MECH 6970: GPS | 1/20/23

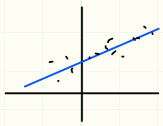
Estimation

$y = Hx + v$ ⇐ Measurement model (linear)

- Estimate x ($\hat{x}$)
- y = measurement
- v = noise / uncertainty of measurement $N(0,\sigma^2)$
- $\dot{x} = 0$ (x is constant)

With a bias ($N(b,\sigma^2)$) $y = \begin{bmatrix} H & 1 \end{bmatrix}\begin{bmatrix} x \\ b \end{bmatrix} + v$

$\boxed{\text{Ex}}$ $y = mx + b$



$x = \begin{bmatrix} m \\ b \end{bmatrix}$

$\boxed{\text{Ex}}$ $y = a_0 + b_0 t + c_0 t^2$ $\quad x = \begin{bmatrix} a_0 \\ b_0 \\ c_0 \end{bmatrix}$

Properties:
- $\frac{d}{dx}(y^T x) = y^T$
- $\frac{d}{dx}(x^T A x) = x^T A + x^T A^T$
  - If A is symmetric, $A = A^T$
- $(AB)^T = A^T B^T$

$e_y = y - \hat{y} = (Hx + v) - (H\hat{x})$

$e_y = y - H\hat{x}$

$J = \frac{1}{2} e_y e_y^T = \frac{1}{2}(y - H\hat{x})(y - H\hat{x})^T$

$J = y^T y - \hat{x}^T H^T y - y^T H \hat{x} + \hat{x}^T H^T H \hat{x}$

$\frac{\partial J}{\partial \hat{x}} = \frac{1}{2}(-2y^T H + 2\hat{x} H^T H) = 0$

$= H^T H \hat{x} - H^T y$

$\hat{x} = \underbrace{(H^T H)^{-1} H^T}_{\text{pinv}(H) \Rightarrow n \times m} y$ ⇐ least squares fit

$\frac{\partial^2 J}{\partial \hat{x}^2} = H^T H$ ⇐ Hessian
- positive definite } if H is rank n
- invertible

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Estimation

$y_{m\times 1} = H_{m\times n} x_{n\times 1} + v$ $\quad v = N(0, \sigma^2)$

x : $n\times 1$

y : $m\times 1$ measurement vector

H : $m\times n$ observation matrix

GOAL: Find estimate of x (call $\hat{x}$)
- $e_x = x - \hat{x}$ (error)
- $E\{e_x\} = 0$
- $E\{e_x^T e_x\} = P_{n\times n}$ ← minimize

you've just gotta believe

Instead calculate $e_y = y - \hat{y}$
- $E\{e_y\} = 0$
- $E\{e_y^2\} = \sigma_y^2$ ← minimize

– works IFF $y = Hx + v$ is correct model

Cost Function

$J = \frac{1}{2} e_y e_y^T = \frac{1}{2}\sum e_y^2$

- chosen because $y = x^2$ has closed form solution for minimum ($0 = 2x$)
- $\frac{\partial J}{\partial x} = 0$
- $\underbrace{\frac{\partial^2 J}{\partial x^2}}_{\text{determines minima or maxima}} = ??$ ← hessian

Weighted Least Squares

$y = Hx + v$

$E\{v_1\} = 0$

$E\{v_1^2\} = \sigma_1^2$

$E\{v_2\} = 0$

$E\{v_2^2\} = \sigma_2^2$

$E[vv^T] = R = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_m^2 \end{bmatrix}$

$J = \frac{1}{2} e_y^T R^{-1} e_y$ $\quad$ * R = measurement uncertainty

$\frac{\partial J}{\partial \hat{x}} = 0$

$\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$

* $y = Hx + v$

$E\{y = Hx + v\}$

$E\{y\} = E\{Hx + v\}$

$= E\{Hx\} - \cancel{E\{v\}}^{\,0}$

* $E(x - \hat{x}) = E\{x - (H^T R^{-1} H)^{-1} H^T R^{-1} y$

$= E\{x\} - E\{(H^T R^{-1} H)^{-1} H^T R^{-1}(Hx + v)\}$

$= E(x) - E\{\cancel{(H^T R^{-1} H)^{-1}(H^T R^{-1} H)}x + \underbrace{(H^T R^{-1} H)^{-1} H^T R}_{A} v\}$

$= \cancel{E\{x\} - E\{x\}} - E\{\cancel{Av}\}^{\,0}$

$= 0$ (Estimate is unbiased)

$E\{e_y^2\} = E\{(y-H\hat{x})^2\}$

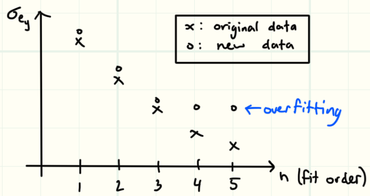
$= E\{((Hx+v)+H\hat{x})^2\}$

$= E\{\cancel{(Hx)^2} + v^2 + \cancel{(H\hat{x})^2} - \cancel{2(Hx+v)(H\hat{x})}\}$

$= E\{v^2\} = \sigma_v^2$

Checking Fit

- e_y ⇒ residual
  - 0 mean, gaussian, $\sigma = \sigma_v$
  - "white" (random), >> fft
- check fit on a new dataset
  - ensures didn't fit something specific in data values



$y = a$

$y = at + b$

$y = at^2 + bt + c$

$\vdots$

$E\{x-\hat{x}\} = 0$ → unbiased estimate

$E\{(x-\hat{x})(x-\hat{x})^T\} = E\{(x-(H^TRH)^{-1}H^TR^{-1}\underbrace{y}_{H\hat{x}+v})(x-(H^TR^{-1}H)^{-1}H^TR^{-1}y)^T\}$

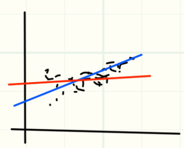
$\vdots$

$= \underbrace{(H^TR^{-1}H)^{-1}}_{n\times n} = P_{WLS}$

$\sigma^2(H^TH)^{-1} = P_{LS}$

not the accuracy
- statistical prediction of the accuracy
- does not involve the measurements

Ex



$P = \begin{bmatrix} \sigma_{\hat{m}}^2 & \# \\ \# & \sigma_{\hat{b}}^2 \end{bmatrix}$ $\quad H = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}$

$y = mx + b$

$\hat{m}$ and $\hat{b}$ are correlated (error in one results in error in other)

1/30/23

$\hat{x}_{WLS} = \underbrace{(H^TR^{-1}H)^{-1}H^TR^{-1}}_{A}\underbrace{y}_{Hx+v}$ $\qquad \hat{x}_{LS} = (H^TH)^{-1}H^Ty$

$E\{(x-\hat{x})(x-\hat{x})^T\} = E\{(x-A(Hx+v))(x-A(Hx+v))^T\}$

$= E\{(x-x-Av)(x-x-Av)^T\}$

$= E\{(Av)(Av)^T\}$

$= E\{Avv^TA^T\}$

$= A\underbrace{E\{vv^T\}}_{R}A^T$

$= [(H^TR^{-1}H)^{-1}H^TR^{-1}]R[(H^TR^{-1}H)^{-1}H^TR^{-1}]^T$

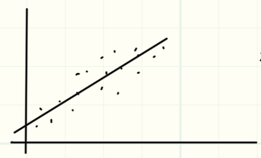
$= [(H^TR^{-1}H)^{-1}H^TR^{-1}]\cancel{R}[\cancel{R^{-1}}H(H^TR^{-1}H)^{-1}]$

$= \cancel{(H^TR^{-1}H)^{-1}H^TR^{-1}H}(H^TR^{-1}H)^{-1}$

$P_{WLS} = (H^TR^{-1}H)^{-1} = \begin{bmatrix} \sigma^2_{x_1-\hat{x}_1} & & & \\ & \sigma^2_{x_2-\hat{x}_2} & & \\ & & \ddots & \\ & & & \sigma^2_{x_n-\hat{x}_n} \end{bmatrix}$

$P_{LS} = \sigma^2(H^TH)^{-1}$

Ex



$\hat{x} = \begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix}$ $\qquad P = \begin{bmatrix} 2 & 0.5 \\ 0.5 & 4 \end{bmatrix}$

$\sigma^2_{m-\hat{m}} = \sqrt{2}$

$\sigma^2_{b-\hat{b}} = \sqrt{4} = 2$

$\rho(\sqrt{2})(2) = \frac{1}{2} \Rightarrow \rho = \frac{1}{4\sqrt{2}}$

$H = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \\ x_n & 1 \end{bmatrix}$

- σ^2 : more accurate meas.
- $(H^TH)^{-1}$: more meas., spaced farther apart

Ex Table Length

$y = \ell + v \Rightarrow H = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$ $\qquad E\{v^Tv\} = \sigma^2$

$\hat{x} = \hat{\ell} = (H^TH)^{-1}H^Ty = \frac{1}{n}\sum y$

$P = \sigma^2_{\ell-\hat{\ell}} = \sigma^2(H^TH)^{-1} = \frac{\sigma^2}{n}$

* $\sigma = 1$, $n = 10$
  $\sigma^2_{\ell-\hat{\ell}} = \frac{1}{10}$

$\sigma_{AVG} = \frac{\sigma_{MEAS}}{\sqrt{n}}$

Ex

$y = x^2 + v$

$\cancel{y = \begin{bmatrix} x & 1 \end{bmatrix}\begin{bmatrix} \hat{m} \\ \hat{b} \end{bmatrix} + v}$

Finding Solutions to NL functions

Newton-Raphson Method search for non-linear solution using slope to get next iteration

- More generically linearize the NL function to iteratively find solution

① Choose $\hat{x}_k$
② Linearize about $\hat{x}_k$
③ Solve for $\delta\hat{x}$
④ $\hat{x}_{k+1} = \hat{x}_k + \delta\hat{x}$
⑤ repeat until $\delta\hat{x}$ < tolerance

$$f(x) \approx f(x_0) + \frac{\partial f}{\partial x}\Big|_{x=x_0}(x-x_0)$$

$$f(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}\Big|_{\substack{x=x_0\\y=y_0}}(x-x_0) + \frac{\partial f}{\partial y}\Big|_{\substack{x=x_0\\y=y_0}}(y-y_0)$$

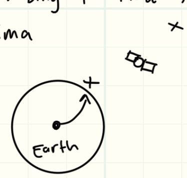
$\boxed{\text{Ex}}$ $y = x^2 + 2x + 5$

$$y \approx \hat{x}_k^2 + 2\hat{x}_k + 5 + (2\hat{x}_k + 2)\,\delta x$$

$$\delta x \approx \frac{y - (\hat{x}_k^2 + 2\hat{x}_k + 5)}{2\hat{x}_k + 2}$$

K	y	$\hat{x}_n$	$\delta\hat{x}_k$
1	29	3	1.125
2	29	4.125	-0.1235
3	29	4.015	-0.0015
4	29	4.000	$<10^{-4}$

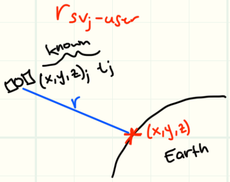
- Linearizing to find NL solution can result in Local minima



$\boxed{\text{Ex}}$ $f(x,y) = x^2 + y^2 = 10$

$$\approx (\hat{x}_k^2 + \hat{y}_k^2) + 2\hat{x}_k\,\delta x_k + 2\hat{y}_k\,\delta y_k$$

$$\underbrace{\rho^j}_{\text{psuedorange}} = \underbrace{\sqrt{(x_j - x)^2 + (y_j - y)^2 + (z_j - z)^2}}_{r_{sv_j\text{-user}}} + \underbrace{b}_{c\,t_{err} = c(t_j - t)} + \nu$$



$$\rho^j = \hat{\rho}_k^j + \frac{\partial \rho^j}{\partial \hat{x}_k}\delta x_k + \frac{\partial \rho^j}{\partial \hat{y}_k}\delta y_k + \frac{\partial \rho^j}{\partial \hat{z}_k}\delta z_k + \frac{\partial \rho^j}{\partial b}\delta b$$

$\downarrow$

$$\hat{\rho}_k^j = \sqrt{(x^j - \hat{x}_k)^2 + (y^j - \hat{y}_k)^2 + (z^j - \hat{z}_k)^2} + \hat{b}_k$$

$$\frac{\partial \rho^j}{\partial \hat{x}_k} = \cancel{\tfrac{1}{2}}\Big[\underbrace{(x^j - \hat{x}_k)^2 + (y^j - \hat{y}_k)^2 + (z^j - \hat{z}_k)^2}_{r_k^j}\Big]^{-1/2} * \cancel{2}(x^j - \hat{x}_k) * -1$$

$$= \frac{-(x^j - \hat{x}_k)}{\sqrt{r_k^j}} = \frac{\hat{x}_k - x^j}{\sqrt{r_k^j}}$$

* same for y & z $\quad \frac{\partial \rho^j}{\partial \hat{y}_k} = \frac{-(y^j - \hat{z}_k)}{\sqrt{r_k^j}} \quad \frac{\partial y^j}{\partial \hat{z}_k} = \frac{-(z^j - \hat{z}_k)}{\sqrt{r_k^j}}$

$$\frac{\partial \rho^j}{\partial \hat{b}_k} = 1$$

$$\rho^j - \hat{\rho}_k^j \approx \begin{bmatrix} \frac{\partial \rho^j}{\partial \hat{x}_k} & \frac{\partial \rho^j}{\partial \hat{y}_k} & \frac{\partial \rho_j}{\partial \hat{z}_k} & 1 \end{bmatrix} \begin{bmatrix} \delta x_k \\ \delta y_k \\ \delta z_k \\ \delta b_k \end{bmatrix} + \nu$$

$$\Delta\hat{\rho}_k^j \approx \underbrace{\begin{bmatrix} \frac{-\Delta\hat{x}_k^j}{r_k^j} & \frac{-\Delta\hat{y}_k^j}{r_k^j} & \frac{-\Delta\hat{z}_k^j}{r_k^j} & 1 \end{bmatrix}}_{G} \begin{bmatrix} \delta x_k \\ \delta y_k \\ \delta z_k \\ \delta b_k \end{bmatrix} + \nu$$

$$= (G^TG)^{-1}G^T(\Delta\hat{\rho}_k^j)$$

$$\hat{x}_{k+1} = \hat{x}_k + \delta x$$