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Estimation of Constant With Known Measurement and unknown Measurement Statistics

$$\underset{m\times1}{Y} = \underset{m\times n}{H}\,\underset{n\times1}{X} + \underset{n\times1}{\eta}$$

meas. → Y; meas. matrix → H; state → X; meas. noise → η

Least Squares

- Assumining number of linearly independant rows of $Y \geq n$
- $\hat{x} = (H^T H)^{-1} H^T Y$
- Properties:
  1) $\dfrac{\partial f(x)}{\partial x} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1} & \dfrac{\partial f}{\partial x_2} & \cdots & \dfrac{\partial f}{\partial f_n} \end{bmatrix}$
     - where $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$
     - $\dfrac{\partial u^T x}{\partial x} = \dfrac{\partial x^T u}{\partial x} = u^T$
  2) $\dfrac{\partial (x^T x)}{\partial x} = 2x^T$
  3) $\dfrac{\partial (x^T A x)}{\partial x} = x^T A + A^T x^T$
     - If $A = A^T$, then $= 2x^T A$

- Least squares minimizes $J = \frac{1}{2} e_y^T e_y$
  - $e_y = y - \hat{y} = y - H\hat{x}$

$$J = \tfrac{1}{2}(y - H\hat{x})^T (y - H\hat{x})$$
$$= \tfrac{1}{2}\left[y^T y - y^T H\hat{x} - (H\hat{x})^T y + (H\hat{x})^T H\hat{x}\right]$$

$$\frac{\partial J}{\partial \hat{x}} = 0 = \tfrac{1}{2}\left[0 - y^T H - y^T H + 2\hat{x}^T H^T H\right]$$
$$0 = -H^T y + H^T H \hat{x}$$
$$\hat{x} = (H^T H)^{-1} H^T y$$

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Ex Estimate initial height, velocity, constant acceleration of rocket given time stamped Height measurements.

t	$\tilde{h}$
0	1380
1	1223
2	1001
3	974

$$h = h_0 + v_0 t + \tfrac{1}{2} a t^2$$
$$\tilde{h} = h + v \quad (v \text{ ↳ noise})$$

$$x = \begin{bmatrix} h_0 \\ v_0 \\ a \end{bmatrix} \quad y = \begin{bmatrix} 1380 \\ 1223 \\ 1001 \\ 974 \end{bmatrix} \quad H = \begin{bmatrix} 1 & t & t^2 \\ 1 & t & t^2 \\ 1 & t & t^2 \\ 1 & t & t^2 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \qquad y = Hx$$
$$\hat{x} = (H^T H)^{-1} H^T \tilde{y}$$

$$\frac{\partial J}{\partial \hat{x}} = -y^T H + \hat{x}^T H^T H = 0$$

$$\frac{\partial^2 J}{\partial \hat{x}^2} = H^T H \quad \rightarrow \text{Hessian} \rightarrow \text{covariance}$$
- Positive semi definite
- symmetric

* For $m > n \Rightarrow \hat{x} = (H^T H)^{-1} H^T y$
* For $m = n \Rightarrow \hat{x} = H^{-1} y$

Weighted Least Squares

↳ accounts for relative "noise power" (σ^2) on each measurement

- New Cost Function

$$J = \tfrac{1}{2} e_y^T W e_y \qquad W\text{: Weighting matrix}$$
$$= \tfrac{1}{2}[y - H\hat{x}] W [y - H\hat{x}]$$

$$\frac{\partial J}{\partial \hat{x}} = 0 \Rightarrow \hat{x} = (H^T W^{-1} H)^{-1} H^T W^{-1} y$$
$$W = \text{diag}([\sigma_1^2 \;\; \sigma_2^2 \; \cdots \; \sigma_m^2])$$
→ $W \geq 0$
→ can be normalized $\sum W_i = 1$
→ smaller variance (noise power) → larger weight

* $W^{-1} = \text{diag}\left(\left[\frac{1}{\sigma_1^2} \cdots \frac{1}{\sigma_n^2}\right]\right)$

Ex $W = \frac{1}{\sigma^2} I_{m\times m}$

$$\hat{x} = \left(H^T \left[\cancel{\tfrac{1}{\sigma^2} I_{m\times m}}\right] H\right)^{-1} H^T \left[\cancel{\tfrac{1}{\sigma^2} I_{m\times m}}\right] y$$
$$\hat{x} = (H^T H)^{-1} H^T y$$

Ex $y_i = x + v_i \qquad v_i \sim N(0, \sigma^2) \; \forall i$

$$\hat{x} = \frac{1}{m} \sum_{i=1}^{m} y_i$$

$$H = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times1} \qquad H^T H = m \qquad \hat{x} = m^{-1} \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix}_{1\times n} y_{m\times1}$$

$\boxed{Ex}$ $y_i = x + v_i$ Where: $y_1 \sim \sigma^2$, $y_{2-m} \sim 1$

$$W = \begin{bmatrix} \frac{1}{\sigma^2} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$$

$$H^T W H = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{\sigma^2} + (m-1)$$

$$H^T W y = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sigma^2} & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \frac{1}{\sigma^2} y_1 + \sum_{i=2}^{m} y_i$$

$$\hat{x} = \frac{1}{\frac{1}{\sigma^2} + (m-1)} \left(\frac{1}{\sigma^2} y_1 + \sum_{i=2}^{m} y_i \right) = \frac{y_1 + \frac{1}{1/\sigma^2} \sum_{i=2}^{m} y_i}{1 + \frac{m-1}{1/\sigma^2}}$$

* if $\sigma^2 >> 1$ then $\hat{x} = \frac{1}{m-1} \sum_{i=2}^{m} y_i$
* if $\sigma^2 << 1$ then $\hat{x} = y_1$

What happens when $m < n$?

Minimum Norm Solution

↳ minimize $\hat{x}$ subject to $e_y = 0$

$$J(x, \lambda) = \frac{1}{2} \hat{x}^T \hat{x} + \lambda^T (y - H\hat{x})$$

1) $\frac{\partial J}{\partial \hat{x}} = 0 = \hat{x}^T - \lambda^T H$
$\rightarrow \hat{x}^T = \lambda^T H$

2) $\frac{\partial J}{\partial \lambda} = 0 = y - H\hat{x}$
$= y - H(\lambda^T H)^T$
$\rightarrow \lambda = (HH^T)^{-1} y$

$\boxed{\hat{x} = H^T (HH^T)^{-1} y}$

↳ not as accurate as having more measurements than states

Stochastic Least Squares

↳ probabilistic derivation

$y_i = H_i x + v_i$
$v_i \sim N(0, \sigma_i^2)$

given $v_i = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$ $E(vv^T) = \begin{bmatrix} \sigma_1^2 & & \\ & \sigma_2^2 & \\ & & \ddots \\ & & & \sigma_n^2 \end{bmatrix} = R$

DIAGONAL
↓ (v is uncorrelated in time, i.e. white)

→ Derive best estimate of $\hat{x}$ from a maximum likelihood prospective

* Set $\hat{x}$ that maximizes probability of getting our measurements, $y_{m \times 1}$

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1) Find y conditioned on x $f_{y|x}(y|x)$
2) Select $\hat{x}$ to maximize $f_{y|x}(y|x)$

$y = Hx + v$ $v \sim N(0, R)$ $R = E\{vv^T\}$

$M_y = E\{y\} = E\{Hx + \cancel{v}\} = H E\{x\}$
↳ knowing / given $X = x$
$E\{y\} = Hx$

$cov\{y | X = x\} = E\{(y - \bar{y})(y - \bar{y})^T\} = E\{(\cancel{Hx} + v - \cancel{Hx})(\cancel{Hx} + v - \cancel{Hx})^T\}$
$= E\{vv^T\} = R$

$$* \; f_{y|x}(y|x) = \frac{1}{(2\pi)^{n/2} |R|^{1/2}} e^{-\frac{1}{2}(y - Hx)^T R^{-1} (y - Hx)}$$

Maximize $f_{y|x}(y|x) \longrightarrow$ minimize $\frac{1}{2}(y - Hx)^T R^{-1} (y - Hx)$
↳ SAME RESULT
↳ $\hat{x} = (H^T R^{-1} H)^{-1} H^T R^{-1} y$

Estimator Metrics

1) Biased / Unbiased

* Unbiased $\rightarrow E\{x - \hat{x}\} = 0$

$= E\{x - (H^T R^{-1} H)^{-1} H^T R^{-1} y\}$
$= E\{x - (H^T R^{-1} H)^{-1} H^T R^{-1} (Hx + v)\}$
$= E\{\cancel{x - x} + (H^T R^{-1} H)^{-1} H^T R^{-1} v\}$
$= (H^T R^{-1} H)^{-1} H^T R^{-1} * E\{v\}$
$= 0$

* What if $v \sim N(b, R)$
↳ $y = Hx + b + v^*$ $v^*(0, R)$
↳ $y = \begin{bmatrix} H & 1 \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} + v^*$

2) Consistency

* Consistant Estimator $\rightarrow Q_1 \triangleq \lim_{n \to \infty} E\{(x - \hat{x})^T (x - \hat{x})\} = 0$

$Q_2 = E\{(x - \hat{x})(x - \hat{x})^T\}$

$Q_1 = trace\{Q_2\}$

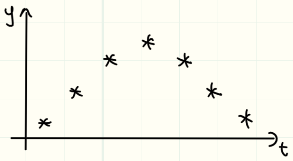
$= E\{\underbrace{(H^T R^{-1} H)^{-1} H^T R^{-1} \overbrace{v v^T}^{R} R^{-1} H}_{I} (H^T R^{-1} H)^{-1}\}$

COVARIANCE MATRIX $\longrightarrow Q_2 = (H^T R^{-1} H)^{-1}$

Ex $y_i = x + v_i \qquad v \sim N(0,R) \qquad R = \mathrm{diag}([\sigma^2 \cdots \sigma^2])$

$$H = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{m\times 1} \qquad Q_2 = (H^T R^{-1} H)^{-1} = \left(\begin{bmatrix} 1 & \cdots & 1 \end{bmatrix} \sigma^2 I \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)^{-1} = \left(\frac{m}{\sigma^2}\right)^{-1} = \frac{\sigma^2}{m}$$

Ex Classic Problem



– Polynomial fit

$y = a + bt + ct^2 + \ldots$

$$y = \begin{bmatrix} 1 & t & t^2 & \cdots \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ \vdots \end{bmatrix}$$

→ To check model fit, look at residuals $(r = y - H\hat{x})$

* WE WANT $E\{y - H\hat{x}\} = 0$

$\sigma^2 = \frac{r^T r}{m-n} = \text{const.}$

* TWO OPTIONS:
  1) Brute Force
     – Increase n, create residuals, check to see when $\sigma^2 \to c$
  2) Take FFT of r and examine frequency content

Recursive Estimation

Data in two batches:

$y_1 = H_1 x + v_1 \qquad y_2 = H_2 x + v_2$

$\hat{x} = (H_1^T H_1)^{-1} H_1^T y_1$

$$\hat{x} = \left(\begin{bmatrix} H_1 & H_2 \end{bmatrix} I \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \right)^{-1} \begin{bmatrix} H_1 & H_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \leftarrow \text{batched}$$

$\hat{x} = \text{new info} + \hat{x}_1 \leftarrow$ recursive

GOAL: solve recursively without losing precision of batches

$\hat{x} = \hat{x}_0 + \delta x$

If we want:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{y} = \underbrace{\begin{bmatrix} H_1 \\ H_2 \end{bmatrix}}_{H} x + \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{v}$$

$x = (H^T R^{-1} H)^{-1} H^T R^{-1} y$

$(H^T R^{-1} H) x = H^T R^{-1} y$

$$* \; H^T R^{-1} H = \begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} = H_1^T R_1^{-1} H_1 + H_2 R_2^{-1} H_2$$

↑ batches are uncorrelated

$$* \; H^T R^{-1} = \begin{bmatrix} H_1^T & H_2^T \end{bmatrix} \begin{bmatrix} R_1^{-1} & 0 \\ 0 & R_2^{-1} \end{bmatrix} = H_1^T R_1^{-1} + H_2^T R_2^{-1}$$

GOAL: $\hat{x} = x_0 + \delta x \leftarrow$ correction based on new data

$Q = Q_{batch} = [H_1^T R_1^{-1} H_1 + H_2^T R_2^{-1} H_2]^{-1}$

DEFINE: $\hat{x}_1 = (H_1^T R_1^{-1} H_1)^{-1} H_1^T R_1^{-1} y_1$

$Q_1 = (H_1^T R_1^{-1} H_1)^{-1}$

$$\begin{bmatrix} H_1^T R_1^{-1} H_1 & H_2^T R_2^{-1} H_2 \end{bmatrix} \hat{x} = \begin{bmatrix} H_1^T R_1^{-1} & H_2^T R_2^{-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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cont.

$\hat{x} = x_1 + \delta x$

$$\underbrace{\begin{bmatrix} H_1^T R_1^{-1} H_1 & H_2^T R_2^{-1} H_2 \end{bmatrix}}_{Q^{-1}} (\hat{x}_1 + \delta x) = H_1^T R_1^{-1} y_1 + H_2^T R_2^{-1} y_2$$

$\hookrightarrow = H_1^T R_1^{-1} H_1 = Q_1$

$Q^{-1} \hat{x}_1 + Q^{-1} \delta x = Q_1^{-1} \hat{x}_1 + H_2^T R_2^{-1} y_2$

$Q^{-1} \delta x = [Q_1^{-1} - Q^{-1}] \hat{x}_1 + H_2 R_2^{-1} y_2$

$= [\cancel{H_1^T R_1^{-1} H_1} - H_2 R_2^{-1} H_2 - \cancel{H_1^T R_1^{-1} H_1}]^{-1} \hat{x}_1 + H_2^T R_2^{-1} y_2$

$[H_1^T R_1^{-1} H_1 \quad H_2^T R_2^{-1} H_2] \delta x = -H_2^T R_2^{-1} H_2 \hat{x}_1 + H_2 R_2^{-1} y^{-1}$

$$\delta x = \underbrace{[H_1^T R_1^{-1} H_1 + H_2^T R_2^{-1} H_2]^{-1}}_{\text{normalization}} \underbrace{H_2^T R_2^{-1}}_{\text{projection}} \underbrace{(y_2 - H_2 \hat{x}_1)}_{\text{innovation}}$$

- Innovation: old estimate mapped to new measurement domain subtracted from new measurement
- Projection: back to state domain
- Normalization: by total uncertainty

- $E\{(x - \hat{x}_1)(x - \hat{x}_1)^T\} = Q_1 = (H^T R_1^{-1} H_1)^{-1}$
- $E\{(x - \hat{x})(x - \hat{x})^T\} = Q = (H_2^T R_2^{-1} H_2 + H_1^T R_1^{-1} H_1)^{-1}$

Ex 2 batches → Batch 1 has bad meas.

$\hookrightarrow R_1 = \sigma_1^2 \quad \sigma_1^2 \gg \sigma_2^2$

$\hookrightarrow R_2 = \sigma_2^2 \quad R_1 \gg R_2$

$Q^{-1} = H_1^T \frac{1}{\sigma_1^2} H_1 + H_2^T \frac{1}{\sigma_2^2} H_2 \approx H_2^T \frac{1}{\sigma_2^2} H_2$

$\hat{x} = \hat{x}_1 + \delta x = x_1 + [\cancel{H_1^T \frac{1}{\sigma_1^2} H_1} + H_2^T \frac{1}{\sigma_2^2} H_2]^{-1} H_2^T \frac{1}{\sigma_2^2} (y_2 - H_2 \hat{x}_1)$

$= x_1 + (H_2^T R_2^{-1} H_2)^{-1} H_2^T R_2^{-1} y_2 - \cancel{(H_2^T R_2^{-1} H_2)(H_2^T R_2 H_2)}^{I} \hat{x}$

$= (H_2^T R_2^{-1} H_2)^{-1} H_2^T R_2^{-1} y_2$

$Q^{-1} \approx H_1^T \frac{1}{\sigma_1^2} H_1$

$\hat{x} = \hat{x}_1 + \frac{H_2 \cancel{\frac{1}{\sigma_2^2}}^{\infty}}{\cancel{H_1^T} \frac{1}{\sigma_1^2} H_1} (y_2 - H_2 x_1)$

$\hat{x} = x_1$

$\boxed{\text{Ex}}$
- 1^{st} batch has m measurements
- 2^{nd} batch has l measurements

$$R_1 = \frac{1}{\sigma_1^2} I \qquad y_1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} x + v_1 \qquad v_1 \sim N(0, \sigma_1^2)$$

$$\hat{x}_1 = \frac{1}{m}\sum_{i=1}^{m} y_i \qquad Q_1 = (H_1^T R_1^{-1} H_1)^{-1} = \frac{\sigma_1^2}{m}$$

$$R_2 = \frac{1}{\sigma_2^2} \qquad y_2 = x + v_2 \qquad v_2 \sim N(0, \sigma_2^2)$$

$$Q = Q_1 + (H_2^T R_2^{-1} H)^{-1}$$
$$= \frac{\sigma_1^2}{m} + \sigma_2^2$$
$$Q^{-1} = \frac{m}{\sigma_1^2} + \frac{1}{\sigma_2^2}$$

$$\hat{x} = \hat{x}_1 + Q H_2^T R_2^{-1} (y_2 - H_2 \hat{x}_1)$$
$$\hat{x} = \hat{x}_1 + \frac{\sigma_1^2 \sigma_2^2}{m\sigma_2^2 + \sigma_1^2} \frac{1}{\sigma_2^2} (y_2 - \hat{x}_1)$$

$$* \; \hat{x} = \hat{x}_1 + \frac{\sigma_2^2}{m+1} \frac{1}{\sigma_2^2} (y_2 - \hat{x}_1) \leftarrow \text{estimator goes to sleep!}$$

Recursive Least Squares Form

0) Given x_0 and Q_0 (initial guess of state + covariance)

1) UPDATE TOTAL UNCERTAINTY (COVARIANCE)

$$Q_{k+1}^{-1} = Q_{k+1}^{-1} + H_{k+1}^T R_{k+1}^{-1} H_{k+1}$$

2) UPDATE STATE ESTIMATE

$$\hat{x}_{k+1} = \hat{x}_k + Q_{k+1} H_{k+1}^T R_{k+1}^{-1} (y_{k+1} - H_{k+1}\hat{x}_k)$$

$*$ Recall $\dot{x} = A\hat{x} + Bu + L(y - C\hat{x})$

$\uparrow$ estimate gain from desired estimator dynamics

$$* \; L_{RLS} = Q_{k+1} H_{k+1}^T R_{k+1}^{-1}$$

$$* \; Q_{k+1} H_{k+1}^T R_{k+1}^{-1} = \overbrace{[Q_k^{-1} + H_k^T R_{k+1}^{-1} H_{k+1}]^{-1}}^{n \times n} H_{k+1}^T R_{k+1}^{-1}$$
$$= Q_k H_{k+1}^T \underbrace{[H_{k+1} Q_k H_{k+1}^T + R_{k+1}]^{-1}}_{m \times m}$$

Estimating Dynamic States

- Let x vary with time and be "driven" by deterministic and stochastic inputs
- How to propogate $\hat{x}$?
- How to propogate Q (P)?

$*$ Recall $\dot{x} = Ax + Bu$
- deterministic
  - known u, known x_0
  - trajectory is completely specified by $x(t)$
- stochastic
  - x_0 — PDF i.e. $N(\bar{x}, P)$
  - inputs w — PDF i.e. $N(0, Q)$

- KEY IDEA: Gaussian PDF is fully defined by $\bar{x}, P$ and normality maintained through linear transformation

Propogation Derivation in Discrete

- Model $\hat{x}_{k+1} = A_d \hat{x}_k + B_d u + w_k$

A_d State transition matrix } linear transforms, can be time varying
B_d Control matrix
x_k state of random process
u_k deterministic inputs
w_k stochastic inputs
- zero mean
- white
- $E\{w_{k_1} w_{k_2}^T\} = Q\,\delta(k_2 - k_1)$

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$$x_k = A_d x_{k-1} + B_d u_{k-1} + w_{k-1}$$

Initialization:
- $E\{x_0\} = \hat{x}_0$
- $E\{(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)\} = P_0$
- $E\{(x_0 - \hat{x}_0) w_k^T\} = 0$ for all w_k

How to propogate $\hat{x}$ and P?

$$\hat{x}_k = E\{A_d x_{k-1} + B_d u_{k-1} + w_{k-1}\}$$
$$= A_d\{x_{k-1}\} + B_d u_{k-1} + 0$$
$$\hat{x}_k = A_d \hat{x}_{k-1} + B_d u_{k-1}$$

$$P_k = E\{(x_k - \hat{x}_k)(x_k - \hat{x}_k)^T\}$$
$$* \; x_k - \hat{x}_k = (A_d x_{k-1} + \cancel{B_d u_{k-1}} + w_{k-1}) - (A_d \hat{x}_{k-1} + \cancel{B_d u_{k-1}}) \; *$$
$$* \qquad = A_d(x_{k-1} - \hat{x}_{k-1}) + w_{k-1} \; *$$
$$= E\{[A_d(x_{k-1} - \hat{x}_{k-1}) + w_{k-1}][A_d(x_{k-1} - \hat{x}_{k-1}) + w_{k-1}]^T\}$$

3 Types of Terms:
1) $E\{A_d(x_{k-1} - \hat{x}_{k-1})(x_{k-1} - \hat{x}_{k-1})^T A_d^T\} = A_d P_{k-1} A_d^T$
2) $E\{w_{k-1} w_{k-1}^T\} = Q_{k-1}$
3) $E\{A_d(x_{k-1} - \hat{x}_{k-1}) w_{k-1}^T\} = 0$ or $E\{w_{k-1} A_d^T (x_{k-1} - \hat{x}_{k-1})^T\} = 0$
   ↳ uncorrelated x + w ∴ 0

$$P_k = A_d P_{k-1} A_d^T + Q_{k-1}$$

- Given constant noise variance: $Q_k = Q_{k-1} = Q_d$
- Given A_d and Q_d, a steady state covariance may exist as $k \to \infty$:
  - $P_k = P_{k-1}$
  - $P_{ss} = A_d P_{ss} A_d^T + Q_d$

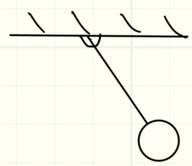
$\boxed{\text{Ex}}$ Scalar x

$$x_k = \alpha x_{k-1} + w_{k-1} \qquad w \sim \mathcal{N}(0,1)$$
$$P_k = \alpha P_{k-1} \alpha + 1$$
$$P_{ss} = \alpha^2 P_{ss} + 1$$
$$P_{ss} = \frac{1}{1-\alpha^2}$$

* Covariance must be positive $\therefore$ $-1 < \alpha < 1$ for P_{ss} to exist (stable eigenvalues – inside unit circle)
* homogeneous solution for x must go to 0 if P_{ss} is stable

* $P = A_d P A_d^T + Q$
  - ↳ discrete Lyapunov equation
  - ↳ solutions exist for stable LTI systems

$\boxed{\text{Ex}}$



$$J\ddot{\theta} + b\dot{\theta} + mgL\theta = w$$
$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{mgL}{J} & -\frac{b}{J} \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} w$$

Continuous Propogation

Model : $\dot{x} = Ax + Bu + B_w w$

1) $E\{w(t)\} = 0$
2) $E\{w(\tau)w(\gamma)^T\} = Q_c\,\delta(\tau-\gamma)$ ← noise PSD
3) $E\{x_0\} = \hat{x}_0$
4) $E\{(x_0-\hat{x}_0)(x_0-\hat{x}_0)^T\} = P_0$

$$* \; x_k = A_d x_{k-1} + \underbrace{\int_{t_{k-1}}^{t_k} A_d(t_{k-1},\tau) B_w w(\tau)\, d\tau}_{w_{k-1}} \; *$$

$$* \; A_d = \text{expm}(A * dt) \; *$$

$$Q_d = E\{w_{k-1} w_{k-1}^T\} = E\Big\{\Big(\int_{t_{k-1}}^{t_k} A_d B_w w(\tau)\, d\tau\Big)\Big(\int_{t_{k-1}}^{t_k} A_d B_w w(\gamma)\, d\gamma\Big)^T\Big\}$$
$$= E\Big\{\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t_k} A_d B_w w(\tau) w(\gamma)^T B_w^T A_d^T\, d\tau\, d\gamma\Big\}$$
$$= E\Big\{\int_{t_{k-1}}^{t_k} A_d B_w Q_c B_w^T A_d^T\, d\tau\Big\}$$

- Q_c could be simple (ie. affecting one state, diagonal)
- Q_d Generally a full matrix

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- $Q_d = \int A_d B_w Q_c A_d \, d\tau$
- $P_{k+1} = A_d P_k A_d^T + Q_d$
- For fast sampling of LTI system:

$$A_d(t_k, t_{k+1}) = e^{A\,dt} \cong I + A\,dt + \text{H.O.T.}$$

$$Q_d \approx (I + A\,dt) B_w Q_c B_w^T (I + A\,dt)^T \int dt$$
$$\approx B_w Q_c B_w^T\, dt + \text{H.O.T}$$

$$P_{k+1} \approx (I + A\,dt) P_k (I + A\,dt)^T + B_w Q_c B_w^T\, dt + \cancel{\text{H.O.T.}}^{\approx 0}$$
$$P_{k+1} - P_k = (AP_k + P_k A^T)dt + \cancel{AP_k A^T dt^2}^{\approx 0} + B_w Q_c B_w^T\, dt$$
$$\frac{P_{k+1} - P_k}{dt} = AP_k + P_k A^T + B_w Q_c B_w^T$$

$$\dot{P} = AP + PA^T + B_w Q_c B_w^T$$ } continuous covariance matrix dynamic / propagation model

$\dot{P}$:
- $n \times n$ matrix of diff. EQ.
- Symmetric → $\frac{n^2+n}{2}$ EQ. to be solved
- $AP + PA^T$ → homogeneous
  - From stable A ⇒ $\dot{P}$ decreases
- $B_w Q_c B_w^T$ → nonhomogeneous / input
  - positive ⇒ causes $\dot{P}$ to increase

Continuous	Discrete
$\dot{x} = Ax + Bu + B_w w$	$x_k = A_d x_{k-1} + B_d u_{k-1} + w_{k-1}$
↳ w : white $\sim \mathcal{N}(0, Q_c)$	↳ w_{k-1} : white $\sim \mathcal{N}(0, Q_d)$
$\dot{\hat{x}} = A\hat{x} + Bu$	$\hat{x}_k = A_d \hat{x}_{k-1} + B_d u_{k-1}$
$\dot{P} = AP + PA^T + B_w Q_c B_w^T$	$P_k = A_d P_{k-1} A_d^T + Q_d$
– continuous lyapunov	$P_{ss} = A_d P_{ss} A_d^T + Q_d$
	– discrete lyapunov

Conversions :
- $A_d = \text{expm}(A\,dt)$
- $B_d = \int_0^{dt} e^{A\tau} B\, d\tau$
- $Q_d = \int_0^{dt} A_d B_w Q_c A_d^T\, d\tau$

Discrete Approximations (Bryson):
- $S \triangleq \begin{bmatrix} -A & B_w Q_c B_w^T \\ 0 & A^T \end{bmatrix}$
- $C = e^{S\Delta t} = \begin{bmatrix} C_{11} & C_{12} \\ 0 & C_{22} \end{bmatrix}$
- $A_d = C_{22}^T$
- $Q_d = C_{22}^T C_{12}$

$\boxed{\text{Ex}}$ $\dot{x} = -x + w$ Scalar, Find P_{ss}

$Q_c = 1$

$$\dot{P} = AP + PA^T + B_w Q_c B_w^T$$
$$\dot{P} = 2AP + B_w^2 Q_c \quad \Rightarrow \quad \dot{P} + P = \frac{B_w^2 Q_c}{-2A}$$

$$P(t) = -\frac{B_w^2 Q_c}{2A}\left(1 - e^{2At}\right) + \underbrace{P_0 e^{2At}}_{\text{I.C.}}$$

* For $A < 0$: $P_{ss} = -\frac{B_w^2 Q_c}{2A}$
* For $B_w = Q_c = A = 1$: $P_{ss} = \frac{1}{2}$

In discrete with $dt = 0.1$: $A_d = 0.905$

$S = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$ ↳ from MATLAB: $Q_d = 0.0906$

$P_{ss,d} = 0.4998$