理论力学

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内容回顾

- 拉格朗日力学快速回顾
- 拉格朗日力学的优势
- 广义坐标、位形空间
- 拉格朗日方程，拉格朗日量的“规范不变性”
- 引入“约束”

  完整约束(Holonomic)、非完整约束(nonholonomic)

拉格朗日量的非唯一性

- 如果 *L* 是描述一个体系的拉格朗日量，则

$$L' = L + \frac{dF(q,t)}{dt}$$

也是体系的拉格朗日量，其中 *F* 是广义坐标和时间的任何可微函数。

证明：

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}}\left(\frac{dF}{dt}\right)\right) - \frac{\partial}{\partial q}\left(\frac{dF}{dt}\right) = 0$$

$$\frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial t}$$

今日目标

- 达朗伯原理
- 哈密顿原理
- 非完整约束
- 拉格朗日乘子法

虚位移

考虑一个受约束的系统

普通坐标 $\boldsymbol{r}_i(i = 1,\ldots,N)$

广义坐标 $q_j(j = 1,\ldots,n)$

$$\begin{cases} \boldsymbol{r}_1 = \boldsymbol{r}_1(q_1, q_2, \ldots, q_n, t) \\ \boldsymbol{r}_2 = \boldsymbol{r}_2(q_1, q_2, \ldots, q_n, t) \\ \boldsymbol{r}_3 = \boldsymbol{r}_3(q_1, q_2, \ldots, q_n, t) \\ \vdots \\ \boldsymbol{r}_N = \boldsymbol{r}_N(q_1, q_2, \ldots, q_n, t) \end{cases}$$

设想将系统所有质点做一个小移动

$$\boldsymbol{r}_i \to \boldsymbol{r}_i + \delta\boldsymbol{r}_i \qquad q_j \to q_j + \delta q_j$$

虚位移

满足瞬时约束 ($\delta t = 0$) 的位移。
只受约束的限制，无运动方程无关。

实位移

同时满足约束和运动方程的真实位移。

虚位移必须满足约束方程

3N个不完全独立坐标

$$\delta\boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j$$

n个独立坐标

实位移

$$d\boldsymbol{r}_i = \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}dq_j + \frac{\partial \boldsymbol{r}_i}{\partial t}dt$$

达朗伯原理

- 牛顿运动方程 $$\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i = 0$$

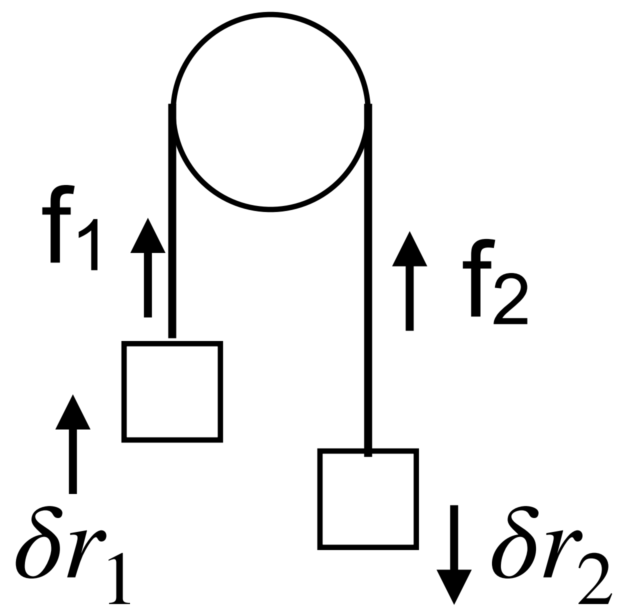
- 受力由外力和约束力组成 $$\boldsymbol{F}_i = \boldsymbol{F}_i^{(a)} + \boldsymbol{f}_i$$

  其中外力是已知的 $$\boldsymbol{F}_i^{(a)} = \boldsymbol{F}_i^{(a)}(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_i, \ldots, \boldsymbol{r}_N, t)$$

- 约束力所做的净虚功通常为零 **理想约束** 滑动摩擦除外

  虚位移与约束力垂直 $$\boldsymbol{f}_i\, \delta\boldsymbol{r}_i = 0$$

  虚功之和为零 $$\sum_i \boldsymbol{f}_i\, \delta\boldsymbol{r}_i = 0$$



我们在牛顿方程两边同乘虚位移，且对 i 求和，可得...

达朗伯原理

理想约束下，每个质点的外力和倒转有效力**所做虚功之和为零**。

$$\sum_i (\boldsymbol{F}_i^{(a)} - \dot{\boldsymbol{p}}_i)\delta \boldsymbol{r}_i = 0$$

消去了约束力，以后略去上标 “(a)”

达朗伯原理 (1743)

$$\sum_i \boldsymbol{f}_i \, \delta \boldsymbol{r}_i = 0$$

注意，系数 $\boldsymbol{F}_i - \dot{\boldsymbol{p}}_i$ 不一定为零，因为变量 $\delta \boldsymbol{r}_i$ 不完全独立，将 r_i 转化为 q_j

广义力

$$1\text{st term} = \sum_i \boldsymbol{F}_i \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j = \sum_j Q_j \delta q_j$$

$$Q_j \equiv \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

广义力 Q_j 的单位不一定是 [力]　　$Q_j q_j$ 的单位总是 [功]

达朗伯原理

将 $\boldsymbol{r}_i$ 转化为 q_j

$$\text{2nd term} = \sum_i \dot{\boldsymbol{p}}_i \delta \boldsymbol{r}_i = \sum_i \dot{\boldsymbol{p}}_i \sum_j \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j = \sum_{i,j} m_i \ddot{\boldsymbol{r}}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}\delta q_j$$

$$T \equiv \sum_i \frac{m v_i^2}{2}$$

可以证明

$$\ddot{\boldsymbol{r}}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} \to \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\frac{v_i^2}{2}\right)\right] - \frac{\partial}{\partial q_j}\left(\frac{v_i^2}{2}\right)$$

$$\sum_j \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right\}\delta q_j$$

$$\dot{\boldsymbol{r}} = \frac{\partial \boldsymbol{r}}{\partial q}\dot{q} + \frac{\partial \boldsymbol{r}}{\partial t}$$

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{r}}{\partial q}\right) = \frac{\partial^2 \boldsymbol{r}}{\partial q \partial q'}\dot{q}' + \frac{\partial^2 \boldsymbol{r}}{\partial q \partial t}$$

达朗伯原理

$$\sum_j \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right\}\delta q_j = 0$$

拉格朗日方程

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

独立变量

接近结果中...

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$Q_j \equiv \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

保守势： $\boldsymbol{F}_i = -\nabla_i V$ 无旋有势 $V(\boldsymbol{r}_1, \boldsymbol{r}_2, \ldots, \boldsymbol{r}_N, t)$

$$Q_j \equiv \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\sum_i \nabla_i V \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

$$\Rightarrow \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

拉格朗日方程

$$\sum_j \left\{ \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right] - Q_j \right\} \delta q_j = 0$$

独立变量

接近结果中…

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$Q_j \equiv \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j}$$

广义势：

$$\boldsymbol{F}_i = -\frac{\partial U}{\partial \boldsymbol{r}_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\boldsymbol{r}}_i}\right)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$Q_j \equiv \sum_i \boldsymbol{F}_i \frac{\partial \boldsymbol{r}_i}{\partial q_j} = \sum_i \left(-\frac{\partial U}{\partial \boldsymbol{r}_i} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{\boldsymbol{r}}_i}\right) \right) \frac{\partial \boldsymbol{r}_i}{\partial q_j} = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

带电粒子在电磁场中的运动

洛仑兹力：

$$\boldsymbol{F} = q\left[\boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B})\right]$$

速度依赖

电场 **E** 和磁场 **B**

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial A}{\partial t}$$

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}$$

Maxwell 方程

速度依赖的势函数

$$U = q\phi - q\boldsymbol{A} \cdot \boldsymbol{v}$$

验证其有效性?

拉格朗日量

$$L = \frac{1}{2}mv^2 - q\phi + q\boldsymbol{A} \cdot \boldsymbol{v}$$

达朗伯原理用到的假设

- **完整约束**

  始终用到的假设

  $$\boldsymbol{r}_i = \boldsymbol{r}_i(q_1, q_2, \ldots, q_n, t)$$

- **约束力所做的净虚功为零**

  不计摩擦力　理想约束

  $$\sum_i \boldsymbol{f}_i\, \delta\boldsymbol{r}_i = 0$$

- **单演系统**

  外力由某一广义势函数给出

  $$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

  $$U = U(q, \dot{q}, t)$$

拉格朗日方程

单演系统： 系统除约束力外的所有力都由某一广义势函数给出

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=0$$

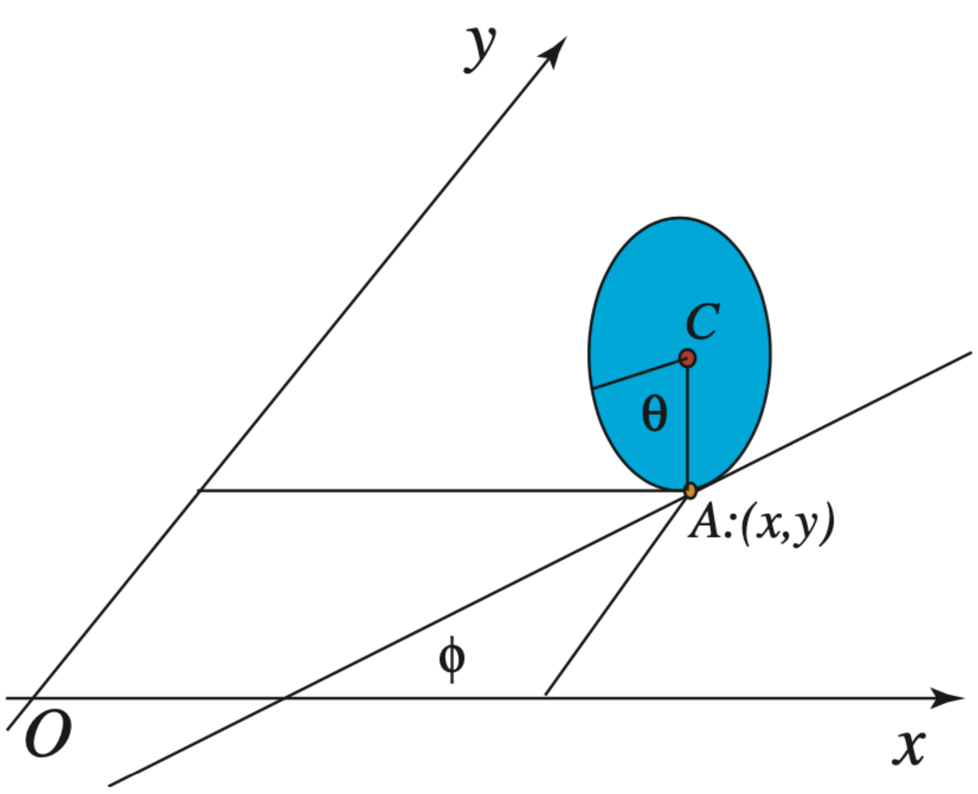
不是所有作用于系统的力都从势导出：

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\frac{\partial L}{\partial q_j}=Q_j$$

非完整约束

二维平面上垂直纯滚的均匀圆盘

考虑一个半径为a的均匀圆盘，它在二维平面上无滑动地纯滚，假定圆盘中心(质心)C点与圆盘和平面的接触点A之间的连线永远垂直于平面。



$$dx = a d\theta \cos\phi$$
$$dy = a d\theta \sin\phi$$

不能写成积分形式，非完整约束！

半完整约束！

非完整体系

完整约束：

$$f_\alpha(q_1, \ldots, q_n, t) = 0$$

积分形式

$$\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0$$

微分形式

我们将处理特定的非完整约束体系，即约束方程形式可写为

$$\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0$$

Pfaffian 形式

$$\sum_k a_{\alpha k} \delta q_k = 0$$

拉格朗日乘子法

同样适用于完整约束情形，如不便将所有 q 化为独立坐标的情形

拉格朗日乘子法

考虑 m 个 约束，n 个广义坐标的保守系统

$a_{\alpha j}\delta q_j = 0$ **重复指标求和**

$\alpha = 1,\ldots,m; \quad j = 1,\ldots,n$

达朗伯原理

$$\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j = 0$$

$$\left\{\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right] - \lambda_\alpha a_{\alpha j}\right\}\delta q_j = 0$$

先对 j 求和；拉氏乘子项为0；
如果先对 α 求和呢?

选择 λ_α

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_{\alpha=1}^{m}\lambda_\alpha a_{\alpha j} = 0$$

δq_j 是独立变量 $\quad j = 1,\ldots,n-m$

δq_j 非独立变量 $\quad j = n-m+1,\ldots,n$

拉格朗日乘子法

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \boxed{\lambda_\alpha a_{\alpha j}} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j$$

一种广义力；称为广义约束力

- 完整约束

$$f_\alpha(q_1, \ldots, q_n, t) = 0$$

$$\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0$$

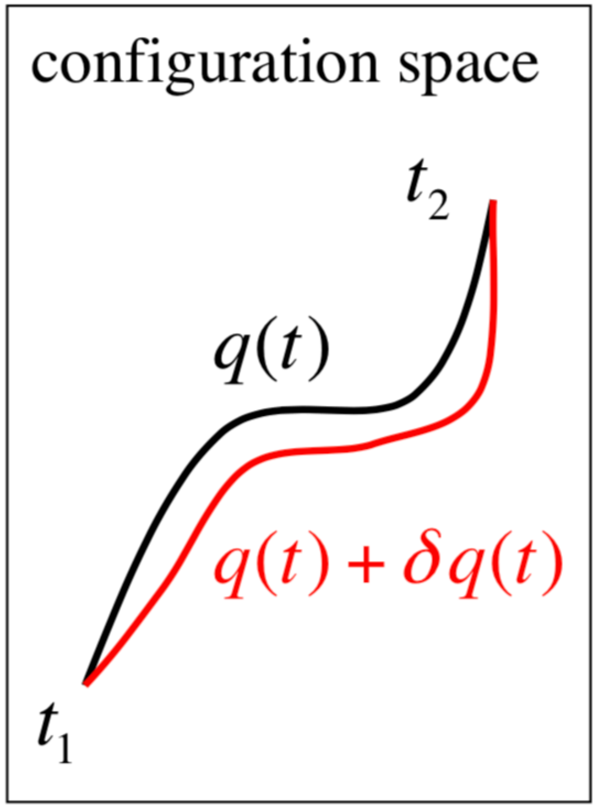
$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = Q_i$$

- 非完整约束

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = -\lambda_\alpha a_{\alpha j} = Q_j$$

哈密顿原理

单演系统在通过位形空间中两个点所能做的各种（相邻）运动中，**<u>真实运动路径使作用量取极值</u>**。



$$\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j dt = 0$$

非单演系统：

$$\delta I = \int_{t_1}^{t_2} \left[\delta L(q_j, \dot{q}_j, t) - Q_\alpha \delta q_\alpha \right] dt = 0$$

American Journal of Physics 34, 1202 (1966)：
先积分后变分：变分路径是所有几何上连接初末态可能的路径。
先变分后积分：变分路径是从真实路径基础上由虚位移构建得来的。

完整、单演系统：等价（连续可微）
非完整、或非单演系统：不等价

作用量极值性质

- 考虑一维空间自由粒子 $$L = \frac{1}{2}m\dot{x}^2$$

$$\delta^2 I[q] = \int_{t_1}^{t_2} \left(\frac{\partial^2 L}{\partial \dot{x}^2}\right)(\delta\dot{x})^2 \, \mathrm{d}t = \int_{t_1}^{t_2} m(\delta\dot{x})^2 \mathrm{d}t > 0$$

极小值

- 考虑一维空间非自由粒子 $$L = \frac{1}{2}m\dot{x}^2 - V(x,t)$$

$$\delta^2 I[q] = \int_{t_1}^{t_2} \left[m(\delta\dot{x})^2 - \left(\frac{\partial^2 V}{\partial x^2}\right)(\delta x)^2 \right] \mathrm{d}t$$

$V'' < 0$ 极小值

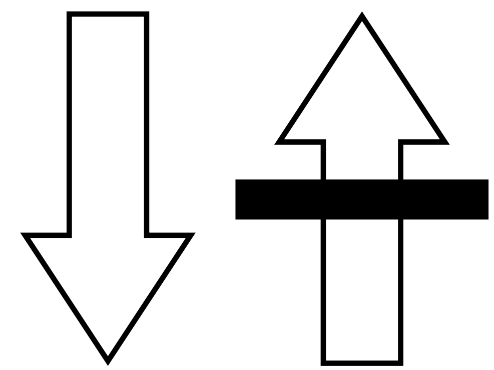
$V'' > 0$ 鞍点

不可能是极大值!

非完整体系

完整约束： $f_\alpha(q_1, \ldots, q_n, t) = 0$ 积分形式

$$\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0$$ 微分形式



我们将处理特定的非完整约束体系，即约束方程形式可写为

$$\sum_k a_{\alpha k} dq_k + a_{\alpha t} dt = 0$$

Pfaffian 形式

$$\sum_k a_{\alpha k} \delta q_k = 0$$

拉格朗日乘子法

同样适用于完整约束情形，如不便将所有 q 化为独立坐标的情形

拉格朗日乘子法

$$\delta I = \delta \int_{t_1}^{t_2} \left[L(q_j, \dot{q}_j, t) - \sum_{\alpha=1}^{m} \lambda_\alpha(t) f_\alpha(q_j, \dot{q}_j, t) \right] dt = 0$$

第一项 $$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

第二项 $$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \sum_{\alpha=1}^{m} \left(\lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} - \lambda_\alpha \frac{d}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_i} - \dot{\lambda}_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_i} \right) \delta q_i$$

选择 λ_α $\Rightarrow$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} + \sum_{\alpha=1}^{m} \left(\lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} - \lambda_\alpha \frac{d}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_i} - \dot{\lambda}_\alpha \frac{\partial f_\alpha}{\partial \dot{q}_i} \right) = 0$$

$i = 1, \ldots, n-m$ $\qquad$ $i = n-m+1, \ldots, n$

拉格朗日乘子法

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}+\sum_{\alpha=1}^{m}\left(\lambda_\alpha\frac{\partial f_\alpha}{\partial q_i}-\lambda_\alpha\frac{d}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_i}-\dot{\lambda}_\alpha\frac{\partial f_\alpha}{\partial \dot{q}_i}\right)=0$$

Pfaffian 形式

$$f_\alpha(q,\dot{q},t)=\sum_k a_{\alpha k}\dot{q}_k+a_{\alpha t}=0$$

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}+\sum_{\alpha=1}^{m}\lambda_\alpha\boxed{\left(\frac{\partial f_\alpha}{\partial q_i}-\frac{d}{dt}\frac{\partial f_\alpha}{\partial \dot{q}_i}\right)}-\sum_{\alpha=1}^{m}\dot{\lambda}_\alpha a_{\alpha i}=0$$

可积完整约束时为零！（拉格朗日量的规范不变性）

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=\sum_{\alpha=1}^{m}\dot{\lambda}_\alpha a_{\alpha i}=Q_i$$

$\dot{\lambda}$ 等价于λ

Q 的物理意义是广义力，对应相应的约束方向不确定，因为λ 的符号可以任意

拉格朗日乘子法

Pfaffian 形式

$$f_\alpha(q,\dot q,t)=\sum_k a_{\alpha k}\dot q_k+a_{\alpha t}=0$$

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}+\sum_{\alpha=1}^{m}\lambda_\alpha\underbrace{\boxed{\left(\frac{\partial f_\alpha}{\partial q_i}-\frac{d}{dt}\frac{\partial f_\alpha}{\partial \dot q_i}\right)}}_{G_{\alpha j}}-\sum_{\alpha=1}^{m}\dot\lambda_\alpha a_{\alpha i}=0$$

不可积非完整约束时不为零！

运动方程与推广的拉格朗日方程不一致！

哈密顿原理可推广至完整约束情形，但不适用于非完整（不可积）约束：

变分路径不一定满足约束！

$$f_\alpha(\boldsymbol{q}^\star+\delta\boldsymbol{q},\dot{\boldsymbol{q}}^\star+\delta\dot{\boldsymbol{q}},t)=\sum_{j=1}^{n}\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(a_{\alpha j}\delta q_j\right)-G_{\alpha j}\delta q_j\right]\neq 0$$

$$\frac{\partial f}{\partial q}\delta q+\frac{\partial f}{\partial \dot q}\delta\dot q$$

非完整约束

$$\delta I := \int_{t_1}^{t_2} \left(\delta L(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t) + \sum_{\alpha=1}^{m} \lambda_\alpha \sum_{i=1}^{n} a_{\alpha i} \delta q_i \right) dt = 0$$

American Journal of Physics 34, 1202 (1966)

第一项 $$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i$$

第二项 $$\int_{t_1}^{t_2} dt \sum_{i=1}^{n} \sum_{\alpha=1}^{m} \lambda_\alpha a_{\alpha i} \delta q_i$$

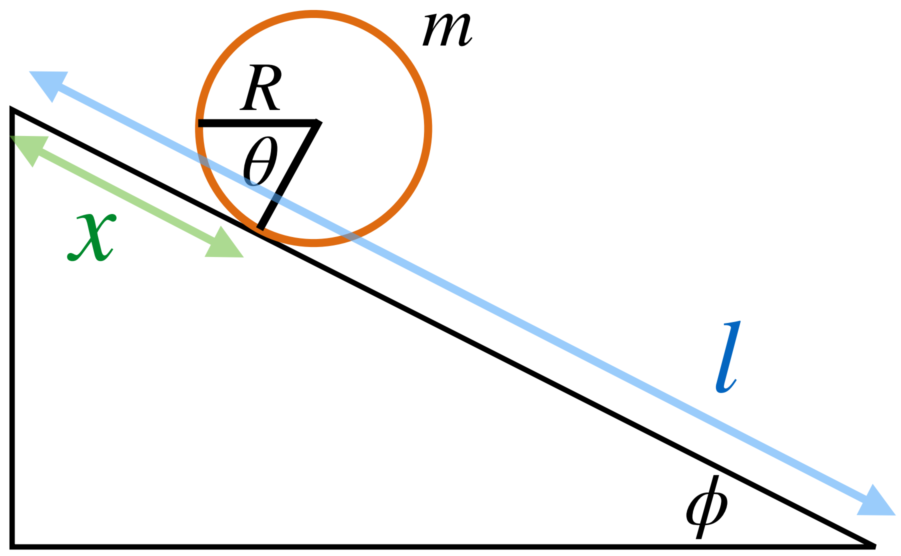
选择 λ_α $\Rightarrow$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} + \sum_{\alpha=1}^{m} \lambda_\alpha a_{\alpha i} = 0$$

$i = n - m + 1, \ldots, n$

$i = 1, \ldots, n - m$

一个示例



一个无滑滚下斜面的铁环：

动能： $T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} mR^2\dot{\theta}^2$

势能： $V = mg(l - x)\sin\phi$

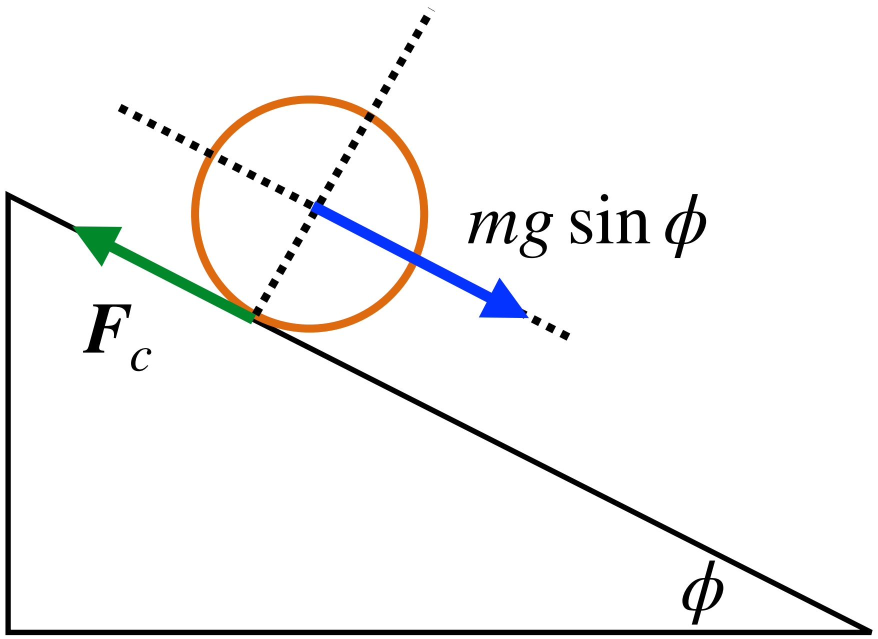
约束： $R\dot{\theta} = \dot{x}$ → 这是一个完全约束，可约化一个变量

$T = m\dot{x}^2$

$$L = T - V = m\dot{x}^2 - mg(l - x)\sin\phi$$

→ $2m\ddot{x} - mg\sin\phi = 0$ → $\ddot{x} = \dfrac{g\sin\phi}{2}$

一个示例



一个无滑滚下斜面的铁环：

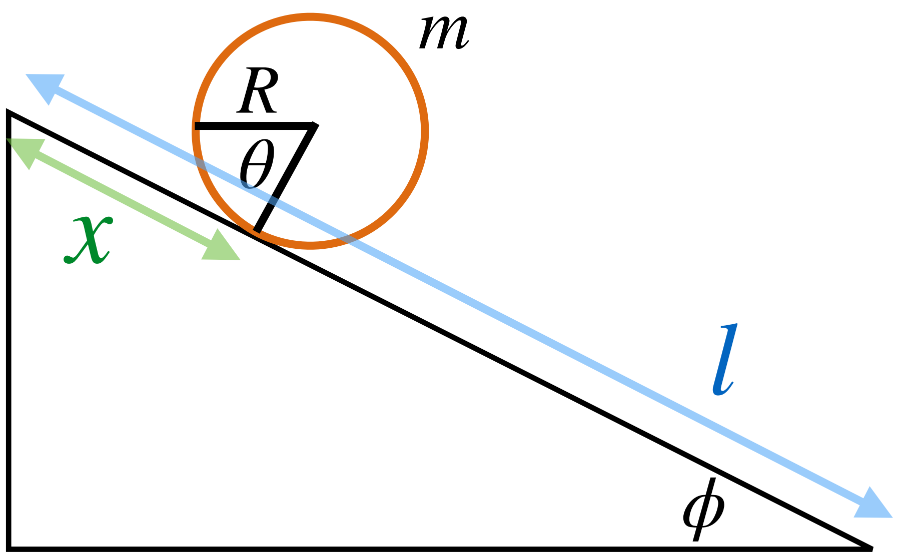
试回到牛顿力学，要保证铁环无滑滚动，有静摩擦力$\boldsymbol{F}_c$，即约束力。

$$mg\sin\phi - \boldsymbol{F}_c = m\ddot{x}$$

$$\Rightarrow \quad \boldsymbol{F}_c = mg\sin\phi - \frac{mg\sin\phi}{2} = \frac{mg\sin\phi}{2}$$

一个示例

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = -\sum_{\alpha=1}^{m} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_i} = Q_i$$



一个无滑滚下斜面的铁环：

现在我们不首先利用约束方程约化变量，而是保持两个变量 x 和 θ，利用拉格朗日乘子法

动能：$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mR^2\dot{\theta}^2$

势能：$V = mg(l-x)\sin\phi$

完全约束：$R\dot{\theta} - \dot{x} = 0$　　$f(\theta, x) = R\theta - x = 0$

$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}mR^2\dot{\theta}^2 - mg(l-x)\sin\phi$$

一个约束方程，故只引入一个λ

$\Rightarrow$ $m\ddot{x} - mg\sin\phi + \lambda = 0$

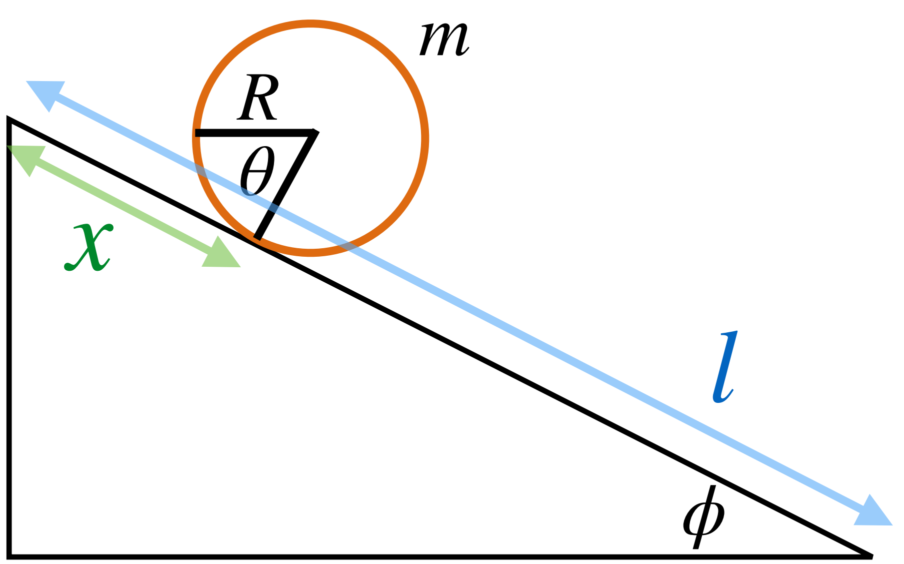
$\Rightarrow$ $mR^2\ddot{\theta} - \lambda R = 0$

$\Rightarrow$ $m\ddot{x} - mg\sin\phi = -mR\ddot{\theta}$

一个示例

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}=-\sum_{\alpha=1}^{m}\lambda_\alpha\frac{\partial f_\alpha}{\partial q_i}=Q_i$$

一个无滑滚下斜面的铁环：



$$m\ddot{x}-mg\sin\phi=-mR\ddot{\theta}$$

$$f(\theta,x)=R\theta-x=0 \Rightarrow R\ddot{\theta}-\ddot{x}=0$$

$$\Rightarrow 2m\ddot{x}-mg\sin\phi=0 \Rightarrow \ddot{x}=\frac{g\sin\phi}{2}$$

现在，我们进一步求得 λ

约束力！

$$\lambda=mg\sin\phi-m\ddot{x}=\frac{mg\sin\phi}{2}$$

$$|Q_x|=\left|\lambda\frac{\partial f}{\partial x}\right|=\frac{mg\sin\phi}{2}$$

$$|\boldsymbol{F}_c|=\frac{mg\sin\phi}{2}$$

总结

- 达朗伯原理
- 哈密顿原理
- 非完整约束
- 拉格朗日乘子法
- 接下来：受限三体问题