理论力学

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内容回顾

- 哈密顿-雅可比方程
- 哈密顿主函数
- 分离变量法
- 哈密顿特征函数与作用变量

谐振子

- 考虑一维简谐振子

$$H(q,p)=\frac{p^2}{2m}+\frac{kq^2}{2}=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)$$

$$\omega^2\equiv\frac{k}{m}$$

列出哈密顿-雅可比方程

$$\frac{\partial S}{\partial t}+H\left(q,\frac{\partial S}{\partial q},t\right)=0$$

$$\frac{\partial S}{\partial t}+\frac{1}{2m}\left[\left(\frac{\partial S}{\partial q}\right)^2+m^2\omega^2q^2\right]=0$$

H 不显含 *t*，故可将哈密顿主函数写为

$$S(q,\alpha,t)=W(q,\alpha)-\alpha t$$

$$\Rightarrow\quad\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2+m^2\omega^2q^2\right]=\alpha$$

分离变量 t

α 的意义即能量！

谐振子

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial q}\right)^2 + m^2\omega^2 q^2\right] = \alpha$$

$$W = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq$$

$$S = \int \sqrt{2m\alpha - m^2\omega^2 q^2}\, dq - \alpha t$$

$$S(q, \alpha, t) = W(q, \alpha) - \alpha t$$

先不积分，因为我们只需偏导数

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$Q = \beta = \frac{\partial S}{\partial \alpha} = \sqrt{\frac{m}{2\alpha}} \int \frac{dq}{\sqrt{1 - m\omega^2 q^2/2\alpha}} - t$$

令

$$x = \sqrt{\frac{m}{2\alpha}}\omega q$$

$$\beta + t = \frac{1}{\omega}\int \frac{dx}{\sqrt{1 - x^2}} = \frac{1}{\omega}\arcsin(x) = \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2\alpha}}\omega q\right)$$

谐振子

$$\beta + t = \frac{1}{\omega}\arcsin\left(\sqrt{\frac{m}{2\alpha}}\omega q\right) \Rightarrow q(t) = \sqrt{\frac{2\alpha}{m\omega^2}}\sin(\omega t + \beta') \qquad \beta' = \omega\beta$$

$$p = \frac{\partial S}{\partial q} = \frac{\partial}{\partial q}\left(\int\sqrt{2m\alpha - m^2\omega^2 q^2}dq - \alpha t\right) = \sqrt{2m\alpha - m^2\omega^2 q^2}$$

$$\Rightarrow p(t) = \sqrt{2m\alpha}\cos(\omega t + \beta')$$

两个常数 α 和 β 需依靠初始条件确定，

令 t = 0 时

$q = q_0$ $\quad$ $p = p_0$

$$\alpha = E = \frac{1}{2m}\left(p_0^2 + m^2\omega^2 q_0^2\right)$$

$$\frac{q_0}{p_0} = \frac{1}{m\omega}\tan(\beta')$$

α，也就是 P， 的意义即能量！

β， 也就是 Q， 的意义即相角！

谐振子的振动频率

不关心运动细节

- 考虑一维简谐振子

$$H(q,p)=\frac{p^2}{2m}+\frac{kq^2}{2}=\frac{1}{2m}\left(p^2+m^2\omega^2q^2\right)=\alpha$$

$$\omega^2\equiv\frac{k}{m}$$

- 写出作用量变量

$$J=\oint pdq=\oint\sqrt{2m\alpha-m^2\omega^2q^2}\,dq$$

变量代换积分：

$$q=\sqrt{\frac{2\alpha}{m\omega^2}}\sin\theta$$

$$\Rightarrow\quad J=\frac{2\alpha}{\omega}\int_0^{2\pi}\cos^2\theta\,d\theta=\frac{2\alpha}{\omega}\pi$$

$$\sqrt{2m\alpha-m^2\omega^2q^2}\;dq=\sqrt{2m\alpha-m^2\omega^2\left(\frac{2\alpha}{m\omega^2}\right)\sin^2\theta}\;\sqrt{\frac{2\alpha}{m\omega^2}}\cos\theta d\theta$$

$$=\sqrt{2m\alpha(1-\sin^2\theta)}\;\sqrt{\frac{2\alpha}{m\omega^2}}\cos\theta d\theta=\frac{2\alpha}{\omega}\cos^2\theta d\theta$$

谐振子

- 容易反解 α

$$\alpha = H = \frac{J\omega}{2\pi}$$

$$J = \frac{2\alpha}{\omega}\pi$$

- 解得频率

$$\frac{\partial K}{\partial J} \equiv \nu(J)$$

$$\nu = \frac{\partial H}{\partial J} = \frac{\partial}{\partial J}\left(\frac{J\omega}{2\pi}\right) = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\omega^2 \equiv \frac{k}{m}$$

Done !

开普勒问题

- 哈密顿量

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\varphi^2}{r^2\sin^2\theta}\right) - \frac{k}{r}$$

H 不显含 t

- 利用哈密顿特征函数以及相应的哈密顿-雅可比方程

$$\frac{1}{2m}\left[\left(\frac{\partial W}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial W}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial W}{\partial \varphi}\right)^2\right] - \frac{k}{r} = E$$

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1$$

- 令 $W(r,\theta,\varphi) = W_1(r) + W_2(\theta) + \alpha_\varphi \varphi$

φ 是循环坐标，必然可分离

$$r^2\left(\frac{\partial W_1}{\partial r}\right)^2 + 2mr^2\left(-\frac{k}{r} - E\right) = -\left(\frac{\partial W_2}{\partial \theta}\right)^2 - \frac{1}{\sin^2\theta}\alpha_\varphi^2$$

===〉两端都等于常数!

开普勒问题

$$r^2\left(\frac{\partial W_1}{\partial r}\right)^2 - 2mr^2\left(\frac{k}{r}+E\right) = -\alpha_\theta^2 = -\left(\frac{\partial W_2}{\partial \theta}\right)^2 - \frac{1}{\sin^2\theta}\alpha_\varphi^2$$

$$W(r,\theta,\varphi) = W_1(r) + W_2(\theta) + \alpha_\varphi \varphi$$

负定

$$S = \int\sqrt{2m\left(E+\frac{k}{r}\right) - \frac{\alpha_\theta^2}{r^2}}dr + \int\sqrt{\alpha_\theta^2 - \frac{\alpha_\varphi^2}{\sin^2\theta}}d\theta + \alpha_\varphi\varphi - Et$$

三个常数

$E, \alpha_\theta, \alpha_\varphi$

$$E \to H, \quad \alpha_\varphi \to L_z, \quad \alpha_\theta^2 = p_\theta^2 + \frac{1}{\sin^2\theta}p_\varphi^2 \to L^2$$

练习证明

$$\boldsymbol{L} = r\,\hat{e}_r \times m\left(\dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_\theta + r\sin\theta\,\dot{\varphi}\,\hat{e}_\varphi\right)$$

$$= mr^2\dot{\theta}\hat{e}_\varphi - mr^2\sin\theta\dot{\varphi}\hat{e}_\theta$$

$$L^2 = m^2r^4\dot{\theta}^2 + m^2r^4\sin^2\theta\dot{\varphi}^2$$

$$p_r = m\dot{r}$$

$$p_\theta = mr^2\dot{\theta}$$

$$p_\varphi = mr^2\sin^2\theta\dot{\varphi} = L_z$$

开普勒问题

$$S=\int\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr+\int\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}d\theta+L_z\varphi-Et$$

$$\beta_1=\frac{\partial S}{\partial E}=\frac{\partial}{\partial E}\int\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr-t$$

r(t) **验证!**

$$\beta_2=\frac{\partial S}{\partial L_z}=\frac{\partial}{\partial L_z}\int\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}d\theta+\varphi$$

$$\beta_3=\frac{\partial S}{\partial (L^2)}=\frac{\partial}{\partial (L^2)}\int\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{(L^2)}{r^2}}dr+\frac{\partial}{\partial (L^2)}\int\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}d\theta$$

分离变量并不自动体现轨道的平面属性?

想一想为什么?

$$p_i=\frac{\partial F_2}{\partial q_i}\qquad Q_i=\frac{\partial F_2}{\partial P_i}$$

开普勒问题的作用—角变量

$$S=\int\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr+\int\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}d\theta+L_z\varphi-Et$$

$$p_\varphi=\frac{\partial S}{\partial\varphi}=L_z \quad\Rightarrow\quad J_\varphi=\oint p_\varphi d\varphi=2\pi L_z$$

$$p_\theta=\frac{\partial S}{\partial\theta}=\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}} \quad\Rightarrow\quad J_\theta=\oint p_\theta d\theta=\oint\sqrt{L^2-\frac{L_z^2}{\sin^2\theta}}d\theta$$

$$p_r=\frac{\partial S}{\partial r}=\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}} \quad\Rightarrow\quad J_r=\oint p_r dr=\oint\sqrt{2m\left(E+\frac{k}{r}\right)-\frac{L^2}{r^2}}dr$$

经过积分计算，得到

$$J_\varphi=2\pi L_z \qquad J_\theta=2\pi(L-L_z) \qquad J_r=2\pi\left(k\sqrt{\frac{m}{-2E}}-L\right)$$

开普勒问题的作用—角变量

$$J_\varphi = 2\pi L_z$$

$$J_\theta = 2\pi(L - L_z)$$

$$J_r = 2\pi\left(k\sqrt{\frac{m}{-2E}} - L\right)$$

$$J_r + J_\theta + J_\varphi = 2\pi k\sqrt{\frac{m}{-2E}} \rightarrow E = -\frac{2\pi^2 mk^2}{(J_r + J_\theta + J_\varphi)^2}$$

- 所有三个作用变量以组合的方式出现在能量中，可知简并！

$$\nu = \frac{\partial H}{\partial J_r} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\varphi} = \frac{4\pi^2 mk^2}{(J_r + J_\theta + J_\varphi)^3}$$

$$\tau = \pi k\sqrt{\frac{m}{-2E^3}} \qquad a = -\frac{k}{2E}$$

开普勒第三定律

- 利用生成函数

$$F = (\omega_\varphi - \omega_\theta)J_1 + (\omega_\theta - \omega_r)J_2 + \omega_r J_3$$

$$(\omega_{\varphi,\theta,r},\ J_{\varphi,\theta,r}) \rightarrow (\omega_{1,2,3},\ J_{1,2,3})$$

开普勒问题的作用—角变量

- 利用生成函数

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

$$F = (\omega_\varphi - \omega_\theta)J_1 + (\omega_\theta - \omega_r)J_2 + \omega_r J_3$$

$$(\omega_{\varphi,\theta,r},\ \ J_{\varphi,\theta,r}) \to (\omega_{1,2,3},\ \ J_{1,2,3})$$

两个方向的频率为零

$$\omega_1 = \frac{\partial F}{\partial J_1} = \omega_\varphi - \omega_\theta = 0 \qquad \omega_2 = \frac{\partial F}{\partial J_2} = \omega_\theta - \omega_r = 0 \qquad \omega_3 = \frac{\partial F}{\partial J_3} = \omega_r$$

$$J_r = \frac{\partial F}{\partial \omega_r} = J_3 - J_2 \qquad J_\varphi = \frac{\partial F}{\partial \omega_\varphi} = J_1 \qquad J_\theta = \frac{\partial F}{\partial \omega_\theta} = J_2 - J_1$$

$$J_1 = J_\varphi \qquad J_2 = J_\varphi + J_\theta \qquad J_3 = J_\varphi + J_\theta + J_r$$

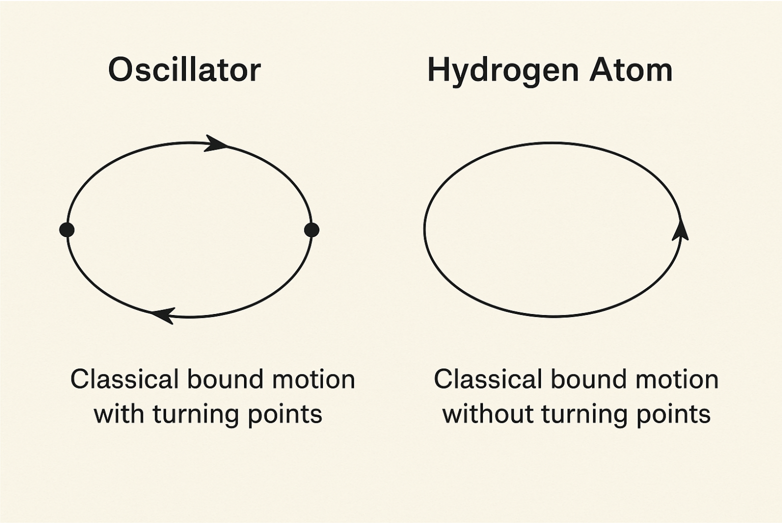
$$E = -\frac{2\pi^2 m k^2}{(J_r + J_\theta + J_\varphi)^2} \quad \Rightarrow \quad E = -\frac{2\pi^2 m k^2}{J_3^2}$$

只含一个非零频率的方向

平面周期运动

开普勒问题的量子化

- 索末菲量子化条件： $J_3 = nh$



- 氢原子能级

$$E = -\frac{2\pi^2 mk^2}{J_3^2} = -\frac{2\pi^2 m Z^2 e^4}{n^2 h^2}$$

与求解薛定谔方程结果一致

- n 为主量子数，是这个简并系统的唯一量子数。

- 考虑相对论修正、外加磁场后会分别解除 J_2, J_1方向的简并度。

哈密顿主函数的波动行为

考虑一个质点在二维空间自由运动，其哈密顿主函数

$$S(x, y, \alpha_1, \alpha_2, t) = W(x, y, \alpha_1, \alpha_2) - \alpha_1 t$$

$$\alpha_1 = E$$

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = 2mE \quad \Rightarrow \quad W(x, y, \alpha_1, \alpha_2) = \alpha_2 x + \sqrt{2mE - \alpha_2^2}\, y$$

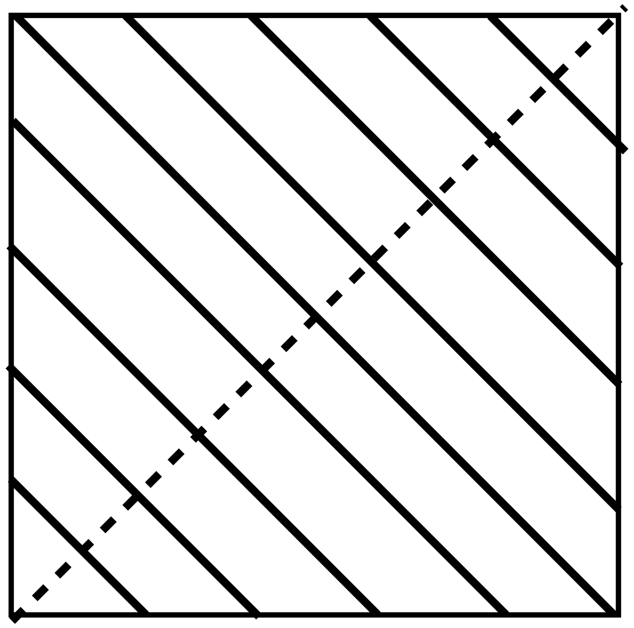
令x, y 方向的初始速度相同

$$W(x, y) = \sqrt{mE}\, x + \sqrt{mE}\, y$$

$$p_i = \frac{\partial F_2}{\partial q_i}$$

$$S(x, y, t) = \sqrt{mE}\, x + \sqrt{mE}\, y - Et$$

$$\boldsymbol{p} = \nabla W(x, y)$$



取 W 为常数的“曲面”集，质点运动方向沿其法线方向

取 S 为常数的“曲面”集，质点运动伴随着 S 波的传播

从波动光学到几何光学

- 惠更斯的波动光学中，波动方程为

$$\nabla^2\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0$$

n 是折射率，c 是真空中的光速

- 若 n 为常数，有平面波解

$$\phi = \phi_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}$$

$\boldsymbol{k}$ 取 z 方向

k_0 是真空中的波数

$$\phi = \phi_0 e^{ik_0(nz-ct)}$$

波数

$$k = \frac{2\pi}{\lambda} = \frac{n\omega}{c}$$

相速度

$$u = \frac{\omega}{k} = \frac{c}{n}$$

- 若 n 不完全是常数，而是在空间缓慢变化的

$$\phi = e^{A(\boldsymbol{r})}e^{ik_0[L(\boldsymbol{r})-ct]}$$

A 是波幅，L是光程、或称光程函、程函

代入波动方程可得

$$\nabla^2 A + (\nabla A)^2 + k_0^2[n^2 - (\nabla L)^2] = 0$$

$$\nabla^2 L + 2\nabla A\cdot\nabla L = 0$$

$$\nabla\phi = \phi\nabla(A + ik_0L)$$

$$\nabla^2\phi = \phi[\nabla^2(A+ik_0L) + (\nabla(A+ik_0L))^2]$$

从波动光学到几何光学

- n 随距离缓慢变化，即 n 在一个波长范围内的变换可忽略，即波长与介质的线度相比很短，即短波近似。

$$\nabla^2 A + (\nabla A)^2 + \boxed{k_0^2[n^2 - (\nabla L)^2]} = 0$$

$$\nabla^2 L + 2\nabla A \cdot \nabla L = 0$$

k_0 很大

$$k = \frac{2\pi}{\lambda} = \frac{n\omega}{c}$$

几何光学的程函方程

$$(\nabla L)^2 = n^2$$

—势场 V 中的单粒子的哈密顿-雅可比方程

$$(\nabla W)^2 = 2m(E - V)$$

程函 L 对应于作用量函数 W;
折射率 n 对应于速度；
费马原理对应于莫佩蒂原理

$$\int n\,ds \Longleftrightarrow \int p\,dq$$

$$H\left(q_i, \frac{\partial W}{\partial q_i}\right) = \alpha_1$$

从“几何力学”到波动力学?

- 光是波，几何光学是波动光学的短波极限
- 经典力学或许是某种“ 几何力学” ?
- 是波动力学的短波极限?

从“粒子”力学 到 “波动”力学

- 作用量函数 *W* 对应于 程函 *L*, 两者仅差一比例常数

- 作用量函数 *S* = *W* - *Et* 对应于 光波总相位 $k_0[L - ct]$

波长 频率 波速

$$\lambda\nu = u$$

相速度!

$$k_0[L - ct] = \frac{2\pi}{\lambda_0}(L - ct) = \frac{2\pi}{h}\left(\boxed{\frac{Lh}{\lambda_0}} - \underline{h\nu t}\right)$$

W　　$E = h\nu = 2\pi\hbar\nu$

光电效应!
能量对应于频率
两者仅差一比例常数

- 计算相速度

$$u = \frac{dr}{dt} = \frac{E}{|\nabla W|} = \frac{E}{p}$$

$$\lambda = u/\nu = \frac{E/p}{E/h} = h/p$$

德布罗意波!

注意 *u* 不是粒子移动速度 *v*

$$S = W - Et = Const$$
$$dS = dW - Edt = \nabla W \cdot d\boldsymbol{r} - Edt = 0$$

$$v = p/m = |\nabla W|/m = \sqrt{2(E - V)/m}$$

从“粒子”力学 到 “波动”力学

- 因此，可以引入一个相位为S 的波函数描述粒子运动

$$\psi(\mathbf{r},t) = A(\mathbf{r})e^{iS(\mathbf{r},t)/\hbar} = A(\mathbf{r})e^{i(W-Et)/\hbar}$$

德布罗意波！物质波！

S(r,t) 的等高线是物质波的波前

- 代入波动方程

$$\nabla^2\psi - \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = 0$$

$$\nabla^2\psi - \frac{1}{u^2}\frac{\partial^2\psi}{\partial t^2} = \nabla^2\psi - \frac{2m(E-V)}{E^2}\cdot\frac{-E^2}{\hbar^2}\psi = 0$$

$$\nabla^2\psi + \frac{2m(E-V)}{\hbar^2}\psi = 0$$

薛定谔方程！

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi = i\hbar\frac{\partial\psi}{\partial t}$$

想一想为什么是一阶导数呢?

量子力学中的薛定谔方程

$$\phi = e^{A(\mathbf{r})} e^{ik_0[L(\mathbf{r})-ct]}$$

- 量子力学中，粒子的运动用波函数来描述

$$\psi(\mathbf{r},t) = A(\mathbf{r},t)e^{iS(\mathbf{r},t)/\hbar}$$

A 是振幅，S 是相位，$\hbar$ 是普朗克常数，具有作用量的量纲

- 假设粒子在一保守势场 V 中运动，波函数的行为满足薛定谔方程

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

左边：
$$i\hbar\frac{\partial\psi}{\partial t} = e^{iS/\hbar}\left(i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right)$$

右边：
$$\nabla\psi = e^{iS/\hbar}\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

$$\nabla^2\psi = \dots$$

量子力学中的薛定谔方程

$$\nabla^2\psi = \left(\frac{i}{\hbar}e^{iS/\hbar}\nabla S\right)\left(\nabla A + \frac{i}{\hbar}A\nabla S\right) + e^{iS/\hbar}\nabla\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

$$= \frac{i}{\hbar}e^{iS/\hbar}\nabla S\cdot\nabla A - \frac{1}{\hbar^2}e^{iS/\hbar}A(\nabla S)^2 + e^{iS/\hbar}\left(\nabla^2 A + \frac{i}{\hbar}\nabla A\cdot\nabla S + \frac{i}{\hbar}A\nabla^2 S\right)$$

$$= e^{iS/\hbar}\left(\nabla^2 A - \frac{1}{\hbar^2}A(\nabla S)^2 + \frac{i}{\hbar}\left(2\nabla A\cdot\nabla S + A\nabla^2 S\right)\right)$$

$$\nabla\psi = e^{iS/\hbar}\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

实部：

$$\left(-A\frac{\partial S}{\partial t}\right) = -\frac{\hbar^2}{2m}\left(\nabla^2 A - \frac{A}{\hbar^2}(\nabla S)^2\right) + VA$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = e^{iS/\hbar}\left(i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right)$$

$$\frac{1}{2m}(\nabla S)^2 + V + \frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}$$

$$\psi = Ae^{iS/\hbar}$$

量子力学中的薛定谔方程

实部：

$$\frac{1}{2m}(\nabla S)^2 + V + \frac{\partial S}{\partial t} = \frac{\hbar^2}{2m}\frac{\nabla^2 A}{A}$$

经典极限：振幅的空间变化很小，波长短到一定程度时，总是可以将振幅的空间变化忽略的。

$\hbar \to 0$

$$\frac{1}{2m}(\nabla S)^2 + V + \frac{\partial S}{\partial t} = 0$$

哈密顿-雅可比方程

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

$$H = \frac{p^2}{2m} + V$$

$$p = \frac{\partial S}{\partial q}$$

若将波函数的相位 S 视为哈密顿主函数，此即为哈密顿-雅可比方程。

量子力学中的薛定谔方程

$$\nabla^2\psi = \left(\frac{i}{\hbar}e^{iS/\hbar}\nabla S\right)\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)e^{iS/\hbar}\nabla\left(\nabla A + \frac{i}{\hbar}A\nabla S\right)$$

$$= e^{iS/\hbar}\left(\nabla^2 A - \frac{1}{\hbar^2}A(\nabla S)^2 + \frac{i}{\hbar}\left(2\nabla A\cdot\nabla S + A\nabla^2 S\right)\right)$$

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$$

$$i\hbar\frac{\partial\psi}{\partial t} = e^{iS/\hbar}\left(i\hbar\frac{\partial A}{\partial t} - A\frac{\partial S}{\partial t}\right)$$

$$\psi = Ae^{iS/\hbar}$$

虚部：

$$\hbar\frac{\partial A}{\partial t} = -\frac{\hbar^2}{2m}\frac{1}{\hbar}\left(2\nabla A\cdot\nabla S + A\nabla^2 S\right)$$

$$\frac{\partial A^2}{\partial t} = -\nabla\cdot\left(A^2\frac{\nabla S}{m}\right)$$

$$(\nabla S)/m = p/m = v$$

$$\rho = A^2 = \psi^*\psi$$

概率密度守恒方程

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho v)$$

总结

- 哈密顿-雅可比方法求解谐振子
- 哈密顿-雅可比方法求解开普勒问题
- 波动力学的构建：

  薛定谔方程在经典极限下对应于哈密顿-雅可比方程和连续性方程