

Homework 1

1. A control law for a simple rotation table is to be designed. The table has a rotational moment of inertia (J) of 10 kg-m² and rotational damping (b) of 1 N-m-s/rad. Torque is commanded to the motor and the table's position is measured using a rotary encoder.
 - (a) Derive the simple differential equation for the system.
 - (b) Convert the system into a state-space format.
 - (c) What are the eigenvalues of the system.

Solution:

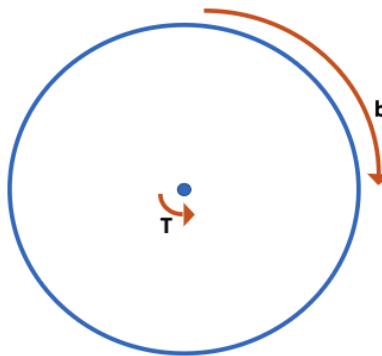


Figure 1: System Diagram.

Summing the moments on the table.

$$\begin{aligned}
 \sum M &= J\ddot{\theta} = T - b\dot{\theta} \\
 &= 10\ddot{\theta} = T - 1\dot{\theta} \\
 T &= 10\ddot{\theta} + \dot{\theta}
 \end{aligned} \tag{1}$$

Linearizing the system and putting it in matrix form.

$$\begin{aligned}
 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} \\
 \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} x_2 \\ 0.1(u - x_2) \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ 0.1(T - \dot{\theta}) \end{bmatrix} \\
 y &= \theta
 \end{aligned} \tag{2}$$

Formulating the state space equations.

$$\begin{aligned}\dot{x} &= Ax + Bu \\ \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u\end{aligned}\quad (3)$$

$$\begin{aligned}y &= Cx \\ y &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}\end{aligned}\quad (4)$$

Solving for the eigenvalues of the open-loop system.

$$\begin{aligned}0 &= \det(sI - A) \\ 0 &= \det \begin{bmatrix} s & -1 \\ 0 & s + 0.1 \end{bmatrix} \\ 0 &= s(s + 0.1) \\ s &= 0, -0.1\end{aligned}\quad (5)$$

2. Design an observer for the above system.

- (a) Show that the system is observable.
- (b) Design L such that the error dynamics have $f_n = 50$ Hz and $\zeta = 0.7$.
- (c) Provide a plot of the step response of the estimator.

Solution:

To determine the observability of the system, the rank of the observability matrix is checked. For this system this must equal 2 (the dimension of A).

$$\mathcal{O} = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\quad (6)$$

$$\text{rank}(\mathcal{O}) = 2\quad (7)$$

The system is observable and the poles of the observer can be placed using the dynamics defined.

$$\begin{aligned}
 0 &= s^2 + 2\omega_n\zeta s + \omega_n^2 \\
 0 &= s^2 + 2(2\pi f_n)\zeta s + (2\pi f_n)^2 \\
 0 &= s^2 + 2(100\pi)0.7s + (100\pi)^2 \\
 s &= -219.91 \pm 224.35i = s_{obsv}
 \end{aligned} \tag{8}$$

$$L = place(A', C', s_{obsv})' = \begin{bmatrix} 439.72 \\ 98652.08 \end{bmatrix} \tag{9}$$

The new closed-loop A matrix.

$$A_{obsv} = A - LC = \begin{bmatrix} -439.72 & 1 \\ -98652.08 & -0.1 \end{bmatrix} \tag{10}$$

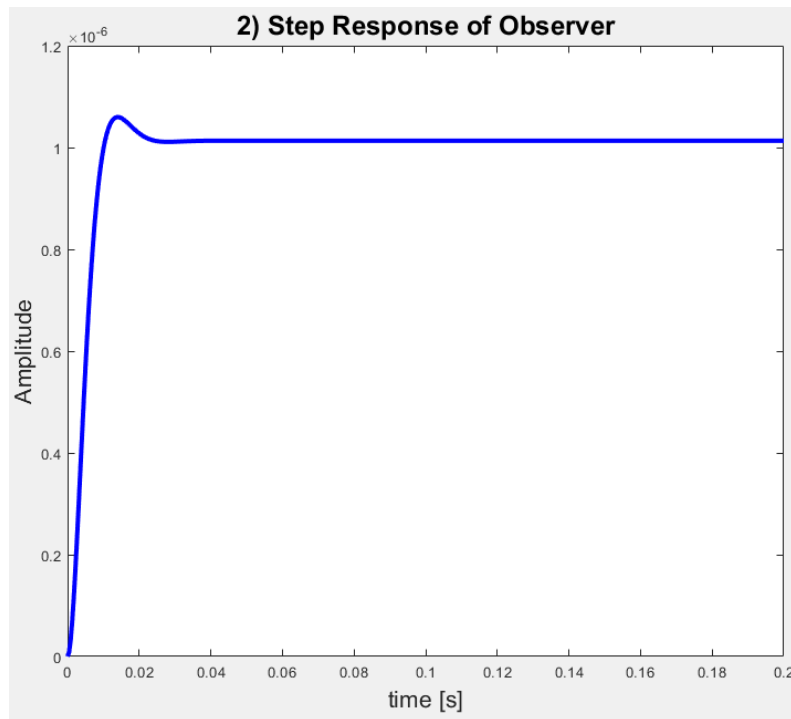


Figure 2: Observer Step Response.

3. Design a state-feedback controller for the table.

(a) Show that the system is controllable.

- (b) Design K such that $f_n = 10$ Hz and $\zeta = 0.7$.
 (c) Provide a plot of the step response of the combined estimator and controller.

Solution:

To determine the controllability of the system, the rank of the controllability matrix is checked. For this system this must equal 2 (the dimension of A).

$$\mathcal{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0.1 \\ 0.1 & -0.01 \end{bmatrix} \quad (11)$$

$$\text{rank}(\mathcal{C}) = 2 \quad (12)$$

The system is controllable and the poles of the controller can be placed using the dynamics defined.

$$\begin{aligned} 0 &= s^2 + 2\omega_n\zeta s + \omega_n^2 \\ 0 &= s^2 + 2(2\pi f_n)\zeta s + (2\pi f_n)^2 \\ 0 &= s^2 + 2(20\pi)0.7s + (20\pi)^2 \\ s &= -43.98 \pm 44.87i = s_{cont} \end{aligned} \quad (13)$$

$$K = \text{place}(A, B, s_{cont}) = \begin{bmatrix} 39478.40 & 878.65 \end{bmatrix} \quad (14)$$

The new combined closed-loop system.

$$\begin{aligned} A &= \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -3947.84 & -87.96 & 3947.84 & 87.86 \\ 0 & 0 & -439.72 & 1 \\ 0 & 0 & -98652.07 & -0.1 \end{bmatrix} \\ B &= \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.1 \\ 0 \\ 0 \end{bmatrix} \\ C &= \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ D &= 0 \end{aligned} \quad (15)$$

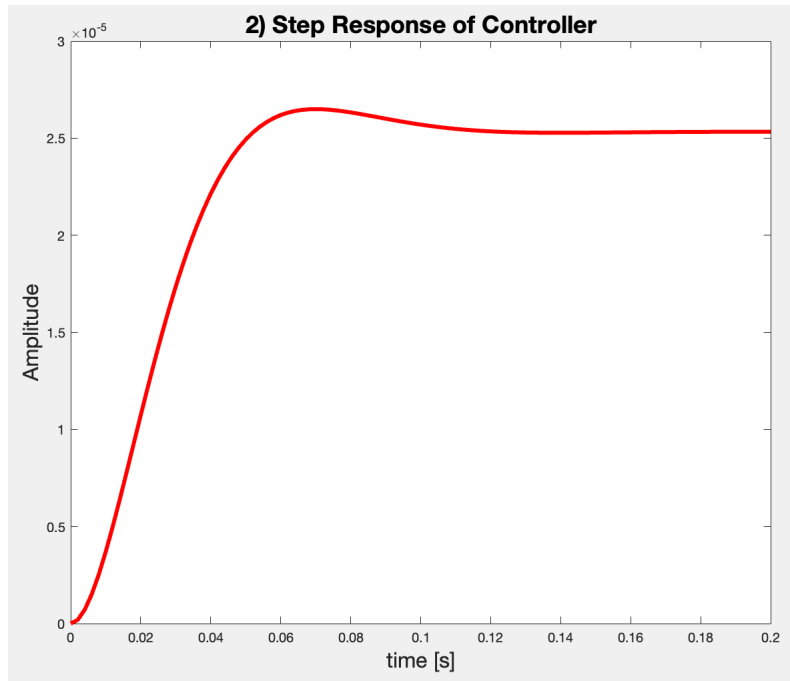


Figure 3: Combined Controller and Observer Step Response.

4. Solve for the equivalent compensator for the system.

- What kind of classical compensator does it resemble?
- Calculate the closed loop transfer function.
- Plot the Bode Plot of the closed-loop system.
- Find the gain and phase margin.

Solution:

Transforming the controller and observer defined above into an equivalent compensator results in the following system.

$$\begin{aligned}
 A_{comp} &= A - BK - LC = \begin{bmatrix} -439.72 & 1 \\ -102599.92 & -87.96 \end{bmatrix} \\
 B_{comp} &= L = \begin{bmatrix} 439.72 \\ 98652.08 \end{bmatrix} \\
 C_{comp} &= K = \begin{bmatrix} 39478.40 & 878.65 \end{bmatrix}
 \end{aligned} \tag{16}$$

With an equivalent transfer function of:

$$\begin{aligned}\frac{U(s)}{Y(s)} &= C_{comp}(sI - A_{comp})^{-1}B_{comp} = -K(sI - A + BK + LC)^{-1}L \\ &= \frac{-1.04(10^8)s - 3.896(10^9)}{s^2 + 527.7s + 1.413(10^5)}\end{aligned}\quad (17)$$

This resembles a "lead-lag" compensator because of the extra pole in the denominator (first order numerator, second order denominator). The new system looks like the following when $r = 0$ which in turn makes $e = -y$ and $\frac{U(s)}{E(s)} = -\frac{U(s)}{Y(s)}$.

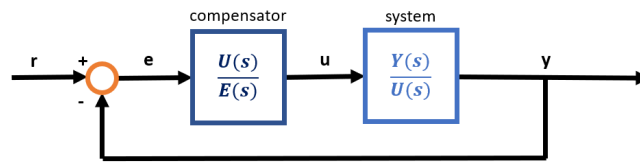


Figure 4: System diagram with Compensator.

To close the loop between the compensator and the plant the following state space system is applied.

$$\begin{aligned}A_{cl} &= \begin{bmatrix} A & BC_{comp} \\ -B_{comp}C & A_{comp} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -0.1 & 3947.84 & 87.86 \\ -439.72 & 0 & -439.72 & 1 \\ -98652.08 & 0 & 102599.92 & -87.96 \end{bmatrix} \\ B_{cl} &= \begin{bmatrix} 0 \\ B_{comp} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 439.72 \\ 98652.08 \end{bmatrix} \\ C_{cl} &= \begin{bmatrix} C & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \\ D_{cl} &= 0\end{aligned}\quad (18)$$

Plotting the Bode Plot for the closed-loop system.

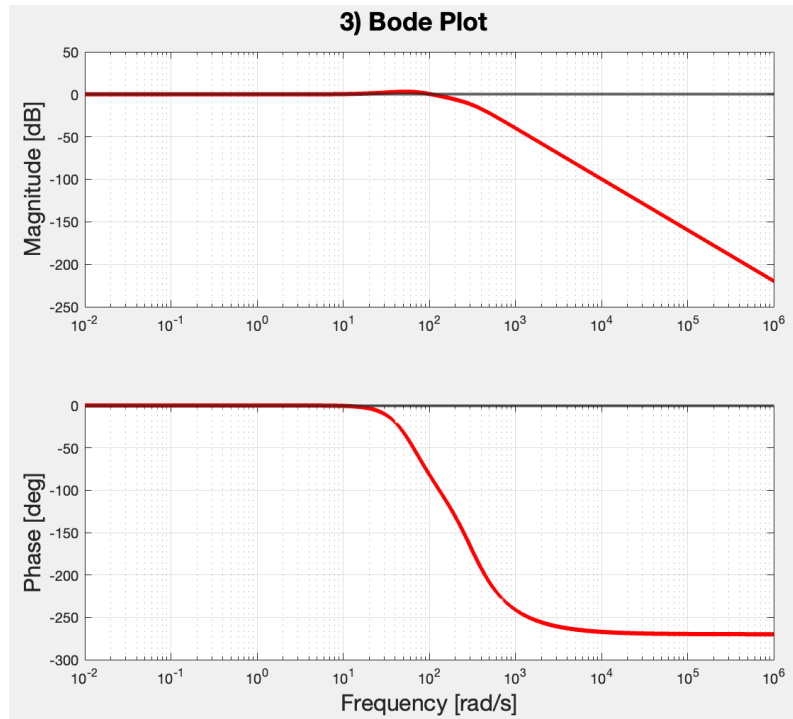


Figure 5: Closed-Loop Compensator Bode Plot.

Using the MATLAB function `margin`, the gain and phase margin are easily calculated and can be confirmed by analyzing the Bode Plot above.

$$\begin{aligned} G_m &= 14.26 \text{ dB} \\ \phi_m &= 94.69^\circ \end{aligned} \tag{19}$$

5. Design the controller in the discrete domain assuming a 1 KHz sample rate.

- (a) Discretize the state space model. Where are the eigenvalues?
- (b) Design L to provide the same response as problem 2.
- (c) Design K to provide the same response as problem 3.
- (d) Where are the closed-loop estimator and controller poles located?
- (e) Solve for the equivalent compensator transfer function.

Solution:

To discretize the continuous model, the MATLAB function `c2d` was used in combination with `ss` to

transform the continuous model into state-space.

$$\begin{aligned}
 sys &= c2d(ss(A, B, C, D), 1/1000) \\
 A_z &= \begin{bmatrix} 1 & 0.001 \\ 0 & 0.999 \end{bmatrix} \\
 B_z &= \begin{bmatrix} 5(10^{-8}) \\ 1(10^{-4}) \end{bmatrix} \\
 C_z &= \begin{bmatrix} 1 & 0 \end{bmatrix} \\
 D_z &= 0
 \end{aligned} \tag{20}$$

To find the discrete poles, s_{obsv} and s_{cont} must be converted to discrete poles.

$$\begin{aligned}
 z &= e^{sT} \\
 z_{obsv} &= e^{s_{obsv} * T} = e^{(-219.91 \pm 224.35i) * 1/1000} = 0.7825 \pm 0.1786i \\
 z_{cont} &= e^{s_{cont} * T} = e^{(-43.98 \pm 44.87i) * 1/1000} = 0.9560 \pm 0.0429i
 \end{aligned} \tag{21}$$

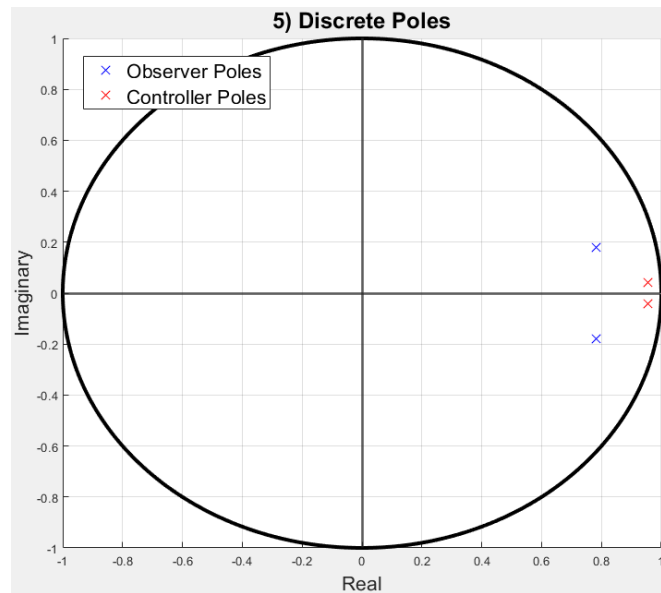


Figure 6: Discrete Pole Placement.

These new poles can then be used in the MATLAB *place* function to create the discrete observer and controller.

$$L = place(A'_z, C'_z, z_{obsv})' = \begin{bmatrix} 0.4349 \\ 79.1604 \end{bmatrix} \quad (22)$$

$$K = place(A_z, B_z, z_{cont}) = \begin{bmatrix} 37781.3243 & 859.9995 \end{bmatrix}$$

Using the same method of finding the compensator transfer function from above:

$$\begin{aligned} \frac{Y(z)}{U(z)} &= -K_z(zI - A_z + B_zK + L_zC_z)^{-1}L_z \\ &= \frac{-8.451(10^4)z + 8.152(10^4)}{z^2 - 1.477z + 0.594} \end{aligned} \quad (23)$$

6. Compare the continuous and discrete response using simulation and using equivalent compensator. Plot the simulated and equivalent compensator responses. Compare the expected and actual response.

Solution:

Closing the loop for the discrete compensator using $c2d$ on the continuous compensator system.

$$sys_{comp_z} = c2d(sys_{comp}, 1/1000) \quad (24)$$

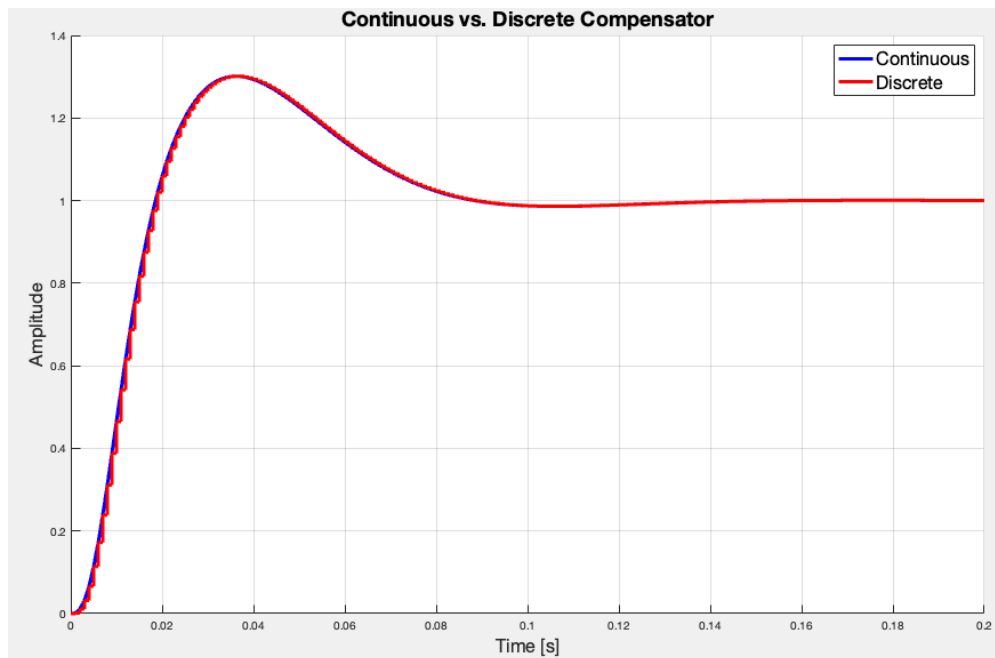


Figure 7: Equivalent Compensator Response.