Problem 13

Let $\mathbf{m}_1$ denote the magnetic moment of the fixed dipole. Let $\mathbf{m}_2$ denote the magnetic moment of the dipole that will move. Let $\mathbf{r}$ be the vector that points from $\mathbf{m}_1$ to $\mathbf{m}_2$. Then the force between the two dipoles is given by

$$\mathbf{F} = \frac{3\mu_0}{4\pi r^5}\left[(\mathbf{m}_1 \cdot \mathbf{r})\mathbf{m}_2 + (\mathbf{m}_2 \cdot \mathbf{r})\mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2)\mathbf{r} - \frac{5(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r})}{r^2}\mathbf{r}\right] \quad (1)$$

which I found on Wikipedia.

Let the magnitude of each magnetic moment be m. In the first configuration (a), the following relations are true: $\mathbf{m}_1 \cdot \mathbf{r} = 0$, $\mathbf{m}_2 \cdot \mathbf{r} = 0$, $\mathbf{m_1} \cdot \mathbf{m}_2 = -m^2$. Thus,

$$\mathbf{F} = -\frac{3\mu_0 m^2}{4\pi r^5}\mathbf{r}. \quad (2)$$

In the second configuration (b), the following relations are true: $\mathbf{m_1} \cdot \mathbf{r} = mr$, $\mathbf{m_2} \cdot \mathbf{r} = mr$, $\mathbf{m_1} \cdot \mathbf{m_2} = m^2$, and we will also define $\hat{\mathbf{r}}$ as the unit vector pointing in the $\mathbf{r}$ direction. We will find that

$$\mathbf{F} = -\frac{3\mu_0 m^2}{2\pi r^5}\mathbf{r}. \quad (3)$$

(2) and (3) are very similar, so we should be able to compute the elapsed times in very similar ways. First, let the mass of the dipole be w. Then define, for ease of notation,

$$\gamma \equiv \frac{3\mu_0 m^2}{4\pi w}.$$

(2) becomes:

$$\ddot{r} = -\gamma r^{-4}. \quad (4)$$

We use the trick $a \; dx = v \; dv$ to write:

$$\begin{aligned}\int_0^{\dot{r}} \dot{r}' \; d\dot{r}' &= \int_{r_0}^{r} \ddot{r} \; dr' \\ &= \int_{r_0}^{r} -\gamma r'^{-4} \; dr' \\ &= \left[\frac{1}{3}\gamma r^{-3}\right]_{r_0}^{r},\end{aligned}$$

which implies that

$$\dot{r} = -\left[\frac{2}{3}\gamma\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right)\right]^{1/2}, \quad (5)$$

where we have taken the negative value because we know that r is decreasing. This is a separable differential equation that allows us to write:

$$
\begin{aligned}
t &= \int_0^{r_0} \left[\frac{2}{3}\gamma\left(\frac{1}{r^3} - \frac{1}{r_0^3}\right)\right]^{-1/2} \, dr \\
&= \left(\frac{3}{2\gamma}\right)^{1/2} \int_0^{r_0} \frac{r^{3/2}}{[1-(r/r_0)^3]^{1/2}} \, dr. \qquad (6)
\end{aligned}
$$

(6) is rather difficult to integrate. I used Mathematica. The answer was

$$
\begin{aligned}
t &= \left(\frac{3}{2\gamma}\right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} \pi^{1/2} r_0^{5/2} \\
&= \left(\frac{2\pi^2 w}{\mu_0 m^2}\right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} r_0^{5/2}. \qquad (7)
\end{aligned}
$$

In part (b), the answer is similar, except $\gamma \to 2\gamma$. This yields

$$
t = \left(\frac{\pi^2 w}{\mu_0 m^2}\right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} r_0^{5/2}. \qquad (8)
$$

Contestant: Jacob H. Nie