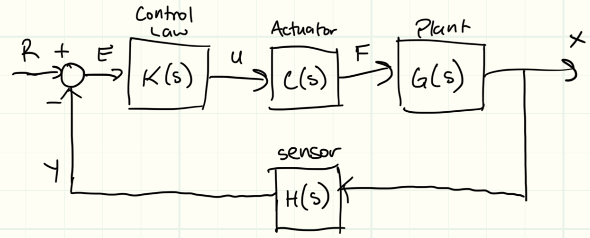
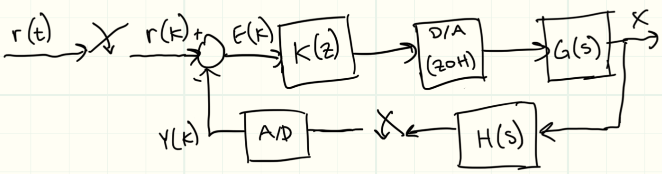
MECH 7710 ~ Optimal 1/11/23

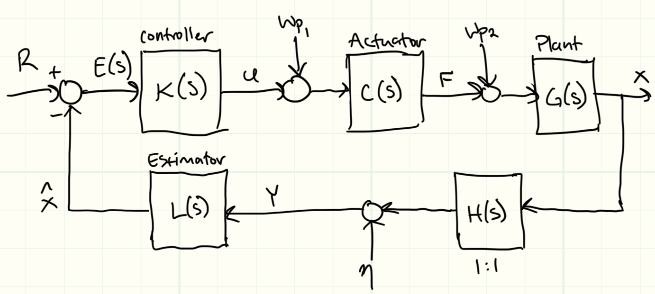
Classical Deterministic Dynamic Feedback Systems



Discrete case:



Continuous with unknown inputs/noise:



* Design of K(s) and L(s) should consider:
  - i) Limited state feedback
  - ii) Bad / noisy measurements
  - iii) Imperfect dynamic models
  - iv) Quantization error / delay

Two Review Topics:
1) Continuous / Discrete Systems
2) State-Space Representation

Continuous

$\dot{x} = Ax + Bu$

$y = Cx$

* $A = F$
* $B = G$
* $C = H$

Discrete

$x_k = A_D x_{k-1} + B u_k$

$y_k = C_D x_k$

* $A_D = \Phi_{k-1,k}$ → state transition matrix

Continuous

$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t) e^{-st} dt$

$\mathcal{L}\{\dot{x}\} = sX(s) + \text{I.C.}$

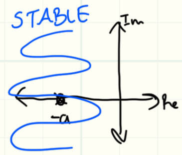
$\dot{x} + ax = u$

$\frac{X(s)}{U(s)} = \frac{1}{s+a}$

CE: $s + a = 0$

$s = -a$

STABLE



$G_{DC} = \frac{1}{a}$

Discrete

$Z\{x(k)\} = X(z) = \sum_{k=0}^{\infty} x(k) z^{-k}$

$Z\{x(k-1)\} = z^{-1} X(z)$

$Z\{x(k+1)\} = z X(z)$

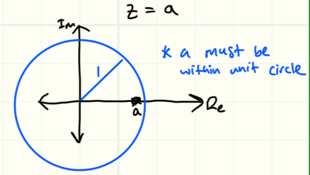
$x(k) = a x(k-1) + u(k)$

$X(z) = a z^{-1} X(z) + U(z)$

$\frac{X(z)}{U(z)} = \frac{1}{1 - a z^{-1}}$

CE: $1 - a z^{-1} = 0$

$z = a$



* a must be within unit circle

$G_{DC} = \frac{1}{1-a} \quad (z=1)$

* discrete eigenvalue is a function of the continuous eigenvalue and the time step

$z = e^{sT}$

T: sample rate

RELATIONSHIP

$Z = e^{sT}$ * T: time step $= (\Delta t, dt)$

- Final Value Theorum

• $\lim_{t \to \infty} x(t) = \lim_{s \to 0} sX(s)$

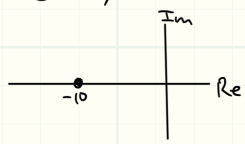
• $\lim_{k \to \infty} x(k) = \lim_{z \to 1} (z-1) X(z)$

1/18/23

$\boxed{\text{Ex}}$ $\dot{x} + 10x = 10u$, $x(0) = 0$

$$x(t) = 10\left(1 - e^{-\frac{t}{10}}\right)$$

$$\frac{X(s)}{U(s)} = \frac{10}{s+10}$$



$s = -10$

$$\left.\begin{aligned} \dot{x}(k) &= 10u(k) - 10x(k) \\ x(k+1) &= x(k) + \dot{x}(k)\,T \end{aligned}\right\} \text{euler integration}$$

$x(k+1) = x(k)[1 - 10T] + 10T\,u(k)$ * if $T = \frac{1}{20}$

$x(k+1) = \frac{1}{2}x(k) + \frac{1}{2}u(k)$

$$\left.\begin{aligned} zX(z) &= \tfrac{1}{2}X(z) + \tfrac{1}{2}U(z) \\ \frac{X(z)}{U(z)} &= \frac{\frac{1}{2}}{z - \frac{1}{2}} = \frac{1}{2z - 1} \end{aligned}\right\} z\text{-transform}$$

* $z = \frac{1}{2}$

$z = e^{sT} = e^{-10\left(\frac{1}{20}\right)} = e^{-\frac{1}{2}} \approx 0.6$

* euler integration ≠ relationship

Methods of Approximating Continuous Functions

1) Matched poles and zeros (MPZ)

a) Calculate equivalent poles & zeros
$z = e^{sT}$

b) Match DC Gain

$\frac{X(s)}{U(s)} = \frac{10}{s+10}$, $T = \frac{1}{20}$

$z = e^{sT} \approx 0.6$

$\frac{X(z)}{U(z)} = \frac{0.4}{z - 0.6} = 1$ * for $z = 1$ $\rightarrow g_{DC} = 1$

*euler: $x(k+1) = \frac{1}{2}x(k) + \frac{1}{2}u(k)$

* MPZ: $x(k+1) = 0.6\,x(k) + 0.4\,u(k)$

$\Updownarrow$

$x(k) = 0.6\,x(k-1) + 0.4\,u(k-1)$

2) Tustin's Method

a) Find TF of continuous system

b) Plugin $s = \frac{2}{T}\,\frac{z-1}{z+1} = \frac{2}{T}\,\frac{1 - z^{-1}}{1 + z^{-1}}$

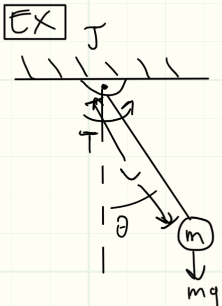
* roughly equivalent to trapezoidal integration

3) MATLAB

a) `>> c2d ( tf , T , 'method')`
`>> c2d ( sys , T , 'method')`

State-Space Control and Estimation

$\boxed{\text{Ex}}$



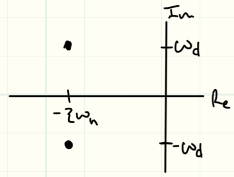
$\sum M = J\ddot{\theta} = T - mgL\sin\theta$

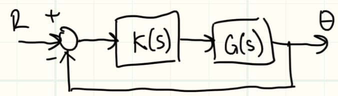
* SAA $T = J\ddot{\theta} + mgL\,\theta$

$T(s) = Js^2\,\Theta(s) + mgL\,\Theta(s)$

$T(s) = \Theta(s)\left[Js^2 + mgL\right]$

$\frac{\Theta(s)}{T(s)} = \frac{1}{Js^2 + mgL}$





$$\frac{\Theta(s)}{R(s)} = \frac{KG}{1 + KG} = \frac{K_1 s + K_2}{Js^2 + K_1 s + mgL + K_2}$$

$T = J\ddot{\theta} + mgL\theta$ $\dot{x} = Ax + Bu$
$y = Cx + Du$

$$\begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{mgL}{J} & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{J} \end{bmatrix} u$$

$$\begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$$

- Assuming full State-Feedback
- Assuming $R = 0$
- $u = T = -Kx$
  - $\Rightarrow K = [K_1 \quad K_2]$
  - $\rightarrow \dot{x} = Ax + B(-Kx) = (A - BK)x$
  - $\rightarrow$ get eigenvalues $0 = \det(sI - A)$
    $0 = \det(sI - (A - BK))$

State Space Estimator

$$\underbrace{\hat{x}}_{\text{estimated value}} = \overbrace{x}^{\text{true value}} + \delta x \qquad \tilde{x} = \text{measured value}$$

Estimator Dynamics: $\dot{\hat{x}} = A\hat{x} + Bu + L(y - \underbrace{C\hat{x}}_{\hat{y}})$

(Controller $\dot{x} = Ax + Bu$, $y_{m\times1} = Cx_{n\times1}$

$$\dot{\hat{x}} = \underbrace{(A - LC)}_{\bar{A}}\hat{x} + Bu + Ly$$

- L must be $n \times m$
  - $\hat{x}_{n\times1}$, $\hat{y}_{m\times1}$, $u_{p\times1}$
  - $A_{n\times n}$, $C_{m\times n}$, $B_{n\times p}$

- eigenvalues $0 = \det(sI - \bar{A})$
- place L using desired eigenvalues
  >> $L = \text{place}(A', C', \text{poles})'$

Equivalent Compensator

$$y \rightarrow \boxed{?} \rightarrow u$$

- Turns output into the input

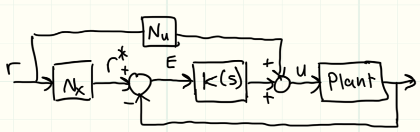
$$\dot{x} = Ax + Bu$$
$$y = Cx$$
$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$
$$u = -K\hat{x}$$

$$\dot{\hat{x}} = \underbrace{(A - BK - LC)}_{A_{comp}}\hat{x} + \underbrace{L}_{B_{comp}}y \qquad u = \underbrace{-K}_{C_{comp}}\hat{x} + \underbrace{D}_{D_{comp}}u$$

$$\frac{U(s)}{Y(s)} = -K[sI - (A - BK - LC)]^{-1}L$$

* for 1 measurement and 1 input *

Feed forward Reference Scaling

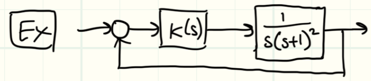


$$r^* = rN_x \qquad \begin{bmatrix} N_x \\ N_u \end{bmatrix} = \begin{bmatrix} A - I & B \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ I \end{bmatrix}$$
$$u_{ss} = rN_u$$

$$\bar{N} = N_u + KN_x \Rightarrow u = -Kx + \bar{N}r$$

Gain and Phase Margin

- Associated with open loop TF



* open loop Bode