# Homework 1

**1.** A control law for a simple rotation table is to be designed. The table has a rotational moment of inertia $(J)$ of 10 kg-m2 and rotational damping $(b)$ of 1 N-m-s/rad. Torque is commanded to the motor and the table's position is measured using a rotary encoder.

(a) Derive the simple differential equation for the system.
(b) Convert the system into a state-space format.
(c) What are the eigenvalues of the system.

**Solution:**

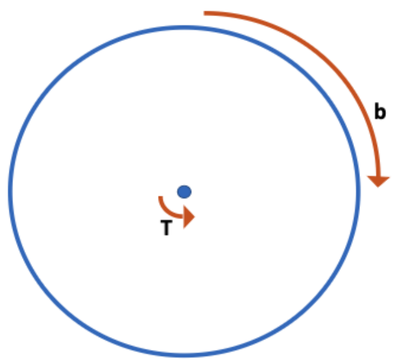


Figure 1: System Diagram.

Summing the moments on the table.

$$\begin{aligned} \sum M &= J\ddot{\theta} = T - b\dot{\theta} \\ &= 10\ddot{\theta} = T - 1\dot{\theta} \\ T &= 10\ddot{\theta} + \dot{\theta} \end{aligned} \tag{1}$$

Linearizing the system and putting it in matrix form.

$$\begin{aligned} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\ \begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} &= \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ 0.1(u - x_2) \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ 0.1(T - \dot{\theta}) \end{bmatrix} \\ y &= \theta \end{aligned} \tag{2}$$

Formulating the state space equations.

$$\dot{x} = Ax + Bu$$
$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} T \tag{3}$$

$$y = Cx$$
$$\begin{bmatrix} y \end{bmatrix} = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \tag{4}$$

Solving for the eigenvalues of the open-loop system.

$$\begin{aligned} 0 &= det(sI - A) \\ 0 &= det \begin{bmatrix} s & -1 \\ 0 & s + 0.1 \end{bmatrix} \\ 0 &= s(s + 0.1) \\ s &= 0, -0.1 \end{aligned} \tag{5}$$

**2.** Design an observer for the above system.

(a) Show that the system is observable.
(b) Design $L$ such that the error dynamics have $f_n = 50$ Hz and $\zeta = 0.7$.
(c) Provide a plot of the step response of the estimator.

**Solution:**

To determine the observability of the system, the rank of the observability matrix is checked. For this system this must equal 2 (the dimension of A).

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{6}$$

$$rank(\mathcal{O}) = 2 \tag{7}$$

The system is observable and the poles of the observer can be placed using the dynamics defined.

$$
\begin{aligned}
0 &= s^2 + 2\omega_n \zeta s + \omega_n^2 \\
0 &= s^2 + 2(2\pi f_n)\zeta s + (2\pi f_n)^2 \\
0 &= s^2 + 2(100\pi)0.7s + (100\pi)^2 \\
s &= -219.91 \pm 224.35i = s_{obsv}
\end{aligned} \tag{8}
$$

$$
L = place(A', C', s_{obsv})' = \begin{bmatrix} 439.72 \\ 98652.08 \end{bmatrix} \tag{9}
$$

The new closed-loop A matrix.

$$
A_{obsv} = A - LC = \begin{bmatrix} -439.72 & 1 \\ -98652.08 & -0.1 \end{bmatrix} \tag{10}
$$

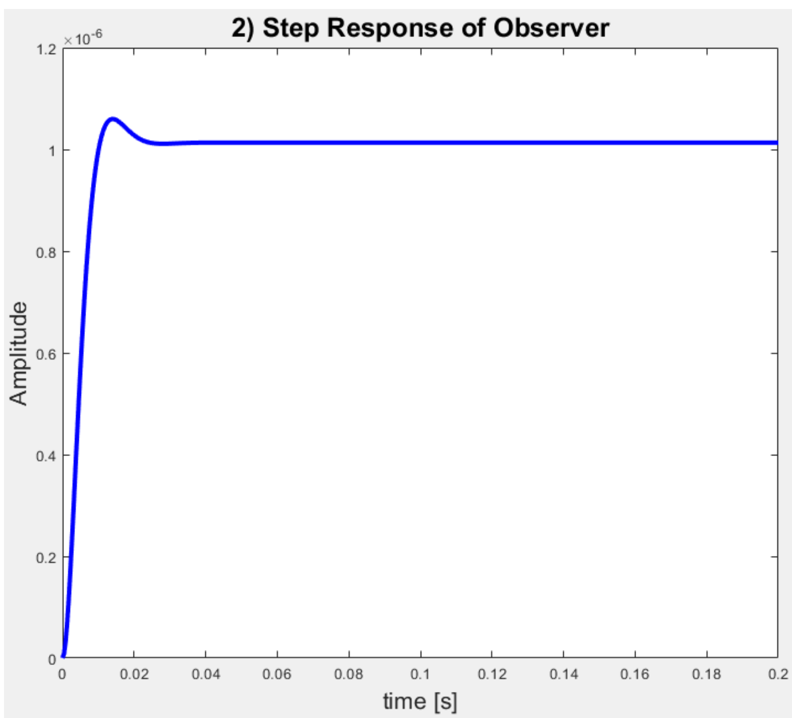


Figure 2: Observer Step Response.

**3.** Design a state-feedback controller for the table.

(a) Show that the system is controllable.

(b) Design $K$ such that $f_n = 10$ Hz and $\zeta = 0.7$.
(c) Provide a plot of the step response of the combined estimator and controller.

**Solution:**

To determine the controllability of the system, the rank of the controllability matrix is checked. For this system this must equal 2 (the dimension of A).

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & -0.01 \end{bmatrix} \tag{11}$$

$$rank(\mathcal{C}) = 2 \tag{12}$$

The system is controllable and the poles of the controller can be placed using the dynamics defined.

$$\begin{aligned} 0 &= s^2 + 2\omega_n \zeta s + \omega_n^2 \\ 0 &= s^2 + 2(2\pi f_n)\zeta s + (2\pi f_n)^2 \\ 0 &= s^2 + 2(20\pi)0.7s + (20\pi)^2 \\ s &= -43.98 \pm 44.87i = s_{cont} \end{aligned} \tag{13}$$

$$K = place(A, B, s_{cont}) = \begin{bmatrix} 39478.40 & 878.65 \end{bmatrix} \tag{14}$$

The new combined closed-loop system.

$$\begin{aligned} A &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3947.84 & -87.96 & 3947.84 & 87.86 \\ 0 & 0 & -439.72 & 1 \\ 0 & 0 & -98652.07 & -0.1 \end{bmatrix} \\ B &= \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ D &= 0 \end{aligned} \tag{15}$$

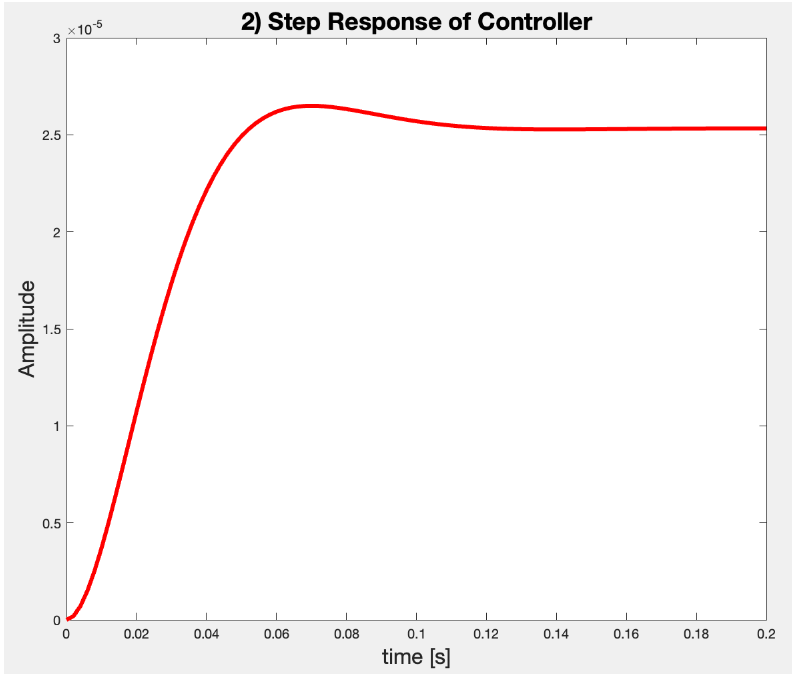


Figure 3: Combined Controller and Observer Step Response.

**4.** Solve for the equivalent compensator for the system.

(a) What kind of classical compensator does it resemble?
(b) Calculate the closed loop transfer function.
(c) Plot the Bode Plot of the closed-loop system.
(d) Find the gain and phase margin.

**Solution:**

Transforming the controller and observer defined above into an equivalent compensator results in the following system.

$$\begin{aligned} A_{comp} &= A - BK - LC = \begin{bmatrix} -439.72 & 1 \\ -102599.92 & -87.96 \end{bmatrix} \\ B_{comp} &= L = \begin{bmatrix} 439.72 \\ 98652.08 \end{bmatrix} \\ C_{comp} &= K = \begin{bmatrix} 39478.40 & 878.65 \end{bmatrix} \end{aligned} \tag{16}$$

With an equivalent transfer function of:

$$\begin{aligned}\frac{U(s)}{Y(s)} &= C_{comp}(sI - A_{comp})^{-1}B_{comp} = -K(sI - A + BK + LC)^{-1}L \\ &= \frac{-1.04(10^8)s - 3.896(10^9)}{s^2 + 527.7s + 1.413(10^5)}\end{aligned} \tag{17}$$

This resembles a ”lead-lag” compensator because of the extra pole in the denominator (first order numerator, second order denominator). The new system looks like the following when $r = 0$ which in turn makes $e = -y$ and $\frac{U(s)}{E(s)} = -\frac{U(s)}{Y(s)}$.

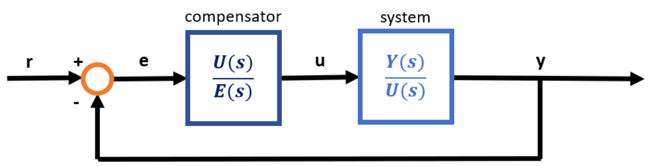


Figure 4: System diagram with Compensator.

To close the loop between the compensator and the plant the following state space system is applied.

$$\begin{aligned}A_{cl} &= \begin{bmatrix} A & BC_{comp} \\ -B_{comp}C & A_{comp} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1 & 3947.84 & 87.86 \\ -439.72 & 0 & -439.72 & 1 \\ -98652.08 & 0 & 102599.92 & -87.96 \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} 0 \\ B_{comp} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 439.72 \\ 98652.08 \end{bmatrix} \\ C_{cl} &= \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ D_{cl} &= 0\end{aligned} \tag{18}$$

This results in the following closed-loop transfer function and Bode Plot.

$$H_{cl} = \frac{1.04(10^7)s + 3.896(10^8)}{s^4 + 527.8s^3 + 1.413(10^5)s^2 + 1.042(10^7)s + 3.896(10^8)}$$

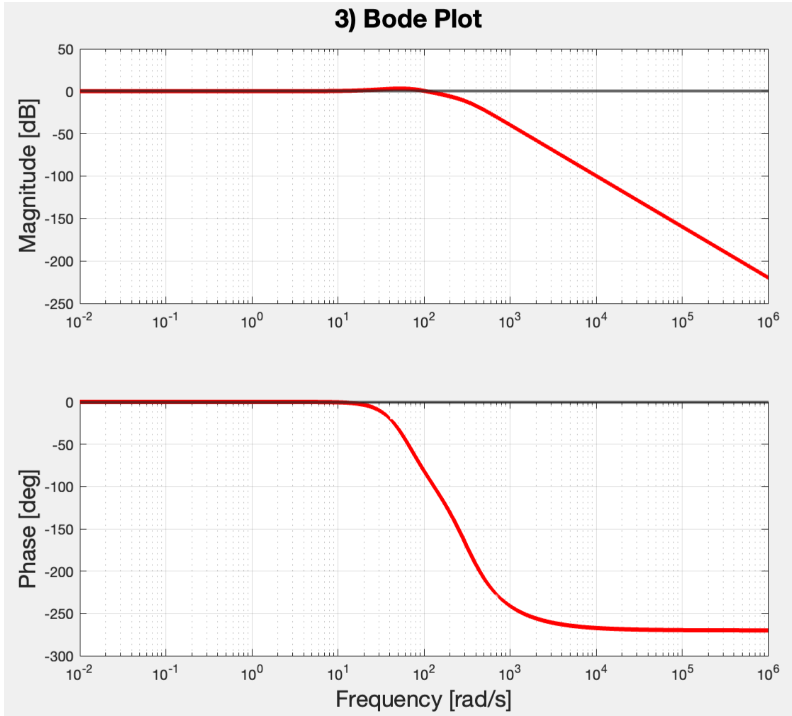


Figure 5: Closed-Loop Compensator Bode Plot.

Using the MATLAB function $margin$, the gain and phase margin are easily calculated and can be confirmed by analyzing the Bode Plot above.

$$\begin{aligned} G_m &= 14.26 \text{ dB} \\ \phi_m &= 94.69\,^\circ \end{aligned} \tag{19}$$

**5.** Design the controller in the discrete domain assuming a 1 KHz sample rate.

(a) Discretize the state space model. Where are the eigenvalues?
(b) Design L to provide the same response as problem 2.
(c) Design K to provide the same response as problem 3.
(d) Where are the closed-loop estimator and controller poles located?
(e) Solve for the equivalent compensator transfer function.

**Solution:**

To discretize the continuous model, the MATLAB function $c2d$ was used in combination with $ss$ to transform the continuous model into state-space.

$$\begin{aligned} &sys = c2d(ss(A,B,C,D),1/1000) \\ &A_z = \begin{bmatrix} 1 & 0.001 \\ 0 & 0.999 \end{bmatrix} \\ &B_z = \begin{bmatrix} 5(10^-8) \\ 1(10^-4) \end{bmatrix} \\ &C_z = \begin{bmatrix} 1 & 0 \end{bmatrix} \\ &D_z = 0 \end{aligned} \tag{20}$$

To find the discrete poles, $s_{obsv}$ and $s_{cont}$ must be converted to discrete poles.

$$\begin{aligned} &z = e^{sT} \\ &z_{obsv} = e^{s_{obsv}*T} = e^{(-219.91 \pm 224.35i)*1/1000} = 0.7825 \pm 0.1786i \\ &z_{cont} = e^{s_{cont}*T} = e^{(-43.98 \pm 44.87i)*1/1000} = 0.9560 \pm 0.0429i \end{aligned} \tag{21}$$

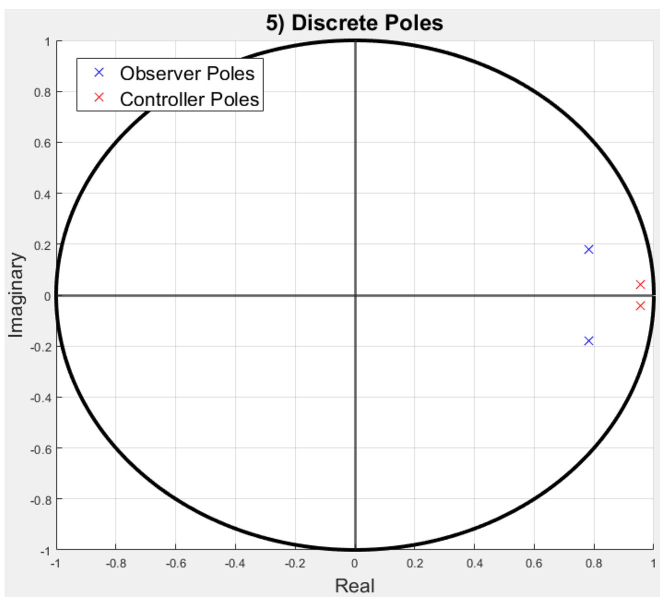


Figure 6: Discrete Pole Placement.

These new poles can then be used in the MATLAB $place$ function to create the discrete observer and controller.

$$\begin{aligned} L &= place(A_z', C_z', z_{obsv})' = \begin{bmatrix} 0.4349 \\ 79.1604 \end{bmatrix} \\ K &= place(A_z, B_z, z_{cont}) = \begin{bmatrix} 37781.3243 & 859.9995 \end{bmatrix} \end{aligned} \tag{22}$$

Using the same method of finding the compensator transfer function from above:

$$\begin{aligned} \frac{Y(z)}{U(z)} &= -K_z(zI - A_z + B_zK + L_zC_z)^{-1}L_z \\ &= \frac{-8.451(10^4)z + 8.152(10^4)}{z^2 - 1.477z + 0.594} \end{aligned} \tag{23}$$

**6.** Compare the continuous and discrete response using simulation and using equivalent compensator. Plot the simulated and equivalent compensator responses. Compare the expected and actual response.

**Solution:**
Closing the loop for the discrete compensator using $c2d$ on the continuous compensator system.

$$sys_{comp_z} = c2d(sys_{comp}, 1/1000) \tag{24}$$

The new transfer function looks like:

$$H_{cl} = \frac{0.001532z^3 + 0.003935z^2 - 0.004013z - 0.001154}{z^4 - 3.477z^3 + 4.522z^2 - 2.665z + 0.5899}$$

The continuous and discrete responses are almost identical except for the stair-stepping caused by the discretization. The only difference is that the compensated system does overshoot more than the original system.

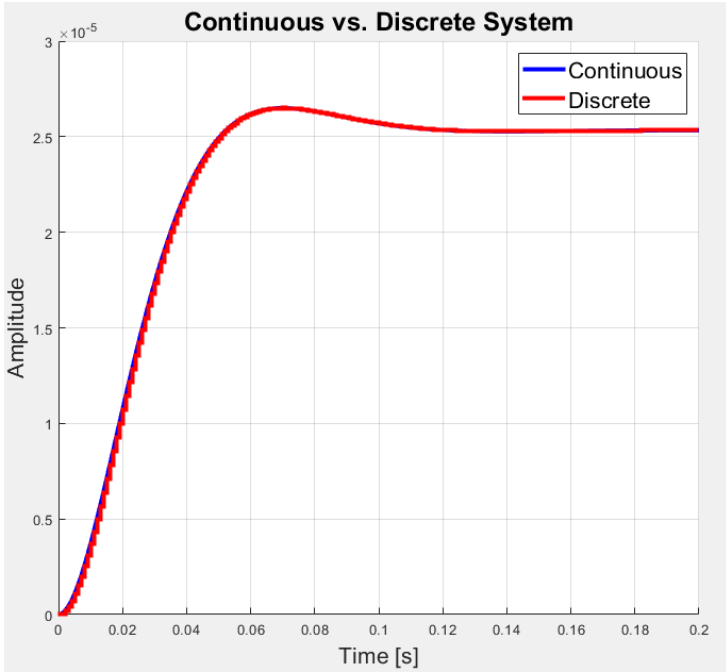


Figure 7: System Response.

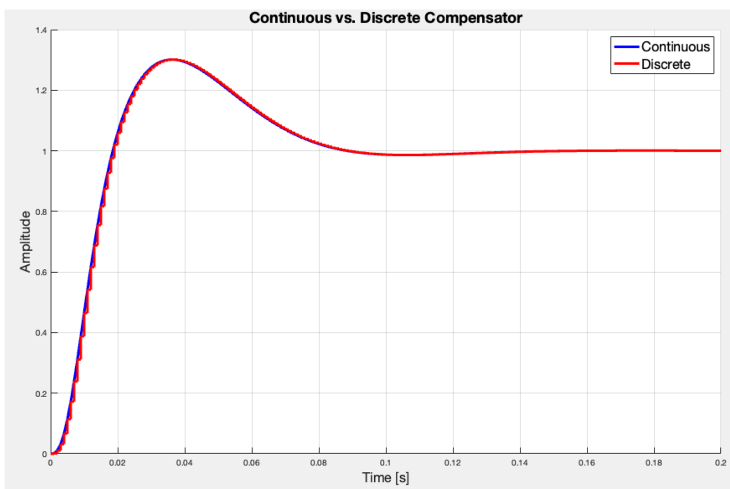


Figure 8: Equivalent Compensator Response.