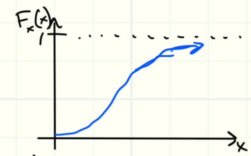
Probability Theory and Random Variables

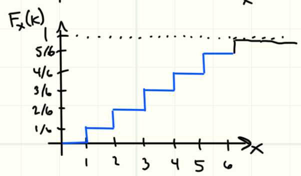
- Random variable X
  - Exact value is unpredictable
  - Characterized by statistical measures

Cummulative Density Function $F_x(x)$ a function of x that describes the probability of X being less than a number

- $F_x(x) = P\{X \le x\}$
- $F_x(b) - F_x(a) = P\{a \le X \le b\}$



- monotonically (strictly) increasing
- continuous



- dice roll
- discrete

1/30/23

Probability Density Functions (PDF)

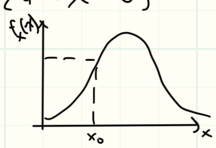
$$f_x(x) = \frac{dF_x(x)}{dx}$$

or

$$F_x(x) = \int_{-\infty}^{\infty} f_x(\eta)\,d\eta \quad \text{(over the range of } x)$$

$$F_x(b) - F_x(a) = \int_a^b f_x(\eta)\,d\eta = P\{a < X < b\}$$

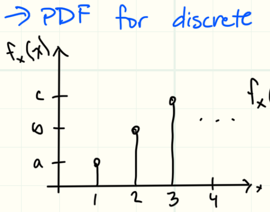
$$P\{x = x_0\} = \lim_{\Delta x \to 0} \int_{x_0}^{x_0+\Delta x} f_x(\eta)\,d\eta$$



Properties: ① $f_x(x) \ge 0 \quad -\infty < x < \infty$
② $\int_{-\infty}^{\infty} f_x(\eta)\,d\eta = 1$

Probability Mass Function

→ PDF for discrete random variable



$$f_x(x) = a\,\delta(x-1) + b\,\delta(x-2) + c\,\delta(x-3) + \dots$$

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & \text{otherwise} \end{cases}$$

- PDF's are difficult to obtain / derive
- Statistical properties can model PDFs

① Mean $E\{x\} = \bar{x}$

expectation, mean (1st moment)

$$E\{x\} \triangleq \int_{-\infty}^{\infty} \eta\, f_x(\eta)\,d\eta$$

$\boxed{Ex}$ Sum of 2 rolled dice

$f_x(x) =$

2	3	4	5	6	7	8	9	10	11	12
1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

$$\bar{x} = 2\left(\tfrac{1}{36}\right) + 3\left(\tfrac{2}{36}\right) + \dots + 12\left(\tfrac{1}{36}\right) = 7$$

Expectation Properties:

$$E\{aX\} = a\,E\{x\}$$

$$E\{g(x)\} = \int_{-\infty}^{\infty} g(\eta)\, f_x(\eta)\,d\eta = \sum g(k)\, f_x(k)$$

$$E\{g(x) + c\} = E\{g(x)\} + c$$

② Variance $E\{(x-\bar{x})^2\} = \sigma^2$

- $var(x) = \int_{-\infty}^{\infty} (\eta - \bar{x})^2 f_x(\eta)\,d\eta = \sum (k-\bar{x})^2 f_x(k)$

- For scalar random variable x:

$$E\{(x-\bar{x})^2\} = E\{x^2 - 2x\bar{x} - \bar{x}^2\}$$
$$= E\{x^2\} - E\{2x\bar{x}\} + \bar{x}^2$$

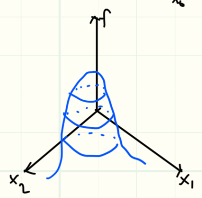
Vector of Random Variables

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

- $n \times 1$
- n = number of variables

Joint Probability $P\{x_0 \le x \le x_0 + \Delta x\}$

$$= \int_{x_0}^{x_0+\Delta x} f_x(\eta)\, d\eta_1 d\eta_2 \cdots$$



Properties : ① $f_x(x) \ge 0$

② Normalized $\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_x(\eta)\, d\eta_1 \cdots d\eta_n = 1$

Joint Statistics $\bar{x}_{n\times 1} = E\{x_{n\times 1}\}$

VECTOR MEAN

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \vec{\eta}\, f_x(\eta)\, d\eta_1 \cdots d\eta_n$$

NOTE: $\vec{x} \sim$ Random Vector

$\vec{y} = A\vec{x}B + C$

$E\{\vec{y}\} = E\{A\vec{x}B + C\} = A E\{\vec{x}\} B + C$

$\boxed{EX}$ Two dice

$D_1 = 1, 1, 3, 4, 4, 6$

$D_2 = 2, 2, 6, 7, 8, 9$

$$\begin{array}{c|ccccc|c}
D_1 \backslash D_2 & 2 & 6 & 7 & 8 & 9 & \\
\hline
1 & 4/36 & 2/36 & 2/36 & 2/36 & 2/36 & \Sigma = \frac{1}{3} \\
3 & 2/36 & 1/36 & 1/36 & 1/36 & 1/36 & \Sigma = \frac{1}{6} \\
4 & 4/36 & 2/36 & 2/36 & 2/36 & 2/36 & \Sigma = \frac{1}{3} \\
6 & 2/36 & 1/36 & 1/36 & 1/36 & 1/36 & \Sigma = \frac{1}{36} \\
\hline
 & \Sigma = \frac{1}{3} & \Sigma = \frac{1}{6} & \Sigma = \frac{1}{6} & \Sigma = \frac{1}{6} & \Sigma = \frac{1}{6} &
\end{array}$$

4×5 } $f_{D_1 D_2}$

Marginal Density PDF of any value in vector

- given 2×1 $f_{xy}(x,y)$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\, dy = \sum_{k=1}^{m} f_{xy}(x, y_k)$$

$$\bar{x} = E\begin{Bmatrix} D_1 \\ D_2 \end{Bmatrix} = \begin{bmatrix} E\{D_1\} \\ E\{D_2\} \end{bmatrix} = \begin{bmatrix} \sum_k k\, f_{D_1}(k) \\ \sum_k k\, f_{D_2}(k) \end{bmatrix} = \begin{bmatrix} \sum_k k \cdot \sum_j f_{D_1 D_2}(D_1, j) \\ \sum_k k \cdot \sum_j f_{D_1 D_2}(j, D_2) \end{bmatrix}$$

$$E\{D_1\} = \begin{bmatrix} 1 & 3 & 4 & 6 \end{bmatrix} \overbrace{f_{D_1 D_2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}^{f_{D_1}(k)}$$

Covariance

- $n \times n$ matrix

$$P = P\{(x - \bar{x})_{n\times 1} (x - \bar{x})^T_{1\times n}\}$$

$$= \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (x-\bar{x})(x-\bar{x})^T d\eta_1 \cdots d\eta_n$$

Properties : ① Symmetric $P = P^T$

② Positive Semi-Definite

- $Y^T P Y \ge 0 \quad \forall\ Y \ne 0$

③ Diagonal is var(x)

- $P_{ii} = \sigma_i^2$

④ Off-Diagonals relate the variables

- $P_{ij} = \rho_{ij} \sigma_i \sigma_j$
- Correlation Coefficient ρ_{ij} , $|\rho_{ij}| \le 1$

$$P_{12} = cov(x_1, x_2)$$
$$= E\{(x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\}$$
$$= E\{x_1 x_2\} - E\{\overset{\bar{x}_1}{\cancel{x_1}}\}\bar{x}_2 - E\{\overset{\bar{x}_2}{\cancel{x_2}}\}\bar{x}_1 + \bar{x}_1 \bar{x}_2$$

$$P_{12} = E\{x_1 x_2\} - \bar{x}_1 \bar{x}_2$$

$$\rho_{12} = \frac{cov(x_1, x_2)}{\sqrt{\sigma_1^2 \sigma_2^2}} = \frac{E\{(x_1 - \bar{x})(x_2 - \bar{x}_2)\}}{\sigma_1 \sigma_2} = \frac{E\{x_1 x_2\} - \bar{x}_1 \bar{x}_2}{\sigma_1 \sigma_2}$$

- If $\rho_{12} = 0$, then $E\{x_1 x_2\} = \bar{x}_1 \bar{x}_2$

⑤ Linear Transformation

- $y = Ax$ $\quad (\bar{y} = A\bar{x})$ mean after linear transform
- Given x has covariance P_x , P_y?

$$P_y = E\{(y - \bar{y})(y - \bar{y})^T\}$$
$$= E\{(Ax - A\bar{x})(Ax - A\bar{x})^T\}$$
$$= E\{A(x - \bar{x})(x - \bar{x})^T A^T\}$$

$$P_y = A P_x A^T$$

Ex Given 2 random variables

- $V_1 \sim \mathcal{N}(\bar{V}_1, \sigma_1^2)$
- $V_2 \sim \mathcal{N}(\bar{V}_2, \sigma_2^2)$
- V_1 and V_2 are uncorrelated
- $X = \begin{bmatrix} V_1 \\ V_2 - V_1 \end{bmatrix}$
- Find P_x and $f_{x_1 x_2}$

$$P_x = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$$

$$P_{11} = E\{(x_1-\bar{x}_1)^2\} = E\{(V_1-\bar{V}_1)^2\} = \sigma_1^2$$

$$\begin{aligned} P_{22} &= E\{(x_2-\bar{x}_2)^2\} = E\{[(V_2-V_1)-(\bar{V}_2-\bar{V}_1)]^2\} \\ &= E\{(V_2-V_1-\bar{V}_2+\bar{V}_1)^2\} \\ &= E\{V_2V_2 - \cancel{V_2V_1} - V_2\bar{V}_2 + \cancel{V_2\bar{V}_1} \\ &\quad - \cancel{V_1V_2} + V_1V_1 + \cancel{V_1\bar{V}_2} - V_1\bar{V}_1 \\ &\quad - \bar{V}_2V_2 + \cancel{\bar{V}_2V_1} + \bar{V}_2\bar{V}_2 - \cancel{\bar{V}_2\bar{V}_1} \\ &\quad + \cancel{\bar{V}_1V_2} - \bar{V}_1V_1 - \cancel{\bar{V}_1\bar{V}_2} + \bar{V}_1\bar{V}_1\} \\ &= E\{V_2^2 - \bar{V}_2^2 + V_1^2 - \bar{V}_1^2\} \\ &= \underbrace{E\{V_2^2\} - \bar{V}_2^2} + \underbrace{E\{V_1^2\} - \bar{V}_1^2} \\ &= \sigma_2^2 + \sigma_1^2 \end{aligned}$$

* uncorrelated

* deterministic

$$\begin{aligned} P_{12} &= E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\} = E\{(V_1-\bar{V}_1)[(V_2-V_1)-(\bar{V}_2-\bar{V}_1)]\} \\ &= E\{\cancel{V_1V_2} - V_1^2 - \cancel{V_1\bar{V}_2} + V_1\bar{V}_1 - \cancel{\bar{V}_1V_2} + \bar{V}_1V_1 + \cancel{\bar{V}_1\bar{V}_2} + \bar{V}_1^2\} \\ &= E\{-V_1^2 + \bar{V}_1^2\} \\ &= -\sigma_1^2 \end{aligned}$$

$$\boxed{P_x = \begin{bmatrix} \sigma_1^2 & -\sigma_1^2 \\ -\sigma_1^2 & \sigma_1^2+\sigma_2^2 \end{bmatrix}}$$

$$\rho_{12} = \frac{cov(x_1,x_2)}{\sqrt{var(x_1)\,var(x_2)}} = \frac{-\sigma_1^2}{\sqrt{\sigma_1^2(\sigma_1^2+\sigma_2^2)}} = \frac{-\sigma_1}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Joint Distributions

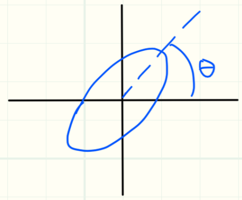
1) Multivariant Gaussian
   - Given Random Vector $X \sim \mathcal{N}(\bar{x}, P_x)$
   - $f_x(x) = \dfrac{1}{(2\pi)^{\frac{n}{2}}\det(P)^{\frac{1}{2}}} e^{-\frac{1}{2}(x-\bar{x})^T P_x^{-1}(x-\bar{x})}$
   - creates an ellipse on 2D
     - Hyperlipsoid for X dimensions
     - Axes & shape determined by P_x

Ex $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $E\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $P_x = \begin{bmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 \\ \rho_{12}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$

$$\det(P_x) = \sigma_1^2\sigma_2^2 - \rho_{12}^2\sigma_1^2\sigma_2^2 = \sigma_1^2\sigma_2^2(1-\rho_{12}^2)$$

$$C = f_x(x) = \frac{1}{(2\pi)^1\,\sigma_1\sigma_2(1-\rho_{12}^2)^{1/2}} e^{\frac{-1}{2(1-\rho_{12})^{1/2}}\left(\frac{x_1^2}{\sigma_1^2} - 2\rho_{12}\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)}$$

$$-2\ln\left[C\,2\pi\sigma_1\sigma_2(1-\rho_{12}^2)^{1/2}\right] = \frac{1}{(1-\rho_{12})^{1/2}}\left(\frac{x_1^2}{\sigma_1^2} - 2\rho_{12}\frac{x_1x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)$$



$$\tan 2\theta = \frac{2\rho_{12}\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2}$$

Intercepts: $x_1 = \pm l\sigma_1$

$x_2 = \pm l\sigma_2$

Properties of Jointly Distributed Random Variable

1) Independance
   - Strong condition
     - $f_{\underline{x}}(\underline{x}) = f_{x_1}(x_1) f_{x_2}(x_2) \dots f_{x_n}(x_n)$
   - implies uncorrelated
   - For all functions $g(\cdot)$ and $h(\cdot)$
     - $E\{g(x_1)h(x_2)\} = E\{g(x_1)\} + E\{h(x_2)\}$

2) Uncorrelated
   - weaker condition
   - not linearly correlated
   - x_1 and x_2 are uncorrelated if second moments are finite and $cov(x_1,x_2) = E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\} = 0$
   - $P_{x_1x_2} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix}$
   - uncorrelated does not imply independance

Ex Given $x \sim$ uniform$(-1,1)$ $\quad y = \begin{bmatrix} x \\ x^2 \end{bmatrix}$

$E\{y_1y_2\} = 0$ by symmetry

$$cov(y_1,y_2) = \overset{\text{symmetry}}{\cancel{E\{y_1y_2\}}} - \overset{x \text{ is uniform}}{\cancel{E\{y_1\}}}E\{y_2\} = 0 \rightarrow \text{uncorrelated}$$

$g(y_1) = y_1 \qquad h(y_2) = \sqrt{y_2} = y_1$

$E\{g(y_1)h(y_2)\} = E\{y_1^2\} = E\{x^2\} = \sigma_x^2$

$\therefore$ $E\{g(y_1)\} = 0$, $E\{g(y_1)h(y_2)\} \neq 0$, y_1 and y_2 are not independant

$\boxed{Ex}$ Consider $\left.\begin{array}{l} x_1 \sim N(0, \sigma_1^2) \\ x_2 \sim N(0, \sigma_2^2) \end{array}\right\}$ uncorrelated

$$f_x(x) = \frac{1}{2\pi\sigma_1\sigma_2(1-\cancel{\rho_{12}^2})^{1/2}} e^{\frac{-1}{2(1-\cancel{\rho_{12}^2})}\left(\frac{x_1^2}{\sigma_1^2} - 2\cancel{\rho_{12}}\frac{x_1 x_2}{\sigma_1\sigma_2} + \frac{x_2^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{2\pi\sigma_1\sigma_2} e^{\frac{-1}{2}\left(\frac{x_1^2}{\sigma_1^2} + \frac{x_2^2}{\sigma_2^2}\right)} = \frac{1}{2\pi\sigma_1\sigma_2} e^{\frac{-1}{2}\left(\frac{x_1^2}{\sigma_1^2}\right)} e^{\frac{-1}{2}\left(\frac{x_2^2}{\sigma_2^2}\right)}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{\frac{-1}{2}\left(\frac{x_1^2}{\sigma_1^2}\right)} \frac{1}{\sqrt{2\pi}\,\sigma^2} e^{\frac{-1}{2}\left(\frac{x_2^2}{\sigma_2^2}\right)}$$

$$= f_{x_1}(x_1) f_{x_2}(x_2) \rightarrow \text{independant}$$

1) If x_1 and x_2 are independant they are uncorrelated
2) If x_1 and x_2 are independant <u>and Gaussian</u> they are independant

2-8-23

3) Transforms

- Suppose x & y are R.V. with $y = g(x)$
- g is invertable such that $x = g^{-1}(y)$
- $g(\cdot)$ & $g^{-1}(y)$ are continuously differentiable

$$f_Y(y) = f_x(g^{-1}(y)) \left\| \frac{\partial g^{-1}(y)}{\partial Y} \right\|$$

$\boxed{Ex}$ $Y = kX \quad X \sim N(0, \sigma_x^2)$

$$g^{-1}(y) = X = \frac{Y}{k}$$

$$\frac{\partial g^{-1}(y)}{\partial Y} = \frac{1}{k} \quad \& \quad \left\|\frac{\partial g^{-1}(y)}{\partial Y}\right\| = \frac{1}{\|k\|}$$

$$f_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{\frac{-1}{2}\frac{x^2}{\sigma_x^2}}$$

$$f_Y(y) = \frac{1}{\|k\|}\frac{1}{\sqrt{2\pi}\,\sigma_x} e^{\frac{-1}{2}\frac{(Y/k)^2}{\sigma_x^2}} = \frac{1}{\sqrt{2\pi}\,\sigma_x\|k\|} e^{\frac{-1}{2}\frac{Y^2}{\sigma_x^2 k^2}}$$

$\therefore Y \sim N(0, k^2\sigma_x^2)$

$\boxed{Ex}$ Consider $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \begin{array}{l} x_1 \sim N(\mu_1, \sigma_1^2) \\ x_2 \sim N(\mu_2, \sigma_2^2) \end{array}$

$$P_x = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \therefore \text{independant}$$

$$Y = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} X \quad \therefore \quad X = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} Y$$

$$f_x(x) = \frac{1}{2\pi \det(P_x)^{1/2}} e^{\frac{-1}{2}\underbrace{(x-\bar{x})^T}_{1\times2}\underbrace{P_x^{-1}}_{2\times2}\underbrace{(x-\bar{x})}_{2\times1}}$$

$$f_Y(y) = f_x\left(\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} Y\right) * \underbrace{\det\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{1}$$

4) Conditional Probability

- Given Jointly Distributed R.V., conditional probability defines PDF of 1 R.V. relative to the rest

$\boxed{Ex}$ Consider $f_{XY}(x,y)$

- X conditioned on Y (X given Y)
  - ↳ $f_{X|Y}(x|y)$, probability $X=x$ given $Y=y$
    - ↳ $f_{X|Y}(x|y) \triangleq \dfrac{f_{XY}(x,y)}{f_Y(y)}$

Recall: - $f_X(x) = \int_y f_{XY}(x,\eta)\,d\eta$

- INDEPENDENT → $f_{XY}(x,y) = f_X(x) f_Y(y)$
  - ↳ $f_{X|Y}(x|y) = \dfrac{f_X(x)\cancel{f_Y(y)}}{\cancel{f_Y(y)}}$
    - ↳ If X and Y are independant, knowing Y tells nothing about X.

$$f_{XY}(x,y) = f_{X|Y}(x|y)\, f_Y(y) = f_{Y|X}(y|x)\, f_X(x)$$

BAYES RULE: $f_{X|Y}(x|y) = \dfrac{f_{Y|X}(y|x)\, f_X(x)}{f_Y(y)}$

5) Normality is Preserved through Linear Transform

- Given X, W as Independant Random Vectors

$$X \sim N(\bar{X}, P_X)$$
$$W \sim N(\bar{W}, P_W)$$
$$Z = AX + BW \Rightarrow \bar{Z} = A\bar{X} + B\bar{W}$$
$$P_Z = A P_X A^T + B P_W B^T$$
$$\therefore Z \sim N(\bar{Z}, P_Z)$$

- For sum of 2 independant R.V.

$$X + Y = Z$$

$$f_Z(z) = \int f_X(\eta)\, f_Y(z-\eta)\, d\eta$$

* Convulsion of PDF

Stochastic Processes

- Family of random vectors indexed by a parameter set (e.g. Time)
  - $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ $\quad \bar{X}(t) = \begin{bmatrix} \bar{x}_1(t) \\ \bar{x}_2(t) \\ \vdots \\ \bar{x}_n(t) \end{bmatrix}$

$P_X(t)$ → covariance at time t

Autocorrelation Matrix:

- $R_X(t_1, t_2) = E\{x(t_1)\, x(t_2)^T\}$
- Measure of smoothness

* $R_X(t_1, t_1) = E\{x(t_1)\, x(t_1)^T\} = \sigma_X^2 + \bar{x}^2$

$P_X = E\{(x(t_1) - \bar{x}(t_1))(x(t_1) - \bar{x}(t_1))^T\}$

Autocovariance Matrix:

- $R'_X(t_1, t_2) = E\{(x(t_1) - \bar{x}(t_1))(x(t_2) - \bar{x}(t_2))^T\}$

Cross Correlation:

- $R_{XY}(t_1, t_2) = E\{x(t_1)\, Y(t_2)^T\}$

Stationary Stochastic Process

1) $E\{x(t)\} = \bar{x} \quad \forall\ t$ (constant mean)

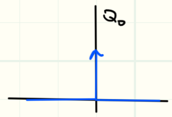
2) $E\{x(t_1)x(t_2)^T\} = E\{x(t_3)x(t_4)^T\} \quad \forall\ t_2 - t_1 = t_4 - t_3 = \tau$

- autocorrelation is invariant with time over the same Δt
- $R_X(\tau) = R_X(t_1, t_2)$

2/13/23

White Noise

- No serial (temporal) correlation
- Contains all frequencies with equal power
- Stationary
- Properties:
  1) Zero Mean
  2) $R_w(\tau) = Q_o\,\delta(\tau)$
     - $\delta(\tau) = \begin{cases} \infty & \tau = 0 \\ 0 & \text{otherwise} \end{cases}$
     - satisfies $\int_{-\infty}^{\infty} \delta(\tau)\,dt = 1$



$* \; R_x(\tau) = Q_o\,\delta(\tau) = (\sigma^2 + \cancel{\mu^2})\,\delta(\tau)$ $\quad 0$

$*$ NOTE: $\int_{-\infty}^{\infty} \delta(x-c) f(x)\,dx = f(c)$

$\int_{-\infty}^{\infty} \delta(t-c)\, e^{-j\omega t}\,dt = e^{-j\omega c}$

Wide Band Noise

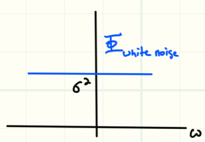
- White Noise over all frequencies we care about

Power Spectral Density

- Given stationary process with

$$R_x(\tau) = E\{x(t)\,x(t+\tau)^T\}$$

- PSD $\Phi_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$
  - Fourier transform of autocorrelation function
- Properties
  1) Real valued
  2) Symmetric about $\omega = 0$
  3) units of Power
- $R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Phi_x(\omega)\, e^{j\omega\tau}\, d\tau$



$* \; \Phi_{wn}(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-j\omega\tau}\, d\tau$

$= \int_{-\infty}^{\infty} \sigma^2 \delta(\tau)\, e^{-j\omega\tau}\, d\tau = \sigma^2$

$\boxed{\text{EX}}$ First Order Markov Process

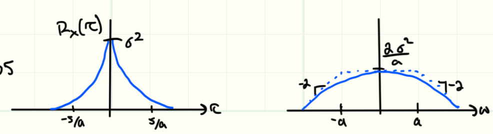
$\dot{x} + ax = w \qquad w$ (white noise) $\sim N(0, \sigma^2)$

$\frac{X(s)}{W(s)} = \frac{1}{s+a}$ $\rightarrow$ 1st order LPF

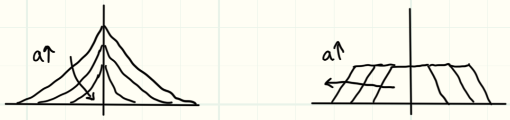
$R_x(\tau) = \sigma^2 e^{-a|\tau|}$

$\Phi_x(\omega) = \int_{-\infty}^{\infty} \sigma^2 e^{-a|\tau|}\, e^{-j\omega\tau}\, d\tau = \frac{2a\sigma^2}{\omega^2 + a^2}$

$e^{-5} \approx 0.005$



$*$ "a" effects the magnitude of the filtering



$*$ as $a \rightarrow \infty$ we get back white noise