Homework 1

1. Use the MATLAB *conv* function to produce discrete PDF's for 6 dice throws. Check that $\sum PDF = 1.0$ and plot each PDF with a normal distribution plot of the same average and sigma.

(a) 6 numbered 1, 2, 3, 4, 5, 6
(b) 6 numbered 4, 5, 6, 7, 8, 9
(c) 6 numbered 1, 1, 3, 3, 3, 5
(d) 3 numbered 1, 2, 3, 4, 5, 6 and 3 numbered 1, 1, 3, 3, 3, 5

Solution:
First the PMF for each dice was created, the *conv* function was then used to combine the remaining 5 rolls with the first, after which the sum of the new PMF was calculated. The following MATLAB code was used.

```
% probability mass functions
prob1.pmf_a = [1/6, 1/6, 1/6, 1/6, 1/6, 1/6];
prob1.pmf_b = [1/6, 1/6, 1/6, 1/6, 1/6, 1/6];
prob1.pmf_c = [1/3, 0, 1/2, 0, 1/6, 0];
prob1.pmf_d_1 = [1/6, 1/6, 1/6, 1/6, 1/6, 1/6];
prob1.pmf_d_2 = [1/3, 0, 1/2, 0, 1/6, 0];

% initialize
prob1.a = prob1.pmf_a;
prob1.b = prob1.pmf_b;
prob1.c = prob1.pmf_c;
prob1.d = prob1.pmf_d_1;

% convolve pdf for 5 more throws
for i = 1:5
    prob1.a = conv(prob1.a, prob1.pmf_a);
    prob1.b = conv(prob1.b, prob1.pmf_b);
    prob1.c = conv(prob1.c, prob1.pmf_c);
    if (i < 3)
        prob1.d = conv(prob1.d, prob1.pmf_d_1);
    else
        prob1.d = conv(prob1.d, prob1.pmf_d_2);
    end
end
```

```
% check sum == 1
prob1.sum_a = sum(prob1.a);
prob1.sum_b = sum(prob1.b);
prob1.sum_c = sum(prob1.c);
prob1.sum_d = sum(prob1.d);
```

This results in a sum of 1 for each of the new PMF's and plotting the distributions shown below:

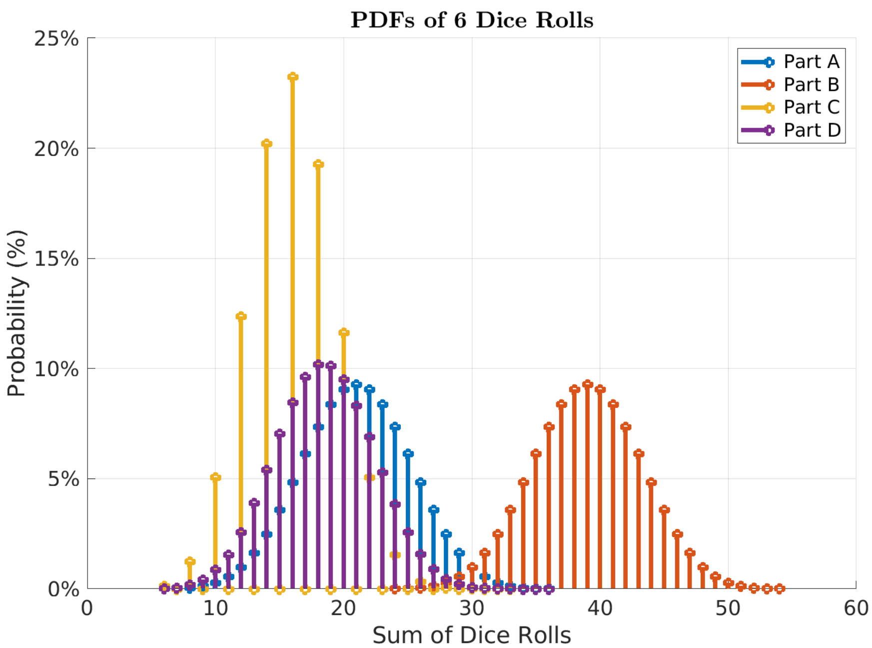

Figure 1: PDF's after Convolution.

2. What is the joint PDF for 2 fair dice (x_1, x_2) (a 6x6 matrix where each row adds to the probability index of x_1 and each column to the probability index for x_2).

(a) What are E$\{x_1\}$, E$\{x_1$-E$\{x_1\}\}$, E$\{x_1^2\}$, E$\{(x_1$-E$\{x_1\})^2\}$ and E$\{(x_1$-E$\{x_1\})(x_2$-E$\{x_2\})\}$.

(b) Form the covariance matrix for x_1 and x_2.

(c) Find the joint PDF for $v_1 = x_1$ and $v_2 = x_1 + x_2$.

(d) Find the mean, E$\{v_1$-E$\{v_1\}\}$, rms, and variance of v_1.

(e) Find the mean, E$\{v_2$-E$\{v_2\}\}$, rms, and variance of v_2.

(f) Find the covariance matrix of v_1 and v_2.

Solution:

To create the joint PDF of two dice, the vector PDF for each dice was multiplied together (each row/column sums to 1):

$$f_{x_1x_2} = \frac{1}{6}.*ones(1,6)^T * \frac{1}{6}.*ones(1,6) = \frac{1}{36}.*ones(6,6)$$

For *part a*:

$$E\{x_1\} = \bar{x}_1 = \int x_1 f_{x_1} dx = \sum_{i=1}^{6} \frac{1}{6} i = 3.5$$
$$E\{x_1 - E\{x_1\}\} = E\{x_1 - \bar{x}_1\} = \bar{x}_1 - \bar{x}_1 = 0$$
$$E\{x_1^2\} = \int x_1^2 f_{x_1} dx = \sum_{i=1}^{6} \frac{1}{6} i^2 = 15.17$$
$$E\{(x_1 - E\{x_1\})^2\} = \sigma_1^2 = E\{x_1^2\} - \bar{x}_1^2 = 15.17 - 3.5^2 = 2.92$$
$$E\{(x_1 - E\{x_1\})(x_2 - E\{x_2\})\} = E\{x_1 x_2\} - \bar{x}_1\bar{x}_2 = E\{x_1\}E\{x_2\} - \bar{x}_1\bar{x}_2 = 0$$

For *part b*:

$$\sigma_1^2 = \sigma_2^2$$
$$P = \begin{bmatrix} 2.92 & 0 \\ 0 & 2.92 \end{bmatrix}$$

For *part c* (multiplying $f_{v_1}^T * f_{v_2}$):

$$v_1 = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{bmatrix}$$
$$v_2 = \begin{bmatrix} 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{bmatrix}$$
$$f_{v_1} = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$
$$f_{v_2} = \begin{bmatrix} \frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{6} & \frac{5}{36} & \frac{1}{9} & \frac{1}{12} & \frac{1}{18} & \frac{1}{36} \end{bmatrix}$$

$$f_{v_1 v_2} = \begin{bmatrix} \frac{1}{216} & \frac{1}{108} & \frac{1}{72} & \frac{1}{54} & \frac{5}{216} & \frac{1}{36} & \frac{5}{216} & \frac{1}{54} & \frac{1}{72} & \frac{1}{108} & \frac{1}{216} \end{bmatrix} .* ones(6, 11)$$

Now each row of $f_{v_1 v_2}$ sums to f_{v_1} and each column sums to f_{v_2}.

For *part d*: v_1 has the same values as x_1 (*part a*).

For *part e*:

$$E\{v_2\} = \bar{v}_2 = \int v_2 f_{v_2} dv = 7$$
$$E\{v_2 - E\{v_2\}\} = E\{v_2 - \bar{v}_2\} = \bar{v}_2 - \bar{v}_2 = 0$$
$$E\{v_2^2\} = \int v_2^2 f_{v_2} dv = 54.83$$
$$E\{(v_2 - \bar{v}_2)^2\} = \sigma_{v2}^2 = E\{v_2^2\} - E\{v_2\}^2 = 54.83 - 7^2 = 5.83$$

For *part f*:

$$E\{((x_2+x_1)-(\bar{x}_2+\bar{x}_1))^2\} = E\{x_2^2-\bar{x}_2^2+x_1^2-\bar{x}_1^2\} = \sigma_2^2+\sigma_1^2 = \sigma_{v2}^2 = 5.83$$
$$E\{(x_1-\bar{x}_1)((x_2+x_1)-(\bar{x}_2+\bar{x}_1))\} = P_{12} = P_{21} = E\{x_1\}-\bar{x}_1 = \sigma_1^2 = 2.92$$

$$P = \begin{bmatrix} \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2+\sigma_2^2 \end{bmatrix} = \begin{bmatrix} 2.92 & 2.92 \\ 2.92 & 5.93 \end{bmatrix}$$

3. Two random vectors X_1 and X_2 are called uncorrelated if:

$$E\{(X_1-\bar{X}_1)(X_2-\bar{X}_2)\} = 0$$

(a) Show that independent random vectors are uncorrelated.
(b) Show that uncorrelated Gaussian random vectors are independent.

Solution:
For independent random vectors:

$$E\{x_1x_2\} = E\{x_1\}E\{x_2\} = \bar{x}_1\bar{x}_2$$
$$E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\} = E\{x_1x_2-x_1\bar{x}_2-x_2\bar{x}_1+\bar{x}_1\bar{x}_2\} = \bar{x}_1\bar{x}_2-\bar{x}_1\bar{x}_2-\bar{x}_1\bar{x}_2+\bar{x}_1\bar{x}_2$$

This sums to 0, therefore the vectors are uncorrelated.

$$E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\} = 0$$

For any random Gaussian vectors:

$$\rho_{12} = \frac{E\{(x_1-\bar{x}_1)(x_2-\bar{x}_2)\}}{\sigma_1\sigma_2}$$

If uncorrelated, the covariance between the vectors is 0, therefore:

$$\rho_{12} = 0$$
$$P_{12} = 0$$

4. Consider a sequence created by throwing a pair of dice and summing the numbers (-2.5, -1.5, -0.5, 0.5, 1.5, 2.5) called $V_0(k)$.

(a) What is the PDF?
(b) What are the mean and variance of this sequence?
(c) If we generate a new random sequence ($V_N(k+1) = (1-r)V_N(k) + rV_0(k)$) ($V_N$ is serially correlated, not white). In steady-state, what are the mean and variance of the new sequence?

(d) What is the covariance function: $R(k) = E\{V_N(k)V_N(k-L)\}$ (Hint: $V_N(k)$ and $V_0(k)$ are uncorrelated).

(e) Are there any practical constraints on r?

Solution:

Using the MATLAB *conv* function two make the PDF for the sum of two dice rolls:

$$V_0 = \begin{bmatrix} -5 & -4 & -3 & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$
$$f_{V_0} = \begin{bmatrix} \frac{1}{36} & \frac{1}{18} & \frac{1}{12} & \frac{1}{9} & \frac{5}{36} & \frac{1}{6} & \frac{5}{36} & \frac{1}{9} & \frac{1}{12} & \frac{1}{18} & \frac{1}{36} \end{bmatrix}$$

Calculating the mean and variance of V_0:

$$\mu_{V_0} = \sum_{i=1}^{11} f_{V_0}(i)V_0(i) = 0$$
$$\sigma_{V_0}^2 = \sum_{i=1}^{11} \left(f_{V_0}(i)V_0^2(i)\right) - \mu_{V_0}^2 = 5.83$$

For the system, $V_N(k+1) = (1-r)V_N(k) + rV_0(k)$, the mean of V_N is most likely zero because it is made entirely from V_0, which is zero mean. An analytical solution for the variance was determined below. An emperical (monte-carlo) approach was also used, described in the code and figure below.

$$\begin{aligned}
\mu(k) &= 0 \\
\sigma^2(k) &= E\{V_N(k)V_N(k)\} \\
&= E\{(V_N(k))^2\} \\
&= E\{((1-r)V_N(k) + rV_0(k))^2\} \\
&= E\{V_0^2(k)r^2 - 2V_0(k)V_N(k)r^2 + 2V_0(k)V_N(k)r + V_N^2(k)r^2 - 2V_N^2(k)r + V_N^2(k)\} \\
&= E\{V_0^2(k)r^2 + V_N^2(k)r^2 - 2V_N^2 r + V_N^2(k)\} \\
&= E\{V_0^2(k)r^2 + V_N^2(k)[r^2 - 2r + 1]\} \\
&= r^2 E\{V_0^2(k)\} + (1-r)^2 E\{V_N^2(k)\} \\
&= r^2(5.83) + (1-r)^2 E\{V_N^2(k)\}
\end{aligned}$$

At steady state:

$$\sigma^2(k) = r^2(5.83)$$

```
% special dice
prob4.dice = [-2.5, -1.5, -0.5, 0.5, 1.5, 2.5];

% initialization
```

```
i = 1;
prob4.VN_var = zeros(100,1);
prob4.VN_mean = zeros(100,1);

% r values
for r = linspace(0,1.9,100)
    % reset
    prob4.V0 = zeros(10000,1);
    prob4.VN = zeros(10000,1);

    % monte carlo runs
    for k = 1:10000
        prob4.V0(k) = sum([prob4.dice(randi(6)), prob4.dice(randi(6))]);
        prob4.VN(k+1) = (1-r)*prob4.VN(k) + r*prob4.V0(k);
    end

    % get statistics
    prob4.VN_var(i) = var(prob4.VN);
    prob4.VN_mu(i) = mean(prob4.VN);
    i = i + 1;
end
```

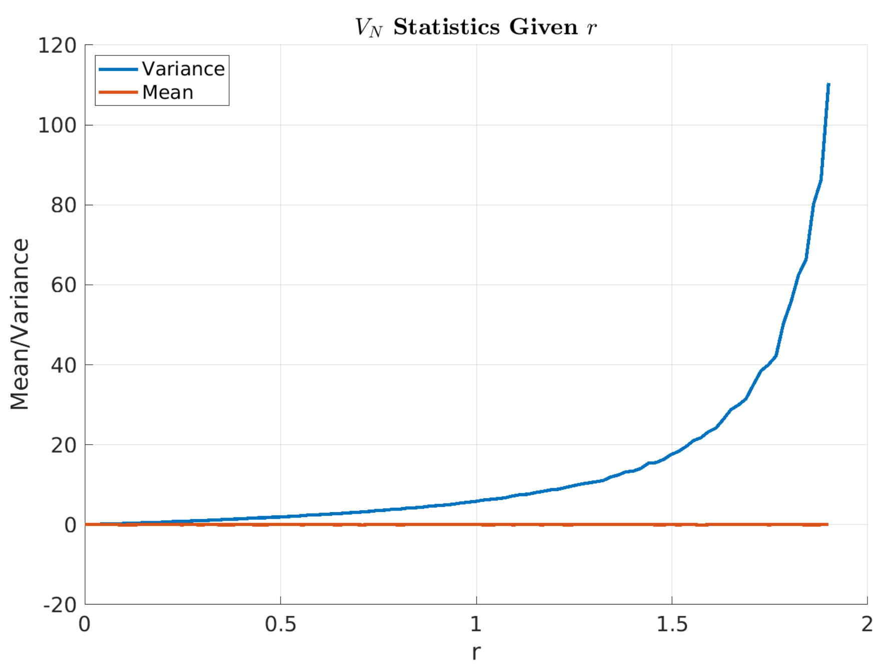

Figure 2: Mean and Variance From Monte-Carlo Simulation.

This proves that the mean is approximately zero and the provides insight that the variance is exponentially related to r.

The autocorrelation function (covariance) for the system is as follows:

$$R(k) = E\{V_N(k)V_N(k+1-L)\}$$

Rewriting for the autocorrelation for $k+1$:

$$\begin{aligned}R(k+1) &= E\{V_N(k+1)V_N(k+1-L)\}\\ &= E\{[(1-r)V_N(k)+rV_0(k)][(1-r)V_N(k-L)+rV_0(k-L)]\}\\ &= E\{(1-r)^2V_N(k)V_N(k-L)+r(1-r)V_N(k)V_0(k-L)+\\ &\quad r(1-r)V_0(k)V_N(k-L)+r^2V_0(k)V_0(k-L)\}\end{aligned}$$

Because the expectation of $V_0(k)=0$ and the two variables are uncorrelated:

$$\begin{aligned}R(k+1) &= (1-r)^2E\{V_N(k)V_N(k-L)\}\\ &= (1-r)^2R(k)\end{aligned}$$

With this, it is obvious that the relationship is of exponential decay as the number of rolls increases.

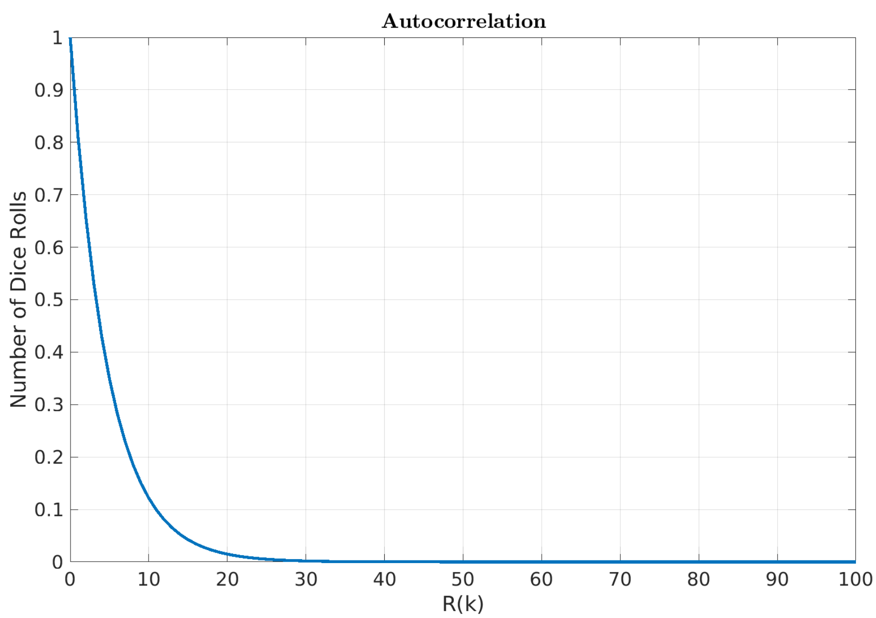

Figure 3: Autocorrelation.

Analyzing this expectation, it can be determined that r must remain between 0 and 2, otherwise the system would have unbounded error. Specifically, $r<0$ or $r>2$ is grows exponentially, quickly.

5. A random variable x has a PDF given by:

$$f_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{2}, & 0 \leq x \leq 2 \\ 0, & x > 2 \end{cases}$$

(a) What is the mean of x?

(b) What is the variance of x?

Solution:

For the piecewise function, x only exists in the second function, therefore the mean and variance of x are taken only from that portion.

$$\mu_x = \int_0^2 x\frac{x}{2}dx = \frac{1}{2}(\frac{1}{3}x^3)\Big|_0^2 = \frac{4}{3}$$

$$\sigma_x^2 = \int_0^2 x^2\frac{x}{2}dx - \mu_x^2 = \frac{1}{2}(\frac{1}{4}x^4)\Big|_0^2 - \mu_x^2 = \frac{2}{9}$$

6. Consider a normally distributed 2D vector x, with a mean of 0 and

$$P_x = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$$

(a) Find the eigenvalues of P_x

(b) The likelihood ellipses are given by and equation of the form: $x^T P_x^{-1} x = k^2$. What are the principal axes in this case.

(c) Plot the likelihood ellipses for k = 0.25, 1, 1.5.

(d) What is the probability of finding x inside of each of these ellipses?

Solution:

Using MATLAB's *eig* function:

$$s = 1.5858, \quad 4.4142$$

$$V = \begin{bmatrix} -0.9239 & 0.3827 \\ 0.3827 & 0.9239 \end{bmatrix}$$

The principal component directions and magnitude are the eigenvalues and eigenvectors solved above where the principal axes are the lines spanned by the eigenvectors. Below is a plot of the specified ellipses:

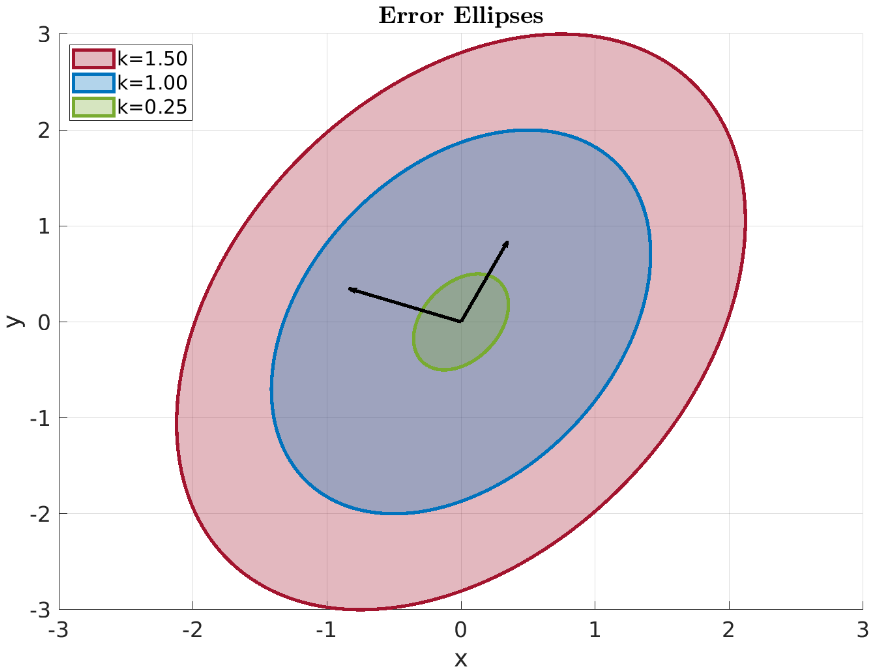

Figure 4: Specified Error Ellipses.

Created using the following steps:

```
% covariance matrix
prob6.P = [2, 1; 1, 4];

% eigenvalues/vectors of covariance
[prob6.v, prob6.s] = eig(prob6.P);

% create the transformation matrix
prob6.Ainv = inv(prob6.s^(-1/2) * prob6.v);

% create a circle
prob6.theta = 0:360;
prob6.a = [cosd(prob6.theta); sind(prob6.theta)];

% transform the circle into an ellipse of specifed size
prob6.b1 = prob6.Ainv * (0.25 .* prob6.a);
prob6.b2 = prob6.Ainv * (1.00 .* prob6.a);
prob6.b3 = prob6.Ainv * (1.50 .* prob6.a);
```

To determine the likelihood of a value falling in any of the ellipses, the likelihood was linearly interpolated using the k values given and the table provided. This results in the following percentages:

$$\%_{k=0.25} = 9.8350\%$$
$$\%_{k=1.00} = 39.3400\%$$
$$\%_{k=1.50} = 63.3333\%$$

7. Given $x\ N(0, \sigma_x^2)$ and $y = 2x^2$:

(a) Find the PDF of y.
(b) Draw the PDF's of x and y on the same plot for $\sigma_x = 2.0$.
(c) How has the density function changed by this transformation?
(d) Is y a normal random variable?

Solution:

Given the statistics of x, its PDF is:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(\frac{-1}{2}\frac{x^2}{\sigma_x^2}\right)$$

Relating x to y:

$$g^{-1}(y) = x = \left(\frac{y}{2}\right)^{\frac{1}{2}}$$
$$\frac{\partial g^{-1}(y)}{\partial y} = \frac{1}{2}\left(\frac{y}{2}\right)^{\frac{-1}{2}}$$

Using the relation:

$$f_Y(y) = f_X(g^{-1}(y))\left\|\frac{\partial g^{-1}(y)}{\partial y}\right\|$$
$$f_Y(y) = \left\|\frac{1}{2}\left(\frac{y}{2}\right)^{\frac{-1}{2}}\right\|\frac{1}{\sqrt{2\pi}\sigma_x} \exp\left(\frac{-1}{2}\frac{\left(\left(\frac{y}{2}\right)^{\frac{1}{2}}\right)^2}{\sigma_x^2}\right)$$

Using these two PDF's to create a plot:

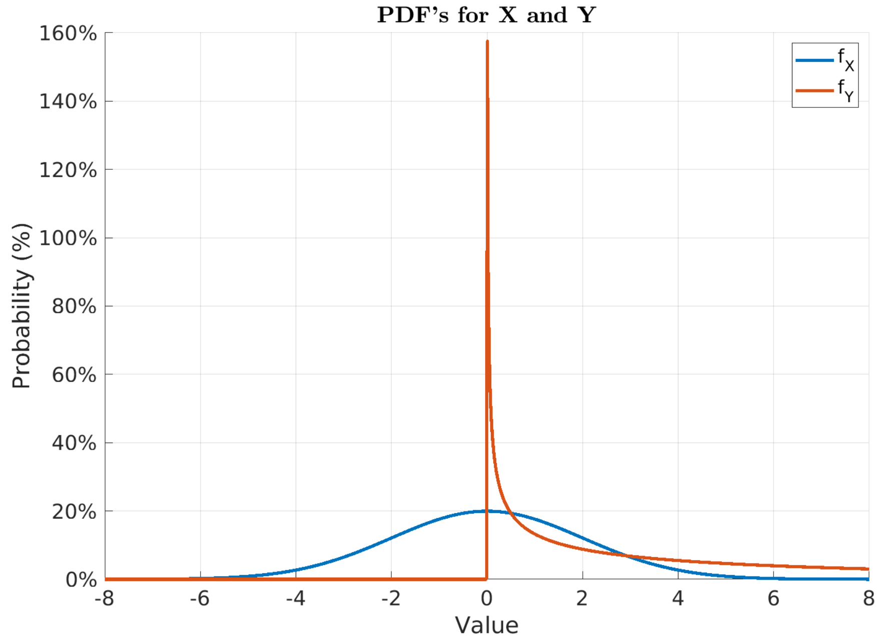

Figure 5: PDFs for x and y.

As shown, the PDF for y is no longer zero mean or Gaussian as the sum of the probability for all possible outcomes can not be equal to one. This is because a nonlinear transformation is used on the PDF of a function assumed to be linear, changing the characteristics of the function. The use of this squared term also causes this to only be positive when rational. There is also no solution at zero since the function approaches infinity as it heads towards zero.