Statistics

Expectation
expected value (mean)

- $E\{x\} = \bar{x}$

$$E\{x\} = \bar{x} = \underbrace{\int_{-\infty}^{\infty} x\, p(x)\, dx}_{\text{continuous}} = \underbrace{\sum_{k=-\infty}^{\infty} k\, p(k)}_{\text{discrete}}$$

- $E\{x+y\} = E\{x\} + E\{y\}$

Probability Density Function
$p(x)$

$$\int_{-\infty}^{\infty} p(x)\,dx = 1 \qquad \sum_{k=-\infty}^{\infty} p(k) = 1$$

$\boxed{\text{Ex}}$ Discrete : Dice

K	1	2	3	4	5	6
$p(k)$	1/6	1/6	1/6	1/6	1/6	1/6

ℓ^{th} Moment
$E\{(x-\bar{x})^{\ell}\}$

- 1^{st} Moment $\Rightarrow$ $E\{x-\bar{x}\} = 0$
- 2^{nd} Moment $\Rightarrow$ $E\{(x-\bar{x})^2\} = \text{var}(x) = \sigma^2$

$$\hookrightarrow \underbrace{\sigma^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 p(x)\,dx}_{\text{continuous}} \qquad \underbrace{\sigma^2 = \sum_{k=-\infty}^{\infty} (k-\bar{k})^2 p(k)}_{\text{discrete}}$$

* if a process is "stationary" the mean and variance are constant

Gaussian Distribution
is fully quantified by 2 variables
1) $E\{x\} = \bar{x}$
2) $E\{(x-\bar{x})^2\} = \sigma^2$

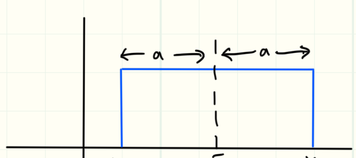
$\pm 1\sigma \Rightarrow 63\%$
$\pm 2\sigma \Rightarrow 85\%$
$\pm 3\sigma \Rightarrow 95\%$

* $p(x) = (\sigma\sqrt{2\pi})^{-1} \exp\left(-\frac{1}{2}(x-M)^2 \sigma^{-1}\right)$

* $\bar{x}$ and σ^2 can be estimated / approximated from sampled data
- as $N \to \infty$ $\bar{x}$ converges to $E\{x\}$

* $E\{x\} \approx \frac{1}{N}\sum_{i=1}^{N} x_i \qquad \sigma^2 \approx \frac{1}{N-1}\sum_{i=1}^{N} (x_i - \bar{x}_{est})^2$

Uniform Distribution
each output is equally likely

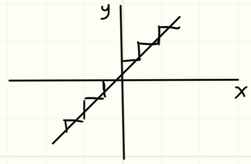


$E\{x\} = \frac{x_2+x_1}{2}$

$p(x) = \frac{1}{2a} \quad (x_1 \le x \le x_2)$

$\sigma = \frac{2a}{\sqrt{12}} = \frac{a}{\sqrt{3}}$

* encoder



Random Variable
$x(t)$ or x_k
$y(t)$ or y_k

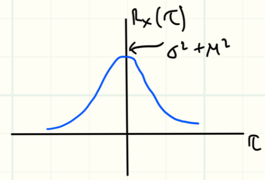
Stochastic Process
is a "family" of random variables indexed by a paramater set (usually time)

- Usually assumed to be a "stationary" process
  - $\hookrightarrow E\{\underline{x}(t)\} = \bar{\underline{x}} = \mu_x$ (for all time)
  - $\hookrightarrow E\{\underline{x}(t_1)x^T(t_2)\} = Q(t_1 - t_2)$
    - for a stationary process $E\{\underline{x}(t_1)\underline{x}^T(t_1+\tau)\} = Q(\tau)$

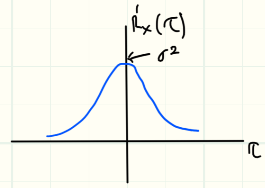
Autocorrelation Matrix
a measure of correlation by a seperation of time

- $R_x(t_1,t_2) = E\{\underline{x}(t_1)x^T(t_2)\}$
  - If stationary $R_x(\tau) = E\{\underline{x}(t_1)\underline{x}^T(t_1+\tau)\}$

* $\tau = 0 \Rightarrow E\{x(t_1)x(t_1)\} = E\{x(t_1)^2\}$



Autocovariance Matrix
- $R_x'(\tau) = E\{(\underline{x}(t)-\underline{M})(\underline{x}(t+\tau)-\underline{M})^T\}$
  - If $\underline{M}=0$ then autocovariance = autocorrelation



$\boxed{\text{Ex}}$ Random sequence

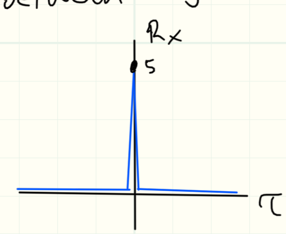
$x = [1\ {-3}\ {-2}\ 3\ 4\ {-2}\ 0\ 1\ {-2}\ 3\ \ldots]$

$E\{x\} = 0$

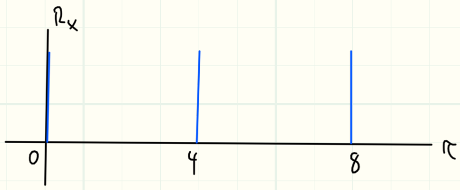
$E\{x^2\} = 5$

* best guess $= 0$

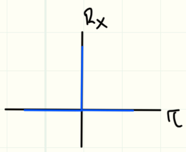
* 63% between ± 5



$\boxed{EX}$ $x = [1\ -2\ 3\ 2\ 1\ -2\ 3\ 2\ \cdots]$



White Noise random, no serial correlation

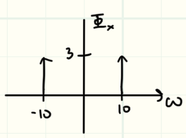


$R_x(\tau) = Q_0\,\delta(t)$

$*$ $\delta(t)$ = Dirac Delta $\Rightarrow \delta(t) = 0 \ \forall\ t \neq 0$

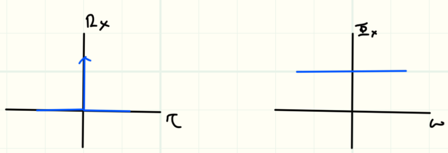
Power Spectral Density (PSD)
- fft → fast fourier transform
- $y = \sum_{i=0}^{N} a_i \sin(\omega_i t)$

$\boxed{EX}$ $3\sin(10t)$



- $\Phi_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau) e^{-j\omega\tau}\, d\tau$
- $R_x(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Phi_x(\omega) e^{j\omega\tau}\, d\omega$

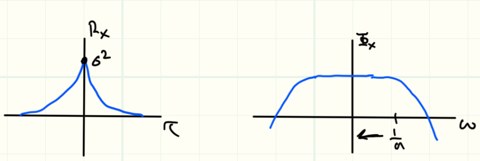
$\boxed{EX}$ White Noise



$\boxed{EX}$ $\dot{x} = ax + w$

$*$ w = random white noise (gaussian, $\mathcal{N}(\mu, \sigma^2)$)
↳ $w \sim \mathcal{N}(0,1)$

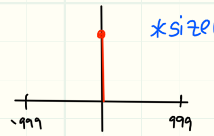
$x_{k+1} = x_k + \dot{x}\Delta t = x_k + (ax_k + w_k)\Delta t$



if $x \sim \mathcal{N}(0, \sigma^2)$
- $x \Rightarrow$ random
- $E\{x\} = 0$
- $E\{x^2\} = \sigma^2$

Computing / Estimating R_x

$$R_x(k) = \frac{1}{N}\sum_{i=1}^{\tilde{N}} a_i a_{i+k} \qquad -(N-1) < k < (N-1)$$



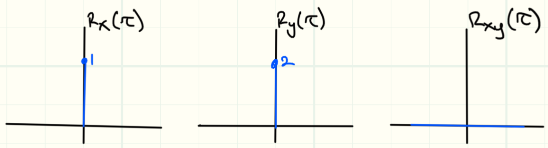
$*size(a) = [1000, 1]$

MATLAB
```
>> sum(a_i .* a_i+k)
>> a_i * a_i+k^T

>> randn(1000,1)   * gauss, N(0,1)
>> rand(1000,1)    * uniform
```

Ex $\quad x \sim \mathcal{N}(0,1)$ $\quad$ x & y are independant

$y \sim \mathcal{N}(0,2)$



$$R_{xy}(\tau) = \frac{1}{N}\sum_{i=1}^{\tilde{N}} x_i y_{i+k} = 0 \quad \forall\, k$$

Independant $\Rightarrow$ also uncorrelated

Uncorrelated $\Rightarrow$ may or may not be independant

* A random variable
1) only correlated with itself at a shift of 0
2) not correlated with any other variable for any τ

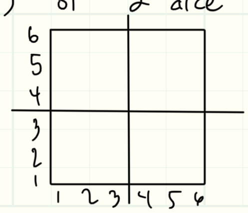
Family of Random Variables

Ex $\quad \underline{x} = \begin{bmatrix} \eta_g \\ \eta_a \\ \eta_t \end{bmatrix} \Rightarrow$ noise on gyro. accel. therm.

Ex $\quad \underline{x} = \begin{bmatrix} X_{dice1} \\ X_{dice2} \end{bmatrix}$

Joint Probability Density Function (PDF)

Ex $\quad p(x,y)$ of 2 dice



$p(x,y) = \frac{1}{36}$

$$\int \cdots \iint p(\underline{x})\, d\underline{x} = 1$$

Multivariable Normal / Gaussian

$$p(x) = \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} \exp\left[-\tfrac{1}{2}(\underline{x}-\bar{\underline{x}})^T P^{-1}(\underline{x}-\bar{\underline{x}})\right]$$

$$E\{\underline{x}\} = \underline{\mu} = \bar{\underline{x}}$$

$$E\{(\underline{x}-\underline{\mu})^T(\underline{x}-\underline{\mu})\} = P = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \cdots \\ \rho_{12}\sigma_1\sigma_2 & \ddots & \\ \vdots & & \ddots \\ & & & \sigma_n^2 \end{bmatrix}$$

Covariance Matrix
↳ diagonal is variance

$\rho_{ij} \Rightarrow$ correlation coefficient
- If uncorrelated, $\rho_{ij} = 0$
- If independant, $\rho_{ij} = 0$

Ex $\quad E\{\underline{x}\} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \qquad P = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 16 \end{bmatrix}$

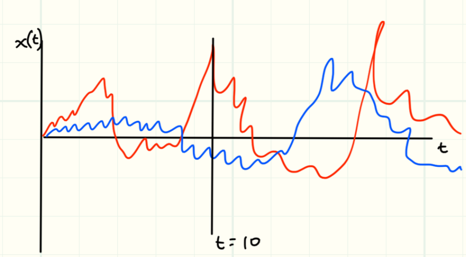
$$\underline{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \pm \sqrt{4}(3) \\ 1 \pm \sqrt{3}(3) \\ -2 \pm \sqrt{16}(3) \end{bmatrix}$$

MATLAB $\quad$ `>> cov()`

Analyzing "Non-Deterministic" Outputs

$\dot{x} + x = 1 \Rightarrow x(t) = 1 - e^{-t}$

$\dot{x} + x = w(t) \Rightarrow w \sim \mathcal{N}(0,1) \Rightarrow x(t) = \text{random}$



* with enough runs mean = 0 with some variance

* "Monte-Carlo Simulation"
↳ running N times with random inputs and measuring statistics