# Homework 1

## 1 Part 1

1-1 Given that 1 minute of latitude is approximately equal to 1 nautical mile (1852 meters), how many significant digits after the decimal must be included for a latitude represented in degrees to describe a fix that is accurate to 1 cm? How many significant digits are required after the decimal in the arc-seconds field if the latitude is represented in degrees, arc-minutes, and arc-seconds to describe a fix that is accurate to 1 cm? Note: 1 degree = 60 arc-minutes, 1 arc-minute = 60 arc-seconds (you may find arc-minutes referred to as 'minutes' and arc-seconds referred to as 'seconds').

**Solution:**

$$\begin{aligned} dms &= 0^\circ \quad 1\ \text{min} \quad 0\ \text{s} \\ deg &= 0^\circ + \frac{1\ \text{min}}{60\ \text{s}} + \frac{0\ \text{s}}{3600\ \text{s}} = 0.0167\,^\circ \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{dms_{1cm}}{0.01\ \text{m}} &= \frac{1\ \text{min}}{1852\ \text{m}} * \frac{60\ \text{s}}{1\ \text{min}} \\ dms_{1cm} &= 3(10^{-4})\ \text{s} \end{aligned} \tag{2}$$

$$\begin{aligned} \frac{deg_{1cm}}{0.01\ \text{m}} &= \frac{0.0167\,^\circ}{1852\ \text{m}} \\ dms_{1cm} &= 9(10^{-8})^\circ \end{aligned} \tag{3}$$

As shown, the decimal precision required for sub-centimeter accuracy are 4 and 8 decimal places in DMS and decimal degrees respectively.

1-2 Suppose you start from location 45° N, 120° W (lat, long) and fly at an altitude of 10 km with ground speed of 885 km/hr for eight hours at a constant heading of 45° from true north. Where would you end up? Assume a spherical earth with radius of 6371 km. Two helpful notes: (i) Ground speed in aviation means the speed of an aircraft relative to the surface of the earth (to be distinguished from air speed, which means the speed of an aircraft relative to its surrounding air mass). (ii) It's safe to say that you'll end up pretty far north. Don't worry too much about precision -you get full credit if your answer is within a kilometer of the exact answer.

**Solution:**

Using eq. 2.111 from Groves "Principles of GNSS, Inertial, and Multisensor Integrated Navigation" simplified with the assumption that the radius of the Earth is constant (spherical):

$$\begin{aligned} \dot{L} &= \frac{v_{north}}{R+h} \\ \dot{\lambda} &= \frac{v_{east}}{(R+h)\cos L} \\ \dot{h} &= v_{down} \end{aligned} \tag{4}$$

Where:

$$\begin{aligned} v &= 885\frac{\text{km}}{\text{hr}} * \frac{1000\ \text{m}}{1\ \text{km}} * \frac{1\ \text{hr}}{3600\ \text{s}} = 245.83\frac{\text{m}}{\text{s}} \\ v_{north} &= v * \sin 45 \\ v_{east} &= v * \cos 45 \\ v_{down} &= 0 \end{aligned} \tag{5}$$

Applying euler integration starting at llh = [45, -120, 10000] with a dt = 1 s:

$$\begin{aligned} L_{k+1} &= L_k + dt * \dot{L}_k \\ \lambda_{k+1} &= \lambda_k + dt * \dot{\lambda}_k \\ h_{k+1} &= h_k + dt * \dot{h}_k \end{aligned} \tag{6}$$

$$llh_{final} = \begin{bmatrix} 89.951535 \\ 123.494191 \\ 10000 \end{bmatrix} \tag{7}$$

Figure 1: Path of the Aircraft.

1-3 An aircraft is carrying a transmitter that is broadcasting a single tone at 100 MHz. The aircraft flies away from you on a straight line at constant altitude with ground speed of 360 km/hr. You measure the following Doppler shifts from 100 MHz spaced 0.1 s apart: -33.1679 Hz, -33.1711 Hz, and -33.1743 Hz. Determine the range rates in m/s that correspond to these Doppler measurements. In the figure below, the aircraft altitude is $Y_0$: and its horizontal distance from observer at the three instants of Doppler measurements is shown as $X_0$, $X_1$, and $X_2$, respectively. Set up two linear equations that relate $X_1$ and $X_2$, to $X_0$. Can you set up two non-linear equations that relate $X_0$ and $Y_0$ to the measurements? For extra credit, solve the equations iteratively.

**Solution:**

$$\begin{aligned} f_R &= f_T(1 - \frac{\dot{r}}{v_s}) \\ \dot{r} &= \frac{v_s}{f_T}(f_R - f_T) \end{aligned} \tag{8}$$

Substituting in $v_s = 3(10^8)$ m/s (speed of light), $f_T = 100(10^6)$ Hz, and $f_R - f_T =$ the measured doppler shifts:

$$\dot{r} = \begin{bmatrix} 99.503700 \\ 99.513300 \\ 99.522900 \end{bmatrix} \frac{\text{m}}{\text{s}} \tag{9}$$

The two linear equations are the equations for a constant velocity vehicle with $v = 360$ km/hr $= 100$ m/s.

$$\begin{aligned} x_1 &= x_0 + v * dt = x_0 + (360 * \frac{1000}{3600}) * 0.1 = x_0 + 10 \\ x_2 &= x_0 + v * dt = x_0 + (360 * \frac{1000}{3600}) * 0.2 = x_0 + 20 \end{aligned} \tag{10}$$

The nonlinear equations were derived as followed:

$$\begin{aligned} r_0 &= \sqrt{x_0^2 + y_0^2} \\ r_0^2 &= x_0^2 + y_0^2 \\ 2r_0\dot{r_0} &= 2x_0\dot{x_0} + 2y_0\dot{y_0} \\ \dot{r_0} &= \frac{x_0\dot{x_0}}{r_0} = \frac{x_0 v}{\sqrt{x_0^2 + y_0^2}} \end{aligned} \tag{11}$$

Similarly the equations for $\dot{r_1}$ and $\dot{r_2}$ as functions of $x_0$ and $y_0$:

$$\begin{aligned} \dot{r_1} &= \frac{x_1\dot{x_1}}{r_1} = \frac{(x_0 + 10)v}{\sqrt{(x_0 + 10)^2 + y_0^2}} \\ \dot{r_2} &= \frac{x_2\dot{x_2}}{r_2} = \frac{(x_0 + 20)v}{\sqrt{(x_0 + 20)^2 + y_0^2}} \end{aligned} \tag{12}$$

To solve for $X_0$ and $Y_0$ iteratively, a jacobian matrix and measurement vector must be made a follows:

$$\begin{aligned} \dot{r_0} &= \hat{\dot{r_0}} + \frac{\partial \dot{r}}{\partial y_0}(y_0 - \hat{y}) + \frac{\partial \dot{r}}{\partial x_0}(x_0 - \hat{x}) \\ \dot{r_1} &= \hat{\dot{r_1}} + \frac{\partial \dot{r}}{\partial y_1}(y_1 - \hat{y}) + \frac{\partial \dot{r}}{\partial x_1}(x_1 - \hat{x}) \\ \dot{r_2} &= \hat{\dot{r_0}} + \frac{\partial \dot{r}}{\partial y_2}(y_2 - \hat{y}) + \frac{\partial \dot{r}}{\partial x_2}(x_1 - \hat{x}) \end{aligned} \tag{13}$$

Using both the linear and nonlinear equations, the resulting system is:

$$H = \begin{bmatrix} \frac{100y_0^2}{(x_0^2+y_0^2)^{\frac{3}{2}}} & \frac{-100x_0y_0}{(x_0^2+y_0^2)^{\frac{3}{2}}} \\ \frac{100y_0^2}{((x_0+10)^2+y_0^2)^{\frac{3}{2}}} & \frac{-100(x_0+10)y_0}{((x_0+10)^2+y_0^2)^{\frac{3}{2}}} \\ \frac{100y_0^2}{((x_0+20)^2+y_0^2)^{\frac{3}{2}}} & \frac{-100(x_0+20)y_0}{((x_0+20)^2+y_0^2)^{\frac{3}{2}}} \end{bmatrix}$$

$$y = \dot{r} - \hat{\dot{r}} = \begin{bmatrix} 99.503700 - \hat{\dot{r}}_0 \\ 99.513300 - \hat{\dot{r}}_1 \\ 99.522900 - \hat{\dot{r}}_2 \end{bmatrix} \tag{14}$$

$$\Delta x = \begin{bmatrix} x_0 - \hat{x_0} \\ y_0 - \hat{y_0} \end{bmatrix}$$

$$\begin{aligned} y &= H\Delta x \\ \Delta x &= (H^T H)^{-1} H^T y \end{aligned}$$

This can be iteratively solved by consecutively adding $\Delta x$ to the current estimate of $x$ until the error is considered negligible.

1-4 A pseudolite (short for pseudo-satellite) consists of a generator of a GPS-like signal and a transmitter. Pseudolites are used to augment the GPS signals. Suppose an observer is constrained to be on the line joining two pseudolites PL1 and PL2, which are separated by 1 km. The pseudolite clocks are perfectly synchronized but the observer's clock may have an unknown bias with respect to the pseudolite clocks. Estimate the observer's position and clock bias given that the pseudoranges to PL1 and PL2 are (a) 550 m and 500 m, respectively, and (b) 400 m and 1400 m, respectively.

**Solution:**

Using the equation for psuedoranges:

$$\rho_1 = \sqrt{(x_1 - x)^2} - b = \sqrt{(0 - x)^2} - b = \sqrt{x^2} - b$$
$$\rho_2 = \sqrt{(x_2 - x)^2} - b = \sqrt{(1000 - x)^2} - b \tag{15}$$

$$-x + b = 0 - \rho_1$$
$$x + b = 1000 - \rho_1$$

Solving a system of two linear equations:

$$\begin{bmatrix} 0 - \rho_1 \\ 1000 - \rho_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix}$$
$$\begin{bmatrix} x \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 - \rho_1 \\ 1000 - \rho_2 \end{bmatrix}$$
$$\begin{bmatrix} x \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 - 550 \\ 1000 - 500 \end{bmatrix} = \begin{bmatrix} 525 \\ -25 \end{bmatrix} \text{m} \tag{16}$$
$$\begin{bmatrix} x \\ b \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 - 400 \\ 1000 - 1400 \end{bmatrix} = \begin{bmatrix} 0 \\ -400 \end{bmatrix} \text{m}$$

# 2 Part 2

Generate two random sequences of length 100 and randomly comprised of +1 and -1.

(a) Plot the histogram on each sequence.
(b) Plot the spectral analysis on each sequence.
(c) Plot the autocorrelation of each sequence with itself.
(d) Plot the cross correlation between the two sequences.
(e) BONUS: Repeat for a sequence that is 1000 long and compare to the above.

**Solution:**

The two sequences were created using the following MATLAB commands:

$$seq1 = 2 * \mathsf{ceil}(\mathsf{rand}(100, 1) - 0.5) - 1$$
$$seq2 = 2 * \mathsf{ceil}(0.1 * \mathsf{randn}(100, 1)) - 1 \tag{17}$$

Resulting in the following histograms:

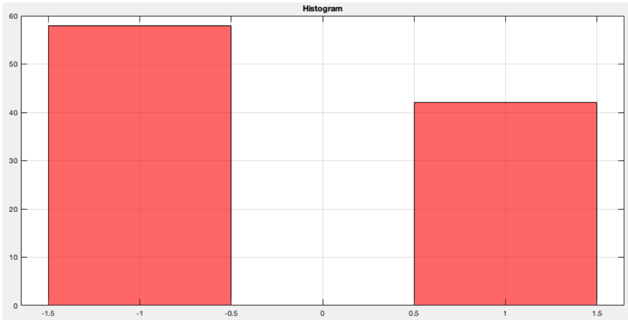


Figure 2: Sequence 1 Histogram.

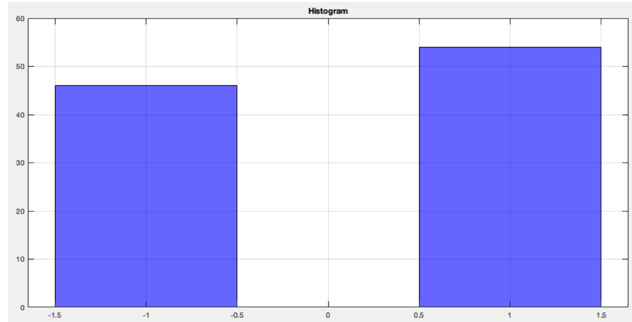


Figure 3: Sequence 2 Histogram.

Using the MATLAB command $abs(fft(seq))$ to get the power spectral density of each sequence and plotting the output:

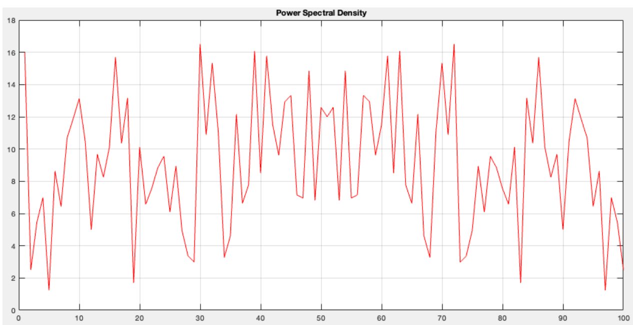


Figure 4: Sequence 1 PSD.

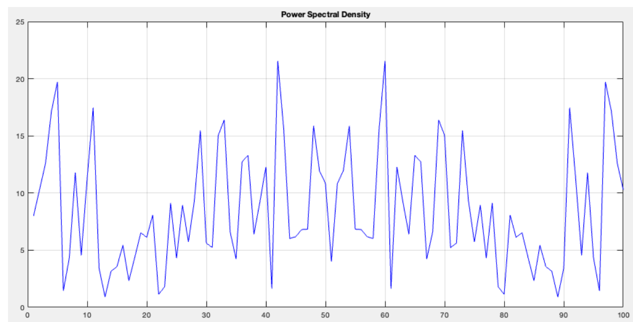


Figure 5: Sequence 2 PSD.

Next a MATLAB function was made for the correlation of different sequences.

```
function [R, shifts] = correlation(seq1, seq2)
  if size(seq1) ~= size(seq2)
    fprintf('ERROR: "correlation" -> sequneces not the same size!');
    return;
  end

  N = length(seq1);
  shifts = -(N-1):(N-1);
  R = zeros([length(shifts),1]);

  for k = 1:length(shifts)
    seq1_ = circshift(seq1, shifts(k));
    R(k) = 1/N * sum(seq1_.*seq2);
  end
```

```
    end
```

This function outputs the following autocorrelations, showing that each sequence is only correlated with itself when there is no shift in the sequence:

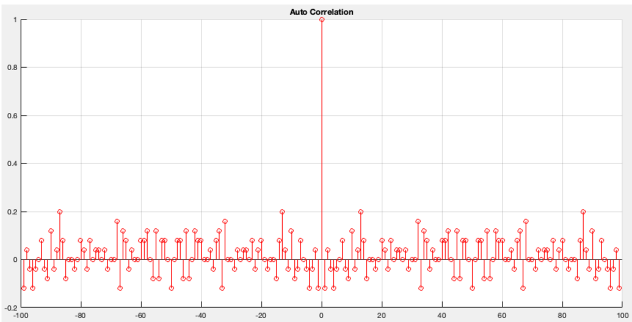


Figure 6: Sequence 1 Autocorrelation.

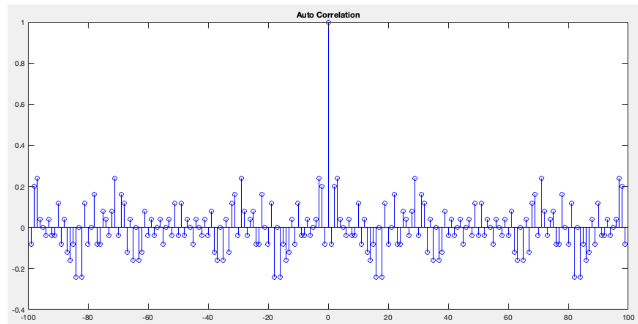


Figure 7: Sequence 2 Autocorrelation.

Using the same function to find the crosscorrelation of the sequences, there is no correlation:

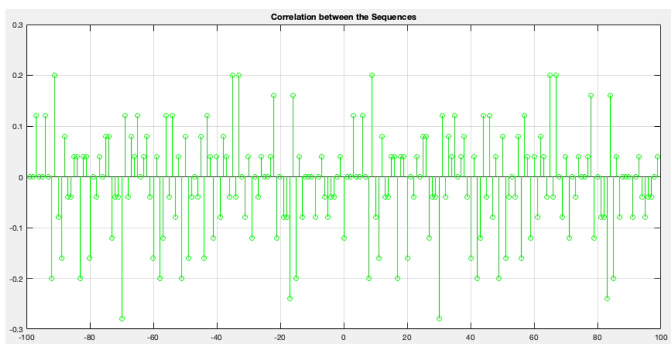


Figure 8: Crosscorrelation of the Sequences.

Repeating all the steps above for a sequence of length 1000.

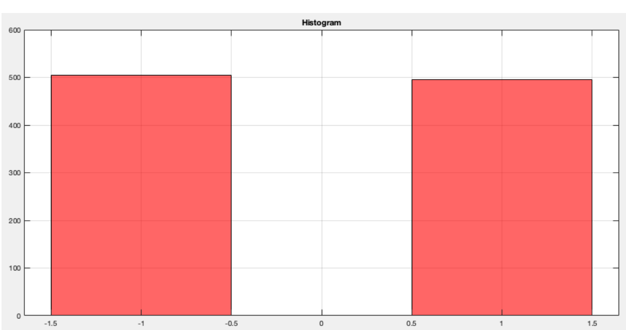


Figure 9: Sequence 1 Histogram.

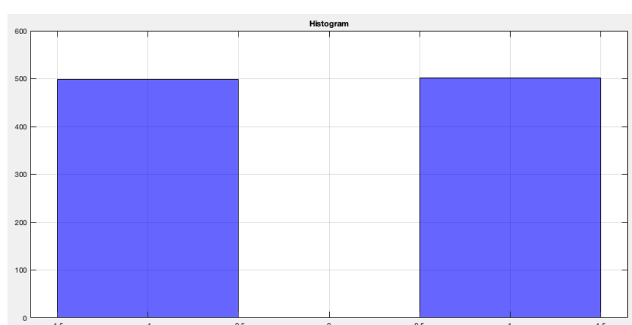


Figure 10: Sequence 2 Histogram.

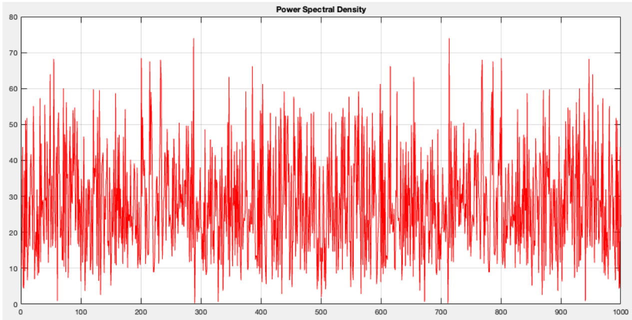


Figure 11: Sequence 1 PSD.

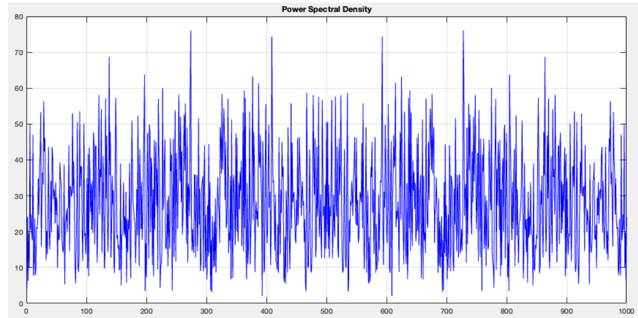


Figure 12: Sequence 2 PSD.

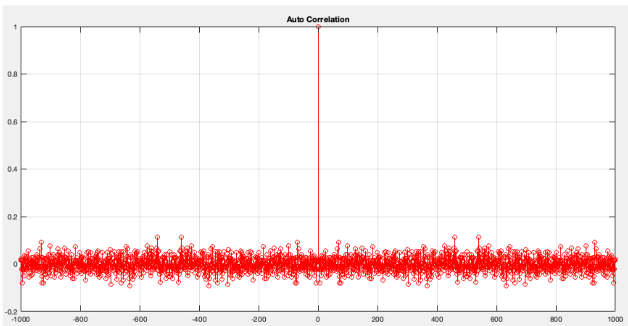


Figure 13: Sequence 1 Autocorrelation.

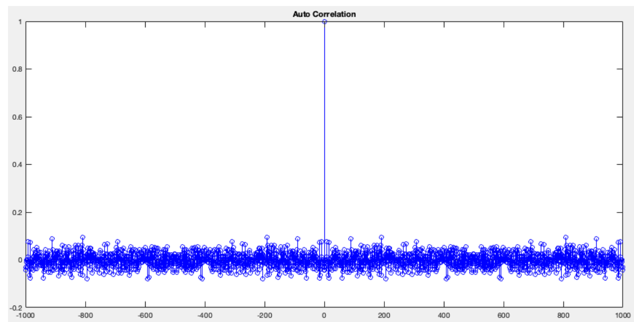


Figure 14: Sequence 2 Autocorrelation.

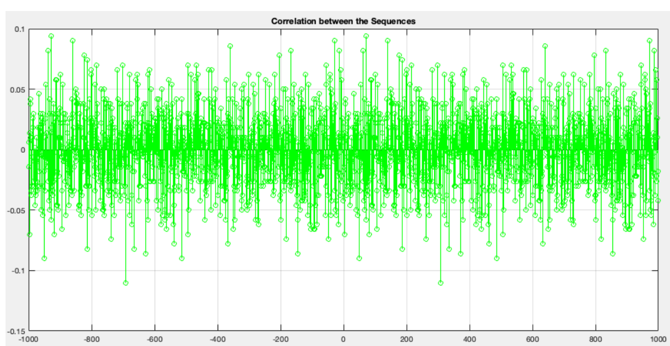


Figure 15: Crosscorrelation of the Sequences.

# 3 Part 3

Generate 3 sequences 1000 long:

- A = 3 + 3*randn(1000,1)
- B = 5 + 5*randn(1000,1)
- C = A + B
- DATA = [A B C]

(a) Find the mean and variance for A, B, and C.
(b) Find the mean of DATA.
(c) Find the covariance of DATA.

**Solution:**

The mean, variance, and covariance can be easily calculated in MATLAB using the $mean$, $var$, and $cov$ functions.

$$mean(A) = 2.995996$$
$$mean(B) = 5.080351$$
$$mean(C) = 8.076347$$
$$mean(DATA) = 5.384231$$

$$var(A) = 8.941887$$
$$var(B) = 26.113027$$
$$var(C) = 33.745355$$
$$cov(DATA) = \begin{bmatrix} 8.941887 & -0.654780 & 8.287107 \\ -0.654780 & 26.113027 & 25.458247 \\ 8.287107 & 25.458247 & 33.745355 \end{bmatrix}$$

# 4 Part 4

Develop the taylor series linearized approximation for the following equation:

$$r(x,y) = \sqrt{(x-a)^2 + (y-b)^2}$$

**Solution:**

Taking the partial derivatives with respect to x and y:

$$\begin{aligned} \frac{\partial r}{\partial x} &= \frac{x-a}{\sqrt{(x-a)^2 + (y-b)^2}} \\ \frac{\partial r}{\partial y} &= \frac{y-b}{\sqrt{(x-a)^2 + (y-b)^2}} \end{aligned} \tag{18}$$

The first order approximation for a function of x and y takes the form:

$$r(x,y) = r(x_0, y_0) + \frac{\partial r}{\partial x}\Delta x + \frac{\partial r}{\partial y}\Delta y \tag{19}$$

$$r(x,y) = \sqrt{(x_0-a)^2 + (y_0-b)^2} + \frac{(x_0-a)(x-x_0)}{\sqrt{(x_0-a)^2 + (y_0-b)^2}} + \frac{(y_0-b)(y-y_0)}{\sqrt{(x_0-a)^2 + (y_0-b)^2}} \tag{20}$$