

Problem 7

Some notation and conventions used:

$$\begin{aligned}\mathcal{F}[x(t)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} x(t) \exp(-i\omega t) dt \\ &= \hat{x}(\omega) \\ \mathcal{F}^{-1}[\hat{x}(\omega)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \hat{x}(\omega) \exp(i\omega t) dt \\ &= x(t)\end{aligned}$$

Theorem 1. *If*

$$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t') dt',$$

then

$$\hat{x}(\omega) = (2\pi)^{1/2} \hat{G}(\omega) \hat{F}(\omega).$$

Proof.

$$\begin{aligned}\hat{x}(\omega) &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} x(t) \exp(-i\omega t) dt \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dt \exp[(-i\omega(t-t'))] G(t-t') \int_{-\infty}^{\infty} dt' \exp(-i\omega t') F(t') \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} d(t-t') \exp[(-i\omega(t-t'))] G(t-t') \int_{-\infty}^{\infty} dt' \exp(-i\omega t') F(t') \\ &= \left(\frac{1}{2\pi}\right)^{1/2} (2\pi)^{1/2} \hat{G}(\omega) \cdot (2\pi)^{1/2} \hat{F}(\omega) \\ &= (2\pi)^{1/2} \hat{G}(\omega) \hat{F}(\omega).\end{aligned}$$

□

Let us calculate $\hat{F}(\omega)$. This is:

$$\begin{aligned}\mathcal{F}[m\ddot{x}(t)] &= \left(\frac{1}{2\pi}\right)^{1/2} m \int_{-\infty}^{\infty} \frac{d^2x}{dt^2} \exp(-i\omega t) dt \\ &= \left(\frac{1}{2\pi}\right)^{1/2} m \left[\dot{x} \exp(-i\omega t) \Big|_{-\infty}^{\infty} + i\omega x \exp(-i\omega t) \Big|_{-\infty}^{\infty} - \omega^2 \int_{-\infty}^{\infty} x \exp(-i\omega t) dt \right] \\ &= -\omega^2 m \hat{x}(\omega),\end{aligned}$$

where we have assumed that x and \dot{x} tend towards 0 as t goes to $\pm\infty$.

This allows us to calculate $\hat{G}(\omega)$:

$$\begin{aligned}\hat{G}(\omega) &= \left(\frac{1}{2\pi}\right)^{1/2} \frac{\hat{x}(\omega)}{\hat{F}(\omega)} \\ &= -\left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{\omega^2 m}\right).\end{aligned}$$

Thus,

$$G(t) = -\left(\frac{1}{2\pi m^2}\right)^{1/2} \mathcal{F}^{-1}[\omega^{-2}]$$

Let us introduce the following theorem:

Theorem 2.

$$\mathcal{F}^{-1}\left[\frac{d}{d\omega}\hat{f}(\omega)\right] = -itf(t).$$

Proof.

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{d}{d\omega}\hat{f}(\omega)\right] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \frac{d}{d\omega}\hat{f}(\omega) \\ &= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} d\omega \exp(i\omega t) \frac{d}{d\omega} \int_{-\infty}^{\infty} dt' \exp(-i\omega t') f(t') \\ &= \left(\frac{-i}{2\pi}\right) \int_{-\infty}^{\infty} dt' t' f(t') \int_{-\infty}^{\infty} d\omega \exp[i\omega(t-t')] \\ &= -i \int_{-\infty}^{\infty} dt' t' f(t') \delta(t-t') \\ &= -itf(t).\end{aligned}$$

□

Now let us also introduce the transform of the sgn function.

Theorem 3.

$$\mathcal{F}[\text{sgn}(t)] = \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{2}{i\omega}\right).$$

Proof. We have the integral

$$\mathcal{F}[\text{sgn}(t)] = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \text{sgn}(t) \exp(-i\omega t) dt.$$

It is also known that

$$\frac{d}{dt} \operatorname{sgn}(t) = 2\delta(0).$$

We can use this fact to integrate by parts as follows:

$$\begin{aligned} \mathcal{F}[\operatorname{sgn}(t)] &= \left(\frac{1}{2\pi}\right)^{1/2} \left[\frac{i}{\omega} \operatorname{sgn}(t) \exp(-i\omega t) \Big|_{-\infty}^{\infty} - \frac{2i}{\omega} \int_{-\infty}^{\infty} \delta(0) \exp(-i\omega t) \, dt \right] \\ &= - \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{2i}{\omega}\right). \end{aligned}$$

□

We can combine these two theorems:

$$\begin{aligned} \mathcal{F}^{-1}[\omega^{-2}] &= \mathcal{F}^{-1} \left\{ \frac{d}{d\omega} \mathcal{F} \left[-(2\pi)^{1/2} \left(\frac{i}{2}\right) \operatorname{sgn}(t) \right] \right\} \\ &= i(2\pi)^{1/2} \left(\frac{i}{2}\right) t \operatorname{sgn}(t) \\ &= - (2\pi)^{1/2} \cdot \frac{t}{2} \operatorname{sgn}(t). \end{aligned}$$

Thus,

$$\begin{aligned} G(t) &= \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{m}\right) (2\pi)^{1/2} \left(\frac{t}{2}\right) \operatorname{sgn}(t) \\ &= \frac{|t|}{2m}. \end{aligned}$$

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