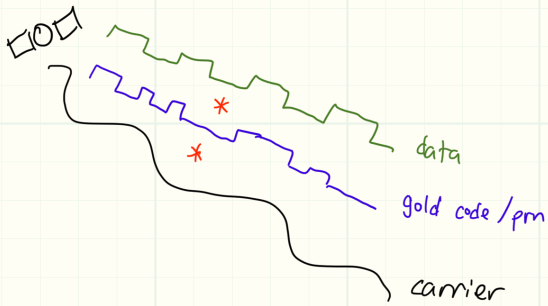
GPS Reciever Measurements

Psuedorange

$$\rho^j = \sqrt{(x^j-x)^2+(y^j-y)^2+(z^j-z)^2} + I + T + c(\delta t - \delta t^j) + \nu$$

I	ionospheric error
T	tropospheric error
δt	user clock error
δt^j	j^{th} SV clock error
ν	noise



- $\rho^j_{corr} = \rho^j + c\,\delta t^j$
- Lump I, T and ν into ν

$$\rho^j_{corr} = r + cb + \nu$$

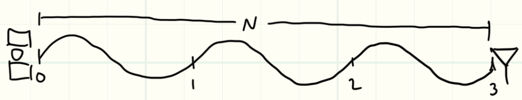
- LS: $\rho^j_{corr} - \hat{\rho}^j = \begin{bmatrix} -u_x & -u_y & -u_z & 1 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta b \end{bmatrix}$

Carrier Phase

$$\phi^j = \frac{r + N^j}{\lambda} + T - I + c(\delta t - \delta t^j) + \varepsilon_\phi$$

$$\Phi^j = \lambda \phi^j = r + \lambda N^j + T - I + c(\delta t - \delta t^j) + \varepsilon_\Phi$$

λ	carrier wavelength (L1 $\approx$ 19 cm)
ϕ^j	phase (rad)
Φ^j	phase (m) — DOPPLER
N^j	Integer ambiguity
ε	Noise



$N\lambda \approx 3(19) = 57$ cm

* $\nu \sim N(0, 1\,m)$

* $\varepsilon_\Phi \sim N(0, 1\,mm)$

$$\dot{\Phi}^j = \dot{r} + \cancel{\dot{N}}^{0} + \cancel{\dot{T}}^{\approx 0} - \cancel{\dot{I}}^{\approx 0} + c(\dot{\delta t} - \cancel{\dot{\delta t}^j}^{\approx 0}) + \varepsilon_\Phi$$

Methods to get $\dot{\Phi}$

① $\dfrac{\Phi_k - \Phi_{k-1}}{\Delta t} = \dfrac{\Delta r}{\Delta t} + \dfrac{\Delta N}{\Delta t} + \dfrac{\Delta T}{\Delta t} - \dfrac{\Delta I}{\Delta t} + c\left(\dfrac{\Delta \delta t}{\Delta t} - \dfrac{\Delta \delta t^j}{\Delta t}\right) + \dfrac{\Delta \varepsilon}{\Delta t}$

* 1/2 phase delay

$N(0, \sqrt{2}\sigma_\varepsilon)$

② Measuring frequency of carrier

* continuous

$$\Phi^j = \sqrt{(x^j-x)^2+(y^j-y)^2} + \lambda N^j + c(\delta t - \delta t^j) + T - I + \varepsilon_\Phi$$

$$\dot{\Phi}^j = \cancel{\tfrac{1}{2}}\left[(x^j-x)^2+(y^j-y)^2\right]^{-1/2}\cancel{(2)}(x^j-x)(\dot{x}^j-\dot{x}) + \cancel{\tfrac{1}{2}}\left[(x^j-x)^2+(y^j-y)^2\right]^{-1/2}\cancel{(2)}(y^j-y)(\dot{y}^j-\dot{y}) + c\dot{\delta t} - c\dot{\delta t}^j + \varepsilon_\Phi$$

$$\begin{bmatrix} \dot{\Phi}^j - c\dot{\delta t}^j - u^j_x \dot{x}^j - u^j_y \dot{y}^j \end{bmatrix} = \begin{bmatrix} \frac{-x}{r} & \frac{-y}{r} & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ c\dot{b} \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \dot{\Phi}^j - c\dot{\delta t}^j & \begin{bmatrix} -u^j_x & -u^j_y & -u^j_z \end{bmatrix} \begin{bmatrix} \dot{x}^j \\ \dot{y}^j \\ \dot{z}^j \end{bmatrix} \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} -u^j_x & -u^j_y & -u^j_z & 1 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{b} \end{bmatrix}}_{x}$$

* Closed-Form

→ linear

$$x = (H^T H)^{-1} H^T y$$

$$\begin{bmatrix} \dot{\Phi}^j - c\dot{\delta t}^j - u^j_x \dot{x}^j - u^j_y \dot{y}^j \end{bmatrix} = \begin{bmatrix} \frac{-(x^j-x)}{\sqrt{(x^j-x)^2+(y^j-y)^2}} & \frac{-(y^j-y)}{\sqrt{(x^j-x)^2+(y^j-y)^2}} & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ c\dot{b} \end{bmatrix}$$

* Assuming static $\Rightarrow \dot{x} = \dot{y} = \dot{z} = 0$

↳ can solve for position using doppler measurements

$$\begin{bmatrix} \dot{\Phi}^j - c\dot{\delta t}^j - u^j_x \dot{x}^j - u^j_y \dot{y}^j \end{bmatrix} = 0$$

Dilution of Precision (DOP)

$$P = \sigma_\rho^2 \underbrace{(G^T G)^{-1}}_{DOP} = \begin{bmatrix} \sigma_x^2 & & & \\ & \sigma_y^2 & & \not\equiv \\ & \not\equiv & \sigma_z^2 & \\ & & & \sigma_b^2 \end{bmatrix}$$

$$G = \begin{bmatrix} -u^j_x & -u^j_y & -u^j_z & 1 \end{bmatrix}$$

$$= \sigma_\rho^2 H$$

$$= \sigma_\rho^2 \begin{bmatrix} H_{11} & & & \\ & H_{22} & & \\ & & H_{33} & \\ & & & H_{44} \end{bmatrix}$$

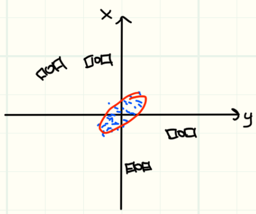
PDOP:	$\sqrt{H_{11}+H_{22}+H_{33}}$	$\sigma_{3D} = \sigma_\rho * \text{PDOP}$
HDOP:	$\sqrt{H_{11}+H_{22}}$	$\sigma_{2D} = \sigma_\rho * \text{HDOP} = \sqrt{\sigma_x^2+\sigma_y^2}$
VDOP:	$\sqrt{H_{11}}$	$\sigma_{up} = \sigma_\rho * \text{VDOP} = \sqrt{\sigma_z^2}$
TDOP:	$\sqrt{H_{44}}$	$\sigma_{time} = \sigma_\rho * \text{TDOP}$
GDOP:	$\sqrt{H_{11}+H_{22}+H_{33}+H_{44}}$	

only dependant on LS model G

DRMS

Distance Root Mean Square

$$DRMS = \sqrt{\sigma_x^2 + \sigma_y^2} \approx HDOP * \sigma_\rho$$



CEP

Circular error probability

↳ Radius of circle that captures X% of data

↳ typically 50% CEP ≈ 0.75 DRMS

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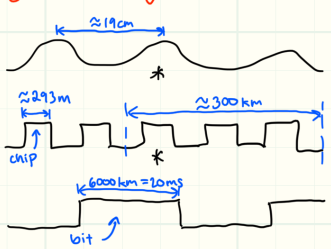
GPS Signal Lengths



1023 "chips"

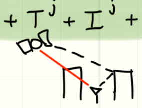
$= \frac{1}{1023}$ ms ≈ 293m

	f (MHz)	λ (cm)
L1	1575.42	~19 cm
L2	1227.60	
L5	1176.45	

*nominal frequency

$$\rho^j = r^j + c(b - b^j) + T^j + I^j + M_\rho^j + \nu_\rho^j$$

M : Multipath



$$\rho_{CORR}^j = \rho^j + cb^j = r^j + cb + \varepsilon$$

↳ lump T, I, M, ν

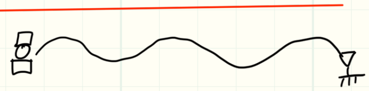
$$\phi^j = r^j + c(b - b^j) + T^j - I_f^j + \lambda N^j + M_\phi^j + \nu_\phi^j$$

Expected Accuracy: $\sigma_\varepsilon^2 (G^T G)^{-1}$

– $\sigma_\varepsilon^2 \approx \sigma_T^2 + \sigma_I^2 + \sigma_M^2 + \sigma_\nu^2$

• "User equivalent range error"

Carrier Measurement



$x(t) = A\sin(\omega t)$

* $\omega = \omega_{L1} + \omega_{dop}$

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} f(t)\,dt \approx \phi(t_0) + f(t_0)(t - t_0)$$

$$\phi(t) = \phi(t) - \phi^j(t - b^j) + N^j$$

$$\phi(t) - \phi^j(t - b^j) = f*(t - t_0)$$

$$\phi(t) = f*(t - t_0) + N = \frac{r}{\lambda} + N$$

$$\phi(t) = \frac{r - I_f^j + T^j}{\lambda} + \frac{c(b - b^j)}{\lambda} + N^j + \nu_\phi^j$$

– $\frac{\Delta\phi}{\Delta t} \approx \omega$ ⇒ doppler (with ½Δt delay)

– ϕ ⇒ integrated doppler (contains N)

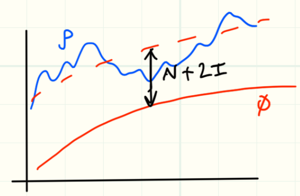
– $\dot{\phi} = \frac{\Delta\phi}{\Delta t}$ ⇒ psuedorange rate

↳ includes clock drift rate!

↳ $\dot{N} = 0$

* $\sigma_{\nu_\rho} \approx 0.5$ m

* $\sigma_{\nu_\phi} \approx 0.001$ m



* N = const.

* I(t) ≠ const

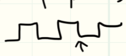
(≈ 10-20min)

$$x = \int v\,dt$$

$$x = V\,dt + \underline{\underline{x_0}} + \int v_\phi\,dt$$

↳ unbounded error

Range Error Sources

1) Satellite PVT
2) Atmospheric (Ionosphere, Tropospheric)
3) Reciever Meas. Error 

SV PVT

• monitored & reported to user in what is called "ephemerides"

↳ updated every 2 hours ("good" for 4 hours)

• New SVs < 3m of error (Position & "Time")

↳ more stable clocks

↳ errors are estimated < 1 m

↳ cross & along tracks > radial error

Time

• $b^j = t^j - t_{gps}$

↳ $b^j = af_0 + af_1(t_{gps} - t_{oc}) + af_2(t_{gps} - t_{oc})^2 + \Delta t_R$

↳ $af_1, af_2, af_0, t_{oc}, \Delta t_R$ in Data message

↳ Δt_R = relativistic clock error

↳ error on $b^j \approx 5$ns

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• $\Delta t_{UTC} = t_{gps} - t_{UTC}$

↳ $\Delta t_{UTC} = A_0 + A_1(t_{gps} - t_{ou}) + \Delta t_{LS}$

↳ $A_0, A_1, t_{ou}, \Delta t_{LS}$ in Data message

* Clock Time based on $\int_0^t f\,dt$ *

• $f_{nominal} = f + f_{error}$

• $t_{err} = \int f_{err}\,dt = f_{err}(t_0 - t) + t_0^{err}$

• If $\dot f_{err} \neq 0 \Rightarrow t_{err} = \iint \dot f_{err}\,dt$

$$= t_0^{err} + \dot f(t-t_0) + \ddot f(t-t_0)^2$$

$$= \underset{\text{clock offset}}{af_0} + \underset{\text{clock drift}}{af_1}(t-t_0) + \underset{\text{clock drift rate}}{af_2}(t-t_0)^2$$

* $P = \begin{bmatrix} & \\ & D \end{bmatrix} = \sigma^2 (G^T G)^{-1}$

↳ $\sigma_{time}^2 \approx \sigma_{UERE}^2 + (TDOP)^2 \approx 25ns$

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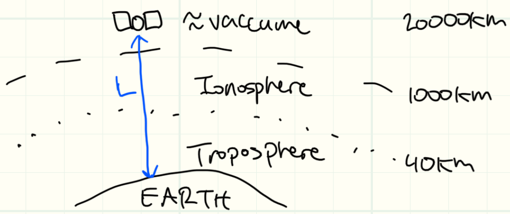
Atmospheric Errors



- L @ zenith ≈ 20,000 km
- L @ horizon ≈ 26,000 km

- Atmosphere refracts (changes speed and direction) of signal
- changes speed more than direction (path length)

$\eta = \frac{c}{v}$ — η refractive index, c speed in vaccume, v speed in median

$$\tau = \frac{1}{c}\int_{sv}^{user} \eta(l)\,dl$$

$$\Delta\tau_{err} = \frac{1}{c}\int_{sv}^{user} (\eta(l)-1)\,dl$$

$$\Delta\rho_{err} = \int_{sv}^{user} (\eta(l)-1)\,dl$$

L-Band Signal

• Ionosphere is dispersive

↳ function of frequency

• Troposphere is not

$V_p = \frac{\omega_{carrier}}{K}$ — V_p Phase velocity (carrier)

$V_g = \frac{d\omega}{dK}$ — V_g Group velocity (code)

$$s = \cos(\omega_1 t)\cos(\omega_2 t) = \tfrac{1}{2}\cos((\omega_1+\omega_2)t) + \tfrac{1}{2}\cos((\omega_1-\omega_2)t)$$

$$s = \cos\left[\omega_1\left(t - \frac{x}{V_g}\right)\right]\cos\left[\omega_2\left(t - \frac{x}{V_p}\right)\right]$$

↳ for dispersive median $V_g \neq V_p$

↳ "code-carrier" divergence

↳ code is delayed

↳ carrier is advanced

Ionospheric Delay

$TEC = \int_{sv}^{user} \eta(l)\,dl$ — TEC Total Electron Content

↳ amount of electrons in $1m^2$ tube cross-section from user to SV

↳ Poles & equator have rapid fluctuation of TEC (effected by solar flares)

$\eta_p \approx 1 - \frac{40.3\eta}{f^2}$ — η_p Phase delay

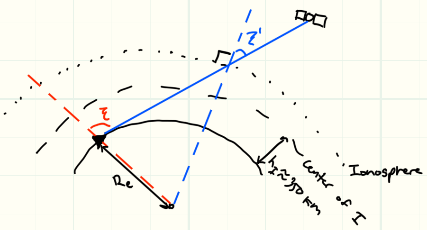
$\eta_g \approx 1 + \frac{40.3\eta}{f^2}$ — η_g Group Delay

$I_\rho \approx \frac{40.3\ TEC}{f^2}$ — I_ρ Psuedorange error

$I_\phi \approx \frac{-40.3\ TEC}{f^2}$ — I_ϕ Phase Error

* ΔTEC of 1 leads to $\Delta\rho \approx 16$ cm

Obliquity Factor



$$TEC = \frac{1}{\cos(\xi')} TECV$$

TECV vertical TEC

$$OF_I = \frac{1}{\cos(\xi')} = \sqrt{1 - \left(\frac{R_e \sin(\xi)}{R_e + h_I}\right)}$$

OF_I obliquity factor

$$1 < OF_I < 3$$

overhead — 5° from horizon

$$I(\xi) = I_z * OF_I$$

I_z: ionospheric error at zenith

$I_z \approx$ 1-3m @ night

$\approx$ 5-15m @ day

(36m in solar flare)

Monitoring of Ionosphere

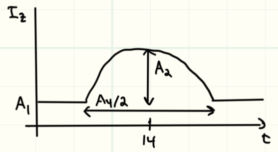
- Some available on internet
- broadcast ephemerides (from SV)

$$\frac{\hat{I}_{z_{L1}}}{c} = \begin{cases} A_1 + A_2 \cos\left(\frac{2\pi(t-A_3)}{A_4}\right) & \text{if } (t-A_3) < \frac{A_4}{4} \\ A_1 \end{cases}$$

$A_1 = 5(10^{-9})$ sec

$A_2 = 50{,}400$ sec

A_3 + A_4 are in data message (P.18, subframe 4)



Dual Frequency

L1 : original civilian

L2 : used to be encrypted (now L2C → civilian)

L5 : New (not on all SV yet)

$$\rho_f^j = r^j + c(b - b^j) + I_f^j + T_\rho^j + \nu_\rho^j$$

$$\rho_f^j = \rho_{IF}^j + \frac{40.3\,TEC}{f_f^2}$$

$$I_{L1} = \frac{40.3\,TEC}{f_{L1}^2} = \frac{f_{L1}^2}{(f_{L1}^2 - f_{L2}^2)}(\rho_{L2} - \rho_{L1})$$

$$\rho_{IF} = \frac{f_{L1}^2}{(f_{L1}^2 - f_{L2}^2)}\rho_{L1} - \frac{f_{L2}^2}{(f_{L1}^2 - f_{L2}^2)} = 2.546\rho_{L1} + 1.546\rho_{L2}$$

$$\frac{40.3\,TEC}{f_{L1}^2} = \rho_{L1} - \rho_{IF} \qquad \frac{40.3\,TEC}{f_{L2}^2} = \rho_{L2} - \rho_{IF}$$

$\rho_{IF} \approx 3\times$ "noisier"

multiplying all other noise

$$\phi_f^j = \frac{r - I_f^j + T_\phi^j}{\lambda} + c(b - b^j) + N_f + \nu_\phi$$

$$I_{L1} = \frac{f_{L2}^2}{f_{L1}^2 - f_{L2}^2}\left[\lambda_{L1}(\phi_{L1} - N_{L1}) - \lambda_{L2}(\phi_{L2} - N_{L2})\right]$$

- can't solve for I_{L1} from ϕ
  - ↳ but if $\dot{N} = 0$, you can monitor $\dot{I}_{L1}$

Troposhere Delay

- Delay due to dry gasses and water vapor
  - ↳ affected by weather

$$\eta = \eta_D + \eta_W$$

$$\hat{T} = \hat{T}_{zD} m_D(el) + \hat{T}_{zW} m_W(el)$$

m mapping as a function of sv elevation (same as OF for ionosphere)

- see book for T_{zD}, T_{zW}, + m models

Reciever Measurement Error

- Measure ≈ ½% - 1% of cycle
  - ϕ (19 cm) ⇒ 1-2 mm
  - ρ (300 m) ⇒ 1.5-3 m
  - Multipath ≈ 1 m in benign
    - ≈ 5-10 m in refractive environment

	Bias (m)	Random (m)	Total (m)
SV clock	2.0	0.7	2.1
Ephemeris	2.1	0	2.1
Iono	4.0	0.5	4.0
Tropo	0.5	0.5	0.7
Meas	0.5	0.2	0.5
Multipath	1.0	1.0	1.4
UERE	5.1	1.4	5.3
Filtered UERE	5.1	0.4	5.1