Homework 3

1. Kalman Filter at its best – simulation (actually the Kalman filter is also quite reliable when we have an excellent model and low noise sensors). Suppose we have a 2^{nd} order system that we are regulating about zero (position and velocity) by wrapping an "optimal" control loop around the system. The new dynamics of the continuous time system are given by the closed-loop A matrix:

$$A_{cl} = \begin{bmatrix} 0 & 1 \\ -1 & -1.4 \end{bmatrix}$$

Suppose our measurement is simply position ($C = \begin{bmatrix} 1 & 0 \end{bmatrix}$). There is a white noise process disturbance (force, $B_w = \begin{bmatrix} 0 & 1 \end{bmatrix}^T$) acting on the controlled system.

(a) Simulate the controlled system with the disturbance force ($1\sigma = 2$) and a sampled sensor noise ($1\sigma = 1$) for 100 seconds at a 10 Hz sample rate.

(b) What is Q, Q_d and R_d?

(c) Calculate the steady state Kalman gain for the system. This can be done in one of many ways: iterate the kalman filter until it converges, dlqe.m, dare.m, kalman.m, dlqr.m (+ predictor to current estimator trick), etc. What is the steady state covariance of the estimates after the time update, $P^{(-)}$, as well as after the measurement update, $P^{(+)}$. Where are the poles of the estimator?

(d) Now use the steady state kalman filter to generate an estimate ($\hat{x}$ and $\hat{\dot{x}}$) of the 2 states over time. Calculate the norm of the standard deviation of the errors for each state.

$$N = \sqrt{(std(\dot{x} - \hat{\dot{x}}))^2 + (std(x - \hat{x}))^2} \tag{1}$$

(e) Change the ratio of the Q_d and R_d weights in the Kalman filter design (repeat *part d* with the new Kalman gain but **DO NOT** regenerate a new x and $\dot{x}$) and determine the effect on the estimation errors. For what ratio of Q_d to R_d are the errors minimized? Note: Often in practice we do not know the actual Q_d and R_d, so these tend to be "tuning" parameters we can use to tune our filter. However, according to Kalman the estimation errors are only minimized if we use the Q_d and R_d of the physical system.

Solution:

To simulate the system we can use the continuous model in *Equation 2* and evaluate over 100 seconds.

$$\begin{aligned} \dot{x} &= A_{cl}x + B_w w \\ y &= Cx + \nu \end{aligned} \tag{2}$$

Where x contains the states of position and velocity, $\dot{x}$ is the rate of change in these systems, w is the disturbance force or process noise, and ν is the measurement noise. *Figure 1* shows the simulated system.

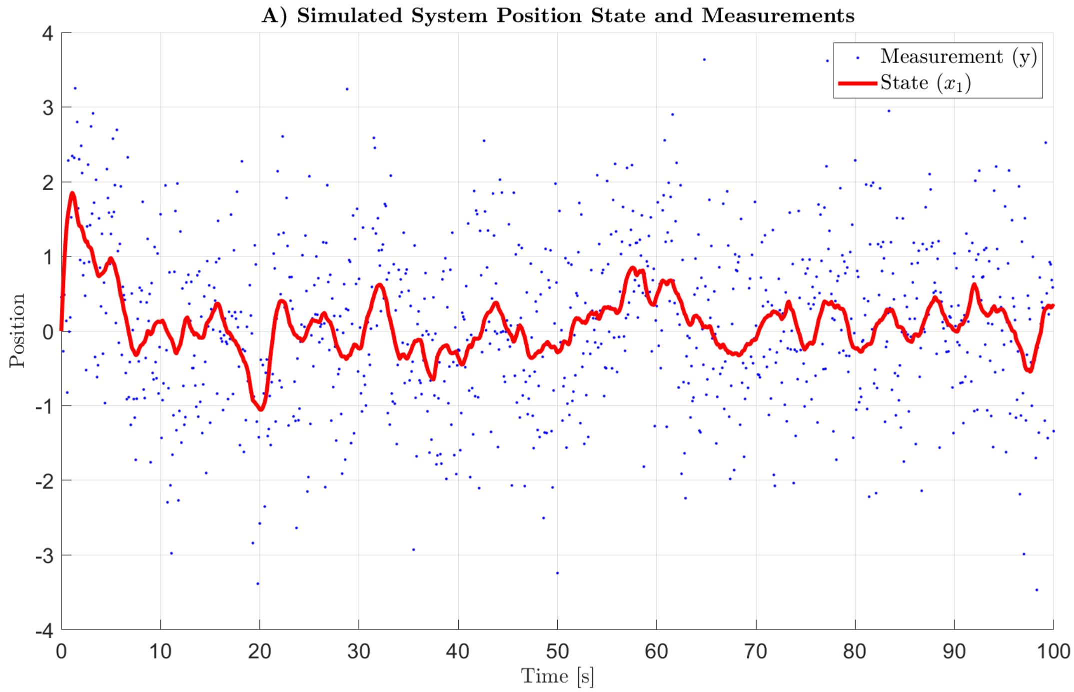

Figure 1: Simulated System Position with Measurements.

Using the desired continuous noise characteristics (*Equation 3*):

$$\begin{aligned} Q_c &= 2^2 = 4 \\ R &= 1^2 = 1 \end{aligned} \tag{3}$$

Discretizing the system can be done using *Equation 4*, known as Bryson's Trick.

$$\begin{aligned} s &= \begin{bmatrix} -A & B_w Q_c B_w^T \\ 0 & A^T \end{bmatrix} \\ c &= e^{s\Delta t} \\ A_d &= c[2,2]^T \\ Q_d &= A_d c[1,2] \end{aligned} \tag{4}$$

Applying this trick to the system:

$$
\begin{aligned}
s &= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 1.4 & 0 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1.4 \end{bmatrix} \\
c \approx eye(4) + s\Delta t &= \begin{bmatrix} 1 & -0.1 & 0 & 0 \\ 0.1 & 1.14 & 0 & 0.4 \\ 0 & 0 & 1 & -0.1 \\ 0 & 0 & 0.1 & 0.86 \end{bmatrix} \\
A_d &= \begin{bmatrix} 1 & 0.1 \\ -0.1 & 0.86 \end{bmatrix} \\
Q_d &= \begin{bmatrix} 0 & 0 \\ 0 & 0.344 \end{bmatrix}
\end{aligned}
$$

Since R is measurement noise, it does not change via discretation and can be assumed equivalent to the continuous measurement noise.

In order to determine the steady-state Kalman Gain and Covariance, the time and measurement updates must be implemented (*Equation 5*).

$$
\begin{aligned}
&\textbf{Time Update} \\
&x_k^- = A_d x_{k-1}^+ \\
&P_k^- = A_d P_{k-1}^+ A_d^T + Q_d \\
&\textbf{Measurement Update} \\
&L_k = P_k^- C^T (C P_k^- C^T + R)^{-1} \\
&P_k^+ = (I - L_k C) P_k^- \\
&x_k^+ = x_k^- + L_k (y_k - C x_k^-)
\end{aligned} \tag{5}
$$

Running the Kalman Filter until it reaches steady state results in *Figure 2*, which includes values of L_{ss}, P_{ss}^-, and P_{ss}^+.

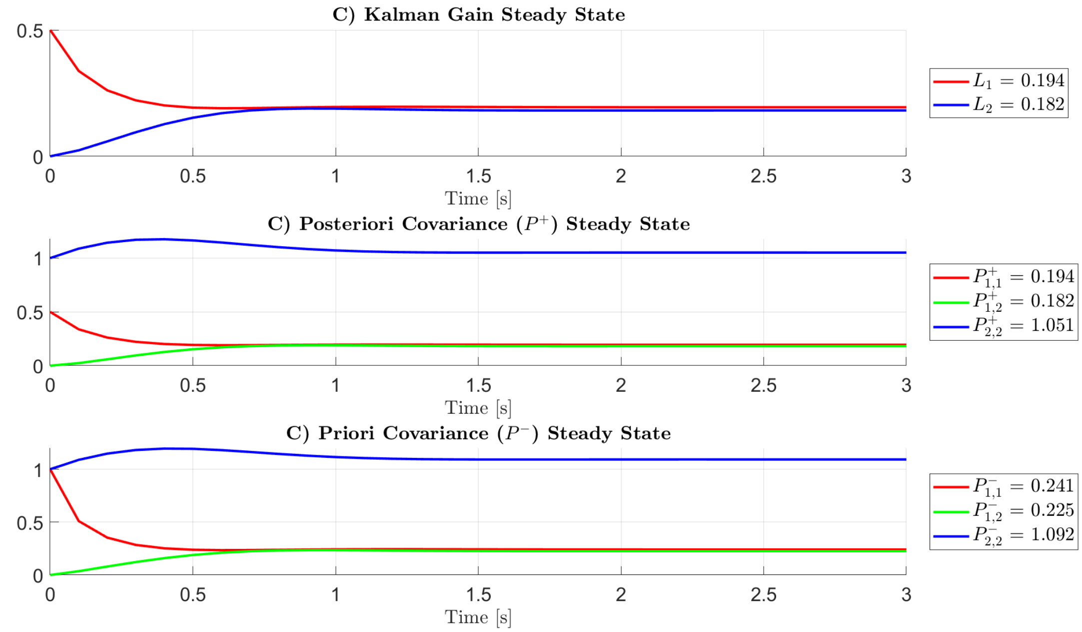

Figure 2: Kalman Filter Values vs. Time.

To find the poles of the new system, the determinant of the new closed-loop system must be evaluated as follows:

$$\begin{aligned} 0 &= det(s-(A_{cl}-L_{ss}C)) \\ s &= \begin{bmatrix} 0.833+0.166i \\ 0.833-0.166i \end{bmatrix} \end{aligned} \tag{6}$$

The position estimate of the Kalman Filter is shown in *Figure 3*.

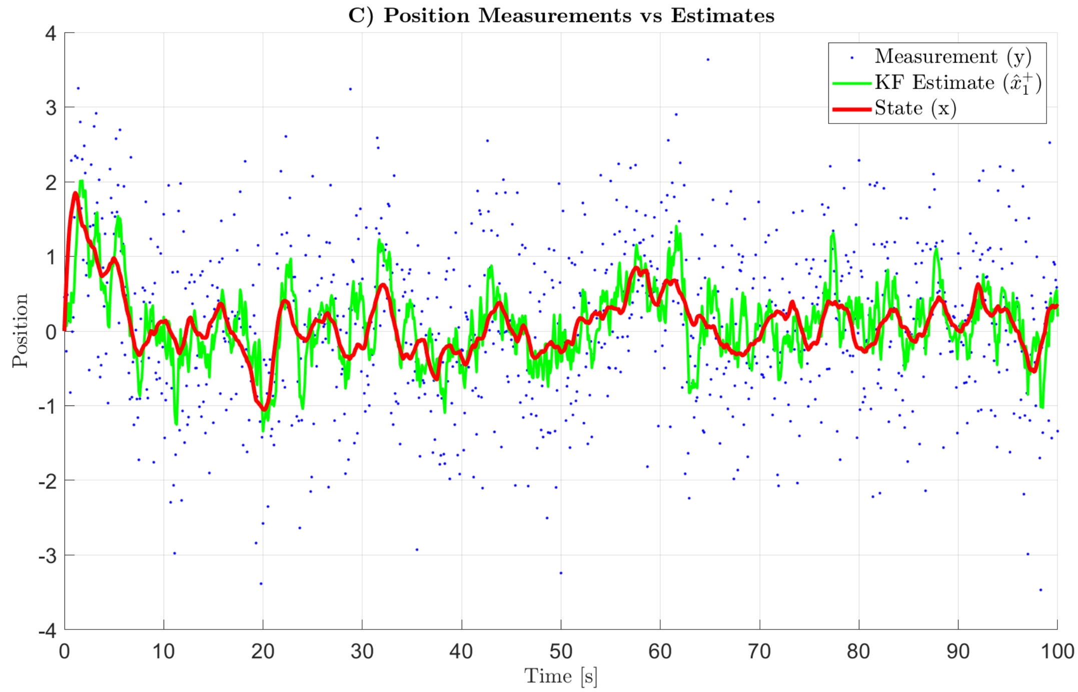

Figure 3: Filtered Position Solution.

Applying *Equation 1* to the difference between the states of the simulated system and the Kalman Filter estimation results in a normalized standard deviation of $N = 0.600$.

To find the optimal ratio of measurement to process noise, measurement noise was kept constant, $R = 1$, and the process noise was changed on a log scale from $10^{-6} \leftrightarrow 10^{6}$ such that the ratio between the values changed on each iteration. A graphical representation is shown in *Figure 4*.

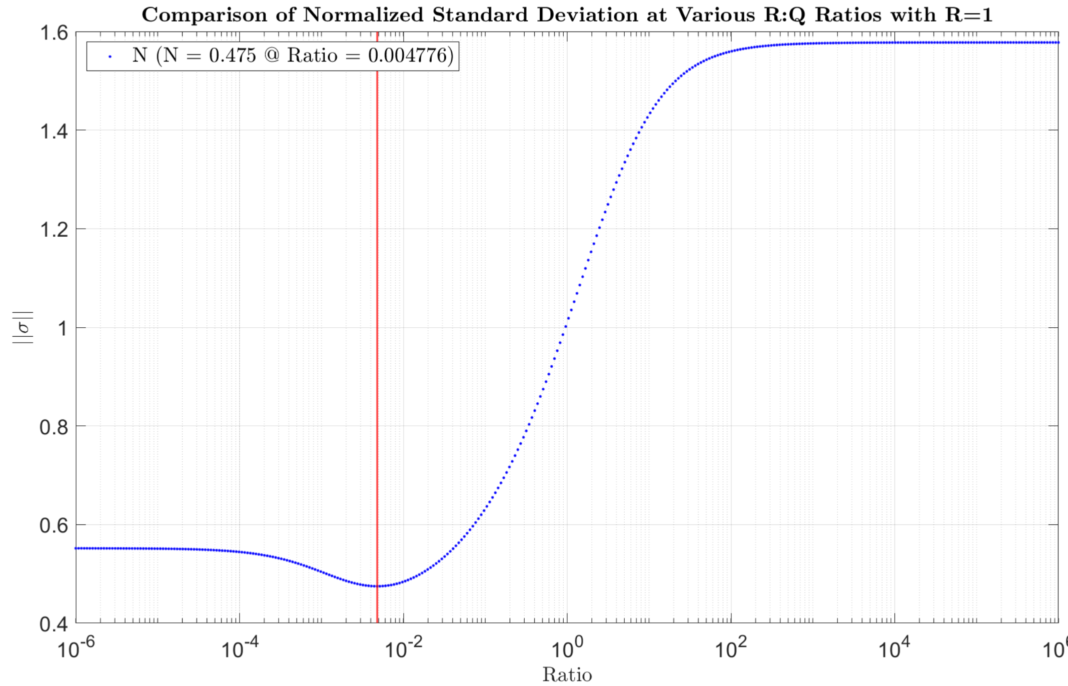

Figure 4: Normalized Standard Deviation at Different R to Q Ratios.

As shown in the *Figure 4*, the optimal ratio occurs at $R : Q = 0.0048$ with a normalized standard deviation of $N = 0.475$ for this simulated system.

2. Download the data *hw3_2* from the website. The data is in the form $\begin{bmatrix} t & y \end{bmatrix}$. Suppose we want to design an estimator to estimate the bias in the measurement y. We believe that the bias (x) is constant, so we use the model given by:

$$\dot{x} = 0$$
$$y_k = x_k + \nu_k$$
$$\nu_k \sim N(0, 1)$$

(a) Run the Kalman filter estimator with $Q_d = 0$. What happens at $t > 100$ seconds? Why? Calculate the steady state Kalman gain L_{ss}. Plot $L(k)$. This is known as the filter "going to sleep" (becomes a least squares estimator).

(b) To offset this problem we will "tune" Q_d to track the bias. What is the effect of changing Q_d on the ability to track the step change in the bias? Try values of Q_d from 0.0001 to 0.01 and plot $L(k)$ as well as the estimate of the bias ($\hat{x}$). What is the tradeoff?

(c) Now filter the measurement using the first order low-pass filter:
(Command: $yf = filter(numd, dend, y, y0)$)

$$H(z) = \frac{\sqrt{Q_d}}{z - (1 - \sqrt{Q_d})}$$

(d) How does this compare to the Kalman filter solution. Why are these two filters the same for this problem?

Solution:

When the time passes 100 seconds, the Kalman Filter's estimate of position fails. This is because there is assumed to be no process noise on this system, therefore after a few iterations of the filter, it believes it has converged to the correct value and both the covariance and Kalman Gain on the estimate are tiny. This leads to the Kalman Filter not being able to catch up ('going to sleep') with the dynamic state as shown in *Figure 5*.

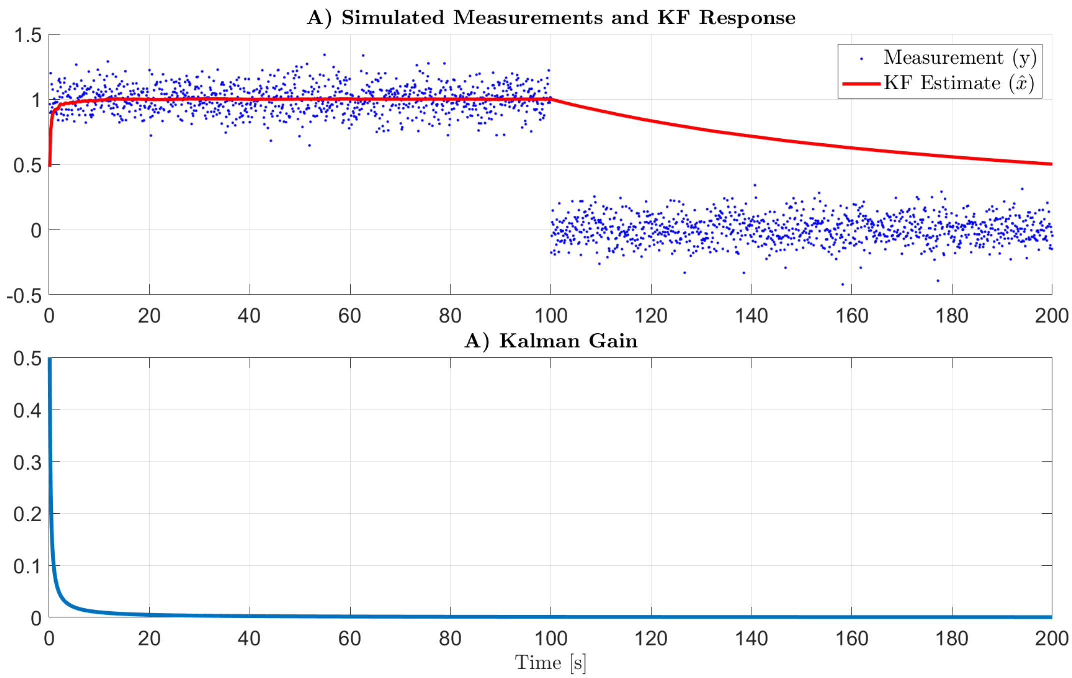

Figure 5: Kalman Filter 'Going to Sleep.'

This figure also shows the Kalman Gain reaching a steady state values of $L_{ss} = 0.0005$ quickly. This can also be calculated and confirmed with *Equation 7*.

$$\begin{aligned} P_{ss}^m &= \left[\left(A_d P_{ss}^m A_d^T + Q_d \right)^{-1} + C R^{-1} C^T \right]^{-1} \\ L_{ss} &= P_{ss}^m C^T R^{-1} \\ &= 0.0005 \end{aligned} \tag{7}$$

To tune the filter, four log spaced values of $Q = \begin{bmatrix} 0.1 & 0.01 & 0.001 & 0.0001 \end{bmatrix}$ were used as the process noise of the system. The larger the process noise value, the quicker the filter was able to adapt to the tracked the change in measured position, shown in *Figure 6*. It also shows that as Q is increased L_{ss} is also increased as a byproduct. This indicates that more filtering (less smoothing) is occurring as the dynamic model is trusted less.

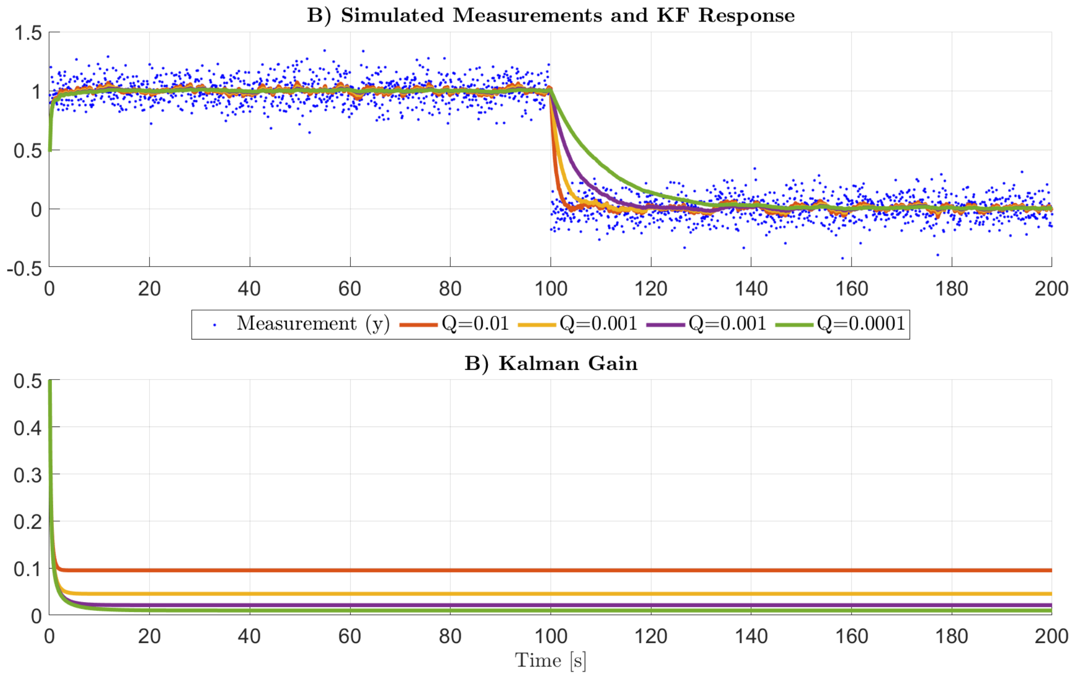

Figure 6: Filtered Position Estimate with a Varied Process Noise.

Choosing a process noise of $Q = 0.1$, the MATLAB command $>> filter$ is used to create a first order low pass filter as a comparison to the Kalman Filter we created. Analyzing *Figure 7*, it is obvious that the Low Pass Filter is identical to the Kalman Filter. This is because once the Kalman Filter reaches steady-state, it assumes the system is stationary and morphs into a first order Low Pass Filter with the same pole placement.

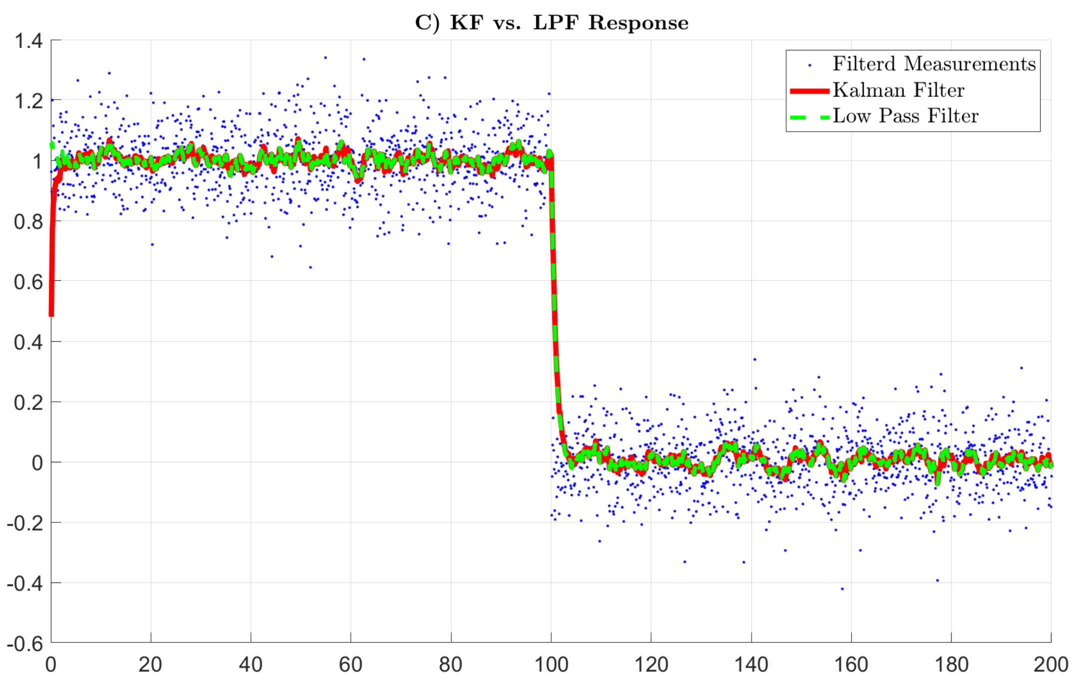

Figure 7: Kalman Filter v. First Order Low Pass Filter.

3. Design a "Navigation" type Kalman filter to estimate the states [East, North, Radar_Bias, Psi, Gyro_Bias]. Note: this is a non-linear problem that requires an Extended Kalman Filter (EKF) to do correctly. However, we can solve the problem in one of two ways:

(i) linearize the equations about the nominal operating point and produce a constant A matrix for that operating point
(ii) simply update the A matrix at every time step with our measurements or estimates

Download the data *hw3_3* from the website and run the filter sampled at 5 Hz.

(a) How did you choose the covariance values for Q_d (especially for the radar and gyro biases).
(b) How does the bias estimation compare to a Least Squares Solution. How does the bias estimate compare to the Recursive Least Squares solution if you make the covariance (Q_d) of the bias estimates equal to zero.
(c) Integrate the last 40 seconds of data to see how well you have estimated the biases. This can simply be done by "turning off" the measurements in the observation matrix! Why do the bias estimates remain constant during the 40 second "outage?"

Solution:
For the time update, option *ii* was chosen. This resulted in a nonlinear state transition matrix that was updated every single iteration, and a linear measurement mapping matrix as described in *Equaion 8* and *Equation 9*.

$$x_k = A_{k-1}x_{k-1}$$

$$\begin{bmatrix} \hat{E} \\ \hat{N} \\ \hat{\psi} \\ \hat{\dot{\psi}} \\ \hat{v} \\ \hat{b}_{\dot{\psi}} \\ \hat{b}_v \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & \Delta t \sin(\hat{\psi}_{k-1}) & 0 & -\Delta t \sin(\hat{\psi}_{k-1}) \\ 0 & 1 & 0 & 0 & \Delta t \cos(\hat{\psi}_{k-1}) & 0 & -\Delta t \cos(\hat{\psi}_{k-1}) \\ 0 & 0 & 1 & \Delta t & 0 & -\Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{E} \\ \hat{N} \\ \hat{\psi} \\ \hat{\dot{\psi}} \\ \hat{v} \\ \hat{b}_{\dot{\psi}} \\ \hat{b}_v \end{bmatrix}_{k-1} \tag{8}$$

$$y_k = Cx_k$$

$$\begin{bmatrix} E \\ N \\ \psi \\ \dot{\psi} \\ v \end{bmatrix}_k = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{E} \\ \hat{N} \\ \hat{\psi} \\ \hat{\dot{\psi}} \\ \hat{v} \\ \hat{b}_{\dot{\psi}} \\ \hat{b}_v \end{bmatrix}_k \tag{9}$$

With these matrices defined, the Kalman Filter can be applied as in *Equation 5*. Both the measurement noise and process noise are assumed to be uncorrelated between different states. The measurement noise was defined as 1 for measurement and the process noise on each estimate of the measurement was 0.5. These were chosen as smaller than the measurement noise because the measurements seem to have little noise on them. For the biases, since they should remain constant and are not measured, no measurement noise was applied and a small process noise of 0.001 was used. Q_d and R are defined in *Equation 10*.

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$Q_d = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.001 \end{bmatrix} \tag{10}$$

Figure 8 shows the output of the Kalman Filter states specified in the problem statement compared to their respective measurements.

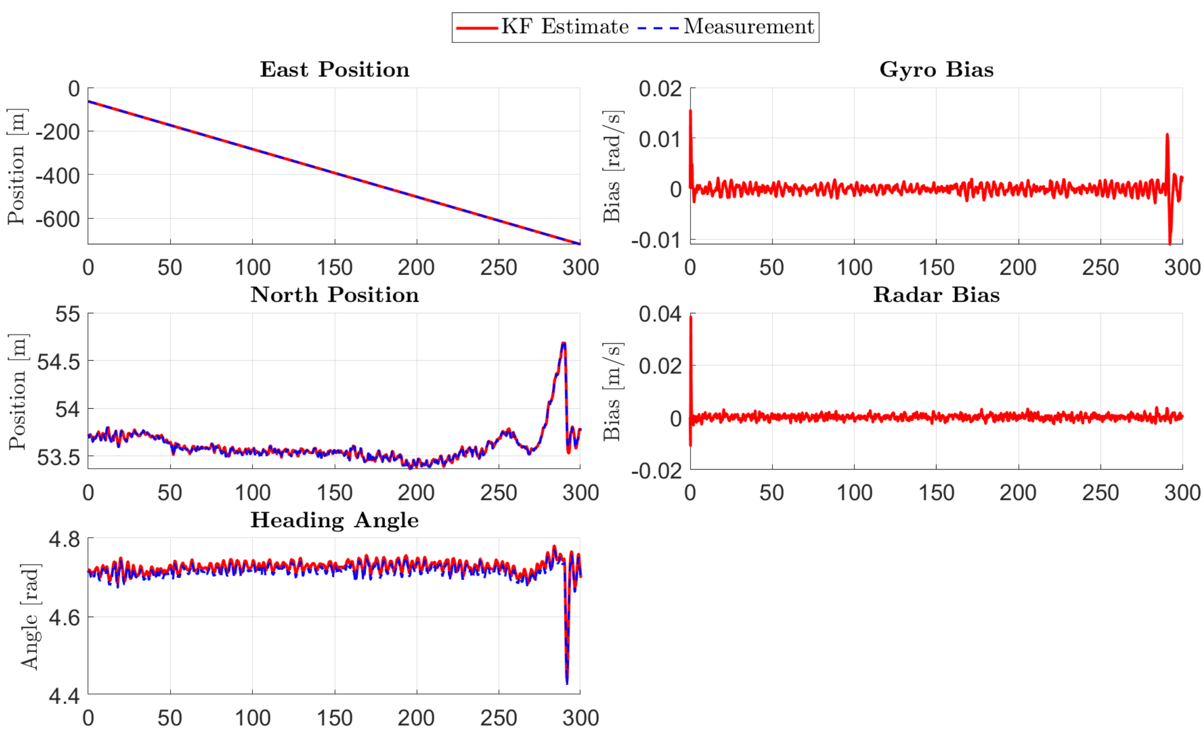

Figure 8: Navigation Kalman Filter Estimates.

To run the system with least squares, the C matrix defined in *Equation 8* as the geometry matrix of the system. From here, the psuedoinverse of the matrix is taken and multiplied by y_k (*Equation 11*).

$$x_k = (C^T C)^{-1} C^T y_k \tag{11}$$

For recursive least squares, a measurement propagation, similar to the Kalman Filter measurement update, is used to recursively update the state *Equation 12*.

$$\begin{aligned} Q_k^{-1} &= Q_{k-1}^{-1} + C^T R^{-1} C \\ x_k &= Q C^T R^{-1}(y_k - C x_{k-1}) \end{aligned} \tag{12}$$

Figure 9 shows the least squares and recurve least squares estimations of the system. The recursive least squares solution is considerably worse than both the Kalman Filter and the regular least squares. This is because recursive least squares assumes no dynamics, whereas the Kalman Filter expects dynamics and least squares just estimates the most likely solution to the information given. Due to this, the Kalman Filter outperforms both least squares options in the estimation of the biases.

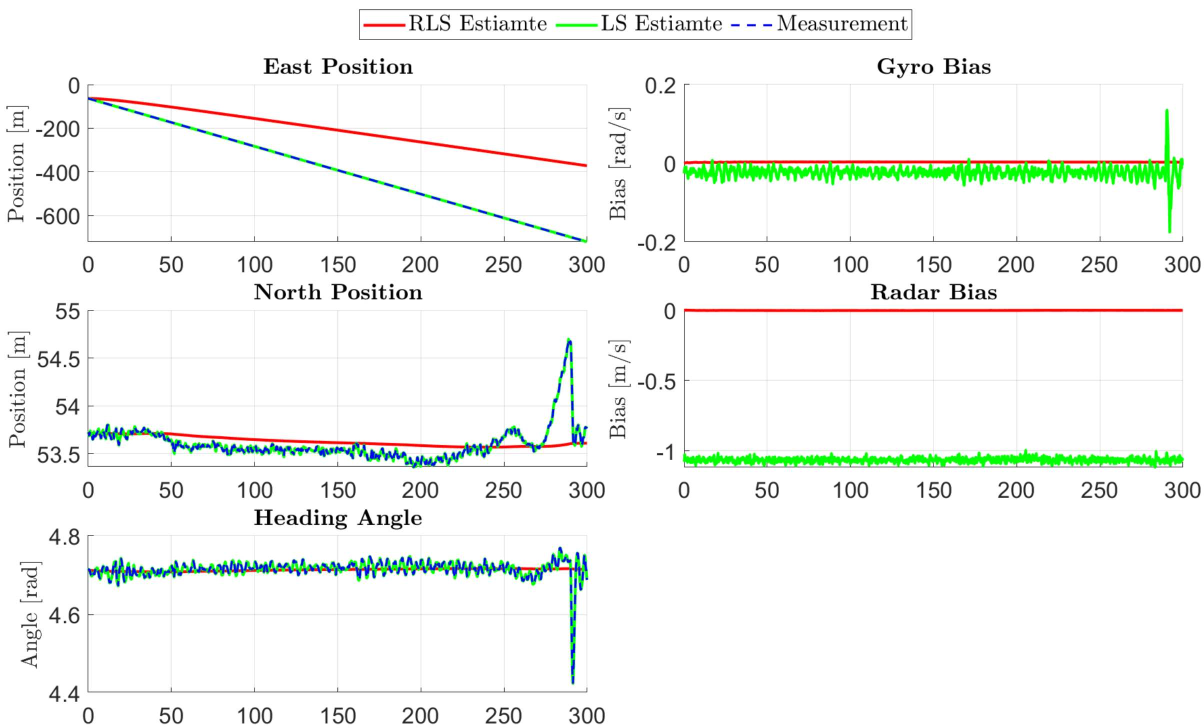

Figure 9: Navigation Least Squares Estimates.

When solely integrating the system during for last 40 seconds, or the 'outage,' there is no measurement update or correction step. The measurement update is crucial because it updates the covariance matrix that utilizes the radar and gyroscope biases while filtering. Therefore, the bias estimates remain constant throughout the 'outage.' *Figure 10* shows the last 50 seconds of the Kalman Filter designed in Part A where the measurement update is turned off for the last 40 seconds.

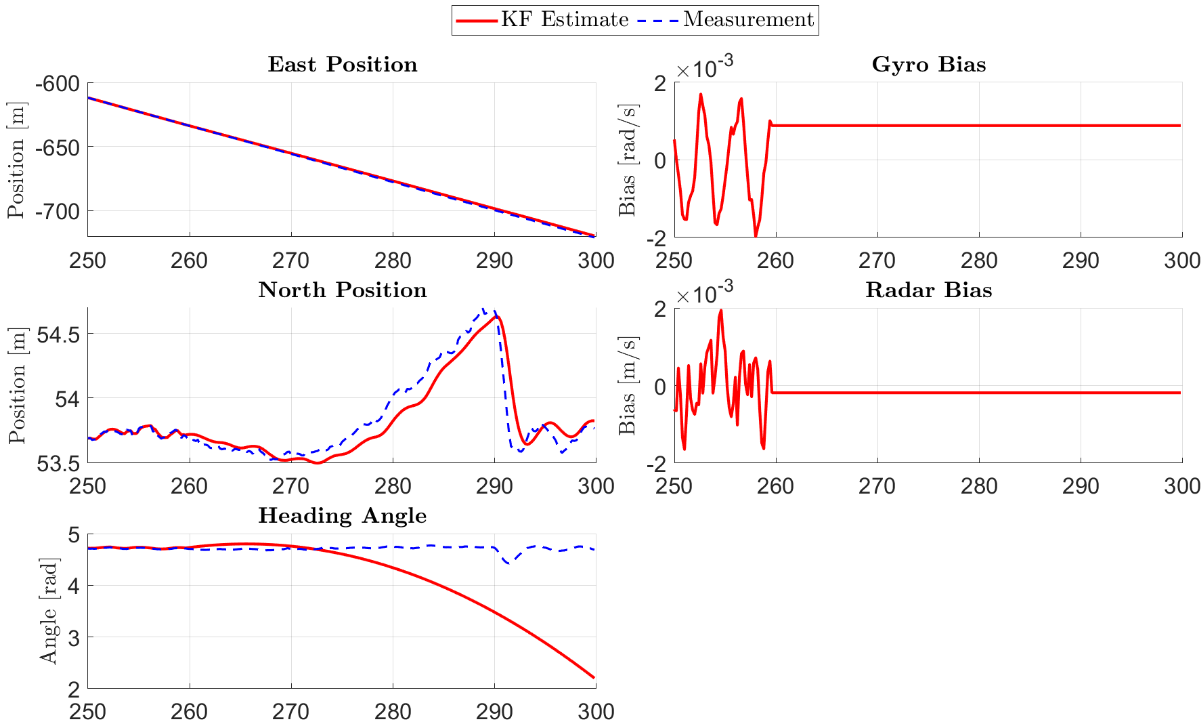

Figure 10: Navigation 'Outage' Estimates.

As shown, the bias estimates were tuned enough to provide good position estimates but provided poor heading measurements.

4. Estimator for Vehicle Dynamics. The yaw dynamics of a car (for a stability control system) can be described by the following model (at $25m/s$):

$$\dot{x} = \begin{bmatrix} -2.62 & 12 \\ -0.96 & -2 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \beta \end{bmatrix} + \begin{bmatrix} 14 \\ 1 \end{bmatrix} \delta$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \nu_k$$

Where:

$$\dot{\psi} = \text{Vehicle Yaw Rate}$$
$$\beta = \text{Vehicle Sideslip Angle}$$
$$\delta = \text{Steer Angle}$$
$$\nu_k = \text{Sample Sensor Noise}$$

(a) Assuming we can only measure the yaw rate ($\nu_k \sim N[0, (0.1)^2]$), design a Kalman filter to do full state estimation (select a reasonable Q_d). Provide a unit step steer input and estimate both states. On one page plot the actual states and estimated states (use $subplot(2, 2, n)$ for each of the two states). Where are the steady state poles of the estimator?

(b) Now, somebody has loaded the trunk of the vehicle with bricks, changing the CG of the vehicle so now the actual model (at 25 m/s) is:

$$\dot{x} = \begin{bmatrix} -2.42 & 4 \\ -0.99 & -2 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \beta \end{bmatrix} + \begin{bmatrix} 18 \\ 1 \end{bmatrix} \delta$$
$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + \nu_k$$

NOTE: We do not know that somebody has loaded the trunk and that the C.G. has changed, therefore we must use the model for *part a* in our Kalman Filter. Redo *part a*. Can you estimate the slip angle correctly? Try various Q_d.

(c) Now lets say we have a noisy measurement of the slip angle ($\eta_k \sim N[0, (0.5)^2]$):

$$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x_k + \begin{bmatrix} \nu_k \\ \eta_k \end{bmatrix}$$

Assuming the sensor noises are uncorrelated, what is R?

(d) Redo *part a*. What is the effect of changing the element of Q_d associated with the slip angle estimate. What must Q_d equal to ensure an unbiased estimate of the states. How much filtering does that provide?

Solution:

To run the Kalman Filter, the system must first be discretized (*Equation 13*).

$$\begin{aligned} A_d &\approx eye(2) + A\Delta t = \begin{bmatrix} 0.738 & 1.20 \\ -0.096 & 0.80 \end{bmatrix} \\ B_d &\approx B\Delta t = \begin{bmatrix} 1.4 \\ 0.1 \end{bmatrix} \\ C_d &= C = \begin{bmatrix} 1 & 0 \end{bmatrix} \end{aligned} \tag{13}$$

The measurement noise is provided as $R = 0.1^2 = 0.01$ and the process noise was tuned to be:

$$Q_d = \begin{bmatrix} 2 & -1 \\ -1 & 0.5 \end{bmatrix}$$

Using these parameters along with the one given in the problem statement, we can estimate the yaw rate and sideslip angle of the car (*Figure 11*). From the Kalman Filter, with a constant Q_d and R, we get the discrete poles of the steady-state Kalman Filter to be:

$$poles = \begin{bmatrix} -0.013 + 0i & 0.533 + 0i \end{bmatrix}$$

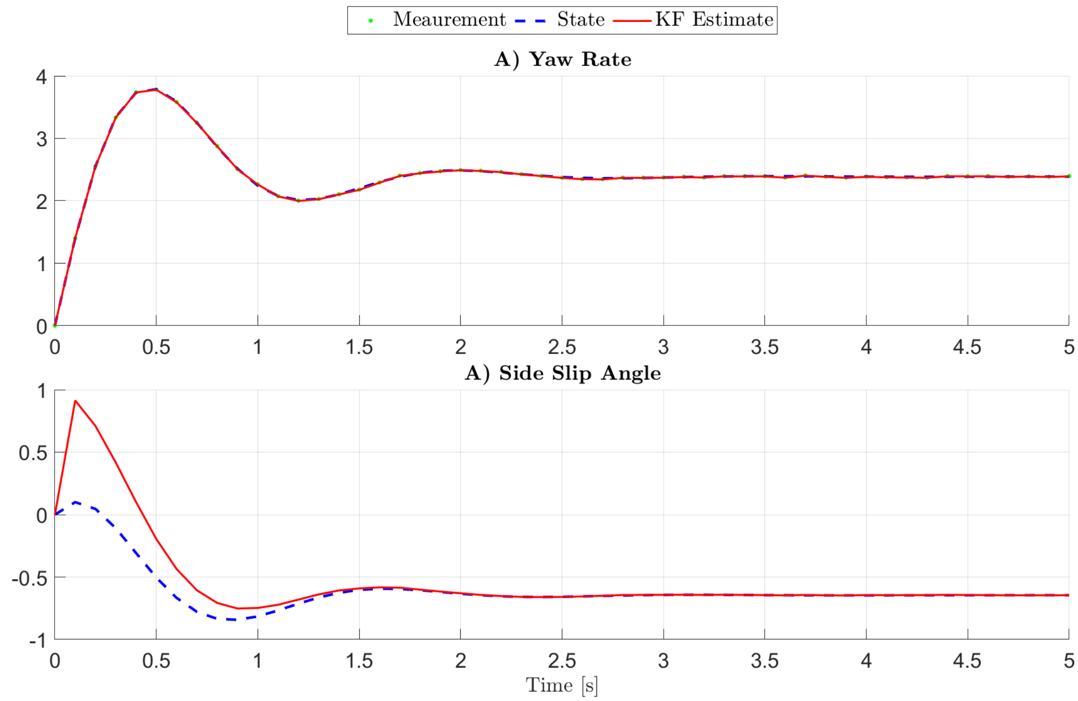

Figure 11: Vehicle Dynamics Kalman Filter Estimation.

It is shown that the estimate of the sideslip angle is of similar accuracy to the yaw rate estimation. This is because the sideslip angle of the car is not directly observable based on the current measurement matrix.

When the dynamic model of the system is slightly altered, the process noise matrix can also be slightly altered to match the new system. However, all other matrices are kept the same as in Part A. The new process noise matrix and discrete poles are:

$$Q_d = \begin{bmatrix} 3 & -1.2 \\ -1.2 & 0.5 \end{bmatrix}$$
$$poles = \begin{bmatrix} -0.099 + 0i & 0.639 + 0i \end{bmatrix}$$

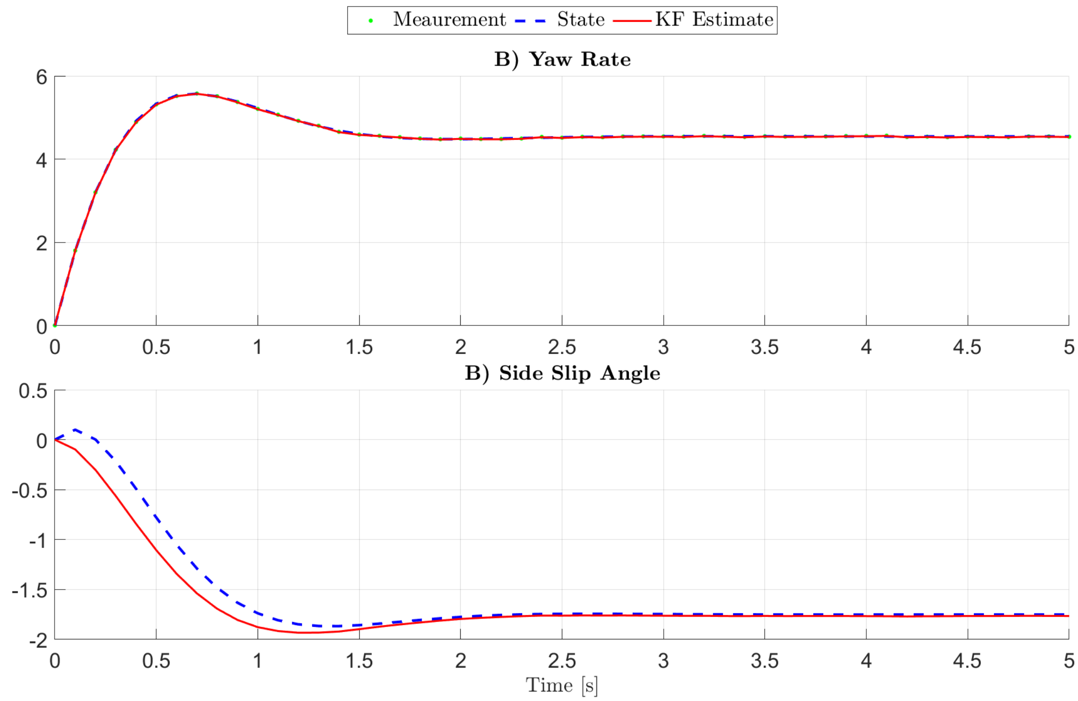

Figure 12: Vehicle Dynamics Kalman Filter Estimation with Altered Dynamic Model.

As shown in *Figure 12*, with the altered process noise, the steady state errors are approximately the same as in Part A.

Adding a noisy measurement of slip angle with the statistics defined in the problem statement changes the measurement noise and new geometry matrix to be:

$$R = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.25 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As we can see from *Figure 13*, the effect that adding a second measurement has on the estimation is substantial. It is obvious that is makes the estimation of the parameter much more noisy. However, the same process noise matrix from Part A does not work such that the process noise values must be significantly smaller to work for this system. The addition of the direct measurement of sideslip allows much more leniency in the design of the system. Decreasing the value of Q_d allows for more smoothing by reducing the apparent noise. The new values of Q_d and the steady state Kalman Filter poles are:

$$Q = \begin{bmatrix} 0.2 & -0.01 \\ -0.01 & 0.02 \end{bmatrix}$$

$$poles = \begin{bmatrix} 0.080 + 0i & 0.362 + 0i \end{bmatrix}$$

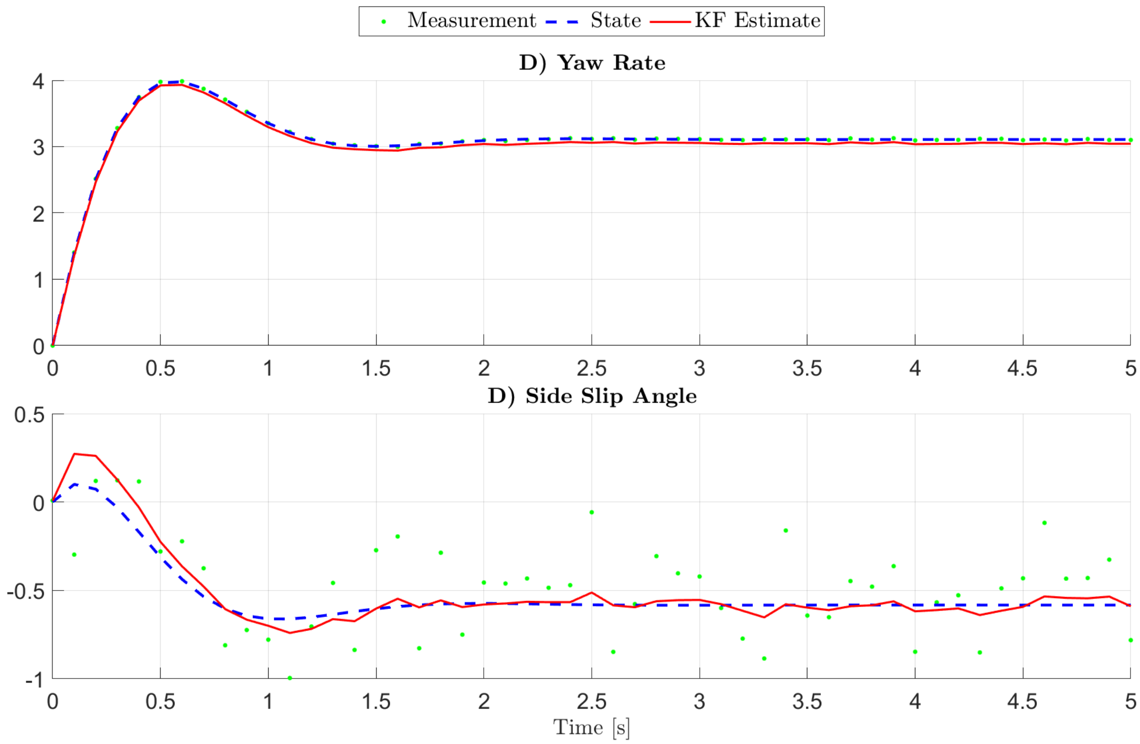

Figure 13: Vehicle Dynamics Kalman Filter with the Addition of Measurements of Sideslip.

Overall, a more realistic estimate of sideslip is achieved but at the cost of increased noise.