When a question asks to *show* a result, it means you have to give a formal proof. Try to give a sufficient amount of details to support your computations and make use of the knowledge you acquired during lectures and recitations. Needless to say, cheating will not be tolerated.

Name:____

1. $\{X_1, \ldots, , X_n\}$ be an IID sample of size n from a distribution F, with finite mean μ and finite variance σ^2. A weighted sample mean takes the form

$$\bar{X}_n^* = \frac{1}{n}\sum_{i=1}^{n} w_i X_i,$$

where the w_i's are constants (not random), such that $\frac{1}{n}\sum_{i=1}^{n} w_i = 1$.

   a) Show that $\bar{X}_n^*$ is an unbiased estimator of μ.
   b) Calculate $Var(\bar{X}_n^*)$.
   c) Show that $\bar{X}_n^* \xrightarrow{p} \mu$. Clearly state the additional conditions that w_i needs to satisfy for this result to hold.
   d) What is the asymptotic distribution of $\bar{X}_n^*$?

2. Let X be a continuous random variable with density equal to

$$f_X(x) = \begin{cases} \frac{e^{-\frac{(x-\lambda)^2}{2}}}{\sqrt{2\pi}\Phi(\lambda)} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the Moment Generating Function of X.
   b) Compute the first three moments of X.
   c) Find the Characteristic function of X.

3. In independent Bernoulli trials with success probability p, let X be the variable counting the number of failures before the m^{th} success. This variable has probability mass function equal to

$$P(X = x) = \begin{cases} \binom{m+x-1}{m-1} p^m (1-p)^x & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

   a) Find the Moment Generating Function of X.
   b) Compute the first three moments of X.
   c) Find the Characteristic function of X.

4. Let $\{X_1, \ldots, , X_n\}$ be an IID sample of size n from the geometric distribution

$$f_X(x; p) = \begin{cases} p^x(1-p) & x \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

   a) Determine the Maximum Likelihood estimator of p, called $\hat{p}_n$. Verify that this maximum likelihood estimator is unique.

b) Show that the estimator is consistent.

c) Find the Information matrix for this problem and use it to determine the asymptotic distribution of $\hat{p}_n$.

5. $\{X_1, \ldots, , X_n\}$ be an IID observations from the from a zero-inflated Poisson distribution. This distribution arises from a mixture model where, with probability p, X is observed from the 'distribution' which has point mass at zero (i.e. P(Y = 0) = 1), and with probability (1?p), X is observed from a Poisson distribution with parameter λ. The zero-inflated Poisson distribution has density function

$$f_X(x; p, \lambda) = \begin{cases} p + (1-p)e^{-\lambda} & x = 0 \\ (1-p)\frac{\lambda^x e^{-\lambda}}{x!} & x > 0 \\ 0 & \text{otherwise} \end{cases} .$$

a) Determine the Maximum Likelihood estimator of (p, λ), called $(\hat{p}_n, \hat{\lambda}_n)$. Is this maximum likelihood estimator unique?

b) Is this estimator consistent?

c) Find the Information matrix for this problem and use it to determine the asymptotic distribution of $(\hat{p}_n, \hat{\lambda}_n)$.

The End.