Homework 6

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1) Consider a Multinomial Probit Model

$$
\begin{aligned}
y_{ij}^* &= V_{ij} + \varepsilon_{ij} \\
y_{ij} &= \mathbb{1}(y_{ij}^* \geq \max_{k \neq j} y_{ik}^*) \\
\varepsilon_{ij} &\sim iid\, N(0, \Omega)
\end{aligned}
$$

In order to reduce the number of intergrals by one, we can fix all of the diagonal elements of Ω to be one and set the base choice to 1. Then replace Ω with $\tilde{\Omega}$

$$
\begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iJ} - \varepsilon_{i1} \end{pmatrix} \sim N(0, \tilde{\Omega})
$$

Proof.

$$
\tilde{\Omega} = E \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iJ} - \varepsilon_{i1} \end{pmatrix} \begin{pmatrix} \varepsilon_{i2} - \varepsilon_{i1} \\ \varepsilon_{i3} - \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iJ} - \varepsilon_{i1} \end{pmatrix}'
$$

$$
= \begin{pmatrix} 2 - 2\Omega_{12} & \Omega_{23} - \Omega_{12} - \Omega_{13} + 1 & \cdots & \Omega_{2J} - \Omega_{12} - \Omega_{1J} + 1 \\ \Omega_{23} - \Omega_{12} - \Omega_{13} + 1 & 2 - 2\Omega_{12} & \cdots & \Omega_{3J} - \Omega_{13} - \Omega_{1J} + 1 \\ \vdots & \vdots & \ddots & \vdots \\ \Omega_{2J} - \Omega_{12} - \Omega_{1J} + 1 & \Omega_{3J} - \Omega_{13} - \Omega_{1J} + 1 & \cdots & 2 - 2\Omega_{1J} \end{pmatrix}
$$

Clearly, the elements on each diagonal are not equal to zero, which implies the covariance matrix for errors is unidentified. $\square$

2) Consider a Ordered Logit Model

$$w = X\beta + u$$
$$u \sim N(0, \sigma^2 I)$$

Denote by $\tau = (\tau_0, \tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (-\infty, 3.5, 5, 8, 15, \infty)$ a supporting vector. D_i becomes

$$D_i = k \text{ iff } \tau_k < w \leq \tau_{k+1}$$

The relevant terms for estimation are

$$\begin{aligned} P_{ik} &= Pr(D_i = k \mid X) \\ &= Pr(\tau_k < w \leq \tau_{k+1} \mid X) \\ &= Pr(\tau_k < X\beta + u \leq \tau_{k+1}) \\ &= Pr(\tau_k - X\beta < u \leq \tau_{k+1} - X\beta) \\ &= \Phi\left(\frac{\tau_{k+1} - X\beta}{\sigma}\right) - \Phi\left(\frac{\tau_k - X\beta}{\sigma}\right) \end{aligned}$$

Applying MLE

$$\hat{\beta} = \underset{\beta}{\operatorname{argmax}} \sum_{k=0}^{4} \mathbb{1}(D_i = k) \log\left[\Phi\left(\frac{\tau_{k+1} - X\beta}{\sigma}\right) - \Phi\left(\frac{\tau_k - X\beta}{\sigma}\right)\right]$$

3) Consider a Nested Logit Model

$$\begin{aligned} y_{ij}^* &= X_{ij}\beta + u_{ij} \\ y_{ij} &= \mathbb{1}(y_{ij}^* \geq \max_{k \neq j} y_{ik}^*) \\ u_{ij} &\sim iid\, EV \end{aligned}$$

The probability of choosing j is

$$P_{ij} = \frac{\exp\{X_{ij}\beta\}}{\sum_j \exp\{X_{ij}\beta\}}$$

The expected value of the best j is

$$E \max_j y_{ij}^* = \log \sum_j \exp\{X_{ij}\beta\} + u$$