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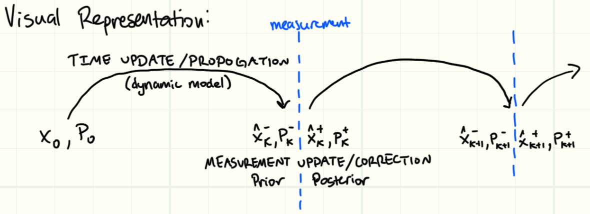
Optimal Estimator

- Discrete Model: $x_k = A_d x_{k-1} + w_{k-1}$ ← dynamic model uncertainty (PROCESS NOISE)
- Measurement Model: $y_k = C_d x_k + v_k$ ← measurement uncertainty (MEASUREMENT NOISE)
- Assumptions:
  - $E\{w_k\} = 0$
  - $E\{w_\tau w_\gamma^T\} = Q_d\,\delta(\tau-\gamma)$
  - $E\{v_k\} = 0$
  - $E\{v_\tau v_\gamma\} = R\,\delta(\tau-\gamma)$
  - $E\{w_\tau v_\gamma\} = 0$ for all τ and γ
  - $E\{x_0\} = \hat{x}_0$
  - $E\{(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\} = P_0$

Visual Representation:



- At any instance:
  - $\tilde{x}_k = x_k - \hat{x}_k$
  - $\tilde{x}_k^+ = x_k + \hat{x}_k^+$
  - $\tilde{x}_k^- = x_k - x_k^-$
- Assuming $E\{\tilde{x}_k\} = 0$: $P_k = E\{\tilde{x}_k \tilde{x}_k^T\}$

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- Assuming update form: $\hat{x}_k^+ = L_k' \hat{x}_k^- + L_k y_k$
- We want an unbiased estimator:
  - Given $E\{\hat{x}_k^-\} = 0 \Rightarrow$ Constrains $E\{\hat{x}_k^+\} = 0$

error

$$
\begin{aligned}
E\{\tilde{x}_k^+\} &= E\{x_k - \hat{x}_k^+\} = 0 \qquad y_k = C_D x_k + v_k \\
&= E\{x_k - (L_k' \hat{x}_k^- + L_k y_k)\} \\
&= E\{x_k - L_k' \hat{x}_k^- + L_k(C_d x_k + v_k)\} \\
&= E\{(I - L_k' - L_k C_d) x_k\} \\
&= \underbrace{(I - L_k' - L_k C_d)}_{\text{must } = 0} E\{x_k\}
\end{aligned}
$$

must $= 0$ ∴ $L_k' = I - L_k C_d$

State Update to guarantee unbiased estimator
$$
\begin{cases}
\hat{x}_k^+ = (I - L_k C_d)\hat{x}_k^- + L_k y_k \\
\hat{x}_k^+ = \hat{x}_k^- + L_k(y_k - C_d \hat{x}_k)
\end{cases}
$$

- Now we need P_k^+, covariance update equation, and the gain equation, L_k!
  - where 1) $P_k^+ = f(P_k^-, C_d, R_k)$
  - 2) $L_k = f(P_k^-, C_d, R_k)$
- RECALL: Find gain equation, L_k, that minimizes sum of squares of estimator error.

Cost Function: $J_k = E\{\tilde{x}_k^{+T} \tilde{x}_k^+\} = \text{TRACE}(P_k^+)$

$$
\begin{aligned}
\tilde{x}_k^+ &= x_k - \hat{x}_k^+ \\
&= x_k - (I - L_k C_d)\hat{x}_k^- - L_k(C_d x_k + v_k) \\
&= (I - L_k C_d)(x_k - \hat{x}_k^-) - L_k v_k \\
&= (I - L_k C_d)\tilde{x}_k^- - L_k v_k
\end{aligned}
$$

$$P_k^+ = E\{[(I - L_k C_d)\tilde{x}_k^- - L_k v_k][(I - L_k C_d)\tilde{x}_k^- - L_k v_k]^T\}$$

* Three Types of Terms *

1) $(I - L_k C_d)\tilde{x}_k \tilde{x}_k^T (I - L_k C_d)^T \longrightarrow (I - L_k C_d) P_k^- (I - L_k C_d)^T$
2) $L_k v_k v_k^T L_k^T \longrightarrow L_k R_k L_k^T$
3) $E\{\tilde{x}_k^- v_k\} = 0$
   - ↳ assumes old estimate error is uncorrelated with the new measurement error

$$P_k^+ = (I - L_k C_d) P_k^- (I - L_k C_d)^T + L_k R_k L_k^T$$ ← for any choice of L_k

- Find optimal $L_k \rightarrow$ Minimizes $\text{TRACE}(P_k^+)$
  - ↳ Two useful properties:
    - a) $\frac{d}{dA} AB = B^T$, where AB is square
    - b) $\frac{d}{dA} (ACA)^T = 2AC$, where C is symmetric

$$
\begin{aligned}
0 &= \frac{\delta}{\delta L_k} \text{TRACE}(P_k^+) \\
&= \frac{\delta}{\delta L_k} \text{TRACE}((I - L_k C_d) P_k^- (I - L_k C_d)^T + L_k R_k L_k^T) \\
&= \frac{\delta}{\delta L_k} \text{TRACE}(-L_k C_d P_k^- - P_k^- C_d^T L_k^T + L_k C_d P_k^- C_d^T L_k + L_k R_k L_k^T) \\
&= -2 P_k^- C_d^T + 2 L_k (C_d P_k C_d^T + R_k)
\end{aligned}
$$

$$L_k = P_k^- C_d^T (C_d P_k C_d^T + R_k)^{-1}$$ ← Kalman Gain

↓

$$P_k^+ = (I - L_k C_d) P_k^-$$ ← Simplified Covariance Matrix

Kalman Filter Summary

$\hat{x}_0, P_0$

PROPAGATE

$$\hat{x}_{k+1}^- = A_d \hat{x}_k^+ + B_d u_k$$
$$P_{k+1}^- = A_d P_k^+ A_d^T + Q_d$$

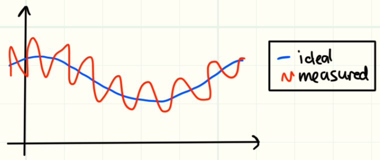
UPDATE

$$L_k = P_k^- C_d^T [C_d P_k^- C_d^T + R_k]^{-1}$$
$$P_k^+ = [I - L_k C_d] P_k^-$$
$$\hat{x}_k^+ = \hat{x}_k^- + L_k(y_k - C_d \hat{x}_k^-)$$

- Alternate Update: Upate Covariance before gain
  - $P_k^+ = [(P_k^-)^{-1} + C_d R_k^{-1} C_d]^{-1}$
  - $L_k = P_k^+ C_d^T R_k^{-1}$

Signal to Noise Ratio (SNR)

- Given an ideal or desired signal and the acual measured signal with added noise

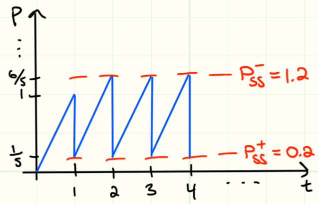


SNR

Power: $\dfrac{\text{Avg. Signal Power}}{\text{Avg. Noise Power}} = \dfrac{S}{N}$ Voltage: $\dfrac{\text{RMS Signal Voltage}}{\text{RMS Noise Voltage}}$

$$* \; SNR_{Pow} = (SNR_{Volt})^2 \; *$$

- Conviently expressed in dB:
  - $SNR_{dB} = 10 \log_{10}(SNR_{Pow})$
    $= 20 \log_{10}(SNR_{Volt})$



Calculating P_{ss}^- and P_{ss}^+

$$P_{ss}^- = A_d P_{ss}^+ A_d^T + Q_d$$
$$= A_d\left[(P_{ss}^-)^{-1} + CR^{-1}C^T\right]^{-1} A_d + Q_d$$
$$= 1\left[(P_{ss}^-)^{-1} + (1)\left(\tfrac{1}{4}\right)^{-1}(1)\right]^{-1} 1 + 1$$
$$= \frac{P_{ss}^-}{4P_{ss}^- + 1} + 1$$
$$0 = P_{ss}^-(4P_{ss}^- + 1) - P_{ss}^- - 1(4P_{ss}^- + 1)$$
$$0 = 4(P_{ss}^-)^2 + \cancel{P_{ss}^-} - \cancel{P_{ss}^-} - 4P_{ss}^- - 1$$
$$0 = 4(P_{ss}^-)^2 - 4P_{ss}^- - 1$$
$$P_{ss} = \tfrac{1}{2}(1 \pm \sqrt{2}) \quad * \text{ must be positive } *$$
$$= \tfrac{1}{2}(1 + \sqrt{2}) \approx 1.2$$

$$P_{ss}^+ = (I - L_{ss}C)P_{ss}^-$$
$$= \left[(A_d P_{ss}^+ A_d + Q_d)^{-1} + CR^{-1}C\right]^{-1}$$
$$= \left[((1)P_{ss}^+(1) + 1)^{-1} + 1\left(\tfrac{1}{4}\right)^{-1} 1\right]^{-1}$$
$$P_{ss}^+(4P_{ss}^+ + 5) = P_{ss}^+ + 1$$
$$P_{ss}^+ = -\tfrac{1}{2}(1 \pm \sqrt{2}) \quad * \text{ Must be Positive } *$$
$$P_{ss}^+ = \tfrac{-1}{2}(1 - \sqrt{2}) \approx 0.2$$

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$\boxed{EX}$ Given $\dot{x} = w(t)$, $w \sim N(0, 1)$; $y = x + v$, $v \sim N(0, 1/4)$ } white

$\hat{x}_0 = 0$, $P_0 = 0$, $\Delta t = 1$

$x_k = \cancel{A_d} x_{k-1} + w_{k-1}$ $y = Cx + v \Rightarrow C = 1$

$A_d = e^{\cancel{A\Delta t}} = 1$ $Q_d = \int_0^1 e^{\cancel{A\Delta t}} Q_c e^{\cancel{A\Delta t}} \, d\tau = \int_0^1 Q_c \, d\tau = 1$ $* \; Q_c = E\{ww^T\} = 1$

1st Propogation

$$\hat{x}_1^- = A_d \cancel{\hat{x}_0} = 0$$
$$P_1^- = A_d \cancel{P_0} A_d^T + Q_d = 1$$

1st Meas. Update

$$L_1 = P_1^- C^T\left[CP_1^-C^T + R\right]^{-1} = (1)(1)\left[(1)(1)(1) + \tfrac{1}{4}\right]^{-1} = \left(\tfrac{5}{4}\right)^{-1} = 4/5$$
$$\hat{x}_1^+ = \hat{x}_1^- + L_1(y_1 - C\hat{x}_1^-) = 0 + \tfrac{4}{5}(y_1 - 1(0)) = \tfrac{4}{5} y_1$$
$$P_1^+ = (I - L_1C)P_1^- = \left(1 - \tfrac{4}{5}(1)\right)(1) = \tfrac{1}{5}$$

2nd Propogation

$$\hat{x}_2^- = A_d \hat{x}_1^+ = (1)\left(\tfrac{4}{5} y_1\right) = \tfrac{4}{5} y_1$$
$$P_2^- = A_d P_1^+ A_d + Q_d = (1)\left(\tfrac{1}{5}\right)(1) + 1 = \tfrac{6}{5}$$

2nd Meas. Update

$$L_2 = P_2^- C^T\left[(CP_2^-C^T + R\right]^{-1} = \left(\tfrac{6}{5}\right)(1)\left[(1)\left(\tfrac{6}{5}\right)(1) + \tfrac{1}{4}\right]^{-1} = 24/29$$
$$\hat{x}_2^+ = \hat{x}_2^- + L_2(y_2 - C\hat{x}_2^-) = \left(\tfrac{4}{5} y_1\right) + \tfrac{24}{29}\left(y_2 - (1)\left(\tfrac{4}{5} y_1\right)\right) = \tfrac{4}{29} y_1 + \tfrac{24}{29} y_2$$
$$P_2^+ = (I - L_2C)P_2^- = \left(1 - \left(\tfrac{24}{29}\right)(1)\right)\left(\tfrac{6}{5}\right) = \tfrac{6}{29}$$

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Continuous Filtering

- $\dot{x} = Ax + B_w w$
- $y = Cx + v$

- $E\{w(t)\} = 0$
- $E\{w(\tau)w(\gamma)^T\} = Q_c \, \delta(\tau - \gamma)$
- $E\{v(t)\} = 0$
- $E\{v(\tau)v(\gamma)^T\} = R \, \delta(\tau - \gamma)$

- GOAL: Minimize $E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\}$

- $v_k = \dfrac{1}{\Delta t}\displaystyle\int_0^{\Delta t} v(\tau)\, d\tau$ ← average measurement noise

- $R_k = E\{v_k v_k^T\} = \dfrac{1}{\Delta t^2}\displaystyle\int_0^{\Delta t}\int_0^{\Delta t} E\{v(\alpha)v(\beta)\}\, d\alpha\, d\beta$
  $\approx \dfrac{1}{\Delta t^2} \Delta t R = \dfrac{R}{\Delta t}$

- So as $\Delta t \to 0$, discrete noise tends to infinite plus covariance with weight $R \to$ looks like $R\,\delta(t)$

Covariance Equations

1) $P_{k+1}^- = A_d P_k^+ A_d^T + Q_d$

2) $P_k^+ = (I - L_k C)P_k^-$

3) $L_k = P_k^- C^T (C P_k^- C^T + R_k)^{-1}$

↳ $P C^T \left(C P C^T + \frac{R}{\Delta t}\right)^{-1} = P C^T R^{-1} \Delta t$

$P_{k+1}^- \to P_k^+ \to P$

• For small Δt

↳ $Q_d \approx B_w Q_c B_w^T \Delta t \to 0$

↳ $A_d = e^{A\Delta t} \to 1$

• Define $L = \frac{L_k}{\Delta t}$

↳ $L = PCR^{-1}$ ← continuous Kalman gain

$$P_{k+1}^- = \underbrace{(I + A\Delta t)}_{\approx e^{A\Delta t}} P_k^+ (I + A\Delta t)^T + B_w Q_c B_w^T \Delta t$$

$$= P_k^+ + \Delta t \left[A P_k^+ + P_k^+ A + B_w Q_c B_w^T\right] + H.O.T.$$

$$P_k^+ = P_k^- - L_k C P_k^-$$

$$↳ \underbrace{\frac{P_{k+1}^- - P_k^-}{\Delta t}}_{\dot P} = \underbrace{\frac{-L_k}{\Delta t}}_{L} C P_k^- + A P_k^- + P_k^- A^T + B_w Q_c B_w^T - \underbrace{A L_k C P_k^-}_{L*\Delta t} - L_k C P_k^- A^T + H.O.T.$$

$\Delta t \to 0$

$$\dot P = AP + PA^T + B_w Q_c B_w^T - LCP$$

$$= AP + PA^T + B_w Q_c B_w - PC^T R^{-1} CP$$ ← matrix riccati equation

· Stable: $AP + PA^T \to P$ decreases

· unstable: $AP + PA^T \to P$ increases

· Process noise causes P to increase (Q_c)

· New info (measurements) cause P to decrease

State Equations

1) $\hat x_{k+1}^- = A_d \hat x_k^+$

2) $\hat x_k^+ = \hat x_k^- + L_k (y_k - C\hat x_k^-)$

$= A_d \hat x_k^- + A_d L_k (y_k - C\hat x_k^-)$

$= (I - A\Delta t)\hat x_k^- + (I - A\Delta t) L_k (y_k - C\hat x_k^-)$

$$\frac{\hat x_{k+1}^- - \hat x_k^-}{\Delta t} = A\hat x_k^- + \frac{L_k}{\Delta t}\left(y_k - C\hat x_k^- + CA\Delta t\, \hat x_k^-\right)$$

$\Delta t \to 0$, $\frac{L_k}{\Delta t} \to L$

$$\dot{\hat x} = A\hat x + L(y - C\hat x)$$

↑ function of covariance (similar to setting eigenvalues)

Continuous Equations ← Kalman-Bucy Filter

$\dot{\hat x} = A\hat x + L(y - C\hat x)$

$\dot P = AP + PA^T + B_w Q_c B_w^T + PC^T R^{-1} CP$

$L = PC^T R^{-1}$

$\boxed{Ex}$ $\dot x = 0$

$y = x + v$

$$\dot P = \cancel{A}^{0} P + P\cancel{A^T}^{0} + B_w \cancel{Q_c}^{0} B_w^T - P\cancel{C^T}^{1} R^{-1} \cancel{C}^{1} P$$

$$\dot P = -\frac{P^2}{R} \qquad \frac{dP}{dt} = -\frac{P^2}{R}$$

$$\int_{P_0}^{P(t)} \frac{1}{P^2}\, dP = \int_0^t \frac{-1}{R}\, dt$$

$$-\frac{1}{P}\Big|_{P_0}^{P(t)} = \frac{-t}{R}\Big|_0^t$$

$$-\left(\frac{1}{P(t)} - \frac{1}{P_0}\right) = -\left(\frac{t}{R} - \cancel{\frac{0}{R}}\right)$$

$$P(t) = \frac{RP_0}{R + P_0 t}$$

$$L = PCR^{-1} = \left(\frac{\cancel{R}P_0}{R + P_0 t}\right)(1)\left(\frac{1}{\cancel{R}}\right) = \frac{P_0}{R + P_0 t}$$

$$\dot{\hat x} = \cancel{A}^{0}\hat x + L(y - \cancel{C}^{1}\hat x) = \frac{P_0}{R + P_0 t}(y - \hat x)$$

$\boxed{Ex}$ Given 2nd order Kinematic Model, Find P_{ss}, L_{ss}.

$Q_c = R = 0.1$.

$$\begin{bmatrix} \dot x \\ \ddot x \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot x \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + v$$

$$\cancel{\dot P}^{0} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} P + P \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} (0.1) \begin{bmatrix} 0 & 1 \end{bmatrix} - P \begin{bmatrix} 1 \\ 0 \end{bmatrix} \left(\frac{1}{0.1}\right) \begin{bmatrix} 1 & 0 \end{bmatrix} P$$

$$P_{ss} = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

$$0 = \begin{bmatrix} P_{12} & P_{22} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P_{12} & 0 \\ P_{22} & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix} - \begin{bmatrix} 10P_{11}^2 & 10P_{11}P_{12} \\ 10P_{11}P_{12} & 10P_{12}^2 \end{bmatrix}$$

Element 2,2: $0 = 0 + 0 + 0.1 - 10P_{12}^2$

$P_{12} = \frac{1}{10}$

Element 1,1: $0 = P_{12} + P_{12} + 0 - 10P_{11}^2$

$P_{11} = \sqrt{\frac{2}{100}} = \frac{\sqrt{2}}{10}$

Element 2,1: $0 = 0 + P_{22} + 0 - 10P_{11}P_{12}$

$P_{22} = \frac{\sqrt{2}}{10}$

$$P_{ss} = \begin{bmatrix} \frac{\sqrt{2}}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{\sqrt{2}}{10} \end{bmatrix}$$

$$L_{ss} = P_{ss} C^T R^{-1} = \begin{bmatrix} \frac{\sqrt{2}}{10} & \frac{1}{10} \\ \frac{1}{10} & \frac{\sqrt{2}}{10} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} (10) = \begin{bmatrix} \sqrt{2} \\ 1 \end{bmatrix}$$

3-27-23

MATLAB

- Continuous ARE
  - $\hookrightarrow [P_{ss}, \lambda_{ss}, L_{ss}^T] = \text{care}(A^T, C^T, B_w Q_c B_w^T, R)$

Comments on Steady State Kalman Filter

- For LTI systems, L_{ss} works fine
  - recursive not needed
  - transient behavior influences start but dies out
- For $\dot{x} = Ax + B_w w$
  $y = Cx + v$

  Assuming (i) $Q_c, R > 0$
  (ii) LTI
  (iii) $[A, C]$ are detectable (all unobservable modes/states are stable)
  (iv) $[A, B_w]$ stabilizable (all uncontrollable modes/states are stable)

  $$0 = AP_{ss} + P_{ss}A^T + B_w Q_c B_w^T + P_{ss} C^T R^{-1} C P_{ss}$$

  - P_{ss} must be PSD and unique
  - IFF (i)-(iv) holds, P_{ss} exists and x is assymtotically stable (Assymtotically $\hat{x} \to x$)

Why bother with recursion??

- Nonlinear
- Time varying
- Modeling errors
- Large number of states can lead to numerical issues

Frequency Domain Analysis

State Dynamics: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x}) = (A - LC)\hat{x} + Bu + Ly$

$y = C\hat{x} + v$

Error Dynamics: $\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} = Ax + \cancel{Bu} + B_w w - A\hat{x} - \cancel{Bu} - L(\cancel{Cx} + v - \cancel{C\hat{x}})$

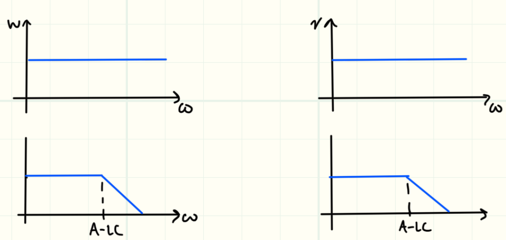
$= A(x - \hat{x}) + B_w w - Lv$

$\dot{\tilde{x}} = (A - LC)\tilde{x} + B_w w + Lv$

1) Filter dynamics determined by $A - LC$
2) Equation shows conflict between fast convergence and good noise rejection

$$\left.\begin{aligned} \frac{E}{w} &= \frac{B_w}{s + (A - LC)} \\ \frac{E}{v} &= \frac{v}{s + (A - LC)} \end{aligned}\right\}$$

want $A - LC$ to be large
- more negative eigenvalues
- faster convergence



- KF optimally weight dynamics and measurements

$\boxed{\text{Ex}}$ $\dot{x} = w \quad w \sim N(0, q) \qquad \frac{x(s)}{w(s)} = \frac{1}{s}$

$y = x + v \quad v \sim N(0, r)$

Optimal Filter: $\dot{\hat{x}} = \frac{P}{r}(y - \hat{x})$

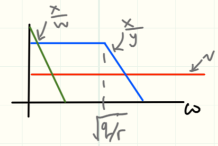
$$P(t) = \sqrt{qr}\left(\frac{1 + \frac{P_0 - \sqrt{rq}}{P_0 + \sqrt{rq}} e^{-2\sqrt{q/r}\,t}}{1 - \frac{P_0 - \sqrt{rq}}{P_0 + \sqrt{rq}} e^{-2\sqrt{q/r}\,t}}\right)$$

* $q \uparrow$ causes system to converge slower
* $r \uparrow$ causes system to converge faster
– small measurement noise $\to$ get to truth faster

$P_{ss} = \sqrt{qr} \qquad L_{ss} = \frac{\sqrt{qr}}{r} = \sqrt{\frac{q}{r}}$

$\dot{\hat{x}} = -\sqrt{\frac{q}{r}}\,\hat{x} + \sqrt{\frac{q}{r}}\,y$

$\frac{X(s)}{Y(s)} = \frac{\sqrt{\frac{q}{r}}}{s + \sqrt{\frac{q}{r}}} \qquad \frac{X(s)}{W(s)} = \frac{1}{s}$



* bandwidth increases as $q \uparrow$
* $\frac{X(s)}{W(s)}$ limits how much filtering can be performed

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Exponential Correlated Noise

$\dot{d} = -\frac{d}{\tau_c} + w \qquad w$: white noise
$\qquad\qquad\qquad\quad d$: total disturbance (i.e. wind)

$\dot{P} = AP + PA^T + B_w Q_c B_w^T$

$0 = \frac{1}{\tau_c} P_{ss} + P_{ss}\frac{1}{\tau_c} + \sigma_w^2 B_w^2$

$P_{ss} = \frac{\tau_c}{2}\sigma_w^2 B_w^2 \quad \text{or} \quad \sigma_w^2 B_w^2 = \frac{2P_{ss}}{\tau_c}$

Discrete Covariance of w: $Q_d = \int_0^{\Delta t} e^{At} B_w Q_c B_w e^{At}\,dt$

$Q_d = \sigma_w^2 B_w^2 \left(1 - e^{-\frac{2\Delta t}{\tau_c}}\right)$

$w_d \sim N(0, Q_d) \to Q_d = \frac{2P_{ss}}{\tau_c}\left(1 - e^{-\frac{2\Delta t}{\tau_c}}\right)$

$w \sim N(0,1)$

$d_{k+1} = e^{-\frac{\Delta t}{\tau_c}} d_k + \sqrt{Q_d}\, w = e^{-\frac{\Delta t}{\tau_c}} d_k + \sqrt{\frac{2P_{ss}}{\tau_c}\left(1 - e^{-\frac{2\Delta t}{\tau_c}}\right)}\, w$

$\boxed{\text{Ex}}$
$$\begin{bmatrix} \dot{p} \\ \dot{v} \\ \dot{d} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{\tau_c} \end{bmatrix} \begin{bmatrix} p \\ v \\ d \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sqrt{\frac{2\sigma_c^2}{\tau_c}} \end{bmatrix} w$$

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Kalman Approximation for Nonlinear Dynamics or Measurement Models

$\dot{x} = f(x,u,t) + g(x,t)\,w(t)$ ↖ process noise, stochastic disturbance

$y = h(x,t) + v(t)$ ↖ measurement noise

- same noise assumptions

$x_k = A_d x_{k-1} + w_{k-1}$

$y_k = C_d x_k + v_k$

GOAL: minimize

$$J = E\left\{\tfrac{1}{2}(x_0-\hat{x}_0)^T P_0(x_0-\hat{x}_0) + \tfrac{1}{2}\sum_{k=1}^{N-1} w_k^T Q_d^{-1} w_k + \tfrac{1}{2}\sum_{k=1}^{N-1}(y_k - C_d x_k)^T R_d^{-1}(y_k - C_d x_k)\right\}$$

↳ subject to $x_k = A_d x_{k-1} + w_{k-1}$

- Can be solved with Lagrangian multipliers using a "sweep" solution
- Results in Kalman filter which is globally optimal for LTI systems

Non-Linear Cost Function

$$J = E\left\{\tfrac{1}{2}(x_0-\hat{x}_0)^T P_0(x_0-\hat{x}_0) + \tfrac{1}{2}\sum_{k=1}^{N-1} w_k Q_k^{-1} w_k + \tfrac{1}{2}(y_k - h(x_k,t_k))^T R^{-1}(y_k - h(x_k,t_k))\right\}$$

↳ subject to $\dot{x} = f(x,u,t) + g(x,t)\,w(t)$

- No closed form solution (non-linear)

One Big Batch

L.S. $\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$

$P = (H^T R^{-1} H)^{-1}$

Multiple Batches

$$J = E\left\{\tfrac{1}{2}(x_k-\hat{x}_k)^T P^{-1}(x_k-\hat{x}_k) + \tfrac{1}{2}(y_k - H_k x_k)^T R_k^{-1}(y_k - H_k x_k)\right\}$$ Linear

$\hat{x}_k^+ = \hat{x}_k^- + P_k H_k^T R_k^{-1}(y_k - H\hat{x}_k^-)$

$P_k^+ = P_k^- + H_k^T R_k^{-1} H_k$

non-linear

$$J = E\left\{\tfrac{1}{2}(x_k-\hat{x}_k)^T P^{-1}(x_k-\hat{x}_k) + \tfrac{1}{2}(y - h(x_k,t_k))^T R^{-1}(y - h(x_k,t_k))\right\}$$

- Need: 1) Need an approximation to $h(x_k,t)$
  2) Rewrite J

Taylor Series

$h(x_k,t_k) \approx h(\hat{x}_k) + \left.\frac{\partial h}{\partial x}\right|_{x=\hat{x}_k}(x_k - \hat{x}_k)$

↖ prior is operating point

$H_k \triangleq \left.\frac{\partial h}{\partial x}\right|_{x=\hat{x}_k}$

$z_k \triangleq y_k - h(\hat{x}_k) + H_k\hat{x}_k$

New Cost Function: $J = E\left\{\tfrac{1}{2}(x_k-\hat{x}_k)^T P^{-1}(x-\hat{x}_k) + \tfrac{1}{2}(z_k - H_k\hat{x}_k)^T R^{-1}(z_k - H_k\hat{x}_k)\right\}$

LS Solution: $\hat{x}_k^+ = \hat{x}_k^- + P_k^- H_k^T R^{-1}(y - h(\hat{x}_k^-, t))$

$(P_k^+)^{-1} = (P_k^-)^{-1} + H_k^T R_k^{-1} H_k$

EKF Solution:

$$\left.\begin{aligned} L_k &= P_k^- H_k^T (H_k P_k^- H_k^T + R_k^{-1}) \\ P_k^+ &= (I - L_k H_k)P_k^- \\ \hat{x}_k^+ &= \hat{x}_k^- + L_k(y_k - h(\hat{x}_k^-, t_k)) \end{aligned}\right\}$$ Extended Kalman Filter Measurement Update

- Time update if dynamics are linear
  ↳ use original update
- Non-Linear Approximation

  $E\{\dot{x} = f(x,t) + g(x,t)\,w(t)\}$

  $E\{\dot{x}\} = E\{f(x,t)\} + E\{g(x,t)\cancel{w(t)}\}$ 0
- Taylor Series of $f(x,t)$

  $f(x,t) \approx f(\hat{x}_k) + \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_k}(x-\hat{x})$

  $\dot{\hat{x}} = f(\hat{x},t)$

  ↳ use nonlinear equations and numerical integration to update state (i.e. mean)

Runge-Kutta

- $\hat{x}_{k+1} = \hat{x}_k + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

  → $k_1 = \Delta t\, f(\hat{x}_k, t_k)$

  → $k_2 = \Delta t\, f(\hat{x}_k + \tfrac{1}{2}k_1, t_k + \tfrac{1}{2}\Delta t)$

  → $k_3 = \Delta t\, f(\hat{x}_k + \tfrac{1}{2}k_2, t_k + \tfrac{1}{2}\Delta t)$

  → $k_4 = \Delta t\, f(\hat{x}_k + k_3, t_k + \Delta t)$

Covariance

- No 4th order approximation
- From 1st order Taylor Series

$F \triangleq \left.\frac{\partial f}{\partial x}\right|_{x=\hat{x}_k}$

$A_d \approx e^{F\Delta t}$

$Q_d = \int_{t_k}^{t_{k+1}} A_d\, g(\hat{x},t)\, Q_c\, g(\hat{x},t) A_d^T\, d\tau$

$P_{k+1}^- = A_d P_k^+ A_d^T + Q_d$