任宇桐 2400011498 理论力学 作业一

1. 作纯滚动时的约束方程：

$$\begin{cases} \dot{x} = a\dot{\varphi}\sin\theta \\ \dot{y} = -a\dot{\varphi}\cos\theta \end{cases} \quad 即： \begin{cases} dx = a\sin\theta\, d\varphi \\ dy = -a\cos\theta\, d\varphi \end{cases}$$

① "纯滚动"不是完整约束，θ 不是定值，微分约束不可积

② "沿直线纯滚动"是完整约束，积分后为：

$$\begin{cases} x = a\varphi\sin\theta + C_1 \\ y = -a\varphi\cos\theta + C_2 \end{cases}$$

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) + \frac{1}{2}\left(\frac{1}{2}ma^2\dot{\varphi}^2 + \frac{1}{4}ma^2\dot{\theta}^2\right) \qquad V = mga$$

$$L = T - V \qquad f_1 = \dot{x} - a\dot{\varphi}\sin\theta = 0 \qquad f_2 = \dot{y} + a\dot{\varphi}\cos\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \lambda_1 \frac{\partial f_1}{\partial \dot{x}}$$

$$\Rightarrow \quad m\ddot{x} = \lambda_1$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} = \lambda_2 \frac{\partial f_2}{\partial \dot{y}}$$

$$\Rightarrow \quad m\ddot{y} = \lambda_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \quad \frac{1}{4}ma^2\ddot{\theta} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \lambda_1 \frac{\partial f_1}{\partial \dot{\varphi}} + \lambda_2 \frac{\partial f_2}{\partial \dot{\varphi}}$$

$$\Rightarrow \quad \frac{1}{2}ma^2\ddot{\varphi} = -\lambda_1 a\sin\theta + \lambda_2 a\cos\theta$$

运动方程：

$$\begin{cases} \ddot{\theta}=0 \\ \frac{1}{2}ma^2\ddot{\varphi}=-ma\sin\theta\,\ddot{x}+ma\cos\theta\,\ddot{y} \end{cases}$$

约束：$\begin{cases} \dot{x}=a\dot{\varphi}\sin\theta \\ \dot{y}=-a\dot{\varphi}\cos\theta \end{cases}$

联立得：$\frac{1}{2}ma^2\ddot{\varphi}+ma^2\ddot{\varphi}=0 \quad \Rightarrow \dot{\varphi}=\dot{\varphi}(t=0)$

$\dot{\theta}=0 \quad \Rightarrow \dot{\theta}=\dot{\theta}(t=0)$

$$\Rightarrow \begin{cases} \dot{x}=a\dot{\varphi}(t=0)\cdot\sin(\theta(t=0)+\dot{\theta}(t=0)t) \\ \dot{y}=-a\dot{\varphi}(t=0)\cdot\cos(\theta(t=0)+\dot{\theta}(t=0)t) \end{cases}$$

质心作速度大小为 $a\dot{\varphi}(t=0)$、角速度为 $\dot{\theta}(t=0)$ 的匀速圆周运动

2. 取质点 m_1 的径向位移与角向位移 r,θ 为广义坐标：

$T=\frac{1}{2}m_1(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}m_2\dot{r}^2$

$V=m_2gr$

$L=T-V=\frac{1}{2}m_1(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}m_2\dot{r}^2-m_2gr$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}}-\frac{\partial L}{\partial r}=0 \quad \Rightarrow \quad m_1\ddot{r}+m_2\ddot{r}-m_1r\dot{\theta}^2+m_2g=0$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}-\frac{\partial L}{\partial \theta}=0 \quad \Rightarrow \frac{d}{dt}(m_1r^2\dot{\theta})=2m_1r\dot{\theta}\dot{r}+m_1r^2\ddot{\theta}=0$

初次积分：

$\frac{\partial L}{\partial \theta}=0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}=0 \Rightarrow p_\theta=m_1r^2\dot{\theta}=0$

$\frac{\partial L}{\partial t}=0 \Rightarrow \frac{\partial L}{\partial \dot{r}}\dot{r}+\frac{\partial L}{\partial \dot{\theta}}\dot{\theta}-L=c \Rightarrow \frac{1}{2}m_1(\dot{r}^2+r^2\dot{\theta}^2)+\frac{1}{2}m_2\dot{r}^2+m_2gr=c$

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3.

电磁场中粒子的拉格朗日量:

$$L=\frac{1}{2}m\vec{v}^2-q(\phi-\vec{v}\cdot\vec{A})$$

令 $\vec{A}\Rightarrow\vec{A}+\nabla\psi(\vec{r},t)$, $\phi\Rightarrow\phi-\frac{\partial\psi}{\partial t}$

$$\Rightarrow L'=\frac{1}{2}m\vec{v}^2-q\left(\phi-\frac{\partial\psi}{\partial t}-\vec{v}\cdot\vec{A}-\vec{v}\cdot\nabla\psi\right)$$

$$=\frac{1}{2}m\vec{v}^2-q(\phi-\vec{v}\cdot\vec{A})+q\left(\frac{\partial\psi}{\partial t}+\vec{v}\cdot\nabla\psi\right)$$

$\delta L=L'-L=q\frac{d\psi}{dt}$ 相差全微分项, 对质点运动无影响

4. 球面上两点间距离:

$$S=\int_{(\theta_1,\varphi_1)}^{(\theta_2,\varphi_2)}\sqrt{(d\theta)^2+(\sin\theta d\varphi)^2}\,R=\int_{\varphi_1}^{\varphi_2}R\sqrt{\sin^2\theta+\left(\frac{d\theta}{d\varphi}\right)^2}\,d\varphi$$

由 $L=\sqrt{\sin^2\theta+\left(\frac{d\theta}{d\varphi}\right)^2}$ 不显含 φ $\Rightarrow$ $\frac{\partial L}{\partial(d\theta/d\varphi)}\cdot\frac{d\theta}{d\varphi}-L=C$

$$\Rightarrow\frac{\left(\frac{d\theta}{d\varphi}\right)^2}{\sqrt{\sin^2\theta+(d\theta/d\varphi)^2}}-\sqrt{\sin^2\theta+\left(\frac{d\theta}{d\varphi}\right)^2}=-c'$$

即 $$\frac{\sin^2\theta}{\sqrt{\sin^2\theta+(d\theta/d\varphi)^2}}=c'$$

不妨将起始位置取在 $\theta_1=0$ 处, 此时 $\sin\theta_1=0\Rightarrow c'=0$

$\Rightarrow d\varphi=0\Rightarrow$ 球面上大圆为短程线

或积分求解一般形式和初始条件(坐标):

$$\frac{d\theta}{d\varphi}=\sqrt{c'\sin^4\theta-\sin^2\theta}\Rightarrow\frac{d\theta}{\sqrt{c'\sin^4\theta-\sin^2\theta}}=d\varphi$$

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积分得：

$$\arccos(C\cdot\cot\theta)=\varphi-\varphi_0$$

$$\Rightarrow C\cdot\cot\theta=\cos(\varphi-\varphi_0)$$

即： $C\cos\theta=\sin\theta\cos(\varphi-\varphi_0)$

取 $C=-\dfrac{\cos\theta_0}{\sin\theta_0}$ 则方程化为： $\cos\theta\cos\theta_0+\sin\theta\sin\theta_0\cos(\varphi-\varphi_0)=0$

对应大圆法矢量为： $\vec{n}=(\sin\theta_0\cos\varphi_0,\ \sin\theta_0\sin\varphi_0,\ \cos\theta_0)$

5. $T=\frac{1}{2}m(\omega l\sin\theta)^2\times2+\frac{1}{2}ml^2\dot\theta^2\times2+\frac{1}{2}\cdot M(2l\sin\theta\,\dot\theta)^2$

$$=m\omega^2l^2\sin^2\theta+ml^2\dot\theta^2+2Ml^2\sin^2\theta\,\dot\theta^2$$

$$V=-2mgl\cos\theta-2Mgl\cos\theta$$

$$L=T-V=ml^2(\omega^2\sin^2\theta+\dot\theta^2)+2Ml^2\sin^2\theta\,\dot\theta^2+2(M+m)gl\cos\theta$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta}-\frac{\partial L}{\partial\theta}=0\Rightarrow\quad\frac{d}{dt}(ml^2\dot\theta+2Ml^2\sin^2\theta\,\dot t)=0$$

$$\Rightarrow ml^2\ddot\theta+4Ml^2\sin\theta\cos\theta\,\dot\theta^2+2Ml^2\sin^2\theta\,\ddot\theta=0$$

初积分： $\frac{\partial L}{\partial t}=0$

$$\Rightarrow\frac{\partial L}{\partial\dot\theta}\dot\theta-L=ml^2\dot\theta^2+2Ml^2\sin^2\theta\,\dot\theta^2-2(M+m)gl\cos\theta-m\omega^2l^2\sin^2\theta=c$$

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6. 采用角度 θ 与径向"位移" ρ 为广义坐标.

纯滚约束: $\omega r = (R+r)\dot\theta \quad \Rightarrow f_1 = \omega r - (R+r+\rho)\dot\theta = 0$

脱离约束: $\dot\rho = 0 \quad \Rightarrow f_2 = \dot\rho = 0$

$$L = T - V = \frac{1}{2}m(R+r)^2\dot\theta^2 + \frac{1}{2}mr^2\omega^2 + \frac{1}{2}m\dot\rho^2 - mg(R+r+\rho)\cos\theta$$

$$= \frac{1}{2}m(R+r)^2\dot\theta^2 + \frac{1}{2}m\dot\rho^2 - mg(R+r+\rho)\cos\theta + \frac{1}{2}mr^2\omega^2$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot\rho} - \frac{\partial L}{\partial\rho} = \lambda$$

$$\Rightarrow \lambda = mg\cos\theta - m(R+r+\rho)\dot\theta^2 \Big|_{\rho=0} = mg\cos\theta - m\dot\theta^2(R+r)$$

代入脱离约束 $\dot\rho=0$ 得到初次积分:

$$m(R+r)^2\dot\theta^2 + mg(R+r)\cos\theta = mg(R+r)$$

$$\Rightarrow m(R+r)\dot\theta^2 = mg\cancel{(R+r)}(1-\cos\theta)$$

$$\Rightarrow \lambda = mg\cos\theta - mg(1-\cos\theta) = mg(2\cos\theta - 1)$$

$\lambda = 0 \Rightarrow \cos\theta = 1/2$ 　分离时 $\theta = \arccos 1/2$

以质点相对转角 θ 为广义坐标

7. $L = T - V = \frac{1}{2}ma^2\dot\theta^2 + \frac{1}{2}ma^2\sin^2\theta\,\omega^2 + mga\cos\theta$

由 $\frac{d}{dt}\frac{\partial L}{\partial\dot\theta} - \frac{\partial L}{\partial\theta} = 0 \Rightarrow ma^2\ddot\theta = m\omega^2a^2\sin\theta\cos\theta - mga\sin\theta$.

~~$\theta\to0$ 时: $\ddot\theta = (\omega^2 a - g)$~~ 平衡位置 $\ddot\theta = 0 \Rightarrow \cos\theta = \frac{g}{\omega^2 a}$

或 $\theta=0$ 或 $\theta=\pi$.

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$\omega_0=\sqrt{g/a}$

① $\omega>\omega_0$时 平衡位置 $\theta=0$, $\arccos\frac{g}{\omega^2 a}$, π

验证位置 $\theta_0=\arccos\frac{g}{\omega^2 a}$ 是否稳定

对 $\ddot\theta=\omega^2 a\sin\theta\cos\theta-g\sin\theta$ 研究

令 $\theta=\theta_0+\delta$

$$\ddot\delta=\omega^2 a\sin(\theta_0+\delta)\cos(\theta_0+\delta)-g\sin(\theta_0+\delta)$$

$$=\cancel{\omega^2 a(\sin\theta_0\cos\theta_0}$$

$$=\omega^2 a(\cos^2\theta_0\cdot\delta-\sin^2\theta_0\cdot\delta)-g\cos\theta_0\cdot\delta$$

$$=\cancel{[\omega^2 a(2\cos^2\theta_0-1)-g\cos\theta_0]\delta}$$

$$=\omega^2 a(2\cos^2\theta_0-1-\cos^2\theta_0)\delta=-\omega^2 a\sin^2\theta_0\delta$$ 是稳定位置.

② $\omega=\omega_0$时. 平衡位置是 $\theta=0,\pi$

对于 $\theta=0$处: $\ddot\delta=\omega^2 a\delta-g\delta=-(g-\omega^2 a)\delta$ 是稳定位置

$\theta=\pi$处: $\ddot\delta=\omega^2 a\delta+g\delta=(\omega^2 a+g)\delta$ 不是稳定位置

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8. 哈密顿原理：

$$\delta I=\delta\int_{t_1}^{t_2}L\,dt=0$$

$$\Rightarrow \delta\int L\,dt=\int \delta L\,dt=\int\left(\frac{\partial L}{\partial q_i}\delta q_i+\frac{\partial L}{\partial \dot q_i}\delta \dot q_i+\frac{\partial L}{\partial \ddot q_i}\delta \ddot q_i\right)dt.$$

$$\int\frac{\partial L}{\partial \dot q_i}\delta\dot q_i\,dt=\int\frac{\partial L}{\partial \dot q_i}d(\delta q_i)=\frac{\partial L}{\partial \dot q_i}\delta q_i\Big|_1^2-\int\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}\delta q_i\,dt=-\int\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_i}\right)\delta q_i\,dt$$

$$\int\frac{\partial L}{\partial \ddot q_i}\delta\ddot q_i=\int\frac{\partial L}{\partial \ddot q_i}d(\delta\dot q_i)=\frac{\partial L}{\partial \ddot q_i}\delta\dot q_i\Big|_1^2-\int\frac{d}{dt}\frac{\partial L}{\partial \ddot q_i}\,d\delta q_i$$

$$=-\frac{d}{dt}\frac{\partial L}{\partial \ddot q_i}\delta q_i\Big|_1^2+\int\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q_i}\delta q_i\,dt$$

$$\Rightarrow\int\left(\frac{\partial L}{\partial q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}+\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q_i}\right)\delta q_i\,dt=0$$

δq_i 任意 $\Rightarrow$

$$\frac{d^2}{dt^2}\frac{\partial L}{\partial \ddot q_i}-\frac{d}{dt}\frac{\partial L}{\partial \dot q_i}+\frac{\partial L}{\partial q_i}=0$$

当 $L=-\frac{m}{2}q\ddot q-\frac{k}{2}q^2$ 时.

$$\frac{\partial L}{\partial \ddot q}=-\frac{m}{2}q\qquad \frac{\partial L}{\partial q}=-kq\qquad \frac{\partial L}{\partial \dot q}=0$$

$$\Rightarrow \frac{d^2}{dt^2}\left(-\frac{m}{2}q\right)+(-kq)=0$$

若 k, m 均为常数 $\Rightarrow$ $\frac{m}{2}\ddot q+kq=0$ 　即 $q=A\cos(\omega t+\varphi)$

$$\omega=\sqrt{\frac{2k}{m}}$$

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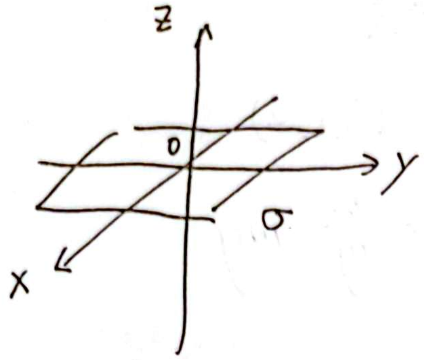
9.

(1) 电荷均匀分布在无限大平面上. 不妨设平面为xOy平面.



用直角坐标(x,y,z)

则拉氏量

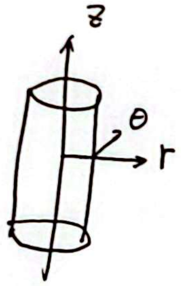
$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-V(z)$$

由对称性, ~~拉氏量~~势能仅与z方向位置相关

$\Rightarrow$ 循环坐标 x $\quad p_x=\frac{\partial L}{\partial \dot{x}}=m\dot{x}$

y $\quad p_y=\frac{\partial L}{\partial \dot{y}}=m\dot{y}$

(2) 电荷均匀分布在无限大圆柱面上. 用柱坐标.



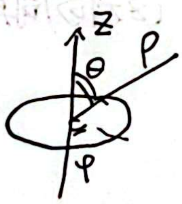
$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)-V(r)$$

由对称性, 势能仅与r相关.

循环坐标 z $\quad p_z=\frac{\partial L}{\partial \dot{z}}=m\dot{z}$

θ $\quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$

(3) 电荷均匀分布在圆环面上, 用球坐标, 不妨设圆环在$\theta=\frac{\pi}{2}$平面

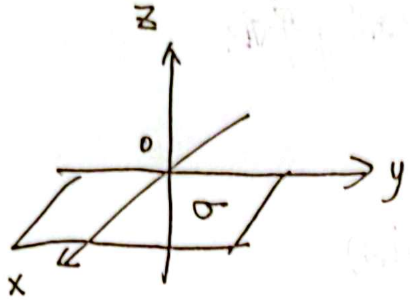


$$L=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\theta}^2+\rho^2\sin^2\theta\dot{\varphi}^2)-V(r,\theta)$$

由对称性, 势能与~~θ~~φ无关.

循环坐标φ $\quad p_\varphi=\frac{\partial L}{\partial \dot{\varphi}}=m\rho^2\sin^2\theta\dot{\varphi}$

(4) 无限大半平面：用直角坐标，不妨设带电平面在 xoy 平面，x 正半轴上

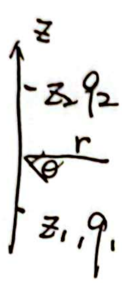


由对称性，势能与 y 无关

$$L=\frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)-V(x,z)$$

循环坐标 y $\quad p_y=\frac{\partial L}{\partial \dot{y}}=m\dot{y}$

(5) 两点. 用柱坐标，不妨设两点均在 z 轴上.

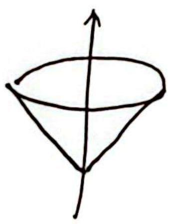


由对称性，势能与 θ 无关

$$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)-V(r,z)$$

循环坐标 θ $\quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$

(6) 圆锥面. 用球坐标.

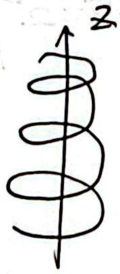


势能与 φ 无关

$$L=\frac{1}{2}m(\dot{\rho}^2+\rho^2\dot{\theta}^2+\rho^2\sin^2\theta\,\dot{\varphi}^2)-V(\rho,\theta)$$

循环坐标 φ $\quad p_\varphi=\frac{\partial L}{\partial \dot{\varphi}}=m\rho^2\sin^2\theta\,\dot{\varphi}$

(7) 螺线. 用柱坐标



~~由对称性，势能有θ方向平移不变性（z方向周期性）~~

$$\cancel{L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)-V(r,\theta,z)}$$

~~循环坐标~~ $\cancel{\theta \quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}}$

由对称性，势能有 θ 方向平移不变性

$L=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2+\dot{z}^2)-V(r,z)$ $\quad$ 循环坐标 θ $\quad p_\theta=\frac{\partial L}{\partial \dot{\theta}}=mr^2\dot{\theta}$

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