

Problem 13

Let \mathbf{m}_1 denote the magnetic moment of the fixed dipole. Let \mathbf{m}_2 denote the magnetic moment of the dipole that will move. Let \mathbf{r} be the vector that points from \mathbf{m}_1 to \mathbf{m}_2 . Then the force between the two dipoles is given by

$$\mathbf{F} = \frac{3\mu_0}{4\pi r^5} \left[(\mathbf{m}_1 \cdot \mathbf{r})\mathbf{m}_2 + (\mathbf{m}_2 \cdot \mathbf{r})\mathbf{m}_1 + (\mathbf{m}_1 \cdot \mathbf{m}_2)\mathbf{r} - \frac{5(\mathbf{m}_1 \cdot \mathbf{r})(\mathbf{m}_2 \cdot \mathbf{r})}{r^2} \mathbf{r} \right] \quad (1)$$

which I found on Wikipedia.

Let the magnitude of each magnetic moment be m . In the first configuration (a), the following relations are true: $\mathbf{m}_1 \cdot \mathbf{r} = 0$, $\mathbf{m}_2 \cdot \mathbf{r} = 0$, $\mathbf{m}_1 \cdot \mathbf{m}_2 = -m^2$. Thus,

$$\mathbf{F} = -\frac{3\mu_0 m^2}{4\pi r^5} \mathbf{r}. \quad (2)$$

In the second configuration (b), the following relations are true: $\mathbf{m}_1 \cdot \mathbf{r} = mr$, $\mathbf{m}_2 \cdot \mathbf{r} = mr$, $\mathbf{m}_1 \cdot \mathbf{m}_2 = m^2$, and we will also define $\hat{\mathbf{r}}$ as the unit vector pointing in the \mathbf{r} direction. We will find that

$$\mathbf{F} = -\frac{3\mu_0 m^2}{2\pi r^5} \mathbf{r}. \quad (3)$$

(2) and (3) are very similar, so we should be able to compute the elapsed times in very similar ways. First, let the mass of the dipole be w . Then define, for ease of notation,

$$\gamma \equiv \frac{3\mu_0 m^2}{4\pi w}.$$

(2) becomes:

$$\ddot{r} = -\gamma r^{-4}. \quad (4)$$

We use the trick $a \, dx = v \, dv$ to write:

$$\begin{aligned} \int_0^{\dot{r}} \dot{r}' \, d\dot{r}' &= \int_{r_0}^r \ddot{r} \, dr' \\ &= \int_{r_0}^r -\gamma r'^{-4} \, dr' \\ &= \left[\frac{1}{3} \gamma r^{-3} \right]_{r_0}^r, \end{aligned}$$

which implies that

$$\dot{r} = - \left[\frac{2}{3} \gamma \left(\frac{1}{r^3} - \frac{1}{r_0^3} \right) \right]^{1/2}, \quad (5)$$

where we have taken the negative value because we know that r is decreasing. This is a separable differential equation that allows us to write:

$$\begin{aligned} t &= \int_0^{r_0} \left[\frac{2}{3} \gamma \left(\frac{1}{r^3} - \frac{1}{r_0^3} \right) \right]^{-1/2} dr \\ &= \left(\frac{3}{2\gamma} \right)^{1/2} \int_0^{r_0} \frac{r^{3/2}}{[1 - (r/r_0)^3]^{1/2}} dr. \end{aligned} \quad (6)$$

(6) is rather difficult to integrate. I used Mathematica. The answer was

$$\begin{aligned} t &= \left(\frac{3}{2\gamma} \right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} \pi^{1/2} r_0^{5/2} \\ &= \left(\frac{2\pi^2 w}{\mu_0 m^2} \right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} r_0^{5/2}. \end{aligned} \quad (7)$$

In part (b), the answer is similar, except $\gamma \rightarrow 2\gamma$. This yields

$$t = \left(\frac{\pi^2 w}{\mu_0 m^2} \right)^{1/2} \frac{\Gamma(5/6)}{\Gamma(1/3)} r_0^{5/2}. \quad (8)$$

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