Problem 7

Some notation and conventions used:

$$\begin{aligned}\mathcal{F}[x(t)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt \\ &= \hat{x}(\omega) \\ \mathcal{F}^{-1}[\hat{x}(\omega)] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \hat{x}(\omega)\exp(i\omega t)\,dt \\ &= x(t)\end{aligned}$$

Theorem 1. *If*

$$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t')\,dt',$$

then

$$\hat{x}(\omega) = (2\pi)^{1/2}\hat{G}(\omega)\hat{F}(\omega).$$

Proof.

$$\begin{aligned}\hat{x}(\omega) &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} x(t)\exp(-i\omega t)\,dt \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} dt\,\exp\left[(-i\omega(t-t'))\right]G(t-t') \int_{-\infty}^{\infty} dt'\,\exp(-i\omega t')F(t') \\ &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} d(t-t')\,\exp\left[(-i\omega(t-t'))\right]G(t-t') \int_{-\infty}^{\infty} dt'\,\exp(-i\omega t')F(t') \\ &= \left(\frac{1}{2\pi}\right)^{1/2} (2\pi)^{1/2}\hat{G}(\omega)\cdot(2\pi)^{1/2}\hat{F}(\omega) \\ &= (2\pi)^{1/2}\hat{G}(\omega)\hat{F}(\omega).\end{aligned}$$

□

Let us calculate $\hat{F}(\omega)$. This is:

$$\begin{aligned}\mathcal{F}[m\ddot{x}(t)] &= \left(\frac{1}{2\pi}\right)^{1/2} m\int_{-\infty}^{\infty} \frac{d^2x}{dt^2}\exp(-i\omega t)\,dt \\ &= \left(\frac{1}{2\pi}\right)^{1/2} m\left[\dot{x}\exp(-i\omega t)\Big|_{-\infty}^{\infty} + i\omega x\exp(-i\omega t)\Big|_{-\infty}^{\infty} - \omega^2\int_{\infty}^{\infty} x\exp(-i\omega t)\,dt\right] \\ &= -\omega^2 m\hat{x}(\omega),\end{aligned}$$

where we have assumed that x and $\dot{x}$ tend towards 0 as t goes to $\pm\infty$.

This allows us to calculate $\hat{G}(\omega)$:

$$\begin{aligned}\hat{G}(\omega) &= \left(\frac{1}{2\pi}\right)^{1/2} \frac{\hat{x}(\omega)}{\hat{F}(\omega)} \\ &= -\left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{1}{\omega^2 m}\right).\end{aligned}$$

Thus,

$$G(t) = -\left(\frac{1}{2\pi m^2}\right)^{1/2} \mathcal{F}^{-1}\left[\omega^{-2}\right]$$

Let us introduce the following theorem:

Theorem 2.

$$\mathcal{F}^{-1}\left[\frac{d}{d\omega}\hat{f}(\omega)\right] = -itf(t).$$

Proof.

$$\begin{aligned}\mathcal{F}^{-1}\left[\frac{d}{d\omega}\hat{f}(\omega)\right] &= \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} d\omega\ \exp(i\omega t)\frac{d}{d\omega}\hat{f}(\omega) \\ &= \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} d\omega\ \exp(i\omega t)\frac{d}{d\omega}\int_{-\infty}^{\infty} dt'\ \exp(-i\omega t')f(t') \\ &= \left(\frac{-i}{2\pi}\right) \int_{-\infty}^{\infty} dt'\ t' f(t') \int_{-\infty}^{\infty} d\omega\ \exp\left[i\omega(t-t')\right] \\ &= -i\int_{-\infty}^{\infty} dt'\ t' f(t')\delta(t-t') \\ &= -itf(t).\end{aligned}$$

$\square$

Now let us also introduce the transform of the sgn function.

Theorem 3.

$$\mathcal{F}[\mathrm{sgn}(t)] = \left(\frac{1}{2\pi}\right)^{1/2} \left(\frac{2}{i\omega}\right).$$

Proof. We have the integral

$$\mathcal{F}[\mathrm{sgn}(t)] = \left(\frac{1}{2\pi}\right)^{1/2} \int_{-\infty}^{\infty} \mathrm{sgn}(t)\exp(-i\omega t)\ dt.$$

It is also known that
$$\frac{d}{dt}\mathrm{sgn}(t) = 2\delta(0).$$
We can use this fact to integrate by parts as follows:
$$\begin{aligned}\mathcal{F}[\mathrm{sgn}(t)] &= \left(\frac{1}{2\pi}\right)^{1/2}\left[\frac{i}{\omega}\mathrm{sgn}(t)\exp(-i\omega t)\Big|_{-\infty}^{\infty} - \frac{2i}{\omega}\int_{-\infty}^{\infty}\delta(0)\exp(-i\omega t)\ dt\right]\\ &= -\left(\frac{1}{2\pi}\right)^{1/2}\left(\frac{2i}{\omega}\right).\end{aligned}$$

$\square$

We can combine these two theorems:
$$\begin{aligned}\mathcal{F}^{-1}[\omega^{-2}] &= \mathcal{F}^{-1}\left\{\frac{d}{d\omega}\mathcal{F}\left[-(2\pi)^{1/2}\left(\frac{i}{2}\right)\mathrm{sgn}(t)\right]\right\}\\ &= i(2\pi)^{1/2}\left(\frac{i}{2}\right)t\ \mathrm{sgn}(t)\\ &= -(2\pi)^{1/2}\cdot\frac{t}{2}\ \mathrm{sgn}(t).\end{aligned}$$

Thus,
$$\begin{aligned}G(t) &= \left(\frac{1}{2\pi}\right)^{1/2}\left(\frac{1}{m}\right)(2\pi)^{1/2}\left(\frac{t}{2}\right)\mathrm{sgn}(t)\\ &= \frac{|t|}{2m}.\end{aligned}$$

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