理论力学

赵鹏巍

四维力

- 相对论运动学 —〉相对论动力学
- 要保证牛顿力学在静止系下成立 $\boldsymbol{F} = \dot{\boldsymbol{p}} = \frac{d\boldsymbol{p}}{dt}$
- 简单类比可知：

  我们需要将动量转换为四动量

  我们需要将时间转换为“原时” 考虑“时间膨胀”
- 自然推广，可知 $\frac{dp^\mu}{d\tau} = K^\mu$

  K^μ 必须是一个四矢量，**四维力**

  τ 是原时 $dt = \gamma d\tau$

如何构造四维力？以电磁力为例！

电磁力

- 大家已经学过 Maxwell 方程，

  不满足伽利略不变性，满足洛伦兹不变性，所以电磁相互作用应该是相对论的。

- 非相对论下，带电粒子感受到的电磁力（洛伦兹力）$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

  可从一个广义势中推导得来

  $$U = q(\phi - \boldsymbol{A} \cdot \boldsymbol{v})$$

  $$Q_j = -\frac{\partial U}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q}_j}\right)$$

- 相对论情况下，

  已知四速度：$(u^0, \boldsymbol{u}) = (\gamma c, \gamma \boldsymbol{v})$

  定义**四维势**：$A = (A^0, \boldsymbol{A}) = (\phi/c, \boldsymbol{A})$

  于是，有标量积

  $$A^\mu u_\mu = \gamma\phi - \boldsymbol{A} \cdot \gamma \boldsymbol{v} = \gamma(\phi - \boldsymbol{A} \cdot \boldsymbol{v}) \Rightarrow \gamma U = q A^\mu u_\mu$$

  相对论情况下的广义势

电磁力

- $A = (A^0, \boldsymbol{A}) = (\phi/c, \boldsymbol{A})$

  四维势 A 包含关于电磁场的所有信息，因此正是协变理论所要确定的物理量。

$$U = q(\phi - \boldsymbol{A} \cdot \boldsymbol{v})$$

$$\gamma U = q A^\mu u_\mu$$

- 三维力 $F^i = \frac{\partial U}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{x}_i}\right)$

- 推广至四维力

$$K^\mu = \frac{\partial(qA^\nu u_\nu)}{\partial x_\mu} - \frac{d}{d\tau}\left(\frac{\partial(qA^\nu u_\nu)}{\partial u_\mu}\right) = q\left(\frac{\partial A^\nu}{\partial x_\mu}u_\nu - \frac{dA^\mu}{d\tau}\right)$$

注意时空度规！

带电粒子在电磁场中的四维力！

电磁力

$$K^\mu = q\left(\frac{\partial A^\nu}{\partial x_\mu}u_\nu - \frac{dA^\mu}{d\tau}\right) = q\left(\frac{\partial A^0}{\partial x_\mu}u_0 + \frac{\partial A^i}{\partial x_\mu}u_i - \frac{\partial A^\mu}{\partial x_0}u_0 - \frac{\partial A^\mu}{\partial x_i}u_i\right)$$

电场 $\boldsymbol{E}$

$$A = (A^0, \boldsymbol{A}) = (\phi/c, \boldsymbol{A})$$

$$E^i = -\nabla\phi - \frac{\partial A^i}{\partial t} = c\left(\frac{\partial A^0}{\partial x_i} - \frac{\partial A^i}{\partial x_0}\right)$$

$$(\boldsymbol{v}\times\boldsymbol{B})^i = (\boldsymbol{v}\times(\nabla\times\boldsymbol{A}))^i = \left(\frac{\partial A^j}{\partial x_i} - \frac{\partial A^i}{\partial x_j}\right)\frac{u_j}{\gamma}$$

注意爱因斯坦求和！

$$K^0 = \frac{\gamma}{c}qv^iE^i$$

$$K^i = \gamma q\left[E^i + (\boldsymbol{v}\times\boldsymbol{B})^i\right]$$

注意时空度规！

Maxwell 方程

$$\nabla\cdot\boldsymbol{B} = 0$$

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}$$

旋度无源

$$\boldsymbol{B} = \nabla\times\boldsymbol{A}$$

$$\nabla\times(\boldsymbol{E} + \frac{\partial\boldsymbol{A}}{\partial t}) = 0$$

梯度无旋

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial\boldsymbol{A}}{\partial t}$$

电磁力

- 空间部分 $$\frac{dp^i}{d\tau} = K^i = \gamma q\left[E^i + (\boldsymbol{v}\times\boldsymbol{B})^i\right]$$

与非相对论一致！ $$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F} = q\left[\boldsymbol{E} + (\boldsymbol{v}\times\boldsymbol{B})\right]$$

- 时间部分 $$\frac{dp^0}{d\tau} = K^0 = \frac{\gamma}{c}\boxed{qv^iE^i}$$ → ***F* • *v*** **电磁力所作功率**

$$\Rightarrow \frac{dp^0}{dt} = \frac{W}{c} \xrightarrow{\text{积分}} p^0 = \frac{E}{c}$$

四动量的时间分量对应能量！

电磁场张量

$$K^{\mu} = q\left(\frac{\partial A^{\nu}}{\partial x_{\mu}}u_{\nu} - \frac{dA^{\mu}}{d\tau}\right) = q\left(\frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}}\right)u_{\nu} \equiv qF^{\mu\nu}u_{\nu}$$

电磁场张量

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

四维电磁力也可以用一个二阶电磁场张量与四维速度乘积的形式表示~

协变的拉格朗日表述

- 哈密顿原理应该具有明显的协变性

  1. 作用量积分必须是一个洛伦兹标量
  2. 时空坐标对等处理，不应该仅对时间 t 积分

     应该找一个洛伦兹不变量来描述系统在四维时空的演变，取代时间 t

     原时 τ 是一个自然的选择?
  3. 拉格朗日量必须是一个洛伦兹标量

$$q' = \frac{dq}{d\tau} \qquad t' = \frac{dt}{d\tau}$$

$$\delta \int_{t_a}^{t_b} L(q^j, \dot{q}^j, t)dt = 0 \quad \Rightarrow \quad \delta \int_{\tau_a}^{\tau_b} L_1(q^j, (q^j)', t, t')d\tau = 0$$

- 拉格朗日方程形式

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L_1}{\partial \boldsymbol{q}'}\right) - \frac{\partial L_1}{\partial \boldsymbol{q}} = 0 \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L_1}{\partial t'}\right) - \frac{\partial L_1}{\partial t} = 0 \qquad \frac{d}{d\tau}\frac{\partial L_1}{\partial (q^\mu)'} - \frac{\partial L_1}{\partial q^\mu} = 0$$

自由度变多了吗?

协变的拉格朗日表述

- 哈密顿原理应该始终保持

$$\delta\int_{t_a}^{t_b} L(q^j,\dot q^j,t)dt = \delta\int_{\tau_a}^{\tau_b} L_1(q^j,(q^j)',t,t')d\tau = 0$$

$$L_1 = Lt'$$

仅差一个忽略的规范

- 易知，

$$\boldsymbol{q}' = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau} \qquad t' = \frac{\mathrm{d}t}{\mathrm{d}\tau} \qquad \dot{\boldsymbol{q}} = \frac{\boldsymbol{q}'}{t'}$$

$$\frac{\partial L_1}{\partial (q^\mu)'}(q^\mu)' = \left[L - \frac{\partial L}{\partial \dot q^i}\dot q^i\right]t' + \frac{\partial L}{\partial \dot q^i}(q^i)' = L_1 \Rightarrow L_1 - \frac{\partial L_1}{\partial (q^\mu)'}(q^\mu)' = 0$$

约束条件!

$$\frac{\partial L_1}{\partial t'} = \frac{\partial (Lt')}{\partial t'} = L + t'\frac{\partial L}{\partial t'} = L + t'\sum_j \frac{\partial L}{\partial \dot q_j}\frac{\partial \dot q_j}{\partial t'} = L + t'\sum_j \frac{\partial L}{\partial \dot q_j}\left(-\frac{1}{t'}\dot q_j\right) = L - \sum_j \frac{\partial L}{\partial \dot q_j}\dot q_j$$

$$\frac{\partial L_1}{\partial q_j'} = \frac{\partial (Lt')}{\partial q_j'} = t'\frac{\partial L}{\partial q_j'} = t'\sum_k \frac{\partial L}{\partial \dot q_k}\frac{\partial \dot q_k}{\partial q_j'} = t'\sum_k \frac{\partial L}{\partial \dot q_k}\frac{\partial}{\partial q_j'}\left(\frac{q_k'}{t'}\right) = \sum_k \frac{\partial L}{\partial \dot q_k}\delta_{jk} = \frac{\partial L}{\partial \dot q_j}$$

一个相对论自由粒子

- 拉格朗日量

$$L_1 = -\frac{1}{2}m_0 u_\mu u^\mu - \frac{1}{2}m_0 c^2$$

$$L_1 = \frac{1}{2}m_0 c^2\left[\frac{1}{c^2}\sum_{i=1}^{3}\left(\frac{dq^i}{d\tau}\right)^2 - \left(\frac{dt}{d\tau}\right)^2 - 1\right]$$

$$\xrightarrow{\frac{dt}{d\tau}\to 1} \quad L_{\text{nr}} = \frac{1}{2}m_0\sum_{i=1}^{3}(\dot{q}^i)^2 - m_0 c^2$$

- 约束条件

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\sum_{i=1}^{3}\left(\frac{dq^i}{d\tau}\right)^2 - 1 = 0$$

$$L_1 - \frac{\partial L_1}{\partial (q^\mu)'}(q^\mu)' = 0$$

$$\xrightarrow{\dot{\boldsymbol{q}} = \frac{\boldsymbol{q}'}{t'}} \quad \frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\sum_{i=1}^{3}\left(\frac{dq^i}{dt}\right)^2} = \sqrt{1-\beta^2}$$

$$d\tau = \sqrt{1-\beta^2}\,dt$$

正是原时的定义!

- 传统的拉格朗日量

$$L = L_1\frac{d\tau}{dt} = -m_0 c^2\sqrt{1-\beta^2}$$

练习

时空位形空间+约束

—〉 位形空间

电磁场中的带电粒子

- 拉格朗日量

$$L(x^\mu, u^\mu) = \frac{1}{2} m u_\mu u^\mu + q u^\mu A_\mu$$

$$L_1 = L_{\text{free}} + q \sum_{i=1}^{3} A^i \frac{dq^i}{d\tau} - q\phi \underline{\frac{dt}{d\tau}}$$

$$\frac{dt}{d\tau} \to 1$$

$$L_{\text{nr}} = L_{\text{nr}}^{\text{free}} - q\phi + q\boldsymbol{A} \cdot \boldsymbol{v}$$

注意这一项在速度不依赖的势中也出现！

- 约束条件

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2} \sum_{i=1}^{3} \left(\frac{dq^i}{d\tau}\right)^2 - 1 = 0$$

与自由粒子一致！

$$L_1 - \frac{\partial L_1}{\partial (q^\mu)'} (q^\mu)' = 0$$

$$\dot{\boldsymbol{q}} = \frac{\boldsymbol{q}'}{t'}$$

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2} \sum_{i=1}^{3} \left(\frac{dq^i}{dt}\right)^2} = \sqrt{1 - \beta^2}$$

$$d\tau = \sqrt{1 - \beta^2}\, dt$$

正是原时的定义！

- 传统的拉格朗日量

$$L = L_1 \frac{d\tau}{dt} = -m_0 c^2 \sqrt{1 - \beta^2} - q\phi + q\boldsymbol{A} \cdot \boldsymbol{v}$$

练习

时空位形空间+约束

—〉位形空间

协变的哈密顿表述

- 勒让德变换

时间 t 的共轭动量

$$H_1(q^j, p_j, t, E) = p_i \frac{dq^i}{d\tau} - E\frac{dt}{d\tau} - L_1(q^j, (q^j)', t, t')$$

$$H = p\dot{q} - L$$

$$\frac{\partial L_1}{\partial q_j'} = \frac{\partial L}{\partial \dot{q}_j} \qquad \frac{\partial L_1}{\partial t'} = L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j$$

- 共轭动量

$$p_i = \frac{\partial L_1}{\partial (q^i)'} = \frac{\partial L}{\partial \dot{q}^i} \qquad p_0(\tau) = \frac{\partial L_1}{\partial t'} = L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = -H(q_j(\tau), p_j(\tau), t(\tau)) = -E(\tau)$$

- 约束条件

$$L_1 - \frac{\partial L_1}{\partial (q^\mu)'}(q^\mu)' = 0 \quad \Longrightarrow \quad H_1(q^j(\tau), p_j(\tau), t(\tau), E(\tau)) = 0$$

- 正则方程

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H_1}{\partial q^\mu} \qquad \frac{dq^\mu}{d\tau} = \frac{\partial H_1}{\partial p_\mu} \qquad \frac{d}{d\tau}\frac{\partial L_1}{\partial (q^\mu)'} - \frac{\partial L_1}{\partial q^\mu} = 0$$

协变的哈密顿表述

- 哈密顿量

$$H_1(q^j, p_j, t, E) = p_i \frac{dq^i}{d\tau} - E\frac{dt}{d\tau} - L_1$$

$$\frac{dH_1}{d\tau} = \left[\frac{\partial H_1}{\partial q^\mu}\frac{dq^\mu}{d\tau} + \frac{\partial H_1}{\partial p_\mu}\frac{dp_\mu}{d\tau}\right] = \left[\frac{\partial H_1}{\partial q^\mu}\frac{\partial H_1}{\partial p_\mu} - \frac{\partial H_1}{\partial p_\mu}\frac{\partial H_1}{\partial q^\mu}\right] = 0$$

H_1 为常数

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H_1}{\partial q^\mu}$$

$$H_1 = pq' - Et' - Lt' = (H - E)t'$$

$$L_1 = Lt'$$

$$\dot{\boldsymbol{q}} = \frac{\boldsymbol{q}'}{t'}$$

$$\frac{dq^\mu}{d\tau} = \frac{\partial H_1}{\partial p_\mu}$$

$$t' = \frac{\mathrm{d}t}{\mathrm{d}\tau}$$

$$\boldsymbol{q}' = \frac{\mathrm{d}\boldsymbol{q}}{\mathrm{d}\tau}$$

- 考虑一特殊情况，

$$H_1(q, p, t, E) = H(q, p, t) - E$$

相应的正则方程

$$\frac{dt}{d\tau} = \frac{\partial H_1}{\partial(-E)} = 1$$

$$\delta t = \delta\tau + \mathrm{Const}$$

时间平移

一个相对论自由粒子的哈密顿量

- 勒让德变换

$$H_1(q^j, p_j, t, E) = p_i \frac{dq^i}{d\tau} - E\frac{dt}{d\tau} - L_1(q^j, (q^j)', t, t')$$

$$L_1 = \frac{1}{2} m_0 c^2 \left[\frac{1}{c^2} \sum_{i=1}^{3} \left(\frac{dq^i}{d\tau} \right)^2 - \left(\frac{dt}{d\tau} \right)^2 - 1 \right]$$

- 由拉氏量可知

$$p_i = \frac{\partial L_1}{\partial (q^i)'} = m_0 (q^i)'$$

$$E = -\frac{\partial L_1}{\partial t'} = m_0 c^2 t'$$

$$H_1(\boldsymbol{p}, E) = \frac{\boldsymbol{p}^2}{m_0} - \frac{E^2}{m_0 c^2} - L_1 = \frac{\boldsymbol{p}^2}{2m_0} - \frac{E^2}{2m_0 c^2} + \frac{1}{2} m_0 c^2$$

约束条件就是色散关系

$$E^2 = \boldsymbol{p}^2 c^2 + m_0^2 c^4$$

- 正则方程

$$\frac{dp_\mu}{d\tau} = -\frac{\partial H_1}{\partial q^\mu} = 0 \qquad \frac{dq^i}{d\tau} = \frac{\partial H_1}{\partial p_i} = \frac{p_i}{m_0} \qquad \frac{dt}{d\tau} = \frac{\partial H_1}{\partial (-E)} = \frac{E}{m_0 c^2}$$

一个相对论自由粒子的哈密顿量

- 正则方程

$$\boxed{\frac{dp_\mu}{d\tau} = -\frac{\partial H_1}{\partial q^\mu} = 0} \qquad \frac{dq^i}{d\tau} = \frac{\partial H_1}{\partial p_i} = \frac{p_i}{m_0} \qquad \boxed{\frac{dt}{d\tau} = \frac{\partial H_1}{\partial(-E)} = \frac{E}{m_0 c^2}}$$

能动量守恒

$$\frac{dt}{d\tau} = \frac{E}{m_0 c^2} = \frac{\sqrt{\boldsymbol{p}^2 c^2 + m_0^2 c^4}}{m_0 c^2}$$

色散关系

$$\frac{dq^i}{dt} = \frac{dq^i}{d\tau}\frac{d\tau}{dt} = \frac{p_i}{m_0}\frac{m_0 c^2}{E} = \frac{p_i c^2}{\sqrt{\boldsymbol{p}^2 c^2 + m_0^2 c^4}} = \frac{\partial H_R}{\partial p_i}$$

- 回到传统的哈密顿量

$$H_R = e = \sqrt{\boldsymbol{p}^2 c^2 + m_0^2 c^4}$$

$$H_{\rm nr}(\boldsymbol{p}, E) = \frac{\boldsymbol{p}^2}{2m_0}$$

物理上与 H_1 等价，虽然没有显式的Lorentz对称性

相对论谐振子

- 考虑一个一维谐振子，哈密顿量如何写呢?

$$H_1 = H_1^{\text{free}} + V$$ ✘ 需要正则动量!

$$H_1^{\text{free}}(\boldsymbol{p}, E) = \frac{\boldsymbol{p}^2}{2m_0} - \frac{E^2}{2m_0c^2} + \frac{1}{2}m_0c^2$$

- 易从勒让德变换中看到这一点

$$L_1 = L_{\text{free}} - V(q,t)\frac{dt}{d\tau}$$

$$p_i = \frac{\partial L_1}{\partial (q^i)'} = m_0 (q^i)'$$

$$E = -\frac{\partial L_1}{\partial t'} = m_0c^2 t' + V(q,t)$$

- 故一维相对论谐振子的哈密顿量应为

$$H_1(p,E) = \frac{1}{2m_0}\left[p^2 - \left(\frac{E - \frac{1}{2}kx^2}{c}\right)^2\right] + \frac{1}{2}m_0c^2$$

类似电磁场中带电粒子的哈密顿量

机械动量不是正则动量!

- 相应的约束条件

$$p^2c^2 - \left(E - \frac{1}{2}kx^2\right)^2 + m_0^2c^4 = 0$$

→

传统相空间的相对论哈密顿量

$$E = H_R = \sqrt{p^2c^2 + m_0^2c^4} + \frac{1}{2}kx^2$$

相对论谐振子

$$E = H_R = \sqrt{p^2c^2 + m_0^2c^4} + \frac{1}{2}kx^2$$

- 正则方程

$$\dot{x} = \frac{\partial H_R}{\partial p} = \frac{pc^2}{\sqrt{p^2c^2 + m_0^2c^4}}$$

$$\dot{p} = -\frac{\partial H_R}{\partial q} = -kx$$

- 可得运动方程

$$\ddot{x} + \frac{k}{m_0}\left(1 - \frac{\dot{x}^2}{c^2}\right)^{\frac{3}{2}} x = 0 \quad \Rightarrow \quad \ddot{x} + \frac{k}{m_0\gamma^3}x = 0$$

$\gamma \to 1$ 时回到非相对论情况

- 若利用哈密顿量 H_1，可得到和 H_R 完全等价的动力学方程。此外，还可得

$$\frac{dt}{d\tau} = -\frac{\partial H_R}{\partial E} = \frac{E - \frac{1}{2}kx^2}{m_0c^2}$$

$$H_1(p, E) = \frac{1}{2m_0}\left[p^2 - \left(\frac{E - \frac{1}{2}kx^2}{c}\right)^2\right] + \frac{1}{2}m_0c^2$$

电磁场中的相对论粒子

- 勒让德变换

$$H_1(q^j, p_j, t, E) = p_i \frac{dq^i}{d\tau} - E\frac{dt}{d\tau} - L_1(q^j, (q^j)', t, t')$$

$$L_1 = L_{\text{free}} + q\sum_{i=1}^{3} A^i \frac{dq^i}{d\tau} - q\phi\frac{dt}{d\tau}$$

- 由拉氏量可知

$$p_i = \frac{\partial L_1}{\partial (q^i)'} = m_0 (q^i)' + qA^i(\boldsymbol{q}, t)$$

$$E = -\frac{\partial L_1}{\partial t'} = m_0 c^2 t' + q\phi(\boldsymbol{q}, t)$$

$$H_1(\boldsymbol{q}, \boldsymbol{p}, t, E) = \frac{1}{2m_0}\left[\left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 - \left(\frac{E - q\phi(\boldsymbol{q}, t)}{c}\right)^2\right] + \frac{1}{2}m_0 c^2$$

- 相应的约束条件 $\left(E - q\phi(\boldsymbol{q}, t)\right)^2 = \left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 c^2 + m_0^2 c^4$

$$E = H_R = \sqrt{\left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 c^2 + m_0^2 c^4} + q\phi(\boldsymbol{q}, t)$$

传统相空间的相对论哈密顿量

$$H_{nr} = \frac{1}{2m_0}\left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 + q\phi(\boldsymbol{q}, t)$$

电磁场中的相对论粒子

$$A = (A^0, \boldsymbol{A}) = (\phi/c, \boldsymbol{A})$$

- 我们已知$(\phi, \boldsymbol{A}c)$是闵氏空间的四矢量

  观察约束条件，不难发现，$(E - q\phi, \boldsymbol{p}c - q\boldsymbol{A}c)$ 也是闵氏空间的四矢量

  故 $(E, \boldsymbol{p}c)$ 亦为四矢量。

$$\left(E - q\phi(\boldsymbol{q}, t)\right)^2 = \left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 c^2 + m_0^2 c^4$$

注意，这里正则动量并非普通的机械动量、线动量

- 所以，我们得到：**能量 + 正则动量 = 某个洛伦兹系内的四矢量**

- 考察相对论哈密顿量 H_1，不难发现，其也可写为一个协变形式

$$H_1 = \frac{1}{2m_0}(p_\mu - qA_\mu)(p^\mu - qA^\mu) + \frac{1}{2}m_0c^2$$

$$H_1(\boldsymbol{q}, \boldsymbol{p}, t, E) = \frac{1}{2m_0}\left[\left(\boldsymbol{p} - q\boldsymbol{A}(\boldsymbol{q}, t)\right)^2 - \left(\frac{E - q\phi(\boldsymbol{q}, t)}{c}\right)^2\right] + \frac{1}{2}m_0c^2$$

电磁场中的一个带电粒子

- 哈密顿量

$$H = \frac{1}{2} m u_\mu u^\mu = \frac{(p_\mu - qA_\mu)(p^\mu - qA^\mu)}{2m}$$

- 哈密顿正则方程

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p_\mu} = \frac{p^\mu - qA^\mu}{m}$$

$$\frac{dp^\mu}{d\tau} = -\frac{\partial H}{\partial x_\mu} = q\frac{p_\nu - qA_\nu}{m}\frac{\partial A^\nu}{\partial x_\mu}$$

通过一些运算，可得

$$m\frac{du^\mu}{d\tau} = q\left(\frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}\right)u_\nu = K^\mu$$

这正是前面导出的四维电磁力!

电磁场与哈密顿量

- 在哈密顿力学表述中，电磁场的出现总是将正则动量改变为

$$\underbrace{p_A^\mu}_{\text{含电磁场！}} = \underbrace{p_0^\mu}_{\text{不含电磁场！}} + qA^\mu$$

- 这是一个非常有用的技巧：
  要给出含电磁场体系的哈密顿量，只需将不含电磁场的自由粒子哈密顿量中 p^μ 替换为 $p^\mu - qA^\mu$

- 这一技巧在量子力学中经常使用！

协变拉格朗日表述的局限性

- 我们目前仅了解四维的电磁力

  所以，许多其他问题还不能利用协变的拉格朗日方程来表述

- 推广至多粒子系统

$$\delta I = \delta \int L d\tau \qquad \frac{\partial L}{\partial x^\mu} - \frac{d}{d\tau}\frac{\partial L}{\partial u^\mu} = 0$$

这里应该使用哪个粒子的原时?

- 在广义坐标变换下，拉格朗日方程形式是不变的，而这里的每一个广义坐标甚至可能根本不对应一个单粒子 —〉“什么的原时？ ”

- 粒子间的直接相互作用在非相对论下很平常，而在相对论下，不可能构建协变的、非接触的直接相互作用，因为相对论不允许超距作用，“光速最大””

- 这些问题需要我们放弃“粒子”的图像，而引入“场”的概念。

总结

- 四维力：电磁力
- 扩展的位形空间：协变的拉格朗日表述
- 扩展的相空间：协变的哈密顿表述
- 局限性：构建很有限的几个体系，如电磁场中的单粒子
- —〉场论