When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during the lectures and the recitations. Solutions should be your own. Please update your solutions in a single pdf file, with the first page indicating your name and SBU ID. Needless to say, cheating will not be tolerated.

Name:____

1. Six letters are selected at random one after another from the 26 letters of the English alphabet. Find the probabilities that the words formed: (i) contains exactly one "s"; (ii) consists of the first six letters (that is, only contains 'a', 'b', 'c', 'd', 'e','f'); (iii) is the word "reader", when

   - Drawing with replacement.
   - Drawing without replacement.

2. Jane has three children, each of which is equally likely to be a boy or a girl independently of the others. Define the events

$$\begin{aligned}A =& \{\text{all the children are of the same sex}\}\\ B =& \{\text{there is at most one girl}\}\\ C =& \{\text{the family includes a boy and a girl}\}.\end{aligned}$$

   a) Show that A is independent of B, and that B is independent of C.
   b) Is A independent of C?
   c) Suppose that the probability of having a boy is now 1/3. Do the results above still hold?

3. Let G be the event that an accused is guilty, and T the event that some testimony is true. Some lawyers have argued that $P(G|T) = P(T|G)$. Show that this holds if and only if $P(G) = P(T)$. (*Hint*: You have to show the implication in both directions. That is, that $P(G|T) = P(T|G)$ implies $P(G) = P(T)$, and that $P(G) = P(T)$ implies $P(G|T) = P(T|G)$)

4. In your pocket, there is a random number of coins, N, where N has the Poisson distribution with parameter λ. You toss each coin once, with heads showing with probability p each time. Show that the total number of heads has the Poisson distribution with parameter λp.

5. A drunken sailor walks into a bar, itching to pick a fight. He sees a man sitting at the bar and he considers the choice of whether to pick a fight him. The man could either be a wimp or a tough-guy, but wimps always dress like tough guys, so the sailor can't tell them apart. However, he thinks that wimps and tough-guys hang out in bars with equal probability. When wimps are antagonized, they fight with probability w and they win with probability 0.4. When tough-guys are antagonized, they fight with probability t and they win with probability 0.7. The sailor's ship will be in port for two days, so he will have the opportunity to return the next day and fight the man again, regardless of what happens on the first day. For the next three questions, assume that the sailor decides to pick a fight on the first day. Each fight has no tie, so the sailor either wins or loses.

   a) What is the probability that the sailor's opponent is a tough-guy, given that the opponent fought and lost on the first day?

b) What is the sailor's probability of losing a fight on the second day, conditional on having fought against an opponent who lost on the first day? (*Hint*: Consider the same game with updated beliefs on the distribution of types in the population)

c) Assume that the sailor antagonized the man at the bar on the first day, and that the man fought and lost. On the second day, the sailor's payoff is 0 if there is no fight, 10 if he fights and wins, and -10 if he fights and looses. What choice gives a higher expected payoff on the second day, challenging the man at the bar to a re-match or backing down?

6. The amount of bread (in hundreds of pounds) that a bakery sells in a day is a random variable with density

$$f_X(x) = \begin{cases} cx & \text{for } 0 \leq x < 3 \\ c(6-x) & \text{for } 3 \leq x < 6 \\ 0 & \text{otherwise} \end{cases}$$

a) Find the value of c that would make f_X a pdf.

b) What is the probability that the total amount of bread sold in a day is (i) more than 300 pounds; (ii) between 150 and 450 pounds.

c) Denote by A and B the events in b(i) and b(ii), respectively. Are A and B independent?

7. Suppose X follow an exponential distribution with density

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

Determine the density function, f_Y, of the random variable

$$Y = \begin{cases} X & X \leq 1 \\ 1/X & X > 1 \end{cases}$$

8. The joint density of (X, Y) is given by

$$f_{XY}(x, y) = ye^{-y(x+1)}, \text{ with } x, y > 0.$$

Find

a) The marginal densities of X and Y.

b) The conditional distribution $F_{X|Y}(x|y)$.

9. Suppose $P(Y = 0) = P(Y = 1) = 0.5$, that $Z \sim N(0, 1)$; and that Y and Z are independent. Set $X = YZ$. What is the distribution of X?

10. Let $X_1, X_2, \ldots$ be mutually independent random variables with the same mean μ and variance σ^2, and let N be an integer-valued random variable with mean n and variance ν, with N independent of all the X_i. Let

$$S = \sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i \mathbb{1}(N \geq i),$$

where $\mathbb{1}(\cdot)$ is the indicator function. Compute $Var(S)$ in terms of $\{\mu, \sigma^2, n, \nu\}$.

The End.