1. Given $P(A) = 1/3$, $P(B) = 1/4$, and $P(A \cap B) = 1/6$, find the following probabilities

$$P(A^c), \quad P(A^c \cup B), \quad P(A \cap B^c), \quad P(A \cup B^c), \quad P(A^c \cup B^c).$$

Solution.

$$\begin{aligned}
P(A^c) =& 1 - P(A) = 2/3 \\
P(A^c \cup B) =& P(A^c) + P(B) - P(A^c \cap B) = 2/3 + 1/4 - (1/4 - 1/6) = 5/6 \\
P(A \cap B^c) =& P(A) - P(A \cap B) = 1/3 - 1/6 = 1/6 \\
P(A \cup B^c) =& P(A) + P(B^c) - P(A \cap B^c) = 1/3 + 3/4 - 1/6 = 11/12 \\
P(A^c \cup B^c) =& P(A^c) + P(B^c) - P(A^c \cap B^c) = 2/3 + 3/4 - (2/3 - 1/4 + 1/6) = 5/6.
\end{aligned}$$

2. For any three events A, B, and C, show that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Use the relationship above to solve the following problem. The percentages of students who passed courses A, B, and C are as follows: A : 50%, B : 40%, C = 30%, $A \cap B$: 35%, $B \cap C$: 20%, and 15% passed all three courses. What is the percentage of students who passed at least one of the three courses?

Solution. *Use the fact that* $P(A) = P(A \cap B) + P(A \cap B^c)$*, thus*

$$\begin{aligned}
P(A \cup B \cup C) =& P(A \cup B \cup C \cap A) + P(A \cup B \cup C \cap A^c) \\
=& P(A) + P(B \cup C \cap A^c) = P(A) + P((B \cap A^c) \cup (C \cap A^c)) \\
=& P(A) + P(B \cap A^c) + P(C \cap A^c) - P(B \cap C \cap A^c) \\
=& P(A) + P(A) - P(A \cap B) + P(C) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).
\end{aligned}$$

This concludes the proof. The application should be straightforward.

3. Three boxes contain two rings each, but in one of them they are both gold, in the second both silver, and in the third one of each type. You have the choice of randomly extracting a ring from one of the boxes, the content of which is unknown to you. You look at the extracted ring, and then you have the possibility to extract a second ring again from any of the three boxes. Suppose you can select each box with equal probability. Let us assume the first ring you extract is a gold one. Is it preferable to extract the second one from the same or from a different box?

Solution. *Bayes' theorem. Define the following events:*

- B_1*: Extracting from box 1.*
- B_2*: Extracting from box 2.*
- B_3*: Extracting from box 3.*
- A*: The first ring extracted is a gold ring.*

Then $P(B_1) = P(B_2) = P(B_3) = 1/3$.

$$P(A|B_1) = 1, \qquad P(A|B_2) = 0, \qquad P(A|B_3) = 0.5,$$

and

$$P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + P(A|B_3)P(B_3)$$

Finally:

$$P(B_1|A) = \frac{P(A|B_1)P(B_1)}{P(A)} = \frac{2}{3}, \qquad P(B_2|A) = \frac{P(A|B_2)P(B_2)}{P(A)} = 0$$

$$P(B_3|A) = \frac{P(A|B_3)P(B_3)}{P(A)} = \frac{1}{3}$$

So that, with probability 0.66*, you are extracting from box* 1*, and with probability* 0.33*, from box* 2*. Then, it is better to extract the ring from the same box.*

4. Recall that the pmf of the Poisson distribution can be written as

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}.$$

Show that the pmf satisfies

$$p_X(k) = \frac{\lambda}{k}p_X(k-1),$$

and hence determine the values of k for which the terms $p_X(k)$ reach their maximum (for a given λ).

Solution. *It is easy to prove that*

$$p_X(k) = e^{-\lambda}\frac{\lambda^k}{k!} = \frac{\lambda}{k}e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}.$$

Using this property, we aim at finding the value of k such that

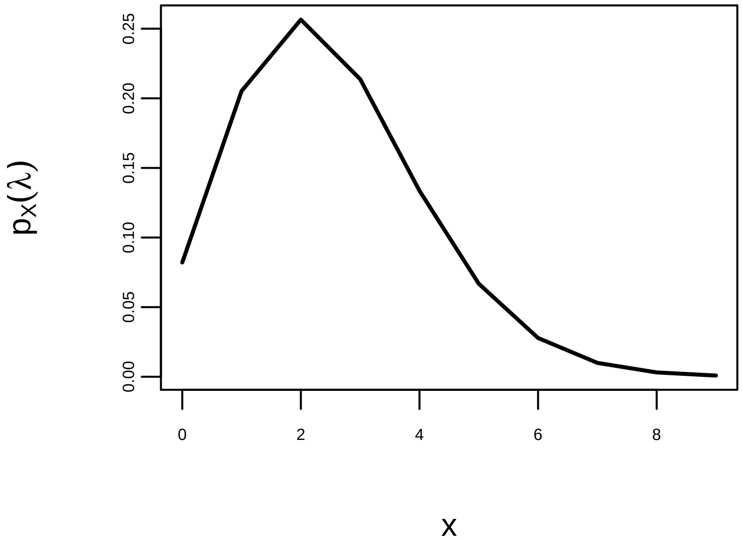
$$p_X(k-1) \leq p_X(k) \geq p_X(k+1).$$

That is

$$\begin{aligned} p_X(k) \geq& p_X(k-1) \quad \Rightarrow \quad \frac{\lambda}{k} \geq 1 \quad \Rightarrow \quad k \leq \lambda \\ p_X(k) \geq& p_X(k+1) \quad \Rightarrow \quad 1 \geq \frac{\lambda}{k+1} \quad \Rightarrow \quad k \geq \lambda - 1. \end{aligned}$$

Therefore, the density reaches its maximum for $k \in [\lambda-1, \lambda]$. In the figure below, I depict the Poisson distribution with $\lambda = 2.5$. It is easy to see that it reaches its maximum at 2*, which is right in between* 1.5 *and* 2.5*.*

```r
lower<-qpois(0.001, lambda=2.5)
upper<-qpois(0.999, lambda=2.5)
n<-seq(lower,upper,1)
q<-seq(0.001,0.999,0.001)
dPoisson25 <- data.frame(N=n,
                         Density=dpois(n, lambda=2.5),
                         Distribution=ppois(n, lambda=2.5))
qPoisson25 <- data.frame(Q=q, Quantile=qpois(q, lambda=2.5))
par(mar = c(4,4,1,1))
plot(n,dPoisson25$Density,type="l",lwd = 2,xlab="x",
     ylab =expression(p[X](lambda)),xaxt="n",yaxt="n")
axis(1, mgp=c(3, .5, 0),cex.axis = 0.5)
axis(2, mgp=c(3, .5, 0),cex.axis = 0.5)
```



5. Suppose that the duration (in minutes) of long-distance telephone calls follows an exponential density function

$$f_X(x) = \frac{1}{5}e^{-x/5}, \text{ for } x > 0.$$

Find the probability that the duration of a conversation

```
p1 <- 1 - pexp(5, rate = 0.2)
p2 <- pexp(6, rate = 0.2) - pexp(5, rate = 0.2)
p3 <- pexp(3, rate = 0.2)
# Notice that this is equal to p3
p4 <- (pexp(6, rate = 0.2) - pexp(3, rate = 0.2))/(1- p3)
```

a) Will exceed 5 minutes.

Solution. *First notice that*

$$F_X(x) = P(X \leq x) = \int_0^x f_X(t)dt = \int_0^x \frac{1}{5}e^{-t/5}dt$$
$$= \left[-e^{-t/5}\right]_0^x = 1 - e^{-x/5}.$$

In this question we have that

$$P(X > 5) = 1 - F_X(5) = 1 - \left(1 - e^{-1}\right) = e^{-1} = 0.3679.$$

b) Will be between 5 and 6 minutes.

Solution. *Here*

$$P(5 \leq X < 6) = F_X(6) - F_X(5) = \left(1 - e^{-6/5}\right) - \left(1 - e^{-1}\right) = e^{-1} - e^{-6/5} = 0.0667.$$

c) Will be less than 3 minutes.

Solution. *Here*

$$F_X(3) = 1 - e^{-3/5} = 0.4512.$$

Cont.

d) Will be less than 6 minutes given that it was greater than 3 minutes.

Solution. *Use the memory-less properties of the exponential distribution to claim that this is the same as the answer to c. Since,*

$$P(X<6|X\geq 3)=\frac{P(3\leq X<6)}{P(X\geq 3)}=\frac{e^{-3/5}-e^{-6/5}}{e^{-3/5}}=1-e^{-3/5}=F_X(3).$$

6. Let (X,Y) be *uniform* in the triangle $x\geq 0$, $y\geq 0$, and $x+y\leq 2$ (*Hint*: Uniform means it is constant over the entire support). Find:

a) The joint pdf of (X,Y).

Solution. *We must have that*

$$\begin{aligned}\int_0^2\int_0^{2-x} f_{XY}(x,y)dydx =&1\\ c\int_0^2\int_0^{2-x} dydx =&c\int_0^2(2-x)dx=c\,(4-2)=1\\ \Rightarrow\quad c=&1/2=f_{XY}(x,y).\end{aligned}$$

b) The marginal pdf of X.

Solution. *We use the relationship between the joint and marginal distribution to get*

$$f_X(x)=\frac{1}{2}\int_0^{2-x}dy=1-\frac{x}{2}.$$

c) The conditional pdf of Y given $X=x$. Is this conditional distribution defined for every $x\in[0,2]$? Explain.

Solution. *The conditional pdf is the ratio between the joint and the marginal. Therefore*

$$f_{Y|X}(y|x)=\frac{1}{2}\frac{2}{2-x}=\frac{1}{2-x}.$$

Where $x=2$, the conditional distribution is not defined.

d) $E\,[Y|X=x]$.

Solution. *Use the properties of the uniform distribution*

$$E\,[Y|X=x]=\frac{2-x}{2}.$$

7. Let X and Z be independent, each following a standard normal distribution. Let $a,b\in\mathbb{R}$ (not simultaneously equal to 0), and let $Y=aX+bZ$.

a) Compute $Corr(X,Y)$.

Solution. *We have that*

$$Corr(X,Y)=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=\frac{a}{\sqrt{a^2+b^2}}.$$

b) Show that $|Corr(X,Y)|\leq 1$ in this case.

Solution. *Since b^2 is either positive or zero, then the absolute value of the correlation is either equal to 1 (if $b=0$) or lower than 1 (if $b\neq 0$).*

c) Give necessary and sufficient conditions on the values of a and b such that $Corr(X,Y)=1$.

Solution. *Necessary and sufficient conditions for the correlation to be equal to 1 are: $b=0$, and $a>0$.*

8. Show that for a continuous random variable X with density function f_X and cumulative distribution function F_X

$$\mu = E(X) = \int_0^\infty (1 - F_X(x))\, dx - \int_{-\infty}^0 F_X(x)dx.$$

(*Hint*: Use the definition of the mean and integration by parts).

Solution. *Recall that*

$$\mu = \int_{-\infty}^\infty x f_X(x)dx = \int_{-\infty}^\infty x dF_X(x) = \int_{-\infty}^0 x dF_X(x) - \int_0^\infty x d(1 - F_X(x))$$
$$= \int_0^\infty (1 - F_X(x))\, dx - \int_{-\infty}^0 F_X(x)dx$$

The End.