}}}{\sqrt{\dfrac{[T-(n+1)]s^2}{\sigma_u^2}/[T-(n+1)]}}$$
$$\sim \frac{N(0,1)}{\sqrt{\dfrac{\chi^2_{T-(n+1)}}{T-(n+1)}}}$$
$$\sim t_{T-(n+1)}$$

3. To get a consistent estimator of β, it must be true that $\alpha = 1, Var(e) = 0$.

4. Ordered Logit Model

$$\begin{aligned} y_{it} &\sim Poisson(\lambda_{it}) \\ \log(\lambda_{it}) &= OAge_{it} + Race_i + Gender_i + u_{it} \\ Age_{it} &= 10(Race_i + Gender_i + e_{it}) \\ OAge_{it} &= k \text{ iff } k-1 \leq \frac{Age_{it}}{10} < k \end{aligned}$$

5. RLS with omitted variables($J > 1$)

Rewritting $H_0 : R\beta = c$

where $R = \begin{pmatrix} 0 & 0 & 1 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & \cdots & 0 \end{pmatrix}, c = (0,0,0)'$

Let $e = y - X\hat{\beta}$.

The test statistic is

$$\frac{T(R\hat{\beta}-c)'\left[R\left(\frac{X'X}{T}\right)^{-1}R'\right]^{-1}(R\hat{\beta}-c)}{Js^2}$$

where $s^2 = \dfrac{e'e}{T-(n+1)}$, $J = Rank(R)$

Its distribution

$$\begin{aligned}
F &= \frac{T(R\hat{\beta}-c)'\left[R\left(\dfrac{X'X}{T}\right)^{-1}R'\right]^{-1}(R\hat{\beta}-c)}{Js^2} \\
&= \frac{\dfrac{T(R\hat{\beta}-c)'\left[R\left(\dfrac{X'X}{T}\right)^{-1}R'\right]^{-1}(R\hat{\beta}-c)}{\sigma_u^2}/J}{\dfrac{[T-(n+1)]s^2}{\sigma_u^2}/[T-(n+1)]} \\
&\sim \frac{\dfrac{\chi_J^2}{J}}{\dfrac{\chi^2_{T-(n+1)}}{T-(n+1)}} \\
&\sim F_{J,T-(n+1)}
\end{aligned}$$

The distribution remains unchanged if there were some omitted variables.

6. Random effect model

$$\begin{aligned}
Ee_{it}^2 &= \frac{1}{1-\rho^2} \\
E(e_{it}e_{it-1}) &= 0 \\
E(e_{it}e_{it-2}) &= \frac{\rho}{1-\rho^2} \\
&\vdots \\
E(e_{it}e_{it-2n+1}) &= 0 \\
E(e_{it}e_{it-2n}) &= \frac{\rho^n}{1-\rho^2}
\end{aligned}$$

Let Ω be the covariance matrix of v. WLOG, T is even.

$$\begin{aligned}\Omega &= E(vv') \\ &= \begin{pmatrix} A & 0_T & \cdots & 0_T \\ 0_T & A & \cdots & 0_T \\ \vdots & \vdots & \ddots & \vdots \\ 0_T & 0_T & \cdots & A \end{pmatrix}\end{aligned}$$

where

$$A_{T\times T} = \begin{pmatrix} \frac{1}{1-\rho^2}+\sigma_u^2 & \sigma_u^2 & \frac{\rho}{1-\rho^2}+\sigma_u^2 & \cdots & \frac{\rho^{\frac{T}{2}}}{1-\rho^2}+\sigma_u^2 \\ \sigma_u^2 & \frac{1}{1-\rho^2}+\sigma_u^2 & \sigma_u^2 & \cdots & \sigma_u^2 \\ \frac{\rho}{1-\rho^2}+\sigma_u^2 & \sigma_u^2 & \frac{1}{1-\rho^2}+\sigma_u^2 & \cdots & \frac{\rho^{\frac{T}{2}-1}}{1-\rho^2}+\sigma_u^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\rho^{\frac{T}{2}}}{1-\rho^2}+\sigma_u^2 & \sigma_u^2 & \frac{\rho^{\frac{T}{2}-1}}{1-\rho^2}+\sigma_u^2 & \cdots & \frac{1}{1-\rho^2}+\sigma_u^2 \end{pmatrix}$$

7. The expectation of $\left(X'\hat{\Omega}^{-1}X\right)^{-1} X'\hat{\Omega}^{-1}y$ does not exist.

8. Endogeneity

   For the first equation, it violates the order condition because $4-4 \ngeq 2-1$, i.e. β_{12} cannot be identified. Alternatively, the reduced form is

$$\begin{cases} y_{1i} = \pi_{10} + \pi_{11}x_{11} + \pi_{12}x_{12} \\ y_{2i} = \pi_{20} + \pi_{21}x_{11} + \pi_{22}x_{12} \end{cases}$$

   Applying OLS, we have six equations $(\pi_{10}, \pi_{11}, \pi_{12}, \pi_{20}, \pi_{21}, \pi_{22})$ with 7 parameters to estimate. Therefore, the model is under identified, i.e. β_{12} cannot be estimated.

9. Find a instrument variable Z with the following properties

   - $plim\left(\frac{Z'X}{T}\right)$ is invertible

- $plim\left(\frac{Z'u}{T}\right) = 0$

Let $Z^* = (Z \mid Q), X^* = (X \mid Q), \beta^* = (\beta, \gamma)'$

Rewritting the model

$$y = X^*\beta^* + u$$

Multiplying both sides by $Z^{*\prime}$

$$Z^{*\prime}y = Z^{*\prime}X^*\beta^* + Z^{*\prime}u$$

The orthogonality condition is

$$E[Z^{*\prime}y - Z^{*\prime}X^*\beta^*] = 0$$

10. Orthogonality condition from log-likelihood function

$$\begin{aligned} P\{y_i = 1 \mid x_i\} &= Pr\{y_i^* > 0 \mid x_i\} \\ &= Pr\{x_i\beta + u_i > 0\} \\ &= 1 - \Phi(-x_i\beta) \\ &= \Phi(x_i\beta) \end{aligned}$$

The log-likelihood function is

$$l(y \mid x, \beta) = y_i \log \Phi(x_i\beta) + (1 - y_i) \log \Phi(-x_i\beta)$$

Noting that $\phi(x_i\beta) = \phi(-x_i\beta)$

$$\begin{aligned} \nabla_\beta l &= x_i\phi(x_i\beta)\left[\frac{y_i}{\Phi(x_i\beta)} - \frac{1 - y_i}{\Phi(-x_i\beta)}\right] \\ &= w(x_i\beta)x_i[y_i - \Phi(x_i\beta)] \end{aligned}$$

where $w(x_i\beta) = \dfrac{\phi(x_i\beta)}{\Phi(x_i\beta)\Phi(-x_i\beta)}$

The orthogonality condition can be simplified as

$$E[y_i - \Phi(x_i\beta)]x_i = 0$$

11. Consider a Nested Logit Model.

$$
\begin{aligned}
y_{ijt}^* &= x_{ijt}\beta + z_{it}\gamma_j + u_{ij} + \varepsilon_{ijt} \\
y_{ijt} &= \mathbb{1}(y_{ijt}^* > y_{ikt}^* \, \forall k \neq j) \\
u_i &= (u_{i1}, u_{i2}, \dots, u_{iJ})' \sim iid\, N(0, \Omega) \\
\varepsilon_{ijt} &\sim \; EV \\
i &= 1, 2, \dots n \\
j &= 1, 2, \dots J \\
t &= 1, 2, \dots T
\end{aligned}
$$

Define

$$
Pr(y_{ijt} \mid x_{ijt}, z_{it}, u_{ij}) = \frac{\exp\{x_{ijt}\beta + z_{it}\gamma_j + u_{ij}\}}{\sum\limits_k \exp\{x_{ikt}\beta + z_{it}\gamma_k + u_{ik}\}}
$$

Then, ijt-specific conditional likelihood contribution is

$$
L_{ijt}(u_{ij}) = \left(\frac{\exp\{x_{ijt}\beta + z_{it}\gamma_j + u_{ij}\}}{\sum\limits_k \exp\{x_{ikt}\beta + z_{it}\gamma_k + u_{ik}\}} \right)^{y_{ijt}}
$$

Once we condition on u_{ij}, the observations over t for i, j are independent. Therefore, the ij-specific conditional likelihood contribution is

$$
L_{ij}(u_{ij}) = \prod_{t=1}^{T} \left(\frac{\exp\{x_{ijt}\beta + z_{it}\gamma_j + u_{ij}\}}{\sum\limits_k \exp\{x_{ikt}\beta + z_{it}\gamma_k + u_{ik}\}} \right)^{y_{ijt}}
$$

and the ij-specific unconditional likelihood contribution is

$$
L_{ij} = \int L_{ij}(u_{ij}) \mathrm{d}F(u_{ij} \mid \Omega)
$$

The likelihood function is

$$
\begin{aligned}
L &= \prod_{i=1}^{n} \prod_{j=1}^{J} L_{ij} \\
&= \prod_{i=1}^{n} \prod_{j=1}^{J} \int \prod_{t=1}^{T} \left(\frac{\exp\{x_{ijt}\beta + z_{it}\gamma_j + u_{ij}\}}{\sum\limits_k \exp\{x_{ikt}\beta + z_{it}\gamma_k + u_{ik}\}} \right)^{y_{ijt}} \mathrm{d}F(u_{ij} \mid \Omega)
\end{aligned}
$$

The covariation of $u_i = (u_{i1}, u_{i2}, \dots, u_{iJ})'$ in the data indenfies Ω.

12. Binary model with dummy endogenous regressors ($\alpha < 0$).

    Since we have 25 (5 by 5) feasible regions in this case. and the middle 9 (3 by 3) regions are multiple equilibria.